An Introduction to Quantum Field Theory and Feynman Rules

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1 Introduction

A central theme in studying particle physics is calculating matrix element M_{fi}, which is governed by a set of Feynman rules. In many standard textbooks,e.g.Griffiths[1], Feynman rules are introduced without explanation. Quantum field theory(QFT) offers a rigorous derivation of these rules. In this article, we aim low.**We will focus exclusively on real scalar fields. The logic flow of this project is**: Section 2 and section 3 focus on the basic language of QFT/particle physics and the connection between them. Section 4,5,6.1 are essential tools in deriving Feynman rules. The rest of section 6 is devoted to deriving Feynman rules. Section 7 is for fixing some tiny subtlety we skipped for continuity of logic flow. It is assumed that readers have a moderate knowledge in quantum mechanics. c=$\hbar$=1 in this article[1].

2 Lagrangian Approach to Second Quantization

2.1 Lagrangian and Equation of Motion

In classical mechanics, we typically deal with Lagrangian expressed in terms of generalized coordinates. In field theory, we deal with Lagrangian (density) expressed in terms of field variable ϕ. For a generic Lagrangian (density) $\mathscr{L}[\phi, \partial_\mu\phi, \partial_\mu\partial_\nu\phi, ...]$. By employing $\delta\int d^4x\mathscr{L} = 0$,Euler-Lagrangian equation is:

$$\frac{\partial\mathscr{L}}{\partial\phi} - \partial_\mu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\phi)} + \partial_\mu\partial_\nu\frac{\partial\mathscr{L}}{\partial(\partial_\mu\partial_\nu\phi)} + \cdots = 0 \tag{1}$$

As mentioned in Griffiths, getting the correct Lagrangian of field theory in the past is almost a guess work, viz. knowing equation of motion beforehand, physicist reverse engineering the Lagrangian that would produce the equation of motion[1]. For a scalar field, start from the $E^2 = \vec{p}^2 + m^2$, and apply[2] the usual quantization procedure $p_\mu \to i\partial_\mu$,as we did in deriving *Schrödinger* equation. We expect **free** scalar field ϕ to satisfy Klein-Gordon equation $(\Box^2 + m^2)\phi$=0.

We can reverse engineering the free Lagrangian $\mathscr{L}_0 = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2$. It is trivial to check Euler-Langrangian equation of $\mathscr{L}_0$ will yield K-G equation. we define conjugate momentum as $\pi \equiv \frac{\partial\mathscr{L}}{\partial(\partial_t\phi)} = \partial_t\phi$. Doing Legendre transformation will yield:

$$H_0 = \int\mathscr{H}_0 d^3x = \int d^3x(\pi\partial_t\phi - \mathscr{L}_0) = \int d^3x\frac{1}{2}(\pi^2 + (\nabla\phi)^2) + \frac{1}{2}m^2\phi^2 \tag{2}$$

[1]so we can use p and k interchangeably

[2]When we write $\vec{p}$ or $\vec{x}$,we mean three-vector,while p and x typically mean four vector.

2.2 Why Do We Need Free Field Theory?

If the real world is described by an interacting theory, why are we interested in the free theory? Answer: we normally do not calculate the full interacting theory analytically, but only perturbatively. **As will be clear after proving Schwinger-Dyson equation, the basis of doing perturbation theory, is the solution to the free field KG equation, no matter how complicated the interaction of full Lagrangian may be.** This is just an analogy to fine-structure of hydrogen atom. We need an exactly solvable system upon which we can do perturbation.

2.3 What to Quantize?

Physical meaning of KG equation is explained in 2.5. For a real scalar function $\phi_0(\vec{x},t)$ satisfying KG equation, the general solution[2] is below. We omitted derivation, *but one may easily verify this is indeed the solution*, because of $\Box\phi_0 = -p^2\phi_0$ and the mass-momentum relation $p^2 = m^2$. The complex conjugate in the solution is just to ensure the reality.

$$\begin{aligned}&(\Box^2 + m^2)\phi_0 = 0\\ \Rightarrow&\phi_0(\vec{x},t) = \int d^3p \frac{a(\vec{p})exp(-ip_\mu x^\mu)) + a^*(\vec{p})exp(ip_\mu x^\mu)}{(2\pi)^3\sqrt{2\omega_p}}\\ &\pi_0(\vec{x},t) = \partial_t\phi_0\end{aligned} \tag{3}$$

where $p^\mu = (\omega_p, \vec{p})$, coefficients are chosen out of convention. In Non-Relativistic Quantum Mechanics, $\hat{x}$ and $\hat{p}$ are promoted to non-commutitive operators. One might have guessed from the Euler-Lagrangian equation that ϕ is the QFT version of x, and correspondingly π should be the QFT version of p. **We thus promote π_0 and ϕ_0 to operators satisfying:**(canonical quantization)

$$\begin{aligned}[\phi_0(\vec{x},t), \phi_0(\vec{y},t)] = [\pi_0(\vec{x},t), \pi_0(\vec{y},t)] = 0\\ [\phi_0(\vec{x},t), \pi_0(\vec{y},t))] = i\delta^3(\vec{x}-\vec{y})\end{aligned} \tag{4}$$

To achieve this goal, we promote $a(\vec{p})$ and $a^*(\vec{p})$ to operators a_p and $a_p^\dagger$, **with no time dependence**. It took some straight forward algebric[3][2] work to show that equation (4) is equivalent to

$$\begin{aligned}&[a_k, a_p^\dagger] = (2\pi)^3\delta^3(\vec{p}-\vec{k})\\ &[a_k, a_p] = [a_k^\dagger, a_p^\dagger] = 0\\ \Rightarrow\ &\phi_0(\vec{x},t) = \int d^3p \frac{a_p exp(-ip_\mu x^\mu)) + a_p^\dagger exp(ip_\mu x^\mu)}{(2\pi)^3\sqrt{2\omega_p}}\end{aligned} \tag{5}$$

Note that above equations hold for **free fields** ϕ_0. In an interacting theory, one should solve ϕ first classically,and then get π via definition. Then one should promote both of them to non-commuting operators, and get the corresponding relation among creation/annihilation operators. However,this is rarely realistic. But as we will discuss in section 7, equation (4)/(5) and $\pi = \partial_t\phi$ will also hold for interacting theory with $a/a^\dagger$ **gaining complicated time dependence,viz. $a_p \to a_p(t)$, as determined by Euler-Langrangian equation.**

2.4 Hamiltonian and Interpretation of Operators

Using the expression of ϕ_0 and π_0 , free Hamiltonian[4] has the form $H_0 = \int d^3p(2\pi)^{-3}\omega_p(a_p^\dagger a_p + \frac{1}{2}) \approx \int d^3p(2\pi)^{-3}\omega_p(a_p^\dagger a_p)$. We neglect the term that integrates to give some constant, which is *ground state energy*. Some subtlety of this constant energy is disscussed in [2].

