Chapter 4

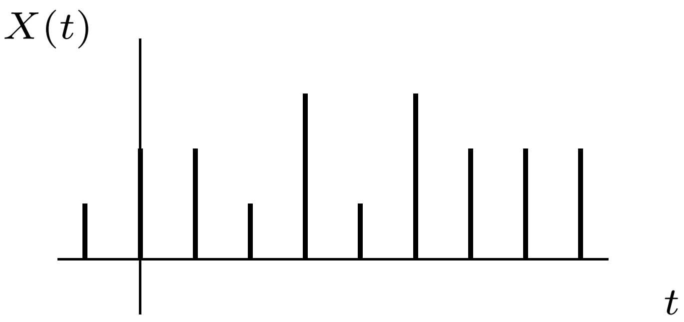
Specification of Stochastic Processes

Consider a probability space $(\Omega, \mathcal{F}, \mathcal{P})$ A *stochastic process*, also called *random process* and denoted by $X(t)$, $t \in \mathcal{T}$, is a map that associates to each outcome ω and each t that belongs to a set $\mathcal{T}$ a value $X(t, \omega)$. Typically, t denotes time, $\mathcal{T}$ is called the *index set*, and $X(t)$ is the *state* of the process at time t.

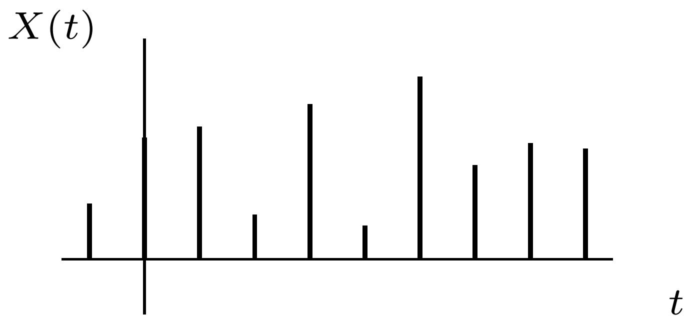
The process is *discrete-time* if $\mathcal{T}$ is discrete and *continuous-time* if $\mathcal{T}$ is an interval of the real line.

A stochastic process is *discrete-valued* if $X(t)$ assumes discrete values and *continuous-valued* if $X(t)$ assumes continuous values.

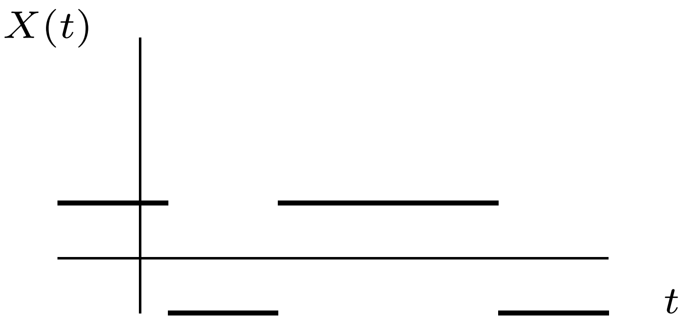
In particular, for a fixed t, the stochastic process is a random variable called the *state* of the process at time t and for a fixed ω the stochastic process is a deterministic function of time called a *sample path*.



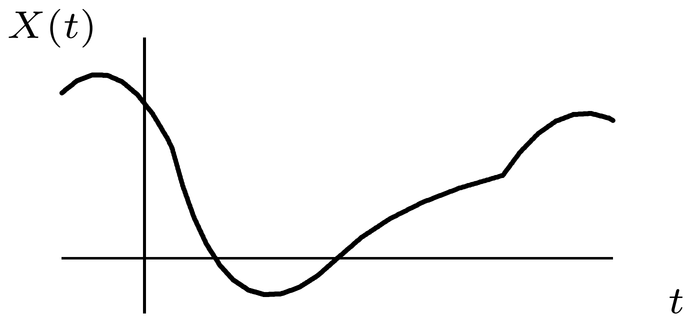
Discrete-time, discrete-valued



Discrete-time, continuous-valued



Continuous-time, discrete-valued



Continuous-time, continuous-valued

For any positive integer N and any distinct $t_1, t_2, \ldots, t_N$ in $\mathcal{T}$, $X(t_1), X(t_2), \ldots, X(t_N)$ are random variables which are characterized by a joint PMF

$$p_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1, x_2, \ldots, x_N)$$

if the process is discrete-valued, and by a joint CDF

$$F_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1, x_2, \ldots, x_N)$$

or a joint PDF

$$f_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1, x_2, \ldots, x_N)$$

if the process is continuous-valued.

