Chapter 6

Wiener Processes

The Wiener process is a mathematical model of Brownian motion which describes the erratic movement of a particle in a liquid or gas as it is subjected to collisions with molecules and atoms. Nobert Wiener gave a rigorous mathematical formulation of Brownian motion.

Here we consider a one-dimensional Brownian motion. Such motion is used to model randomness in many diverse fields such as stock market fluctuations, population growth, etc.

Consider a particle starting at position 0. In each unit of time of duration Δt it takes a step of length Δx either upward with probability $1/2$ or downward with probability $1/2$, independently of all previous steps, where Δt and Δx are positive real numbers. Let $W(t)$ be the position of the particle at time t. Then, $W(0) = 0$ and

$$W(t) = X(1) + X(2) + \cdots + X(\lfloor t/\Delta t \rfloor),$$

where $X(1), X(2), \ldots, X(\lfloor t/\Delta t \rfloor)$ are independent and

$$X(i) = \begin{cases} \Delta x & \text{with probability } 1/2 \\ -\Delta x & \text{with probability } 1/2. \end{cases}$$

We have

$$
\begin{aligned}
\mathrm{E}[X(i)] &= 0 \\
\mathrm{E}[X^2(i)] &= (\Delta x)^2 \\
K_{X(i)} &= \mathrm{E}[X^2(i)] - (\mathrm{E}[X(i)])^2 = (\Delta x)^2.
\end{aligned}
$$

Then, $\mathrm{E}[W(t)] = 0$ and, since $X(1), X(2), \ldots, X(\lfloor t/\Delta t \rfloor)$ are independent,

$$
K_{W(t)} = \lfloor t/\Delta t \rfloor (\Delta x)^2.
$$

From the Central Limit Theorem, for a fixed t, we have

$$\frac{W(t)}{\sqrt{\lfloor t/\Delta t \rfloor (\Delta x)^2}} \xrightarrow{d} \text{Gaussian}(0,1)$$

as $\Delta t \to 0$. Let $\alpha = \frac{\Delta x}{\sqrt{\Delta t}}$. Then, $\Delta x = \alpha \sqrt{\Delta t}$ and the Central Limit Theorem implies that

$$\frac{W(t)}{\alpha \sqrt{t}} \xrightarrow{d} \text{Gaussian}(0,1),$$

i.e.,

$$W(t) \xrightarrow{d} \text{Gaussian}(0, \alpha^2 t)$$

as $\Delta t \to 0$.

Since $X(1), X(2), \ldots$ are independent, it follows that $W(t_2) - W(t_1), W(t_3) - W(t_2), \ldots$ are independent for all $t_1 < t_2 < \cdots$. Therefore, the process $W(t)$ has independent increments.

Furthermore, since $X(1), X(2), \ldots$ are identically distributed, it follows that for $0 \leq t_1 < t_2$, $W(t_2) - W(t_1)$ depends only on the difference $t_2 - t_1$, i.e., $W(t)$ has stationary increments and the distribution of $W(t_2) - W(t_1)$ is the same as that of $W(t_2 - t_1) - W(0) = W(t_2 - t_1)$.

The Wiener Process A continuous-valued stochastic process $W(t)$, $t \geq 0$, is a *Wiener process* if

1. $W(0) = 0$.
2. $W(t)$ has stationary independent increments.
3. $W(t)$ is Gaussian$(0, \alpha^2 t)$, i.e., a Gaussian random variable with zero mean and variance $\alpha^2 t$.

The parameter α^2, where $\alpha > 0$, is called the *variance rate*.

