Chapter 7

Poisson Processes

A stochastic process, $N(t)$, $t \geq 0$, is said to be a *counting process* if it counts the number of events that occur in the time interval $[0, t]$. For example, the number of customers who visit a bank is a counting process.

A counting process, $N(t)$, has the following properties:

1. $N(0) = 0$.
2. $N(t)$ assumes nonnegative integer values.
3. If $t_1 < t_2$, then $N(t_1) \leq N(t_2)$.
4. If $t_1 < t_2$, then $N(t_2) - N(t_1)$ counts the number of events that occur in the interval $(t_1, t_2]$.

Arrival and Interarrival Times Let $N(t)$, $t \geq 0$ be a counting process. The k^{th} *arrival time*, denoted by T_k, is the occurrence time of the k^{th} event [a]. In particular, the random variables $T_1, T_2, \ldots$ are the times in which the counting process increases.

We define the k^{th} *interarrival time*, denoted by ΔT_k, as follows [b]. Let $\Delta T_1 = T_1$ and $\Delta T_k = T_k - T_{k-1}$ for $k \geq 2$. Clearly, the interarrival times are random variables and they are related to the arrival times through

$$T_k = \sum_{i=1}^{k} \Delta T_i.$$

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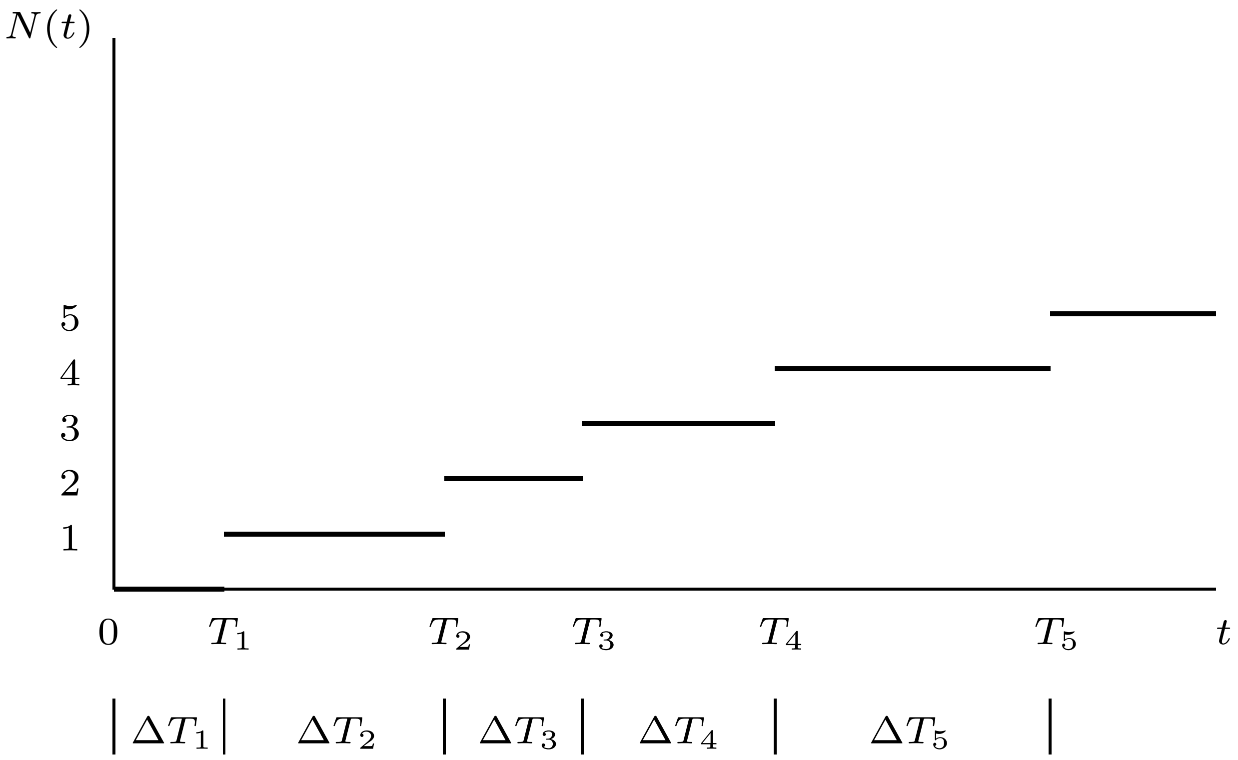
[a] The arrival times are denoted in the textbook by S_k and called epochs.