The stochastic process is characterized by the above for all positive integers N, all distinct $t_1, t_2, \ldots, t_N$, and all $x_1, x_2, \ldots, x_N$.

We say that the process is *strict sense stationary* (SSS) if all its joint PMFs (if discrete-valued) or joint CDFs and PDFs (if continuous-valued) are invariant to time-shifts, i.e.,

$$\begin{aligned} p_{X(t_1+h),X(t_2+h),\ldots,X(t_N+h)}&(x_1,x_2,\ldots,x_N) \\ &= p_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1,x_2,\ldots,x_N), \end{aligned}$$

$$\begin{aligned} F_{X(t_1+h),X(t_2+h),\ldots,X(t_N+h)}&(x_1,x_2,\ldots,x_N) \\ &= F_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1,x_2,\ldots,x_N), \end{aligned}$$

$$\begin{aligned} f_{X(t_1+h),X(t_2+h),\ldots,X(t_N+h)}&(x_1,x_2,\ldots,x_N) \\ &= f_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1,x_2,\ldots,x_N), \end{aligned}$$

for all h, all N, all distinct $t_1, t_2, \ldots, t_N$ in $\mathcal{T}$ such that $t_1+h, t_2+h, \ldots, t_N+h$ are also in $\mathcal{T}$, and all $x_1, x_2, \ldots, x_N$.

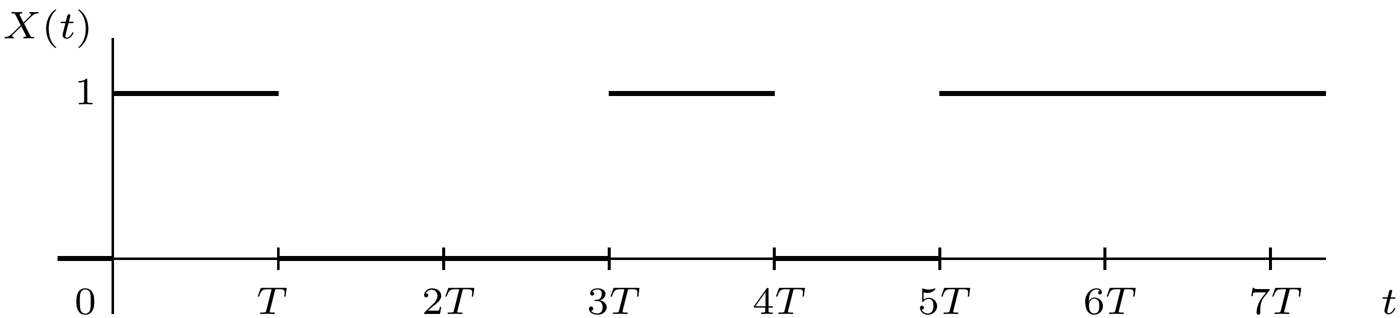
Example 1 Let $\Pi_T(t)$ be the pulse function defined by

$$\Pi_T(t) = \begin{cases} 1 & 0 \le t < T \\ 0 & \text{otherwise.} \end{cases}$$

Let B_n, as $n \in \mathbb{Z}$ (the set of integers), be independent Bernoulli(p), $0 < p < 1$, random variables. Then,

$$X(t) = \sum_{n \in \mathbb{Z}} B_n \Pi_T(t - nT)$$

defines a discrete-valued continuous-time stochastic process. At any time t, $X(t)$ is either 0 or 1.


