# EEC 133 HW1

(a) (i) $U(\theta,\phi) = U_0 \qquad \eta = \eta_0 = 377\Omega, \quad |E_0| = 5\,\frac{V}{m}, \quad R = 100^m$

$$|\vec{S}_{av}| = \frac{1}{2\eta}|E_0|^2 = \boxed{0.033\ \frac{W}{m^2}}$$

(ii)

$$P_{rad} = |\vec{S}_{av}|\, 4\pi R^2 = \boxed{4146.9^W}$$

(b) $e = \frac{P_{rad}}{P_{in}} = 90\%, \quad P_{rad} = (0.9)(P_{in}) = 113.1^{mW}, \quad U_{max} = 200\,\frac{mW}{sr}$

(i) $P_{in} = 125.66^{mW}$

$$D_0 = \max\{D(\theta,\phi)\} = \max\left\{\frac{4\pi}{P_{rad}}\left(U(\theta,\phi)\right)\right\} = \frac{4\pi}{P_{rad}} U_{max} = 22.2^{\frac{1}{sr}} = \boxed{13.5\text{ dB}}$$

$$G_0 = eD_0 = 19.98^{\frac{1}{sr}} = \boxed{13\text{ dB}}$$

(ii) $P_{rad} = 125.66^{mW}$

$$D_0 = \frac{4\pi}{P_{rad}} U_{max} = 20^{\frac{1}{sr}} = \boxed{13^{dB}} \qquad G_0 = eD_0 = 18^{\frac{1}{sr}} = \boxed{12.6^{dB}}$$

(c)(i)

$$A = \int_{45^\circ}^{60^\circ}\int_{30^\circ}^{60^\circ} r^2\, d\Omega, \quad \Omega_A = \frac{A}{r^2} = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}}\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \sin(\theta)\, d\theta\, d\phi = \boxed{0.096^{sr}}$$

$$\beta_\theta = \frac{\pi}{6}, \quad \beta_\phi = \frac{\pi}{12}, \quad \beta_\theta\beta_\phi = \frac{\pi^2}{72} = \boxed{0.137}$$

(ii)

$$\Omega_A \approx \frac{4\pi}{D_0} \Rightarrow \boxed{D_0 \approx \frac{4\pi}{\Omega_A} = 130.9 = 21.2^{dB}}$$

$$1(d)$$

We know :

$$\int D(\theta, \phi) d\Omega = 4\pi$$

$$F(\theta, \phi) = \frac{U(\theta, \phi)}{U_{max}}$$

$$D(\theta, \phi) = \frac{4\pi}{P_{rad}} U(\theta, \phi)$$

Therefore,

$$U(\theta, \phi) = U_{max} F(\theta, \phi)$$

and

$$D(\theta, \phi) = \frac{4\pi U_{max}}{P_{rad}} F(\theta, \phi)$$

But what is $\frac{4\pi U_{max}}{P_{rad}}$?

$$\int \frac{4\pi U_{max}}{P_{rad}} F(\theta, \phi) d\Omega = 4\pi$$

$$\int F(\theta, \phi) d\Omega = \frac{P_{rad}}{U_{max}}$$

As a result,

$$D(\theta, \phi) = \frac{4\pi F(\theta, \phi)}{\int F(\theta, \phi) d\Omega}$$

(d)(i) $F(\theta,\phi) = \cos^2\theta$

Half Power Point: $\cos^2\theta = \frac{1}{2} \Rightarrow \theta_1 = -\frac{\pi}{4}, \theta_2 = \frac{\pi}{4}$

$HPBW = \frac{\pi}{4} - \left(-\frac{\pi}{4}\right) = \boxed{\frac{\pi}{2}}$

$$D(\theta,\phi) = \frac{4\pi F(\theta,\phi)}{\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \sin\theta \, d\theta \, d\phi} = \boxed{3\cos^2\theta}$$

(ii) $F(\theta,\phi) = \cos^3\theta \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

Half Power Point: $\cos^3\theta = \frac{1}{2} \Rightarrow \theta_2 = 0.654 \; \theta_2 = -0.654$

