A Second Course on Quantum Field Theory: Path Integral

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Contents

1 Introduction

This is assumed to be the second part of the series, following my previous note *An Introduction to Quantum Field Theory and Feynman Rules* [1]. Feynman once said that a person fully masters a certain piece of knowledge only when he or she can derive it in several different ways. In this note, we are going to derive the Feynman rules for Feynman diagrams using path integral approach. We will again only focus on real scalars in this note, but leave the case of spinful particles to later notes, for the sake of simplicity.We adopt identical convention with our last note, e.g.$\hbar = c = 1$. Let us get started.

2 Deriving Path Integral for Free Field Theory

2.1 trivial math

We compute some Gaussian integrals[2], which we will need later, here:

$$I = \int_{-\infty}^{\infty} dp e^{-1/2ap^2+Jp} = \int_{-\infty}^{\infty} dp e^{-\frac{1}{2}a(p-\frac{J}{a})^2+\frac{J^2}{2a}} = \sqrt{\frac{2\pi}{a}} e^{\frac{J^2}{2a}} \tag{1}$$

The trick we used in the above equation is called *complete the square*, which can be easily generalized to multivariblae case. **A** is real symmetric, thus diagonalizable by an orthogonal matrix S :

$$\begin{aligned} A &= S^{\dagger}\Lambda S \qquad |det(S)| = 1 \\ I &= \int_{-\infty}^{\infty} d\vec{p} e^{-\frac{1}{2}p^{\dagger}Ap+J^{\dagger}p} \\ &= \int_{-\infty}^{\infty} d\vec{p} e^{-\frac{1}{2}p^{\dagger}\Lambda p+J^{\dagger}S^{\dagger}p} \\ &= \prod_i \int_{-\infty}^{\infty} dp_i e^{-\frac{1}{2}\lambda_i(p_i-\frac{(J^{\dagger}S^{\dagger})_i}{\lambda_i})^2+\frac{(J^{\dagger}S^{\dagger})_i^2}{2\lambda_i}} \\ &= \sqrt{\frac{2\pi}{\prod_i \lambda_i}} exp(\frac{1}{2}\sum_i \frac{(J^{\dagger}S^{\dagger})_i(SJ)_i}{\lambda_i}) \\ &= \sqrt{\frac{2\pi}{det(A)}} exp(\frac{1}{2}J^{\dagger}A^{-1}J) \end{aligned} \tag{2}$$

We have made use of the fact that S and J are both real.

2.2 path integral for quantum mechanics

Consider a system with Hamiltonian, where $[Q,P]=i$:

$$H(P,Q) = \frac{1}{2m}P^2 + V(Q) \tag{3}$$

Let $|q\rangle$ be the eigenstate of Q. In Heisenberg picture $Q(t) = e^{iHt}Qe^{-iHt}$, with instantaneous eigenstate[1] $|q,t\rangle = e^{iHt}|q\rangle$[2]. The quantity we are interested in is the transition amplitude:

$$\begin{aligned} \langle q'', t''|q', t'\rangle &= \langle q''|exp(iH(t''-t'))|q\rangle \\ = \int \prod_{j=1}^{N\to\infty} dq_j \langle q''|exp(-iH\delta t)|q_N\rangle &\langle q_N|exp(-iH\delta t)|q_{N-1}\rangle ... \langle q_1|exp(-iH\delta t)|q'\rangle \end{aligned} \tag{4}$$

We have made use of completeness relation $\int_{-\infty}^{\infty}|q\rangle\langle q| = 1$ and insert it at each infinitesimal time interval $\delta t = \frac{T}{N+1}$. Notice that because each piece of time is infinitesimal, the above equation applies even when the Hamiltonian is time-dependent.

Make use of the Campbell-Baker-Hausdorff formula for matrices A and B:

$$exp(A+B) = exp(A)exp(B)exp(-\frac{1}{2}[A,B]+...) \tag{5}$$

Thus, given H(P,Q)

$$exp(-iH\delta t) = exp(-i\frac{\delta t}{2m}P^2)exp(-i\delta t V(Q))exp(O(\delta t^2)) \tag{6}$$

$$\begin{aligned} \langle q_2|exp(-iH\delta t)|q_1\rangle &= \int dp_1 \langle q_2|exp(-i\frac{\delta t}{2m}P^2)|p_1\rangle \langle p_1|exp(-i\delta t V(Q))|q_1\rangle \\ &= \int \frac{dp_1}{2\pi} exp(-iH(p_1,q_1)\delta t)exp(ip_1(q_2-q_1)) \end{aligned} \tag{7}$$

[1] $Q(t)|q,t\rangle = q$

[2] The relationship between various representation was carefully discussed in Ref.[1].

