TAMING THE INFINITE SNAKE

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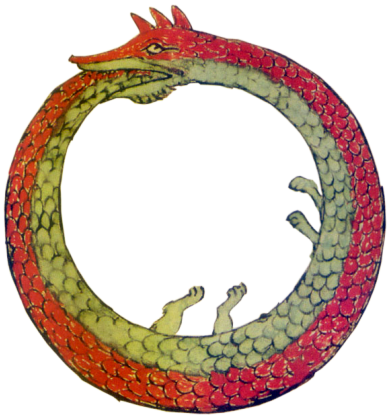

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1. Introduction

The 'infinite snake' is a non-holonomic planar mechanical system: an infinite chain of rigid segments of unit length, with kinematics constrained by the requirement that the instantaneous velocity of a segment's midpoint be parallel to the segment itself.

Geometrically, these constraints define a non-integrable vector distribution $D \subset T\mathscr{M}$ (a sub-bundle of the tangent bundle) on the system's configuration space $\mathscr{M}$ (an infinite-dimensional manifold). It has rank 2, and is spanned by a distinguished pair of vector fields $X, Y \in \mathfrak{X}(\mathscr{M})$. Their iterated Lie brackets satisfy non-trivial relations, and the main result of this paper is an *explicit isomorphism* between the Lie subalgebra generated by X, Y in $\mathfrak{X}(\mathscr{M})$, and the positively graded ideal of a *twisted loop algebra* of $\mathfrak{so}(2,1)$. Furthermore, this Lie algebra can also be identified with the *symbol* of D at a generic point, a fundamental invariant of a non-integrable distribution.

In the remainder of this introductory section, I will sketch the broader context of this work, state the main results in a precise form, and outline the contents of the article.

1.1. The infinite snake has its origins in a 'geometric robots' project proposed by Pawel Nurowski around 2014. I will briefly explain its two fundamental ideas:

(1) certain non-integrable distributions can be viewed as sources of rich geometry,
(2) a large supply of interesting non-integrable distributions is offered by simple, toy-like mechanical systems.

Traditionally, non-integrable distributions arise in areas such as PDEs, non-holonomic mechanics, and control theory. Pursuing their intrinsic properties, one quickly arrives at the problem of local equivalence (including auto-equivalence, i.e. symmetry). That in turn forces one to consider local invariants of a non-integrable distribution, in particular its *symbol*: a bundle of graded Lie algebras, associated with an increasing filtration of the tangent bundle induced by the distribution (possibly after removing a singular subset).

The filtration $F^\bullet TM$ induced by $D \subset TM$ satisfies $F^0TM = 0$, $F^1TM = D$ and is minimal with the property that $[\Gamma(F^iTM), \Gamma(F^jTM)] \subset \Gamma(F^{i+j}TM)$. There is a smallest $k > 0$ such that F^kTM is involutive, and we shall assume that actually $F^kTM = TM$ (i.e. the filtration is exhaustive); otherwise, we would restrict to its leaves. The sequence $(\operatorname{rk} F^iTM)_{1 \leq i \leq k}$ is called the *growth vector* of D.

Now, the symbol is the associated graded object of $F^\bullet TM$, a graded vector bundle over M. Its fibres have a natural structure of graded nilpotent Lie algebras, generated in degree one, with graded components in degrees $1, \dots, k$ of dimensions encoded by the increments of the growth vector (padded by zero on the left). We may visualise this data as a map from M into the moduli space of such Lie algebras. Distributions for which this map is constant are of particular interest: by the work of Tanaka and others, they allow one to construct a Cartan geometry over the base manifold. Having fixed the isomorphism class of symbol algebras, this construction can be made functorial, thus reducing the equivalence problem for such distributions to the equivalence problem for certain Cartan geometries.

From this point of view geometry is a means to the end of classifying non-integrable distributions, with further practical applications in mind. The first idea underlying the 'geometric robots' project is to take the opposite view: non-integrable distributions, through the process described above, can be used to generate geometric structures of much richer nature. For example, generic rank two distributions in dimension five

induce on their underlying fivefold a Cartan geometry modeled on a homogeneous space for the split real form of the exceptional group G_2, along with a wealth of associated objects, including a conformal class of signature $(3,2)$.

How then does one produce interesting distributions? Nurowski was inspired by the work of Masato Ishikawa, exploring certain robotic systems from an engineering perspective, yet revealing remarkably deep geometric properties. In particular, Ishikawa studied (also experimentally!) planar robots built from rigid elements (segments or polygons), connected at adjacent vertices by active rotating joints, and possibly equipped with passive wheels. The condition that the wheels roll without skipping or skidding introduces non-holonomic constraints, cutting out a non-integrable distribution in the tangent bundle of the robot's configuration space. One such robot, the *three-segment snake*, gives rise to a generic rank two distribution in dimension five, and thus to a G_2 Cartan geometry. Another, the *trident snake*, induces a generic rank three distribution in dimension six, corresponding to an $\mathsf{SO}(4,3)$ Cartan geometry. One may easily imagine other 'snakes', i.e. planar robots composed of rigid segments and passive wheels with a no-skidding-or-skipping constraint. These systems have a combinatorial flavour, but also include continuous parameters, namely segment lenghts and wheel positions, giving rise to families of geometries (perhaps with special, exceptionally symmetric, members). That is the second idea underlying the 'geometric robots' project.

1.2. How much can be deduced from a 'snake's combinatorial structure? Certainly the dimension of its configuration space, and the rank of the constraint distribution – the initial and final entries of the growth vector (assuming the filtration of the tangent bundle is exhaustive). What about the entire growth vector? I was approaching this question in 2014, attempting to understand what happens as a 'snake' is built inductively, by appending new segments one by one. Arguably the simplest construction of this sort is the one that resembles an actual snake: a sequence of segments, each attached to the end of the previous one. With each segment equipped with a wheel in its very middle, these systems are what we'll call n-segment snakes (including the 3-segment snake mentioned above in relation to G_2).

Here are the growth vector increments (i.e. ranks of graded components of the symbol) for the first few:

n	growth vector increments						
1	2	1					
2	2	1	1				
3	2	1	2				
4	2	1	2	1			
5	2	1	2	1	1		
6	2	1	2	1	2		
7	2	1	2	1	2	1	
8	2	1	2	1	2	1	1
9	2	1	2	1	2	1	2

The pattern is evident: the growth vector increments are an alternating sequence of 2 and 1, except perhaps for the terminal entry, where a 1 can appear instead of 2. The initial entry is 2, and that's the rank of the constraint distribution. The sum of the entries is $n+2$, and that's the dimension of the configuration space. Explaining this pattern had been the main motivation behind this paper.

1.3. Let us quickly sketch the connection between Theorem 1 and the above table of growth vector increments. We need to consider n-segment snakes for $n \in \mathbb{N} \cup \{\infty\}$. The n-segment snake is described by a configuration space $\mathscr{M}_n$ together with a distribution $D_n \subset T\mathscr{M}_n$. In particular, $\mathscr{M}_0 \simeq \mathbb{R}^2$ and $D_0 = T\mathscr{M}_0$. We may view each finite snake as a subsystem of the infinite snake, consisting of the snake's 'head', along with a finite number of subsequent segments. Shorter finite snakes are in turn subsystems of longer finite snakes, so that eventually we have projection maps

$$\pi^n_{n'} : \mathscr{M}_n \to \mathscr{M}_{n'}, \quad 0 \leq n' \leq n \leq \infty$$

such that $T\pi^n_{n'} : T\mathscr{M}_n \to T\mathscr{M}_{n'}$ maps D_n into $D_{n'}$. It turns out that in fact the latter map induces isomorphisms between the fibres of D_n and $D_{n'}$. Ultimately, taking $n' = 0$ we find that $\operatorname{rk} D_n = 2$ for all $n \in \mathbb{N} \cup \{\infty\}$.

The translation group $\mathbb{R}^2$ acts on all the $\mathscr{M}_n$, so that the projections $\pi^n_{n'}$ are equivariant, and the distributions D_n are invariant. In particular, the action trivialises $T\mathscr{M}_0 \simeq \mathscr{M}_0 \times \mathbb{R}^2$; combining this with isomorphisms $D_n \simeq \pi^{n*}_{n'} D_{n'}$ induced by the projections, we obtain trivialisations $D_n \simeq \mathscr{M}_n \times \mathbb{R}^2$. Eqiuvalently, for each $n \in \mathbb{N} \cup \{\infty\}$ we get a map

$$\mathbb{R}^2 \to \mathfrak{X}(\mathscr{M}_n)$$

whose image spans D_n at each point. Letting $\mathsf{FL}(\mathbb{R}^2)$ denote the free Lie algebra on the vector space $\mathbb{R}^2$ (i.e. the free Lie algebra on two generators), the above map extends to a Lie algebra homomorphism

$$\phi_n : \mathsf{FL}(\mathbb{R}^2) \to \mathfrak{X}(\mathscr{M}_n)$$

by the universal property of FL.

The algebra $\mathsf{FL}(\mathbb{R}^2)$ is naturally graded; we may also view it as filtered, with $F^\ell \mathsf{FL}(\mathbb{R}^2)$ spanned by elements of degree $\leq \ell$. On the other hand, letting $m \in \mathscr{M}_n$ be a regular point with an open neighbourhood $U \subset \mathscr{M}_n$ over which D_n induces a well-defined filtration $F^\bullet TU$ by vector sub-bundles, we may set $F^\ell \mathfrak{X}(U) = \Gamma(F^\ell TU)$. Then the map

$$\operatorname{res}_U \circ \phi_n : \mathsf{FL}(\mathbb{R}^2) \to \mathfrak{X}(U)$$

is compatible with the filtrations, and the image of $F^\ell \mathsf{FL}(\mathbb{R}^2)$ spans $F^\ell TU$ at each point. Thus, via evaluation at m, we have a surjective homomorphism of *graded* Lie algebras

$$\phi^{\mathrm{symb}}_{n,m} : \mathsf{FL}(\mathbb{R}^2) \to \operatorname{symb}_m D_n$$

onto the *symbol* of D_n at m. To understand the latter, we need to control $\ker \phi^{\mathrm{symb}}_{n,m}$. So far, we have:

$$\ker \phi_n \subset \ker \phi^{\mathrm{symb}}_{n,m}.$$

In fact, the image of ϕ_n is contained in the Lie subalgebra $\mathfrak{X}^{\mathrm{pr}}(\mathscr{M}_n)$ of *projectable* vector fields with respect to all the projections $\pi^n_{n'}$, $n' \leq n$: these are the vector fields on $\mathscr{M}_n$ whose action as derivations preserves each subalgebra $\pi^{n*}_{n'} C^\infty(\mathscr{M}_{n'}) \subset C^\infty(\mathscr{M}_n)$. Projection maps $\pi^n_{n'}$ induce well-defined Lie algebra homomorphisms

$$\pi^n_{n'*} : \mathfrak{X}^{\mathrm{pr}}(\mathscr{M}_n) \to \mathfrak{X}^{\mathrm{pr}}(\mathscr{M}_{n'}),$$

and we have by construction that[1]

$$\phi_{n'} = \pi^n_{n'*}\phi_n \quad 0 \le n' \le n \le \infty.$$

Hence $\ker \phi_n \subset \ker \phi_{n'}$ for $n' \le n$, and in particular

$$\ker \phi_\infty \subset \ker \phi_n \subset \ker \phi^{\mathrm{symb}}_{n,m}.$$

Even further, ϕ_n does in fact factor through $\mathfrak{X}^{\mathrm{pr}}(\mathscr{M}_n)^{\mathbb{R}^2}$, the subspace of *translation-invariant* projectable vector fields on $\mathscr{M}_n$ (of course preserved by the $\pi^n_{n'*}$). Specialising to $n = \infty$ and omitting projectability, let us consider the quotient map

$$0 \to \mathbb{R}^2 \to \mathfrak{X}(\mathscr{M}_\infty)^{\mathbb{R}^2} \xrightarrow{q} \mathfrak{X}(\mathscr{M}_\infty/\mathbb{R}^2).$$

Theorem 1 states that the image of $q \circ \phi_\infty$ is isomorphic to the graded Lie algebra

$$\mathfrak{L}_+ = \bigoplus_{i>0} \mathfrak{L}_i, \quad \mathfrak{L}_{2i} \simeq \mathfrak{so}(2), \quad \mathfrak{L}_{2i+1} \simeq \mathbb{R}^2$$

with Lie brackets induced by the Cartan decomposition

$$\mathfrak{so}(1,2) = \mathfrak{so}(2) \oplus \mathbb{R}^2.$$

We may restate it as follows.

