Algebra - MATH1241

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- *Definitions, Propositions, Corrolary & Theorems are collected from the coursepack unless noted.*
- *All Solutions are written by me.*

Table of contents

- Algebra - MATH1241
  - Table of contents
  - Vector Spaces
    - Definitions and examples of vector spaces
    - Subspaces
    - Linear Combinations & Spans
    - Linear independence
    - Basis and dimension
    - Coordinate vectors
    - Further important examples of vector spaces
  - Linear Transformations
    - Introduction to linear maps
    - Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices
    - Subspaces associated with linear maps
    - Further applications and examples of linear maps
      - Laplace Transform
    - Representation of linear maps by matrices
    - Matrix arithmetic and linear maps
    - One-to-one, onto and invertible linear maps and matrices
    - Proof of the Rank-Nullity Theorem
    - One-to-one, onto and inverses for functions
  - Eigenvalues & Eigenvectors
    - Definitions & Examples
    - Eigenvectors, bases and diagonalisation
      - Solution of first-order linear differential equations
      - Markov chains
    - Applications of eigenvalues and eigenvectors
      - Power of a matrix
    - Eigenvalues & MAPLE
      - Finding eigenvalues and eigenvectors of a matrix
- Practice
  - Past Papers
    - 2019T2
    - 2018S2
    - 2017S2
    - 2016S2
    - 2015S2

    - $*$ 2014S2
    - $*$ 2013S2
    - $*$ 2012S2
    - $*$ 2011S2
    - $*$ 2010S2
  - Exercises

Vector Spaces

Definitions and examples of vector spaces

Definition. A **vector space** V over the field $\mathbb{F}$ is a non-empty set V of vectors on which addition of vectors is defined and multiplication by a scalar is defined in such a way that the following ten fundamental properties are satisfied:

1. **Closure under Addition**. If $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} \in V$.
2. Associative Law of Addition. If $\vec{u}, \vec{v}, \vec{w} \in V$, then $(\vec{u}+\vec{v})+\vec{w} = \vec{u}+(\vec{v}+\vec{w})$.
3. Commutative Law of Addition. If $\vec{u}, \vec{v} \in V$, then $\vec{u} + \vec{v} = \vec{v} + \vec{u}$.
4. **Existence of Zero**. There exists an element $0 \in V$ such that, for all $\vec{v} \in V, \vec{v} + 0 = \vec{v}$.
5. Existence of Negative. For each $\vec{v} \in V$ there exists an element $\vec{w} \in V$ (usually written as $-\vec{v}$), such that $\vec{v} + \vec{w} = 0$.
6. **Closure under Multiplication by a Scalar**. If $\vec{v} \in V$ and $\lambda \in F$, then $\lambda\vec{v} \in V$.
7. Associative Law of Multiplication by a Scalar. If $\lambda, \mu \in \mathbb{F}$ and $\vec{v} \in V$, then $\lambda(\mu\vec{v}) = (\lambda\mu)\vec{v}$.
8. If $\vec{v} \in V$ and $1 \in \mathbb{F}$ is the scalar one, then $1\vec{v} = \vec{v}$.
9. Scalar Distributive Law. If $\lambda, \mu \in \mathbb{F}$ and $\vec{v} \in V$, then $(\lambda + \mu)\vec{v} = \lambda\vec{v} + \mu\vec{v}$.
10. Vector Distributive Law. If $\lambda \in \mathbb{F}$ and $\vec{u}, \vec{v} \in V$, then $\lambda(\vec{u} + \vec{v}) = \lambda\vec{u} + \lambda\vec{v}$.

Proposition. In any vector space V , the following properties hold for addition.

1. ***Uniqueness of Zero***. There is one and only one zero vector.
2. ***Cancellation Property***. If $\vec{u}, \vec{v}, \vec{w} \in V$ satisfy $\vec{u}+\vec{v} = \vec{u}+\vec{w}$, then $\vec{v} = \vec{w}$.
3. ***Uniqueness of Negatives***. For all $\vec{v} \in V$, there exists only one $\vec{w} \in V$ such that $\vec{v} + \vec{w} = 0$.

Subspaces

Definition A subset S of a vector space V is called a ***subspace*** of $ $V if S is itself a vector space over the same field of scalars as V and under the same rules for addition and multiplication by scalars.

In addition if there is at least one vector in V which is not contained in S, the subspace S is called a proper subspace of V .

Theorem (Subspace Theorem). A subset S of a vector space V over a field $\mathbb{F}$, under the samerules for addition and multiplication by scalars, is a subspace of V if and only if

- i) The vector 0 in V also belongs to S.
- ii) S is closed under vector addition, and
- iii) S is closed under multiplication by scalars from F.

Linear Combinations & Spans

Definition. Let $S = \{\vec{v_1}, \cdots, \vec{v_n}\}$ be a finite set of vectors in a vector space V over a field $\mathbb{F}$. Then a ***linear combination*** of S is a sum of scalar multiples of the form

$$\lambda_1\vec{v_1} + \cdots + \lambda_n\vec{v_n} \text{ with } \lambda_1, \cdots, \lambda_n \in \mathbb{F}$$

Definition Let $S = \{\vec{v_1}, \cdots, \vec{v_n}\}$ be a finite set of vectors in a vector space V over a field $\mathbb{F}$. Then the ***span*** of the set S is the set of *all* linear combinations of S, that is,

$$span(S) = span(\vec{v_1}, \cdots, \vec{v_n}) = v \in V : v = \lambda_1\vec{v_1} + \cdots + \lambda_n v_n \text{ for some } \lambda_1, \cdots, \lambda_n \in \mathbb{F}.$$

Technique. How can we tell $\vec{u} \in \mathbb{S}$?

1. $\vec{u} \in \mathbb{S}$
2. $\vec{u}$ is linear combination of elements of $\mathbb{S}$
3. We can find scalar α_1, α_2 such that

$$\vec{u} = \alpha_1 \begin{pmatrix} 1 \\ 2 \\ 0 \end{pmatrix} + \alpha_2 \begin{pmatrix} 3 \\ -2 \\ -1 \end{pmatrix}$$

Definition A finite set S of vectors in a vector space V is called a ***spanning*** set for V if span $(S) = V$ or equivalently, if every vector in V can be expressed as a linear combination of vectors in S.

Definition The subspace of R^m spanned by the columns of an $m \times n$ matrix A is called the ***column space*** of A and is denoted by $col(A)$.

Linear independence

Definition. Suppose that $S = \{\vec{v_1}, \cdots, \vec{v_n}\}$ is a subset of a vector space. The set S is a ***linearly independent set*** if the only values of the scalars $\lambda_1, \lambda_2, ..., \lambda_n$ for which

$\lambda_1 v_1 + \cdots + \lambda_n\vec{v_n} = 0$ are $\lambda_1 = \lambda_2 = \cdots = \lambda_n = 0$.

Definition. Suppose that $S = \{\vec{v_1}, \cdots, \vec{v_n}\}$ is a subset of a vector space. The set $S = \{\vec{v_1}, ..., \vec{v_n}\}$ is a ***linearly dependent set*** if it is not a linearly independent set, that is, if there exist scalars $\lambda_1, ..., \lambda_n$, not all zero, such that

$$\lambda_1 v_1 + \cdots + \lambda_n \vec{v_n} = 0.$$

Basis and dimension

Definition. A set of vectors B in a vector space V is called a ***basis*** for V if:

1. B is a linearly independent set, and
2. B is a spanning set for V (that is, $span(B) = V$).

Theorem. The number of vectors in any spanning set for a vector space V is always greater than or equal to the number of vectors in any linearly independent set in V .

Theorem. If a vector space V has a finite basis then every set of basis vectors for V contains the same number of vectors, that is, if $B_1 = \{\vec{u_1}, \cdots, \vec{u_m}\}$ and $B_2 = \{\vec{v_1}, \cdots, \vec{v_n}\}$ are two bases for the same vector space V then $m = n$.

Definition. If V is a vector space with a finite basis, then the **dimension** of V , denoted by $dim(V)$, is the number of vectors in any basis for V . Such a V is called a **finite dimensional vector space**.

Theorem. Suppose that V is a finite dimensional vector space.

1. the number of vectors in any spanning set for V is greater than or equal to the dimension of V ;
2. the number of vectors in any linearly independent set in V is less than or equal to the dimension of V ;
3. if the number of vectors in a spanning set is equal to the dimension then the set is also a linearly independent set and hence a basis for V ;
4. if the number of vectors in a linearly independent set is equal to the dimension then the set is also a spanning set and hence a basis for V .