The factor 2π comes from choosing $\langle q|p\rangle = \frac{1}{\sqrt{2\pi}} exp(ipq)$. Bring this back:

$$\begin{aligned}\langle q'', t''|q', t'\rangle &= \int \prod_{k=1}^{N} dq_k \prod_{j=0}^{N} \frac{dp_j}{2\pi} exp(ip_j(q_{j+1} - q_j)) exp(-iH(p_j, q_j)\delta t) \\ &= \int Dq\, Dp\, exp[i \int_{t'}^{t''} dt(p(t)\dot{q}(t) - H(p(t), q(t)))]\end{aligned} \tag{8}$$

$Dq\ Dp$ just says that out integration should be done over all paths with correct boundary condition. Now, it turns our that if H is of the form stated at the beginning of this subsection, then the integration on p can be done in closed form(Gaussian Integral), where M is some irrelevant normalization factor. **We will detemine normalization later in this note.**

$$\int dp_j\, exp(ip_j(q_{j+1} - q_j))\, exp(-i\delta t(\frac{p_j^2}{2m} + V(q_j))) = M exp(iL\delta t) \tag{9}$$

Thus

$$\langle q'', t''|q', t'\rangle = \int Dq\, exp(i \int_{t'}^{t''} Ldt) \tag{10}$$

We have appropriately absorbed M into Dq as needed. This is indeed a very nice formula, but the crucial question is what can we calculate with this formula? In general, observable is calculated using matrix element. For example, $Q(t_1)$ with $t' < t_1 < t''$.

$$\langle q'', t''|Q(t_1)|q', t'\rangle = \langle q''|exp(-iH(t'' - t_1))Q exp(-iH(t_1 - t'))|q'\rangle \tag{11}$$

Do exactly the same thing as we do before, i.e. insert complete set of eigenstate of Q. We will obtain:

$$\langle q'', t''|Q(t_1)|q', t'\rangle = \int DqDp\, q(t_1)\, exp(i \int_{t'}^{t''} dt(p\dot{q} - H)) \tag{12}$$

However, for $Q(t_1)$ and $Q(t_2)$, time ordering is important, i.e.

$$\langle q'', t''|T\{Q(t_1)Q(t_2)\}|q', t'\rangle = \int DqDp\, q(t_1)\, q(t_2)\, exp(i \int_{t'}^{t''} dt(p\dot{q} - H)) \tag{13}$$

T is the time ordering parameter which pulls the later time operator to the left. It arises because of the non-commutative $Q(t_1)$ and $Q(t_2)$. Actually we would love to calculate $\langle q'', t''|Q(t_1)Q(t_2)|q', t'\rangle$ for general t_1 and t_2, but it seems that there is no way to do so in path integral approach. Instead, notice that in LSZ formula, most of the things we need to calculate are time-ordered. So we are satisfied with what we can get from path integral.

We can further develop these ideas by defining the so-called functional derivative $\frac{\delta f(t_2)}{\delta f(t_1)} = \delta(t_1 - t_2)$. It will enable us to calculate the time-ordered correlation function more easily. Let us define the following thing by replacing $H \to H - f(t)q(t) - h(t)p(t)$ in equation 8:

$$\langle q'', t''|q', t'\rangle_{fh} \equiv \int Dq\, Dp\, exp[i \int_{t'}^{t''} dt(p(t)\dot{q}(t) - H(p(t), q(t)) + f(t)q + h(t)p)] \tag{14}$$

It is easy to observe that time-ordered product is expressible in terms of functional derivative, because each functional derivative will pull down a factor of p or q from the exponential into the integral, giving the formula for correlation function:

$$\langle q'', t''|T\{Q(t_1)...P(t_n)...\}|q', t'\rangle = \frac{1}{i} \frac{\delta}{\delta f(t_1)} ... \frac{1}{i} \frac{\delta}{\delta h(t_n)} \langle q'', t''|q', t'\rangle_{fh} |_{f=h=0} \tag{15}$$

Sometimes we are interested in ground state transition amplitude between initial time $t' \to -\infty$ and final time $t'' \to \infty$ instead of position eigenstate transition amplitude. For later convenience, we also include f and h term.

$$
\begin{aligned}
\langle 0, t''|0, t'\rangle_{fh} &= \int dq'' dq' \langle 0, t''|q'', t''\rangle \langle q'', t''|q', t'\rangle_{fh} \langle q', t'|0, t'\rangle \\
&\equiv \int dq'' dq' \psi_0(q'')^* \langle q'', t''|q', t'\rangle_{fh} \psi_0(q')
\end{aligned} \tag{16}
$$

At this point, we can surely bring the path integral formula for $\langle q'', t''|q', t'\rangle_{fh}$ we derived earlier. However, it would be very tedious to do so. Instead, we derive the following technique(energy is assumed to be bounded below, with $|0\rangle$ having lowest energy.):

$$
|q', t'\rangle = exp(iHt')|q'\rangle = \sum_{n=0}^{\infty} exp(iHt')|n\rangle\langle n|q'\rangle = \sum_{n=0}^{\infty}\sum_{n=0}^{\infty} exp(iE_n t')|n\rangle\langle n|q'\rangle \tag{17}
$$