Theorem. *There is a commutative diagram of Lie algebra homomorphisms*

$$\begin{array}{ccc} \mathsf{FL}(\mathbb{R}^2) & \xrightarrow{\phi_\infty} & \mathfrak{X}(\mathscr{M}_\infty)^{\mathbb{R}^2} \\ \downarrow{\scriptstyle \psi} & & \downarrow{\scriptstyle q} \\ \mathfrak{L}_+ & \xrightarrow{\iota} & \mathfrak{X}(\mathscr{M}_\infty/\mathbb{R}^2) \end{array}$$

where ψ is a graded surjection induced by the identification $\mathfrak{L}_1 = \mathbb{R}^2$ and the universal property of FL*, while ι is an injection.*

In particular, $\ker \phi_\infty \subseteq \ker(q \circ \phi_\infty) = \ker \psi$. In fact, it turns out that the inclusion is an equality, so that eventually

$$\ker \psi = \ker \phi_\infty \subset \ker \phi^{\mathrm{symb}}_{n,m}.$$

Put differently, the symbol of D_n at a general point $m \in \mathscr{M}_n$ is a *graded quotient* of $\mathfrak{L}_+$. Now the puzzle is essentially reduced to the following observation:

Proposition. *For each $n \in \mathbb{N}$, there is, up to isomorphism, precisely one $(n+2)$-dimensional graded quotient Lie algebra of $\mathfrak{L}_+$ generated in degree 1. The ranks of its graded components follow the pattern of the table in Subsection 1.2.*

[1]By the way, if the reader is uncomfortable working with the infinite-dimensional $\mathscr{M}_\infty \simeq \varprojlim_{n<\infty} \mathscr{M}_n$, limit of a diagram of spaces (manifolds?), it is enough accept the more palatable $\mathfrak{X}^{\mathrm{pr}}(\mathscr{M}_\infty) \simeq \varprojlim_{n<\infty} \mathfrak{X}^{\mathrm{pr}}(\mathscr{M}_n)$, limit of a diagram of Lie algebras.

1.4. The proof of Theorem 1 is, ultimately, computational. The calculation is however greatly simplified by exploiting *symmetries* of the main system of interest: the infinite snake. We've already employed *translational* symmtery, working with vector fields on the reduced configuration space, which we'll now denote simply by $M = \mathscr{M}_\infty/\mathbb{R}^2$. We'll also write $\xi = q \circ \phi_\infty$. Of course, the original system enjoyed full *Euclidean* symmetry, whence we're still left with the rotation group $\mathsf{O}(2)$ acting on M, so that the homomorphism

$$\xi : \mathsf{FL}(\mathbb{R}^2) \to \mathfrak{X}(M)$$

becomes $\mathsf{O}(2)$-equivariant with respect to the natural induced actions on the domain and codomain.

It is natural to diagonalise the $\mathsf{SO}(2)$-actions. This requires complexifying the above situation:

$$\xi : \mathsf{FL}(\mathbb{C}^2) \to \mathfrak{X}_\mathbb{C}(M) = \mathbb{C} \otimes \mathfrak{X}(M)$$

where we extend ξ to a complex-linear map. Let $v^\pm = e_1 \pm ie_2$ be an isotropic basis of $\mathbb{C}^2$, with a suitable generator of $\mathfrak{so}(2,\mathbb{C})$ acting as $v^\pm \mapsto \pm v^\pm$. We are interested in the Lie subalgebra generated by

$$\zeta^\pm = \xi(v^\pm) \in \mathfrak{X}_\mathbb{C}(M).$$

These turn out to satisfy certain remarkable identities.

There's a further semi-symmetry we haven't yet exploited: the tail of an infinite snake is itself an infinite snake – and it behaves as such. That is, the resulting map $\mathrm{tail} : \mathscr{M}_\infty \to \mathscr{M}_\infty$ sends $D_\infty \subset T\mathscr{M}_\infty$ to itself. Observe that the data corresponding to tail at the finite n level is a map $\mathscr{M}_n \to \mathscr{M}_{n-1}$; this only becomes a self-map at $n = \infty$. Of course, tail is not invertible – hence the prefix *semi*.[2]

Much of Sections 2 and 3 is devoted to working out how the tail map interacts with ξ, and thus $\zeta^\pm$. While this is not difficult, and quite entertaining to see intuitively, some care has to be taken to write down correct equations for ξ and $\zeta^\pm$. In short, one considers a *twist* $\widetilde{\mathsf{tl}} : M \to M$ of the tail map descended to M, along with the head map $\mathsf{hd} : M \to \mathbb{T}$: these define an isomorphism

$$\langle \mathsf{hd}, \widetilde{\mathsf{tl}} \rangle : M \xrightarrow{\simeq} \mathbb{T} \times M.$$

Using its inverse, we consider the pair of injective homomorphisms

$$\mathfrak{X}_\mathbb{C}(\mathbb{T}) \xrightarrow{\iota} \mathfrak{X}_\mathbb{C}(M) \xleftarrow{\sigma} \mathfrak{X}_\mathbb{C}(M)$$

of Lie algebras. In particular, the 'shift operator' σ sends an infinitesimal motion of the infinte snake, now viewed as the tail of another infinite snake, to an infinitesimal motion of the latter, keeping the head immobile. Now, we have the relation:

$$\zeta^+ = z^2\partial_z + z\sigma\left(\zeta^+ + z\partial_z\right) \tag{1}$$

where $z \in C^\infty(\mathbb{T},\mathbb{C})$ is the standard complex coordinate on the torus, and we've omitted the maps $\mathsf{hd}^* : C^\infty(\mathbb{T},\mathbb{C}) \to C^\infty(M,\mathbb{C})$ and $\iota : \mathfrak{X}_\mathbb{C}(\mathbb{T}) \to \mathfrak{X}_\mathbb{C}(M)$ from notation.

Here's a number of observations regarding (1).

(1) The operator $1 - z\sigma : \mathfrak{X}_\mathbb{C}(M) \to \mathfrak{X}_\mathbb{C}(M)$ is invertible, so that (1) determines ζ^+ uniquely.
(2) There is a reflection in $\mathsf{O}(2)$ inducing involutions on $\mathfrak{X}_\mathbb{C}(M)$ and related spaces, denoted by τ, such that $\tau\sigma = \sigma\tau$, $\tau z = z^{-1}\tau$ as operators on $\mathfrak{X}_\mathbb{C}(M)$, and $\zeta^- = \tau\zeta^+$. Hence (1) determines ζ^- as well.

[2]One could go further and consider doubly infinite snakes, with tail becoming an honest symmetry.

(3) The vector fields $z^2\partial_z$, $z\partial_z$ and $\tau(z^2\partial_z) = \partial_z$, identified with their images under ι, form a Lie algebra isomorphic to $\mathfrak{sl}(2,\mathbb{C})$, a complexification of the $\mathfrak{so}(1,2)$ giving rise to the loop algebra $\mathfrak{L}_+$ eventually controlling the entire situation. This is a first glimpse of the role of $\mathfrak{so}(1,2)$ in the infinite snake's kinematics.

In light of the latter observation, one may contemplate three factors in the infinite snake's generalised symmetries: obvious Euclidean symmetry, 'shift' semi-symmetry, and 'hidden' $\mathfrak{so}(1,2)$.

One may view (1) as constructing an object over the infinite-dimensional space M in terms of finite-dimensional data – a pair of elements of $\mathfrak{sl}(2,\mathbb{C})$ embedded in $\mathfrak{X}_{\mathbb{C}}(M)$. That is the 'taming' of the snake's infinite aspect by means of the shift semi-symmetry and associated (co)recursion (cf. the picture of Ouroboros eating its tail on the title page). The strategy for proving Theorem 1 is now easy to guess: we seek relations analogous to (1) satisfied by iterated Lie brackets of $\zeta^{\pm}$, and use these to determine $\ker\xi \subset \mathsf{FL}(\mathbb{C}^2)$.

1.5. Section 2 constructs the reduced configuration space M along with the map $\mathbb{R}^2 \to \mathfrak{X}(M)$, and derives an identity relating the latter with the tail map. Section 3 passes to the complexification and then to a further abstract setting, defined by algebraic relations between σ, z, etc. Theorem 1 is reduced to its algebraic analogue, Theorem 2. The latter is proved in Section 4. Finally, Section 5 fills in some details regarding the symbols of finite snakes, and discusses several other aspects of our result.

Acknowledgements.

2. Deriving snake kinematics.

In what follows, we will work with certain infinite-dimensional manifolds in a rather synthetic manner. Formally, this involves an extension of the usual category of C^∞ finite-dimensional manifolds and C^∞ maps, with suitable properties. There are many possible choices of such a category,[3] and the resulting algebraic object we're interested in – the Lie algebra of vector fields generated by lifts of Euclidean translations to the snake's reduced configuration space – is invariant. Thus, on one hand, we will proceed with its construction assuming the spaces and maps make sense in a suitable category; on the other hand, in section 3 we'll reformulate the Main Theorem in a purely algebraic setting; section 4 of the paper is devoted to proving this purely algebraic formulation.

2.1. The snake lives in a Euclidean plane E. Its basic building block is a single segment. The configuration of a single segment is parameterised by a pair of points at unit distance. These pairs form a double circle fibration:

$$\mathrm{pr}_1, \mathrm{pr}_2 : \mathcal{T} \rightrightarrows E; \quad \mathcal{T} = \{(p, q) \mid |p - q| = 1\} \subset E \times E.$$

The motion of the segment is uniquely determined by the motion of one of its endpoints. Correspondingly, there is a rank two distribution

$$D_{\mathcal{T}} \subset T\mathcal{T}$$

fitting into a commutative diagram

$$\begin{array}{ccc} & D_{\mathcal{T}} & \\ {\scriptstyle \mathrm{pr}_{1*}}\swarrow & & \searrow{\scriptstyle \mathrm{pr}_{2*}} \\ \mathrm{pr}_1^* TE & \xrightarrow{\quad\Phi\quad} & \mathrm{pr}_2^* TE \end{array}$$

of isomorphisms of vector bundles over $\mathcal{T}$. Evaluating Φ at $s = (p, q) \in \mathcal{T}$, we have that

$$\Phi_s : T_p E \to T_q E$$

is a *reflection* in the line $\overline{pq}$, followed by parallel transport (this can be taken as an axiom).

2.2. Now, the configuration space of an infinite snake is an infinite fibre product

$$\mathcal{M} = \mathcal{T} \times_E \mathcal{T} \times_E \mathcal{T} \times_E \cdots$$

where $\mathcal{T}$ is viewed as fibred over E using pr_1 on the left and pr_2 on the right. It follows that $\mathcal{M}$ carries a rank two distribution

$$\begin{array}{ccc} D & = & D_{\mathcal{T}} \times_{TE} D_{\mathcal{T}} \times_{TE} \cdots \\ \downarrow & & \downarrow \\ T\mathcal{M} & = & T\mathcal{T} \times_{TE} T\mathcal{T} \times_{TE} \cdots \end{array}$$

Consider the projections

$$\mathcal{T} \xleftarrow{\text{head}} \mathcal{M} \xrightarrow{\text{tail}} \mathcal{M}$$

[3] In particular, any well-adapted model of Synthetic Differential Geometry (see Moerdijk and Reyes, *Models for Smooth Infinitesimal Analysis*). In fact, it is enough to have limits of sequences of locally trivial fibrations of finite-dimensional manifolds, and then to work with Lie algebras of projectable vector fields. Indeed, one could express the entire construction in terms of sequences of finite-dimensional manifolds, at the cost of index bookkeeping completely obscuring the intuitive geometric picture.

$$\begin{aligned} \mathrm{head}(s_1, s_2, s_3, \dots) &= s_1 \\ \mathrm{tail}(s_1, s_2, s_3, \dots) &= (s_2, s_3, \dots) \end{aligned}$$

together with

$$\mathrm{nose} = \mathrm{pr}_1 \circ \mathrm{head} : \mathscr{M} \to E$$

$$\mathrm{neck} = \mathrm{pr}_2 \circ \mathrm{head} : \mathscr{M} \to E$$

so that

$$\mathrm{neck} = \mathrm{nose} \circ \mathrm{tail}\,.$$

Note that we get a coinductive[4] description of $\mathscr{M}$:

$$\langle \mathrm{head}, \mathrm{tail} \rangle : \mathscr{M} \xrightarrow{\simeq} \mathscr{T} \times_E \mathscr{M}$$

Now, in particular the map

$$\mathrm{nose}_* : T\mathscr{M} \to \mathrm{nose}^* TE$$

restricts to an *isomorphism* on the rank 2 distribution $D \subset T\mathscr{M}$.

2.3. Consider the translation action of $\mathbb{R}^2$ on E, and the induced actions on $\mathscr{T}$, $\mathscr{M}$, their tangent bundles and so on. Note that the distributions $D_{\mathscr{T}}$ and D are translation invariant, and the maps head, tail, etc. defined above are equivariant. We may consider the reduced configuration space

$$M = \mathscr{M}/\mathbb{R}^2,$$

along with its coinductive description

$$\langle \mathsf{hd}, \mathsf{tl} \rangle : M \xrightarrow{\simeq} \mathbb{T} \times M$$

where $\mathbb{T} \subset \mathbb{R}^2$ is the unit circle, identified with $\mathscr{T}/\mathbb{R}^2$. This in particular exhibits M as an infinite torus, parameterising orientations of the infinite snake while forgetting the position of its nose.