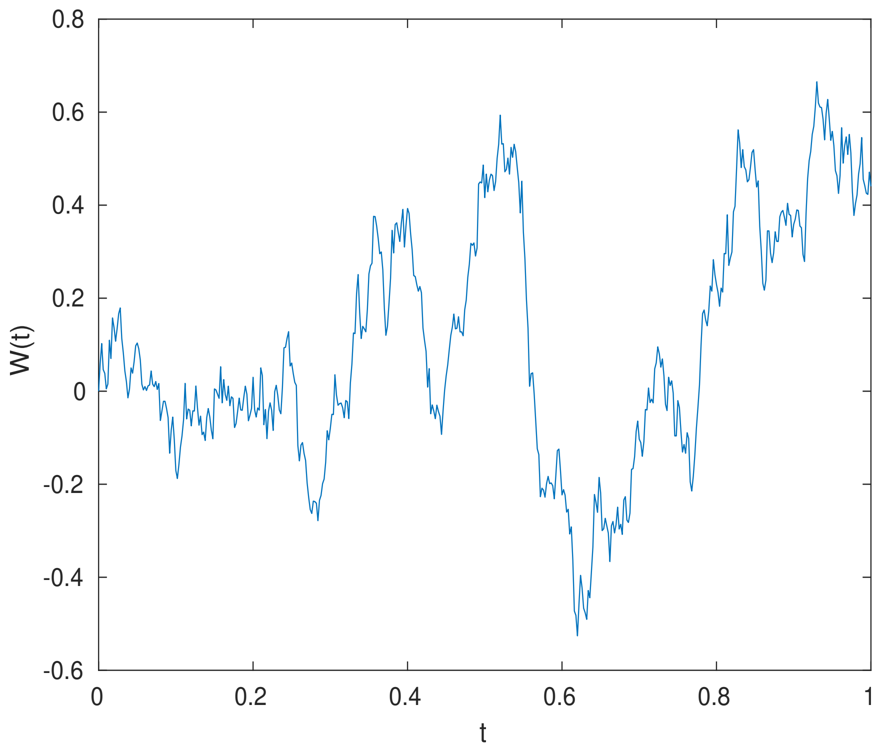


Figure 1: A sample path of Wiener Process.

Mean and autocorrelation function of the Wiener process Since $W(t)$ is Gaussian$(0, \alpha^2 t)$, $m_W(t) = \mathrm{E}[W(t)] = 0$ for $t \geq 0$. For $0 \leq t_1 \leq t_2$, we have

$$\begin{aligned} R_W(t_1, t_2) &= \mathrm{E}[W(t_1)W(t_2)] \\ &= \mathrm{E}[W(t_1)(W(t_1) + (W(t_2) - W(t_1))] \\ &= \mathrm{E}[(W(t_1))^2] + \mathrm{E}[W(t_1)(W(t_2) - W(t_1)] \\ &= \mathrm{E}[(W(t_1))^2] + \mathrm{E}[W(t_1)]\mathrm{E}[W(t_2) - W(t_1)] \\ &\qquad \text{(independent increments)} \\ &= \mathrm{E}[(W(t_1))^2] = \alpha^2 t_1. \end{aligned}$$

By symmetry, it follows that for all t_1, t_2,

$$R_W(t_1, t_2) = \alpha^2 \min\{t_1, t_2\}.$$

The Wiener Process is a Gaussian Process For $0 < t_1 < t_2 < \cdots < t_N$, $W(t_1), W(t_2) - W(t_1), \ldots,$ $W(t_N) - W(t_{N-1})$ are independent Gaussian random variables with zero mean and variances $\alpha^2 t_1, \alpha^2(t_2 - t_1), \ldots, \alpha^2(t_N - t_{N-1})$, respectively. Hence, the joint PDF of $W(t_1), W(t_2), \ldots, W(t_N)$ is

$$
\begin{aligned}
& f_{W(t_1),W(t_2),\ldots,W(t_N)}(w_1, w_2, \ldots, w_N) \\
&= f_{W(t_1)}(w_1) f_{W(t_2)-W(t_1)}(w_2 - w_1) \cdots \\
&\qquad \times f_{W(t_N)-W(t_{N-1})}(w_N - w_{N-1}) \\
&= \frac{1}{\sqrt{(2\pi\alpha^2)^N (t_1(t_2 - t_1) \cdots (t_N - t_{N-1}))}} \\
&\qquad \times e^{-\frac{1}{2\alpha^2}\left(\frac{w_1^2}{t_1} + \frac{(w_2 - w_1)^2}{t_2 - t_1} + \cdots + \frac{(w_N - w_{N-1})^2}{t_N - t_{N-1}}\right)}.
\end{aligned}
$$