The key observation[3] is that if we replace H by $(1-i\epsilon)H$, where ϵ some small positive constant that will eventually be taken to be zero at the end of everything, then the leading term would be $exp(iE_0 t')|0\rangle\langle 0|q'\rangle$, with all other terms infinitely suppressed compared with it. Thus, up to some unimportant normalization factor:

$$
\langle 0, t''|0, t'\rangle_{fh} = \langle q'', t''|q', t'\rangle_{fh}|_{H\to(1-i\epsilon)H} = \int DpDq exp(i\int_{-\infty}^{\infty} dt(p\dot{q} - (1-i\epsilon)H + fq + hp)) \tag{18}
$$

Notice that in equation 17, we didn't specify anything on the boundary condition. Thus, in the above equation, we do not need to be careful about the boundary condition of the action. Any reasonable boundary condition will do the job.

More often is the case that path integral can not be solved exactly for an interacting theory. Let us divide our Hamiltonian into two parts $H(p, q) = H_0(p, q) + H_1(p, q)$. We **hope** that the path integral containing H_0 is exactly solvable. Notice that the $|0, t''\rangle$ and $|0, t'\rangle$ appearing in the following equation should still be understood as ground states of the full Hamiltonian, not just of the H_0.

$$
\begin{aligned}
\langle 0, t''|0, t'\rangle_{fh} &= \int Dp\, Dq\, exp(i\int_{-\infty}^{\infty} dt(p\dot{q} - (1-i\epsilon)(H_0 + H_1) + fq + hp)) \\
&= \int Dp\, Dq\, exp(-i\int_{-\infty}^{\infty}(1-i\epsilon)H_1 dt)\, exp(i\int_{-\infty}^{\infty} dt(p\dot{q} - (1-i\epsilon)H_0 + fq + hp))
\end{aligned} \tag{19}
$$

We notice that H_1 is a function of p and q, which can be pulled down by functional derivative as introduced earlier. Thus replace $H_1(p, q)$ with $H_1(p \to \frac{1}{i}\frac{\delta}{\delta h(t)}, q \to \frac{1}{i}\frac{\delta}{\delta f(t)})$:

$$
\langle 0, t''|0, t'\rangle_{fh} = \int Dp\, Dq\, exp(-i\int_{-\infty}^{\infty}(1-i\epsilon)H_1(\frac{1}{i}\frac{\delta}{\delta h(t)}, \frac{1}{i}\frac{\delta}{\delta f(t)})dt)\, exp(i\int_{-\infty}^{\infty} dt(p\dot{q} - (1-i\epsilon)H_0 + fq + hp)) \tag{20}
$$

For notational simplicity, we will often drop the ϵ, but its presence should be kept in mind. if H_1 is only a function of q, but not of p, and if we are only interested time time-ordered products of q, i.e. $\langle 0, t''|T\{Q(t_1)...Q(t_n)\}|0, t'\rangle$, and if H(or H_0) is no more than quadratic in P with the term quadratic in P does not involve Q, then[3]

$$\langle 0,t''|0,t'\rangle_f = \int Dq\, exp(i\int_{-\infty}^{\infty} L_1(\frac{1}{i}\frac{\delta}{\delta f(t)})dt)\, exp(i\int_{-\infty}^{\infty} dt(L_0(\dot q,q)+fq)) \tag{21}$$

Where $L_0(\dot q,q) = p\dot q - H_0(p,q)$ and $L_1(q) = -H_1(q)$ per definition.

2.3 Path Integral for free quantum field theory

Free field theory turns to be exactly solvable, as in the case of canonical quantization discussed in [1]. Free theory's Lagrangian and Hamiltonian are:

$$\begin{aligned}\mathscr{L}_0 &= \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2\\ \mathscr{H}_0 &= \frac{1}{2}(\nabla\phi)^2 + \frac{1}{2}\Pi^2 + \frac{1}{2}m^2\phi^2\end{aligned} \tag{22}$$

The transition from quantum mechanics to quantum field theory is by doing the following replacement, where J is called sources, and is merely something used to calculate correlation function.:

$$q(t)\to\phi(x) \quad f(t)\to J(x) \tag{23}$$

We employed the $i\epsilon$ trick earlier to pick out the ground state. Notice that the entire essence is to give H a small negative imaginary part.[4] Thus we can just replace m^2 in the $\mathscr{H}_0$ by $m^2 - i\epsilon$. For notational simplicity, we will still use m^2 in the following, but what we actually mean is $m^2 - i\epsilon$, which we will recover when necessary.

$$Z_0[J] \equiv \langle 0,t''|0,t'\rangle_J = \int D\phi\, exp[i\int d^4x(\mathscr{L}_0 + J\phi)] \tag{24}$$

It turns out that $Z_0[J]$ can be evaluated by simply using Gaussian integral.

$$Z_0[J] = \int D\phi exp(i\int d^4x(-\frac{1}{2}\phi(\Box + m^2)\phi + J\phi)) \tag{25}$$