[b] The interarrival times are denoted in the textbook by X_k.


$N(t)$
5
4
3
2
1
0
T_1
T_2
T_3
T_4
T_5
t
ΔT_1
ΔT_2
ΔT_3
ΔT_4
ΔT_5

A counting process, $N(t)$, $t \geq 0$, is a *Poisson process* if

1. $N(0) = 0$.

2. $N(t)$ has stationary independent increments.

3. $N(t)$, for any given $t > 0$, is Poisson(λt), i.e., a Poisson random variable with mean λt, where $\lambda > 0$ is constant. In particular, for $k = 0, 1, 2, \ldots$,

$$P(N(t) = k) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}.$$

The parameter λ is called the *rate* of the Poisson process.

A Motivation of Poisson Processes

Example 1 Traffic accidents occur in a busy intersection with an average rate of λ accidents a day. Let $N(t)$, $t \geq 0$, be the random process that counts the number of accidents in t days starting from some time denoted by 0. We will argue that it is reasonable to assume that $N(t)$ is a Poisson process of rate λ.

Clearly, the probability that an accident occurs exactly at time 0 is 0. So, we may assume that $N(0) = 0$. It is understandable to assume that the probability that a number of accidents occur in a given time interval depends only on the length of the interval but not on the time it starts. Hence, $N(t)$ has stationary increments. It also make sense to assume that the numbers of accidents occurring in nonoverlapping time intervals are independent. It remains to argue that it is reasonable to assume that $N(t)$ is Poisson(λt).

We divide the time interval t into n time intervals of equal duration t/n. Then, on average $\lambda t/n$ accidents occur in a given interval and the number of accidents occurring in different intervals are independent. If the intervals are short enough, i.e., n is large enough, then it is reasonable to also assume that the probability of more than one accident in a given interval is negligible.

With these assumptions, in each interval either one accident occurs or no accident occurs. Since the average number of accidents in an interval is $\lambda t/n$, the probability of an accident is $\lambda t/n$. Hence, the number of accidents in t days is the number of intervals among the n intervals in which accidents occur. This implies that $N(t)$ is Binomial$(n, \lambda t/n)$. For large n, Binomial$(n, \lambda t/n)$ is approximated by Poisson(λt).

An equivalent definition of a Poisson process is the following. First, recall the small-o notation: for a function $f(t)$ we write $f(t) = o(t)$ if $\lim_{t\to 0} f(t)/t = 0$.

Theorem 1 *A counting process, $N(t)$, $t \geq 0$, is a* Poisson process *of rate λ if and only if*

1. $N(0) = 0$.

2. $N(t)$ *has stationary independent increments.*

3'. *For any given $t > 0$,*

- $P(N(t) = 1) = \lambda t + o(t)$.
- $P(N(t) \geq 2) = o(t)$.

Proof. First we show Condition 3 implies Condition 3'. Notice that

$$e^{-\lambda t} = \sum_{i=0}^{\infty} \frac{(-1)^i (\lambda t)^i}{i!} = 1 - \lambda t + o(t).$$

Hence,

$$\begin{aligned} P(N(t)=1) &= e^{-\lambda t}(\lambda t) = (1-\lambda t + o(t))\lambda t = \lambda t + o(t) \\ P(N(t) \geq 2) &= 1 - (P(N(t)=0) + P(N(t)=1)) \\ &= 1 - e^{-\lambda t}(1+\lambda t) \\ &= 1 - (1-\lambda t + o(t))(1+\lambda t) = o(t). \end{aligned}$$

Next, we will show that Condition 3', together with Conditions 1 and 2, implies Condition 3.

Let $P_k(t) = P(N(t) = k)$ and $h > 0$. We have

$$\begin{aligned}
P_k(t+h) &= P(N(t+h) = k) \\
&= \sum_{i=0}^{k} P(N(t) = k-i, N(t+h) - N(t) = i) \\
&= \sum_{i=0}^{k} P(N(t) = k-i)\ P(N(t+h) - N(t) = i) \\
&\qquad \text{(Condition 2: independent increments)} \\
&= \sum_{i=0}^{k} P(N(t) = k-i)\ P(N(h) - N(0) = i) \\
&\qquad \text{(Condition 2: stationary increments)} \\
&= \sum_{i=0}^{k} P_{k-i}(t)\ P(N(h) = i) \\
&\qquad \text{(Condition 1).}
\end{aligned}$$

