Chapter 8

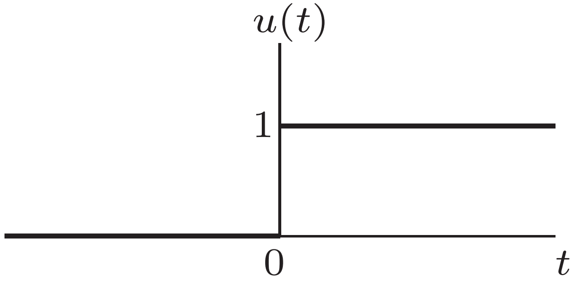
Processing and Frequency Analysis of Random Signals

Signals A *signal* is a (real) function of time, t.

Below are some common functions that are used frequently as signals.

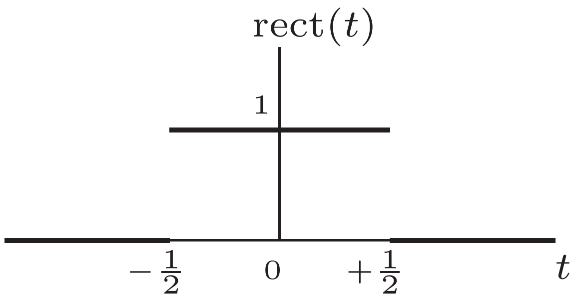
Step Function

$$u(t) = \begin{cases} 0 & t < 0 \\ 1 & t \geq 0. \end{cases}$$



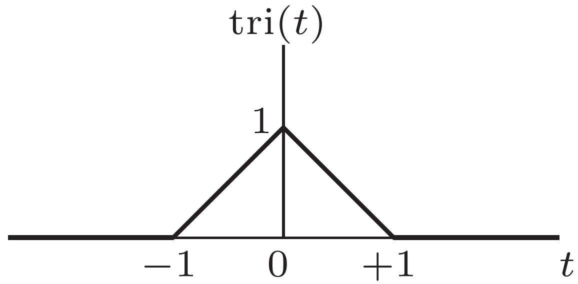
Rectangular Function

$$\mathrm{rect}(t) = \begin{cases} 1 & |t| \leq 1/2 \\ 0 & |t| > 1/2. \end{cases}$$



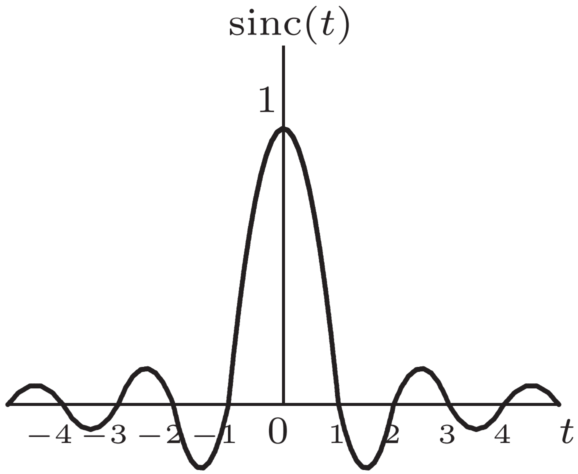
Triangular Function

$$\mathrm{tri}(t) = \begin{cases} 1 - |t| & |t| \le 1 \\ 0 & |t| > 1. \end{cases}$$



Sinc Function

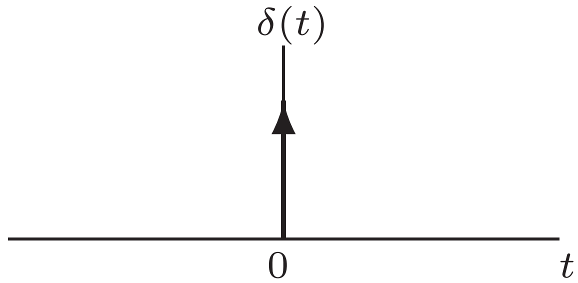
$$\mathrm{sinc}(t) = \frac{\sin(\pi t)}{\pi t}.$$



Delta Function The *delta function*, $\delta(t)$, also called the *impulse function*, is a "generalized" function defined as follows. If $x(t)$ is a smooth function (i.e., it has derivatives of all orders) with bounded support, then

$$(x * \delta)(t) = x(t),$$

where $(x * h)(t) = \int_{-\infty}^{\infty} x(u)h(t-u)du$ is the *convolution* of $x(t)$ and $h(t)$.