Recall the multi-variable Gaussian integration in equation 2, what we need here is just the inverse of the operator $(\Box + m^2)$. At a physicist level rigor, it is more or less important to ask what the precise meaning of inverse of $\Box$ is. A function $f(x)$ is mapped by $\Box$ to $g(x)$, i.e. $\Box f(x) = g(x)$. We want to have an operator L such that $Lg(x) = f(x)$. It would not hard to verify that the following Green's function will do the job:

$$\int dyG(x,y)g(y) = f(x) \tag{26}$$

provided the following relation is satisfied:

$$\Box_y G(x,y) = \delta(x-y) \tag{27}$$

Due to the symmetry of the problem, we can guess G to be of the form $G(x-y)$, which would further make the equation of G be:

$$\Box_x G(x-y) = \delta(x-y) \tag{28}$$

[3] Actually, all these assumptions are quite valid in the model we will meet.

[4] Surely when the Hamiltonian has negative energy, we can have positive imaginary part, but this is not important as long as it is bounded from below.

Let us go back to our original problem, it is obvious that the inverse $\Pi(x-y)$ we need is:

$$(\Box_x + m^2)\Pi(x-y) = -\delta(x-y) \tag{29}$$

This is lucky, we learned in the last note [1] that this equation is just solved by the Feynman propagator:

$$\Pi(x-y) = \int \frac{d^4p}{(2\pi)^4}\frac{1}{p^2-m^2+i\epsilon}exp(ip(x-y)) \tag{30}$$

Bring this result back in according to equation 2:

$$Z_0[J] = Mexp\{-i\int d^4xd^4y\frac{1}{2}J(x)\Pi(x-y)J(y)\} \tag{31}$$

M is some unimportant normalization factor, which we can set to 1.

2.4 a note on ground state

We said just above that M can be set to 1. The reason is subtler than it looks: the thing we have been calling $|0,t''\rangle$ and $|0,t'\rangle$ are actually the same thing, viz. $|\Omega\rangle$ as used in Ref.[1]. The vaccum is assumed to have **zero** energy, and thus do not evolve. Actually smart readers should already have sensed this fact : we claimed that we can use functional derivative to calculate correlation function $\langle 0,t''|T\{Q(t_1)...\}|0,t'\rangle$. But as derived in Ref.[1], the thing we are interested in is $\langle\Omega|T\{\phi(x_1)...\}|\Omega\rangle$. The only possibility is then $|0,t''\rangle = |0,t'\rangle = |\Omega\rangle$. This been clarified, it is then easy to see that since M is just $\langle 0,t''|0,t'\rangle$, then by normalization condition, it must be one.

From here on, we will just use $|\Omega\rangle$ for simplicity.

2.5 correlation function

We have established $M = \langle\Omega|\Omega\rangle = 1 = Z_0[0]$. And given equation 25, it is easy to observe that the correlation function is given by[5]

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)...\}|\Omega\rangle = \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}...Z_0[J]|_{J=0} \tag{32}$$

Let us test it by calculating a two-point function,using equation 31:

$$\begin{aligned}\langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\frac{1}{i}\frac{\delta}{\delta J(x_2)}Z_0[J] \\ &= \frac{1}{i}\frac{\delta}{\delta J(x_1)}\{-(\int d^4x'\Pi(x_2-x')J(x'))Z_0[J]\}|_{J=0} \\ &= \frac{-1}{i}\Pi(x_2-x_1)\end{aligned} \tag{33}$$

Where in the last line we make use of the fact that everything should eventually be evaluated at $J=0$. Four point function is more or less the same and can be obtained by straight forward calculation:

$$\begin{aligned}&\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle = (\frac{-1}{i})^2[\Pi(x_1-x_2)\Pi(x_3-x_4)+\Pi(x_1-x_3)\Pi(x_2-x_4)+\Pi(x_1-x_4)\Pi(x_2-x_3) \\ &= [D_F(x_1-x_2)D_F(x_3-x_4)+D_F(x_1-x_3)D_F(x_2-x_4)+D_F(x_1-x_4)D_F(x_2-x_3)]\end{aligned} \tag{34}$$

[5] though it is assumed that we are discussing free theory, I still use $|\Omega\rangle$.

