Calculus - MATH1241

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- *Definitions, Propositions, Corrolary & Theorems are collected from the coursepack unless noted.*
- *All Solutions are written by me.*

Taylor Series

Taylor's polynomial

Definition. Suppose that f is n-times differentiable at x_0. Then the ***Taylor polynomial*** p_n of degree n for f about x_0 is given by

$$p_n(x) = f(x_0) - \frac{f^{(1)}(x_0)}{1!}(x - x_0) + \frac{f^2(x_0)}{2!}(x - x_0)^2 + \cdots + \frac{f^{(n)}(x_0)}{n!}(x - x_0)^n.$$

We also call p_n the n th Taylor polynomial for f about x_0.

Taylor's theorem

Sequences

A ***sequence*** is a real-valued function defined on (a subset of) the natural numbers.

Describing the limiting behaviour of sequences

Suppose that $\{a_n\}$ is a sequence. Our primary objective is to describe the behaviour of an as $n \to \infty$. There are two main types of behaviour. Either

- an approaches some finite number L, in which case we say that the sequence $\{a_n\}$ is ***convergent*** and write $lim_{n\to\infty} a_n = L$; or
- the sequence $\{a_n\}$ is not convergent, in which case we say that $\{a_n\}$ is ***divergent***.

Divergent sequences can be further classified according to the list below.

- If $a_n \to \infty$ as $n \to \infty$ (that is, an grows without bound) then we say that the sequence - $\{a_n\}$ ***diverges to infinity***.
- If $a_n \to -\infty$ as $n \to \infty$ then we say that the sequence $\{a_n\}$ ***diverges to negative infinity***.
- If $\{a_n\}$ has no limit as $n \to \infty$ but remains bounded then we say that $\{a_n\}$ ***boundedly divergent***.
- If $\{a_n\}$ exhibits none of the above behaviour then we say that $\{a_n\}$ ***unboundedly divergent***.

Techniques for calculating limits of sequences

Definition. A sequence $\{a_n\}_{\infty}^{n} = 0$ of real numbers is said to be

- ***increasing*** if $a_n < a_n + 1$ for each natural number n
- ***non-decreasing*** if $a_n \leq a_n + 1$ for each natural number n,
- ***decreasing*** if $a_n > a_n + 1$ for each natural number n, and
- ***non-increasing*** if $a_n \geq a_n + 1$ for each natural number n.

If any of these four properties holds then the sequence is said to be ***monotonic***.

Theorem. If $\{a_n\}_{\infty}^{n} = 0$ is a bounded monotonic sequence of real numbers then it is converges to some real number L.

Suprema and infima

Definition 4.3.18. Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers.

- We say that M is an ***upper bound*** for $\{a_n\}_{n=0}^{\infty}$ if an $\{a_n\} \leq M$ for every natural number n.
- We say that M is a ***lower bound*** for $\{a_n\}_{n=0}^{\infty}$ if an $\{a_n\} \geq M$for every natural number n.
- We say that K is the ***least upper bound*** for $\{a_n\}_{n=0}^{\infty}$ if K is an upper bound for $\{a_n\}_{n=0}^{\infty}$ and $K \leq M$ whenever M is an upper bound for $\{a_n\}_{n=0}^{\infty}$.
- We say that K is the ***greatest lower bound*** for $\{a_n\}_{n=0}^{\infty}$ if K is a lower bound for $\{a_n\}_{n=0}^{\infty}$ and $K \geq M$ whenever M is a lower bound for $\{a_n\}_{n=0}^{\infty}$.

Definition. Suppose that $\{a_n\}_{n=0}^{\infty}$ is a sequence of real numbers.

- If $\{a_n\}_{n=0}^{\infty}$ has a least upper bound M, then M is also called the ***supremum*** of $\{a_n\}_{n=0}^{\infty}$ and is denoted by

$$supanorsupan : n \geq 0.n \geq 0$$

- If $\{a_n\}_{n=0}^{\infty}$ has a greatest lower bound M, then M is also called the ***infimum*** of $\{a_n\}_{n=0}^{\infty}$ and is denoted by

$$infanorinfan : n \geq 0.n \geq 0$$

Infinite series

Definition. Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers. For each natural number n, let sn denote the nth partial sum given by

$$s_n = a_0 + a_1 + a_2 + \cdots + a_n = \sum_{k=0}^{n} a_k$$

- If the sequence $\{s_n\}_{n=0}^{\infty}$ of partial sums converges to a number L then we say that ak. k=0 In this case we also say that the series is summable.

- If the sequence $\{s_n\}_{n=0}^{\infty}$ of partial sums diverges then we say that the

Tests for series convergence

Lemma. Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of positive numbers and let s_n denote the partial sum given by

$$s_n = \sum_{k=0}^{n} a_k.$$

If $\{s_n\}_{n=0}^{\infty}$ is a bounded sequence then the infinite series $\sum_{k=0}^{n} a_k$ is convergent.

Theorem. (The kth term test for divergence.). If $a_k \nrightarrow 0$ as $k \to \infty$ then $\sum_{k=0}^{\infty} a_k$ diverges.

