ARE142: Annuity Formula

Tao Wang

April 8, 2025

1 Introduction

There are three types of personal savings strategies in ARE142, each has its own formula. Some of them revolve around the concept of *annuity*, which is a contract that provides a series of regular payments over a period.

They are:

1. One-Time payment now and compound after n years
2. Save money every year and compound after n years
3. Withdraw $X annually for n years from an initial saving that compound

They correspond to three formula (1.2, 1.3, 1.4) to calculate their future and present values:

1.1 Definition

$$\text{FV} = \text{Future Value} \tag{1}$$
$$\text{PV} = \text{Present Value} \tag{2}$$
$$\text{PMT} = \text{Periodic Payment in an Annuity} \tag{3}$$
$$\text{i} = \text{Interest Rate} \tag{4}$$
$$FV_{year=0} = PV \tag{5}$$
$$FV_{year=k+1} = FV_{year=k} \times (1+i) \tag{6}$$
$$\tag{7}$$

1.2 Future value of initial savings compounded after n years

1.2.1 Formula

$$FV_{year=n} = PV(1+i)^n$$

1.2.2 Proof

We will show that this is true through the strong induction.
At year 0, we have

$$FV_{year=0} = PV(1+i)^0 = PV$$

This is true because future value at year 0 is the present value.
We can assume that right after year k, we have

$$FV_{year=k} = PV(1+i)^k$$

Right after year k + 1, by we have

$$FV_{year=k+1} = PV(1+i)^{k+1} = (1+i)PV(1+i)^k = FV_{year=k} \times (1+i)$$

By Definition,

$$FV_{year=k} \times (1+i) = FV_{year=k+1}$$

Therefore, $FV_{year=n} = PV(1+i)^n$

1.3 Future value of n year annuity with payment starting at the end of the first year

1.3.1 Formula

$$FV_{year=n} = \frac{\text{PMT}}{i}\left[(1+i)^n - 1\right]$$

1.3.2 Derivation

We will pay annuity at the end of each year from year 1 to n.
Moreover, each annuity will compound until after n years.
According to Formula 1.2.1, we can express the future value of the compounded kth annuity (PV_k) at the end of year n as

$$FV_{year=n} = PV_k(1+i)^{n-i}$$

To find the total future value of n year annuity, we will combine all of the compounded kth annuity from the end of year 1 to n. We assume all annuity have the same PMT.

$$FV_{year=n} = \sum_{k=1}^{n} \text{PMT}(1+i)^{k-1} = \text{PMT}(1+i)^0 + \text{PMT}(1+i)^1 + \ldots + \text{PMT}(1+i)^{n-1}$$

This is a divergent geometric series whose sum is

$$FV_{year=n} = \frac{\text{PMT}(1-(1+i)^n)}{(1-(1+i))}$$

As a result,

$$FV_{year=n} = \frac{\text{PMT}}{-i}(1-(1+i)^n) = \boxed{\frac{\text{PMT}}{i}\left[(1+i)^n - 1\right]}$$

1.4 Present value of n year annuity with payments start at the end of the first year

1.4.1 Formula

$$PV_{year=0} = \frac{\text{PMT}}{i}\left[1 - \frac{1}{(1+i)^n}\right]$$

1.4.2 Derivation

The annuity allows us to withdraw the the same PMT at end of the kth year.
We can get the present value of this withdrawal at the beginning of year 1 with

$$PV_{year=0} = \frac{\text{PMT}}{(1+i)^k} = \text{PMT}\left(\frac{1}{1+i}\right)^k$$

Similar to how we found Formula 1.3, we will add all of the kth present values and find the sum of the geometric series

$$\begin{aligned}
PV_{year=0} = \sum_{k=1}^{n} \text{PMT}\left(\frac{1}{1+i}\right)^k &= \left(\frac{1}{1+i}\right)\left(\sum_{k=1}^{n}\text{PMT}\left(\frac{1}{1+i}\right)^{k-1}\right) \\
&= \left(\frac{1}{1+i}\right)\frac{\text{PMT}(1-(\frac{1}{1+i})^n)}{(1-(\frac{1}{1+i}))} \\
&= \left(\frac{1}{(1+i-1)}\right)\text{PMT}\left(1-\left(\frac{1}{1+i}\right)^n\right) \\
&= \boxed{\frac{\text{PMT}}{i}\left[1-\frac{1}{(1+i)^n}\right]}
\end{aligned}$$