Fourier Transform The *Fourier transform* of a signal $x(t)$ is given by

$$\mathcal{F}[x(t)](\omega) = \int_{-\infty}^{\infty} x(t) e^{-\jmath\omega t} dt.$$

In particular, the Fourier transform is a function, typically denoted here by $\hat{x}(\omega)$, of ω, which is called the *frequency*.

We say that $x(t)$ is the *inverse Fourier transform* of $\hat{x}(\omega)$ and express this as $x(t) = \mathcal{F}^{-1}[\hat{x}(\omega)](t)$. We have

$$x(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{x}(\omega) e^{\jmath\omega t} d\omega.$$

The functions $x(t)$ and $\hat{x}(\omega)$ form a *Fourier pair* and this is expressed as

$$x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \hat{x}(\omega).$$

Properties of Fourier Transforms

Linearity: $a_1 x_1(t) + a_2 x_2(t) \stackrel{\mathcal{F}}{\longleftrightarrow} a_1 \hat{x}_1(\omega) + a_2 \hat{x}_2(\omega)$

Time Shift: $x(t - t_0) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-\jmath\omega t_0} \hat{x}(\omega)$

Frequency Shift: $e^{\jmath\omega_0 t} x(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \hat{x}(\omega - \omega_0)$

Modulation: $x(t)\cos(\omega_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2}(\hat{x}(\omega - \omega_0) + \hat{x}(\omega + \omega_0))$

$x(t)\sin(\omega_0 t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\jmath}(\hat{x}(\omega - \omega_0) - \hat{x}(\omega + \omega_0))$

Differentiation: $x'(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \jmath\omega\hat{x}(\omega)$

Integration: $\int_{-\infty}^{t} x(\tau)d\tau \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{\hat{x}(\omega)}{\jmath\omega} + \pi\hat{x}(0)\delta(\omega)$

Convolution: $(x * h)(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \hat{x}(\omega)\hat{h}(\omega)$

Multiplication: $x(t)h(t) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{2\pi}(\hat{x} * \hat{h})(\omega)$

Scaling: $x(at) \stackrel{\mathcal{F}}{\longleftrightarrow} \frac{1}{|a|}\hat{x}\left(\frac{\omega}{a}\right), a \neq 0$

Duality: $\hat{x}(t) \stackrel{\mathcal{F}}{\longleftrightarrow} 2\pi x(-\omega)$

Area of Function: $\int_{-\infty}^{\infty} x(t)dt = \hat{x}(0)$.

Area of Transform: $\int_{-\infty}^{\infty} \hat{x}(\omega)d\omega = 2\pi x(0)$.

Hermitian Symmetry: If $x(t)$ is real, then $\hat{x}(-\omega) = \hat{x}^*(\omega)$,

i.e., the magnitude of $\hat{x}(\omega)$ is even

and the phase is odd.

Parseval's Theorem: $\int_{-\infty}^{\infty} |x(t)|^2 dt = \frac{1}{2\pi} \int_{-\infty}^{\infty} |\hat{x}(\omega)|^2 d\omega$.

The following are common Fourier transform pairs:

$$
\begin{aligned}
\delta(t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad 1 \\
1 \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad 2\pi\delta(\omega) \\
\cos(\omega_0 t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad \pi(\delta(\omega-\omega_0)+\delta(\omega+\omega_0)) \\
\sin(\omega_0 t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad \tfrac{\pi}{\jmath}(\delta(\omega-\omega_0)-\delta(\omega-\omega_0)) \\
\mathrm{rect}(t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad \mathrm{sinc}\left(\tfrac{\omega}{2\pi}\right) \\
\mathrm{tri}(t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad \mathrm{sinc}^2\left(\tfrac{\omega}{2\pi}\right) \\
\mathrm{sinc}(t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad \mathrm{rect}\left(\tfrac{\omega}{2\pi}\right) \\
e^{-at}u(t) \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad \tfrac{1}{a+\jmath\omega} \\
e^{-a|t|} \quad &\overset{\mathcal{F}}{\longleftrightarrow} \quad \tfrac{2a}{a^2+\omega^2},
\end{aligned}
$$

where a has a positive real part.