2.4. Let $\mathbb{R}^2 \to \mathfrak{X}(E)$ be the infinitesimal action map, an isomorphism onto the space of translation-invariant vector fields. Inverting the isomorphism

$$\mathrm{nose}_* : D \to \mathrm{nose}^* TE$$

gives rise to a map of vector fields

$$\nabla : \mathfrak{X}(E) \to \Gamma(\mathscr{M}, D), \quad (T_m \,\mathrm{nose})(\nabla v) = v_{\mathrm{nose}\, m}$$

restricting to a lifting of translation-invariant fields. Following with projection $\mathscr{M} \to M$, we have a map

$$\xi : \mathbb{R}^2 \to \mathfrak{X}(E)^{\mathbb{R}^2} \xrightarrow{\nabla} \Gamma(\mathscr{M}, D)^{\mathbb{R}^2} \to \mathfrak{X}(\mathscr{M})^{\mathbb{R}^2} \to \mathfrak{X}(M).$$

The group $\mathsf{O}(2)$ acts on $\mathbb{R}^2$, $\mathbb{T}$ as well as E, $\mathscr{T}$, $\mathscr{M}$, etc. so that all the relevant maps are equivariant. In particular, the action descends to M so that ξ is $\mathsf{O}(2)$-equivariant.

[4] *Coinductive*, because it exhibits $\mathscr{M}$ as a *final coalgebra* for the endofunctor $\mathscr{T} \times_E -$ on the category of locally trivial fibre bundles over E

Theorem 1. *Consider $G = \mathsf{SO}(1,2)$, with maximal compact subgroup $\mathsf{O}(2) \subset G$, and a Cartan involution θ acting on the Lie algebra $\mathfrak{g}$ with even eigenspace $\mathfrak{so}(2)$ and odd eigenspace $\mathbb{R}^2$ (as $\mathsf{O}(2)$-modules). Let*

$$\mathfrak{L}_+ = \{f(t) \in t\mathfrak{g}[t] \mid f(-t) = \theta f(t)\} = \bigoplus_{i>0} \mathfrak{L}_i$$

be the positive part of the corresponding twisted loop algebra, with $\mathfrak{L}_{2i} \simeq \mathfrak{so}(2)$, $\mathfrak{L}_{2i+1} \simeq \mathbb{R}^2$. Then there exists an $\mathsf{O}(2)$-equivariant Lie algebra homomorphism

$$\xi^+ : \mathfrak{L}_+ \to \mathfrak{X}(M)$$

such that the following diagram commutes:

$$\begin{array}{ccc} \mathfrak{L}_1 & \hookrightarrow & \mathfrak{L}_+ \\ \downarrow \simeq & & \downarrow \xi^+ \\ \mathbb{R}^2 & \xrightarrow{\ \xi\ } & \mathfrak{X}(M). \end{array}$$

Furthermore, ξ^+ is injective, and thus an isomorphism onto the sub-algebra generated by the image of ξ in $\mathfrak{X}(M)$.

2.5. Consider the 2:1 map

$$\gamma : \mathbb{T} \to \mathsf{O}(2)$$

parameterising reflections so that $\gamma_u u = u$ for all $u \in \mathbb{T}$. We may view γ as a translation quotient of Φ.

Lemma 1. $\forall m \in M, v \in \mathbb{R}^2.\ (T_m \,\mathsf{tl})\xi(v)_m = \xi(\gamma_{\mathsf{hd}\, m} v)_{\mathsf{tl}\, m}$.

Proof. Identifying $\mathbb{R}^2$ with $\mathfrak{X}(E)^{\mathbb{R}^2}$, let $\upsilon = \nabla v$, and let $m' \in \mathscr{M}$ be a lift of $m \in M$. Then

$$(T_{m'} \,\mathrm{head})(\upsilon_{m'}) \in D_{\mathcal{T}, \mathrm{head}\, m'}$$

and thus

$$\begin{aligned} (T_{\mathrm{tail}\, m'} \,\mathrm{nose})(T_{m'} \,\mathrm{tail})(\upsilon_{m'}) &= (T_{m'} \,\mathrm{neck})(\upsilon_{m'}) \\ &= \Phi_{\mathrm{head}\, m'}(T_{m'} \,\mathrm{nose})(\upsilon_{m'}) \\ &= \Phi_{\mathrm{head}\, m'} v \end{aligned}$$

by commutativity of the diagram defining $D_{\mathcal{T}}$. Since

$$\mathrm{tail}_* : D \to \mathrm{tail}^* D, \quad \mathrm{nose}_* : D \to \mathrm{nose}^* TE$$

are isomorphisms over M, it follows that

$$(T_{m'} \,\mathrm{tail})(\nabla v)_{m'} = \nabla(\gamma_{\mathsf{hd}\, m} v)_{\mathrm{tail}\, m'},$$

descending along $\mathscr{M} \to M$ to the desired identity at $m \in M$. □

The lemma is equivalent to commutativity of the following diagram

$$\begin{array}{ccccc} M \times \mathbb{R}^2 & \xrightarrow{\mathsf{hd}^*\gamma} & M \times \mathbb{R}^2 & \xrightarrow{\text{triv.}} & \mathsf{tl}^*(M \times \mathbb{R}^2) \\ \downarrow \xi & & & & \downarrow \mathsf{tl}^* \xi \\ TM & & \xrightarrow{\qquad \mathsf{tl}_* \qquad} & & \mathsf{tl}^* TM \end{array}$$

of vector bundles over M, where $\mathsf{hd}^* \gamma$ is an endomorphism of the trivial $\mathbb{R}^2$-bundle induced by the composite $\gamma \circ \mathsf{hd} : M \to \mathsf{O}(2)$, and we identify ξ with a bundle morphism $M \times \mathbb{R}^2 \to TM$.

FIGURE 1. $(\frac{\pi}{4}, \frac{\pi}{4}, \dots) \in \mathbb{T}^\infty \simeq M$ using (a) $\langle \mathsf{hd}, \mathsf{tl} \rangle$ and (b) $\langle \mathsf{hd}, \widetilde{\mathsf{tl}} \rangle$.

2.6. Lemma 1 provides a corecursive description of ξ, with respect to the coinductive description of M in terms of hd, tl. It expresses the tail part of ξ as a twist of ξ by $\mathsf{hd}^* \gamma$. Our strategy for proving the main theorem will be to use similar corecursive descriptions of iterated Lie brackets of vector fields in the image of ξ. More succintly, we seek recursion relations for the Lie algebra homomorphism

$$\widetilde{\xi} : \mathsf{FL}(\mathbb{R}^2) \to \mathfrak{X}(M)$$

from the free Lie algebra on $\mathbb{R}^2 \simeq \mathfrak{L}_1$, in order to show that it descends along the quotient map $\mathsf{FL}(\mathfrak{L}_1) \to \mathfrak{L}_+$.

This task will be simplified if we twist the tail map tl by a reflection: then the recursion relation will be expressed in terms of rotations only, and these can be simultaneously diagonalised upon complexification. To this end, let

$$\rho : \mathbb{T} \to \mathsf{SO}(2)$$

be the 1:1 parameterisation of rotations, so that $\rho_u(e_1) = u$. Consider the *twisted tail map*

$$\widetilde{\mathsf{tl}} : M \to M, \quad \widetilde{\mathsf{tl}}(m) = \gamma_{e_1} \rho^{-1}_{\mathsf{hd}(m)} \, \mathsf{tl}(m).$$

Note that it still leads to a coinductive description

$$\langle \mathsf{hd}, \widetilde{\mathsf{tl}} \rangle : M \xrightarrow{\simeq} \mathbb{T} \times M.$$

Intuitively, if we used $\langle \mathsf{hd}, \mathsf{tl} \rangle$ iteratively to identify M with an intinite torus, we'd be parameterising the snake's oriention in terms of the angle each segment makes with e_1. On the other hand, if we used $\langle \mathsf{hd}, \widetilde{\mathsf{tl}} \rangle$, we would be using the angle the head makes with e_1, and then the angle each subsequent segment makes with the previous one, measured with alternating orientations. For instance, the point on the infinite torus with all angular coordinates equal corresponds to: a straight-line snake in the former case; and a zig-zag snake in the latter (Fig. 1).

Lemma 2. $\widetilde{\mathsf{tl}} \circ \gamma_{e_1} = \gamma_{e_1} \circ \widetilde{\mathsf{tl}}$

Proof. By $\mathsf{O}(2)$-equivariance of the head map hd and untwisted tail map tl,

$$\begin{aligned} \widetilde{\mathsf{tl}}(\gamma_{e_1} m) &= \gamma_{e_1} \rho^{-1}_{\gamma_{e_1} \mathsf{hd}(m)} \gamma_{e_1} \, \mathsf{tl}(m) \\ &= \rho^{-1}_{\mathsf{hd}(m)} \, \mathsf{tl}(m) \\ &= \gamma_{e_1} \, \widetilde{\mathsf{tl}}(m) \end{aligned}$$

for $m \in M$. □

2.7. Consider the action map

$$\mathsf{act} \circ \rho : \mathbb{T} \to \mathsf{Aut}(M),$$

its differential

$$\alpha : T\mathbb{T} \to \mathfrak{X}(M)$$

and the associated evaluation map

$$\alpha_\bullet : M \times T\mathbb{T} \to TM.$$

Lemma 3. $\forall m \in M, v \in T_mM.\ (T_m\,\widetilde{\mathsf{tl}})(v) = \gamma_{e_1}\rho^{-1}_{\mathsf{hd}(m)} \cdot [(T_m\,\mathsf{tl})(v) - \alpha_{\mathsf{tl}\,m}(T_m\,\mathsf{hd})(v)]$.

Observe that the right hand side involves an element of $\mathsf{O}(2)$ acting on the bracketed expression $[\cdots] \in T_{\mathsf{tl}\,m}M$, and belongs to $T_{\widetilde{\mathsf{tl}}\,m}M$.

Proof. This is an immediate application of the chain rule to $\widetilde{\mathsf{tl}}$, where we note that $-\alpha$ is the differential of act $\circ\, \rho^{-1}$. □

This result may be understood intuitively as follows. Consider a snake whose nose is located at the origin of a Cartesian frame with unit vectors e_1, e_2. The map tl may be viewed as cutting off the head of the snake, and moving the frame by translation to the new nose. On the other hand, the map $\widetilde{\mathsf{tl}}$ constructs a *new* frame with origin at the new nose: its orientation is opposite to the previous one, and its first axis is in the direction of the severed head. Thus, given an infinitesimal motion of the snake, its images under tl and $\widetilde{\mathsf{tl}}$, besides being expressed in different frames, differ by a term arising from an infinitesimal rotation of the frame itself. The latter term is the $\alpha_{\mathsf{tl}\,m}(T_m\,\mathsf{hd})(v)$ in the Lemma.

2.8. Let $\omega = e_1 \wedge e_2$, viewed interchangeably as an element of $\Lambda^2\mathbb{R}^2$, its dual, or $\mathfrak{so}(2)$; in the latter case, $\langle \omega(u), v\rangle = \omega(u,v)$ for $u, v \in \mathbb{R}^2$. Write $\widetilde{\omega} \in \mathfrak{X}(M)$ for the corresponding infinitesimal action.

Lemma 4. $\forall m \in M, v \in \mathbb{R}^2.\ (T_m\,\widetilde{\mathsf{tl}})(\xi(v)_m) = [\xi(\rho^{-1}_{hd(m)}v) - \omega(\mathsf{hd}\,m, v)\widetilde{\omega}]_{\widetilde{\mathsf{tl}}\,m}$.