What is the physical meaning of $a_p^\dagger$ and a_p? As their names imply, they have nice physical interpretations.

Define

$$|\vec{p_1}...\vec{p_n}\rangle \equiv (2\omega_{p_1}...2\omega_{p_n})^{\frac{1}{2}} a_{p_1}^\dagger...a_{p_n}^\dagger |0\rangle$$

[3]Then:

$$\begin{aligned}
&[(a_p^\dagger a_p), a_{p_1}^\dagger...a_{p_n}^\dagger] = a_p^\dagger[a_p, a_{p_1}^\dagger]a_{p_2}^\dagger...a_{p_n}^\dagger + ... + a_p^\dagger a_{p_1}^\dagger...a_{p_{n-1}}^\dagger[a_p, a_{p_n}^\dagger] \\
&= a_{p_1}^\dagger...a_{p_n}^\dagger(2\pi)^3(\sum_{k=1}^{n}\delta^3(p-p_k)) \\
\Rightarrow &H_0 |\vec{p_1}\vec{p_2}...\vec{p_n}\rangle = (\omega_1 + ... + \omega_n) |\vec{p_1}\vec{p_2}...\vec{p_n}\rangle
\end{aligned} \tag{6}$$

Also note that

$$a_p^\dagger a_p a_{p_1} a_{p_1}^\dagger = a_p^\dagger a_{p_1} a_{p_1}^\dagger a_p + a_p^\dagger a_{p_1}(2\pi)^3\delta^3(p-p_1) \Rightarrow H_0 a_{p_1} a_{p_1}^\dagger |0\rangle = 0 \tag{7}$$

These two equations suggest we should interpret $a_p^\dagger/a_p$ as an operator that creates/annihilates a particle with energy ω_p.

2.5 What is Field?

There is a very common question a starter in QFT may ask: **what is the physical meaning of field? Specifically what is ϕ_0?** Classically, field is a function of spacetime,it assigns a number to everypoint in spacetime. In QFT, ϕ_0 is an operator-valued function, which assigns an operator to every point in spacetime.

Since we can interpret $a_p^\dagger$ as creating a free particle with momentum $\vec{p}$. **It is expected** that state $|p\rangle = \sqrt{2\omega_p}a_p^\dagger |0\rangle$ will have a plane wave form in coordinate space. i.e. $\langle\vec{x}|\vec{p}\rangle \overset{unproven}{\propto} exp(i\vec{p}\cdot\vec{x})$(t=0). Indeed,we observe the following:

$$\begin{aligned}
&\langle\vec{p}| \phi_0(\vec{x}, t=0) |0\rangle = \langle 0| \sqrt{2\omega_p}a_p \int \frac{d^3k}{(2\pi)^3\sqrt{2\omega_k}}(a_k^\dagger exp(-i\vec{k}\vec{x})) |0\rangle \\
&= \int \frac{d^3k}{(2\pi)^3}\sqrt{\frac{\omega_p}{\omega_k}} exp(-i\vec{k}\vec{x}) \langle 0| a_p a_k^\dagger |0\rangle \\
&= exp(-i\vec{p}\cdot\vec{x})
\end{aligned} \tag{8}$$

So, $\langle 0|\phi_0(\vec{x}, t=0)|\vec{p}\rangle = exp(i\vec{p}\cdot\vec{x})$, then $\phi_0(\vec{x}, t=0) |0\rangle$ **is the QFT version of** $|\vec{x}\rangle$**.**We can thus interpret $\phi_0(\vec{x})$ as creating a particle at position $\vec{x}$. This is hardly surprising,since canonical quantization already gave some hint on this interpretation.

Notice that KG equation is an equation of motion of fields, while fields are just operators. So KG equation is an equation of motion of operators. This is the default setting in QFT:**Heisenberg picture instead of Schrodinger picture is used in QFT, in which operators carry time dependence.**[4] We can regard equation of motion in QFT as the equation descibing how the influence at one point travels through the spacetime. Specifically, KG equation descrbies *free operator wave* in spacetime as it does in Electrodynamics.

[3]$|0\rangle$ is the vacuum state,state of no particles, it is defined to be $a|0\rangle = 0$,similar to the case in harmonic oscillator,where the energy has to be bounded from below.

[4]KG equation is very simple so creation/annihilation operators are time-independent,but it is not the case for a general Euler-Langrangian euqation.

2.6 Normalization of States

It is better we fix the normalization before it causes any problem when we get into something more subtle. Different from what one might naively expect,$\langle\vec{p}|\vec{p}\rangle$, in our convention, is generally different from 1.

$$\langle\vec{p}|\vec{p}\rangle = 2\omega_p \langle 0| a_p a_p^\dagger |0\rangle = 2\omega_p(2\pi)^3\delta^3(0) \tag{9}$$

How should we deal with this annoying delta function? Readers should have already encountered it once when dealing with free particle in Non-Relativistic Quantum Mechanics. Recall there we fix the problem by using a factor of $\frac{1}{\sqrt{V}}$,and take V to be infinity at the end. The solution is similar here. Recall from Fourier transformation that

$$\begin{aligned} \delta^3(p) &= \frac{1}{(2\pi)^3}\int d^3x exp(i\vec{p}\vec{x}) \\ \Rightarrow \delta^3(0) &= \frac{1}{(2\pi)^3}\int d^3x = \frac{V}{(2\pi)^3} \end{aligned} \tag{10}$$

V will be taken to be infinity in the end. But for intermidiate steps, the normalization of $|p\rangle$ is notationally: $\langle\vec{p}|\vec{p}\rangle = 2EV$, where V is the volume of experiment. By exactly the same logic, we can take $\delta^4(0)=\frac{TV}{(2\pi)^4}$, where T is the duration of experiments. It is very important to observe that when we calculate matrix elements later, all V and T will cancel out, so there is no problem to take T/V to infinity in the end.