The delta function has a constant Fourier transform. In practice, it idealizes functions that have constant Fourier transforms over the frequency band of interest. It can be approximated by $\frac{1}{2}ae^{-a|t|}$ as $a \to \infty$. Indeed, we have the Fourier pair

$$\frac{1}{2}ae^{-a|t|} \overset{\mathcal{F}}{\longleftrightarrow} \frac{1}{1+(\omega/a)^2}.$$

As $a \to \infty$, the right hand side approaches 1 for all frequencies and we can take $\frac{1}{2}ae^{-a|t|}$ as an approximation of $\delta(t)$. The function $\frac{1}{2}ae^{-a|t|}$ looks like an impulse as a becomes large.

Power of a Signal The *power* of the signal $x(t)$ is

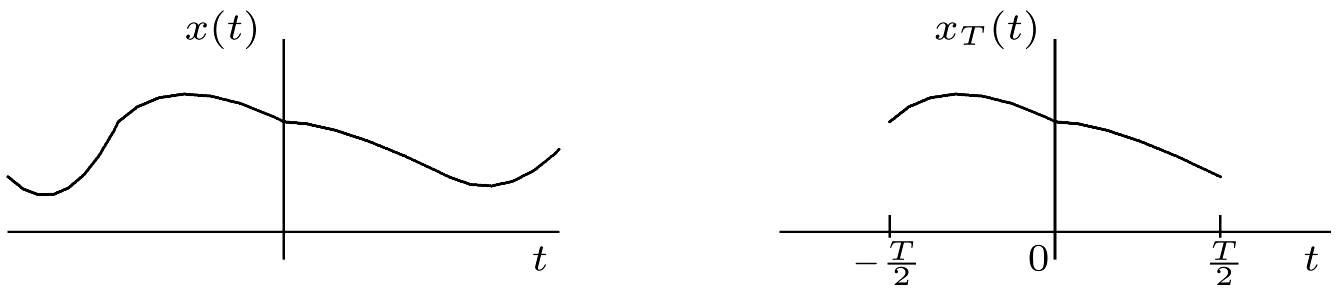
$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-T/2}^{T/2} x^2(t)dt.$$

We define the truncated signal $x_T(t)$, where $T > 0$, as follows:

$$x_T(t) = \begin{cases} x(t) & |t| \le T/2 \\ 0 & \text{otherwise.} \end{cases}$$

Then,

$$P_x = \lim_{T\to\infty} \frac{1}{T} \int_{-\infty}^{\infty} x_T^2(t)dt.$$



Power Spectral Density Let $\hat{x}_T(\omega)$ be the Fourier transform of $x_T(t)$. Parseval's Theorem gives

$$\begin{aligned} P_x &= \lim_{T\to\infty} \frac{1}{2\pi T} \int_{-\infty}^{\infty} |\hat{x}_T(\omega)|^2 d\omega \\ &= \frac{1}{2\pi} \int_{-\infty}^{\infty} \left(\lim_{T\to\infty} \frac{1}{T} |\hat{x}_T(\omega)|^2 \right) d\omega. \end{aligned}$$

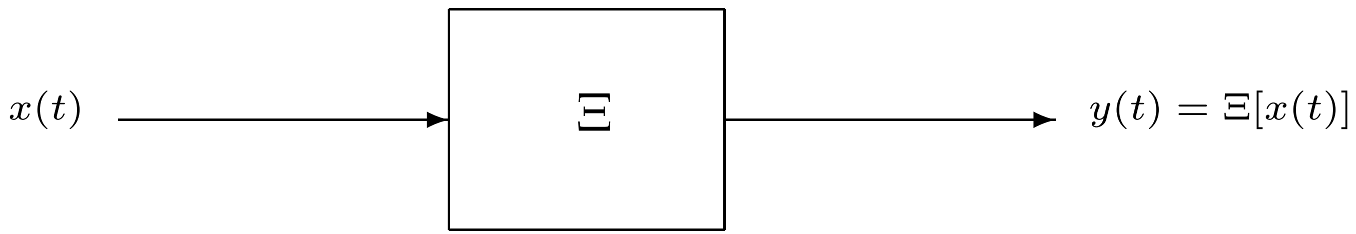
The *power-spectral density* of the signal $x(t)$ is