$HPBW = 0.654 - (-0.654) = \boxed{1.308}$

$$D(\theta,\phi) = \frac{4\pi F(\theta,\phi)}{\int_0^{2\pi}\int_0^{\frac{\pi}{2}} \cos^3\theta \sin\theta \, d\theta \, d\phi} = \boxed{8\cos^3\theta}$$

(iii) $F(\theta,\phi) = \cos^2\theta \cos^2(2\theta)$

Half Power Point: $\cos^2\theta \cos^2(2\theta) = \frac{1}{2} \Rightarrow \theta_1 = -0.358$

$\theta_2 = 0.358$

$HPBW = 0.358 - (-0.358) = \boxed{0.72}$

$$D(\theta,\phi) = \frac{4\pi F(\theta,\phi)}{\int_0^{2\pi}\int_0^{\pi} \cos^2\theta \cos^2(2\theta) \sin\theta \, d\theta \, d\phi} = \boxed{9.5\cos^2\theta \cos^2(2\theta)}$$

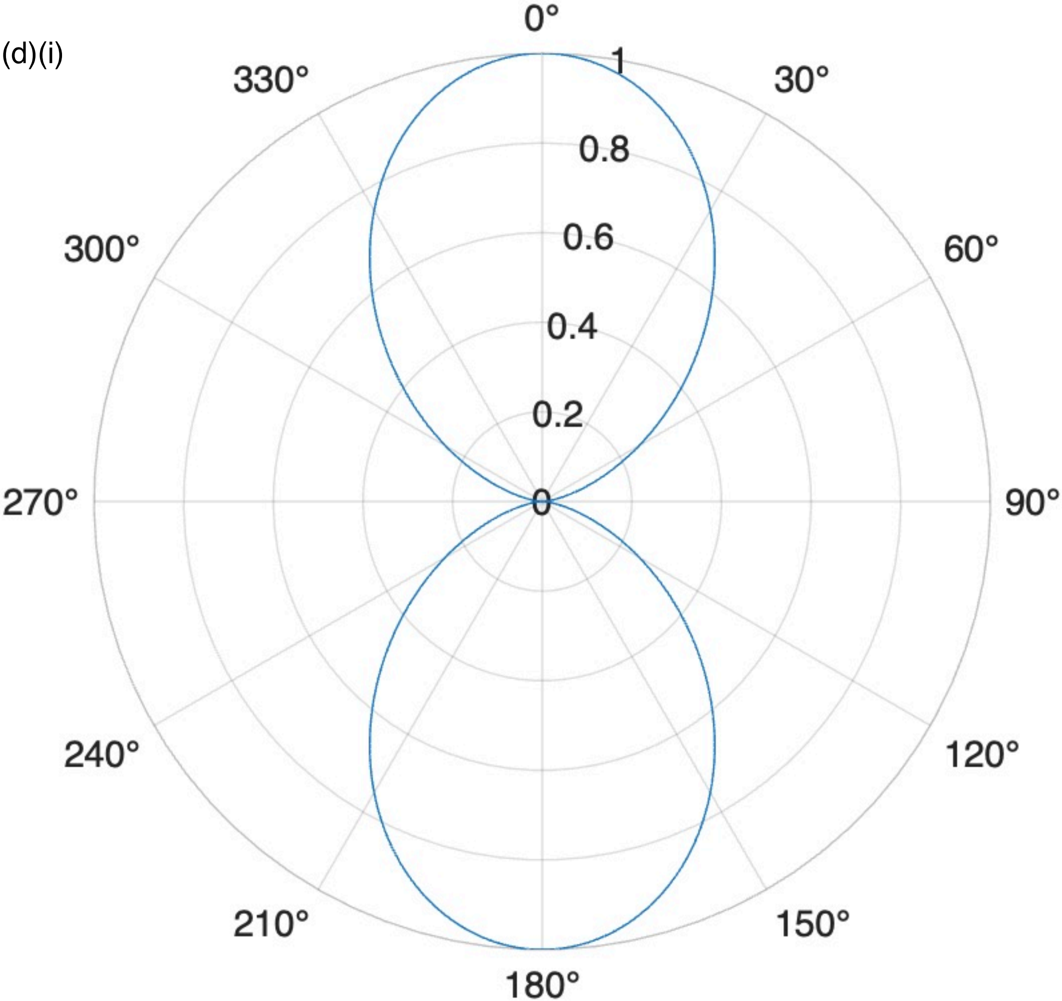
(e)