This joint PDF can be written as

$$f_{\mathbf{W}}(\mathbf{w}) = \frac{1}{\sqrt{(2\pi)^N \det(\mathbf{K}_{\mathbf{W}})}} e^{-\frac{1}{2}\mathbf{w}\mathbf{K}_{\mathbf{W}}^{-1}\mathbf{w}^{\mathsf{T}}},$$

where

$$\begin{aligned}
\mathbf{w} &= (w_1, w_2, \ldots, w_N), \\
\mathbf{W} &= (W(t_1), W(t_2), \ldots, W(t_N)), \\
\mathbf{K}_{\mathbf{W}} &= \alpha^2 \begin{bmatrix} t_1 & t_1 & t_1 & \ldots & t_1 \\ t_1 & t_2 & t_2 & \ldots & t_2 \\ t_1 & t_2 & t_3 & \ldots & t_3 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ t_1 & t_2 & t_3 & \ldots & t_N \end{bmatrix}.
\end{aligned}$$

Time for first hit For a real number $w \neq 0$, let T_w be the random variable giving the time it takes for $W(t)$ to reach w for the first time, i.e.,

$$T_w = \min\{t \geq 0 : W(t) = w\}.$$

This random variable is called the time for the *first hit*. We will derive the CDF and the PDF of T_w. First, we assume that $w > 0$. We can write for $t \geq 0$,

$$\begin{aligned} P(W(t) \geq w) &= P(W(t) \geq w | T_w \leq t) P(T_w \leq t) \\ &\quad + P(W(t) \geq w | T_w > t) P(T_w > t). \end{aligned}$$

We make two observations:

1. If $T_w \le t$, then the process hits w at time at most t. By symmetry, $W(t)$ is equally likely to be above or below w. Hence,

$$P(W(t) \ge w | T_w \le t) = \frac{1}{2}.$$

2. If $T_w > t$, then the process has not reached yet w at time t, and

$$P(W(t) \ge w | T_w > t) = 0.$$

We conclude that

$$P(W(t) \ge w) = \frac{1}{2} P(T_w \le t).$$

Hence,

$$
\begin{aligned}
F_{T_w}(t) &= P(T_w \le t) = 2P(W(t) \ge w) \\
&= 2\,\frac{1}{\sqrt{2\pi\alpha^2 t}} \int_w^\infty e^{-x^2/(2\alpha^2 t)} dx \\
&= \frac{2}{\sqrt{2\pi}} \int_{w/\sqrt{\alpha^2 t}}^\infty e^{-z^2/2} dz,
\end{aligned}
$$

by change of variables. By Leibnitz rule,

$$
\begin{aligned}
f_{T_w}(t) &= \frac{d}{dt} F_{T_w}(t) \\
&= -\frac{2}{\sqrt{2\pi}} e^{-w^2/(2\alpha^2 t)}\ \frac{d}{dt}\left(\frac{w}{\sqrt{\alpha^2 t}}\right) \\
&= \frac{w}{\sqrt{2\pi\alpha^2 t^3}} e^{-w^2/(2\alpha^2 t)}.
\end{aligned}
$$

Since

$$P(T_w < t) = \frac{2}{\sqrt{2\pi}} \int_{w/\sqrt{\alpha^2 t}}^{\infty} e^{-z^2/2} dz,$$

then

$$P(T_w < \infty) = \frac{2}{\sqrt{2\pi}} \int_0^{\infty} e^{-z^2/2} dz = 1,$$

i.e., with probability 1, a particle undergoing Brownian motion hits w. However, the average time to hit w is

$$\begin{aligned}
\mathrm{E}[T_w] &= \int_0^{\infty} t f_{T_w}(t) dt = \frac{w}{\sqrt{2\pi\alpha^2}} \int_0^{\infty} \frac{e^{-w^2/(2\alpha^2 t)}}{\sqrt{t}} dt \\
&> \frac{w}{\sqrt{2\pi\alpha^2}} \int_1^{\infty} \frac{e^{-w^2/(2\alpha^2 t)}}{\sqrt{t}} dt \\
&> \frac{w}{\sqrt{2\pi\alpha^2}} e^{-w^2/(2\alpha^2)} \int_1^{\infty} \frac{1}{\sqrt{t}} dt = \infty.
\end{aligned}$$