$$S_x(\omega) = \lim_{T\to\infty} \frac{1}{T} |\hat{x}_T(\omega)|^2.$$

It gives the distribution of power over various frequencies as

$$P_x = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_x(\omega) d\omega.$$

Linear Time-Invariant Systems A *system*, denoted by Ξ (uppercase ξ, pronounced xi), transforms an *input* signal, $x(t)$, into an *output* signal $y(t) = \Xi[x(t)]$. Both $x(t)$ and $y(t)$ are real functions of time t.



The system is *linear* if for all signals $x_1(t)$ and $x_2(t)$ and all real numbers a_1 and a_2,

$$\Xi[a_1 x_1(t) + a_2 x_2(t)] = a_1 \Xi[x_1(t)] + a_2 \Xi[x_2(t)].$$

The system is *time-invariant* if for every signal $x(t)$ and every real number t_0, $\Xi[x(t)] = y(t)$ implies that $\Xi[x(t - t_0)] = y(t - t_0)$.

In the following, we only consider linear time-invariant (LTI) systems.

Impulse Response The *impulse response*, $h(t)$, of an LTI system is the output of the system if the input signal is the delta function $\delta(t)$, i.e., $h(t) = \Xi[\delta(t)]$.

The Fourier transform, $\hat{h}(\omega)$ but usually denoted by $H(\omega)$, of $h(t)$ is given by

$$H(\omega) = \int_{-\infty}^{\infty} h(t) e^{-\jmath\omega t} dt.$$

The function $H(\omega)$ is called the *frequency response* of the LTI system.

If $x(t)$ is the input to an LTI system with impulse response $h(t)$, then the output is

$$y(t) = (x * h)(t).$$

Indeed, we have $\Xi[\delta(t)] = h(t)$. Hence, by time-invariance, $\Xi[\delta(t - t_0)] = h(t - t_0)$ and, by linearity,

$$\Xi \left[\int_{-\infty}^{\infty} x(t_0)\delta(t - t_0)dt_0 \right] = \int_{-\infty}^{\infty} x(t_0)h(t - t_0)dt_0,$$

i.e.,

$$\Xi[(x * \delta)(t)] = (x * h)(t).$$

Since $(x * \delta)(t) = x(t)$, we conclude that

$$y(t) = \Xi[x(t)] = (x * h)(t).$$

Random Processes Let $X(t)$ be a WSS continuous-time random process. The *power of the random process* $X(t)$ is

$$P_X = \mathrm{E}[X^2(t)] = R_X(0).$$

We define the truncated random process $X_T(t)$, where $T > 0$, as follows:

$$X_T(t) = \begin{cases} X(t) & |t| \leq T/2 \\ 0 & \text{otherwise.} \end{cases}$$

The *power-spectral density* of the process $X(t)$ is

$$S_X(\omega) = \lim_{T\to\infty} \frac{1}{T} \mathrm{E}\left[\left| \widehat{X_T}(\omega) \right|^2 \right],$$

where $\widehat{X_T}(\omega) = \mathcal{F}[X_T(t)](\omega)$ is the Fourier transform of $X(t)$.

Wiener-Khintchine Theorem Consider a WSS random process $X(t)$ with auto-correlation function $R_X(\tau)$ such that

$$\int\limits_{-\infty}^{\infty} |\tau R_X(\tau)| d\tau < \infty.$$

Then, the power-spectral density, $S_X(\omega)$, of the random process $X(t)$ is the Fourier transform of its auto-correlation function, $R_X(\tau)$, i.e.,

$$S_X(\omega) = \int_{-\infty}^{\infty} R_X(\tau) e^{-\jmath\omega\tau} d\tau.$$

It follows from Wiener-Khintchine Theorem that

$$P_X = R_X(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} S_X(\omega) d\omega,$$

i.e., the power-spectral density, $S_X(\omega)$, of $X(t)$ gives the distribution of the power of $X(t)$ over various frequencies.