X(t)
1
0
T
2T
3T
4T
5T
6T
7T
t

For distinct $t_1, t_2, \ldots, t_N$, we determine the joint PMF

$$p_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1, x_2, \ldots, x_N),$$

where $x_1, x_2, \ldots, x_N$ are either 0 or 1.

First, suppose $nT \le t_{i_1}, t_{i_2}, \ldots, t_{i_\ell} < (n+1)T$ for some $n \in \mathbb{Z}$. Then,

$$p_{X(t_{i_1}),X(t_{i_2}),\ldots,X(t_{i_\ell})}(x_{i_1}, x_{i_2}, \ldots, x_{i_\ell})$$
$$= \begin{cases} 1-p & \text{if } x_{i_1} = x_{i_2} = \cdots = x_{i_\ell} = 0 \\ p & \text{if } x_{i_1} = x_{i_2} = \cdots = x_{i_\ell} = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define for each $n \in \mathbb{Z}$,

$$\begin{aligned} \mathcal{I}_n^0 &= \{i : 1 \leq i \leq N, nT \leq t_i < (n+1)T, x_{t_i} = 0\} \\ \mathcal{I}_n^1 &= \{i : 1 \leq i \leq N, nT \leq t_i < (n+1)T, x_{t_i} = 1\}, \end{aligned}$$

and

$$r_n = \begin{cases} 1-p & \text{if } \mathcal{I}_n^0 \text{ is nonempty and } \mathcal{I}_n^1 \text{ is empty} \\ p & \text{if } \mathcal{I}_n^1 \text{ is nonempty and } \mathcal{I}_n^0 \text{ is empty} \\ 1 & \text{if both } \mathcal{I}_n^0 \text{ and } \mathcal{I}_n^1 \text{ are empty} \\ 0 & \text{if both } \mathcal{I}_n^0 \text{ and } \mathcal{I}_n^1 \text{ are nonempty.} \end{cases}$$

Then,

$$p_{X(t_1),X(t_2),\ldots,X(t_N)}(x_1, x_2, \ldots, x_N) = \prod_{n \in \mathbb{Z}} r_n.$$

We conclude that

$$p_{X(t_1)}(x_1) = \begin{cases} 1-p & \text{if } x_1 = 0 \\ p & \text{if } x_1 = 1. \end{cases}$$

For $\lfloor t_1/T \rfloor = \lfloor t_2/T \rfloor$,

$$p_{X(t_1),X(t_2)}(x_1, x_2) = \begin{cases} 1-p & \text{if } x_1 = x_2 = 0 \\ p & \text{if } x_1 = x_2 = 1 \\ 0 & \text{if } x_1 \neq x_2. \end{cases}$$

For $\lfloor t_1/T \rfloor \neq \lfloor t_2/T \rfloor$,

$$p_{X(t_1),X(t_2)}(x_1, x_2) = \begin{cases} (1-p)^2 & \text{if } x_1 = x_2 = 0 \\ p^2 & \text{if } x_1 = x_2 = 1 \\ p(1-p) & \text{if } x_1 = 0, x_2 = 1 \\ p(1-p) & \text{if } x_1 = 1, x_2 = 0. \end{cases}$$