Theorem. If the series $\sum_{k=0}^{\infty} a_k$ converges then $a_k \to$ as $k \to \infty$.

The integral test

Theorem. **(The integral test)**. Suppose that $\sum a_k$ is an infinite series with positive terms. Suppose $f(x)$ is a positive integrable function decreasing on $[1, \infty)$ such that for each positive integer k, $f(k) = a_k$.

- If $\int_1^{\infty} f(x)dx$ converges then so does $\sum_{k=1}^{\infty} a_k$
- If $\int_1^{\infty} f(x)dx$ diverges then so does $\sum_{k=1}^{\infty} a_k$

The comparison test

Theorem. **(The comparison test)**. Suppose that $\{a_k\}_{k=0}^{\infty}$ and $\{b_k\}_{k=0}^{\infty}$ are two positive sequences such that $a_k \leq b_k$ for every natural number k.

- If $\sum_{k=0}^{\infty} b_k$ converges then $\sum_{k=0}^{\infty} a_k$ also converges.
- If $\sum_{k=0}^{\infty} b_k$ diverges then $\sum_{k=0}^{\infty} a_k$ also diverges.

Proposition. Suppose a_n, b_n are sequences with positive terms and suppose $\lim_{n\to\infty} \frac{a_n}{b_n}$ is finite ***and*** not zero, then $\sum^{\infty} a_n$ converges if and only if $\sum^{\infty} b_n$ converges.

Proposition. (Convergence and divergence of p-series). The series

$$\sum_{k=1}^{\infty} \frac{1}{k^p}$$

converges if $p > 1$ and diverges if $p \leq 1$.

The ratio test

Theorem. **(The ratio test)**. Suppose that $\sum a_k$ is an infinite series with positive terms and that

$$\lim_{k\to\infty} \frac{a_{k+1}}{a_k} = r$$

- If $r < 1$ then $\sum a_k$ converges.
- If $r > 1$ then $\sum a_k$ diverges.

Remark. The ratio test does not specify what happens if $r = 1$. In this case, the test is inconclusive; the series may converge or diverge.

Alternating series test

Definition. If $\{a_k\}_{k=0}^{\infty}$ is a sequence of positive real numbers, then the series

$$\sum_{k=0}^{\infty}(-1)^k a_k = a_0 - a_1 + a_2 - a_3 + a_4 - a_5 + a_6 - a_7 + a_8 - a_9 + \cdots$$

is called an ***alternating series***.

Theorem. **(Alternating series test)**. Suppose that $\{a_k\}_{k=0}^{\infty}$ is a sequence of real numbers satisfying the following properties:

- $a_k \geq 0$;
- $a_k \geq a_{k+1}$ for all k (that is, the sequence is nonincreasing); and
- $lim_{k\to\infty} a_k = 0$.

Then the alternating series $\sum_{k=0}^{\infty}(-1)^k a_k$ converges.

Corollary.

Absolute and conditional convergence

Definition.

- A series $\sum_{k=0}^{\infty} a_k$ is said to be ***absolutely convergent*** if the series $\sum_{k=0}^{\infty} \mid a_k \mid$ is convergent.
- A series $\sum_{k=0}^{\infty} a_k$ is said to be ***conditionally convergent*** if the series $\sum_{k=0}^{\infty} \mid a_k \mid$ diverges.
- A ***rearrangement*** of a series $\sum a_k$ is a series that has exactly the same terms but that is summed in a different order.

Taylor Series

Definition. Suppose that a function f has derivatives of all orders at a. Then the series f(a)+f'(a)(x-a)+ f"(a)(x-a)2 + f(k)(a)(x - a)k, is called the Taylor series for f about a. In the case when $a = 0$, the series is also called the ***Maclaurin series*** for f.

Definition. Suppose that I is an interval and that f has derivatives of all orders at some point a. We say that (a) the Taylor series for f about a converges on I if the series

Power Series

Definition. Suppose that $a_k{}_{k=0}^{\infty}$ is a sequence of real numbers and that $a \in \mathbb{R}$. A series of the form

$\sum_{k=0}^{\infty} a_k x^k$

is called a ***power series in powers*** of x. A series of the form

$\sum_{k=0}^{\infty} a_k (x-a)^k$

is called a ***power series in powers*** of $x - a$.

Manipulation of Power Series

Suppose that a power series $\sum_{n=0}^{\infty} a_k x^k$ converges in the interval $(-R, R)$, where R is its radius n=0 of convergence. Then one can define a function $f : (-R, R) \to \mathbb{R}$ by the formula

$$f(x) = \sum_{n=0}^{\infty} a_k x^k \text{ where } \mid x \mid< R$$

Thus the value of f at each point x is a convergent sum of real numbers. Sometimes it is possible to find a closed form for f, but other times we must approximate each value $f(x)$ by using partial sums.

Theorem.