Proof. Combining Lemmas 3 and 4, we have:

$$\begin{aligned}(T_m\,\widetilde{\mathsf{tl}})\xi(v)_m &= \gamma_{e_1}\rho^{-1}_{\mathsf{hd}(m)} \cdot [\xi(\gamma_{\mathsf{hd}\,m}v)_{\mathsf{tl}\,m} - \alpha_{\mathsf{tl}\,m}(T_m\,\mathsf{hd})(\xi(v)_m)] \\ &= \xi(\rho^{-1}_{\mathsf{hd}(m)}v)_{\widetilde{\mathsf{tl}}\,m} + \alpha_{\widetilde{\mathsf{tl}}\,m}(T_m\,\mathsf{hd})(\xi(v)_m)\end{aligned}$$

for all $m \in M$, $v \in \mathbb{R}^2$, where we used $\gamma_u = \rho_u\gamma_{e_1}\rho_u^{-1}$. Now, using

$$(T_m\,\mathsf{hd})\xi(v)_m = \frac{\gamma_{\mathsf{hd}\,m} - 1}{2}v$$

and

$$(T_u\rho)(v) = \omega(u,v)\omega$$

we have

$$\alpha((T_m\,\mathsf{hd})(\xi(v)_m)) = \omega(\mathsf{hd}\,m, -v)\widetilde{\omega}$$

and the claim follows. □

3. Abstracting the algebra.

Recall that the point of 'twisting' the snake's tail was to have a recursion relation (namely that of Lemma 4) involving rotations rather than reflections. The next step is then to complexify $\mathfrak{X}(M)$, so that the action of $\mathsf{SO}(2)$ becomes diagonalisable, simplifying the relation. Furthermore, we'll replace the 'geometric' formula of Lemma 4, expressed in terms of head and tail maps, with an 'algebraic' one, involving certain operators on $\mathfrak{X}_{\mathbb{C}}(M)$. Then, the structure on $\mathfrak{X}_{\mathbb{C}}(M)$ necessary to express the recursion relation becomes simple enough to be abstracted in the (purely algebraic) notion of an *ambient module* (Definition 1). We will rephrase Theorem 1 as a statement about

ambient modules (Theorem 2), and prove it in this context, improving clarity and escaping any uneasiness about the infinite-dimensional geometric setting.

3.1. Define the Lie algebra homomorphism

$$\sigma : \mathfrak{X}(M) \to \mathfrak{X}(M)$$

by

$$\mathsf{hd}_*(\sigma v) = 0, \quad \widetilde{\mathsf{tl}}_*(\sigma v) = \widetilde{\mathsf{tl}}^*\, v$$

for all $v \in \mathfrak{X}(M)$. This is a 'shift map', sending an infinitesimal motion of the snake, to that same motion applied to the (twisted) tail, with head immobile.

Complexify everything, and introduce the notation

$$z : M \to \mathbb{C}, \quad z(m) = \langle e_1 + ie_2, \mathsf{hd}\, m\rangle$$
$$\partial_z \in \mathfrak{X}_{\mathbb{C}}(M), \quad \partial_z z = 1, \quad \widetilde{\mathsf{tl}}_*\, \partial_z = 0$$

where $\langle -,-\rangle$ is the complex bilinear extension of the usual scalar product on $\mathbb{R}^2$.

That is, z is the usual unimodular complex coordinate on $\mathbb{T}$, pulled back by hd to M, while ∂_z is the corresponding complex vector field on $\mathbb{T}$, realised as an infinitesimal motion of the head, with (twisted) tail immobile.

Let

$$\zeta^{\pm} = \xi(e_1 \pm ie_2) \in \mathfrak{X}_{\mathbb{C}}(M).$$

Proposition 1. $\zeta^{\pm} = z^{\pm 1}[\pm z\partial_z + \sigma(\zeta^{\pm} \pm z\partial_z)]$.

Proof. Let $v^{\pm} = e_1 \pm ie_2$. For all $u \in \mathbb{T}$ we claim:

i. $\rho_u v^{\pm} = w(u)^{\mp 1} v^{\pm}$
ii. $\omega(u, v^{\pm}) = \pm i w(u)^{\pm 1}$
iii. $\omega \in \mathfrak{so}(2)$ acts on $\mathbb{T}$ as $iw\partial_w$

where $w(u) = \langle v^+, u\rangle$. Indeed, letting $u = (\cos\phi, \sin\phi)$, we have

$$w(u) = e^{i\phi}, \quad \rho_u e_1 = u, \quad \rho_u e_2 = (-\sin\phi, \cos\phi),$$

so that

$$\rho_u v^{\pm} = (e^{\mp i\phi}, \pm ie^{\mp i\phi}) = w^{\mp 1} v^{\pm}.$$

Also, $\omega(u, v^{\pm}) = \pm ie^{\pm i\phi} = \pm iw^{\pm 1}$. Lastly, $dw = iwd\phi$ so that $\partial_\phi = iw\partial_w$.

Now, in $\mathfrak{X}_{\mathbb{C}}(M)$, we claim that

iv. $\widetilde{\omega} = iz\partial_z$.

Indeed, $\widetilde{\mathsf{tl}}_*\, \widetilde{\omega} = 0$ by Lemma 3 and $\mathsf{hd}_*\, \widetilde{\omega} = iw\partial_w$ by (iii).

Let us rewrite Lemma 4 in the above notation: for all $m \in M$, $v \in \mathbb{C}^2$,

$$\begin{aligned}(T_m \widetilde{\mathsf{tl}})\xi(v^{\pm})_m &= z^{\pm 1}(m)[\xi(v^{\pm}) \mp i\widetilde{\omega}]_{\widetilde{\mathsf{tl}}\, m} \\ (T_m \widetilde{\mathsf{tl}})\zeta^{\pm}_m &= z^{\pm 1}(m)[\zeta^{\pm} \pm z\partial_z]_{\widetilde{\mathsf{tl}}\, m}.\end{aligned}$$

On the other hand,

$$\begin{aligned}(T_m \mathsf{hd})\xi(v^{\pm})_m &= \omega(\mathsf{hd}\, m, -v^{\pm})\partial_\phi \\ (T_m \mathsf{hd})\zeta^{\pm}_m &= \pm w(\mathsf{hd}\, m)^{1\pm 1}\partial_w\end{aligned}$$

where $w = e^{i\phi} : \mathbb{T} \to \mathbb{C}$ as before. Finally, recalling $z = \mathsf{hd}^*\, w$ along with $\widetilde{\mathsf{tl}}_*\, \partial_z = 0$ and $\mathsf{hd}_* \circ \sigma = 0$, we have

$$\begin{aligned}\zeta^{\pm} &= \pm z^{1\pm 1}\partial_z + z^{\pm 1}\sigma(\zeta^{\pm} \pm z\partial_z) \\ &= z^{\pm 1}[\pm z\partial_z + \sigma(\zeta^{\pm} \pm z\partial_z)].\end{aligned}$$

□

3.2.

Definition 1. An *ambient module* is a complex Lie algebra Ω together with the following structure:

elements	$\mathsf{e},\mathsf{h},\mathsf{f} \in \Omega$
maps	$\tau,\sigma,z : \Omega \to \Omega$
properties	
sl_2	$\mathsf{e},\mathsf{h},\mathsf{f}$ are linearly independent and $[\mathsf{h},\mathsf{e}] = \mathsf{e}$, $[\mathsf{h},\mathsf{f}] = -\mathsf{f}$, $[\mathsf{f},\mathsf{e}] = 2\mathsf{h}$
flip	τ is an involutive Lie algebra automorphism
shift	σ is an injective Lie algebra endomorphism
twist	z is an invertible vector space endomorphism, transcendental over $\mathbb{C}$
relations	
flip-sl_2	$\tau\mathsf{e} = -\mathsf{f}$, $\tau\mathsf{f} = -\mathsf{e}$, $\tau\mathsf{h} = -\mathsf{h}$
flip-twist	$\tau \circ z = z^{-1} \circ \tau$
flip-shift	$\tau \circ \sigma = \sigma \circ \tau$
shift-twist	$\forall \varphi \in \mathbb{C}[z,z^{-1}], x \in \Omega.\ [\mathrm{ad}_{\varphi\sigma x}, z] = 0$
shift-sl_2	$\forall \varphi \in \mathbb{C}[z,z^{-1}], x \in \{\mathsf{e},\mathsf{h},\mathsf{f}\}.\ \mathrm{ad}_{\varphi x} \circ \sigma = 0$
twist-sl_2	$\forall \varphi \in \mathbb{C}[z,z^{-1}], x \in \{\mathsf{e},\mathsf{h},\mathsf{f}\}.\ [\mathrm{ad}_{\varphi x}, z] = \varphi z^{1+\lambda(x)}$

where $\lambda(\mathsf{e}) = 1$, $\lambda(\mathsf{h}) = 0$, $\lambda(\mathsf{f}) = -1$, and by transcendence of z we identify the abstract Laurent algebra $\mathbb{C}[z,z^{-1}]$ with the subalgebra generated by z and its inverse in $\mathrm{End}_{\mathbb{C}}\,\Omega$ (vector space endomorphisms).

Note that an ambient module is simultaneously a module for the Laurent algebra $\mathbb{C}[z,z^{-1}]$, and for the Lie algebra $\mathfrak{sl}_2(\mathbb{C}) \simeq \mathbb{C}\langle \mathsf{e},\mathsf{h},\mathsf{f}\rangle$ (where our h is half of the usual one!). These two structures are compatible with respect to the usual action of $\mathfrak{sl}_2(\mathbb{C})$ on Laurent polynomials: e as $z^2\partial_z$, h as $z\partial_z$, f as ∂_z. Furthermore, the involution τ is compatible with an involution on the Laurent algebra sending z to z^{-1}. Finally, relations with the shift map mean that $\sigma\Omega$ commutes with $\mathbb{C}[z,z^{-1}] \cdot \mathfrak{sl}_2(\mathbb{C})$, while the adjoint action of $\mathbb{C}[z,z^{-1}] \cdot \sigma\Omega$ on Ω is $\mathbb{C}[z,z^{-1}]$-linear in its passive argument.

Lemma 5. *$\mathfrak{X}_{\mathbb{C}}(M)$ has the structure of an ambient module as suggested by notation, with $\mathsf{e} = z^2\partial_z$, $\mathsf{h} = z\partial_z$, $\mathsf{f} = \partial_z$, and with τ induced by the action of $\gamma_{e_1} \in \mathsf{O}(2)$.*

Proof. Note that $\gamma_{e_1}^* z = z^{-1}$. Then relations *flip-$sl_2$* and *flip-twist* follow immediately. Next, *flip-shift* is a consequence of Lemma 2, while *shift-twist* and *shift-sl_2* hold because $\mathsf{hd}_* \circ \sigma = 0$ and $\widetilde{\mathsf{tl}}_*\,\partial_z = 0$, respectively. Finally, *twist-sl_2* follows by letting $\mathsf{e},\mathsf{h},\mathsf{f}$ as defined in terms of ∂_z act on z. □

We will consider this ambient module structure on $\mathfrak{X}_{\mathbb{C}}(M)$ as canonical.

Definition 2. An element ζ of an ambient module Ω is called *serpentine* if it satisfies $\zeta = \mathsf{e} + z\sigma(\zeta + \mathsf{h})$.

Note that a serpentine element is non-zero: for if zero were serpentine, we would have $0 = z^{-2}[\mathsf{e},0] = z^{-2}[\mathsf{e},\mathsf{e} + z\sigma\mathsf{h}] = \sigma\mathsf{h}$, contradicting injectivity of σ. We will refer to the τ-invariant Lie subalgebra generated by a serpentine element as a *serpentine subalgebra.*

Lemma 6. *ζ^+ is serpentine in $\mathfrak{X}_{\mathbb{C}}(M)$, and $\zeta^- = \tau\zeta^+$.*

Proof. By Prop. 1, we have

$$\zeta^+ = z^2\partial_z + z\sigma(\zeta^+ + z\partial_z) = \mathsf{e} + z\sigma(\zeta^+ + \mathsf{h})$$

so that ζ^+ is serpentine. Furthermore, since ξ is $\mathsf{O}(2)$-equivariant, it follows that

$$\zeta^- = \xi(v^-) = \xi(\gamma_{e_1} v^+) = \gamma_{e_1}\xi(v^+) = \tau\zeta^+.$$

□

3·3·

Definition 3. A *σ-filtration* on an ambient module Ω is a decreasing filtration $F^\bullet\Omega$ by $\mathbb{C}[z, z^{-1}]$-submodules such that:

(1) $F^0\Omega = \Omega$,
(2) $\mathfrak{sl}_2(\mathbb{C}) \to \Omega/F^1\Omega$ is injective,
(3) $\sigma F^k\Omega \subset F^{k+1}\Omega$ for all $k \geq 0$.

We say that Ω is *separated* (respectively *complete*) if it admits a σ-filtration such that the natural map

$$\Omega \to \varprojlim_k \Omega/F^k\Omega$$

is injective (respectively an isomorphism).