Coordinate vectors

Definition. Let V be an n-dimensional vector space and let the ordered set of vectors $B = \{\vec{v_1}, \cdots, \vec{v_n}\}$ be a basis for V. If then the vector $\vec{v} = x_1\vec{v_1} +$ ůůů $+ x_n\vec{v_n}$

then

$$[\vec{v}]_B = \vec{x} = \begin{pmatrix} x_1 \\ \cdots \\ x_n \end{pmatrix}$$

is called the **coordinate vector of $\vec{v}$ with respect to the ordered basis** B.

Theorem. If B is an ordered basis for a vector space V over a field $\mathbb{F}$ and $\vec{u}, \vec{v} \in V$ and $\lambda \in \mathbb{F}$, then

- (a) $\vec{u} = v$ if and only if $[\vec{u}]_B = [\vec{v}]_B$, that is, two vectors are equal if and only if the corresponding coordinate vectors are equal.

- (b) $[\vec{u}+\vec{v}]_B = [\vec{u}]_B + [\vec{v}]_B$, that is, the coordinate vector of the sum of two vectors is equal to the sum of the two corresponding coordinate vectors.
- (c) $[\lambda\vec{v}]_B = \lambda[\vec{v}]_B$, that is, the coordinate vector of a scalar multiple of a vector is equal to the same scalar multiple of the corresponding coordinate vector.

Further important examples of vector spaces

Theorem (Alternative Subspace Theorem). A subset S of a vector space V over a field $\mathbb{F}$ is a subspace of V if and only if S contains the zero vector and it satisfies the closure condition:

If $v_1, v_2 \in S$, then $\lambda_1 v_1 + \lambda_2 v_2 \in S$ for all $\lambda_1, \lambda_2 \in \mathbb{F}$.

Linear Transformations

Introduction to linear maps

Definition. Let V and W be two vector spaces over the same field $\mathbb{F}$. A function $T : V \to W$ is called a linear map or linear transformation if the following two conditions are satisfied.

Addition Condition. $T(\vec{v} + \vec{v}') = T(\vec{v}) + T(\vec{v}')$ for all $\vec{v}, \vec{v}' \in \vec{V}$, and

Scalar Multiplication Condition. $T(\lambda\vec{v}) = \lambda T(\vec{v})$ for all $\lambda \in \mathbb{F}$ and $\vec{v} \in V$.

Proposition. If $T : V \to W$ is a linear map, then

1. $T(\vec{0}) = \vec{0}$ and
2. $T(-\vec{v}) = -T(\vec{v})$ for all $\vec{v} \in V$.

Theorem. A function $T : V \to W$ is a linear map if and only if for all $\lambda_1, \lambda_2 \in \mathbb{F}$ and $\vec{v_1}, \vec{v_2} \in V$

$$T(\lambda_1\vec{v_1} + \lambda_2\vec{v_2}) = \lambda_1 T(\vec{v_1}) + \lambda_2 T(\vec{v_2})$$

.

Theorem. If T is a linear map with domain V and S is a set of vectors in V , then the function value of a linear combination of S is equal to the corresponding linear combination of the function values of S, that is, if $S = v1, \cdots, vn$ and $\lambda_1, \cdots, \lambda_n$ are scalars, then

$$T(\lambda_1 v1 + \cdots + \lambda_n vn) = \lambda_1 T(v1) + \cdots + \lambda_n T(vn).$$

Theorem. For a linear map $T : V \to W$, the function values for every vector in the domain are known if and only if the function values for a basis of the domain are known. Further, if $B = \{\vec{v_1}, \cdots, \vec{v_n}\}$ is a basis for the domain V then for all $v \in V$ we have $T(\vec{v}) = x_1 T(\vec{v_1}) + \cdots + x_n T(\vec{v_n})$, where $x_1, \cdots, x_n$

are the scalars in the unique linear combination $v = x_1\vec{v_1} + \cdots + x_n\vec{v_n}$ of the basis B.

Linear maps from $\mathbb{R}^n$ to $\mathbb{R}^m$ and $m \times n$ matrices

Theorem. For each $m \times n$ matrix A, the function $T_A : \mathbb{R}^n \to \mathbb{R}^m$, defined by

$$T_A(\vec{x}) = A\vec{x} \text{ for } \vec{x} \in \mathbb{R}^n,$$

is a linear map.

Theorem (Matrix Representation Theorem). Let $T : \mathbb{R}^n \to \mathbb{R}^m$ be a linear map and let the vectors ej for $1 \leq j \leq n$ be the standard basis vectors for $\mathbb{R}$. Then the $m \times n$ matrix A whose columns are given by

$$\vec{a}_j = T(\vec{e}_j) \text{ for } 1 \leq j \leq n$$

has the property that

$$T_A(\vec{x}) = A\vec{x} \text{ for } \vec{x} \in \mathbb{R}^n.$$

Subspaces associated with linear maps

Definition. Let $T : V \to W$ be a linear map. Then the ***kernel*** of T (written $ker(T)$) is the set of all zeroes of T , that is, it is the subset of the domain V defined by

$$ker(T) = \{v \in V : T(v) = 0\}$$

.

Definition. For an $m \times n$ matrix A, the ***kernel*** of A is the subset of $\mathbb{R}^n$ defined by

$$ker(A) = \{\vec{x} \in R^n : A\vec{x} = 0\}$$

, that is, it is the set of all solutions of the homogeneous equation $A\vec{x} = 0$.

Definition. The ***nullity*** of a linear map T is the dimension of $ker(T)$. The nullity of a matrix A is the dimension of $ker(A)$.

Another **Definition**. If A is an $m \times n$ matrix, then a vector $x \in \mathbb{R}^n$ is in the kernel of A precisely when $A\vec{x} = \vec{0} \in \mathbb{R}^n$, then another term for the ***kernel*** of A is the ***null space*** of A .

Definition. Let $T : V \to W$ be a linear map. Then the image of T is the set of all function values of T, that is, it is the subset of the codomain W defined by

$$im(T) = \{\vec{w} \in W : \vec{w} = T(\vec{v})\} \text{ for some } \vec{v} \in V$$

Definition. The image of an $m \times n$ matrix A is the subset of $\mathbb{R}^m$ defined by

$$im(A) = \{\vec{b} \in \mathbb{R}^m : \vec{b} = A\vec{x} \text{ for some } \vec{x} \in \mathbb{R}^n\}$$

Theorem. Let $T : V \to W$ be a linear map between vector spaces V and W. Then $im(T)$ is a subspace of the codomain W of T.

Definition. The ***rank*** of a linear map T is the dimension of $im(T)$. The rank of a matrix A is the dimension of $im(A)$.

Proposition. For a matrix A, $rank(A)$ = maximal number of linearly independent columns of A = number of leading columns in a row-echelon form U for A

Theorem (Rank-Nullity Theorem for Matrices).

For any matrix A, rank(A) + nullity(A) = number of columns of A.

Theorem. The equation $A\vec{x} = \vec{b}$ has:

1. no solution if $rank(A) \neq rank([A \mid \vec{b}])$, and
2. at least one solution if $rank(A) = rank([A \mid \vec{b}])$. Further,

- if $nullity(A) = 0$ the solution is unique, whereas,
- if $nullity(A) = \nu > 0$, then the general solution is of the form

$$\vec{x} = \vec{x}_p + \lambda_1 \vec{k_1} + \cdots + \lambda_\nu \vec{k_\nu} for \lambda_1, \cdots, \lambda_\nu \in \mathbb{R}$$

  , where $\vec{x}_p$ is any solution of $A\vec{x} = \vec{b}$, and where $\{\vec{k_1}, \cdots, \vec{k_\nu}\}$ is a basis for $ker(A)$.

Further applications and examples of linear maps

Laplace Transform

Representation of linear maps by matrices

Theorem (General Matrix Representation Theorem). Let $T : V \to W$ be a linear map from an n-dimensional vector space V to an m-dimensional vector space W , and let $B_V = \{\vec{v_1}, \cdots, \vec{v_n}\}$ be an ordered basis for V and $B_W = \{\vec{w_1}, \cdots, \vec{w_m}\}$ be an ordered basis for W. Then, there is a unique $m \times n$ matrix A such that

$$[T(\vec{v})]_{B_W} = A[\vec{v}]_{B_V}.$$

Further, A is the matrix whose columns are

$\vec{a_j} = [T(\vec{v_j})]_{B_W}$ for $1 \leq j \leq n$.