Suppose that the functions $f : I \to \mathbb{R}$ and $g : I \to \mathbb{R}$ are defined by

$$f(x) = \sum_{n=0}^{\infty} a_k (x-a)^k \text{ and } f(x) = \sum_{n=0}^{\infty} b_k (x-a)^k,$$

where both power series converges on the interval I. Then wher $x \in I$,

- $(f + g)(x) = \sum_{k=0}^{\infty} (a_k + b_k)(x - a)^k$
- $(fg)(x) = \sum_{k=0}^{\infty} \left(\sum_{j=0}^{k} a_j b_{k-j} \right) (x - a)^k$

Theorem.

Remark.

Corollary.

Averages, arc length, speed and surface area

The average value of a function

Definition. Suppose that f is integrable on a closed interval $[a, b]$. Then the average value $\overline{f}$ of f on $[a, b]$ is defined by the formula

$$\overline{f} = \frac{1}{b-a} \int_a^b f(x)dx.$$

The arc length of a curve

Suppose that a curve $\mathcal{C}$ can be expressed in parametric form as

$$\mathcal{C} = \{(x(t), y(t)) \in \mathbb{R}^2 : a \leq t \leq b\},$$

where x and y are differentiable functions of t. Then its arc length l is given by the formula

$$l = \int_a^b \sqrt{[x'(t)]^2 + [y'(t)]^2} dt$$

Arc length for a polar curve

Suppose that a curve is described using polar coordinates by $r = f(\theta)$, $\theta_0 \leq \theta \leq \theta_1$.

The arc length l of a polar curve is given by

$$l = \int_a^b \sqrt{r^2 + (\frac{dr}{d\theta})^2} d\theta.$$

The speed of a moving particle

In summary, the speed $v(t)$ of a particle P at time t is given by

$$v(t) = \sqrt{[x'(t)]^2 + [y'(t)]^2}.$$

Surface area

Given a frustum of slant height s and radii r, R and the slant height of s, the surface area A of the 'curved surface' is given by

$$A = \pi(r + R)s.$$

Total surface area of a conical frustum is

$$A = \pi(r^2 + R^2 + (r+R)s).$$

Practice

Exams

2019T2

1. a) Evaluate the following integral

$$I = \int \frac{2x^3 + x^2 + 4}{(x^2+4)^2}\, dx.$$

b) Use appropriate tests to determine whether each of the following series converges or diverges

i) $\displaystyle\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

ii) $\displaystyle\sum_{n=2}^{\infty} (-1)^{n-1} \frac{1}{3^{1/n}}$

c) Find the general solution to the differential equation

$$\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 27x$$

3. a) In a biological experiment, bacteria and phages (organisms that eat bacteria) grow in a container together. Initially, there are b_0 bacteria and p_0 phages in the container, and after t hours, the number of bacteria is $b(t)$ and the number of phages is $p(t)$. The bacteria and phages each reproduce at a rate of 10% per hour. Each phage eats on average two bacteria per hour. Nothing eats the phages, and in the course of the experiment there are no further limitations on the growth of the populations.

i) Write down a pair of differential equations which express the growth of the populations and their interactions.

ii) Solve the equations to find expressions for p and b as functions of time t.

iii) Under what conditions on b_0 and p_0 do the phages eventually eat all the bacteria, and if so, when does this occur?

b) A surface is formed by rotating the curve $z = x^2 + 1$ around the z-axis.

i) Write down the Cartesian equation of the surface.

ii) Find the area of the part of the surface that lies between $z = 2$ and $z = 5$.

c) Suppose that y is a function of x satisfying the initial value problem

$$y'' + y' + xy^2 = 0 \quad \text{with} \quad y(0) = 1,\ y'(0) = 0\,.$$

You may assume that this initial value problem has a solution.

i) Write down $p_1(x)$, the Taylor polynomial of degree 1 for y about 0.

ii) Find $p_3(x)$, the Taylor polynomial of degree 3 for y about 0, and use it to estimate the smallest positive zero of y.

d) A sequence of real numbers $\{a_k\}_{k=1}^{\infty}$ is defined recursively by $a_1 = 0.55$ and

$$a_{k+1} = \frac{5}{2}a_k(1 - a_k), \qquad \text{for } k \geq 1.$$

The Maple session below is needed for the questions on the following page.

```
> f := x -> 5/2*x*(1-x);
```

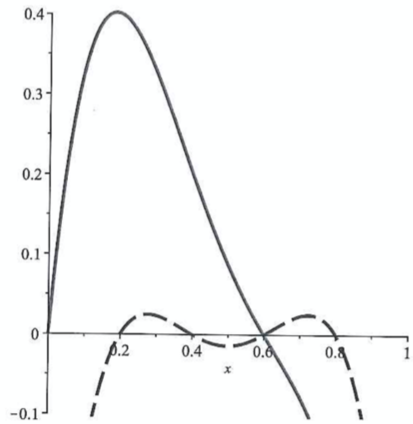
$$f := x \to \frac{5}{2}x(1-x) \tag{1}$$

```
> p := factor(f(f(x))-x);
```

$$p := -\frac{x(5x-3)(25x^2-35x+14)}{8} \tag{2}$$

```
> q := factor(f(f(x))-3/5);
```

$$q := -\frac{(5x-4)(5x-3)(5x-2)(5x-1)}{40} \tag{3}$$




```
> a[1] := 0.55;
```

$$a_1 := 0.55 \tag{4}$$

```
> for i to 7 do a[i+1] := f(a[i]) end do;
```

$$\begin{aligned} a_2 &:= 0.6187500000 \\ a_3 &:= 0.5897460938 \\ a_4 &:= 0.6048640964 \\ a_5 &:= 0.5975088032 \\ a_6 &:= 0.6012300832 \\ a_7 &:= 0.5993811756 \\ a_8 &:= 0.6003084548 \end{aligned} \tag{5}$$

Note that $0.4 < a_1 < 0.8$ and if $0.4 < a_k < 0.8$ then $0.4 < a_{k+1} < 0.8$. Hence by induction all terms in the sequence are between 0.4 and 0.8.

