EEC 133 HW2

(a)

$$g(x,y,z) = F(x,y,z)$$

$$\nabla g = \frac{\partial F}{\partial x}\hat{x} + \frac{\partial F}{\partial y}\hat{y} + \frac{\partial F}{\partial z}\hat{z}$$

$$\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$$

$$\vec{\nabla}\times\vec{\nabla}g = \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\hat{z} - \frac{\partial}{\partial z}\frac{\partial F}{\partial x}\hat{y} - \frac{\partial}{\partial x}\frac{\partial F}{\partial y}\hat{z} + \frac{\partial}{\partial z}\frac{\partial F}{\partial y}\hat{x} + \frac{\partial}{\partial x}\frac{\partial F}{\partial z}\hat{y} - \frac{\partial}{\partial y}\frac{\partial F}{\partial z}\hat{x}$$

Since $F(x,y,z)$ is continuous, we could switch the order of differentiation.

$$\nabla\times\nabla g = \left(\frac{\partial}{\partial y}\frac{\partial F}{\partial x} - \frac{\partial}{\partial y}\frac{\partial F}{\partial x}\right)\hat{z} + \left(\frac{\partial}{\partial z}\frac{\partial F}{\partial y} - \frac{\partial}{\partial z}\frac{\partial F}{\partial y}\right)\hat{x} + \left(\frac{\partial}{\partial x}\frac{\partial F}{\partial z} - \frac{\partial}{\partial x}\frac{\partial F}{\partial x}\right)\hat{y} = 0$$

(b)

(i) $\vec{A} = A_0 \sin(kz-\omega t)\hat{x}$, $\phi = 0$

$$\nabla\cdot\vec{A} = \frac{\partial}{\partial x}A_0\sin(kz-\omega t) = 0$$

Coulomb gauge

(ii) $$\vec{H} = \frac{1}{\mu_0}\nabla\times\vec{A} = \frac{1}{\mu_0}\left(\frac{-\partial}{\partial z}A_0\sin(kz-\omega t)\right)\hat{y}$$

$$= \boxed{\frac{A_0 k}{\mu_0}\cos(kz-\omega t)\ \hat{y}}$$

(iii) $\phi = 0 \quad , \quad \vec{A} = A_0 \sin(kz - \omega t)\,\hat{x}$

$$\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t}$$

$$= -\frac{\partial \vec{A}}{\partial t} = -\frac{\partial}{\partial t}\left(A_0 \sin(kz-\omega t)\,\hat{x}\right)$$

$$= \boxed{\omega A_0 \cos(kz-\omega t)\,\hat{x}}$$

(iv)

$$\vec{E} = \omega A_0 \cos(kz-\omega t)\,\hat{x}$$
$$\vec{H} = \frac{A_0 k}{\mu_0} \cos(kz-\omega t)\,\hat{y}$$

$$\tilde{E} = -j\omega A_0 e^{jkz} \quad \hat{x}$$
$$\tilde{H} = -j\,\frac{A_0 k}{\mu_0} e^{jkz} \quad \hat{y}$$

Wave Eq. for Plain $\tilde{E}$ wave

$$\frac{d^2}{dz^2}\tilde{E} + \omega^2 \mu_0 \epsilon_0 \tilde{E} = 0$$

$$\hat{x}\, j k^2 \omega A_0 e^{jkz} + \hat{x}\,\omega^2 \mu_0 \epsilon_0 \tilde{E} = 0$$

$$\hat{x}\, j\,\omega^3 \mu_0 \epsilon_0 A_0 e^{jkz} - \hat{x}\,\omega^2 \mu_0 \epsilon_0 j\omega A_0 e^{jkz} = 0$$

$\boxed{\text{This is a Plain EM Wave}}$

(c)

(i) $\phi(\vec{r},t) = 0, \quad \vec{A}(\vec{r},t) = -\frac{1}{4\pi\epsilon_0}\frac{qt}{r^2}\,\hat{r}$

$$\vec{H} = \frac{1}{\mu_0}(\nabla \times \vec{A}) = \frac{1}{\mu_0}(\nabla \times \vec{A}) = \boxed{0}$$

(ii) $\vec{E} = -\nabla\phi - \frac{\partial \vec{A}}{\partial t} = -\frac{\partial \vec{A}}{\partial t} = \boxed{\frac{1}{4\pi\epsilon_0}\frac{q}{r^2}\,\hat{r}}$

(iii) A point charge at $r=0$ with charge q.
No current.