3 Cross Sections and Decay Rates

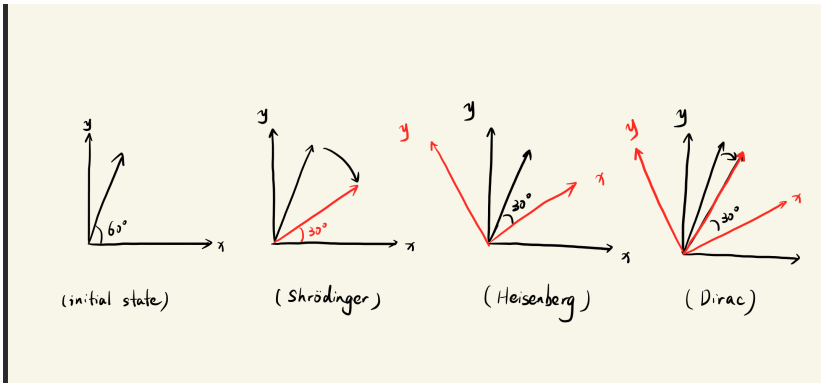
What observables can we calculate using second quantization? There are plenty of them, cross sections, decay rates, etc. They are measured in scattering experiments, whose data can be used to test the preditions from QFT.

3.1 Be Careful With S-Matrix!

It seems that Schwartz[4] used a confusing notation for S matrix[5][2]. Thus the author decides to give a detailed dicusssion. **The conclusion of this section is given in equation (11).**

QFT deals with Heisenberg picture by default. Rigorous discussion on three pictures of quantum mechanics is omitted since readers are assumed to be familiar with those[6]. **Instead, we give an heuristic analogy. Refer to the diagram below,** imagine we have a two dimensional plane,and a vector that points 60 degrees from the x axis. There are three ways we can make this vector point 30 degrees from the x axis, corresponding to *Schrödinger*, Heisenberg, Dirac picture,respectively. In Schrodinger picture,state vector carries the time dependence ,while in Heisenberg picture, axis(operators) carries time dependence. In Dirac picture, time dependence is split between operators and state vectors.

A profound result immediately follows from this cartoonish drawing: Measurement results do not depend upon picture. Just assume the state vector is of length 1. If we measure the x coordinate of the red/black state vector according to black/red x axis in Schrödinger/Heisenberg picture, both are $\frac{\sqrt{3}}{2}$! However, pushing this analogy too far entails confusion, since strictly speaking operator is a tensor, not vector.



The quantity we are interested in, in a scattering experiment, is : Assume the system is prepared in a certain state A($t = -\infty$), and it goes through some interaction. We measure the system again afterwards($t = \infty$). It is found to be in state B. **what is the probability for this to happen?** This question can be formulated more rigorously as follow:

The system's full Hamiltonian is $H = H_0 + V$,with H_0 the free part. Let H be time-independent. For H we can define two sets of basis **in Heisenberg picture**, the *in* states $|i; -\infty\rangle$,[5] and the out states $|f; \infty\rangle$. Practically i and f will contain quantum number such as spin and 4-momentum of particles.

It is a subtle point worth stressing: Though states in Heisenberg picture do not carry time-dependence, operators do. *Out/in states* are created by operators at $t = \pm\infty$. $|p_1; \infty\rangle$ is entirely different from $|p_1; -\infty\rangle$

Question:**How is transition amplitude calculated?** It is simply given by inner product between states. Let us again refer to the above diagram. Assume after the experiment, it is found that the final state is (1,0). In Schrödinger picture,(1,0) refers to a state lying on the **black x axis**,forming a 30° angle with the rotated(red) state vector. In Heisenberg picture, (1,0) refers to a state on the **red x axis**, forming a 30° angle from the black state vector. In either case, the inner product is the same. **By basic quantum mechanics,** inner product in Schrodinger picture is transition amplitude. Thus inner product in Heisenberg basis also gives transition amplitude.

These words can be summarized succinctly as follow:

$$\langle f(t = t_1)|exp(-iH(t_1 - t_2))|i(t = t_2)\rangle_{Schrödinger} = \langle f; t_1|i; t_2\rangle_{Heisenberg}$$

Let us also define a basis for H_0 at an arbitrary time,e.g. $t_0 = 0$. The basis of H_0 are $|i\rangle$ / $|f\rangle$. $|f\rangle$ is assumed to have the same 4-momentum, particle species etc., with $|f; -\infty\rangle$. We **define** an operator S in this basis:

$$\langle f|S|i\rangle \equiv \langle f; \infty|i; -\infty\rangle \tag{11}$$

S records the transition probability, so our task is reduced to calculating components of S. **How is it calculated?** It is just inner product of states in Heisenberg picture as discussed. *In* states should be created by creation operators at t=-∞, and *out* states are created by operators at t=∞. $|\Omega\rangle$ is ground state of interacting theory,which is generally different from $|0\rangle$.

$$\langle f|S|i\rangle \equiv \langle f; +\infty|i; -\infty\rangle = \sqrt{2^n\omega_1...\omega_n}\,\langle\Omega|\,a_{p_3}(\infty)...a_{p_n}(\infty)a^\dagger_{p_1}(-\infty)a^\dagger_{p_2}(-\infty)\,|\Omega\rangle$$

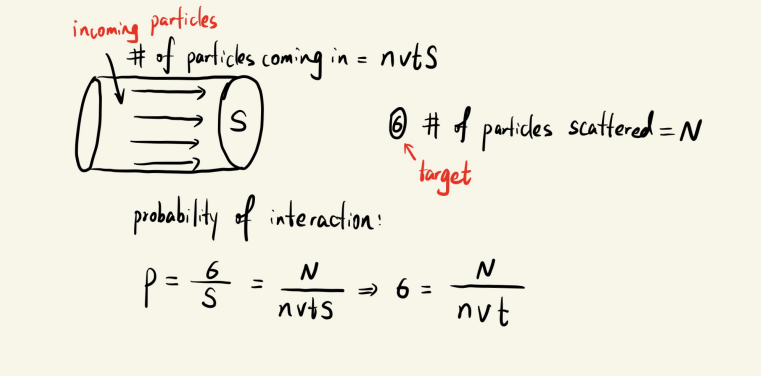
A starter may ask: **Why introduce a new basis for H_0?** Why not use the in states or out states? The answer is that $|i; -\infty\rangle$ is generally not orthogonal to $|f; \infty\rangle$, even if f and i are two sets of different quantum numbers. However, $|f\rangle$ and $|i\rangle$ are orthogonal to each other by definition,

[5]corresponding to two sets of coordinate system

and we always would like to work with an orthogonal basis. Besides, this is the convention most widely used in the literature[5][7]. **Another question is**:Why introduce S operator? It seems redundant because we can calculate the inner product without it. It turns out that S operator is more fundamental than its components, it is time-evlution operator in Dirac picture, a topic we will not pursue here.[6]

3.2 Cross Sections

Cross section is defined as $\sigma \equiv \frac{N}{T\Phi}$, intuitively it measures the probility an interaction can happen. N is total number of particle scattered(different from incoming particles), T is time of experiment(tend to infinity), Φ is number density of beam $\times$ (relative) velocity of incoming beam. $\Phi = nv_r$.