$\theta_2 - \theta_1 = (10^\circ)\left(\frac{\pi}{180^\circ}\right) \quad \cos^n(\theta_2) = \frac{1}{2} \quad \cos^n(\theta_1) = \frac{1}{2}$

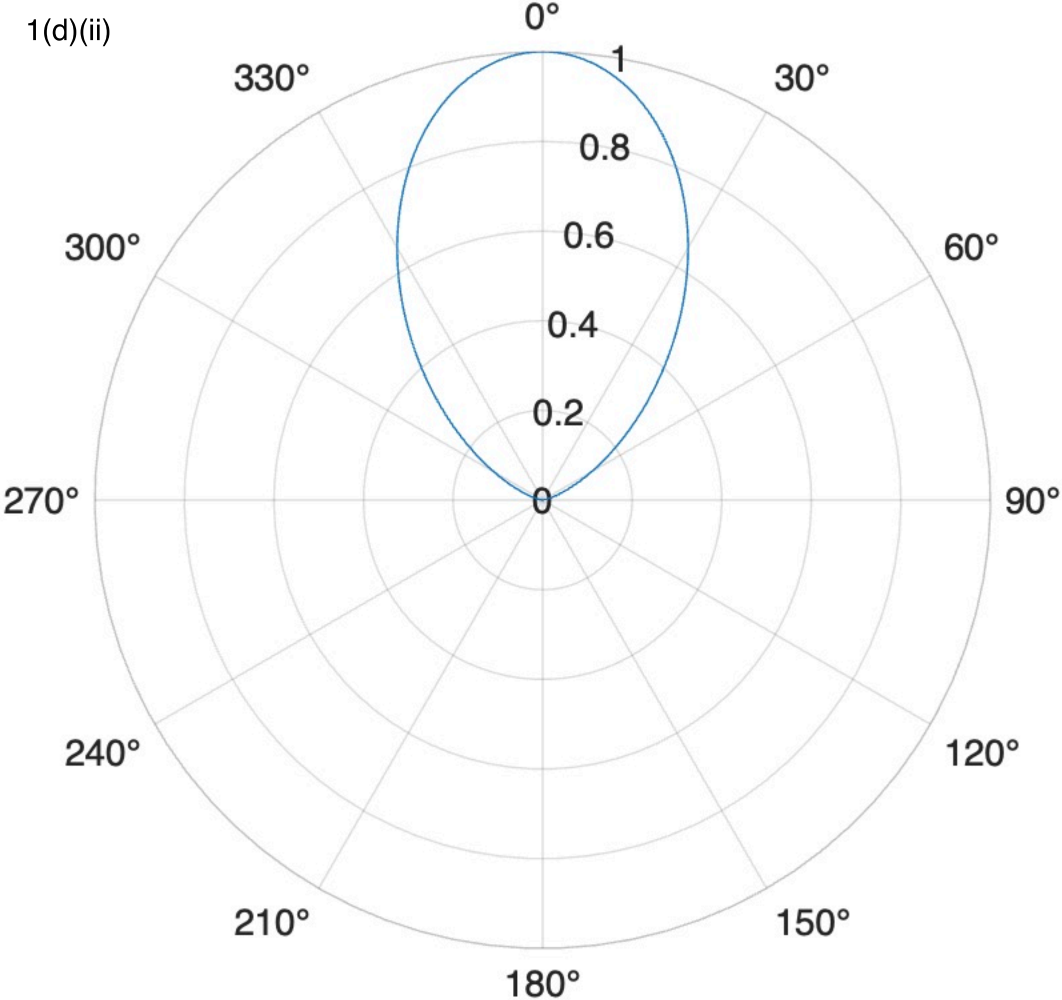
$-\frac{\pi}{2} \le \theta_1, \theta_2 \le \frac{\pi}{2}$