Lemma 7. *Let Ω be an ambient module. If Ω is separated (respectively complete), it has at most one (respectively exactly one) serpentine element.*

Proof. Let $F^\bullet\Omega$ be the σ-filtration exhibiting Ω as an separated ambient module. Then two serpentine elements differ by an element of

$$\bigcap_k F^k\Omega = 0.$$

Furthermore if Ω is complete, the operator $1 - z\sigma : \Omega \to \Omega$ is invertible and

$$\zeta = (1 - z\sigma)^{-1}(\mathsf{e} + z\sigma\mathsf{h})$$

is serpentine. □

Lemma 8. *$\mathfrak{X}_{\mathbb{C}}(M)$ is a complete ambient module.*

Proof. Write $\Omega = \mathfrak{X}_{\mathbb{C}}(M)$. Recalling $\mathscr{M} = \mathcal{T} \times_E \mathcal{T} \times_E \cdots$ and passing to the quotient by the translation group, we have $M = \mathbb{T} \times \mathbb{T} \times \cdots$ with factor projections

$$\mathsf{seg}_k = \mathsf{hd} \circ \mathsf{tl} \circ \cdots \circ \mathsf{tl} : M \to \mathbb{T} \quad (k-1 \text{ compositions}).$$

Define a decreasing filtration $F^\bullet TM$ by $F^k TM = \bigcap_{1 \leq l \leq k} \ker T\,\mathsf{seg}_l$. By construction, the natural map

$$TM \to \varprojlim_k TM/F^k TM$$

is an isomorphism (of vector bundles over M). The same then holds for the twisted version

$$TM \to \varprojlim_k TM/\widetilde{F}^k TM$$

where $\widetilde{F}^k TM = \bigcap_{1 \leq l \leq k} \ker T\,\widetilde{\mathsf{seg}}_l$ for

$$\widetilde{\mathsf{seg}}_k = \mathsf{hd} \circ \widetilde{\mathsf{tl}} \circ \cdots \circ \widetilde{\mathsf{tl}} : M \to \mathbb{T} \quad (k-1 \text{ compositions}).$$

Applying the exact functor $\Gamma(M, \mathbb{C} \otimes -)$ of complexified smooth sections, we obtain an *isomorphism*

$$\Gamma(M, \mathbb{C} \otimes TM) \simeq \Omega \to \varprojlim_k \Omega/F^k\Omega \simeq \Gamma(M, \mathbb{C} \otimes \varprojlim_k TM/\widetilde{F}^k TM)$$

where $F^k\Omega = \Gamma(M, \mathbb{C} \otimes \widetilde{F}^k TM)$ define a filtration $F^\bullet\Omega$ of Ω. By the defining properties of σ, we have $\sigma F^k\Omega \subset F^{k+1}\Omega$. Furthermore, $\Omega/F^1\Omega \simeq \Gamma(M, \mathbb{C} \otimes \mathsf{hd}^* T\mathbb{T})$ and the projection map from Ω maps $\mathsf{e}, \mathsf{h}, \mathsf{f}$ to a triple of linearly independent elements of $\mathfrak{X}_{\mathbb{C}}(\mathbb{T}) \hookrightarrow \Omega/F^1\Omega$. Finally, by construction, each $F^k\Omega$ is a $\mathbb{C}[z, z^{-1}]$-submodule (being the space of sections of a vector bundle over M). Hence $F^\bullet\Omega$ is a σ-filtration. □

3.4.

Theorem 2. *Let Ω be an separated ambient module. Identify $\langle \mathsf{e}, \mathsf{h}, \mathsf{f} \rangle \subset \Omega$ with $\mathfrak{sl}_2(\mathbb{C})$, and consider the positive (twisted) loop algebra*

$$\mathfrak{L}_+^{\mathbb{C}} = \{x(t) \in t\mathfrak{sl}_2(\mathbb{C})[t] \mid x(-t) = (1 - 2\,\mathrm{ad}_{\mathsf{h}}^2)x(t)\} = \bigoplus_{i>0} \mathfrak{L}_i^{\mathbb{C}}$$

together with the induced involution τ. Then, for any serpentine $\zeta \in \Omega$, there is a unique τ-equivariant Lie algebra homomorphism

$$\Psi : \mathfrak{L}_+^{\mathbb{C}} \to \Omega$$

sending $\mathsf{e} \in L_1^{\mathbb{C}}$ to ζ. Furthermore, Ψ is injective, and thus an isomorphism onto the Lie subalgebra generated by ζ and $\tau\zeta$ in Ω.

3.5. Theorem 2 states that a serpentine subalgebra in an separated ambient module is isomorphic to the positive loop algebra of $\mathfrak{sl}_2(\mathbb{C})$. We'll prove this in the next section. Now, let's use it to prove Theorem 1.

Proof of Theorem 1. Recall $\mathfrak{g} = \mathfrak{so}(1,2)$ with Cartan involution θ inducing an $\mathsf{O}(2)$-equivariant decomposition

$$\mathfrak{g} = \mathfrak{so}(2) \oplus \mathbb{R}^2, \quad \mathfrak{so}(2) = \mathbb{R}\langle\omega\rangle, \quad \mathbb{R}^2 = \mathbb{R}\langle e_1, e_2\rangle.$$

Complexifying and extending θ by linearity, we have

$$\mathfrak{g}_{\mathbb{C}} = \mathfrak{so}_2(\mathbb{C}) + \mathbb{C}^2$$

and there is a complex Lie algebra isomorphism

$$\beta : \mathfrak{g}_{\mathbb{C}} \to \mathfrak{sl}_2(\mathbb{C})$$

such that $\beta(\omega) = ih$, $\beta(e_1) = (\mathsf{e} + \mathsf{f})/2$, $\beta(e_2) = (\mathsf{e} - \mathsf{f})/2i$.

Restricting β to $\mathbb{C}^2 = \ker(\theta + 1)$ we have a vector space isomorphism

$$\beta_1 : \mathbb{C} \otimes \mathfrak{L}_1 \to \mathfrak{L}_1^{\mathbb{C}}$$

extending to a complex Lie algebra isomorphism

$$\beta_+ : \mathbb{C} \otimes \mathfrak{L}_+ \to \mathfrak{L}_+^{\mathbb{C}}.$$

This aligns the definition of $\mathfrak{L}_+^{\mathbb{C}}$ in Theorem 2 with the definition of $\mathfrak{L}_+$ in Theorem 1.

Now, using Theorem 2 along with Lemmas 5 and 6, we have an injective Lie algebra homomorphism

$$\Psi : \mathfrak{L}_+^{\mathbb{C}} \to \mathfrak{X}_{\mathbb{C}}(M)$$

such that $\Psi(\mathsf{e}) = \zeta^+$, $\Psi(\mathsf{f}) = \zeta^-$. The composite

$$\xi_{\mathbb{C}}^+ = \Psi \circ \beta_+ : \mathbb{C} \otimes \mathfrak{L}_+ \to \mathfrak{X}_{\mathbb{C}}(M)$$

then fits into a diagram

$$\begin{array}{ccc} \mathbb{C}\otimes\mathfrak{L}_1 & \hookrightarrow & \mathbb{C}\otimes\mathfrak{L}_+ \\ \downarrow\simeq & & \downarrow\xi^+_{\mathbb{C}} \\ \mathbb{C}^2 & \xrightarrow{\mathbb{C}\otimes\xi} & \mathfrak{X}_{\mathbb{C}}(M). \end{array}$$

All four spaces in the above diagram carry a natural action of the Galois group Γ of $\mathbb{C}/\mathbb{R}$, generated by complex conjugation. The generator's action on $\mathfrak{X}_{\mathbb{C}}(M)$ interchanges e with f, and sends h to $-\mathsf{h}$. This induces an action of Γ on $\mathfrak{L}_+^{\mathbb{C}}$ such that Ψ is Γ-equivariant. Next, the map

$$\beta_1 : \mathbb{C}\otimes\mathfrak{L}_1 \to \mathfrak{L}_1^{\mathbb{C}}$$

is immediately seen to be Γ-equivariant. It follows that so is β_+, and thus also $\xi^+_{\mathbb{C}} = \Psi\circ\beta_+$. As a consequence, $\xi^+_{\mathbb{C}}$ is the complex-linear extension of a homomorphism

$$\xi^+ : \mathfrak{L}_+ \to \mathfrak{X}(M),$$

fitting into the diagram defined in the statement of Theorem 1 (a real form of the diagram above). □

4. Proving Theorem 2.

We are now in the purely algebraic setting of an separated ambient module. We will examine the relations between iterated Lie brackets of a serpentine element and its conjugate under the involution, using only the defining equation $\zeta = \mathsf{e} + z\sigma(\zeta+\mathsf{h})$. It will be fun.

Throughout this secion we will work with various objects (in various categories) equipped with a distinguished involutive automorphism. We will denote the latter genericaly by τ; if we want to be more specific, we shall write e.g. $\tau_A : A \to A$, etc.

We erase all notation of the previous sections, save for that of Definition 1 and Theorem 2, so that we can reuse symbols such as ω.

4.1. Fix the alphabet $A = \{\mathsf{a},\mathsf{b}\}$ and let $W \supset A$ be the *free semigroup* on A (i.e. the set of finite, non-empty sequences of elements of A). We refer to elements of A as *generators* and denote them generically by α; we refer to elements of W as *words* and denote them generically by w. We equip A with the non-identity involution τ_A, and extend it to an involutive semigroup automorphism $\tau_W : W \to W$.

Note the following induction principle: if $P \subset W$ is a subset (i.e. a predicate on words) such that $A \subset P$ (base case) and $AP \subset P$ (inductive step), then $P = W$; this is what we mean by 'induction on $w \in W$'. We'll also use an induction scheme exploiting the involution τ:

Lemma 9. *Let $P \subset W$ be a subset satisfying:*

(1) $\tau P = P$,
(2) $\mathsf{a} \in P$,
(3) $\mathsf{a}P \subset P$.

Then $P = W$.

Proof. Use (1) and induction on w to show $w \in P$ for all $w \in W$. □

Define the following semigroup homomorphisms (for the additive structure on the codomain):

$$\begin{array}{rl} \ell : W \to \mathbb{N}, & \ell(\mathsf{a}) = 1 = \ell(\mathsf{b}) \quad \text{(length)}; \\ \mathrm{wt} : W \to \mathbb{Z}, & \mathrm{wt}(\mathsf{a}) = 1 = -\mathrm{wt}(\mathsf{b}) \quad \text{(weight)}. \end{array}$$

Define also the function

$$\mu : W \to \mathbb{Z}, \quad \mu(\alpha) = \mathrm{wt}(\alpha), \quad \mu(\alpha w) = (\mathrm{wt}(\alpha)\,\mathrm{wt}(w) - 1)\mu(w).$$

Lemma 10. *$\ell(\tau w) = \ell(w)$, $\mathrm{wt}(\tau w) = -\mathrm{wt}(w)$, $\mu(\tau w) = -\mu(w)$.*

Put differently, ℓ, wt and μ commute with involution τ, where we put $\tau_{\mathbb{N}} = 1$ and $\tau_{\mathbb{Z}} = -1$.

Proof. The first two are immediate. For the third, we have

$$\mu(\tau\alpha) = \mathrm{wt}(\tau\alpha) = -\mathrm{wt}(\alpha) = -\mu(\alpha)$$

as the base step. Assuming $\mu(\tau w) = -\mu(w)$, compute

$$\begin{array}{rcl} \mu(\tau(\alpha w)) & = & \mu((\tau\alpha)(\tau w)) \\ & = & (\mathrm{wt}(\tau\alpha)\,\mathrm{wt}(\tau w) - 1)\mu(\tau w) \\ & = & -(\mathrm{wt}(\alpha)\,\mathrm{wt}(w) - 1)\mu(w) \\ & = & -\mu(\alpha w). \end{array}$$

Then the claim follows by induction on w. □

4.2. From now on fix an separated ambient module Ω, and identify $\mathfrak{sl}_2$ with the subalgebra spanned by $\mathsf{e}, \mathsf{h}, \mathsf{f}$ in Ω.

Given $x \in \Omega$, the evaluation bracket

$$[\![-]\!]_x : W \to \Omega$$

is defined inductively by:

$$[\![\mathsf{a}]\!]_x = x, \quad [\![\mathsf{b}]\!]_x = \tau x, \quad [\![\alpha w]\!]_x = [[\![\alpha]\!]_x, [\![w]\!]_x]$$

for $\alpha \in A$, $w \in W$.

Lemma 11. *$[\![\tau w]\!]_x = \tau [\![w]\!]_x$ for all $w \in W$, $x \in \Omega$.*

Proof. By induction on w. □

We will use two such maps: $[\![-]\!]_{\mathsf{e}} : W \to \mathfrak{sl}_2$ and $[\![-]\!]_\zeta : W \to \Omega$ for a serpentine $\zeta \in \Omega$. The former interprets a word of W as an element of $\mathfrak{sl}_2$: an iterated, right-associating Lie bracket reading every a as e, and every b as $-\mathsf{f}$. The latter interprets the word in the serpentine Lie algebra generated by ζ. In particular, the image $[\![W]\!]_\zeta$ spans the serpentine Lie algebra as a vector space.