We notice that the stochastic process is not SSS. Indeed,

$$p_{X(0.2T),X(0.7T)}(1,1) = p$$

but with a time shift of $0.4T$, we get

$$p_{X(0.6T),X(1.1T)}(1,1) = p^2.$$

Example 2 Let $\Pi_T(t)$ be the pulse function defined by

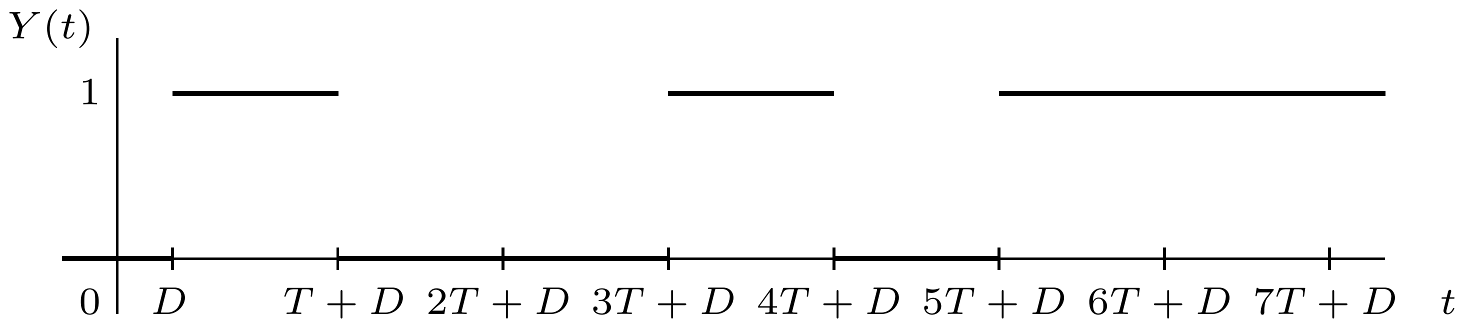
$$\Pi_T(t) = \begin{cases} 1 & 0 \leq t < T \\ 0 & \text{otherwise.} \end{cases}$$

Let B_n, as $n \in \mathbb{Z}$, be independent Bernoulli(p), $0 < p < 1$, random variables. Let D, which represents random delay, be Uniform$(0, T)$ random variable independent of the B_n's.

Then,

$$Y(t) = \sum_{n \in \mathbb{Z}} B_n \Pi_T(t - nT - D)$$

defines a discrete-valued continuous-time stochastic process.



$Y(t)$
1
0
D
$T+D$
$2T+D$
$3T+D$
$4T+D$
$5T+D$
$6T+D$
$7T+D$
t

At any time t, $Y(t)$ is either 0 or 1 with probabilities $1-p$ and p, respectively. This determines $p_{Y(t_1)}(y_1)$ for $y_1 = 0$ and 1.

Next, we consider $p_{Y(t_1),Y(t_2)}(y_1, y_2)$ for $t_1 < t_2$. Clearly, if $t_2 - t_1 \geq T$, then the random variables $Y(t_1)$ and $Y(t_2)$ are independent and

$$p_{Y(t_1),Y(t_2)}(y_1, y_2) = p_{Y(t_1)}(y_1)\, p_{Y(t_2)}(y_2).$$

Hence, for $t_2 - t_1 \geq T$, we have

$$p_{Y(t_1),Y(t_2)}(y_1, y_2) = \begin{cases} (1-p)^2 & \text{if } y_1 = y_2 = 0 \\ p^2 & \text{if } y_1 = y_2 = 1 \\ p(1-p) & \text{if } y_1 = 0, y_2 = 1 \\ p(1-p) & \text{if } y_1 = 1, y_2 = 0. \end{cases}$$

Next, suppose that $0 \leq t_2 - t_1 < T$. Then, for some unique integer n, we either have

- $D + nT \leq t_1 \leq t_2 < D + (n+1)T$, i.e. $t_2 - (n+1)T < D \leq t_1 - nT$, an event which we denote by A_1 and occurs with probability $1 - (t_2 - t_1)/T$, or
- $t_1 < D + nT \leq t_2$, i.e., $t_1 - nT < D \leq t_2 - nT$, an event which we denote by A_2 and occurs with probability $(t_2 - t_1)/T$.

Hence, for $0 \le t_2 - t_1 < T$, we have

$$
\begin{aligned}
p_{Y(t_1),Y(t_2)}(y_1, y_2) &= P(Y(t_1) = y_1, Y(t_2) = y_2) \\
&= P(Y(t_1) = y_1, Y(t_2) = y_2 | A_1) P(A_1) \\
&\quad + P(Y(t_1) = y_1, Y(t_2) = y_2 | A_2) P(A_2) \\
&= P(Y(t_1) = y_1, Y(t_2) = y_2 | A_1) \left(1 - \frac{t_2 - t_1}{T}\right) \\
&\quad + P(Y(t_1) = y_1, Y(t_2) = y_2 | A_2) \frac{t_2 - t_1}{T}.
\end{aligned}
$$