To argue that the Wiener-Khintchine Theorem holds, we notice that

$$\begin{aligned}
\mathrm{E}\left[|\widehat{X_T}(\omega)|^2\right] &= \mathrm{E}\left[\left|\int_{-\infty}^{\infty} X_T(t)e^{-\jmath\omega t}dt\right|^2\right] \\
&= \mathrm{E}\left[\left|\int_{-T/2}^{T/2} X(t)e^{-\jmath\omega t}dt\right|^2\right] \\
&= \mathrm{E}\left[\int_{-T/2}^{T/2} X(t_1)e^{-\jmath\omega t_1}dt_1 \int_{-T/2}^{T/2} X(t_2)e^{\jmath\omega t_2}dt_2\right] \\
&= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} \mathrm{E}[X(t_1)X(t_2)]e^{-\jmath\omega(t_1-t_2)}dt_1 dt_2 \\
&= \int_{-T/2}^{T/2}\int_{-T/2}^{T/2} R_X(t_1-t_2)e^{-\jmath\omega(t_1-t_2)}dt_1 dt_2.
\end{aligned}$$

We make change of variables as follows. Let $\tau = t_1 - t_2$ and $\sigma = t_1 + t_2$. Then, $t_1 = \frac{1}{2}(\sigma + \tau)$ and $t_2 = \frac{1}{2}(\sigma - \tau)$. Hence,

$$\begin{aligned}
\mathrm{E}\left[|\widehat{X_T}(\omega)|^2\right] &= \frac{1}{2} \iint\limits_{-T \le \sigma+\tau, \sigma-\tau \le T} R_X(\tau) e^{-\jmath\omega\tau} d\sigma d\tau \\
&= \frac{1}{2} \int_{-T}^{T} \int_{\max\{-T-\tau, -T+\tau\}}^{\min\{T-\tau, T+\tau\}} R_X(\tau) e^{-\jmath\omega\tau} d\sigma d\tau \\
&= \frac{1}{2} \int_{-T}^{T} \int_{-T+|\tau|}^{T-|\tau|} R_X(\tau) e^{-\jmath\omega\tau} d\sigma d\tau \\
&= \int_{-T}^{T} (T - |\tau|) R_X(\tau) e^{-\jmath\omega\tau} d\tau.
\end{aligned}$$

Next, we consider $S_X(\omega)$.

$$
\begin{aligned}
S_X(\omega) &= \lim_{T\to\infty} \frac{1}{T} \mathrm{E}\left[\left|\widehat{X_T}(\omega)\right|^2\right] \\
&= \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} (T-|\tau|) R_X(\tau) e^{-\jmath\omega\tau}\, d\tau \\
&= \lim_{T\to\infty} \int_{-T}^{T} R_X(\tau) e^{-\jmath\omega\tau}\, d\tau \\
&\qquad - \lim_{T\to\infty} \frac{1}{T} \int_{-T}^{T} |\tau| R_X(\tau) e^{-\jmath\omega\tau}\, d\tau \\
&= \int_{-\infty}^{\infty} R_X(\tau) e^{-\jmath\omega\tau}\, d\tau
\end{aligned}
$$

as the last limit vanishes based on the assumption that $\int\limits_{-\infty}^{\infty} |\tau R_X(\tau)| d\tau < \infty$. This completes the argument.

Cross-spectral Density Recall from Chapter 4 that the cross-correlation function of jointly WSS processes $X(t)$ and $Y(t)$ is given by

$$R_{XY}(\tau) = \mathrm{E}[X(t+\tau)Y(t)]$$

and it satisfies

$$R_{XY}(\tau) = R_{YX}(-\tau).$$

We define the *cross-spectral density* of the jointly WSS processes $X(t)$ and $Y(t)$ by

$$S_{XY}(\omega) = \mathcal{F}[R_{XY}(\tau)](\omega) = \int_{-\infty}^{\infty} R_{XY}(\tau)e^{-\jmath\omega\tau}\,d\tau.$$

From the properties of Fourier transform,

$$S_{XY}(\omega) = S_{YX}(-\omega).$$

Response of LTI Systems to WSS Processes Let $X(t)$, a WSS continuous-time random process, be the input to an LTI system with impulse response $h(t)$. Then, the output, $Y(t)$, is a random process given by $Y(t) = (X * h)(t)$.