Lemma 12. *$[\mathrm{ad}_{[\![w]\!]_{\mathsf{e}}}, z] = \mu(w) z^{1+\mathrm{wt}(w)}$, for all $w \in W$.*

Proof. We use the induction scheme of Lemma 9. Let $P \subset W$ be the set of words $w \in W$ satisfying the above equality. Note that if $w \in P$, then

$$\begin{array}{rcl} [\mathrm{ad}_{[\![\tau w]\!]_{\mathsf{e}}}, z] & = & \tau[\mathrm{ad}_{[\![w]\!]_{\mathsf{e}}}, z^{-1}] \\ & = & -\tau\mu(w) z^{-1+\mathrm{wt}(w)} \\ & = & -\mu(w) z^{1-\mathrm{wt}(w)} \\ & = & \mu(\tau w) z^{1+\mathrm{wt}(\tau w)} \end{array}$$

so that $\tau w \in P$. Furthermore, if $w \in P$, then

$$\begin{aligned}
[\mathrm{ad}_{[\![\mathrm{a}w]\!]_\mathrm{e}}, z] &= [\mathrm{ad}_{[\mathrm{e},[\![w]\!]_\mathrm{e}]}, z] \\
&= [[\mathrm{ad}_\mathrm{e}, \mathrm{ad}_{[\![w]\!]_\mathrm{e}}], z] \\
&= [\mathrm{ad}_\mathrm{e}, [\mathrm{ad}_{[\![w]\!]_\mathrm{e}}, z]] - [\mathrm{ad}_{[\![w]\!]_\mathrm{e}}, [\mathrm{ad}_\mathrm{e}, z]] \\
&= [\mathrm{ad}_\mathrm{e}, \mu(w) z^{1+\mathrm{wt}(w)}] - [\mathrm{ad}_{[\![w]\!]_\mathrm{e}}, z^2] \\
&= \mu(w)(1+\mathrm{wt}(w)) z^{2+\mathrm{wt}(w)} - 2\mu(w) z^{2+\mathrm{wt}(w)} \\
&= \mu(w)(\mathrm{wt}(w)-1) z^{1+\mathrm{wt}(\mathrm{a}w)} \\
&= \mu(\mathrm{a}w) z^{1+\mathrm{wt}(\mathrm{a}w)}
\end{aligned}$$

so that $\mathrm{a}w \in P$. □

4.3. Let $\mathbb{C}W$ denote the semigroup algebra of W; that is, a free vector space on the set W, with a non-unital algebra structure is induced by the semigroup structure on W. The involution τ_W induces an involutive algebra automorphism $\tau_{\mathbb{C}W}$ of $\mathbb{C}W$.

Define a linear map

$$\omega : \mathbb{C}W \to \mathbb{C}W$$

by $\omega(w) = \mathrm{wt}(w)w$ for $w \in W$.

Lemma 13. $\omega \circ \tau = -\tau \circ \omega$.

Proof. $\omega(\tau w) = \mathrm{wt}(\tau w)\tau w = -\tau\, \mathrm{wt}(w) w = -\tau\omega(w)$. □

Next, we introduce the crucial *rest map*

$$\mathsf{R} : W \to \mathbb{C}W$$

defined inductively by

$$\begin{aligned}
\mathsf{R}(\alpha) &= 0, \\
\mathsf{R}(\alpha w) &= \alpha \mathsf{R}(w) + \mathrm{wt}(\alpha)\, [\mathrm{wt}(w)\mathsf{R}(w) + \omega \mathsf{R}(w) + 2\,\mathrm{wt}(w) w - 2\mu(w)\alpha]
\end{aligned}$$

for $\alpha \in A$, $w \in W$.

Lemma 14. $\mathsf{R} \circ \tau = \tau \circ \mathsf{R}$.

Proof. We proceed by induction on $w \in W$. The base case is trivial. Assume $\mathsf{R}(\tau w) = \tau \mathsf{R}(w)$. Then

$$\begin{aligned}
\mathsf{R}(\tau(\alpha w)) &= \mathsf{R}((\tau\alpha)(\tau w)) \\
&= \tau\alpha \mathsf{R}(\tau w) + \mathrm{wt}(\tau\alpha)\, [\mathrm{wt}(\tau w)\mathsf{R}(\tau w) + \omega \mathsf{R}(\tau w) + 2\,\mathrm{wt}(\tau w)\tau w - 2\mu(\tau w)\tau\alpha] \\
&= (\tau\alpha)\tau \mathsf{R}(w) - \mathrm{wt}(\alpha)\, [-\mathrm{wt}(w)\tau \mathsf{R}(w) - \tau\omega \mathsf{R}(w) - 2\,\mathrm{wt}(w)\tau w + 2\mu(w)\tau\alpha] \\
&= \tau\alpha \mathsf{R}(w) + \tau\, \mathrm{wt}(\alpha)\, [\mathrm{wt}(w)\mathsf{R}(w) + \omega \mathsf{R}(w) + 2\,\mathrm{wt}(w) w - 2\mu(w)\alpha] \\
&= \tau \mathsf{R}(w)
\end{aligned}$$

completing the inductive step. □

4.4. Given $x \in \Omega$, we extend the bracket evaluation map $[\![-]\!]_x : W \to \Omega$ to a linear map

$$[\![-]\!]_x : \mathbb{C}W \to \Omega.$$

It is obviously τ-equivariant.

Proposition 2. *Let $\zeta \in \Omega$ be serpentine. Then*

$$[\![w]\!]_\zeta = [\![w]\!]_{\mathsf{e}} + z^{\mathrm{wt}\, w}\sigma\left([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta\right)$$

for all $w \in W$.

Let us first collect some results on the action of $\mathfrak{sl}_2$.

Lemma 15. *Let $\zeta \in \Omega$ be serpentine. Then:*

(1) $[\mathsf{h}, \zeta] = \zeta$, $[\mathsf{e}, \zeta] = z(\zeta - \mathsf{e})$, $[\mathsf{f}, \zeta] = z^{-1}(\zeta - \mathsf{e}) + 2\mathsf{h}$
(2) $[\mathsf{h}, [\![w]\!]_\zeta] = \mathrm{wt}(w)[\![w]\!]_\zeta$
(3) $[\mathsf{h}, [\![r]\!]_\zeta] = [\![\omega r]\!]_\zeta$

for all $w \in W$, $r \in \mathbb{C}W$.

Proof. For (1), we use the serpentine equation:

$$[\mathsf{e}, \zeta] = [\mathsf{e}, \mathsf{e} + z\sigma(\zeta + \mathsf{h})] = z^2\sigma(\zeta + \mathsf{h}) = z(\zeta - \mathsf{e})$$

$$[\mathsf{h}, \zeta] = [\mathsf{h}, \mathsf{e} + z\sigma(\zeta + \mathsf{h})] = \mathsf{e} + z\sigma(\zeta + \mathsf{h}) = \zeta$$

$$[\mathsf{f}, \zeta] = [\mathsf{f}, \mathsf{e} + z\sigma(\zeta + \mathsf{h})] = 2\mathsf{h} + \sigma(\zeta + \mathsf{h}) = z^{-1}(\zeta - \mathsf{e}) + 2\mathsf{h}.$$

For (2), we use $[\mathsf{h}, \zeta] = \zeta$ and $[\mathsf{h}, \tau\zeta] = \tau[-\mathsf{h}, \zeta] = -\tau\zeta$ and proceed by induction on w. The base case is follows from (1), and for the inductive step, assuming the formula holds for $w \in W$, calculate:

$$\begin{aligned}
[\mathsf{h}, [\![\alpha w]\!]_\zeta] &= [\mathsf{h}, [[\![\alpha]\!]_\zeta, [\![w]\!]_\zeta]] \\
&= [[\mathsf{h}, [\![\alpha]\!]_\zeta], [\![w]\!]_\zeta] + [[\![\alpha]\!]_\zeta, [\mathsf{h}, [\![w]\!]_\zeta]] \\
&= [\mathrm{wt}(\alpha)[\![\alpha]\!]_\zeta, [\![w]\!]_\zeta] + [[\![\alpha]\!]_\zeta, \mathrm{wt}(w)[\![w]\!]_\zeta] \\
&= \mathrm{wt}(\alpha w)[\![\alpha w]\!]_\zeta
\end{aligned}$$

for $\alpha \in A$. Finally, (3) follows from (2) by linearity. □

Proof of Proposition 2. Let $P \subset W$ be the subset of elements $w \in W$ satisfying the identity

$$[\![w]\!]_\zeta = [\![w]\!]_{\mathsf{e}} + z^{\mathrm{wt}\, w}\sigma\left([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta\right).$$

We claim $P = W$. First let us check that $\tau P = P$: assuming $w \in P$, we have

$$\begin{aligned}
[\![\tau w]\!]_\zeta &= \tau\left([\![w]\!]_{\mathsf{e}} + z^{\mathrm{wt}\, w}\sigma([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta)\right) \\
&= [\![\tau w]\!]_{\mathsf{e}} + z^{-\mathrm{wt}\, w}\sigma([\![\tau w]\!]_\zeta - \mu(\tau w)\tau\mathsf{h} + \tau[\![\mathsf{R}(w)]\!]_\zeta) \\
&= [\![\tau w]\!]_{\mathsf{e}} + z^{\mathrm{wt}\, \tau w}\sigma([\![\tau w]\!]_\zeta + \mu(\tau w)\mathsf{h} + [\![\mathsf{R}(\tau w)]\!]_\zeta)
\end{aligned}$$

so that $\tau w \in P$. Next, $\mathsf{a} \in P$ by definition of a serpentine element:

$$[\![\mathsf{a}]\!]_\zeta = \zeta = \mathsf{e} + z\sigma(\zeta + \mathsf{h}) = [\![\mathsf{a}]\!]_{\mathsf{e}} + z^{\mathrm{wt}(\mathsf{a})}\sigma([\![a]\!]_\zeta + \mu(\mathsf{a})\mathsf{h} + [\![\mathsf{R}(\mathsf{a})]\!]_\zeta).$$

By Lemma 9 we now only need to show that $w \in P \implies \mathsf{a}w \in P$. Assuming $w \in P$, we have:

$$\begin{aligned}
[\![a]\!]_\zeta &= \mathsf{e} + z\sigma(\zeta + \mathsf{h}) \\
[\![w]\!]_\zeta &= [\![w]\!]_{\mathsf{e}} + z^{\mathrm{wt}\, w}\sigma([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta)
\end{aligned}$$

so that, using Lemma 12,

$$\begin{aligned}
[\![\mathsf{a}w]\!]_\zeta &= [[\![a]\!]_\zeta, [\![w]\!]_\zeta] \\
&= [\mathsf{e} + z\sigma(\zeta + \mathsf{h}),\ [\![w]\!]_\mathsf{e} + z^{\mathrm{wt}\, w}\sigma([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta)] \\
&= [\mathsf{e}, [\![w]\!]_\mathsf{e}] \\
&+ z^{1+\mathrm{wt}\, w}\sigma[\zeta + \mathsf{h},\ [\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta] \\
&+ \mathrm{wt}(w) z^{1+\mathrm{wt}\, w}\sigma([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta) \\
&- \mu(w) z^{1+\mathrm{wt}\, w}\sigma(\zeta + \mathsf{h}) \\
&= [\![\mathsf{a}w]\!]_\mathsf{e} + z^{\mathrm{wt}(\mathsf{a}w)}\sigma\mathscr{E}
\end{aligned}$$

where

$$\begin{aligned}
\mathscr{E} &= [\zeta + \mathsf{h},\ [\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta] \\
&+ \mathrm{wt}(w)\left([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta\right) \\
&- \mu(w)\,(\zeta + \mathsf{h})\,.
\end{aligned}$$

We use Lemma 15 to compute the Lie bracket:

$$[\zeta + \mathsf{h},\ [\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta] = [\![\mathsf{a}w]\!]_\zeta - \mu(w)\zeta + [\![\mathsf{a}\mathsf{R}(w)]\!]_\zeta + \mathrm{wt}(w)[\![w]\!]_\zeta + [\![\omega\mathsf{R}(w)]\!]_\zeta.$$

Substituting, we get:

$$\begin{aligned}
\mathscr{E} &= [\![\mathsf{a}w]\!]_\zeta - \mu(w)\zeta + [\![\mathsf{a}\mathsf{R}(w)]\!]_\zeta \\
&+ \mathrm{wt}(w)[\![w]\!]_\zeta + [\![\omega\mathsf{R}(w)]\!]_\zeta \\
&+ \mathrm{wt}(w)\left([\![w]\!]_\zeta + \mu(w)\mathsf{h} + [\![\mathsf{R}(w)]\!]_\zeta\right) \\
&- \mu(w)\,(\zeta + \mathsf{h}) \\
&= [\![\mathsf{a}w]\!]_\zeta + \mu(w)(\mathrm{wt}(w) - 1)\mathsf{h} \\
&+ [\![\mathsf{a}\mathsf{R}(w)]\!]_\zeta - 2\mu(w)\zeta + 2\,\mathrm{wt}(w)[\![w]\!]_\zeta + [\![\omega\mathsf{R}(w)]\!]_\zeta \\
&+ \mathrm{wt}(w)[\![\mathsf{R}(w)]\!]_\zeta \\
&= [\![\mathsf{a}w]\!]_\zeta + \mu(w)(\mathrm{wt}(w) - 1)\mathsf{h} + [\![R']\!]_\zeta
\end{aligned}$$

where

$$\begin{aligned}
R' &= \mathsf{a}\mathsf{R}(w) + \mathrm{wt}(w)\mathsf{R}(w) + \omega\mathsf{R}(w) + 2\,\mathrm{wt}(w)w - 2\mu(w)\mathsf{a} \\
&= \mathsf{R}(\mathsf{a}w).
\end{aligned}$$

The latter equality follows from the inductive definition of R. Similarly recognising

$$(\mathrm{wt}(w) - 1)\mu(w) = \mu(\mathsf{a}w),$$

we then have

$$\mathscr{E} = [\![\mathsf{a}w]\!]_\zeta + \mu(\mathsf{a}w)\mathsf{h} + [\![\mathsf{R}(\mathsf{a}w)]\!]_\zeta$$

and thus

$$[\![\mathsf{a}w]\!]_\zeta = [\![\mathsf{a}w]\!]_\mathsf{e} + z^{\mathrm{wt}(\mathsf{a}w)}\sigma\left([\![\mathsf{a}w]\!]_\zeta + \mu(\mathsf{a}w)\mathsf{h} + [\![\mathsf{R}(\mathsf{a}w)]\!]_\zeta\right)$$

proving $\mathsf{a}w \in P$. □

4.5. Let us from now on fix a serpentine element $\zeta \in \Omega$; there is at most one by separatedness of Ω. We may then speak of *the* serpentine Lie subalgebra $\mathfrak{S} \subset \Omega$.