Algorithm. Constructing a matrix representation for a linear map.

1. Find a basis $B_V = \{\vec{v_1}, \cdots, \vec{v_n}\}$ for the domain V and a basis $B_W = \{\vec{w_1}, \cdots, \vec{w_m}\}$ for the codomain W.
2. Find the function values $T(\vec{v_j})$, $1 \leq j \leq n$, of the domain basis vectors.
3. Find the coordinate vectors $[T(\vec{v_j})]_{B_W}$ of the function values $T(\vec{v_j})$ with respect to the codomain basis B_W .

Matrix arithmetic and linear maps

Proposition (Addition). Let A and B be real $m \times n$ matrices and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ be the linear maps given by $T_A(\vec{x}) = A\vec{x}$ and $T_B(\vec{x}) = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Then the sum $T = T_A + T_B$ is the linear map $T : R \to R$ given by $T(\vec{x}) = (A + B)\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Proposition (Multiplication by a Scalar). Let A be a real $m \times n$ matrix and $T_A : \mathbb{R}^n \to \mathbb{R}^m$ be the linear map defined by $T_A(\vec{x}) = A\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Then the scalar multiple $T = \lambda T_A$ is the linear map $T : \mathbb{R}^n \to \mathbb{R}^m$ given by $T(\vec{x}) = (\lambda A)\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Proposition (Multiplication and Composition). Let A and B be real $m \times n$ matrices and let $T_A : \mathbb{R}^n \to \mathbb{R}^m$ and $T_B : \mathbb{R}^n \to \mathbb{R}^m$ be the linear maps given by $T_A(\vec{x}) = A\vec{x}$ and $T_B(\vec{x}) = B\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$. Then the composite $T = T_A \circ T_B$ is the linear map $T : R \to R$ given by $T(\vec{x}) = (AB)\vec{x}$ for all $\vec{x} \in \mathbb{R}^n$.

Proposition. Let U, V and W be finite-dimensional vector spaces with bases B_U, B_V and B_W respectively. Let $T : U \to V$ and $S : V \to W$. Then

- $S \circ T$ is a linear transformation from $U \to W$.
- If the matrix for T with respect to B_U and B_V is A_T , the matrix for S with respect to B_V and B_W is A_S and the matrix for $S \circ T$ with respect to B_U and B_W is A_{ST},then $A_{ST} = A_S A_T$.

One-to-one, onto and invertible linear maps and matrices

Proof of the Rank-Nullity Theorem

Theorem *(Rank-Nullity Theorem)*. If V is a finite dimensional vector space and $T : V \to W$ is linear then $rank(T) + nullity(T) = dim(V)$.

Proof

One-to-one, onto and inverses for functions

(See MATH1081 for this)

Definition. The ***range*** or image of a function is the set of all function values, that is, for a function $f : X \to Y$,

$$im(f) = \{y \in Y : y = f(x) \text{for some } x \in X\}.$$

Definition. A function is said to be ***onto*** (or surjective) if the codomain is equal to the image of the function, that is, a function $f : X \to Y$ is onto if for all $y \in Y$ there exists an $x \in X$ such that $y = f(x)$.

Definition. A function is said to be ***one-to-one*** (or injective) if no point in the codomain is the function value of more than one point in the domain, that is, a function $f : X \to Y$ is one-to-one if $f(x_1) = f(x_2)$ if and only if $x_1 = x_2$.

Eigenvalues & Eigenvectors

Definitions & Examples

Definition. Let $T : V \to V$ be a linear map. Then if a scalar λ and non-zero vector $\vec{v} \in V$ satisfy

$$T(\vec{v}) = \lambda\vec{v},$$

then λ is called an ***eigenvalue*** of T and $\vec{v}$ is called an ***eigenvector*** of T for the eigenvalue λ.

Theorem. A scalar λ is an eigenvalue of a square matrix A if and only if $det(A - \lambda I) = 0$, and then v is an eigenvector of A for the eigenvalue λ if and only if v is a non-zero solution of the homogeneous equation $(A - \lambda I)v = 0$, i.e., if and only if $v \in ker(A - \lambda I)$ and $v \neq 0$.

Theorem. If A is an $n \times n$ matrix and $\lambda \in C$, then $det(A - \lambda I)$ is a complex polynomial of degree n in λ.

Definition. For a square matrix A,the polynomial $p(\lambda) = det(A - \lambda I)$ is called the ***characteristic polynomial*** for the matrix A.

Theorem. An $n \times n$ matrix A has exactly n eigenvalues in $\mathbb{C}$ (counted according to their multiplicities). These eigenvalues are the zeroes of the characteristic polynomial $p(\lambda) = det(A - \lambda I)$.

Eigenvectors, bases and diagonalisation

Theorem. If an $n \times n$ matrix has n distinct eigenvalues then it has n linearly independent eigenvectors.

Theorem. If an $n \times n$ matrix A has n linearly independent eigenvectors, then there exists an invertible matrix M and a diagonal matrix D such that

$$M^{-1}AM = D.$$

Further, the diagonal elements of D are the eigenvalues of A and the columns of M are the eigenvectors of A, with the jth column of M being the eigenvector corresponding to the jth element of the diagonal of D. Conversely if $M^{-1}AM =$

D with D diagonal then the columns of M are n linearly independent eigenvectors of A.

Definition. A square matrix A is said to be a ***diagonalisable*** matrix if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$.

Definition (of Matrix function). If the $n \times n$ matrix A is diagonalizable and a function $f : \mathbb{R} \to \mathbb{R}$, then $f(A) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \times \mathbb{R}^n$ is defined by

$$f(A) = M \begin{pmatrix} f(\lambda_1) & \cdots & 0 \\ \cdots & \cdots & \cdots \\ 0 & \cdots & f(\lambda_n) \end{pmatrix} M^{-1}$$

where $\lambda_1, \cdots, \lambda_n$ are the diagonal entries of the diagonal matrix D and M is the invertible matrix (such that $A = MDM^{-1}$).

(From Wikipedia: https://en.wikipedia.org/wiki/Matrix_function and https://archive.siam.org/books/ot104/OT104HighamChapter1.pdf)

Solution of first-order linear differential equations

Proposition. $\mathbf{y}(t) = \mathbf{v}e^{\lambda t}$is a solution of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}$$

if and only if λ is an eigenvalue of A and $\mathbf{v}$ is eigenvectors of A of the eigenvalue of λ.

Proposition. If $\mathbf{u_1}(t)$ and $\mathbf{u_2}(t)$ is a solution of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}$$

then any linear combination of $\mathbf{u_1}$ and $\mathbf{u_2}$ is also a solution.

Markov chains

Dynamical systems are ones where the state of the system at stage $k+1$ depends solely on the state at stage k.

Lemma. If λ is an eigenvalue of A, then λ is also a eigenvalue of A^T .

Theorem. Suppose that A is $n \times n$ matrix and that the sum of each of the columns of A is 1. Then A has 1 as an eigenvalue.

Applications of eigenvalues and eigenvectors

Power of a matrix

Proposition. Let D be the diagonal matrix

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2 & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3 & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n \end{pmatrix}$$

Then for $k \geq 1$,

$$D^k = \begin{pmatrix} \lambda_1^k & 0 & 0 & \cdots & 0 \\ 0 & \lambda_2^k & 0 & \cdots & 0 \\ 0 & 0 & \lambda_3^k & \cdots & 0 \\ \cdots & \cdots & \cdots & \cdots & \cdots \\ 0 & 0 & 0 & \cdots & \lambda_n^k \end{pmatrix}.$$

Proposition. If A is diagonalisable, that is, if there exists an invertible matrix M and diagonal matrix D such that $M^{-1}AM = D$, then

$$A^k = MD^kM^{-1} \text{for integer} k \geq 1.$$

Example. We are told that

$$A = \begin{matrix} 7 & 2 \\ -4 & 1 \end{matrix}$$

has eigenvalues 5 and 3 with respective eigenvectors

$\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ *and* $\begin{pmatrix} 1 \\ -2 \end{pmatrix}$. So we can diagonalise$A$by finding an invertible matrix$M$and diagonal matrix$D$such that

$$M^{-1}AM = D$$

Eigenvalues & MAPLE

Finding eigenvalues and eigenvectors of a matrix

```
with(LinearAlgebra);
M:=<<6|2|2>,<-2|8|4>,<0|1|7>>;
I3:=IdentityMatrix(3);
```

```
p:=Determinant(M-t*I3);
solve(p,t);

NullSpace(M-6*I3);
NullSpace(M-7*I3);
NullSpace(M-8*I3);
```