Clearly,

$$P(Y(t_1) = y_1, Y(t_2) = y_2 | A_1) = \begin{cases} 1 - p & \text{if } y_1 = y_2 = 0 \\ p & \text{if } y_1 = y_2 = 1 \\ 0 & \text{if } y_1 \neq y_2. \end{cases}$$

$$P(Y(t_1) = y_1, Y(t_2) = y_2 | A_2) = \begin{cases} (1 - p)^2 & \text{if } y_1 = y_2 = 0 \\ p^2 & \text{if } y_1 = y_2 = 1 \\ p(1 - p) & \text{if } y_1 = 0, y_2 = 1 \\ p(1 - p) & \text{if } y_1 = 1, y_2 = 0. \end{cases}$$

$$
\begin{aligned}
&p_{Y(t_1),Y(t_2)}(0,0) \\
&\qquad = \left(1 - \frac{t_2 - t_1}{T}\right)(1-p) + \frac{t_2 - t_1}{T}(1-p)^2 \\
&\qquad = (1-p)\left(1 - \frac{t_2 - t_1}{T} p\right) \\
&p_{Y(t_1),Y(t_2)}(1,1) \\
&\qquad = \left(1 - \frac{t_2 - t_1}{T}\right) p + \frac{t_2 - t_1}{T} p^2 \\
&\qquad = p\left(1 - \frac{t_2 - t_1}{T}(1-p)\right) \\
&p_{Y(t_1),Y(t_2)}(y_1, y_2) \\
&\qquad = p(1-p)\frac{t_2 - t_1}{T}
\end{aligned}
$$

for $y_1 \neq y_2$, where $0 \leq t_2 - t_1 < T$.

We notice that $p_{Y(t_1)}(y_1)$ is constant and $p_{Y(t_1),Y(t_2)}(y_1, y_2)$ depends only on the difference $t_2 - t_1$ and, hence, is the same as $p_{Y(t_1+h),Y(t_2+h)}(y_1, y_2)$ for all h.

Actually, for any N, $t_1, t_2, \ldots, t_N, y_1, y_2, \ldots, y_N$,

$$p_{Y(t_1),Y(t_2),\ldots,Y(t_N)}(y_1, y_2, \ldots, y_N)$$

depends only on the differences between $t_1, t_2, \ldots, t_N$ and, therefore, is invariant to time-shifts. We conclude that the process $Y(t)$ is SSS.

Mean, Auto-correlation, and Auto-covariance Let $X(t)$ be a stochastic process. Its *mean* is

$$m_X(t) = \mathrm{E}[X(t)]$$

and its *auto-correlation function* is

$$R_X(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)].$$

The *auto-covariance* function of $X(t)$ is

$$\begin{aligned} K_X(t_1, t_2) &= \mathrm{E}[(X(t_1) - m_X(t_1))(X(t_2) - m_X(t_2))] \\ &= R_X(t_1, t_2) - m_X(t_1)m_X(t_2). \end{aligned}$$

Notice that the auto-covariance function of a stochastic process is the auto-correlation function of the stochastic process $X(t) - m_X(t)$.

The stochastic process $X(t)$ is said to be *wide sense stationary* (WSS) if

1. $m_X(t)$ does not depend on t, in which case it will be denoted by m_X, and
2. $R_X(t_1, t_2)$ depends only on $t_1 - t_2$, in which case it will be denoted by $R_X(t_1 - t_2)$. Typically, τ is used as the argument.

Clearly, if $X(t)$ is SSS, then it is WSS. The converse is not necessarily true.

Example 3 For the stochastic process $X(t)$ in Example 1,

$$\begin{aligned} m_X(t) &= \mathrm{E}[X(t)] = p \\ R_X(t_1, t_2) &= \mathrm{E}[X(t_1)X(t_2)] = \begin{cases} p & \text{if } \lfloor t_1/T \rfloor = \lfloor t_2/T \rfloor \\ p^2 & \text{if } \lfloor t_1/T \rfloor \neq \lfloor t_2/T \rfloor. \end{cases} \end{aligned}$$

Since $R_X(0.2T, 0.7T) = p$ and, with a time shift of $0.4T$, $R_X(0.6T, 1.1T) = p^2$, the stochastic process $X(t)$ is not WSS.