Lemma 16.

(1) $[\![-]\!]_\zeta : \mathbb{C}W \to \mathfrak{S}$ *is surjective,*
(2) ω *acts diagonalisably on* $\mathbb{C}W$*,*
(3) ad_h *acts diagonalisably on* $\mathfrak{S}$*,*
(4) *eigenspaces of* ω *are mapped onto eigenspaces of* ad_h *(with identical eigenvalues).*

Proof. By the Jacobi identity, every nested bracket of copies of $\zeta, \tau\zeta$ in Ω is a linear combination of right-associated iterated brackets of these elements, proving (1). Statement (2) is immediate by construction of ω. Now, recalling part (3) of Lemma 15, we have a linear surjection $\mathbb{C}W \to \mathfrak{S}$ intertwining a pair of operators. Since the one acting on the domain is diagonalisable, it follows that so is the one acting on the codomain, and eigenspaces of the latter are images of those of the former. □

Lemma 17. *Suppose* $w_1, \dots, w_n \in W$*;* $c_1, \dots, c_n \in \mathbb{C}$*;* $\lambda \in \mathbb{Z}$ *satisfy*

(1) $\mathrm{wt}(w_i) = \lambda$ *for all* i*,*
(2) $\sum_i c_i [\![w_i]\!]_\mathsf{e} = 0$*,*
(3) $\sum_i c_i [\![\mathsf{R}(w_i)]\!]_\zeta = 0$.

Then $\sum_i c_i [\![w_i]\!]_\zeta = 0$.

Proof. Note that by Lemma 12, $\sum_i c_i \mu(w_i) = 0$. Then, letting $x = \sum_i c_i [\![w_i]\!]_\zeta \in \Omega$, we have $x = z^\lambda \sigma x$ by Proposition 2. It follows that $x \in \bigcap_k F^k\Omega$ for every σ-filtration $F^\bullet\Omega$, and thus $x = 0$ by separatedness of Ω. □

4.6. Proposition 2 provides a corecursive expression for each of the elements $[\![w]\!]_\zeta$, $w \in W$ spanning the serpentine Lie algebra $\mathfrak{S}$ over $\mathbb{C}$. By Lemma 17, this evaluation bracket is essentially determined by the rest $\mathsf{R}(w)$, along with easily computable data. Now, observe that $\mathsf{R}(w)$ is contained in the subspace of $\mathbb{C}W$ spanned words of length at most $\ell(w) - 1$: this follows from the inductive definition of R. That will allow us to find relations between evaluation brackets, i.e. elements in the kernel of $[\![-]\!]_\zeta : \mathbb{C}W \to \Omega$, by induction on word length.

Let us first introduce a sequence of elements $\epsilon_k \in W$, $k \geq 1$:

$$\epsilon_1 = \mathsf{a}, \quad \epsilon_2 = \mathsf{ba}, \quad \epsilon_3 = \mathsf{aba}, \quad \epsilon_k = \cdots\mathsf{ababa}$$

so that $\ell(\epsilon_k) = k$, and $\mathrm{wt}(\epsilon_k) = k - 2\lfloor k/2 \rfloor$.

Lemma 18. $\mu(\epsilon_k) = (-1)^{k-1} 2^{\lfloor k/2 \rfloor}$.

Proof. We proceed by induction on $k > 0$. Base case follows from $\mu(\epsilon_1) = \mu(\mathsf{a}) = 1$. Split the inductive step into even and odd cases, and use the inductive definition of μ. Assuming the formula holds for $k = 2m$, we have:

$$\mu(\epsilon_{2m+1}) = \mu(\mathsf{a}\epsilon_{2m}) = -\mu(\epsilon_{2m}) = 2^m = (-1)^{(2m+1)-1} 2^{\lfloor (2m+1)/2 \rfloor}.$$

Assuming the formula holds for $k = 2m + 1$, we have:

$$\mu(\epsilon_{2m+2}) = \mu(\mathsf{b}\epsilon_{2m+1}) = -2\mu(\epsilon_{2m+1}) = -2 \cdot 2^m = (-1)^{(2m+2)-1} 2^{\lfloor (2m+2)/2 \rfloor}.$$

This completes the inductive step. □

Proposition 3. *For all* $k > 0$*:*

(1) $[\![\mathsf{a}\epsilon_k]\!]_\zeta = 0$ *if* k *is odd,*

(2) $[\![(1+\tau)\epsilon_k]\!]_\zeta = 0$ *if* k *is even,*
(3) $[\![\mathsf{R}(\epsilon_k)]\!]_\zeta = (1-\tau)\sum_{1\le i\le k-1}(-1)^{k-i}2^{\lfloor k/2\rfloor-\lfloor i/2\rfloor}[\![\epsilon_i]\!]_\zeta$.

Proof. Introduce the following sets:

- P: positive integers k for which (1) and (2) hold,
- Q: positive integers k for which (3) holds.

We will prove the following claims:

(A) $\forall k>0.\ \{1,\dots,k-1\}\subset P \wedge k-1\in Q \implies k\in Q$,
(B) $\forall k>0.\ \{1,\dots,k-1\}\subset P \wedge k\in Q \implies k\in P$.

By mutual induction, these will imply $P=Q=\mathbb{N}\setminus\{0\}$, completing the proof of the Proposition (note that $1\in Q$ trivially).

We will compute in $\mathbb{C}W$ modulo $\ker[\![-]\!]_\zeta$, instead of Ω, to keep notation cleaner.

- **(B)**, $k=2m-1$:

$$\mathrm{wt}(\mathsf{a}\epsilon_{2m-1})=2 \implies [\![\mathsf{a}\epsilon_{2m-1}]\!]_\mathsf{e}=0$$

and

$$\begin{aligned}\mathsf{R}(\mathsf{a}\epsilon_{2m-1}) &= \mathsf{a}\mathsf{R}(\epsilon_{2m-1})+\mathsf{R}(\epsilon_{2m-1})+\omega\mathsf{R}(\epsilon_{2m-1})+2\epsilon_{2m-1}-2^m\mathsf{a}\\ &= (\mathsf{a}+1+\omega)\mathsf{R}(\epsilon_{2m-1})+2\epsilon_{2m-1}-2^m\mathsf{a};\end{aligned}$$

so that, using $\{1,\dots,2m-2\}\subset P$ and $2m-1\in Q$,

$$\begin{aligned}\mathsf{R}(\mathsf{a}\epsilon_{2m-1}) &= (\mathsf{a}+1+\omega)(1-\tau)\sum_{1\le i\le 2m-2}(-1)^{2m-1-i}2^{m-1-\lfloor i/2\rfloor}\epsilon_i+2\epsilon_{2m-1}-2^m\mathsf{a}\\ &\cong \sum_{1\le i\le 2m-3}^{\text{odd}}2^{m-1-(i-1)/2}(2\epsilon_i+\epsilon_{i+1})\\ &- 2\sum_{2\le i\le 2m-2}^{\text{even}}2^{m-1-i/2}(\epsilon_{i+1}+\epsilon_i)\\ &+ 2\epsilon_{2m-1}-2^m\mathsf{a}\\ &= 2^m\epsilon_1+\sum_{3\le i\le 2m-3}^{\text{odd}}2^{\lfloor m-1\rfloor-(i-1)/2}(2\epsilon_i-2\epsilon_i)\\ &- 2\epsilon_{2m-1}+\sum_{2\le i\le 2m-2}^{\text{even}}2^{\lfloor m-1\rfloor-i/2}(-2\epsilon_i+2\epsilon_i)\\ &+ 2\epsilon_{2m-1}-2^m\epsilon_1\\ &= 0 \pmod{\ker[\![-]\!]_\zeta}\end{aligned}$$

whence $[\![\mathsf{a}\epsilon_{2m-1}]\!]_\zeta=0$ by Lemma 17.

- **(B)**, $k=2m$:

$$\mathrm{wt}(\epsilon_{2m})=\mathrm{wt}(\tau\epsilon_{2m})=0 \implies [\![(1+\tau)\epsilon_{2m}]\!]_\mathsf{e}=c(1+\tau)\mathsf{h}=0$$

and

$$[\![\mathsf{R}((1+\tau)\epsilon_{2m})]\!]_\zeta\in(1+\tau)(1-\tau)\Omega=0$$

whence $[\![(1+\tau)\epsilon_{2m-1}]\!]_\zeta=0$ by Lemma 17.

- **(A)**, $k = 2m$:

$$\begin{aligned}\mathsf{R}(\epsilon_{2m}) &= \mathsf{R}(\mathsf{b}\epsilon_{2m-1})\\ &= \mathsf{b}\mathsf{R}(\epsilon_{2m-1}) - \mathsf{R}(\epsilon_{2m-1}) - \omega\mathsf{R}(\epsilon_{2m-1}) - 2\epsilon_{2m-1} + 2^m\mathsf{b}\\ &= (\mathsf{b} - 1 - \omega)\mathsf{R}(\epsilon_{2m-1}) - 2\epsilon_{2m-1} + 2^m\mathsf{b}\end{aligned}$$

so that, using $\{1,\dots,2m-1\} \subset P$ and $2m-1 \in Q$,

$$\begin{aligned}\mathsf{R}(\epsilon_{2m}) &= (\mathsf{b} - 1 - \omega)(1-\tau)\sum_{1\le i\le 2m-2}(-1)^{2m-1-i}2^{m-1-\lfloor i/2\rfloor}\epsilon_i - 2\epsilon_{2m-1} + 2^m b\\ &\cong \sum_{1\le i\le 2m-3}^{\text{odd}} 2^{m-1-(i-1)/2}(\epsilon_{i+1} - 2\epsilon_i)\\ &- \sum_{2\le i\le 2m-2}^{\text{even}} 2^{m-1-i/2}(-2\tau\epsilon_{i+1} - 2\epsilon_i)\\ &- 2\epsilon_{2m-1} + 2^m b\\ &= -2^m\epsilon_1 + 2\tau\epsilon_{2m-1} + \sum_{3\le i\le 2m-3}^{\text{odd}} 2^{m-1-(i-1)/2}(-2\epsilon_i + 2\tau\epsilon_i)\\ &+ \sum_{2\le i\le 2m-2}^{\text{even}} 2^{m-1-i/2}(2\epsilon_i + 2\epsilon_i)\\ &- 2\epsilon_{2m-1} + 2^m\tau\epsilon_1\\ &\cong \sum_{1\le i\le 2m-1} 2^{m-\lfloor i/2\rfloor}(1-\tau)(-1)^i\epsilon_i \pmod{\ker[\![-]\!]_\zeta}\end{aligned}$$

whence $[\![\mathsf{R}(\epsilon_{2m})]\!]_\zeta = (1-\tau)\sum_{1\le i\le 2m-1}(-1)^{2m-i}2^{m-\lfloor i/2\rfloor}[\![\epsilon_i]\!]_\zeta$.