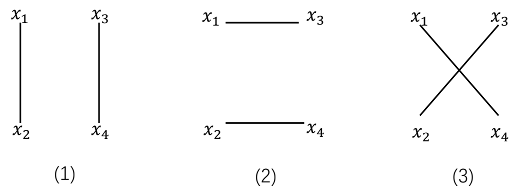

Figure 1: All pairings contributing to 4-point correlation functions

Where we have used the fact that Feynman propagator $D_F = i\Pi$. In retrospect, if we look at equation 33 more closely, we will observe that

$$\langle\Omega|\phi(x_2)|\Omega\rangle = -(\int d^4x'\Pi(x_2 - x')J(x'))Z_0[J]|_{J=0} = 0 \tag{35}$$

This is a general feature. Correlation function of odd number of fields is always 0, because there is always uncanceled J factor, and we always evaluate it at $J = 0$. On the other hand, we can give a diagrammatic representation of the four-point correlation function $\langle\Omega|T\{\phi(x_1)\phi(x_2)\phi(x_3)\phi(x_4)\}|\Omega\rangle$, as in Fig.1. We find that it actually covers all possible pairing of these four points x_1 to x_4. This is also a general feature for correlation function of even number of fields, known as *Wick's theorem*, where $(i_1, i_2...i_{2n})$ is a permutation of $(1, 2, ...2n)$.

$$\langle\Omega|T\{\phi(x_1)\phi(x_2)...\phi(x_{2n-1})\phi(x_{2n})\}|\Omega\rangle = \sum_{all\ permutation} D_F(x_{i_1} - x_{i_2})...D_F(x_{i_{2n-1}} - x_{i_{2n}}) \tag{36}$$

3 Quantum Field Theory For Interacting Fields

3.1 two loop holes

The world is more complicated than free field theory, and is rarely exactly solvable. Let us start by revisiting our old friend,ϕ^3 theory:

$$L = \frac{-1}{2}\phi(\Box + m^2)\phi + \frac{g}{3!}\phi^3 \tag{37}$$

A loop hole for this Lagrangian is that the corresponding Hamiltonian contains a $-\frac{g}{3!}\phi^3$ term, which can be made arbitrary negative for sufficiently large ϕ. It implies that the Hamiltonian is not bounded from below, and probably will ruin all of our previous efforts. We will overlook this problem, because perturbation theory does not know it since $\phi = 0$ is still a local minimum.

Another loop hole is the following. Recall the equation we used earlier:

$$Z_0[J] \equiv \langle 0, t''|0, t'\rangle_J = \int D\phi\, exp[i\int d^4x(\mathscr{L}_0 + J\phi)] \tag{38}$$

The second equality follows from this fact:The part not involving J is exactly integrable, which will produce the normalization factor M times something containing a factor of J. We argued that M should be set to 1, as ground state should remain ground state,i.e. $Z_0[0] = 1$. However, here we have $\mathscr{L}_0 + \mathscr{L}_{int}$, which are not integrable. Thus, we do not have the M for us to set to 1. **We still**

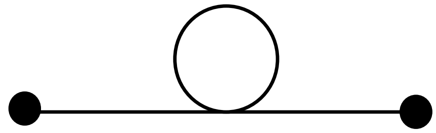

Figure 2: The only diagram of E=1,P=2,V=1.

want the normalization that $Z[0]$, which is just $\langle\Omega|\Omega\rangle$[6], to be 1. For this reason,we write $\propto$ instead of $=$ in the following to remind ourselves of this fact. Once we fix the normalization, we will recover $=$:

$$Z[J] \equiv \langle\Omega|\Omega\rangle_J \propto \int D\phi\, exp[i\int d^4x(\mathscr{L}_0 + \mathscr{L}_{int} + J\phi)] \tag{39}$$

3.2 our old friend

Assume that L_{int} is only a function of ϕ, we can mimic what we did back in section 2.2.

$$\int D\phi\, exp[i\int d^4x(\mathscr{L}_0 + \mathscr{L}_{int} + J\phi)] = exp(i\int d^4x\mathscr{L}_{int}(\frac{1}{i}\frac{\delta}{\delta J(x)}))Z_0[J] \tag{40}$$

Thus, by making Taylor expansion:

$$Z[J] \propto \sum_{V=0}^{\infty}\frac{1}{V!}[\frac{ig}{6}\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(x)})^3]^V \times \sum_{P=0}^{\infty}\frac{1}{P!}[-\frac{i}{2}\int d^4yd^4zJ(y)\Pi(y-z)J(z)]^P \tag{41}$$

If we focus on a term in the above equation for particular values of V and P, then the number of surviving sources after we take all the function derivative is $E = 2P - 3V \geq 0$. There are $A_{2P}^{3V} = \frac{2P!}{(2P-3V)!}$ ways for these 3V vertices to act on 2P sources. However, many of these terms are algebraically identical. This is where we introduce *Feynman diagram* to organize everything. For each E and V, we can plot some connected diagrams,i.e. diagrams in which every point can be reached from another point. It may not be immediately clear what each symbol in these diagrams represent. So let us expand a particular example $E = 1, P = 2, V = 1$, whose only possible diagram is presented in Fig.2.