Example 4 For the stochastic process $Y(t)$ in Example 2,

$$m_{Y(t)} = \mathrm{E}[Y(t)] = p.$$

We also have for $t_2 - t_1 \geq T$,

$$R_Y(t_1, t_2) = \mathrm{E}[Y(t_1)Y(t_2)] = p^2$$

and for $0 \leq t_2 - t_1 < T$,

$$R_Y(t_1, t_2) = \mathrm{E}[Y(t_1)Y(t_2)] = p\left(1 - \frac{t_2 - t_1}{T}(1-p)\right).$$

Since $R_Y(t_1, t_2) = R_Y(t_2, t_1)$, we can write

$$R_Y(t_1, t_2) = p\left(1 - \frac{\min\{|t_1 - t_2|, T\}}{T}(1-p)\right),$$

which is valid for all t_1 and t_2. Since $m_Y(t)$ is constant and $R_Y(t_1, t_2)$ depends only on $t_1 - t_2$, the stochastic process is WSS. This should be indeed the case since it is SSS.

Properties of the Auto-correlation Function

Since $R_X(t_1, t_2) = \mathrm{E}[X(t_1)X(t_2)]$, the auto-correlation function is symmetric in t_1 and t_2, i.e., $R_X(t_1, t_2) = R_X(t_2, t_1)$.

If $X(t)$ is WSS, then $R_X(\tau) = R_X(-\tau)$, i.e., $R_X(\tau)$ is an even function.

The auto-correlation function is nonnegative definite, i.e., for any N, any distinct $t_1, t_2, \ldots, t_N$, and any real vector $\mathbf{a} = (a_1, a_2, \ldots, a_N)$,

$$\sum_{k=1}^{N} \sum_{l=1}^{N} a_k a_l R_X(t_k, t_l) = \mathbf{a}[R_X(t_k, t_l)]_{1 \le k,l \le N} \mathbf{a}^{\mathsf{T}} \ge 0,$$

where T denotes transpose.

Indeed, let $Z = \sum_{l=1}^{N} a_l X(t_l)$. Then,

$$\begin{aligned} \mathrm{E}[Z^2] &= \mathrm{E}\left[\left(\sum_{k=1}^{N} a_k X(t_k)\right)\left(\sum_{l=1}^{N} a_l X(t_l)\right)\right] \\ &= \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l \mathrm{E}[X(t_k)X(t_l)] = \sum_{k=1}^{N}\sum_{l=1}^{N} a_k a_l R_X(t_k, t_l). \end{aligned}$$

The result follows from the fact that $\mathrm{E}[Z^2] \geq 0$ for any random variable Z.

Recall that an $N \times N$ real symmetric matrix $\mathbf{M}$ is nonnegative definite if any of the following three properties hold:

1. $\mathbf{a}\mathbf{M}\mathbf{a}^{\mathsf{T}} \geq 0$ for any real row vector $\mathbf{a} \in \mathbb{R}^N$.
2. All the eigenvalues are nonnegative.
3. The determinants of all principal minors of $\mathbf{M}$, i.e., square submatrices obtained by deleting rows and column with the same indices, are nonnegative.

By setting $N = 1$ and $N = 2$ in last property, it follows, with $\tau = t_1 - t_2$, that if X is WSS, then $R_X(0) \geq 0$ and

$$\det \begin{pmatrix} R_X(0) & R_X(\tau) \\ R_X(-\tau) & R_X(0) \end{pmatrix} \geq 0.$$

Since $R_X(-\tau) = R_X(\tau)$, then $|R_X(\tau)| \leq R_X(0)$.