We will determine the mean the auto-correlation function of $Y(t)$ and conclude that $Y(t)$ is a WSS process and $X(t)$ and $Y(t)$ are jointly WSS. We will also determine the power-spectral density of $Y(t)$.

For the mean of $Y(t)$,

$$\begin{aligned}
m_Y(t) &= \mathrm{E}[Y(t)] = \mathrm{E}[(X * h)(t)] \\
&= \mathrm{E}\left[\int_{-\infty}^{\infty} X(u)h(t-u)du\right] \\
&= \int_{-\infty}^{\infty} \mathrm{E}[X(u)]h(t-u)du \\
&= \int_{-\infty}^{\infty} m_X h(t-u)du \\
&= m_X \int_{-\infty}^{\infty} h(t-u)du \\
&= m_X \int_{-\infty}^{\infty} h(v)dv = H(0)m_X.
\end{aligned}$$

For the cross-correlation function of $Y(t)$ and $X(t)$,

$$\begin{aligned}
R_{YX}(t_1,t_2) &= \mathrm{E}[Y(t_1)X(t_2)] = \mathrm{E}[(X*h)(t_1)X(t_2)] \\
&= \mathrm{E}\left[\left(\int_{-\infty}^{\infty} X(t_1-u)h(u)du\right) X(t_2)\right] \\
&= \int_{-\infty}^{\infty} \mathrm{E}[X(t_1-u)X(t_2)]h(u)du \\
&= \int_{-\infty}^{\infty} R_X(t_1-t_2-u)h(u)du.
\end{aligned}$$

It follows that $R_{YX}(t_1,t_2)$ depends only on the difference t_1-t_2. Hence, we can write

$$R_{YX}(\tau) = \int_{-\infty}^{\infty} R_X(\tau-u)h(u)du = (R_X*h)(\tau).$$

For the auto-correlation function of $Y(t)$,

$$\begin{aligned} R_Y(t_1, t_2) &= \mathrm{E}[Y(t_1)Y(t_2)] = \mathrm{E}[Y(t_1)(X * h)(t_2)] \\ &= \mathrm{E}\left[Y(t_1)\int_{-\infty}^{\infty} X(t_2 - u_2)h(u_2)du_2\right] \\ &= \int_{-\infty}^{\infty} \mathrm{E}[Y(t_1)X(t_2 - u_2)]h(u_2)du_2 \\ &= \int_{-\infty}^{\infty} R_{YX}(t_1 - t_2 + u_2)h(u_2)du_2. \end{aligned}$$

It follows that $R_Y(t_1, t_2)$ depends only on the difference $t_1 - t_2$ and we can write

$$\begin{aligned} R_Y(\tau) &= \int_{-\infty}^{\infty} R_{YX}(\tau + u_2) h(u_2) du_2 \\ &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} R_X(\tau + u_2 - u_1) h(u_1) h(u_2) du_1 du_2. \end{aligned}$$

This can also be written as

$$\begin{aligned} R_Y(\tau) &= \int_{-\infty}^{\infty} R_{YX}(\tau + u_2) h(u_2) du_2 \\ &= \int_{-\infty}^{\infty} R_{YX}(\tau - v) h_-(v) dv \\ &= (R_{YX} * h_-)(\tau) = (R_X * h * h_-)(\tau), \end{aligned}$$

where $h_-(t) = h(-t)$.