First, we consider the case $k = 0$ in which the sum involves only one term corresponding to $i = 0$. We have from Condition 3':

$$\begin{aligned} P_0(t+h) &= P_0(t)\ P(N(h) = 0) \\ &= P_0(t)(1 - (P(N(h) = 1) + P(N(h) \geq 2)) \\ &= P_0(t)(1 - \lambda h + o(h)) \end{aligned}$$

Hence,

$$\frac{P_0(t+h) - P_0(t)}{h} = -\lambda P_0(t) + \frac{o(h)}{h}.$$

As $h \to 0$, we notice that $P_0(t)$ satisfies the differential equation $\frac{d}{dt} P_0(t) = -\lambda P_0(t)$, the solution of which is $P_0(t) = ae^{-\lambda t}$. Together with $P_0(0) = 1$, which follows from Condition 1, we deduce that $P_0(t) = e^{-\lambda t}$.

Next, we consider the case $k \geq 1$ and use induction on k to prove Condition 3, i.e., $P_k(t) = e^{-\lambda t}\frac{(\lambda t)^k}{k!}$. We have just shown that this holds for $k = 0$. We have from Condition 3':

$$
\begin{aligned}
P_k(t+h) &= \sum_{i=0}^{k} P_{k-i}(t)\ P(N(h)=i) \\
&= P_k(t)P(N(h)=0) + P_{k-1}(t)P(N(h)=1) \\
&\qquad + \sum_{i=2}^{k} P_{k-i}(t)P(N(h)=i) \\
&= P_k(t)(1-\lambda h + o(h)) + P_{k-1}(t)(\lambda h + o(h)) \\
&\qquad + \sum_{i=2}^{k} P_{k-i}(t)o(h).
\end{aligned}
$$

Hence,

$$\frac{P_k(t+h) - P_k(t)}{h} + \lambda P_k(t) = \lambda P_{k-1}(t) + \frac{o(h)}{h}.$$

As $h \to 0$, we notice that $P_k(t)$ satisfies the differential equation

$$\frac{d}{dt} P_k(t) + \lambda P_k(t) = \lambda P_{k-1}(t).$$

Therefore,

$$\frac{d}{dt} \left(e^{\lambda t} P_k(t) \right) = \lambda e^{\lambda t} P_{k-1}(t).$$

Using the induction hypothesis $P_{k-1}(t) = e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!}$, it follows that

$$\frac{d}{dt}\left(e^{\lambda t} P_k(t)\right) = \frac{\lambda^k t^{k-1}}{(k-1)!}.$$

This gives

$$e^{\lambda t} P_k(t) = \frac{\lambda^k t^k}{k!} + b,$$

for some constant b. Since Condition 1 implies that $P_k(0) = 0$ for $k \geq 1$, it follows that $b = 0$ and $P_k(t) = e^{-\lambda t} \frac{(\lambda t)^k}{k!}$. $\square$

Distribution of Interarrival Times of Poisson Processes

First, recall that an Exponential(λ), $\lambda > 0$, random variable, X, is a continuous random variable with PDF $f_X(x) = \lambda e^{-\lambda x}$ and CDF $F_X(x) = 1 - e^{-\lambda x}$ for $x \geq 0$. Hence, a random variable X is Exponential(λ) if and only if $P(X > x) = e^{-\lambda x}$ for $x \geq 0$.

The mean of an Exponential(λ) random variable, X, is $1/\lambda$ and its characteristic function is given by

$$\Phi_X(\omega) = \frac{\lambda}{\lambda - \jmath\omega}.$$

Theorem 2 *Let $N(t)$ be a Poisson process of rate λ. Then, the interarrival times ΔT_k, $k = 1, 2, \ldots$, are independent Exponential(λ) random variables.*