(iv) $\psi = \frac{-1}{4\pi\epsilon_0} \frac{qt}{r}$

$$\nabla \psi = \frac{\partial \psi}{\partial r} \hat{r} = \frac{-1}{4\pi\epsilon_0} \frac{qt}{r^2} \hat{r}$$

$$\boxed{\begin{aligned} \vec{A}_{new} &= \vec{A} + \nabla \psi = 0 \\ \phi_{new} &= 0 - \frac{\partial \psi}{\partial t} = \frac{1}{4\pi\epsilon_0} \frac{q}{r} \end{aligned}}$$

(2)

$$\frac{\partial^2 f(z,t)}{\partial z^2} - \frac{1}{v^2} \frac{\partial^2 f(z,t)}{\partial t^2} = 0$$

$$f(z,t) = g_+(z - vt) + g_-(z + vt)$$

Let $x = z \pm vt$. Then,

$$f(x) = g_+(x) + g_-(x)$$

$$\frac{\partial f(x)}{\partial z} = \frac{df(x)}{dx} \frac{\partial x}{\partial z}$$

$$\frac{\partial^2 f(x)}{\partial z} = \frac{d^2 f(x)}{dx^2}$$

$$\frac{\partial f(x)}{\partial t} = \frac{df(x)}{dx} \frac{\partial x}{\partial t}$$

$$\frac{\partial^2 f(x)}{\partial t^2} = \frac{d}{dx} \frac{df(x)}{dx} \frac{\partial x}{\partial t} = (\pm v)^2 \frac{d^2 f(x)}{dx^2}$$

$$\boxed{\frac{d^2 f(x)}{dx^2} - \frac{1}{v^2} v^2 \frac{d^2 f(x)}{dx^2} = 0}$$

Equation is satisfied

g_+ is the rightward traveling wave and g_- is the leftward traveling wave. The solution has two wave components moving in opposite directions.

③ Ampere's Law:

(a) $\nabla \times \vec{H} = \vec{J} + \epsilon_0 \left(\frac{\partial \vec{E}}{\partial t}\right)$

$\Rightarrow \nabla \times \left(\frac{1}{\mu_0} \nabla \times \vec{A}\right) = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\Rightarrow \frac{1}{\mu_0} \nabla \times \nabla \times \vec{A} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\Rightarrow \frac{1}{\mu_0}\left(\nabla(\nabla \cdot \vec{A}) - \nabla^2 \vec{A}\right) = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\Rightarrow \frac{-1}{\mu_0} \nabla^2 \vec{A} = \vec{J} + \epsilon_0 \frac{\partial \vec{E}}{\partial t}$ In Coulomb gauge

$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J} - \mu_0 \epsilon_0 \frac{\partial \vec{E}}{\partial t}$

$\Rightarrow \nabla^2 \vec{A} = -\mu_0 \vec{J}$

magnetostatics $\frac{\partial \vec{E}}{\partial t} = 0$

(b) $\vec{A}(\vec{r}) = \frac{\mu_0}{4\pi} \int \frac{\vec{J}(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} dV'$ ← Solution of $\nabla^2 \vec{A} = -\mu_0 \vec{J}$, $\vec{J}(\vec{r}\,') = I_0 \delta(x') \delta(y') \hat{z}$

$\vec{A}(r, z=0) = \frac{\mu_0}{4\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{I_0 \delta(x') \delta(y') \hat{z}}{\sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}} dx'\, dy'\, dz'$

$= \frac{\mu_0}{4\pi} \int_{-\frac{l}{2}}^{\frac{l}{2}} \frac{I_0 \hat{z}}{\sqrt{(x)^2 + (y)^2 + (z')^2}} dz' = \frac{2\mu_0 I_0 \hat{z}}{4\pi} \int_0^{\frac{l}{2}} \frac{1}{\sqrt{r^2 + z'^2}} dz'$

$= \boxed{\frac{\mu_0 I_0}{2\pi} \ln\left(\frac{\frac{l}{2} + \sqrt{r^2 + \left(\frac{l}{2}\right)^2}}{r}\right) \hat{z}}$

(c) $\vec{H} = \frac{1}{\mu_0}(\nabla \times \vec{A}) = \frac{1}{\mu_0} \frac{-\partial}{\partial r}\left(\frac{\mu_0 I_0}{2\pi} \ln\left(\frac{\frac{l}{2} + \sqrt{r^2 + \left(\frac{l}{2}\right)^2}}{r}\right)\right) \hat{\phi}$