Introduce differential probability $dP \equiv \frac{dN}{N}$, where dN means the number of particles scattered into certain area(solid angle)/energy range. We have:

$$d\sigma = \frac{dN}{T\Phi} = \frac{NdP}{T\Phi}$$

Notice that $\frac{\Phi}{N} = \frac{v_r}{V}$, v_r is relative velocity between colliding particles, V is the volume of experiment(tend to infinity).

dP,which is a probability,is something we are familiar in Quantum Mechanics, compared with cross sections. It **should be proportional to two factors,** one is the phase space volumn around the final state,d^3p, since the number of states in the range[6] is proportional to this volumn(see below). The other factor is the transition probability $\langle f|\, S\, |i\rangle$. Upon normalization:

$$\begin{aligned} dP &= \frac{|\langle f|\, S\, |i\rangle|^2}{\langle f;+\infty|f;+\infty\rangle \langle i,-\infty|i,-\infty\rangle} d\Pi \\ d\Pi &= \prod_{j=final} \frac{V}{(2\pi)^3} d^3p_j \end{aligned} \tag{12}$$

In (12), **we can explain the factor before d^3p as following**: recall in Quantum Mechanics, if we have a box of volume V,in which we confine the free particle, then each allowed momentum will occupy a momentum-space volume of $\frac{(2\pi)^3}{V}$. So the number of allowed momentum in d^3p is

[6]for example, $d\sigma$ corresponds to particles scattered into a small solid angle range, this range contains a number of final state momentum(not just one momentum, since one momentum corresponds to zero solid angle range!)

$d\Pi$. The factor before $d\Pi$ is transition probability to a single allowed momentum, thus the product gives dP.

Still,nothing is said about the S operator. In free theory, there is no transition, so S is identity matrix. In an interacting theory, we can define T/M to be the deviation of S from identity, which is the non-trivial part of the interacting theory. T is defined by S, **M is defined from T by factoring out a delta function.**[7] Notice that typically the final state and initial state are different, so the trivial part $\langle f|i\rangle$ vanishes.

$$\begin{aligned} iT &\equiv S - I \\ T \equiv (2\pi)^4\delta^4(\Sigma p_i^\mu - \Sigma p_f^\mu)M &\equiv (2\pi)^4\delta^4(\Sigma p)M \end{aligned} \tag{13}$$

Combining (10),(12) and (13), we arrive at the final form of differential cross section:

$$\begin{aligned} d\sigma &= \frac{1}{(2E_1)(2E_2)|\vec{v_r}|}|M|^2 d\Pi_{LIPS} \\ d\Pi_{LIPS} &\equiv \prod_{j=final} d^3p_j \frac{(2\pi)^4\delta^4(\Sigma p)}{(2\pi)^3 2E_{p_j}} \end{aligned} \tag{14}$$

$d\Pi_{LIPS}$ is called the Lorentz-invariant phase space. We will not verify it is Lorentz-invariant, since it is an irrelevant detail for our purpose. The procedure is outlined in Schwartz Chapter2 exercises[4]. **Take-away message: Cross section is determined by $M_{fi} = \langle f|M|i\rangle$. which is something to be calculated by QFT**

3.3 Decay Rates

A differential decay rate dΓ is the probability, in the rest frame of particle, that particle decays into many-particle state, normalized by the duration of experiment.

$$d\Gamma \equiv \frac{dP}{T}$$

This decay rate $\Gamma = \int d\Gamma$ determines the half life of particles we observe[1].Note that Γ for different decay modes are additive, i.e. $\Gamma_{total} = \sum_i \Gamma_i$

As a seperate note, by using $dP = \frac{dN}{N}$, we observe that ΓNdt is the number of particles decayed in time dt. This is the reason why Γ is named decay rates.

Bring in the expression of dP, we arrive at the following formula:

$$d\Gamma = \frac{|M_{fi}|^2 d\Pi_{LIPS}}{2E_1} \tag{15}$$

This formula is in the rest frame of particle. To calculate decay rate in other frames, we need to include a time-dilation factor.

4 LSZ Reduction Formula

4.1 What is LSZ Formula and How is it Important?

Is there a way to calculate the S-matrix by using second quantization formulation, so that we can make some theoretical predictions on the cross sections? Yes, it is done by LSZ-reduction,Feynman propagator, Schwinger-Dyson equation combined.

[7]This convention has its roots in cluster decomposition, which we will not pursue here.

In this section, we prove the LSZ reduction formula,**as written below.** Notice $\pm i$ is in the exponent of final/initial states. Usually there are at most 2 particles in the initial state,and many more particles in the final state. $|\Omega\rangle$ is the ground state or vacuum state of an interacting theory. It is generally different from the free theory vacuum state $|0\rangle$. However, it is also annihilated[2, 4] by the annihilation operator at t=$\pm\infty$ and normalized:$\langle\Omega|\Omega\rangle = 1$. p_i^μ is the initial/final state four-momentum. T is **time-ordered product**,which will be explained below. It is worth stressing that $\phi(x_i)$ is interacting fields instead of free fields, otherwise $\Box + m^2$ acting on them will just give zero.

$$\begin{aligned}\langle f|\, S\,|i\rangle \overset{unproven}{=\!=} [i\int d^4x_1 exp(-ip_1^\mu x_{1\mu})(\Box + m^2)]...[i\int d^4x_n exp(ip_n^\mu x_{n\mu})(\Box + m^2)] \\ \times \langle\Omega|\, T\{\phi(x_1)...\phi(x_n)\}\,|\Omega\rangle\end{aligned} \tag{16}$$

The importance of LSZ formula can be observed as following: on the LHS is S operator, from which cross sections and decay rates can be calculated(see last section). On the RHS is time-ordered product of quantum fields, for which a whole set of tools in QFT is developed to calculate it perturbatively. In fancy words, LSZ equation **is the bridge** between experiment and QFT.