So far we assumed that $w > 0$. By symmetry, for all w,

$$
\begin{aligned}
F_{T_w}(t) = P(T_w \le t) &= \frac{2}{\sqrt{2\pi}} \int_{|w|/\sqrt{\alpha^2 t}}^{\infty} e^{-z^2/2} dz \\
f_{T_w}(t) &= \frac{|w|}{\sqrt{2\pi\alpha^2 t^3}} e^{-w^2/(2\alpha^2 t)}.
\end{aligned}
$$

Maximum Value of a Wiener Process Let $W_{\max}(t)$ be the maximum value of the process over the interval $[0, t]$. Then, $W_{\max}(t)$ is a random variable. For $w > 0$, $W_{\max}(t) \geq w$ if and only if $T_w \leq t$. Hence,

$$P(W_{\max}(t) \geq w) = P(T_w \leq t) = \frac{2}{\sqrt{2\pi}} \int_{w/\sqrt{\alpha^2 t}}^{\infty} e^{-z^2/2} dz.$$

Probability of hitting a positive value before a negative value Let w^+ and w^- be positive real numbers. The probability that the Wiener process goes up to w^+ before going down to $-w^-$ is $\frac{w^-}{w^++w^-}$.

Indeed, recall from the study of random walks using Markov chains in Chapter 5 that if a particle performs a random walk with $p = q = 1/2$ on states $1, 2, \ldots, N$ and with absorbing barriers at states 1 and N, then the probability that the particle gets absorbed at state N if it starts at state i, where $i = 2, 3 \ldots, N - 1$, is $f_{i,N} = \frac{i-1}{N-1}$. This probability is the ratio of the number of steps from the starting position of the particle to the other barrier divided by the number of steps between the two barriers. With steps of equal size Δx, if the particle starts at position 0, with a barrier at position w^+ and another at position $-w^-$, then the probability of being absorbed at the barrier in position w^+ is $\frac{w^-}{w^++w^-}$.

Arcsine Law For $0 < t_1 < t_2$, let $O(t_1, t_2)$ be the event that the Wiener process vanishes at least once in the open interval (t_1, t_2), i.e., $W(t) = 0$ for some t, where $t_1 < t < t_2$. We are interested in the probability $P(O(t_1, t_2))$ of this event. To find this probability, we condition on the value of $W(t_1)$:

$$P(O(t_1, t_2)) = \int_{-\infty}^{\infty} P(O(t_1, t_2)|W(t_1) = w_1) f_{W(t_1)}(w_1) dw_1.$$

Because of symmetry, the probability that the process goes from w_1 at time t_1 to 0 before time t_2 is the same as the probability that the process goes from 0 at time t_1 to w_1 before time t_2. This probability equals the probability that, starting from 0, the process hits w_1 before time $t_2 - t_1$.

Hence,

$$\begin{aligned} P(O(t_1,t_2)|W(t_1)=w_1) &= P(T_{w_1} < t_2 - t_1) \\ &= \frac{2}{\sqrt{2\pi}} \int_{|w_1|/\sqrt{\alpha^2(t_2-t_1)}}^{\infty} e^{-x^2/2} dx. \end{aligned}$$