$= \boxed{\frac{I_0}{4\pi} \frac{l \hat{\phi}}{r\sqrt{r^2 + \left(\frac{l}{2}\right)^2}}}$

Solution from EEC130A: $\vec{B} = \frac{\mu_0 I_0}{4\pi r} \frac{l}{\sqrt{r^2 + \left(\frac{l}{2}\right)^2}} \hat{\phi}$

Result matches, because $\mu_0 \vec{H} = \vec{B}$, the solutions match

④ $\vec{J}(\vec{r}) = I_0 \Delta z \, \delta(\vec{r}) \sin(\omega t)\, \hat{z}$

(a)

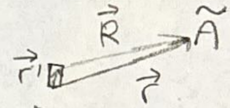
$$I = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} I_0 \Delta z \, \delta(x)\delta(y)\delta(z) \sin(\omega t)\, \hat{z}\cdot\hat{z} \, dx\, dy$$

$$= I_0 \Delta z \, \delta(z) \sin(\omega t)$$

(b)

$$\vec{J}(\vec{r}') = I_0 \Delta z \, \delta(\vec{r}') \cos(\omega t - \tfrac{\pi}{2})\, \hat{z}$$

$$\tilde{J}(\vec{r}') = -jI_0 \Delta z \, \delta(\vec{r}')\, \hat{z}$$



$\vec{R} = \vec{r} - \vec{r}'$

$\vec{R} = (x-x')\hat{x} + (y-y')\hat{y} + (z-z')\hat{z}$

$|\vec{R}| = \sqrt{(x-x')^2 + (y-y')^2 + (z-z')^2}$

$$\tilde{A} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{\mu_0 \tilde{J}(\vec{r}')e^{-jkr}}{4\pi|\vec{R}|}\, dz'dx'dy'$$

$$= \hat{z}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \frac{-\mu_0 j I_0 \Delta z \, \delta(x')\delta(y')\delta(z') e^{-jkr}}{4\pi\sqrt{(x-x')^2+(y-y')^2+(z-z')^2}}\, dx'dy'dz'$$

$$= \hat{z}\left(\frac{-\mu_0 j I_0 \Delta z \, e^{-jkr}}{4\pi\sqrt{x^2+y^2+z^2}}\right) = j\left(\frac{-\mu_0 I_0 \Delta z \, e^{-jkr}}{4\pi r}\right)\hat{z}$$

$$\vec{A} = \mathrm{Re}\{\tilde{A}e^{j\omega t}\} = \mathrm{Re}\left\{e^{j(\omega t - kr - \frac{\pi}{2})}\left(\frac{-\mu_0 I_0 \Delta z}{4\pi r}\right)\hat{z}\right\}$$

$$= \boxed{\frac{\mu_0 I_0 \Delta z}{4\pi r}\sin(\omega t - kr)\hat{z} \qquad r = |\vec{r}|}$$

$$\hat{z} = (\cos\theta)\hat{r} - (\sin\theta)\hat{\theta}$$

$$\vec{A} = \left(\frac{\mu_o I_o \Delta z}{4\pi r}\right)\sin(\omega t - kr)\,(\cos\theta\,\hat{r} - \sin\theta\,\hat{\theta})$$

(c)

$$\vec{H} = \frac{1}{\mu_o}(\nabla\times\vec{A}) = \frac{1}{\mu_o r}\left(\frac{\partial}{\partial r}(rA_\theta) - \frac{\partial A_r}{\partial\theta}\right)\hat{\phi}$$

$$= \frac{\hat{\phi}}{\mu_o r}\,\frac{\mu_o I_o \Delta z}{4\pi}\left(\frac{-\partial}{\partial r}(\sin(\omega t - kr)\sin\theta) - \frac{1}{r}\frac{\partial}{\partial\theta}(\sin(\omega t - kr)\cos\theta)\right)$$

$$= \frac{\hat{\phi} I_o \Delta z}{4\pi r}\left(k\cos(\omega t - kr)\sin\theta + \frac{1}{r}\sin(\omega t - kr)\sin\theta\right)$$

$$\boxed{= \hat{\phi}\left(\frac{I_o \Delta z k^2}{4\pi}\sin\theta\right)\left(\frac{1}{kr}\cos(\omega t - kr) + \frac{1}{(kr)^2}\sin(\omega t - kr)\right)}$$