Practice

Past Papers

2019T2

d) Using the Maple output below, or otherwise, answer the questions that follow.

```
> with(LinearAlgebra):
> N := <<4,0,0,1>|<0,1,-1,0>|<0,-1,1,0>|<1,0,0,4>>;
```

$$N := \begin{bmatrix} 4 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 0 & 0 & 4 \end{bmatrix}$$

```
> Eigenvalues(N);
```

$$\begin{bmatrix} 5 \\ 2 \\ 3 \\ 0 \end{bmatrix}$$

```
> Eigenvectors(N);
```

$$\begin{bmatrix} 5 \\ 2 \\ 3 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 & -1 & 0 \\ 0 & -1 & 0 & 1 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{bmatrix}$$

i) Write down a basis for $\ker N$.

ii) Write down a basis for the column space of N.

iii) State the rank and nullity of N.

iv) What are the eigenvalues of N^2?

d) Consider the linear transformation

$$F : M_{22} \to M_{22} \quad \text{where} \quad F(X) = X + aX^T .$$

Here a is a real constant, and X^T denotes the transpose of the matrix X. You **do not need to prove** that F is a linear transformation.

i) Show that $\ker(F) = \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}$ if and only if $a \neq \pm 1$.

ii) Prove that if $a \neq \pm 1$ then every matrix $B \in M_{22}$ can be written in the form $B = X + aX^T$ for some $X \in M_{22}$.

2018S2

1. i) Prove that

$$S = \{\mathbf{x} \in \mathbb{R}^3 : x_1^2 = x_2 x_3\}$$

is **not** a subspace of $\mathbb{R}^3$.

ii) If $M_{22}(\mathbb{R})$ is the set of 2×2 matrices with real entries, prove that the transpose operation $T : M_{22}(\mathbb{R}) \to M_{22}(\mathbb{R})$, which is given by,

$$T\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \begin{pmatrix} a & c \\ b & d \end{pmatrix},$$

is a linear map.

iv) Consider the matrix $M = \begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$.

a) Find a basis for $\ker(M)$.

b) Find a basis for $\mathrm{im}(M^T)$.

c) Give a geometric description of $\ker(M)$ and $\mathrm{im}(M)$ as subspaces of $\mathbb{R}^2$.

2017S2

iii) Let $\mathbf{v}_1 = \begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$, $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 1 \end{pmatrix}$, and $\mathbf{v}_3 = \begin{pmatrix} -2 \\ -1 \\ 1 \end{pmatrix}$.

a) Write $\mathbf{v}_3$ as a linear combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.

b) Does there exist a linear map $T : \mathbb{R}^3 \mapsto \mathbb{R}^3$ such that

$$T(\mathbf{v}_1) = \begin{pmatrix} 1 \\ 6 \\ 3 \end{pmatrix}, \quad T(\mathbf{v}_2) = \begin{pmatrix} -2 \\ 16 \\ 2 \end{pmatrix}, \quad \text{and} \quad T(\mathbf{v}_3) = \begin{pmatrix} -6 \\ -3 \\ -8 \end{pmatrix}?$$

c) What is the relationship between $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2)$ and $\operatorname{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$?

v) Read the following Maple output and use it to answer the questions below.

```
> with(LinearAlgebra):
> A := <<1,2,7,4,3>|<-1,6,2,8,1>|<2,-4,5,-4,2>|
  <2,3,-1,5,7>|<-1,14,11,20,5>>;
```

$$A := \begin{bmatrix} 1 & -1 & 2 & 2 & -1 \\ 2 & 6 & -4 & 3 & 14 \\ 7 & 2 & 5 & -1 & 11 \\ 4 & 8 & -4 & 5 & 20 \\ 3 & 1 & 2 & 7 & 5 \end{bmatrix}$$

```
> GaussianElimination(A);
```

$$\begin{bmatrix} 1 & -1 & 2 & 2 & -1 \\ 0 & 8 & -8 & -1 & 16 \\ 0 & 0 & 0 & -\frac{111}{8} & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

```
> b := <-2,35,16,49,15>:
```

$$b := \begin{bmatrix} -2 \\ 35 \\ 16 \\ 49 \\ 15 \end{bmatrix}$$

```
> LinearSolve(A,b);
```

$$\begin{bmatrix} 1 - \_t_3 - \_t_5 \\ 5 + \_t_3 - 2\,\_t_5 \\ \_t_3 \\ 1 \\ \_t_5 \end{bmatrix}$$

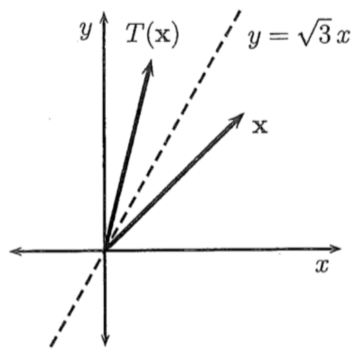
Let A be the matrix A defined in the Maple code above.

a) Give a basis for the kernel of the matrix A.

b) Find one vector in $\mathbf{x} \in \mathbb{R}^5$ such that

$$A\mathbf{x} = \begin{pmatrix} -2 \\ 35 \\ 16 \\ 49 \\ 15 \end{pmatrix}.$$

iv) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which reflects a vector in the line $y = \sqrt{3}\,x$ as show in the diagram.



a) Show that

$$T(\mathbf{e}_1) = \begin{pmatrix} -\dfrac{1}{2} \\ \dfrac{\sqrt{3}}{2} \end{pmatrix}, \quad T(\mathbf{e}_2) = \begin{pmatrix} \dfrac{\sqrt{3}}{2} \\ \dfrac{1}{2} \end{pmatrix}, \quad \text{where } \mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \ \mathbf{e}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

b) Find the matrix A such that

$$A\mathbf{x} = T(\mathbf{x}), \quad \text{for all } \mathbf{x} \in \mathbb{R}^2.$$

c) Find $T\left(\begin{pmatrix} 4 \\ 5 \end{pmatrix}\right)$.

3. i) Let $\mathbb{P}_2$ be the vector space of all real polynomials of degree at most 2.

a) Find three polynomials f_1, f_2, f_3 in $\mathbb{P}_2$ such that $f_i(0) = 1$ for $i = 1, 2, 3$ and $\{f_1, f_2, f_3\}$ is linearly independent.

b) Suppose that $P = \{p_1, p_2, p_3\}$ is a subset of $\mathbb{P}_2$ such that $p_i(1) = 0$ for $i = 1, 2, 3$. Show that P is a linearly dependent set.

ii) Let $A = \begin{pmatrix} 1 & 1 & 0 \\ 0 & 2 & 1 \\ 0 & 0 & 3 \end{pmatrix}$.

a) Given that the eigenvalues of A are $1, 2, 3$, explain why A is diagonalisable.

b) Find an eigenvector of A for the eigenvalue $\lambda = 3$.

c) Let $D = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 2 & 0 \\ 0 & 0 & 3 \end{pmatrix}$ and

$$f(x) = (x-1)(x-2)(x-3) = x^3 - 6x^2 + 11x - 6.$$

Show that $f(D) = D^3 - 6D^2 + 11D - 6I$ is the zero matrix $\mathbf{0}$.

d) Hence, prove that $f(A) = \mathbf{0}$.

e) Compute A^{-1} as a linear combination of A^2, A, I.

2016S2

1. i) Suppose that

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 + 2x_3 \geq 0 \right\}.$$

a) Prove that S is closed under addition.

b) Either prove that S is a subspace of $\mathbb{R}^3$ or explain why it is not a subspace of $\mathbb{R}^3$.

ii) Let $A = \begin{pmatrix} 1 & 3 & 1 & 1 \\ 0 & 1 & 1 & -1 \\ 2 & 5 & 1 & 3 \end{pmatrix}$ and $\mathbf{b} = \begin{pmatrix} 1 \\ 0 \\ 0 \\ -1 \end{pmatrix}$.

a) Is $\mathbf{b}$ in $\ker(A)$? Give reasons.

b) Find a basis for $\operatorname{im}(A)$.

iv) A function $T : \mathbb{R}^2 \to \mathbb{R}^3$ is defined by

$$T(\mathbf{x}) = \begin{pmatrix} x+y \\ 2x \\ -y \end{pmatrix}, \quad \text{for } \mathbf{x} = \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2.$$

a) Prove that T is linear.

b) Determine the nullity of T. Give reasons.

vi) Consider the following polynomials in the vector space of polynomials of degree 3 or less, $\mathbb{P}_3$.

$$\begin{aligned}
p_1(x) &= 1 + 2x + 3x^2 + x^3 \\
p_2(x) &= 1 + x + 3x^2 + x^3 \\
p_3(x) &= 1 + 2x + 4x^2 + x^3 \\
p_4(x) &= 1 - x + 3x^2 + 2x^3 \\
p_5(x) &= 2 + x + x^2 - 4x^3
\end{aligned}$$

Which of the following statements are true and which are false? Explain your answer.

a) The set $\{p_1, p_2, p_3\}$ is a basis for $\mathbb{P}_3$.

b) The set $\{p_1, p_2, p_3, p_4, p_5\}$ is a linearly independent set in $\mathbb{P}_3$.