Use the results from Maple on the previous page to answer the following questions.

i) For $k \in \mathbb{Z}^+$, show that if $0.4 < a_k < 0.6$ then $a_{k+2} < 0.6$ and if $0.6 < a_k < 0.8$ then $a_{k+2} > 0.6$.

ii) Prove that the odd terms $(a_1, a_3, a_5, \ldots)$ are increasing and the even terms $(a_2, a_4, a_6, \ldots)$ are decreasing.

iii) Prove that a_k is convergent and find its limit.

2018S2

Question 2ii). Use appropriate tests to determine whether each of the following series converges or diverges

- $\sum_{n=2}^{\infty}(-1)^n \frac{n}{4n^2-3}$
- $\sum_{n=1}^{\infty} \frac{\sin^2(2n)}{n^2}$

Question 4iii). Determine the interval of convergence of the power series (including the endpoints):

$$\sum_{n=1}^{\infty} \frac{3^n}{n2^n}(x-3)^n$$

Question 4v).

Suppose that $\sum_{n=1}^{\infty} a_n$ is a convergent series with $a_n > 0$ for all n.

- Does $\sum_{n=1}^{\infty} ln(a_n)$ converge or diverge? Explain your answer.
- Does $\sum_{n=1}^{\infty} ln(1+a_n)$ converge or diverge? Explain your answer.

2017S2

Question 4ii).

Question 4iii).

Question 4iv)a). Consider the sequence $\{a_n\}$ given by

$$a_n = \frac{\cos n + n}{n^3 - e^{-n}}$$

Does the series $\sum_{n=1}^{\infty} a_n$ coverges? Give reasons for your answer.

Question 4iv)b). Determine the interval of convergence of the power series (including the endpoints):

$$\sum_{n=1}^{\infty} a_n(x+2)^n$$

You may use the fact that the sequence $\{a_n\}$ is monotonically decreasing.

2016S2

iv) Evaluate each of the following integrals.

a) $I_1 = \int \cos^3 x \ \sin^4 x \, dx$

b) $I_2 = \int \frac{3x^2 + 11x + 12}{(x+1)(x+3)^2} \, dx$

v) a) Find the general solution of the equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 0.$$

b) What form of solution should you try in order to find a particular solution of the equation,

$$\frac{d^2y}{dx^2} + 4\frac{dy}{dx} + 5y = 8 \sin 2x.$$

[Note that you are NOT asked to find the particular solution.]

vi) Determine the open interval of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{2^n}{(n^2 - n + 1)3^n} (x - 3)^n.$$

2. i) a) Determine whether the sequence

$$\sqrt{n + \sqrt{n}} - \sqrt{n}$$

converges or diverges as $n \to \infty$. If it converges, find its limit.

b) Does the series

$$\sum_{n=2}^{\infty} \sqrt{n + \sqrt{n}} - \sqrt{n}$$

converge? Give a reasons for your answer.

ii) By using an appropriate test, determine whether each of the following series converges or diverges.

a) $\sum_{n=1}^{\infty} \frac{n^2}{2^n}$

b) $\sum_{k=3}^{\infty} \frac{1}{k(\ln k)^2}$

iii) The area A of a rectangle with length x and width y is $A = xy$. Use the total differential approximation for A as a function of x and y to estimate the percentage increase in A when x increases by 5% and y decreases by 6%.

iv) Let X be a continuous random variable with probability density function

$$f(x) = \begin{cases} \dfrac{c}{x(1+x^2)} & \text{if } x \geq 1 \\ 0 & \text{if } x < 1, \end{cases}$$

where

$$c = \frac{2}{\ln 2}\ .$$

You are **not** required to prove that f is a probability density function.

a) Find the mean of X.

b) Show that X does not have a finite variance.

c) The median of X is defined to be the real number m such that $P(X \leq m) = \frac{1}{2}$. Show that

$$m > \tan\left(\frac{\pi + \ln 2}{4}\right)\ .$$

2015S2

1. i) Given that $z = x^2y^4 + 3y + 2$, find $\dfrac{\partial^2 z}{\partial y^2}$.

ii) Evaluate $\displaystyle\int \sin^2(x)\cos^3(x)\,dx$.

iii) Suppose that $z = a^2 + b^3 + c^4$ where

$$\begin{aligned} a &= u - v + w, \\ b &= u + v - w, \\ c &= uvw. \end{aligned}$$

Use the chain rule to find $\dfrac{\partial z}{\partial u}$ at the point $(u, v, w) = (1, 0, 1)$.

iv) A surface S in $\mathbb{R}^3$ has equation $z = 3xy^2$. The point $P(1, 2, 12)$ is a point on the surface S.

a) Find a normal vector to S at P.

b) Find a Cartesian equation for the tangent plane to S at P.