By directly expanding the $E = V = 1$ term in equation 41

$$\begin{aligned}\frac{1}{1!}[\frac{ig}{6}\int d^4x(\frac{1}{i}\frac{\delta}{\delta J(x)})^3] \times \sum_{P=0}^{\infty}\frac{1}{2!}[-\frac{i}{2}\int d^4yd^4zJ(y)\Pi(y-z)J(z)]^2 \\ = \frac{ig}{2}\int d^4xd^4w[D_{xx}D_{wx}iJ_w]\end{aligned} \tag{42}$$

Where we have adopted the abbreviation $D_{xy} = D_F(x-y)$ and $J_x = J(x)$. Compare this with the figure, it would not be a surprise if we claim that (apart from some tricky numerical factor) each line stand for Feynman propagator D_{xy}, with xy being the vertices of two ends. A filled circles

[6] $|\Omega\rangle$ is the ground state of the full interacting theory

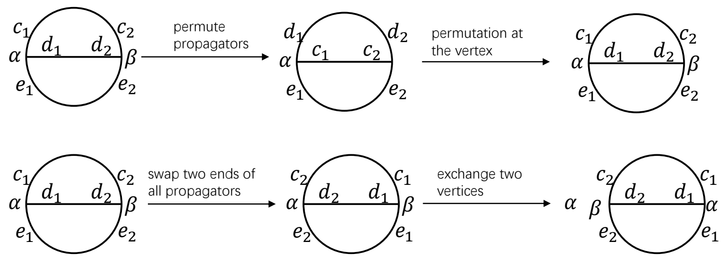

Figure 3: This illustrate two ways of overcounting. The final symmetry factor is 12.

at one end of the Feynman propagator stand for a source $i\int d^4xJ_x$, where x is the vertex. Each vertex (not the filled circle) joins three line, and stands for $ig\int d^4x$, with x the vertex. But again, how would we explain the additional factor of $\frac{1}{2}$? Readers familiar with our last note [1] should know that this is just the so-called symmetry factor. Let us now take a different perspective from the angle of functional derivative. We ask the question: how many ways are there to produce the same diagram?

For a term with fixed V and P in equation 41. The effect of $(\frac{1}{i}\frac{\delta}{\delta J(x)})^3$ is joining three propagator's end together by acting on three J terms. Thus, it does not make a difference to permute these three derivatives, which will yield a factor of 3^V(3 for each vertex). We can also make a permutation among all these V vertices, which yield a factor of V!. Also, for $J(x)\Pi(x-y)J(y)$, it does not make a difference whether a functional derivative is acting on J(x) or J(y), corresponding to permute two ends of the propagator. This yields a factor of 2^P. We can also permute the propagator themselves, which will yield a factor of $P!$. Overall, there are *seemingly* $3^V V!2^P P!$ to produce the same diagram, which neatly cancels the numerical factor in equation 41.

But it still does not explain the factor of $\frac{1}{2}$ we had. In fact, it results from overcounting. Some of the operations we mentioned above will yield identical diagrams,i.e. if operation 1 is followed by operation 2, we are back to the starting point. Some examples are illustrated in the Fig. 3 and Fig.4. In these two figures, each propagator is labeled by a Latin letter, with subscript indicating which end of the propagator it is. Each vertex/source is labeled by a Greek letter. In Fig.3, there are 3! ways to permute these 3 propagator, yielding a factor of 6 of overcounting. And swapping two ends of all propagators will yield a factor of 2 in overcounting. Thus, the overall symmetry factor is 12. For Fig.4, each of the process shown will yield a factor of 2 in overcounting, thus resulting in a symmetry factor of 4.

From my personal perspective, this explanation is much better than the hand-waving explanation we gave for the symmetry factor back in Ref.[1].

3.3 connected diagram and normalization

Let me clear: a connected diagram is a diagram in which every point can be reached from every other point by traveling on the propagators. It is obvious that if we directly expand equation 41, there will be many terms corresponding to unconnected diagram, each of which is made of connected sub-diagrams. For example, that in Fig.5 is unconnected. A general diagram D can be calculated as[3]:

$$D = \frac{1}{S_D}\prod_I (C_I)^{n_I} \tag{43}$$

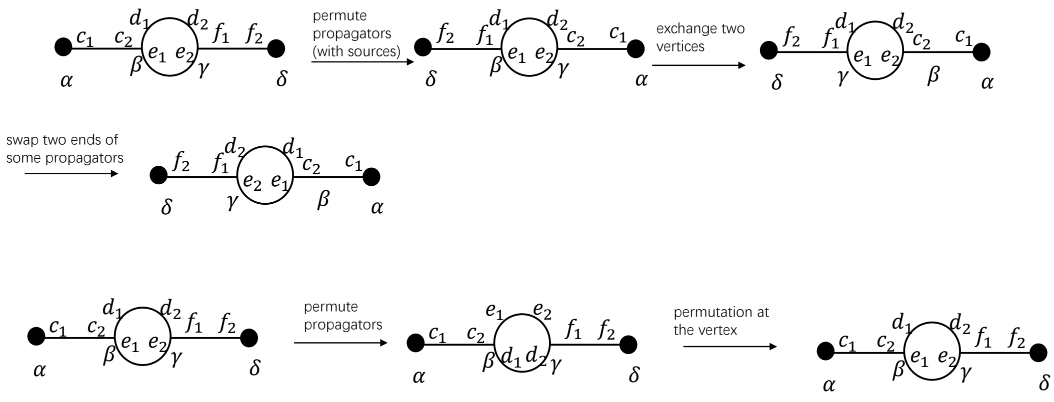

Figure 4: This illustrate two ways of overcounting. The final symmetry factor is 4.