3. i) Let

$$A = \begin{pmatrix} 9 & -1 \\ 4 & 5 \end{pmatrix} .$$

a) Find all eigenvalues and eigenvectors of A.

b) Explain carefully why A is not diagonalisable.

c) Find the matrix of the linear mapping

$$T : \mathbb{R}^2 \to \mathbb{R}^2 \quad \text{where} \quad T(\mathbf{x}) = A\mathbf{x}$$

with respect to the ordered basis $B = \{\mathbf{v}_1, \mathbf{v}_2\}$ in both domain and codomain, where

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \end{pmatrix} .$$

ii) Let V and W be vector spaces, let $T : V \to W$ be a linear transformation, and let $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ be vectors in V.

a) Prove, giving detailed reasons, that if $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_m)$ are linearly independent, then $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent.

b) State, giving reasons, whether the following statement is true or false: if $\mathbf{v}_1, \mathbf{v}_2, \ldots, \mathbf{v}_m$ are linearly independent, then $T(\mathbf{v}_1), T(\mathbf{v}_2), \ldots, T(\mathbf{v}_m)$ are linearly independent.

iii) The field $\mathbb{F} = \mathrm{GF}(4)$ has elements $\{0, 1, \alpha, \beta\}$, with addition and multiplication defined by the following tables.

$+$	0	1	α	β
0	0	1	α	β
1	1	0	β	α
α	α	β	0	1
β	β	α	1	0

$\times$	0	1	α	β
0	0	0	0	0
1	0	1	α	β
α	0	α	β	1
β	0	β	1	α

For the vectors

$$\mathbf{b}_1 = \begin{pmatrix} 1 \\ \alpha \\ \beta \end{pmatrix}, \quad \mathbf{b}_2 = \begin{pmatrix} \beta \\ 1 \\ 1 \end{pmatrix}, \quad \mathbf{b}_3 = \begin{pmatrix} 1 \\ 0 \\ \alpha \end{pmatrix} \quad \text{and} \quad \mathbf{v} = \begin{pmatrix} \alpha \\ 0 \\ 0 \end{pmatrix},$$

a) show that $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ is a basis for $\mathbb{F}^3$;

b) explain without calculation why $\{\mathbf{b}_1, \mathbf{b}_1 + \mathbf{b}_2, \mathbf{b}_2 + \mathbf{b}_3, \mathbf{b}_3\}$ is a spanning set but not a basis for $\mathbb{F}^3$;

c) find the coordinate vector of $\mathbf{v}$ with respect to the ordered basis $\{\mathbf{b}_1, \mathbf{b}_2, \mathbf{b}_3\}$ of $\mathbb{F}^3$.

2015S2

v) Prove that

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1 - 2x_2 + 4x_3 = 0 \right\}$$

is a subspace of $\mathbb{R}^3$.

vi) Consider the vectors in $\mathbb{R}^3$,

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 1 \\ 2 \\ 5 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 2 \\ -3 \\ 3 \end{pmatrix}, \ \mathbf{b} = \begin{pmatrix} -1 \\ 6 \\ 3 \end{pmatrix}.$$

Prove that $\mathbf{b}$ is in $\mathrm{span}(\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$.

vii) Using the following Maple output, answer the questions below. Give reasons.

```
> with(LinearAlgebra):
> A := <<1,1,3,1,2>|<1,2,3,1,3>|<1,-3,3,1,-2>|<2,4,3,-3,5>|
         <2,-8,12,12,-4>>;
```

$$A := \begin{bmatrix} 1 & 1 & 1 & 2 & 2 \\ 1 & 2 & -3 & 4 & -8 \\ 3 & 3 & 3 & 3 & 12 \\ 1 & 1 & 1 & -3 & 12 \\ 2 & 3 & -2 & 5 & -4 \end{bmatrix}$$

```
> LinearSolve(A,<0,0,0,0,0>);
```

$$\begin{bmatrix} -5\_t_3 - 12\_t_5 \\ 4\_t_3 + 6\_t_5 \\ \_t_3 \\ 2\_t_5 \\ \_t_5 \end{bmatrix}$$

```
> B := GaussianElimination(<A|IdentityMatrix(5)>);
```

$$B := \begin{bmatrix} 1 & 1 & 1 & 2 & 2 & 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & -4 & 2 & -10 & -1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -3 & 6 & -3 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 4 & 0 & -\frac{5}{3} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & -\frac{1}{3} & 0 & 1 \end{bmatrix}$$

a) Find a basis for $\ker(A)$.

b) Write down the value of $\operatorname{rank}(A)$.

c) Find a basis for $\mathbb{R}^5$ containing as many columns of A as possible.

v) Let V be a vector space and $T : V \to V$ be a linear map. Suppose that $S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ is a subset of V and $R = \{T(\mathbf{v}_1), T(\mathbf{v}_2), T(\mathbf{v}_3)\}$.

a) State what it means to say that "R is a linearly independent set".

b) Prove that if R is a linearly independent set then S is a linearly independent set.

ii) Consider the mapping $T : \mathbb{P}_2 \to \mathbb{P}_3$ defined by

$$(Tp)(x) = (x^2 + 1)p'(x) - 2xp(x).$$

a) Prove that T is a linear transformation.

b) Find the nullity of T.

c) Find the matrix of T with respect of the standard bases of $\mathbb{P}_2$ and $\mathbb{P}_3$.

iii) The matrix

$$A = \begin{pmatrix} 5 & 0 & -4 \\ 4 & 9 & 4 \\ -8 & -12 & -5 \end{pmatrix}$$

has characteristic polynomial $p(\lambda) = -\lambda^3 + 9\lambda^2 + 9\lambda - 81$.

a) Given that two of the eigenvalues of A are $\lambda_1 = -3$ and $\lambda_2 = 3$, find the remaining eigenvalue λ_3.

b) Find an eigenvector for λ_3.

c) Given that $\mathbf{v}_1 = \begin{pmatrix} 2 \\ -2 \\ 1 \end{pmatrix}$ and $\mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}$ are eigenvectors of A, write down a 3×3 matrix M and a diagonal matrix D such that $D = M^{-1}AM$.

iv) A linear transformation $P : V \to V$ is said to be **idempotent** if $P(P(\mathbf{v})) = P(\mathbf{v})$ for all $\mathbf{v} \in V$.

a) Show that the only possible eigenvalues for an idempotent linear transformation are 0 and 1.

b) Show that if P is idempotent and P is neither the zero nor the identity transformation on V, then both 0 and 1 are eigenvalues.

2014S2

iv) Let $S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 \ : \ z^2 = x^2 + y^2 \right\}$.

a) Prove that S is closed under scalar multiplication.
b) Prove that S is **not** a subspace of $\mathbb{R}^3$.

v) Let $A = \begin{pmatrix} 0 & -1 \\ 2 & 3 \end{pmatrix}$.

a) Find the eigenvalues and eigenvectors for the matrix A.
b) Write down an invertible matrix M and a diagonal matrix D such that

$$D = M^{-1}AM.$$

vi) Let

$$A = \begin{pmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{pmatrix}$$

Using the MAPLE output below, find a basis for ker(A).

```
> with(LinearAlgebra):
> A := <<1,2,3,1>|<2,4,-3,2>|<2,10,-3,1>|<-1,-44,24,6>>;
```

$$\begin{bmatrix} 1 & 2 & 2 & -1 \\ 2 & 4 & 10 & -44 \\ 3 & -3 & -3 & 24 \\ 1 & 2 & 1 & 6 \end{bmatrix}$$

```
> ReducedRowEchelonForm(A);
```

$$\begin{bmatrix} 1 & 0 & 0 & 5 \\ 0 & 1 & 0 & 4 \\ 0 & 0 & 1 & -7 \\ 0 & 0 & 0 & 0 \end{bmatrix}$$

iv) Consider the set S consisting of the vectors $\mathbf{v_1} = \begin{pmatrix} 1 \\ 1 \\ 2 \end{pmatrix}, \mathbf{v_2} = \begin{pmatrix} 2 \\ 3 \\ -1 \end{pmatrix}$ from $\mathbb{R}^3$ and let $\mathbf{u} = \begin{pmatrix} 1 \\ -1 \\ 12 \end{pmatrix}$.

a) Find scalars λ and μ such $\mathbf{u} = \lambda \mathbf{v_1} + \mu \mathbf{v_2}$.
b) A linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^3$ has $\mathbf{v_1}, \mathbf{v_2}$ as eigenvectors with eigenvalues 2 and -1, respectively.
   α) Find $T(\mathbf{u})$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$.
   β) Denote $T(T(\mathbf{u}))$ by $T^2(\mathbf{u})$, $T(T(T(\mathbf{u})))$ by $T^3(\mathbf{u})$, and so on. Express $T^n(\mathbf{u})$ as a linear combination of $\mathbf{v_1}, \mathbf{v_2}$, where n is a positive integer.