4.2 Proof of LSZ

Since the field is interacting, creation/annihilation operators will have time dependence. But the commutation relation of them is preserved at every instant of time. See section 2.3.

$$\phi(x) = \phi(\vec{x}, t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2\omega_p}}[a_p(t)exp(-ip_\mu x^\mu) + a_p^\dagger(t)exp(ip_\mu x^\mu)] \tag{17}$$

4.2.1 Step 1

we first prove the following equation. **Assume** all interactions happen within some finite time interval T(though eventually we will take T to be very large), **so the particles are free at t=$\pm\infty$**

$$i\int d^4x exp(ip^\mu x_\mu)(\Box + m^2)\phi(x) \overset{unproven}{=\!=} \sqrt{2\omega_p}[a_p(\infty) - a_p(-\infty)] \tag{18}$$

Notice that:

$$\begin{aligned} i\int d^4x exp(ip^\mu x_\mu)(\Box + m^2)\phi(x) =& i\int d^4x exp(ip^\mu x_\mu)(\partial_t^2 - \nabla^2 + m^2)\phi(x) \\ =& i\int d^4x exp(ip^\mu x_\mu)(\partial_t^2 + \omega_p^2)\phi(x) \\ =& \int d^4x \partial_t[exp(ip_\mu x^\mu)(i\partial_t + \omega_p)\phi(x)] \quad (verify \quad by \quad direct \quad expansion)\end{aligned} \tag{19}$$

We have used integration by parts(assuming the field to die off at spatial infinity[8]) and energy-momentum relation in (19). By the last line of (19), it is only dependent on fields at t=$\pm\infty$, when $a_p(t)$ and $a_p^\dagger(t)$ are nearly time-independent, since the theory is free then. **For the particular**

[8]See Srednicki[2] chapter 5 for explanation. Schwartz impliciltly postulated a wave packet,which may lead to some confusion

form of field we had in (17),bring it into (19),then we will have(with some algebra)

$$\begin{aligned}
&\int d^3x exp(ip_\mu x^\mu)(i\partial_t + \omega_p)\phi(x) \\
&= exp(i\omega_p t)\int \frac{d^3k}{(2\pi)^3}\int d^3x[\frac{\omega_k+\omega_p}{\sqrt{2\omega_k}}a_k(t)exp(-ik_\mu x^\mu)exp(-i\vec{p}\cdot\vec{x}) + \frac{-\omega_k+\omega_p}{\sqrt{2\omega_k}}a_k^\dagger(t)exp(ik_\mu x^\mu)exp(-i\vec{p}\cdot\vec{x})] \\
&= \sqrt{2\omega_p}a_p(t) \qquad (t=\pm\infty)
\end{aligned} \tag{20}$$

Bring this back to (19),we have

$$i\int d^4x exp(ip_\mu x^\mu)(\Box + m^2)\phi(x) = \sqrt{2\omega_p}[a_p(\infty) - a_p(-\infty)] \tag{21}$$

Take Hermitian conjugate(notice that ϕ is Hermitian)

$$-i\int d^4x exp(-ip_\mu x^\mu)(\Box + m^2)\phi(x) = \sqrt{2\omega_p}[a_p^\dagger(\infty) - a_p^\dagger(-\infty)] \tag{22}$$

This finishes step 1 of the proof.

4.2.2 Step 2 and Voila!

By definition:

$$\begin{aligned}
\langle f|S|i\rangle \equiv \langle f;+\infty|i;-\infty\rangle &= \sqrt{2^n\omega_1...\omega_n}\,\langle\Omega|\,a_{p_3}(\infty)...a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)\,|\Omega\rangle \\
&= \sqrt{2^n\omega_1...\omega_n}\,\langle\Omega|\,T\{a_{p_3}(\infty)...a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)\}\,|\Omega\rangle
\end{aligned} \tag{23}$$

Going from the first line to the second line, we have **defined time-ordering operator T{...}**, which pulls later time operator to the left of earlier time operator. Notice that operators at $t=\infty$ are commutative, so ordering among them is not important. This time-ordering operator enables us to write:

$$\begin{aligned}
&\langle\Omega|\,T\{a_{p_3}(\infty)...a_{p_n}(\infty)a_{p_1}^\dagger(-\infty)a_{p_2}^\dagger(-\infty)\}\,|\Omega\rangle \\
&= \langle\Omega|\,T\{[a_{p_3}(\infty) - a_{p_3}(-\infty)]...[a_{p_n}(\infty) - a_{p_n}(-\infty)][a_{p_1}^\dagger(-\infty) - a_{p_1}^\dagger(\infty)][a_{p_2}^\dagger(-\infty) - a_{p_2}^\dagger(\infty)]\}\,|\Omega\rangle
\end{aligned} \tag{24}$$

Notice that all the *additional* operators in the second line of (24) will either be pulled to the left side or right side, depending upon their time argument, where they will act upon the ground state $|\Omega\rangle/\langle\Omega|$ to give zero. Bring in the results of step 1, we have the desired LSZ formula, as written at the beginning of this section.

5 Feynman Propagator

LSZ formula relates S matrix to time-ordered product. But there is still some difficulty. It is the time-ordered product of *interacting fields*, whose dynamics we cannot solve. As will be clear in the next section, S-D equation enables us to do perturbation expansion of interacting fields **using Feynman propagator, which is something concerning free theory.** This makes it possible to calculate transition probability to any precision we want. One analogy the author finds suitable

is that Feynman propagator is the building blocks and S-D equation is the blueprint of Feynman diagrams.

Using the expression of free field,we can calculate $\langle\phi(x_1)\phi(x_2)\rangle$.

$$\phi_0(\vec{x},t)=\int d^3p\frac{a_p exp(-ip_\mu x^\mu))+a_p^\dagger exp(ip_\mu x^\mu)}{(2\pi)^3\sqrt{2\omega_p}} \tag{25}$$

$$\begin{aligned}&T\{\phi_0(x_1)\phi_0(x_2)\}=\phi_0(x_1)\phi_0(x_2)\theta(t_1-t_2)+\phi_0(x_2)\phi_0(x_1)\theta(t_2-t_1)\\ \overset{(25)}{\Rightarrow}\ &\langle 0|\,T\{\phi_0(x_1)\phi_0(x_2)\}\,|0\rangle=\int\frac{d^3p}{(2\pi)^3(2\omega_p)}exp(-i\vec{p}\cdot(\vec{x_1}-\vec{x_2}))[exp(i\omega_p\tau)\theta(-\tau)+exp(-i\omega_p\tau)\theta(\tau)]\end{aligned} \tag{26}$$

Where $\tau\equiv t_1-t_2$. It is a standard application of Residue Theorem to show that

$$exp(i\omega_p\tau)\theta(-\tau)+exp(-i\omega_p\tau)\theta(\tau)=\lim_{\epsilon\to0^+}\frac{-2\omega_p}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{\omega^2-\omega_p^2+i\epsilon}exp(i\omega\tau) \tag{27}$$

Putting everything together,and let the limit of ϵ be implicit, this is the **Feynman propagator**

$$\begin{aligned}D_F(x_1,x_2)&\equiv\langle 0|\,T\{\phi_0(x_1)\phi_0(x_2)\}\,|0\rangle\\&=\int\frac{d^4p}{(2\pi)^4}\frac{i}{p^2-m^2+i\epsilon}exp(ip_\mu(x_1^\mu-x_2^\mu))\end{aligned} \tag{28}$$

It should be pointed out that four components of p^μ are not on shell, viz $p^2\neq m^2$, they are **four independent integration variable.** Starters in QFT are often confused by the statement **virtual particle is not on shell.** In a sense, there is no physical meaning of this sentence,it is only a mathematical result.