We also have

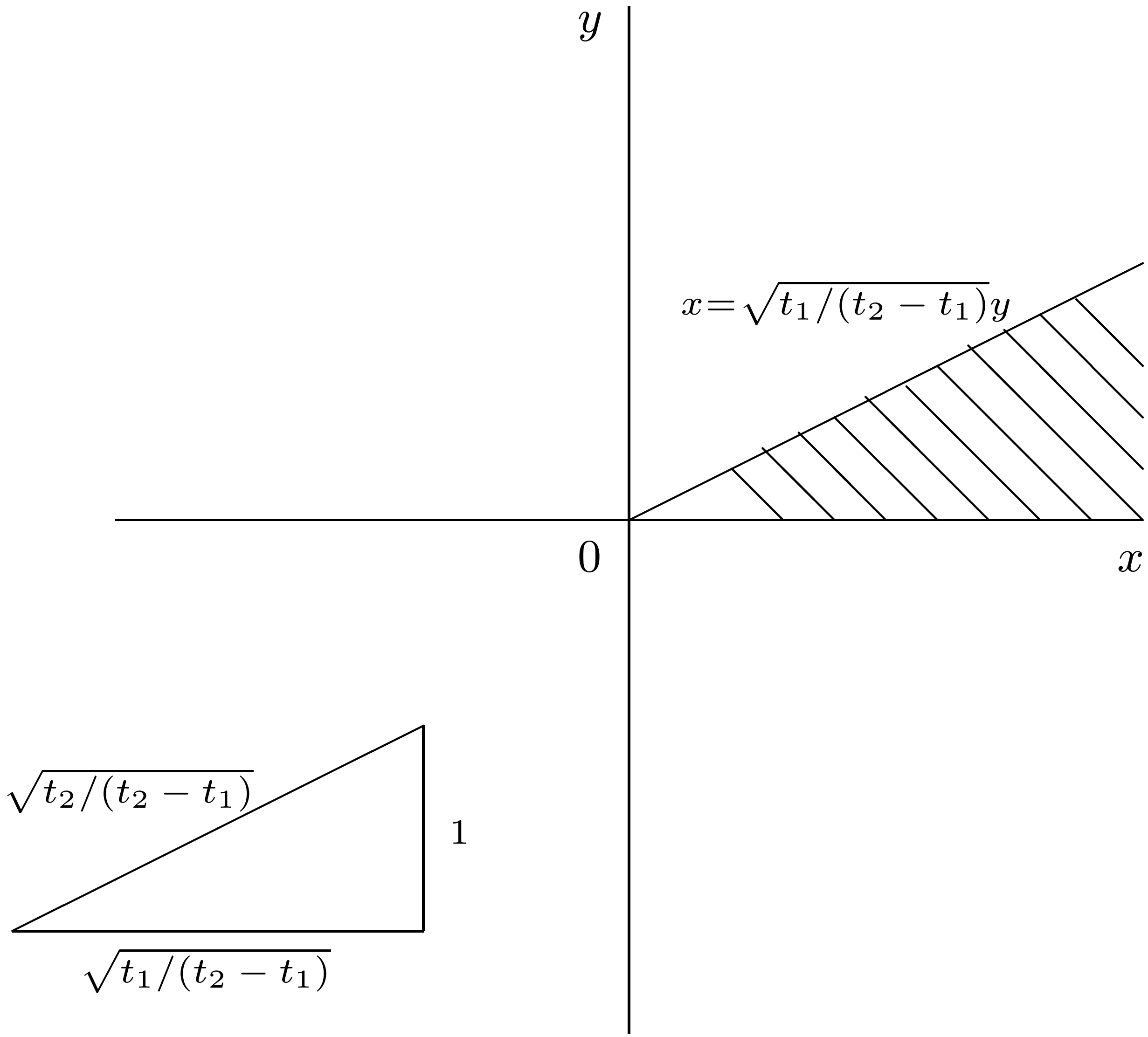
$$f_{W(t_1)}(w_1) = \frac{1}{\sqrt{2\pi\alpha^2 t_1}} e^{-w_1^2/(2\alpha^2 t_1)}.$$

We conclude that

$$\begin{aligned} P(O(t_1,t_2)) &= \int_{-\infty}^{\infty} P(O(t_1,t_2)|W(t_1)=w_1) f_{W(t_1)}(w_1) dw_1 \\ &= 2\int_0^{\infty} P(O(t_1,t_2)|W(t_1)=w_1) f_{W(t_1)}(w_1) dw_1 \\ &= \frac{2}{\pi\alpha\sqrt{t_1}} \int_0^{\infty} \int_{w_1/\sqrt{\alpha^2(t_2-t_1)}}^{\infty} e^{-\left(\frac{x^2}{2} + \frac{w_1^2}{2\alpha^2 t_1}\right)} dx dw_1. \end{aligned}$$