2. i) Find

$$\int \frac{2x^2+6x+12}{(x^2+4)(x-6)}\,dx.$$

ii) Solve the initial value problem

$$\frac{dN}{dt} + \frac{1}{1+t}N = \frac{2t}{1+t}, \qquad N(0) = 2.$$

iii) The intensity I of light at a depth of x metres below the surface of a lake is modelled by the differential equation

$$\frac{dI}{dx} = -2xI^2, \qquad \text{for } x \geq 0.$$

a) By referring to the differential equation explain why the light intensity is a decreasing function of depth.

b) Assuming an intensity of $I = 1$ at the surface, solve the differential equation to find a formula for I in terms of x.

c) No plant life is possible when the light intensity falls to 1% of its value at the surface. Find the depth at which this occurs.

4. i) Use the substitition $y(x) = 1/v(x)$ to transform

$$\frac{dy}{dx} + 2xy - e^x y^2 = 0$$

into a first order differential equation for $v(x)$. Identify the type of differential equation obtained for $v(x)$. [Do **not** solve the equation.]

ii) The charge, $Q(t)$, in a certain circuit satisfies the differential equation

$$Q'' + Q' - 6Q = 0 \quad \text{with} \quad Q(0) = 3.$$

For what value, if any, of $Q'(0)$, will the charge tend to 0 as t tends to infinity?

iii) Use appropriate tests to determine whether each of the following series converges or diverges.

a) $\displaystyle\sum_{n=1}^{\infty} n^2 e^{-n^3}$

b) $\displaystyle\sum_{n=1}^{\infty} \frac{n \ln n}{n^4 - 2n + 1}$

iv) Given that the power series

$$\sum_{n=1}^{\infty} a_n x^n$$

converges at $x = 4$ and *diverges* at $x = 5$, for which of the points $x = -3$, $x = -4$, $x = -5$ and $x = -6$ can you say for sure whether the series converges or diverges? Justify your answer(s).

v) Consider the curve given parametrically by

$$x(t) = 2t + 3, \quad y(t) = t^{3/2}, \quad 0 \leq t \leq 1$$

Find the area of the surface obtained when this curve is revolved around the y-axis.

vi) Let $\{a_n\}$ be a sequence of positive terms such that $\sqrt[n]{a_n} \to r < 1$ as $n \to \infty$.

a) Explain why this implies that there is a constant $R < 1$ and an integer N such that $a_n < R^n$ for all $n > N$.

b) Hence or otherwise prove that $\displaystyle\sum_{n=1}^{\infty} a_n$ converges.

2014S2

1. i) By expanding $\sin(A+B) + \sin(A-B)$, or otherwise, find $I_1 = \displaystyle\int \sin(5x)\cos(x)\,dx$.

ii) Evaluate the integral $I_2 = \displaystyle\int_0^1 \frac{x^2}{\sqrt{4-x^2}}\,dx$.

iii) Use appropriate tests to determine whether each of the following series converges or diverges

a) $\displaystyle\sum_{n=3}^{\infty} \frac{1}{n(\ln\ n)^3}$,

b) $\displaystyle\sum_{n=1}^{\infty} \frac{2}{2^n + 3^n}$.

2. i) Consider the initial value problem $\dfrac{dy}{dx} + (2 + \dfrac{1}{x})y = \dfrac{2}{x}$, with $y(1) = 0$, defined for $x > 0$.

a) Show that an integrating factor for this equation is xe^{2x}.

b) Hence solve the initial value problem.

ii) Find the general solution to $\dfrac{d^2y}{dx^2} + 2\dfrac{dy}{dx} + 2y = 20e^{2x}$.

iii) Consider the MAPLE session:

```
> a:=n->n^n/n!*(x-1)^n;
```

$$a := n \to \frac{n^n (x-1)^n}{n!}$$

```
> a(n+1);
```

$$\frac{(n+1)^{(n+1)}(x-1)^{(n+1)}}{(n+1)!}$$

```
> limit(a(n+1)/a(n),n=infinity);
```

$$ex - e$$

Using MAPLE session above, or otherwise, find the open interval of convergence $I = (a, b)$ for the power series

$$\sum_{n=1}^{\infty} \frac{n^n (x-1)^n}{n!}.$$

iii) Let $\mathcal{R}[\mathbb{R}]$ denote the vector space of real-valued functions defined on $\mathbb{R}$. Let S be the subspace of $\mathcal{R}[\mathbb{R}]$ that is spanned by the **ordered** basis $\mathcal{B} = \{\cos(x), \sin(x)\}$. Define the linear map $T : S \to S$ by

$$T(f) = f - 2f' \qquad \text{where} \quad f' = \frac{df}{dx}.$$

a) Calculate the matrix C that represents T with respect to the basis $\mathcal{B}$.

b) State the rank of the matrix C found in part (a).

c) From part b), what can be deduced about the solutions $y \in S$ of

$$y - 2y' = g$$

where g is a given function in S?

d) Using parts a) and c), or otherwise, find all non-zero solutions $y \in S$ satisfying

$$y - 2y' = \cos x\,.$$

A company finds that on average 0.05 of its customers return a particular product as faulty within the six year warranty period. Assuming that the number of returned products follows an exponential distribution, find, to 3 decimal places, the probability that for a given product the first claim occurs within the warranty period.