Proof. Indeed,

$$P(\Delta T_1 > \Delta t_1) = P(N(\Delta t_1) = 0) = e^{-\lambda \Delta t_1}.$$

Define $u(t) = 1$ for $t \geq 0$ and $u(t) = 0$ for $t < 0$. Then, for $k \geq 2$, $N(t)$ has interarrival times $\Delta t_1, \ldots, \Delta t_{k-1}$ if and only if

$$N(t) = \sum_{i=1}^{k-1} u(t - (\Delta t_1 + \cdots + \Delta t_i)) \text{ for } 0 \leq t \leq \Delta t_1 + \cdots + \Delta t_{k-1}.$$

$$\begin{aligned}
&P(\Delta T_k > \Delta t_k | \Delta T_1 = \Delta t_1, \ldots, \Delta T_{k-1} = \Delta t_{k-1}) \\
&\quad = P(N(\Delta t_1 + \cdots + \Delta t_k) - N(\Delta t_1 + \cdots + \Delta t_{k-1}) = 0 \\
&\qquad\qquad |N(t) = \sum_{i=1}^{k-1} u(t - (\Delta t_1 + \cdots + \Delta t_i)), 0 \leq t \leq \Delta t_1 + \cdots + \Delta t_{k-1}) \\
&\quad = P(N(\Delta t_1 + \cdots + \Delta t_k) - N(\Delta t_1 + \cdots + \Delta t_{k-1}) = 0) \\
&\qquad\qquad \text{(Condition 2: independent increments)} \\
&\quad = P(N(\Delta t_k) - N(0) = 0) \qquad \text{(Condition 2: stationary increments)} \\
&\quad = P(N(\Delta t_k)) = 0) \qquad \text{(Condition 1)} \\
&\quad = e^{-\lambda \Delta t_k} \qquad \text{(Condition 3).}
\end{aligned}$$

Hence, the interarrival times are independent Exponential(λ) random variables. □

Distribution of Arrival Times of Poisson Processes

Theorem 3 *Let $N(t)$ be a Poisson process of rate λ. Then, the k^{th} arrival time, T_k, has a PDF given by*

$$f_{T_k}(t) = \begin{cases} 0 & t < 0 \\ \lambda e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} & t \geq 0. \end{cases}$$

Proof. We have $T_k = \Delta T_1 + \Delta T_2 + \cdots + \Delta T_k$. By Theorem 2, T_k is the sum of k independent Exponential(λ) random variables. Hence, its characteristic function is

$$\Phi_{T_k}(\omega) = \prod_{i=1}^{k} \Phi_{\Delta T_i}(\omega) = \frac{\lambda^k}{(\lambda - \jmath\omega)^k}.$$

The inverse Fourier transform of $\Phi_{T_k}(\omega)$ is $f_{T_k}(t)$ as given above.

Here is another proof.

$$
\begin{aligned}
& P(t < T_k \leq t+h) \\
&\quad = \quad P(N(t) = k-1, N(t+h) = k) \\
&\quad = \quad P(N(t) = k-1, N(t+h) - N(t) = 1) \\
&\quad = \quad P(N(t) = k-1)\ P(N(t+h) - N(t) = 1) \\
&\qquad\qquad\qquad \text{(condition 2: independent increments)} \\
&\quad = \quad P(N(t) = k-1)\ P(N(h) = 1) \\
&\qquad\qquad\qquad \text{(condition 2: stationary increments)} \\
&\quad = \quad e^{-\lambda t} \frac{(\lambda t)^{k-1}}{(k-1)!} (\lambda h + o(h)) \\
&\qquad\qquad\qquad \text{(condition 3').}
\end{aligned}
$$

The result follows by dividing by h and pushing it to zero. $\square$

A random variable with PDF $\lambda e^{-\lambda t}\frac{(\lambda t)^{k-1}}{(k-1)!}$ for $t \geq 0$, where k is a positive integer and $\lambda > 0$, is said be an *Erlang* random variable with shape k and rate λ, and denoted by Erlang(k, λ). It is the sum of k independent Exponential(λ) random variables.

From Theorem 3, it follows that the k^{th} arrival time of a Poisson process of rate λ is Erlang(k, λ).

Conditional Distribution of Arrival Times of Poisson Processes

We consider the arrival times $T_1, T_2, \ldots, T_n$ given that $N(t) = n$. Notice that $0 \le T_1 \le T_2 \le \cdots \le T_n \le t$.