3. i) Prove that the function $T : \mathbb{P}(\mathbb{R}) \to \mathbb{R}^2$ defined by

$$T(p) = \begin{pmatrix} p(0) \\ p(1) \end{pmatrix}, \quad \text{for all polynomials } p \in \mathbb{P}(\mathbb{R}),$$

is a linear transformation.

ii) Given a square matrix A, the *matrix exponential* e^A is defined by

$$e^A = \sum_{k=0}^{\infty} \frac{1}{k!} A^k = I + A + \frac{1}{2!} A^2 + \frac{1}{3!} A^3 + \cdots.$$

(You may assume without proof that this series always converges.)

Let $N = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$.

a) Find N^2.

b) Hence or otherwise, calculate the matrix exponential e^N.

c) Prove that if a matrix P is idempotent (that is, $P^2 = P$), then $e^P = I + (e-1)P$.

iii) Let $\mathcal{R}[\mathbb{R}]$ denote the vector space of real-valued functions defined on $\mathbb{R}$. Let S be the subspace of $\mathcal{R}[\mathbb{R}]$ that is spanned by the **ordered** basis $\mathcal{B} = \{\cos(x), \sin(x)\}$. Define the linear map $T : S \to S$ by

$$T(f) = f - 2f' \qquad \text{where} \quad f' = \frac{df}{dx}.$$

a) Calculate the matrix C that represents T with respect to the basis $\mathcal{B}$.

b) State the rank of the matrix C found in part (a).

c) From part b), what can be deduced about the solutions $y \in S$ of

$$y - 2y' = g$$

where g is a given function in S?

Exercise *(November, 2014, question 1 iv)*

Let

$$S = \left\{ \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^3 : z^2 = x^2 + y^2 \right\}$$

- a) Prove that S is closed under scalar multiplication. Suppose $x \in S$ and λ is a scalar, then $x + y = z$. Consider the components of λx. We have

$$(\lambda x)^2 + (\lambda y)^2 = \lambda^2 (x^2 + y^2) = (\lambda z)^2$$

so $\lambda x \in S$.

- b) Prove that S is not a subspace of $\mathbb{R}^3$ (use counter example).

Exercise *(November, 2014, question 2, iv, a)*

2013S2

1. i) Let

$$S = \left\{ \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} \in \mathbb{R}^3 : x_1^3 + x_2^3 + x_3^3 = 0 \right\}.$$

a) Prove that S is closed under scalar multiplication.

b) Show that S is **not** a subspace of $\mathbb{R}^3$.

ii) Suppose that

$$A = \begin{pmatrix} -23 & 3 & -5 & 37 \\ -49 & 8 & -10 & 73 \\ -87 & 9 & -15 & 129 \\ -26 & 3 & -5 & 40 \end{pmatrix}.$$

Using the following Maple output, answer the questions below.

```
> with(LinearAlgebra):
> A := <<-23,-49,-87,-26>|<3,8,9,3>|<-5,-10,-15,-5>
    |<37,73,129,40>>;
```

$$A := \begin{bmatrix} -23 & 3 & -5 & 37 \\ -49 & 8 & -10 & 73 \\ -87 & 9 & -15 & 129 \\ -26 & 3 & -5 & 40 \end{bmatrix}$$

```
> Eigenvectors(A);
```

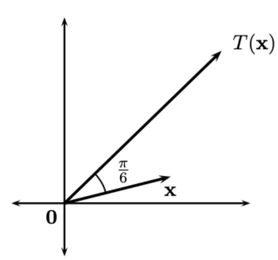
$$\begin{bmatrix} 2 \\ 3 \\ 5 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 & 2 & 1 & 1 \\ 1 & 5 & 2 & 2 \\ 3 & 0 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{bmatrix}$$

a) Without performing any further computations, explain why the matrix A has four linearly independent eigenvectors.

b) Write down the general solution of

$$\frac{d\mathbf{y}}{dt} = A\mathbf{y}.$$

iv) Let $T : \mathbb{R}^2 \to \mathbb{R}^2$ be the linear map which rotates a vector $\mathbf{x}$ about the origin through $\frac{\pi}{6}$ anticlockwise and doubles its length, as shown in the diagram.



a) Show that $T(\mathbf{e}_1) = \begin{pmatrix} \sqrt{3} \\ 1 \end{pmatrix}$, where $\mathbf{e}_1 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

b) Find the matrix A such that $T(\mathbf{x}) = A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^2$.

3. i) The polynomials $p_1,\ p_2,\ p_3,\ p_4$ and p_5 are defined, for all $x \in \mathbb{R}$, by

$$\begin{aligned} p_1(x) &= x^3 + 3x^2 + 2x - 1 \\ p_2(x) &= 2x^3 - 3x^2 + 4x + 2 \\ p_3(x) &= 11x^3 - 3x^2 + 22x + 5 \\ p_4(x) &= 2x^3 - 21x^2 + 4x + 3 \\ p_5(x) &= 9x^3 + 18x^2 + 18x + 2 \end{aligned}$$

Let $S = \{p_1,\ p_2,\ p_3,\ p_4,\ p_5\}$.

a) Explain why S cannot be a basis for $\mathbb{P}_3(\mathbb{R})$. Using the following Maple output, answer the questions below.

```
> with(LinearAlgebra):
> A := <<-1,2,3,1>|<2,4,-3,2>|<5,22,-3,11>|<3,4,-21,2>
    |<2,18,18,9>|<1,0,0,0>|<0,1,0,0>|<0,0,1,0>
    |<0,0,0,1>>;
```