6 Feynman rules

All cards on the table, we are ready to derive Feynman rules. **The goal is clear: To derive a formula of transition probability for interacting fields.** We know how to calculate Feynman propagator, which is something concerning free theory. How does it help in interacting theory? Recall from the first course in QM, where pertuabtion theory allows us to solve unknown system from a known system. Schwinger-Dyson Equation enables us to do similar things QFT,at least in the range where pertubation theory is valid[9].

6.1 Schwinger-Dyson Equation(S-D equation)

Since we are dealing with time-ordered product, we need to be extremely careful when dealing with time-derivative on the RHS of LSZ. Specifiacally speaking:(the absence of subscript 0 stresses they are not free fields)

$$\begin{aligned}\partial_t\langle\Omega|\,T\{\phi(x)\phi(x')\}\,|\Omega\rangle&=\langle\Omega|\,T\{(\partial_t\phi(x))\phi(x')\}\,|\Omega\rangle+\delta(t-t')\langle\Omega|\,[\phi(x),\phi(x')]\,|\Omega\rangle\\ \Rightarrow\partial_t^2\langle\Omega|\,T\{\phi(x)\phi(x')\}\,|\Omega\rangle&=\langle\Omega|\,T\{(\partial_t^2\phi(x))\phi(x')\}\,|\Omega\rangle-i\delta^4(x-x')\end{aligned} \tag{29}$$

[9]QCD is notoriously non-perturbative. In fact,all theories are only perturbative within certain energy range

Notice that we have employed canonical quantization $[\phi(\vec{x},t),\partial_t\phi(\vec{x'},t)] = i\delta^3(\vec{x}-\vec{x'})$ for **interacting fields.** A justification of this in interacting theory is discussed at the end of this article.[10] **Question:**Why not set the third term on the first line of (29) to be zero,since fields at equal time commute by canonical quantization? Answer: if we set it to 0, since delta function forces t=t', we cannot take time derivative derivative of t. Because when we take time derivative of t, **t' is taken to be fixed**,not the same as t.

it follows that (time-ordered product will only bring subtlety when taking time-derivative, but spatial derivative is free to move around) :

$$(\Box + m^2)\langle\Omega|T\{\phi(x)\phi(x')\}|\Omega\rangle = \langle\Omega|T\{(\Box+m^2)\phi(x)\phi(x')\}|\Omega\rangle - i\delta^4(x-x') \tag{30}$$

In a **free** theory, in which $(\Box^2+m^2)\phi_0(x)=0$, equation (30) implies:

$$(\Box_{x_1}+m^2)D_F(x_1,x_2) = -i\delta^4(x_1-x_2)$$

A natural generalization to multiple fields is the following,where we **define** the notation $<...>\equiv\langle\Omega|T\{...\}|\Omega\rangle$

$$\Box_{x_1}<\phi(x_1)...\phi(x_n)>=<\Box_{x_1}\phi(x_1)...\phi(x_n)> -i\sum_{j=2}^{n}\delta^4(x_1-x_j)<\phi(x_2)...\phi(x_{j-1})\phi(x_{j+1})...\phi(x_n)> \tag{31}$$

Remember that a generic interacting Lagrangian is $\mathscr{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) - \frac{1}{2}m^2\phi^2 + \mathscr{L}_{int}[\phi]$,by which the equation of motion is $(\Box+m^2)\phi - \partial_\phi\mathscr{L}_{int}=0$.Bring this into equation (31), it will yield **Schwinger-Dyson Equation (32)**[4]. I know that there are doubts about what would happen when $\mathscr{L}_{int}$ depends upon derivative. Put it aside, and I promise to come back to it later.

$$\begin{aligned}(\Box_{x_1}+m^2)\langle\phi(x_1)...\phi(x_n)\rangle &= \langle\partial_{\phi_1}\mathscr{L}_{int}[\phi_1]...\phi(x_n)\rangle \\ &\quad - i\sum_{j=2}^{n}\delta^4(x_1-x_j)\langle\phi(x_2)...\phi(x_{j-1})\phi(x_{j+1})...\phi(x_n)\rangle\end{aligned} \tag{32}$$

What does S-D equation do? Observe that there are two terms on the RHS, one involving more fields compared with the left, and one involving fewer. **It seems that we are not improving anything, but are making it worse.** Originally we only need to calculate one time-ordered product, but now we need to calculate two! However, observe that there is a term $\partial_{\phi_1}\mathscr{L}_{int}$ on the RHS, this term will introduce a coupling constant g, which describes the strength of interaction. Typically this coupling constant g is small enough for us to do the perturbation expansion. So as we introduce more and more fields in the first term on RHS, this term becomes less and less important due to high order in g. At some point,we can truncate the process. **As we will find out later, by employing S-D equation iteratively, in the end everything is reduced to Feynman propagator.**

6.2 Position-Space Feynman rules

6.2.1 A Simple Toy Model

For concreteness[11],let us work with Lagrangian $\mathscr{L} = \frac{1}{2}(\partial^\mu\phi)(\partial_\mu\phi) + \frac{g}{3!}\phi^3$. For convenience,we introduce **notation** $\delta_{xi} = \delta^4(x-x_i)$,and $D_{ij} = D_{ji} = D_F(x_i,x_j)$. **We set m=0,since non-zero case is a trivial generalization.** Feynman propagator satisfies$(\Box_x)D_{x1} = -i\delta_{x1}$. The

[10] a different justification can also be found in Schwartz,section 7.1

[11] Real scalr field is not purely artificial, Higgs boson is an example

recipe presented here can easily be generalized to more complicated Lagrangian. **Start with the simplest time-ordered product of interacting fields:**

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int d^4x(\Box_x D_{x1})\,\langle\phi_x\phi_2\rangle \\ &= i\int d^4x(D_{x1})\Box_x\,\langle\phi_x\phi_2\rangle \\ &= i\int d^4x(D_{x1})(\frac{g}{2}\,\langle\phi_x^2\phi_2\rangle - i\delta_{x2}) \\ &= D_{12} - \frac{g}{2}\int d^4xd^4yD_{x1}D_{y2}\Box_y\,\langle\phi_x^2\phi_y\rangle \\ &= D_{12} - \frac{g^2}{4}\int d^4xd^4yD_{x1}D_{2y}\,\langle\phi_x^2\phi_y^2\rangle + ig\int d^4xD_{1x}D_{2x}\,\langle\phi_x\rangle\end{aligned} \tag{33}$$