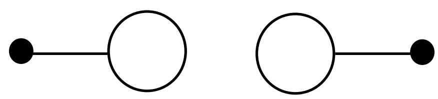

Figure 5: An example of unconnected diagram with E=2, V=2, P=1.

Where the product is over all constituting connected sub-diagrams, with symmetry factor already included in C_I. n_I is the number of sub-diagrams of kind I in the total diagram. S_D is the additional symmetry factor that is not included in individual sub-diagrams. Consider two identical sub-diagrams C_I. operation 1 is to exchange all vertices in these two diagrams. operation 2 is to exchange all propagators in these two diagrams. After these two operations, voila, we are back to the starting point. Generally, there are n_I diagrams to permute, and thus $S_D = n_I!$. Overall, $Z[J]$ is given by summation over all possible diagrams D, each of which is labeled by a set of integers n_I.

$$\begin{aligned} Z[J] &\propto \sum_{\{n_I\}} D \\ &= \sum_{\{n_I\}} \prod_I \frac{1}{n_I!}(C_I)^{n_I} \\ &= \prod_I \sum_{n_I=0}^{\infty} \frac{1}{n_I!}(C_I)^{n_I} \\ &= exp(\sum_I C_I) \end{aligned} \tag{44}$$

This result is remarkable, since Z[J] is given by only connected diagrams. A major benefit of this result is, our nightmare, the normalization of Z[J] is particularly easy to fix right now.

$$Z[J] \propto exp(\sum_{I\in\{E\geq 1\}} C_I)exp(\sum_{I\in\{E=0\}} C_I) \tag{45}$$

$E \geq 1$ refers to those connected diagrams with at least one external sources, while $E = 0$ refers to those without sources, which are referred to as vacuum diagrams. The key point is that when $J = 0$, $E \geq 1$ diagrams are automatically 0, since external sources always give a factor of $\int d^4xJ(x)$. Normalization of $Z[0] = 1$ tell us that we should simply omit the vacuum diagrams. Thus

$$Z[J] = exp(\sum_{I\in\{E\geq 1\}} C_I) \tag{46}$$

3.4 some examples

It will be absolutely justifiable if we stop our note in the previous subsection, since all the principle things have already been covered. However, we still want to use this formalism to calculated something as a test for our formalism by comparing the result in canonical quantization approach, as done in Ref.[1]. For notational simplicity, we define:

$$iW[J] \equiv \sum_{I\in\{E\geq 1\}} C_I \qquad \delta_j \equiv \frac{1}{i}\frac{\delta}{\delta J(x_j)} \tag{47}$$

$$\begin{aligned} \langle\Omega|T\{\phi(x_1)\phi(x_2)\}|\Omega\rangle &= \delta_1\delta_2 Z[J]|_{J=0} \\ = \delta_1\delta_2 iW[J]|_{J=0} &+ (\delta_1 iW[J])|_{J=0}(\delta_2 iW[J])|_{J=0} \end{aligned} \tag{48}$$

An important observation is that δ_j removes one source, and label that point as x_j. Thus $\delta_1\delta_2 iW[J]|_{J=0}$ is given by those diagrams with two sources, while $(\delta_1 iW[J])|_{J=0}$ is given by those diagrams with one source. Because if we do not have the right number of source, either the functional derivative or the remaining J will kill the term's contribution. Up to order g^2, the terms involved for $\delta_1\delta_2 iW[J]|_{J=0}$ are $E = 2, V = 0$ and $E = 2, V = 2$. Up to order g^2, the terms involved for $(\delta_1 iW[J])|_{J=0}$ are $E = 1, V = 1$. They are presented at the left column in Fig.6.

After removing the source, and follow the rules we introduced in section 3.2. We would obtain:

$$\begin{aligned} &\delta_1\delta_2 iW[J]|_{J=0} = D_{x_1x_2} + \frac{(ig)^2}{2}\int d^4wd^4zD_{x_1w}D_{x_2z}(D_{wz})^2 + \frac{(ig)^2}{2}\int d^4zd^4wD_{x_1w}D_{x_2z}D_{wz}^2 + O(g^3) \\ &(\delta_1 iW[J])|_{J=0} = \frac{ig}{2}\int d^4wD_{x_1w}D_{ww} \end{aligned} \tag{49}$$

When we add these things together, we neatly obtain what we had back in Ref.[1]. Some final remarks are in place. When we are calculating the n-point correlation function $\langle\Omega|T\{\phi(x_1)\phi(x_2)...\phi(x_n)\}|\Omega\rangle$. Essentially we are just eliminating the n external sources from a diagram, and label the resulting point as x_i, $i = 1...n$. There are n! ways to do this. However, some of the results are algebraically identical. For example in Fig.6, it does not matter which point we are labeling as x_1 or x_2. Because all the internal vertices are integrated over, so it does not matter how we label them. Simple relabeling of the internal vertices shows that: for the diagrams of $\delta_1\delta_2 iW[J]|_{J=0}$ in Fig.6, which have two branches in labeling x_1 and x_2 , the two branches are identical . This will produce a factor of 2, partially canceling the symmetry factor of the original diagram.