For the power-spectral density of $Y(t)$, we have from the Wiener-Khintchine Theorem

$$\begin{aligned} S_Y(\omega) &= \mathcal{F}[R_Y(\tau)](\omega) \\ &= \mathcal{F}[(R_X * h * h_-)(\tau)](\omega) \\ &= \mathcal{F}[R_X(\tau)](\omega) \times \mathcal{F}[h(\tau)](\omega) \times \mathcal{F}[h_-(\tau)](\omega), \end{aligned}$$

where we used the convolution property of Fourier transform. Again, from the Wiener-Khintchine Theorem, $\mathcal{F}[R_X(\tau)](\omega) = S_X(\omega)$. We also have $H(\omega) = \mathcal{F}[h(\tau)](\omega)$ is the frequency response of the LTI system. Since $h_-(\tau) = h(-\tau)$, from the scaling property, $\mathcal{F}[h_-(\tau)](\omega) = \mathcal{F}[h(-\tau)](\omega) = H(-\omega)$, which equals $H^*(\omega)$ based on the Hermitian symmetry property. We conclude that

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega).$$

Since the mean of $Y(t)$ is constant and its auto-correlation function depends only on the time difference, then $Y(t)$ is WSS. Furthermore, since the cross-correlation function of $Y(t)$ and $X(t)$ depends only on the time difference, $Y(t)$ and $X(t)$ are jointly WSS.

In summary,

$$\begin{aligned}
m_Y &= H(0)m_X \\
R_{YX}(\tau) &= \int_{-\infty}^{\infty} R_X(\tau - u)h(u)du \\
&= (R_X * h)(\tau) \\
R_Y(\tau) &= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} R_X(\tau + u_2 - u_1)h(u_1)h(u_2)du_1du_2 \\
&= (R_X * h * h_-)(\tau) \\
S_Y(\omega) &= |H(\omega)|^2 S_X(\omega).
\end{aligned}$$

White Noise A WSS random process $X(t)$ is called *white noise* if $m_X = 0$ and the power-spectral density $S_X(\omega)$ is a nonzero constant, say $\frac{N_0}{2}$, where $N_0 > 0$. From the Wiener-Khintchine Theorem, it follows that the auto-correlation function is $R_X(\tau) = \frac{N_0}{2}\delta(\tau)$.

White noise is unrealistic for the following reasons:

- The power of white noise is infinite.
- The random variables $X(t)$ and $X(t + \tau)$, $\tau > 0$, are uncorrelated no matter how small τ is.

White noise can be viewed as a "generalized" random process that idealizes random processes that have constant power-spectral density over the frequency band of interest.

Suppose that the white noise, $X(t)$, is passed through an LTI system with frequency response

$$H(\omega) = \begin{cases} 1 & |\omega - \omega_0| \le \pi \\ 0 & \text{otherwise.} \end{cases}$$

This LTI system is an ideal band-pass filter of bandwidth 2π rad/s, i.e., 1 Hz, centered at $\omega_0 > \pi$. Then, the output is a WSS random process, $Y(t)$, with zero mean and power-spectral density

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega).$$

The power of $Y(t)$ is

$$\begin{aligned} P_Y &= \frac{1}{2\pi}\int_{-\infty}^{\infty} |H(\omega)|^2 S_X(\omega) d\omega \\ &= \frac{1}{2\pi}\left(\int_{-\omega_0-\pi}^{-\omega_0+\pi} \frac{N_0}{2} d\omega + \int_{\omega_0-\pi}^{\omega_0+\pi} \frac{N_0}{2} d\omega\right) \\ &= N_0. \end{aligned}$$

This explains why it is convenient to have the factor $\frac{1}{2}$ in $S_X(\omega) = \frac{N_0}{2}$ so that N_0 is the power of the white noise per "positive" frequency unit.

If in addition to being white, $X(t)$ is a Gaussian process, then the random variables $X(t)$ and $X(t+\tau)$, $\tau > 0$, are independent. In this case, we say that $X(t)$ is *white Gaussian noise*.

Example 1 The random motion of electrons in a resistor produces *thermal noise* which is a zero-mean WSS Gaussian process, $X(t)$ in Amps, with auto-correlation function

$$R_X(\tau) = \frac{kT}{Rt_0} e^{-|\tau|/t_0} \text{ Amps}^2,$$

where R is the resistance in Ohms, $k = 1.38 \times 10^{-23}$ Joule/Kelvin is Boltzmann's constant, T is the ambient temperature in Kelvin, and t_0 is the statistical average of time intervals between collisions of free electrons in the resistor, which is in the order of 10^{-12} sec.