Theorem 4 *We have for* $0 < t_1 < t_2 \ldots < t_n < t$,

$$f_{T_1,\ldots,T_n|N(t)}(t_1, \ldots, t_n|n) = \frac{n!}{t^n}.$$

Proof. Define $t_0 = 0$ and choose h_i such that $0 < h_i < t_i - t_{i-1}$ for $i = 1, 2, \ldots, n$. Then,

$$\begin{aligned} &P(t_1 - h_1 < T_1 \le t_1, \ldots, t_n - h_n \le T_n \le t_n|N(t) = n) \\ &\quad = \frac{P(t_1 - h_1 < T_1 \le t_1, \ldots, t_n - h_n < T_n \le t_n, N(t) = n)}{P(N(t) = n)}. \quad (1) \end{aligned}$$

The numerator is the probability that a single event occurs in each interval $(t_i - h_i, t_i]$ for $i = 1, 2 \ldots, n$ but no event occurs in any of the intervals $(t_{i-1}, t_i - h_i]$ for $i = 1, 2, \ldots, n$ or $(t_n, t]$. As these intervals do not overlap, by Condition 2, this equals

$$\prod_{i=1}^{n} P(N(t_i) - N(t_i - h_i) = 1)$$

$$\times \prod_{i=1}^{n} P(N(t_i - h_i) - N(t_{i-1}) = 0) \times P(N(t) - N(t_n) = 0)$$

$$= \prod_{i=1}^{n} P(N(h_i) = 1)$$

$$\times \prod_{i=1}^{n} P(N(t_i - t_{i-1} - h_i) = 0) \times P(N(t - t_n) = 0)$$

$$= \prod_{i=1}^{n} \lambda h_i e^{-\lambda h_i} \times \prod_{i=1}^{n} e^{-\lambda(t_i - t_{i-1} - h_i)} \times e^{-\lambda(t - t_n)} = h_1 \cdots h_n \lambda^n e^{-\lambda t}.$$

By dividing by $P(N(t) = n) = (\lambda t)^n e^{-\lambda t}/n!$, (1) gives

$$P(t_1 - h_1 < T_1 \leq t_1, \dots, t_n - h_n \leq T_n \leq t_n | N(t) = n) = \frac{n!}{t^n} h_1 \cdots h_n.$$

Dividing by $h_1 \cdots h_n$, and taking the limits as $h_1, \dots h_n \to 0$, we get $f_{T_1,\dots,T_n|N(t)}(t_1, \dots, t_n|n)$ as stated. $\square$

In the above, we considered the joint PDF of the arrival times $T_1, T_2, \ldots, T_n$ satisfying $T_1 \leq T_2 \leq \cdots \leq T_n$. Now let us write $\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n$ to denote the arrival times without any order imposed. Hence, $(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$ is a permutation of $(T_1, T_2, \ldots, T_n)$. In total, there are $n!$ permutations. Notice that $(T_1, T_2, \ldots, T_n) = (t_1, t_2, \ldots, t_n)$ if and only if $(\tilde{T}_1, \tilde{T}_2, \ldots, \tilde{T}_n)$ equals any of the $n!$ permutations of $(t_1, t_2, \ldots, t_n)$. Hence,

$$f_{\tilde{T}_1, \ldots, \tilde{T}_n | N(t)}(t_1, \ldots, t_n | n) = \frac{1}{t^n}$$

for $0 < t_1, t_2, \ldots, t_n < t$. In particular, the unordered arrival times are independent Uniform$(0, t)$ random variables.

Example 2 Riders arrive at a bus stop according to a Poisson process. A bus arrives every 10 minutes. Assuming that three riders wait for the bus, what is the probability that none of them waits more that 5 minutes?

Let T_1, T_2, T_3 be the arrival times of the three riders, where $T_1 \leq T_2 \leq T_3$. Then, from Theorem 4,