$\boxed{n = 181.8}, \; -\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$

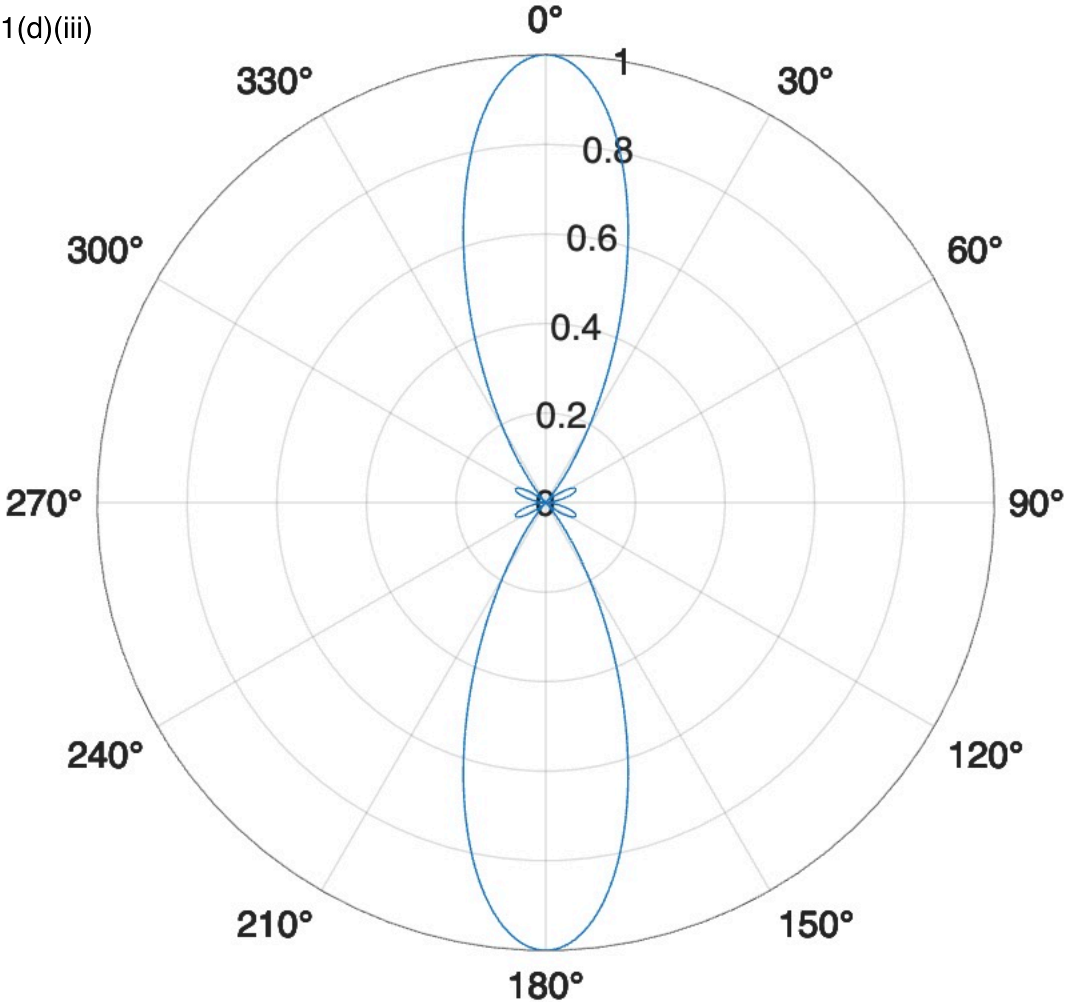
1(d)(i)



