MAT115A HW3

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Exercise 1

Question: Find all positive integer m for which $-7 \equiv 6 \mod m$ is true.

Answer:

Find all $m \in \mathbb{Z}^+$ such that $m \mid -7-6$.
$m = 1, 13$.

Exercise 2

Question: Let $a, b \in \mathbb{Z}$ such that $a \equiv b \mod m$. Prove that $(a, m) = (b, m)$.

Answer:

Since $a \equiv b \mod m$, we have $m \mid a - b$, this implies $a - b = nm$ and $a - nm = b$ for $n \in \mathbb{Z}$.

$$
\begin{aligned}
&(a, m) \\
&= (a - nm, m), \text{ for } n \in \mathbb{Z} && \text{(The Euclidean Algorithm)} \\
&= (b, m) && \text{(Assumption)}
\end{aligned}
$$

Exercise 3

Question: Let $a \in \mathbb{Z}$. Show that a is odd if and only if $a^2 \equiv 1 \mod 8$.

Answer:

We will first show that a is odd implies $a^2 \equiv 1 \mod 8$. a is odd implies $a = 2m + 1$, for $m \in \mathbb{Z}$. Therefore,

$$a^2 = 4m^2 + 4m + 1$$

$$a^2 - 1 = 4m^2 + 4m = 4(m^2 + m)$$

If m is even $(m = 2n)$, then

$$a^2 - 1 = 8(2n^2 + n)$$

If m is odd $(m = 2n + 1)$, then

$$a^2 - 1 = 8(2n^2 + 3n + 1)$$

In both cases, $8 \mid a^2 - 1$, which means $a^2 \equiv 1 \mod 8$.

Now, we will show $a^2 \equiv 1 \mod 8$ implies a is odd.

$a^2 \equiv 1 \mod 8$ implies $a^2 - 1 = 8m$, for $m \in \mathbb{Z}$. Therefore, $a^2 = 2(4m) + 1$.

Since a^2 is odd, a must be odd.

Exercise 4

Find all incongruent solutions to the following congruences.

Exercise 4 (a)

Question: $12x \equiv 16 \mod 32$

Answer:

$$(12, 32) = 4$$

Since $16 = (4)(4)$, there are 4 incongruent solution.

$$4 + \left(\frac{32}{4}n\right), n = 0, 1, 2, 3$$

Incongruent Solution: $4, 12, 20, 28$

Exercise 4 (b)

Question: $481x \equiv 627 \mod 703$

Answer:

$$(481, 703) = 37$$

$$\text{Since } 37 \nmid 627, \text{ there is no solution.}$$

Exercise 5

Find an inverse modulo m for the integer n.

Exercise 5(a)

Question: $m = 17, n = 4$

Answer:

$$4x \equiv 1 \mod 17$$

$$(4, 17) = 1$$

$$13 + 17n, n = 0$$

$$4^{-1} \equiv 13 \mod 17$$

Exercise 5(b)

Question: $m = 35, n = 8$

Answer:

$$8x \equiv 1 \mod 35$$

$$(35, 8) = 1$$

$$1 \mid 1$$

$$22 + 35n, n = 0$$

$$8^{-1} \equiv 22 \mod 35$$