$$f_{T_1,T_2,T_3|N(10)}(t_1, t_2, t_3|3) = \frac{3!}{10^3}.$$

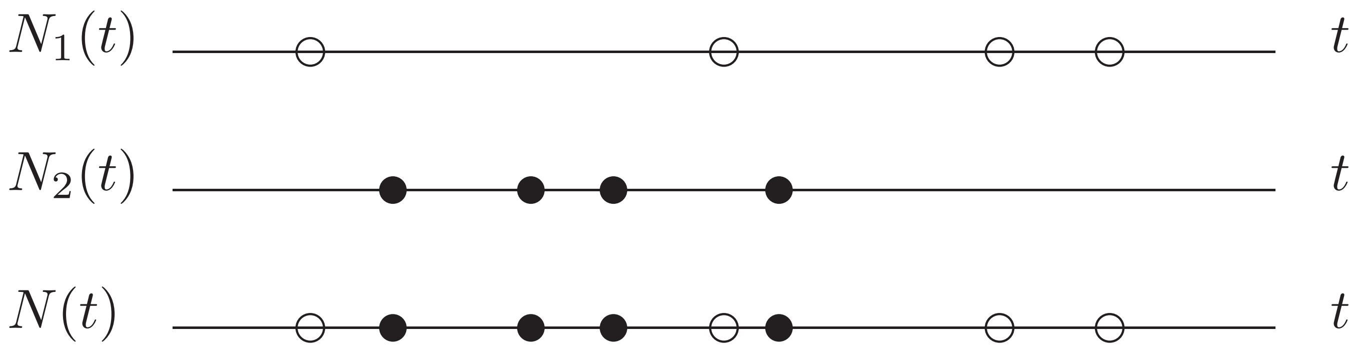
The probability that none of the three riders waits more than 5 minutes is

$$\begin{aligned}
P &= \int_5^{10}\int_{t_1}^{10}\int_{t_2}^{10} f_{T_1,T_2,T_3|N(10)}(t_1,t_2,t_3|3)dt_3dt_2dt_1 \\
&= \int_5^{10}\int_{t_1}^{10}\int_{t_2}^{10} \frac{3!}{10^3}dt_3dt_2dt_1 \\
&= \frac{6}{1000}\int_5^{10}\int_{t_1}^{10}\int_{t_2}^{10} dt_3dt_2dt_1 \\
&= \frac{6}{1000}\int_5^{10}\int_{t_1}^{10}(10-t_2)dt_2dt_1 \\
&= \frac{6}{1000}\int_5^{10}\left(50-10t_1+\frac{t_1^2}{2}\right)dt_1 \\
&= \frac{6}{1000}\times\frac{125}{6} = 1/8.
\end{aligned}$$

This probability can be determined easily as follows. Let $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ be the arrival times of the riders not ordered in time, e.g., ordered alphabetically. Then, $\tilde{T}_1, \tilde{T}_2, \tilde{T}_3$ are independent Uniform$(0, 10)$ random variables. The probability that each is at least 5 is $5/10 = 1/2$. Hence, the probability that the three of them are at least 5 is $(1/2)^3 = 1/8$. This is the probability that none of the three riders waits more than 5 minutes.

Merging Poisson Processes

Theorem 5 *Let $N_1(t)$ and $N_2(t)$ be two independent Poisson processes of rates λ_1 and λ_2, respectively. Then, $N(t) = N_1(t) + N_2(t)$ is a Poisson process of rate $\lambda = \lambda_1 + \lambda_2$.*



Proof. We check that $N(t)$ satisfies Conditions 1–3.

1. We have $N(0) = N_1(0) + N_2(0) = 0$.

2. Since $N_1(t)$ and $N_2(t)$ have stationary and independent increments, then so does $N(t) = N_1(t) + N_2(t)$.

3. For a given t, $N_1(t)$ and $N_2(t)$ are Poisson$(\lambda_1 t)$ and Poisson$(\lambda_2 t)$ random variables, respectively. The characteristic functions of $N_1(t)$ and $N_2(t)$ are $e^{\lambda_1 t(e^{\jmath\omega}-1)}$ and $e^{\lambda_2 t(e^{\jmath\omega}-1)}$, respectively. Since $N(t)$ is the sum of $N_1(t)$ and $N_2(t)$, which are independent, it follows that the characteristic function of $N(t)$ is the product of the characteristic functions $N_1(t)$ and $N_2(t)$:

$$e^{\lambda_1 t(e^{\jmath\omega-1})} \times e^{\lambda_2 t(e^{\jmath\omega-1})} = e^{(\lambda_1+\lambda_2)t(e^{\jmath\omega-1})}.$$

This is the characteristic function of a Poisson$((\lambda_1 + \lambda_2)t)$ random variable. $\square$

Splitting (Thinning) Poisson Processes

Theorem 6 *Let* $N(t)$ *be a Poisson process of rate* λ. *Each event is assigned the label 1 with probability* p_1 *and the label 2 with probability* p_2 *independently of other assignments, where* $p_1 + p_2 = 1$. *Let* $N_1(t)$ *and* $N_2(t)$ *be the processes that count events with label 1 and label 2, respectively. Then,* $N_1(t)$ *and* $N_2(t)$ *are independent Poisson processes of rates* λp_1 *and* λp_2, *respectively.*

Proof. We check that $N_1(t)$ and $N_2(t)$ satisfy Conditions 1–3.