$$A := \begin{bmatrix} -1 & 2 & 5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 2 & 4 & 22 & 4 & 18 & 0 & 1 & 0 & 0 \\ 3 & -3 & -3 & -21 & 18 & 0 & 0 & 1 & 0 \\ 1 & 2 & 11 & 2 & 9 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> GaussianElimination(A);
```

$$\begin{bmatrix} -1 & 2 & 5 & 3 & 2 & 1 & 0 & 0 & 0 \\ 0 & 8 & 32 & 10 & 22 & 2 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\frac{63}{4} & \frac{63}{4} & \frac{9}{4} & -\frac{3}{8} & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -\frac{1}{2} & 0 & 1 \end{bmatrix}$$

b) Find a basis for span(S).

c) Find a basis for $\mathbb{P}_3(\mathbb{R})$ which contains as many of the polynomials of S as possible.

ii) Let

$$A = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 5 & 3 \\ 1 & 3 & 4 & 9 \end{pmatrix}.$$

a) Find a basis for $\ker(A)$.

b) Hence state the value of nullity(A). Give a reason.

iii) Let

$$C = \begin{pmatrix} 1 & 2 \\ -2 & 5 \end{pmatrix}.$$

a) Find the eigenvalue(s) of C and for each eigenvalue find the corresponding eigenvectors.

b) Is C diagonalisable? Give a reason for your answer.

iv) A real $n \times n$ matrix A has n distinct eigenvalues $\lambda_1, \ldots, \lambda_n$ and can be expressed as $M^{-1}DM$, where M is an invertible matrix and D is the diagonal matrix with $\lambda_1, \ldots, \lambda_n$ as its diagonal entries.

a) Show that $A^k = M^{-1}D^kM$.

b) Explain why D^k is a diagonal matrix with diagonal entries $\lambda_1^k, \ldots, \lambda_n^k$.

c) Write the characteristic polynomial $p(x)$ of A as

$$p(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0\,.$$

By replacing the variable x by the matrix D, and a_0 with a_0I, where I is the $n \times n$ identity matrix, prove that $p(D) = \mathbf{0}$ where $\mathbf{0}$ is the $n \times n$ zero matrix.

d) Hence or otherwise, prove that $p(A) = \mathbf{0}$.

e) Use the result in (d) to show that the matrix $A = \begin{pmatrix} 2 & 1 \\ 4 & -1 \end{pmatrix}$ satisfies $A^2 = A + 6I$.

2012S2

1. i) Let $S = \{\mathbf{x} \in \mathbb{R}^2 : x_1x_2 \geq 0\}$.

a) Show that S is closed under multiplication by a scalar.

b) Show that S is a not a subspace.

ii) Consider the set $S = \{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ of vectors in $\mathbb{R}^4$ given by

$$S = \left\{ \begin{pmatrix} 1 \\ -2 \\ 1 \\ 2 \end{pmatrix}, \begin{pmatrix} 2 \\ -4 \\ 2 \\ 4 \end{pmatrix}, \begin{pmatrix} 1 \\ 0 \\ 5 \\ 0 \end{pmatrix}, \begin{pmatrix} 3 \\ -4 \\ 8 \\ 7 \end{pmatrix} \right\},$$

and let $\mathbf{b} = \begin{pmatrix} -2 \\ 1 \\ -10 \\ -7 \end{pmatrix}$.

a) Does $\mathbf{b}$ belong to span(S)? Give reasons.

b) Does S span $\mathbb{R}^4$? Give reasons.

c) Write down a basis for span(S).

iii) A mapping $T : \mathbb{R}^3 \to \mathbb{R}^2$ is defined by

$$T(\mathbf{x}) = \begin{pmatrix} x_1 - x_3 \\ 2x_1 + x_2 \end{pmatrix}.$$

a) Show that T is linear.

b) Write down a matrix A such that $T(\mathbf{x}) = A\mathbf{x}$.

iv) The linear map $T : \mathbb{R}^5 \to \mathbb{R}^5$ has matrix

$$A = \begin{pmatrix} 1 & 3 & -3 & 1 & 4 \\ -1 & 3 & -9 & 2 & -1 \\ 2 & -4 & 14 & 3 & -3 \\ -1 & -3 & 3 & -1 & -4 \\ 0 & 6 & -12 & 3 & 3 \end{pmatrix}.$$

Using the Maple below, or otherwise, answer the following questions, giving brief reasons.

a) Find $\operatorname{rank}(T)$.

b) Find $\operatorname{nullity}(T)$.

c) Find a basis for $\ker(T)$.

d) Write down conditions on $\mathbf{b} \in \mathbb{R}^5$ for $\mathbf{b}$ to be in $\operatorname{im}(T)$.

e) Find a basis for $\mathbb{R}^5$ containing as many columns of A as possible.

```
> with(LinearAlgebra):
> A:=<<1,-1,2,-1,0>|<3,3,-4,-3,6>|<-3,-9,14,3,-12>|
> <1,2,3,-1,3>|<4,-1,-3,-4,3>>:
> Id5:=IdentityMatrix(5):
> AI:=<A|Id5>;
```

$$AI := \begin{bmatrix} 1 & 3 & -3 & 1 & 4 & 1 & 0 & 0 & 0 & 0 \\ -1 & 3 & -9 & 2 & -1 & 0 & 1 & 0 & 0 & 0 \\ 2 & -4 & 14 & 3 & -3 & 0 & 0 & 1 & 0 & 0 \\ -1 & -3 & 3 & -1 & -4 & 0 & 0 & 0 & 1 & 0 \\ 0 & 6 & -12 & 3 & 3 & 0 & 0 & 0 & 0 & 1 \end{bmatrix}$$

```
> ReducedRowEchelonForm(AI);
```

$$\begin{bmatrix} 1 & 0 & 3 & 0 & 2 & 0 & 0 & \frac{1}{12} & -\frac{5}{6} & -\frac{13}{36} \\ 0 & 1 & -2 & 0 & 1 & 0 & 0 & -\frac{1}{12} & -\frac{1}{6} & \frac{1}{36} \\ 0 & 0 & 0 & 1 & -1 & 0 & 0 & \frac{1}{6} & \frac{1}{3} & \frac{5}{18} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -1 & -1 \end{bmatrix}$$

iv) Consider the following vectors in $\mathbb{R}^4$:

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix}, \quad \mathbf{v}_2 = \begin{pmatrix} 2 \\ 3 \\ 4 \\ 1 \end{pmatrix}, \quad \mathbf{v}_3 = \begin{pmatrix} 0 \\ 1 \\ -2 \\ 5 \end{pmatrix}, \quad \mathbf{v}_4 = \begin{pmatrix} 5 \\ 7 \\ 11 \\ 0 \end{pmatrix}, \quad \mathbf{a} = \begin{pmatrix} a \\ b \\ c \\ d \end{pmatrix}.$$

Let S be the set of vectors $\{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\}$.

a) Find a subset of S that is linearly independent, containing as many vectors from S as possible.

b) Give conditions on $\mathbf{a}$ such that $\mathbf{a} \in \operatorname{span}(S)$.

c) Find all ways of representing $\mathbf{a}$ as a linear combination of $\mathbf{v}_1$, $\mathbf{v}_2$, and $\mathbf{v}_4$.

d) Describe $\operatorname{span}(S)$ geometrically.

iv) Let M be a 3×3 matrix and M^T be its transpose. Assume that M is orthogonal, that is, $M^T M = I$. Let $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ be the three columns of M.

a) Explain why $\det M = \pm 1$.

b) Explain why $\mathbf{m}_1$, $\mathbf{m}_2$ and $\mathbf{m}_3$ are mutually perpendicular vectors of length 1.

c) Show that for any square matrix A,

$$(A\mathbf{v}) \cdot \mathbf{w} = \mathbf{v} \cdot (A^T \mathbf{w})$$

for all $\mathbf{v}, \mathbf{w} \in \mathbb{R}^3$, where $\cdot$ denotes the dot product.

d) Deduce that $|M\mathbf{v}| = |\mathbf{v}|$ for all $\mathbf{v} \in \mathbb{R}^3$.

e) Explain why M must have at least one real eigenvalue and at least one real eigenvector $\mathbf{v} \in \mathbb{R}^3$.

f) Show that if $\mathbf{u} \in \mathbb{R}^3$ is an eigenvector of M, then the corresponding eigenvalue is 1 or -1.

Exercise. *(November, 2012, question 2 iv)*

Consider the following vectors in $\mathbb{R}^4$:

$$\vec{v_1} = \begin{pmatrix} 1 \\ 2 \\ 1 \\ 3 \end{pmatrix},$$

Let S be the set of vectors $\{\vec{v_1}, \vec{v_2}, \vec{v_3}, \vec{v_4}\}$.

- a) Find a subset of S that is linearly independent, containing as many vectors from S as possible.
- b) Give conditions on a such that $a \in span(S)$.
- c) Find all ways of representing a as a linear combination of $\vec{v_1}$, $\vec{v_2}$, and $\vec{v_4}$.
- d) Describe $span(S)$ geometrically.

2011S2

1. i) Consider a set of vectors

$$S = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \mathbf{v}_4\} \subset \mathbb{R}^3$$

a) Can S be a spanning set for $\mathbb{R}^3$? Give a reason for your answer.

b) Will all such sets S be spanning sets? Give a reason for your answer.

c) Suppose S consists of the vectors

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 3 \\ 3 \\ 8 \end{pmatrix}, \mathbf{v}_3 = \begin{pmatrix} 1 \\ -1 \\ 2 \end{pmatrix}, \ \mathbf{v}_4 = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}.$$

Determine a subset of S that forms a basis for $\mathbb{R}^3$.

ii) Suppose A is a fixed matrix in $\mathbb{M}_{m,n}$. Apply the subspace theorem to show that

$$S = \{\mathbf{x} \in \mathbb{R}^n : A\mathbf{x} = \mathbf{0}\}$$

is a subspace of $\mathbb{R}^n$.

iii) A linear map $T : \mathbb{R}^m \to \mathbb{R}^n$ has rank k. State the value of the nullity of T.

4. i) Let $A = \begin{pmatrix} 22 & -100 \\ 5 & -23 \end{pmatrix}$.

a) Determine the eigenvalues and corresponding eigenvectors for the matrix A.

b) Write down matrices P and D such that $A = PDP^{-1}$.

c) Hence evaluate A^nP, for any positive integer n.

ii) A discrete random variable X has the probability distribution given by $p_k = \dfrac{c}{2^k}$ for $k = 0, 1, 2, 3, 4$, where c is a constant.

a) Find the value of the constant c.

b) Calculate $P(X = 2)$.

c) Calculate $P((X - 2)^2 < 4)$.

iii) Let V be the vector space $M_{22}(\mathbb{R})$ of all 2×2 real matrices.
Define the linear map $T : M_{22} \to M_{22}$ by $T(A) = A^T$, where A^T denotes the transpose of the matrix A.

a) Show that for any non-zero $n \times n$ matrix C, if $C^T = \lambda C$ then $\lambda = \pm 1$.

b) Find the matrix representation B for T with respect to the ordered basis $\{\mathbf{v_1}, \mathbf{v_2}, \mathbf{v_3}, \mathbf{v_4}\}$ given by

$$\left\{ \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \right\}.$$

c) State the rank of T.

d) Explain why the eigenvalues of T are ± 1.

e) Find bases for each of the eigenspaces of T.

Exercise. *(November, 2011, question 2 iii)*

Consider the vectors

$$\vec{v_1} = \begin{pmatrix} 1 \\ -1 \\ 3 \end{pmatrix}, \vec{v_2} = \begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}, \vec{v_3} = \begin{pmatrix} 7 \\ 2 \\ 3 \end{pmatrix}, \vec{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix}, \vec{u} = \begin{pmatrix} 5 \\ 6 \\ 0 \end{pmatrix}$$

- a) Does $\{\vec{v_1}, \vec{v_2}, \vec{v_3}\}$ span $\mathbb{R}^3$? If so, give reasons; if not, then find condition(s) on a, b, c such that $\vec{w}$ belongs to $span(\vec{v_1}, \vec{v_2}, \vec{v_3})$.

- b) Determine whether or not $\vec{u}$ is in $span(\vec{v_1}, \vec{v_2}, \vec{v_3})$.

2010S2

1. i) Show that the set

$$S = \left\{ \mathbf{x} = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \in \mathbb{R}^2 : x_1 x_2 \geq 0 \right\}$$

is not a vector subspace of the vector space $\mathbb{R}^2$.

ii) The vectors $\mathbf{v}_1$, $\mathbf{v}_2$, $\mathbf{v}_3$, $\mathbf{u}$ and $\mathbf{w}$ in $\mathbb{R}^3$ are defined by

$$\mathbf{v}_1 = \begin{pmatrix} 1 \\ 3 \\ -2 \end{pmatrix}, \ \mathbf{v}_2 = \begin{pmatrix} 2 \\ -5 \\ 7 \end{pmatrix}, \ \mathbf{v}_3 = \begin{pmatrix} 4 \\ -21 \\ 25 \end{pmatrix}, \ \mathbf{u} = \begin{pmatrix} 3 \\ 1 \\ 2 \end{pmatrix} \text{ and } \mathbf{w} = \begin{pmatrix} a \\ b \\ c \end{pmatrix},$$

with the set D given by $D = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3\}$ and the matrix A defined in the Maple session below.

You may use the following Maple session to assist in answering the questions below.