4. i) Show that the equation of the tangent plane to the paraboloid S given by

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{z}{c}$$

at the point $P(x_0, y_0, z_0)$ on S is

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = \frac{z + z_0}{2c}\,.$$

ii) Current observations of the Universe suggest that the reciprocal, u, of the matter density satisfies a differential equation that can be written as

$$\left(\frac{du}{dt}\right)^2 = k^2(2u + \epsilon u^2),$$

where k is a positive constant, $\epsilon = \pm 1$ and t is time. Which of the two values ϵ takes is still a matter of debate, but u is never negative. The radius of the universe is a positive fractional power of u, so if $u \to 0$ the universe is crushed to a point.
Take time $t = 0$ to be the present and assume that $u(0) = 1$. Measurements tell us that $\frac{du}{dt}$ is currently positive.

a) Solve the differential equation in the $\epsilon = 1$ case and show that in this case there is no value of $t > 0$ for which $u = 0$. (In this case the universe grows without bound.

b) Repeat the previous part with $\epsilon = -1$ and show that in this case there **is** a $t > 0$ for which $u = 0$. (In this case the universe is said to undergo a Big Crunch).

iii) The following MAPLE session may assist you with this question.

```
> x:=-8/3*t^3 + 12*t^2 +2:
> y:=(t+1)^2*(t-3)^2:
> factor( diff(x,t)^2 + diff(y,t)^2 );
```

$$16(t-3)^2(t^2+1)^2$$

Let C be the curve given parametrically by

$$\left(-\frac{8}{3}t^3 + 12t^2 + 2, (t+1)^2(t-3)^2\right).$$

a) A parametric curve $\big(x(t), y(t)\big)$ has vertical tangent when $\dfrac{dx}{dt} = 0$ and $\dfrac{dy}{dt} \neq 0$. Show that the curve C has exactly one vertical tangent.

b) Find the arc length of the curve C from $t = 0$ to $t = 3$.

iv) Suppose that y satisfies the initial value problem

$$\frac{dy}{dx} + y^2 = \cos(x) \quad \text{with} \quad y(0) = 0\,.$$

Using implicit differentiation, or otherwise, find the first two non-zero terms of the Maclaurin series of y.

v) Let c be a positive real number and define the sequence $\{a_n\}$ for $n \geq 1$ by

$$a_n = n(c^{1/n} - 1).$$

Find, with reasons, $\lim\limits_{n\to\infty} a_n$.

2013S2

4. i) Using the substitition $y(x) = xv(x)$, solve the differential equation

$$x^2\frac{dy}{dx} = 4x^2 + xy + y^2.$$

ii) Find the interval of convergence of the power series, (making sure to check the end points),

$$\sum_{n=1}^{\infty} \frac{2^n n^2}{n^3+1}(x-1)^n.$$

iii) A certain chemical reaction creates a substance S from two other chemicals A and B. Let X be the amount in grams of the substance S at a given time t. It can be shown that

$$\frac{dX}{dt} = k(k_1 - X)(k_2 - X),$$

where the constant k depends only on the reaction conditions, while the constants k_1 and k_2 depend on the amount of the original chemicals A and B present at the start of the reaction.

a) The reaction is run with initial conditions set so that $X(0) = 0$ and $k_1 = k_2 = 2\,\mathrm{g}$. If there is 1 gram of S present after 1 hour, show it will take a total of 9 hours to create 1.8 g of S.

b) Chemical A is a lot cheaper than chemical B, so the reaction is re-run under the same conditions, that is, with the same value of k, but with an excess of chemical A available. If we take $X(0) = 0$, $k_1 = 20\,\mathrm{g}$ and $k_2 = 2\,\mathrm{g}$, how long will it now take to create 1.8 g of S? (Give your answer correct to 2 decimal places.)

iv) Let $f : [a, b] \to [a, b]$ be a continuous function such that $f(x) \geq x$ for all $x \in [a, b]$. Let $c_1 \in [a, b)$ and define a sequence $\{c_k\}_{k=1}^{\infty} \subset [a, b]$ by

$$c_{k+1} = f(c_k),$$

for $k \geq 1$.

a) Prove that the sequence is convergent.

b) Let $L = \lim_{n\to\infty} c_n$. Prove that $f(L) = L$.