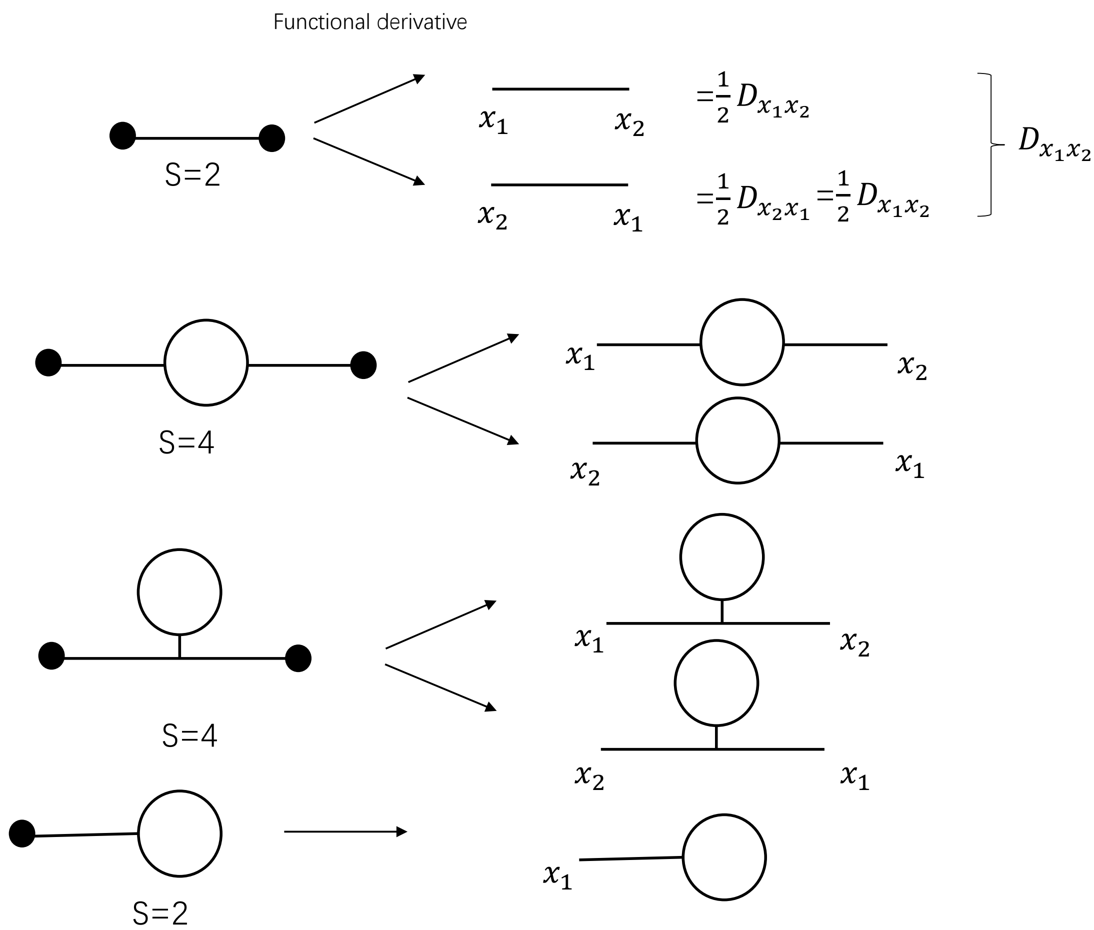

Figure 6: A figurative illustration of the temrs in equation 49

3.5 old question revisited

It may have come across the mind of acute readers that a problem we discussed earlier in Ref.[1], section *Why Leave Him Out?*(ϕ^4). Why do we leave the diagram there out and only consider those diagrams for which every point is at least connected to one external points (the name we use for the relic of the eliminated sources)?

The answer is clear here: for the *generating function* Z[J], we only consider the contribution from connected diagrams. When we calculate any n-point correlation function, any contributing term (a term may correspond to many diagrams though) is a product of $L_i \equiv \delta_1\delta_2...\delta_i iW(J)$. Each L_i must have at least one external points after we eliminated the sources and evaluate at J=0. Thus, we never produce the diagram mentioned in *Why Leave Him Out?*(ϕ^4) of Ref. [1].

References

[1] Tixuan Tan. An introduction to quantum field theory and feynman rules. https://ttx002000.github.io/homepage/files/first_course_QFT.pdf.

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