```
> with(LinearAlgebra):
> A:= <<1,3,-2>|<2,-5,7>|<4,-21,25>>;
```

$$A := \begin{bmatrix} 1 & 2 & 4 \\ 3 & -5 & -21 \\ -2 & 7 & 25 \end{bmatrix}$$

```
> M:=<A|<a,b,c>>;
```

$$M := \begin{bmatrix} 1 & 2 & 4 & a \\ 3 & -5 & -21 & b \\ -2 & 7 & 25 & c \end{bmatrix}$$

```
> GaussianElimination(M);
```

$$\begin{bmatrix} 1 & 2 & 4 & a \\ 0 & -11 & -33 & b-3a \\ 0 & 0 & 0 & b+c-a \end{bmatrix}$$

a) Is D a linearly independent set? Give reasons.

b) State the condition(s) for the vector $\mathbf{w}$ to belong to span(D).

c) Determine all possible real scalars α_1, α_2 and α_3 such that

$$\mathbf{u} = \alpha_1 \mathbf{v}_1 + \alpha_2 \mathbf{v}_2 + \alpha_3 \mathbf{v}_3.$$

d) Find a basis for the kernel of A.

e) State the rank of the matrix A.

f) Find a basis for the image of A.

iv) Let $A = \begin{pmatrix} 5 & -8 \\ 1 & -1 \end{pmatrix}$.

a) Determine the eigenvalues and corresponding eigenvectors for the matrix A.

b) Write down matrices P and D such that $A = PDP^{-1}$.

c) Hence evaluate $A^8 P$.

v) Suppose that the fixed vector $\mathbf{b}$ in $\mathbb{R}^3$ is given by

$$\mathbf{b} = \begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}.$$

You are given that the function $T : \mathbb{R}^3 \to \mathbb{R}^3$, defined by

$$T(\mathbf{x}) = (\mathbf{x} \cdot \mathbf{b})\mathbf{b} \quad \text{for } \mathbf{x} \in \mathbb{R}^3$$

is a linear map.
Find a matrix B which transforms the vector $\mathbf{x}$ into $T(\mathbf{x})$.

iv) A matrix A has eigenvectors

$$\mathbf{v}_1 = \begin{pmatrix} 2 \\ 1 \end{pmatrix} \quad \text{and} \quad \mathbf{v}_2 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}$$

with corresponding eigenvalues $\lambda_1 = 2$ and $\lambda_2 = -3$, respectively.

Determine $A\mathbf{b}$ for the vector $\mathbf{b} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}$.

vi) Suppose that the vectors $\mathbf{v}_j \in \mathbb{R}^n$, for $j = 1, 2, \ldots, n$, are mutually orthogonal, *i.e.* $\mathbf{v}_j \cdot \mathbf{v}_k = 0$ for $j \neq k$ and $j, k = 1, 2, \ldots, n$.
Let $\mathbf{x}$ be an arbitrary vector in $\mathbb{R}^n$.

a) Given that the set $G = \{\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3, \ldots, \mathbf{v}_n\}$ is a basis for $\mathbb{R}^n$, find an expression for $\mathbf{x}$ in terms of the vectors in the basis G.

b) Suppose that the linear map $T : \mathbb{R}^n \to \mathbb{R}$ has the property that

$$T(\mathbf{v}_j) = \gamma_j(\mathbf{v}_j \cdot \mathbf{v}_j) \quad \text{for } j = 1, 2, \ldots, n,$$

where the $\gamma_j \in \mathbb{R}$ are given constants.
Find an expression for $T(\mathbf{x})$ in terms of $\mathbf{x}$, the constants γ_j and the vectors $\mathbf{v}_j$.

3. i) Let $A = \begin{pmatrix} 5 & -8 \\ 1 & -1 \end{pmatrix}$.

a) Determine the eigenvalues and corresponding eigenvectors for the matrix A.

b) Write down matrices P and D such that $A = PDP^{-1}$.

c) Hence evaluate $A^8 P$.

iii) Let $\mathcal{R}[\mathbb{R}]$ denote the vector space of real-valued functions defined on $\mathbb{R}$. Let S be the subspace of $\mathcal{R}[\mathbb{R}]$ that is spanned by the ordered basis $B = (e^{-x}, \sin(x), \cos(x))$. Define the map $T : S \to S$ by

$$T(f) = f' - 2f \qquad \text{where} \quad f'(x) = \frac{df}{dx}.$$

a) Show that the map T is linear.
b) Calculate the matrix C which represents T with respect to the basis B.
c) What is the rank of the matrix C found in part (b)?
d) From part (c), what can be deduced about the solutions $y \in S$ of

$$y' - 2y = g$$

where g is a given function in S?
e) Use your results from parts (b) and (d) to find all non-zero solutions $y \in S$ satisfying

$$y' - 2y = e^{-x}.$$

Exercise. *(November 2010, q2, iv))*

Let $\mathbb{P}_3(\mathbb{R})$ denote the vector space of polynomials of degree 3 or less with real coefficients and let S be given by

$$S = \{p \in \mathbb{P}_3(R) : (x-3)p'(x) - 2p(x) = 0 \text{ for all } x \in \mathbb{R}\}.$$

Show that S is a subspace of $\mathbb{P}_3(\mathbb{R})$.

Exercises

(Problem 6.6.47)