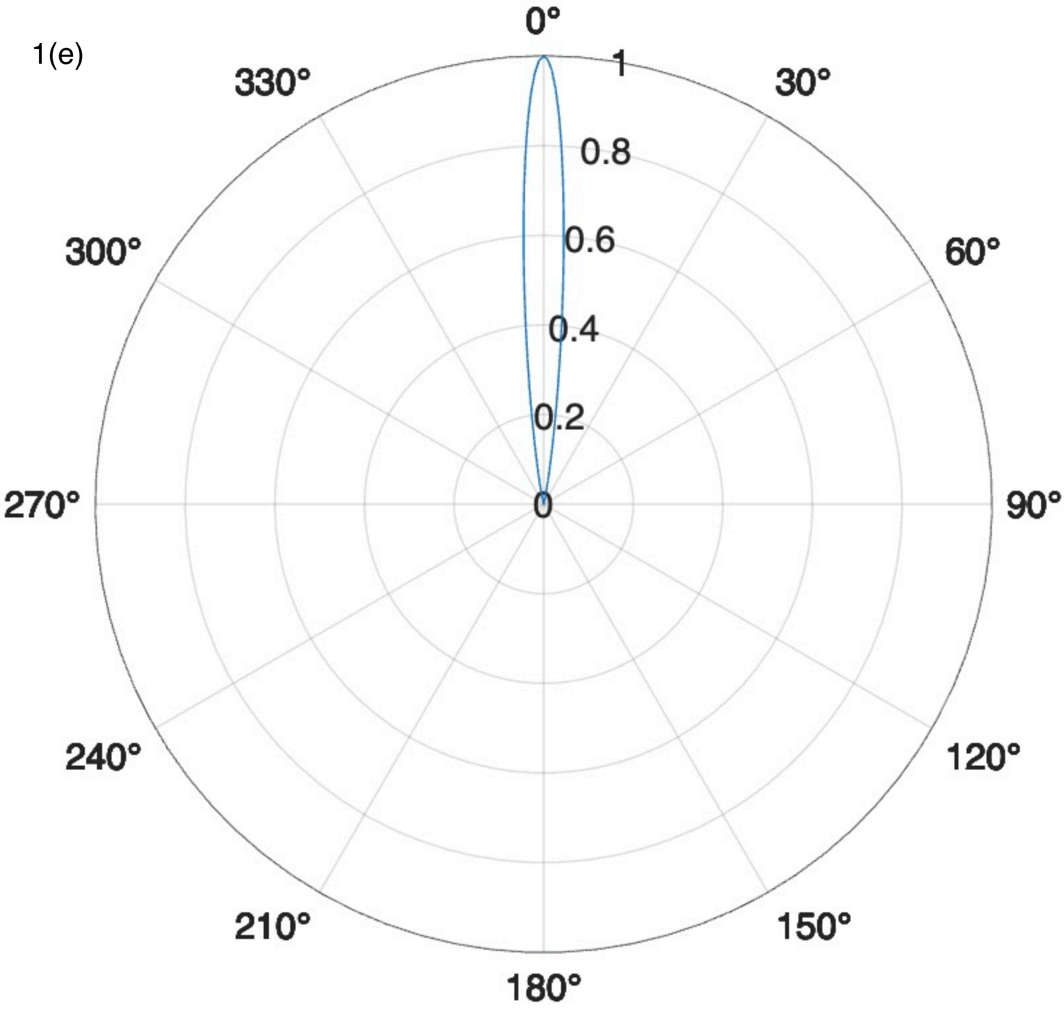
1(d)(ii)



1(d)(iii)



1(e)



②

(a) $Z_S = R_S$, $Z_L = R_L$, $P_{avg} = \frac{1}{2}|\tilde{I}_L|^2 Z_L$

$\tilde{I}_L = \frac{V_S}{Z_S + Z_L}$, $|a(b+jc)|^2 = a^2|b+jc|^2$, $a \in \mathbb{R}$

$$P_{avg_L} = \frac{1}{2}|\tilde{I}_L|^2 Z_L = \frac{1}{2} Z_L \left|\frac{\tilde{V}_S}{Z_S + Z_L}\right|^2 = \frac{1}{2} Z_L \left(\frac{1}{Z_S + Z_L}\right)^2 |\tilde{V}_S|^2$$

$$\frac{dP_{avg_L}}{dZ_L} = \frac{1}{2}|\tilde{V}_S|^2 \left[-2Z_L(Z_S + Z_L)^{-3} + (Z_S + Z_L)^{-2}\right]$$

$$= \frac{1}{2}|\tilde{V}_S|^2 \left[\frac{-2Z_L}{(Z_S + Z_L)^3} + \frac{1}{(Z_S + Z_L)^2}\right]$$

$$= \frac{1}{2}|\tilde{V}_S|^2 \left[\frac{-2Z_L + Z_S + Z_L}{(Z_S + Z_L)^3}\right]$$