1. We have $N(t) = N_1(t) + N_2(t)$ and $N(0) = 0$. This implies that $N_1(0) = N_2(0) = 0$.

2. Since $N(t)$ has stationary and independent increments and the assignments of labels are specified by independent and identically distributed Bernoulli random variables, $N_1(t)$ and $N_2(t)$ have stationary and independent increments.

3. By the Principle of Total Probability,

$$
\begin{aligned}
&P(N_1(t) = n_1, N_2(t) = n_2) \\
&\quad = \sum_{n=0}^{\infty} P(N_1(t) = n_1, N_2(t) = n_2 | N(t) = n) P(N(t) = n) \\
&\quad = P(N_1(t) = n_1, N_2(t) = n_2 | N(t) = n_1 + n_2) P(N(t) = n_1 + n_2) \\
&\qquad\quad \text{(as } P(N_1(t) = n_1, N_2(t) = n_2 | N(t) = n) = 0 \text{ for } n \neq n_1 + n_2) \\
&\quad = P(N_1(t) = n_1, N_2(t) = n_2 | N(t) = n_1 + n_2)\ e^{-\lambda t} \frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!} \\
&\quad = \binom{n_1+n_2}{n_1} p_1^{n_1} p_2^{n_2}\ e^{-\lambda t} \frac{(\lambda t)^{n_1+n_2}}{(n_1+n_2)!} \\
&\quad = e^{-\lambda p_1 t} \frac{(\lambda p_1 t)^{n_1}}{n_1!} \times e^{-\lambda p_2 t} \frac{(\lambda p_2 t)^{n_2}}{n_2!}.
\end{aligned}
$$

By summing over $n_2 = 0, 1, \ldots$, we obtain

$$P(N_1(t) = n_1) = e^{-\lambda p_1 t} \frac{(\lambda p_1 t)^{n_1}}{n_1!},$$

and by summing over $n_1 = 0, 1, \ldots$, we obtain

$$P(N_2(t) = n_2) = e^{-\lambda p_2 t} \frac{(\lambda p_2 t)^{n_2}}{n_2!}.$$

Hence, $N_1(t)$ and $N_2(t)$ are Poisson processes. Since

$$P(N_1(t) = n_1, N_2(t) = n_2) = P(N_1(t) = n_1)P(N_2(t) = n_2),$$

the two processes are independent. $\square$

Mean and Autocorrelation Function of Poisson Process Recall that if X is Poisson(λ), i.e., $P(X = k) = e^{-\lambda}\frac{\lambda^k}{k!}$, then both the mean and variance of X equal λ, i.e., $\mathrm{E}[X] = \lambda$ and $\mathrm{E}[X^2] = \lambda^2 + \lambda$.

Let $N(t)$ be a Poisson process of rate λ. Then, for a given $t > 0$, $N(t)$ is Poisson(λt). Hence,

$$\begin{aligned} m_N(t) &= \lambda t \\ \mathrm{E}[N^2(t)] &= (\lambda t)^2 + (\lambda t). \end{aligned}$$

For $0 \leq t_1 \leq t_2$, we have

$$
\begin{aligned}
R_N(t_1, t_2) &= \mathrm{E}[N(t_1)N(t_2)] \\
&= \mathrm{E}[N(t_1)(N(t_1) + (N(t_2) - N(t_1))] \\
&= \mathrm{E}[(N(t_1))^2] + \mathrm{E}[N(t_1)(N(t_2) - N(t_1)] \\
&= \mathrm{E}[(N(t_1))^2] + \mathrm{E}[N(t_1)]\mathrm{E}[N(t_2) - N(t_1)] \\
&\qquad \text{(independent increments)} \\
&= \mathrm{E}[(N(t_1))^2] + \mathrm{E}[N(t_1)]\mathrm{E}[N(t_2 - t_1)] \\
&\qquad \text{(stationary increments)} \\
&= (\lambda t_1)^2 + \lambda t_1 + \lambda t_1 \times \lambda(t_2 - t_1) \\
&= \lambda t_1 + \lambda^2 t_1 t_2.
\end{aligned}
$$

By symmetry, it follows that for all $t_1, t_2 \geq 0$,

$$R_N(t_1, t_2) = \lambda \min\{t_1, t_2\} + \lambda^2 t_1 t_2.$$