iv) Evaluate each of the following integrals:

a) $I_1 = \displaystyle\int \frac{5x^2 - 6x + 4}{x(x-1)(x-2)}\,dx.$

b) $I_2 = \displaystyle\int_0^{\pi/2} \sin^3\theta \cos^2\theta \, d\theta.$

v) Solve the initial value problem

$$\frac{dy}{dx} = \frac{\sin x}{y^2}, \quad y(0) = 1.$$

vi) Let $f(x) = \ln(1+4x^2)$. The following MAPLE session may assist you with this question.

```
> f := x -> log(4*x^2+1):
> taylor(f(x), x = 0, 6);
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$$4x^2 - 8x^4 + O(x^6)$$

Using the MAPLE output above or otherwise, write down the values of $f''(0)$ and $f''''(0)$.

2. i) Suppose that f is a differentiable function of one variable, and $F(x, y)$ is defined by $F(x, y) = f(2x - 3y^2)$.

a) Show that F satisfies the partial differential equation

$$3y\frac{\partial F}{\partial x} + \frac{\partial F}{\partial y} = 0.$$

b) Given that $F(x, 0) = \cos(x)$ for all x, write down a formula for $F(x, y)$.

ii) The curve C is given parametrically by $x = t^3, y = 2t^2$. Find the arc length of the curve C between $t = 0$ and $t = 1$.

iii) a) Find the general solution $y(x)$ for the following ordinary differential equation:

$$y'' - 5y' + 6y = 0.$$

b) Find the general solution of the following equation:

$$y'' - 5y' + 6y = e^{2x}.$$

2012S2

2. i) For any given differentiable function $f : \mathbb{R} \to \mathbb{R}$, let u be defined by

$$u(x, y) = f(s), \quad \text{where } s = \frac{x}{y}.$$

Determine the partial derivatives $\frac{\partial u}{\partial x}$ and $\frac{\partial u}{\partial y}$.

ii) Consider the function $u(x, y) = \ln(x^2 + y^2)$. Show that u satisfies Laplace's equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0.$$

iii) a) Show that the area A of the planar region between the lines $x = -\frac{a}{2}$ and $x = \frac{a}{2}$ bounded by the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1,$$

where $a, b > 0$, has the form Kab, and find the value of the constant K.

b) If the ellipse is deformed in such a way that a is increased by 4% and b is decreased by 1%, use the total differential approximation to show that the approximate relative change of area $\frac{\Delta A}{A}$ is independent of a and b and calculate its value.

4. i) a) Write down the general solution to the second order differential equation $\frac{d^2y}{dx^2} + 6\frac{dy}{dx} + 9y = 0$, $y(0) = 0$.

b) For what value of x does the solution have a maximum?

ii) a) Consider the solution of the differential equation

$$\frac{dy}{dx} = \frac{xy}{x^2 - a^2}, \qquad a > 0 \tag{7}$$

which passes through the point $(x_0, y_0) = (2a, a)$.
Find the equation of the normal to this solution at (x_0, y_0).

b) Obtain the **general** solution of the differential equation (7) and interpret it geometrically for $|x| < a$ and $|x| > a$.

iii) a) Sketch the graph of the function

$$f(x) = \frac{x}{x^2 - 99}$$

for $x \geq 0$.

b) Carefully investigate the (conditional) convergence of the series

$$\sum_{n=0}^{\infty} \frac{(-1)^n n}{n^2 - 99},$$

justifying the steps in your proof.

iv) a) State the Maclaurin series of $f(t) = 1 - \cos t$.

b) Let $g(t) = \begin{cases} \frac{1-\cos t}{t} & t \neq 0 \\ 0 & t = 0. \end{cases}$

Find the Maclaurin series of the function

$$\operatorname{Cin}(x) = \int_0^x g(t)dt,$$

and state where the series converges.

v) Consider the sequence defined by $a_1 = \frac{1}{4}$, and, for $n \geq 1$,

$$a_{n+1} = 2a_n(1 - a_n).$$

a) Show that the sequence is bounded above by $\frac{1}{2}$.

b) Show that the sequence is strictly increasing.

c) Explain why the sequence is convergent.

d) Find $\lim_{n\to\infty} a_n$.

2011S2

2. i) Let $f : \mathbb{R}^2 \to \mathbb{R}$ and let (x_0, y_0) be a point in $\mathbb{R}^2$.

a) Carefully write down the definition of the derivative $f_y(x_0, y_0)$.

b) If $f(x, y) := x^{1/3}y^{2/3}$ find the value $f_y(0, 0)$.

ii) In a direct-current circuit, the total resistance z, (measured in ohms), produced by two parallel resistors with resistances x and y ohms, is given by

$$\frac{1}{z} = \frac{1}{x} + \frac{1}{y}.$$

The values of x and y are measured to be 6 ohms and 12 ohms respectively and each of the measurements is made with an error whose absolute value is at most 0.1 ohms.

a) Use the given measurements (ignoring the measurement error) to calculate the total resistance z.

b) Show that $\frac{\partial z}{\partial x} = \frac{z^2}{x^2}$ and $\frac{\partial z}{\partial y} = \frac{z^2}{y^2}$.

c) Use the total differential approximation to estimate the maximum absolute error in the calculated value of the total resistance z, correct to three significant figures.