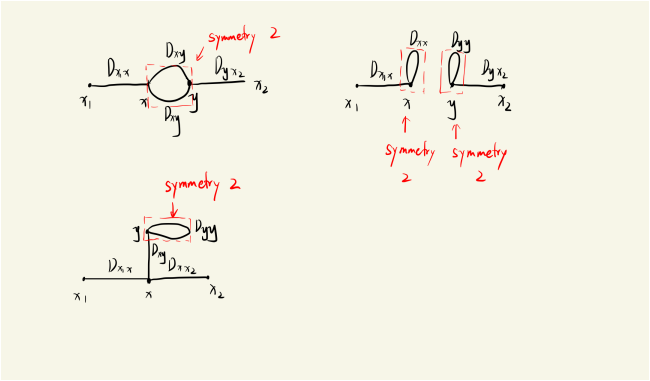
We have used integration by parts and S-D equation. If g is small, e.g. in QED, we are only interested up to certain order of it. Making similar use of S-D equation, one can show that:

$$\begin{aligned}\langle\phi_x^2\phi_y^2\rangle &= 2D_{xy}^2 + D_{xx}D_{yy} + \mathcal{O}(g) \\ \langle\phi_x\rangle &= i\frac{g}{2}\int d^4yD_{xy}D_{yy} + \mathcal{O}(g^2)\end{aligned} \tag{34}$$

Thus,**up to second order in g**,we have

$$\langle\phi_1\phi_2\rangle = D_{12} - g^2\int d^4xd^4y(\frac{1}{2}D_{1x}D_{xy}^2D_{y2} + \frac{1}{4}D_{1x}D_{xx}D_{yy}D_{y2} + \frac{1}{2}D_{1x}D_{2x}D_{xy}D_{yy}) \tag{35}$$

if the theory is free,g=0,we would only have D_{12}. The three new terms in (35) can be diagrammatically represented as below.



Question:Why is there no first order term? We can argue either because the calculation doesn't yield one. Or we can argue that because there is no such diagram with only one internal three-line vertex and two external lines.

The crucial similarity of these **second order diagrams** is that,apart from two external vertices x_1 and x_2,they have **two additional vertices** x and y, each of which has three lines attached(a bubble counts as two). The rules,known as Feynman rules, which connect a diagram to a number are summarized as below[4], assuming we are calculating $\langle\phi_1...\phi_n\rangle$ in ϕ^3 theory(we can check its correctness against our simple example above):

1.Draw n points,label them as x_1 to x_n,from each point, draw a line.

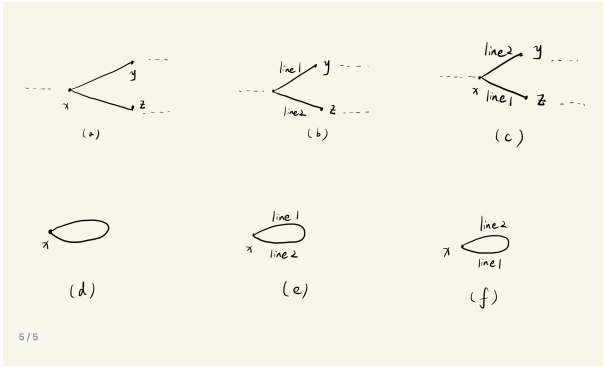
2.A line has two choices: **either** connect with an existing line,which produce a straight line between x_i and x_j **or** split. A split produces a vertex,and from this vertex comes out two new lines. Repeat the procedure for all existing lines.Label every new vertex by y_i **Remeber:**You never connect a line to a vertex,but you connect a line to a line. Usually, these two are the same thing, but connecting a line to a line makes it easier to understand why some vertices connect back to themselves(bubble).

3.Each vertex contribute ig.Each line contributes D_{xy},where x and y are the end points of the line.

4.Integrate all internal vertices y_i,$\int d^4y_1 d^4y_2 ...$

5.Divide each graph by its symmetry factor.

6.2.2 What Symmetry Factor?



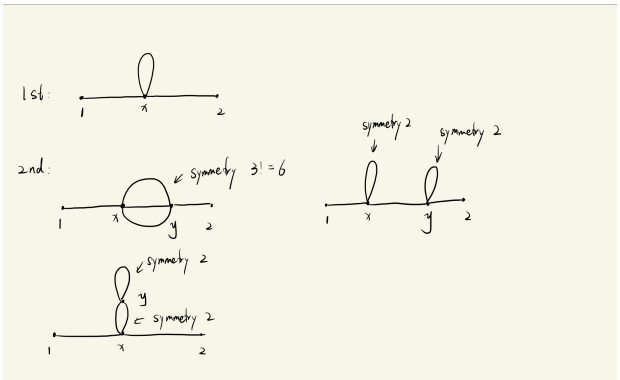
All five rules seem plausible. But some of them need more subtle explanations. It is easy to follow rule (1) and rule (2) to draw a diagram, but we have to ask:**What does this diagram represent?**Generally, interaction is of the form $\mathscr{L}_{int} = \frac{g}{n!}\phi^n$.Let n=3. Then for each **internal** vertex, one line comes in,(3-1)=2 lines come out.We can permutate the lines coming out of a vertex. Diagram (a) is the sum of diagram (b) and (c), we can use the **the sum of** diagram (b) and (c), in which case the left-most vertex would contribute a factor of $i\frac{g}{(3-1)!}$. Since you sum over (b) and (c),the factor $\frac{1}{(3-1)!}$ is canceled. However,one can also use a single diagram (a), in which case we do not distinguish among lines and the left most diagram would contribute ig. These two approaches would yield the same answer

The crucial point is that if we have a bubble, then permutating two lines going out of the vertex will give exactly the same diagram, so we **cannot** sum over diagram (e) and (f), only one of them can be taken into consideration. Then if we treat diagram (d) **as if** it is a summation over (e) and (f), in which case we would assign the vertex a number ig, we have to divide by a symmetry factor of 2 at the end to avoid double counting. The situtaion is the same if we have m lines connecting two vertices, this will yield a symmetry factor of m!.

6.2.3 ϕ^4 model

I must admit that the ϕ^3 model was worked out by Schwartz[12]. However, we can verify the Feynman rules for other types of interaction to make sure that everything is truly general. Take $\mathscr{L}_{int} = \frac{g\phi^4}{4!}$. In last section, we start by direct expansion and assign each term a diagram. **This time we reverse the order:** we start with diagram, and verify that direct expansion agrees with diagram. First we draw the diagram with at most two vertices.