$\frac{dP_{avg_L}}{dZ_L} = 0$ when $Z_S = Z_L = R_L$

$$P_{avg_L}\Big|_{Z_S = R_L} = \frac{1}{2}\frac{R_L}{4R_L^2}|\tilde{V}_S|^2 = \boxed{\frac{1}{8R_L}|\tilde{V}_S|^2}$$

(b)

$Z_S = R_S + jX_S$, $Z_L = R_L + jX_L$, $P_{avg_L} = |\tilde{I}_L|^2 \mathrm{Re}\{Z_L\}$

$\tilde{I}_L = \frac{\tilde{V}_S}{(R_S + R_L) + j(X_S + X_L)}$, $P_{avg_L} = \frac{1}{2}\left|\frac{\tilde{V}_S}{(R_S + R_L) + j(X_S + X_L)}\right|^2 (R_L)$

$$\boxed{P_{avg_L} = \frac{1}{2}\frac{R_L}{(R_S + R_L)^2 + (X_S + X_L)^2}|\tilde{V}_S|^2}$$

(c) $Z_S = Z_L^* \Rightarrow Z_L = R_S - jX_L$, $P_{avg_L} = \frac{1}{2}\frac{R_L}{(2R_L)^2 + (0)^2}|\tilde{V}_S|^2$

The maximum time average power will be dissipated when $Z_S = Z_L^*$ because it gives the maximum power transfer when there's no reactance like in (a)

$$= \boxed{\frac{1}{8R_L}|\tilde{V}_S|^2}$$

③ $P_t = 1^{kW}$, $\lambda = \frac{c}{f} = 0.03^{m}$, $G = 10\log\left(\frac{P_{A3}}{P_{A2}}\right) = 80 \Rightarrow P_{A3} = (10^8)P_{A2}$

$R = 40000^{km}$, $r_{A1} = r_{A4} = 2^m$, $r_{A2} = r_{A3} = 1^m$

$$P_r = H_{recA_4}(10^8)\,H_{recA_2}\;P_t$$

$$H_{recA_1} = H_{recA_2} = \frac{\pi(2)^2\,\pi(1)^2}{(0.03)^2(40000\times10^3)^2} = 2.7\times10^{-11}$$

$$\boxed{P_r = 7.29\times10^{-11}\ W}$$

④ Source free and free space

$$\vec{\nabla}\cdot\tilde{E} = 0$$

$$\vec{\nabla}\times\tilde{E} = -j\omega\mu_0\tilde{H}$$

$$\vec{\nabla}\cdot\tilde{H} = 0$$

$$\vec{\nabla}\times\tilde{H} = j\omega\epsilon_0\tilde{E}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\tilde{E}) = \vec{\nabla}(\vec{\nabla}\cdot\tilde{E}) - \vec{\nabla}^2\tilde{E}$$

(a) $\vec{\nabla}\times(\vec{\nabla}\times\tilde{E}) = -j\omega\mu_0(\vec{\nabla}\times\tilde{H})$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\tilde{E}) - \vec{\nabla}^2\tilde{E} = -j\omega\mu_0\,j\omega\epsilon_0\tilde{E}$$

$$\Rightarrow \vec{\nabla}^2\tilde{E} + \omega^2\epsilon_0\mu_0\tilde{E} = 0$$

$$\Rightarrow \boxed{\vec{\nabla}^2\tilde{E} + k^2\tilde{E} = 0}$$

$$\vec{\nabla}\times(\vec{\nabla}\times\tilde{H}) = j\omega\epsilon_0(\vec{\nabla}\times\tilde{E})$$

$$\Rightarrow \vec{\nabla}(\vec{\nabla}\cdot\tilde{H}) - \vec{\nabla}^2\tilde{H} = \omega^2\epsilon_0\mu_0\tilde{H}$$

$$\Rightarrow \boxed{\vec{\nabla}^2\tilde{H} + \omega^2\epsilon_0\mu_0\tilde{H} = 0}$$