3. i) Consider the ordinary differential equation (ODE)

$$\frac{dy}{dx} = \frac{xy - y^2}{x^2}.$$

a) Determine the general solution $y = y(x)$ to the above ODE.

b) Find the particular solution that passes through the point $(1, 1)$.

ii) a) Determine whether the following series converges or diverges.

$$\sum_{k=2}^{\infty} \frac{\ln(k!)}{k^3}.$$

b) Does the following series converge absolutely or conditionally? Give reasons for your answers.

$$\sum_{k=2}^{\infty} \frac{(-1)^k \log k}{\sqrt{k^2+1}};$$

iii) Consider the ODE

$$\frac{dy}{dx} = e^{x^2} y, \qquad y(0) = 1.$$

a) Write down the Maclaurin series for e^{x^2}.

b) Hence, or otherwise, find a solution (involving a series) to the above initial value problem.

c) Carefully determine the values of x for which the solution is defined.

iv) A sequence of positive numbers $\{a_k\}_{k=1}^{\infty}$ is defined recursively by

$$a_{k+1} = \sqrt{1+a_k}, \qquad a_1 = 1.$$

a) Prove that a_k is strictly increasing.

b) Prove that a_k is bounded above by 3.

c) Hence, explain why there is a number L such that

$$\lim_{k\to\infty} a_k = L.$$

d) Determine the value of L.

2010S2

2. i) A surface with height $z = F(x, y)$ is defined by

$$z = 1 + \frac{25}{x^2+y^2} \quad \text{for } 1 \le x^2 + y^2 \le 25.$$

Find the equation of the plane tangent to the surface at the point $(1, 2, 6)$ on the surface.

ii) A function $u(x, t)$ is defined implicitly by the equation

$$u(x,t) = f(x + tu(x,t)),$$

where f is a differentiable function of one variable. Show that $u(x, t)$ is a solution of the partial differential equation

$$\frac{\partial u}{\partial t} = u\frac{\partial u}{\partial x}.$$

iii) Find the integral

$$I = \int \frac{6x^2 - 11x + 9}{(x-1)(x^2-2x+2)}\, dx.$$

4. i) Find the general solution of the ordinary differential equation

$$(x^2 - xy)\frac{dy}{dx} - 2y^2 + 5xy = 3x^2.$$

ii) Suppose that $a_k = \dfrac{k}{\ln(k!)}$ for $k = 2, 3, 4, \ldots$. In the questions below you may use the inequality

$$\ln(k!) - \ln(k) + k - 1 < k\ln(k) < \ln(k!) + k - 1 \quad \text{for } k = 2, 3, 4, \ldots.$$

a) Find $\lim\limits_{k\to\infty} a_k$, clearly stating your reasons.

b) Does the series $\sum_{k=2}^{\infty} a_k$ converge? Give reasons for your answer.

iii) Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n}} (x-1)^n.$$

iv) a) Write down the first three non-zero terms in the Taylor series of $\sinh x$ about $x = 0$, including an expression for the Lagrange form of the remainder.

b) Hence, or otherwise, write down the first three non-zero terms in the Taylor series, about $x = 0$, of the function

$$\mathrm{Shi}(x) = \int_0^x \frac{\sinh t}{t}\, dt.$$

v) A curve C in the xy-plane is defined parametrically by

$$x(t) = a\cos t, \quad y(t) = b\sin t, \quad \text{for } 0 \leq t \leq \pi, \text{ where } 0 < a < b.$$

Find the surface area of the ellipsoid defined by rotating the curve C about the x-axis.

vi) A tank has a total volume of 200 litres. Initially it holds 40 litres of pure water. Brine containing 2 grams of salt per litre is run into the tank at the rate of 3 litres per minute and the mixture is stirred continuously so that the concentration of the dissolved salt is uniform throughout the tank. At the same time as the brine starts to flow into the tank the mixture is removed from the tank at the rate of 1 litre per minute.
Let t denote time, measured in minutes, from when the brine started to enter the tank and let $x(t)$ denote the mass of salt, in grams, present in the tank after t minutes.

Set up a first order differential equation in x and t which models this system up until the time the tank is full.

Question 4ii).

Suppose that $a_k = \frac{k}{ln(k!)}$ for $k = 2, 3, 4, \cdots$ In the questions below you may use the inequality

$$ln(k!) - ln(k) + k - 1 < kln(k) < ln(k!) + k - 1 \text{ for } k = 2, 3, 4, ...$$

- Find $lim_{k\to\infty} a_k$, clearly stating your reasons.
- Does the series $\sum_{k=2}^{\infty} a_k$ coverge? Give reasons for your answer.

Question 4iii). Determine the radius of convergence of the power series

$$\sum_{n=1}^{\infty} \frac{1}{2^n \sqrt{n}} (x-1)^n$$

Question 4iv).

- Write down the first three non-zero terms in the Taylor series of $sinhx$ about $x = 0$, including an expression for the Lagrange form of the remainder.
- Hence, or otherwise, write down the first three non-zero terms in the Taylor series, about $x = 0$, of the function

$$Shi(x) = \int_0^x \frac{sinht}{t} dt$$