Replacing w_1 by $\sqrt{\alpha^2 t_1}y$ followed by replacing x and y by $r\cos(\theta)$ and $r\sin(\theta)$, respectively, (see next page) gives

$$
\begin{aligned}
P(O(t_1,t_2)) &= \frac{2}{\pi}\int_0^\infty \int_{\sqrt{\frac{t_1}{t_2-t_1}}y}^\infty e^{-\frac{x^2+y^2}{2}}dxdy \\
&= \frac{2}{\pi}\int_0^\infty \int_0^{\arccos(\sqrt{t_1/t_2})} re^{-r^2/2}d\theta dr \\
&= \frac{2}{\pi}\arccos(\sqrt{t_1/t_2})\int_0^\infty re^{-r^2/2}dr \\
&= \frac{2}{\pi}\arccos(\sqrt{t_1/t_2}) \\
&= \frac{2}{\pi}\left(\frac{\pi}{2} - \arcsin(\sqrt{t_1/t_2})\right) \\
&= 1 - \frac{2}{\pi}\arcsin(\sqrt{t_1/t_2}).
\end{aligned}
$$



y
$x = \sqrt{t_1/(t_2 - t_1)}\,y$
0
x
$\sqrt{t_2/(t_2 - t_1)}$
1
$\sqrt{t_1/(t_2 - t_1)}$

Example 1 Let $W(t)$ be a Wiener process with unit variance rate.

1. Find the probability that $W(1) + W(6) > 6$.
2. Find the probability that $W(6) > 5$ given that $W(1) = 1$.
3. Find the probability that $W(1) > 1$ given that $W(6) = 5$.

Solution

1. We have $W(1) + W(6) = 2W(1) + (W(6) - W(1))$, where $W(1)$ is Gaussian$(0, 1)$ and, therefore, $2W(1)$ is Gaussian$(0, 4)$, and $W(6) - W(1)$ is Gaussian$(0, 5)$. Since $W(1)$ and $W(6) - W(1)$ are independent, it follows that $W(1) + W(6)$ is Gaussian$(0, 9)$. Hence,

$$P(W(1)+W(6) > 6) = 1-\varphi\left(\frac{6-0}{\sqrt{9}}\right) = 1-\varphi(2) \approx 0.0228.$$

2. We have

$$
\begin{aligned}
&P(W(6) > 5|W(1) = 1) \\
&\quad = \quad P(W(6) - W(1) > 4|W(1) = 1) \\
&\quad = \quad P(W(6) - W(1) > 4) \quad \text{(independent increments)} \\
&\quad = \quad P(W(5) > 4) \quad \text{(stationary increments)} \\
&\quad = \quad 1 - \varphi(4/\sqrt{5}) \approx 0.0368.
\end{aligned}
$$

3. By Bayes' rule

$$f_{W(1)|W(6)}(w|5) = \frac{f_{W(1)}(w) f_{W(6)|W(1)}(5|w)}{f_{W(6)}(5)},$$

where

$$f_{W(1)}(w) = \frac{1}{\sqrt{2\pi}} e^{-w^2/2}$$

$$f_{W(6)}(5) = \frac{1}{\sqrt{2\pi \times 6}} e^{-25/(2\times 6)}$$

$$f_{W(6)|W(1)}(5|w) = f_{W(5)}(5-w) = \frac{1}{\sqrt{2\pi \times 5}} e^{-(5-w)^2/(2\times 5)}.$$

Hence,

$$
\begin{aligned}
f_{W(1)|W(6)}(w|5) &= \frac{1}{\sqrt{2\pi \times 5/6}} e^{-(w^2/2+(5-w)^2/10-25/12)} \\
&= \frac{1}{\sqrt{2\pi \times 5/6}} e^{-(36w^2-60w+25)/60} \\
&= \frac{1}{\sqrt{2\pi \times 5/6}} e^{-(w-5/6)^2/(2\times 5/6)},
\end{aligned}
$$

which is the PDF of Gaussian$(5/6, 5/6)$. Therefore,

$$
\begin{aligned}
P(W(1) > 1|W(6) = 5) &= 1 - \varphi\left(\frac{1-5/6}{\sqrt{5/6}}\right) \\
&= 1 - \varphi\left(\frac{1}{\sqrt{30}}\right) \approx 0.4276.
\end{aligned}
$$