(d)

$$\tilde{H} = \hat{\phi}\left(\frac{I_o \Delta z k^2}{4\pi}\sin\theta\right)\left(\frac{1}{kr} - \frac{j}{(kr)^2}\right)e^{-jkr}$$

$$\tilde{E} = \frac{1}{j\omega\epsilon_o}\nabla\times\tilde{H} = \frac{1}{j\omega\epsilon_o}\left(\frac{1}{r\sin\theta}\frac{\partial}{\partial\theta}(H_\phi\sin\theta)\hat{r} - \frac{1}{r}\frac{\partial}{\partial r}(rH_\phi)\hat{\theta}\right)$$

$$= \frac{1}{j\omega\epsilon_o r}\frac{I_o \Delta z k^2}{4\pi}\left(\frac{e^{-jkr}}{\sin\theta}\left(\frac{1}{kr} - \frac{j}{(kr)^2}\right)\left(\frac{\partial}{\partial\theta}\sin^2\theta\right)\hat{r} - \sin\theta\frac{\partial}{\partial r}\left(\left(\frac{1}{k} - \frac{j}{k^2 r}\right)e^{-jkr}\right)\hat{\theta}\right)$$

$$= \frac{1}{j\omega\epsilon_o r}\frac{I_o \Delta z k^2}{4\pi}\left(\frac{e^{-jkr}}{\sin\theta}\left(\frac{1}{kr} - \frac{j}{(kr)^2}\right)(2\sin\theta\cos\theta)\hat{r} - \sin\theta\left(\left(\frac{j}{(kr)^2}\right)e^{-jkr} + \left(\frac{1}{k} - \frac{j}{k^2 r}\right)(-jk)e^{-jkr}\right)\hat{\theta}\right)$$

$$= \frac{e^{-jkr}}{j\omega\epsilon_o r}\frac{I_o \Delta z k^2}{4\pi}\left(\left(\frac{1}{kr} - \frac{j}{(kr)^2}\right)2\cos\theta\,\hat{r} - \left(\frac{j}{(kr)^2} - j - \frac{1}{kr}\right)\sin\theta\,\hat{\theta}\right)$$

$$= \frac{I_o \Delta z \eta_o k^2}{4\pi}e^{-jkr}\left(\left(\frac{1}{j(kr)^2} - \frac{1}{(kr)^3}\right)2\cos\theta\,\hat{r} - \left(-\frac{1}{kr} + \frac{j}{(kr)^2} + \frac{1}{(kr)^3}\right)\sin\theta\,\hat{\theta}\right)$$

$$= \frac{I_o \Delta z \eta_o k^2}{4\pi}e^{-jkr}\left(\left(\frac{-j}{(kr)^2} - \frac{1}{(kr)^3}\right)2\cos\theta\,\hat{r} + \left(\frac{1}{kr} - \frac{j}{(kr)^2} - \frac{1}{(kr)^3}\right)\sin\theta\,\hat{\theta}\right)$$

$$\vec{E} = Re\{\tilde{E} e^{j\omega t}\}$$

$$= \frac{I_0 \Delta z \eta_0 k^2}{4\pi} Re\left\{ e^{j(\omega t - kr)} \left(\left(\frac{-j}{(kr)^2} - \frac{1}{(kr)^3} \right) 2\cos\theta \, \hat{r} + \left(\frac{1}{kr} - \frac{j}{(kr)^2} - \frac{1}{(kr)^3} \right) \sin\theta \, \hat{\theta} \right) \right\}$$

$$\boxed{= \frac{I_0 \Delta z \eta_0 k^2}{4\pi} \left[\left(\frac{1}{(kr)^2} \sin(\omega t - kr) - \frac{1}{(kr)^3} \cos(\omega t - kr) \right) 2\cos\theta \, \hat{r} + \left(\frac{1}{kr} \cos(\omega t - kr) + \frac{1}{(kr)^2} \sin(\omega t - kr) - \frac{1}{(kr)^3} \cos(\omega t - kr) \right) \sin\theta \, \hat{\theta} \right]}$$

(e) This is a spherical wave in far field because all of the $\frac{1}{r^2}$ and $\frac{1}{r^3} \approx 0$ in $\vec{E}$ and $\vec{H}$ field. Therefore $\vec{E}$ points in $\hat{\theta}$ and $\vec{H}$ points in $\hat{\phi}$. Their power points in $\hat{r}$.