Cross-correlation and Cross-covariance Let $X(t)$ and $Y(t)$ be stochastic processes. Their *cross-correlation function* is

$$R_{XY}(t_1, t_2) = \mathrm{E}[X(t_1)Y(t_2)]$$

and their *cross-covariance function* is

$$\begin{aligned} K_{XY}(t_1, t_2) &= \mathrm{E}[(X(t_1) - m_X(t_1))(Y(t_2) - m_Y(t_2))] \\ &= R_{XY}(t_1, t_2) - m_X(t_1)m_Y(t_2). \end{aligned}$$

The processes $X(t)$ and $Y(t)$ are *uncorrelated* if $K_{XY}(t_1, t_2) = 0$ for all t_1 and t_2.

We say that the stochastic processes $X(t)$ and $Y(t)$ are *jointly WSS* if both $X(t)$ and $Y(t)$ are WSS and $R_{XY}(t_1, t_2)$ depends only on $t_1 - t_2$, in which case it will be denoted by $R_{XY}(t_1 - t_2)$. Typically, τ is used as the argument. In this case $R_{XY}(\tau) = R_{YX}(-\tau)$.

We say that the stochastic processes $X(t)$ and $Y(t)$ are *independent* if for all $N, M, t_1, \ldots, t_N, t'_1, \ldots, t'_M, x_1, \ldots, x_N, y_1, \ldots, y_M$,

$$
\begin{aligned}
& p_{X(t_1),\ldots,X(t_N),Y(t'_1),\ldots,Y(t'_M)}(x_1, \ldots, x_N, y_1, \ldots, y_M) \\
& \qquad = p_{X(t_1),\ldots,X(t_N)}(x_1, \ldots, x_N)\; p_{Y(t'_1),\ldots,Y(t'_M)}(y_1, \ldots, y_M) \\
& F_{X(t_1),\ldots,X(t_N),Y(t'_1),\ldots,Y(t'_M)}(x_1, \ldots, x_N, y_1, \ldots, y_M) \\
& \qquad = F_{X(t_1),\ldots,X(t_N)}(x_1, \ldots, x_N)\; F_{Y(t'_1),\ldots,Y(t'_M)}(y_1, \ldots, y_M) \\
& f_{X(t_1),\ldots,X(t_N),Y(t'_1),\ldots,Y(t'_M)}(x_1, \ldots, x_N, y_1, \ldots, y_M) \\
& \qquad = f_{X(t_1),\ldots,X(t_N)}(x_1, \ldots, x_N)\; f_{Y(t'_1),\ldots,Y(t'_M)}(y_1, \ldots, y_M)
\end{aligned}
$$

depending on the processes being discrete- or continuous-valued.

If $X(t)$ and $Y(t)$ are independent, then they are uncorrelated.

Example 5 Let $X(t) = \cos(\omega t + \theta_X + \Theta)$ and $Y(t) = \cos(\omega t + \theta_Y + \Theta)$ be stochastic processes where $\omega > 0$, θ_X, and θ_Y are constants, and Θ is Uniform$(0, 2\pi)$ random phase. We have

$$
\begin{aligned}
m_X(t) &= \mathrm{E}[X(t)] = \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t + \theta_X + \theta)d\theta = 0 \\
R_X(t_1, t_2) &= \mathrm{E}[X(t_1)X(t_2)] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t_1 + \theta_X + \theta)\cos(\omega t_2 + \theta_X + \theta)d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} (\cos(\omega(t_1 - t_2)) \\
&\qquad + \cos(\omega(t_1 + t_2) + 2\theta_X + 2\theta))\, d\theta \\
&= \frac{1}{2}\cos(\omega(t_1 - t_2)).
\end{aligned}
$$

Hence, $X(t)$ and, similarly, $Y(t)$ are WSS.

To check if they are jointly WSS, we consider

$$
\begin{aligned}
R_{XY}(t_1, t_2) &= \mathrm{E}[X(t_1)Y(t_2)] \\
&= \frac{1}{2\pi}\int_0^{2\pi} \cos(\omega t_1 + \theta_X + \theta)\cos(\omega t_2 + \theta_Y + \theta)d\theta \\
&= \frac{1}{4\pi}\int_0^{2\pi} (\cos(\omega(t_1 - t_2) + \theta_X - \theta_Y) \\
&\qquad + \cos(\omega(t_1 + t_2) + \theta_X + \theta_Y + 2\theta))\, d\theta \\
&= \frac{1}{2}\cos(\omega(t_1 - t_2) + \theta_X - \theta_Y).
\end{aligned}
$$

Since $R_{XY}(t_1, t_2)$ depends only on the difference $t_1 - t_2$ and both $X(t)$ and $Y(t)$ are WSS, they are jointly WSS and we write $R_{XY}(\tau) = \frac{1}{2}\cos(\omega\tau + \theta_X - \theta_Y)$.