- **(A)**, $k = 2m+1$:

$$\mathsf{R}(\epsilon_{2m+1}) = \mathsf{R}(\mathsf{a}\epsilon_{2m}) = \mathsf{a}\mathsf{R}(\epsilon_{2m}) + \omega\mathsf{R}(\epsilon_{2m}) + 2^{m+1}\mathsf{a}$$

so that, using $\{1,\dots,2m\} \subset P$ and $2m \in Q$,

$$\begin{aligned}\mathsf{R}(\epsilon_{2m+1}) &= (\mathsf{a} + \omega)(1-\tau)\sum_{1\le i\le 2m-1}(-1)^{2m-i}2^{m-\lfloor i/2\rfloor}\epsilon_i + 2^{m+1}\mathsf{a}\\ &\cong \sum_{2\le i\le 2m-2}^{\text{even}} 2^{m-i/2}(2\epsilon_{i+1})\\ &- \sum_{1\le i\le 2m-1}^{\text{odd}} 2^{m-(i-1)/2}(\epsilon_i + \epsilon_{i+1} + \tau\epsilon_i)\\ &+ 2^{m+1}\mathsf{a}\\ &= -2^m(1+\tau)\epsilon_1 + \sum_{3\le i\le 2m-1}^{\text{odd}} 2^{m-(i-1)/2}(2\epsilon_i - (1+\tau)\epsilon_i)\\ &- \sum_{2\le i\le 2m}^{\text{even}} 2^{m-i/2}2\epsilon_i\\ &+ 2^{m+1}\epsilon_1\\ &= \sum_{1\le i\le 2m+1} 2^{m-\lfloor i/2\rfloor}(1-\tau)(-1)^{i+1}\epsilon_i \pmod{\ker[\![-]\!]_\zeta}\end{aligned}$$

whence $[\![\mathsf{R}(\epsilon_{2m+1})]\!]_\zeta = (1-\tau)\sum_{1\le i\le 2m}(-1)^{2m+1-i}2^{m-\lfloor i/2\rfloor}[\![\epsilon_i]\!]_\zeta$. □

4.7.

Lemma 19. *Suppose* $\sum_i c_i[\![\epsilon_i]\!]_\zeta = 0$ *with* $c_1, c_2, \cdots \in \mathbb{C}$ *(almost all zero). Then* $\sum_i c_i[\![\mathsf{R}(\epsilon_i)]\!]_\zeta = 0$.

Proof. Set $\mu_i = \mu(\epsilon_i)$ and note that

$$[\![\epsilon_i]\!]_{\mathsf{e}} = \mu_i \cdot \begin{cases} \mathsf{e} & i \text{ odd} \\ \mathsf{h} & i \text{ even} \end{cases}$$

By Lemma 16 (decomposing $\mathfrak{S}$ into eigenspaces of ad_{h}), we may WLOG assume that all i such that $c_i \neq 0$ have the same parity. In the odd case,

$$\begin{aligned} 0 &= \sum_i c_i[\![\epsilon_i]\!]_\zeta \\ &= \sum_i c_i[\mu_i\mathsf{e} + z\sigma([\![\epsilon_i]\!]_\zeta + \mu_i\mathsf{h} + [\![\mathsf{R}(\epsilon_i)]\!]_\zeta)] \\ &= (\sum_i c_i\mu_i)(\mathsf{e} + z\sigma\mathsf{h}) + z\sigma(\sum_i c_i[\![\mathsf{R}(\epsilon_i)]\!]_\zeta). \end{aligned}$$

Taking its value in $\Omega/F^1\Omega$ for a σ-filtration $F^\bullet\Omega$, we have

$$0 \cong (\sum_i c_i\mu_i)\mathsf{e} \pmod{F^1\Omega}$$

so that $\sum_i c_i\mu_i = 0$. Then, since z is invertible and σ is injective, it follows that $\sum_i c_i[\![\mathsf{R}(\epsilon_i)]\!]_\zeta = 0$.

In the even case,

$$\begin{aligned} 0 &= \sum_i c_i[\![\epsilon_i]\!]_\zeta \\ &= \sum_i c_i[\mu_i\mathsf{h} + \sigma([\![\epsilon_i]\!]_\zeta + \mu_i\mathsf{h} + [\![\mathsf{R}(\epsilon_i)]\!]_\zeta)] \\ &= (\sum_i c_i\mu_i)(\mathsf{h} + \sigma\mathsf{h}) + \sigma(\sum_i c_i[\![\mathsf{R}(\epsilon_i)]\!]_\zeta). \end{aligned}$$

Taking its value in $\Omega/F^1\Omega$ we have once again $\sum_i c_i\mu_i = 0$, and it follows that $\sum_i c_i[\![\mathsf{R}(\epsilon_i)]\!]_\zeta = 0$. □

Define

$$E_0 = \{\epsilon_{2j} \mid j > 0\}, \quad E_1 = \{\epsilon_{2j+1} \mid j \geq 0\}, \quad E_{-1} = \tau E_1$$

and

$$E = E_1 \cup E_0 \cup E_{-1} \subset W.$$

Consider the subspace $\mathbb{C}E \subset \mathbb{C}W$ and the restriction of the evaluation bracket.

Lemma 20. $[\![-]\!]_\zeta : \mathbb{C}E \to \mathfrak{S}$ *is injective.*

Proof. Let $Z \subset \mathbb{C}E$ be the kernel of $[\![-]\!]_\zeta : \mathbb{C}E \to \mathfrak{S}$. By Lemma 16, Z decomposes into eigenspaces of ω:

$$Z = Z_1 \oplus Z_0 \oplus Z_{-1}, \quad Z_d \subset \mathbb{C}E_d$$

with $Z_{-1} = \tau Z_1$. Now, let $F^\bullet Z_0$, $F^\bullet Z_1$ be the filtration by word length, so that

$$F^k Z_d = \mathbb{C}\langle w \in E_d \mid [\![w]\!]_\zeta = 0 \ \wedge \ \ell(w) \leq k\rangle.$$

We will show that there are injective maps

$$F^{2j+1}Z_1 \to F^{2j}Z_0 \to F^{2j-1}Z_1$$

for all $j \geq 1$. It then follows that $Z_1 = F^1 Z_1 = 0$ and thus also $Z_0 = 0$.

Let $P_\lambda : \mathbb{C}E \to \mathbb{C}E$, $\lambda \in \{-1, 0, 1\}$, denote the projections onto eigenspaces of ω with eigenvalue λ. Note that the restriction of $\mathsf{R} : W \to \mathbb{C}W$ to E factors through $\mathbb{C}E \subset \mathbb{C}W$ (by Proposition 3) and extends to a linear map

$$\widetilde{\mathsf{R}} : \mathbb{C}E \to \mathbb{C}E.$$

Define

$$R_1 = P_0 \circ \widetilde{\mathsf{R}}|_{\mathbb{C}E_1} : \mathbb{C}E_1 \to \mathbb{C}E_0, \quad R_0 = P_1 \circ \widetilde{\mathsf{R}}|_{\mathbb{C}E_0} : \mathbb{C}E_0 \to \mathbb{C}E_1.$$

By Lemma 19, $R_1(Z_1) \subset Z_0$ and $R_0(Z_0) \subset Z_1$. In fact, since R maps words of given length to combinations of words of strictly lower length, we have

$$R_1(F^{2j+1}Z_1) \subset F^{2j}Z_0, \quad R_0(F^{2j}Z_0) \subset F^{2j-1}Z_1.$$

Now, we have $\ker(R_1) = \mathbb{C}\epsilon_1$, $\ker(R_0) = 0$ so that $\ker(R_d) \cap Z_d = 0$. It follows that restrictions of R_1, respectively R_0, to $F^{2j+1}Z_1$, respectively $F^{2j}Z_0$, provide the desired injections. □

Proposition 4. *$[\![-]\!]_\zeta : \mathbb{C}E \to \mathfrak{S}$ is an isomorphism of vector spaces.*

Proof. Lemma 20 proves injectivity. Since the full bracket $\mathbb{C}W \to \mathfrak{S}$ is surjective, it is enough to show that for each $w \in W$ there is $\epsilon \in E$ and $c \in \mathbb{C}$ such that $[\![w]\!]_\zeta = c[\![\epsilon]\!]_\zeta$. We use the induction scheme of Lemma 9, letting $P \subset W$ be the set of words satisfying the latter condition. First, $[\![\mathsf{a}]\!]_\zeta = [\![\epsilon_1]\!]_\zeta$ so $\mathsf{a} \in P$. Second, $\tau E = E$ so $\tau P = P$. Finally, $[\![\mathsf{a}E]\!]_\zeta \subset [\![E]\!]_\zeta$ by Proposition 3 so $\mathsf{a}P \subset P$. □

4.8.

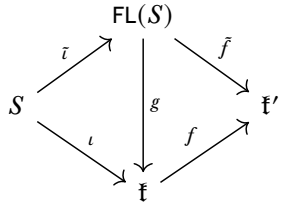
Lemma 21. *Let $\mathfrak{k} \supset S$ be a Lie algebra with a chosen set of generators, $\mathfrak{k}'$ another Lie algebra, and $f : \mathfrak{k} \to \mathfrak{k}'$ a vector space homomorphism such that $[f(s), f(x)] = f([s, x])$ for all $s \in S$, $x \in \mathfrak{k}$. Then f is a Lie algebra homomorphism.*

Proof. Let $\mathsf{FL}(S)$ be the free Lie algebra on S, and let $\iota : S \to \mathfrak{k}$ and $\tilde{\iota} : S \to \mathsf{FL}(S)$ the natural inclusions of generators. By the universal property, there are Lie algebra homomorphisms

$$g : \mathsf{FL}(S) \to \mathfrak{k}, \quad g \circ \tilde{\iota} = \iota$$

$$\tilde{f} : \mathsf{FL}(S) \to \mathfrak{k}', \quad \tilde{f} \circ \tilde{\iota} = \iota \circ f$$

so that the left triangle and the outer square commute in the following diagram:



By the hypothesis on f, the right triangle commutes as well: indeed, for any right-associated iterated bracket $x = [s_n \dots [s_1, s_0] \dots]$ of generators in $\mathsf{FL}(S)$, we have

$$\begin{aligned} \tilde{f}(x) &= [\tilde{f}(\tilde{\iota}s_n) \dots [\tilde{f}(\tilde{\iota}s_1), \tilde{f}(\tilde{\iota}s_0)] \dots]] \\ &= [f(\iota s_n) \dots [f(\iota s_1), f(\iota s_0)] \dots]] \\ &= f(g(x)). \end{aligned}$$

In particular, $\tilde{f}(\ker g) = 0$ so that $\tilde{f}$ descends to a Lie algebra homomorphism $\mathfrak{k} \to \mathfrak{k}'$ necessarily coinciding with f. □

Proof of Theorem 2. By Lemma 21, it will be enough to construct a vector space isomorphism $\Psi : \mathfrak{L}_+^{\mathbb{C}} \to \mathfrak{S}$ such that $\Psi \circ \tau = \tau \circ \Psi$ and

$$\Psi([t\mathsf{e}, x]) = [\mathsf{e}, \Psi(x)]$$

for all $x \in \mathfrak{L}_+^{\mathbb{C}}$. By linearity, it is enough to check x of the form $t^{2k-1}\mathsf{e}$, $t^{2k-1}\mathsf{f}$ and $t^{2k}\mathsf{h}$, with $k > 0$. We define Ψ as a composite

$$\mathfrak{L}_+^{\mathbb{C}} \xrightarrow{\psi} \mathbb{C}E \xrightarrow{[\![-]\!]_\zeta} \mathfrak{S}$$

$$\begin{aligned} \psi(t^{2k}\mathsf{h}) &= -2^{-k}\epsilon_{2k} \\ \psi(t^{2k+1}\mathsf{e}) &= 2^{-k}\epsilon_{2k+1} \\ \psi(t^{2k+1}\mathsf{f}) &= -\tau\psi(t^{2k+1}\mathsf{e}). \end{aligned}$$

Now, computing

$$\begin{aligned} \Psi([t\mathsf{e}, t^{2k}\mathsf{h}]) &= \Psi(-t^{2k+1}\mathsf{e}) \\ &= -2^{-k}[\![\epsilon_{2k+1}]\!]_\zeta \\ &= -2^{-k}[\![\mathsf{a}\epsilon_{2k}]\!]_\zeta \\ &= [\Psi(t\mathsf{e}), \Psi(t^{2k}\mathsf{h})], \end{aligned}$$

and

$$\Psi([t\mathsf{e}, t^{2k+1}\mathsf{e}]) = 0 = [\Psi(t\mathsf{e}), \Psi(t^{2k+1}\mathsf{e})],$$

and

$$\begin{aligned} \Psi([t\mathsf{e}, t^{2k+1}\mathsf{f}]) &= \Psi(-2t^{2k+2}\mathsf{h}) \\ &= 2^{-k}[\![\epsilon_{2k+2}]\!]_\zeta \\ &= -2^{-k}[\![\mathsf{a}\tau\epsilon_{2k+1}]\!]_\zeta \\ &= [\Psi(t\mathsf{e}), \Psi(t^{2k+1}\mathsf{f})], \end{aligned}$$

we have verified that Ψ satisfies the hypotheses of the Lemma. This concludes the proof of the Theorem. □

5. Conclusions