ARE142: Annuity As A Matrix

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1 Introduction

2 General Finance Definition

- Money: a financial asset whose real worth is changing with the period according to the Time Value of Money.
- Payment (PMT): a money. A positive payment means money added to an account, while a negative payment means it's taken out of the account.
- Principal ($\mathcal{P}$): an original sum of money invested or lent.
- Period: an whole number point in time that starts at 0.
- Future Value ($FV_{k,j}$): the worth at period k of a payment made at period j.
- Present Value (PV_j): the worth at period 0 of a payment that is or will be made at period j.

$$PV_j = FV_{0,j} \qquad \text{(The Future at Right Now is Present)}$$

- Time Value of Money: The worth of money is constantly changing

$$FV_{k,j} = \begin{cases} FV_{k-1,j} \times (1 + i_k) & \text{if } k > j \\ FV_{k+1,j} \times (1 + i_k)^{-1} & \text{if } k < j \end{cases}$$

3 Future Value Matrix for Annuity

The future values ($FV_{k,j}$) of a group of payment made to an annuity (from the annuity's perspective) could be described by a Future Value Matrix (FV) that follows the Time Value of Money.

$$\text{FV} = \begin{bmatrix} FV_{0,0} & FV_{0,1} & \cdots & FV_{0,n} \\ FV_{1,0} & FV_{1,1} & \cdots & FV_{1,n} \\ \vdots & \vdots & \ddots & \vdots \\ FV_{n,0} & FV_{n,1} & \cdots & FV_{n,n} \end{bmatrix}$$

where $0 \leq k \leq n$ and $0 \leq j \leq n$

3.1 Definition: Future Value of Annuity

FVA is the total future value at period k of all payments made to the annuity. We could think of it as summing each row of the Future Value Matrix until the diagonal element.

$$\text{FVA}_k = \sum_{j=0}^{k} FV_{k,j} \qquad \text{(Future Value of Annuity)}$$

3.2 Definition: Present Value of Annuity

PVA is the total present value at period 0 of all payments that is or will be made to the annuity.

$$\text{PVA} = \sum_{j=0}^{n} PV_j = \sum_{j=0}^{n} FV_{0,j} \qquad \text{(Present Value of Annuity)}$$

4 Types of Annuity

There are different types of annuity that each fill the Future Value Matrix in a different way.

Using the Definition of each Annuity, we can derive the Growth Function and the Valuation. The Growth Function encapsulates the recursive relationship between sums of each row of the Future Value Matrix. The Valuation is a closed form formula for the sum of row k.

4.1 Single Payment Annuity

A Single Payment Annuity is denoted by FV_{SA}.

It is A single payment, PMT, made to the annuity at period 0.

4.1.1 Definition

$$FV_{k,j} = \begin{cases} \mathcal{P} & \text{if } k = 0 \text{ and } j = 0 \\ 0 & \text{if } k \neq 0 \text{ and } k = j \\ FV_{k-1,j} \times (1 + i_k) & \text{if } k > j \\ FV_{k+1,j} \times (1 + i_k)^{-1} & \text{if } k < j \end{cases}$$

$$0 \leq k \leq n \qquad \text{(Definition of Future Value Matrix)}$$
$$0 \leq j \leq n \qquad \text{(Definition of Future Value Matrix)}$$
$$i_k = i_0 \qquad \text{(Constant Interest)}$$

$$\text{FV}_{SA} = \begin{bmatrix} \mathcal{P} & 0 & \cdots & 0 \\ \mathcal{P}(1+i_0)^1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{P}(1+i_0)^n & 0 & \cdots & 0 \end{bmatrix}$$

4.1.2 Growth Function

$$\text{FVA}_{k+1} = \text{FVA}_k \times (1 + i_0)$$

4.1.3 Valuation

$$\text{PVA} = \text{FVA}_0 = \mathcal{P}$$
$$\text{FVA}_k = \mathcal{P}(1 + i_0)^k$$

4.1.4 Derivation

Derive FVA_k

4.2 Ordinary Annuity

An Ordinary Annuity is represented by FV_{OA}.

It is a regular stream of equal payments to the annuity starting at period 1 to period n (inclusive) with a principal, $\mathcal{P}$.

4.2.1 Definition

$$FV_{k,j} = \begin{cases} \mathcal{P} & \text{if } k = 0 \text{ and } j = 0 \\ \text{PMT} & \text{if } k \neq 0 \text{ and } k = j \\ FV_{k-1,j} \times (1 + i_k) & \text{if } k > j \\ FV_{k+1,j} \times (1 + i_k)^{-1} & \text{if } k < j \end{cases}$$

$$0 \leq k \leq n \qquad \text{(Definition of Future Value Matrix)}$$
$$0 \leq j \leq n \qquad \text{(Definition of Future Value Matrix)}$$
$$i_k = i_0 \qquad \text{(Constant Interest)}$$

$$\text{FV}_{OA} = \begin{bmatrix} \mathcal{P} & \text{PMT}(1+i_0)^{-1} & \cdots & \text{PMT}(1+i_0)^{-n} \\ \mathcal{P}(1+i_0)^1 & \text{PMT} & \cdots & \text{PMT}(1+i_0)^{-(n-1)} \\ \vdots & \vdots & \ddots & \vdots \\ \mathcal{P}(1+i_0)^n & \text{PMT}(1+i_0)^{n-1} & \cdots & \text{PMT} \end{bmatrix}$$

4.2.2 Growth Function

$$\text{FVA}_{k+1} = \text{FVA}_k \times (1 + i_0) + \text{PMT}$$

4.2.3 Valuation

$$\begin{aligned} \text{PVA} &= \mathcal{P} + \sum_{j=1}^{n} \text{PMT} \left(\frac{1}{1+i_0} \right)^j \\ &= \mathcal{P} + \frac{\text{PMT}}{i_0} \left[1 - \frac{1}{(1+i_0)^n} \right] \\ \text{FVA}_k &= \mathcal{P}(1+i_0)^k + \sum_{k=1}^{n} \text{PMT}(1+i)^{k-1} \\ &= \mathcal{P}(1+i_0)^k + \frac{\text{PMT}}{i_0} \left[(1+i_0)^k - 1 \right] \end{aligned}$$

4.2.4 Derivation

Derive FVA_k