The cross-covariance of $X(t)$ and $Y(t)$ is

$$K_{XY}(\tau) = R_{XY}(\tau) - m_X m_Y = \frac{1}{2}\cos(\omega\tau + \theta_X - \theta_Y).$$

Hence, they are correlated.

Notice that if $\theta_X - \theta_Y = \pi/2$, then $K_{XY}(0) = 0$ which implies that $X(t)$ and $Y(t)$, for a common t, are uncorrelated as random variables. However, this does not imply that the stochastic processes $X(t)$ and $Y(t)$ are uncorrelated. For $X(t)$ and $Y(t)$ to be uncorrelated, we should have $K_{XY}(\tau) = 0$ for all τ, which is not the case.

Examples of Random Processes

Gaussian Processes A stochastic process is Gaussian if all joint PDFs are Gaussian. In particular, for all N, all distinct $t_1, t_2, \ldots, t_N$, and all $x_1, x_2, \ldots, x_N$,

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{\sqrt{(2\pi)^n \det(\mathbf{K_X})}} e^{-\frac{1}{2}(\mathbf{x}-\mathbf{m_X})\mathbf{K_X}^{-1}(\mathbf{x}-\mathbf{m_X})^\mathsf{T}},$$

where

$$\begin{aligned}
\mathbf{x} &= (x_1, x_2, \ldots, x_N), \\
\mathbf{X} &= (X(t_1), X(t_2), \ldots, X(t_N)), \\
\mathbf{m_X} &= (m_X(t_1), m_X(t_2), \ldots, m_X(t_N)), \\
\mathbf{K_X} &= [K_X(t_k, t_l)]_{1 \le k,l \le N}.
\end{aligned}$$

Markov Processes A *Markov process*, $X(t)$, is a stochastic process that satisfies the Markov property, i.e., for all N, all $t_1 < \ldots < t_{N-1} < t_N$ in $\mathcal{T}$, and all $x_1, \ldots, x_{N-1}, x_N$,

$$p_{X(t_N)|X(t_1),\ldots,X(t_{N-1})}(x_N|x_1,\ldots,x_{N-1}) = p_{X(t_N)|X(t_{N-1})}(x_N|x_{N-1}),$$

if it is discrete-valued, and

$$F_{X(t_N)|X(t_1),\ldots,X(t_{N-1})}(x_N|x_1,\ldots,x_{N-1}) = F_{X(t_N)|X(t_{N-1})}(x_N|x_{N-1}),$$

$$f_{X(t_N)|X(t_1),\ldots,X(t_{N-1})}(x_N|x_1,\ldots,x_{N-1}) = f_{X(t_N)|X(t_{N-1})}(x_N|x_{N-1})$$

if it is continuous-valued.

In words, if t_{N-1} is identified with present time, then the Markov property says that $X(t)$ is a Markov process if the future, given the present, is independent of the past.

Independent Increment Processes An *independent increment process*, $X(t)$, is a stochastic process for which the random variables $X(t_2) - X(t_1), X(t_3) - X(t_2), \ldots, X(t_N) - X(t_{N-1})$, called *increments*, are independent for all N and all $t_1 < t_2 < \cdots < t_N$.

Stationary Increment Processes A *stationary increment process*, $X(t)$, is a stochastic process for which the random variables $X(t_2) - X(t_1)$ and $X(t_2 + h) - X(t_1 + h)$ have the same PMFs or the same CDFs for all $t_1 < t_2$ and h. This is the case if and only if the distribution of the increment $X(t_2) - X(t_1)$ depends only on the length of the time interval $t_2 - t_1$.