(b) $\vec{\nabla}^2\tilde{E} + k^2\tilde{E} = 0$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{E}}{\partial r}\right) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial\tilde{E}}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2\tilde{E}}{\partial\phi^2} + k^2\tilde{E} = 0$$

$$\tilde{E} = \hat{\theta}\left(\frac{V_0}{r}\right)e^{-jkr}$$

$$\boxed{\frac{1}{r^3}\approx 0 \qquad \vec{\nabla}^2\tilde{E} = \cancel{\frac{-2}{r^2\sin\theta}\frac{\partial(E_\theta\sin\theta)}{\partial\theta}\hat{r}} + \left(\vec{\nabla}^2E_\theta - \cancel{\frac{E_\theta}{r^2\sin^2\theta}}\right)\hat{\theta}}$$

$$\vec{\nabla}^2E_\theta\hat{\theta} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial E_\theta}{\partial r}\right)\hat{\theta} = \frac{\hat{\theta}V_0}{r^2}\frac{\partial}{\partial r}\left(r^2\left(-r^{-2}e^{-jkr} + r^{-1}(-jk)e^{-jkr}\right)\right)$$

$$= \frac{\hat{\theta}V_0}{r^2}\frac{\partial}{\partial r}\left(-e^{-jkr} - (jkr)e^{-jkr}\right)$$

$$= \frac{\hat{\theta}V_0}{r^2}\left[jke^{-jkr} - \left(jke^{-jkr} + k^2re^{-jkr}\right)\right]$$

$$= \frac{-V_0}{r^2}\left(k^2re^{-jkr}\right)\hat{\theta} = k^2\left(-\frac{V_0}{r}e^{-jkr}\right)\hat{\theta}$$

$$\frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial\tilde{E}}{\partial r}\right) + 0 + 0 + k^2\tilde{E} = 0$$

$\boxed{\text{Unit of } V_0 \text{ is in Volts}}$

(c)

$$\vec{\nabla}\times\tilde{E} = -j\omega\mu_0\tilde{H}$$

$$\vec{\nabla}\times\tilde{E} = \frac{-\hat{r}}{r\sin\theta}\left[\frac{\partial F_\theta}{\partial\phi}\right] + \frac{1}{r}\left[\frac{\partial}{\partial r}\left(r\frac{V_0}{r}e^{-jkr}\right)\right]\hat{\phi}$$

$$= \frac{1}{r}\left[-jkV_0e^{-jkr}\right]\hat{\phi} = -jk\frac{V_0}{r}e^{-jkr}\hat{\phi}$$

$$\tilde{H} = \frac{-jk}{-j\omega\mu_0}\frac{V_0}{r}e^{-jkr}\hat{\phi} = \boxed{\sqrt{\frac{\epsilon_0}{\mu_0}}\frac{V_0}{r}e^{-jkr}\hat{\phi}}$$

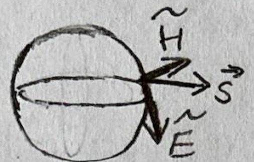
(d)

$$\vec{S}_{av} = \frac{1}{2}\mathrm{Re}\{\tilde{E}\times\tilde{H}^*\}$$

$$= \frac{1}{2}\mathrm{Re}\left\{\frac{V_0}{r}e^{-jkr}\hat{\theta}\times\frac{k}{\omega\mu_0}\frac{V_0}{r}e^{jkr}\hat{\phi}\right\}$$

$$= \frac{1}{2}\hat{r}\,\mathrm{Re}\left\{\left(\frac{V_0}{r}\right)^2\frac{k}{\omega\mu_0}\right\} = \boxed{\frac{1}{2}\left(\frac{V_0}{r}\right)^2\sqrt{\frac{\epsilon_0}{\mu_0}}\,\hat{r}}$$

(e) The wave propagates in the radial direction. The E field points in the $\hat{\theta}$ direction, and The H field points in the $\phi$ direction




(f)

$$\tilde{H} = \frac{1}{\eta_0}\hat{k} \times \tilde{E}, \text{ where } \eta_0 = \sqrt{\frac{\mu_0}{\epsilon_0}}$$

$\tilde{H}$ is related to $\tilde{E}$ by the characteristic impedance, $\eta$. This relationship also holds with the plane electromagnetic wave.

(g)

$$\tilde{E}(r) = (\hat{\theta} a - \hat{\phi} a j) e^{-jkr}, \text{ where } a = \frac{V_0}{r}$$