We have

$$S_X(\omega) = \frac{2kT/R}{1 + (\omega t_0)^2} \quad \text{Amps}^2 \text{ sec.}$$

For $\omega \leq 10^{10}$ rad/sec., $S_X(\omega)$ is almost flat and equals $2kT/R$ Amps^2 sec. Hence, we can approximate thermal noise as white Gaussian noise with $N_0 = 4kT/R$ Amps^2 sec.

Example 2 Let the white noise, $X(t)$, with power-spectral density equal to 2, be the input to an LTI system with impulse response

$$h(t) = e^{-t}u(t) = \begin{cases} 0 & t < 0 \\ e^{-t} & t \geq 0. \end{cases}$$

We determine the auto-correlation function of the output process $Y(t)$.

$$\begin{aligned} R_Y(\tau) &= (R_X * h * h_-)(\tau) = 2(\delta * h * h_-)(\tau) = 2(h * h_-)(\tau) \\ &= 2\int_{-\infty}^{\infty} h(v)h_-(\tau - v)dv = 2\int_{-\infty}^{\infty} h(v)h(v - \tau)dv \\ &= 2\int_{-\infty}^{\infty} e^{-v}u(v)e^{-(v-\tau)}u(v - \tau)dv. \end{aligned}$$

Hence,

$$
\begin{aligned}
R_Y(\tau) &= 2e^{\tau}\int_{-\infty}^{\infty} e^{-2v}u(v)u(v-\tau)dv \\
&= 2e^{\tau}\int_{0}^{\infty} e^{-2v}u(v-\tau)dv \\
&= \begin{cases} 2e^{\tau}\int_0^{\infty} e^{-2v}dv & \tau < 0 \\ 2e^{\tau}\int_{\tau}^{\infty} e^{-2v}dv & \tau \geq 0 \end{cases} \\
&= \begin{cases} e^{\tau} & \tau < 0 \\ e^{-\tau} & \tau \geq 0 \end{cases} \\
&= e^{-|\tau|}.
\end{aligned}
$$

We can also compute $R_Y(\tau)$ using Fourier transforms as follows. We have $S_Y(\omega) = |H(\omega)|^2 S_X(\omega)$, where $S_X(\omega) = 2$ and

$$H(\omega) = \mathcal{F}[e^{-t}u(t)](\omega) = \frac{1}{1 + \jmath\omega}.$$

Hence,

$$S_Y(\omega) = |H(\omega)|^2 S_X(\omega) = \left|\frac{1}{1 + \jmath\omega}\right|^2 \times 2 = \frac{2}{1 + \omega^2}.$$

Therefore,

$$R_Y(\tau) = \mathcal{F}^{-1}\left[\frac{2}{1 + \omega^2}\right](\tau) = e^{-|\tau|}.$$

Example 3 Let the WSS random process $Y(t)$ with $R_Y(\tau) = e^{-|\tau|}$ be the input to an LTI system which is a bandpass filter with frequency response

$$H(\omega) = \begin{cases} 1 & \omega_{\min} \leq |\omega| \leq \omega_{\max} \\ 0 & \text{otherwise,} \end{cases}$$

where $0 < \omega_{\min} < \omega_{\max}$.

Let $Z(t)$ be the output process. Compute the power of $Z(t)$.

From Example 2,

$$S_Z(\omega) = |H(\omega)|^2 S_Y(\omega) = \begin{cases} \frac{2}{1+\omega^2} & \omega_{\min} \le |\omega| \le \omega_{\max} \\ 0 & \text{otherwise.} \end{cases}$$

$$\begin{aligned} P_Z &= \frac{1}{2\pi} \int_{-\infty}^{\infty} S_Z(\omega) d\omega \\ &= \frac{2}{\pi} \int_{\omega_{\min}}^{\omega_{\max}} \frac{1}{1+\omega^2} d\omega \\ &= \frac{2}{\pi} \left(\tan^{-1}(\omega_{\max}) - \tan^{-1}(\omega_{\min}) \right), \end{aligned}$$

where $-\pi/2 \le \tan^{-1}(x) \le \pi/2$ for any real number x.