[12]In chapter 7[4]



The good thing about diagram is that it is more intuitive and can be used to check the result of direct expansion: By following S-D equation, one finds:

$$\begin{aligned}\langle\phi_1\phi_2\rangle &= i\int d^4x D_{x1}\Box_x\langle\phi_x\phi_2\rangle \\ &= D_{12} + i\frac{g}{3!}\int d^4x D_{x1}\langle\phi_x^3\phi_y\rangle \\ &= D_{12} - \frac{g}{3!}\int d^4x d^4y D_{x1}D_{y2}\Box_y\langle\phi_x^3\phi_y\rangle \\ &= D_{12} - (\frac{g}{3!})^2\int d^4x d^4y D_{x1}D_{y2}\langle\phi_y^3\phi_x^3\rangle + \frac{g}{3!}(3i)\int d^4x D_{x1}D_{x2}\langle\phi_x^2\rangle\end{aligned} \tag{36}$$

One can proceed similarly, to show that up to second order in g,we have

$$\langle\phi_y^3\phi_x^3\rangle = 9D_{yy}D_{xy}D_{xx} + 6D_{xy}^3 + \mathcal{O}(g) \tag{37}$$

$$\int d^4x D_{x1}D_{x2}\langle\phi_x^2\rangle = \int d^4x d^4y D_{x1}D_{x2}D_{xx} + \frac{ig}{3!}3\int d^4x d^4y D_{x1}D_{x2}D_{xy}^2D_{yy} + \mathcal{O}(g^2) \tag{38}$$

Bring (37)(38) back to (36) we have first order contribution

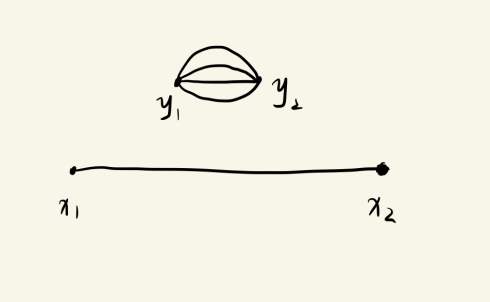
$$\langle\phi_1\phi_2\rangle_{1^{st}} = \frac{g}{3!}(3i)\int d^4x d^4y D_{x1}D_{x2}D_{xx} \tag{39}$$

Second order contribution

$$\begin{aligned}\langle\phi_1\phi_2\rangle_{2^{nd}} = &-\int d^4x d^4y\frac{g^2}{4}D_{x1}D_{y2}D_{yy}D_{xy}D_{xx} \\ &-\int d^4x d^4y\frac{g^2}{6}D_{xy}^3D_{x1}D_{y2} - \int d^4x d^4y\frac{g^2}{4}D_{yy}D_{xy}^2D_{x1}D_{x2}\end{aligned} \tag{40}$$

Three second order terms correspond to three two-vertice graphs. As promised,these three graphs has a symmtry factor of 3!(permutating three internal lines), 2^2(two bubbles), 2^2 (one of 2 come from the bubble,the other come from permutating two internal lines.) Everything agrees!

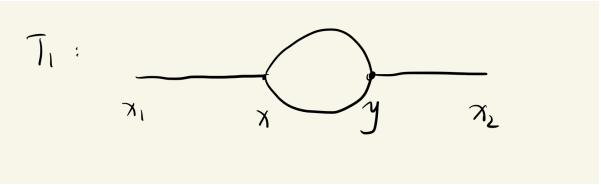
6.2.4 Why Leave Him Out?(ϕ^4)



At the first sight, this seems to be a valid second order diagram. Why did we not include this diagram in the last section? Answer: strictly following the Feynman rules as described, we cannot produce this diagram. Recall that in the above-summarized Feynman rules, an internal vertex does not pop up from vacuum. It is a product of split from an existing line. As a result, we should only include those **connected** diagrams,viz. every internal vertex should be connected to at least one external vertex(directly or indirectly). A useful discussion on proving how these **disconnected** diagrams may be cancelled out can be found in [6, 4, 5].

6.3 Momentum Space Feynman Rules

To derive momentum space Feynman rules, we only need to replace Feynman propagator by its momentum space representation, as in (28). We illustrate the procedure with this diagram. Following position-space Feynman rules, time-ordered product in the diagram is



$$
\begin{aligned}
&T_1 =< \phi(x_1)\phi(x_2) >= -\frac{g^2}{2}\int d^4x \int d^4y D_{1x} D_{xy}^2 D_{y2} \\
&= -\frac{g^2}{2}\int d^4x \int d^4y \int \frac{d^4p_1}{(2\pi)^4}\int \frac{d^4p_2}{(2\pi)^4}\int \frac{d^4p_3}{(2\pi)^4}\int \frac{d^4p_4}{(2\pi)^4} \\
&exp(ip_1^\mu(x_{1\mu}-x_\mu))exp(ip_2^\mu(y_\mu - x_{2\mu}))exp(ip_3^\mu(x_\mu - y_\mu))exp(ip_4^\mu(x_\mu - y_\mu)) \\
&= -\frac{g^2}{2}\int \frac{d^4p_4}{(2\pi)^4}\frac{d^4p_1}{(2\pi)^4}\frac{d^4p_2}{(2\pi)^4} exp(ip_{1\mu}x_1^\mu)exp(-ip_{2\mu}x_2^\mu) \\
&\frac{i}{p_1^2+i\epsilon}\frac{i}{p_2^2+i\epsilon}\frac{i}{(p_1-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon}(2\pi)^4\delta^4(p_1-p_2)
\end{aligned}
\tag{41}
$$

Bring this back to the LSZ reduction formula, in which case there is one particle in, and one particle out. T_1 gives a contribution

$$
\begin{aligned}
\langle f|S|i\rangle_{T_1} &= -[\int d^4x_1 exp(-ip_{i\mu}x_1^\mu)(p_i)^2][\int d^4x_2 exp(ip_{f\mu}x_2^\mu)(p_f)^2]T_1 \\
&= -\frac{g^2}{2}\int d^4p_4(2\pi)^4\frac{i}{(p_i-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon}(2\pi^4)\delta^4(p_i-p_f)
\end{aligned}
\tag{42}
$$

$$\Rightarrow iM_{T_1} = -\frac{g^2}{2}\int \frac{d^4p_4}{(2\pi)^4}\frac{i}{(p_i-p_4)^2+i\epsilon}\frac{i}{p_4^2+i\epsilon} \tag{43}$$

(43) may not be immediately obvious, but observe that in (42) the two integrations on x will yield two delta function. This procedure can be generalized to *Feynman rules in momentum space*[4]:

1.Label each internal line with four momentum q_i

2.Each internal line gets $\frac{i}{q_j^2-m^2+i\epsilon}$(we set m=0 in previous sections),each vertex gets ig,where g is coupling constant(as long as we follow the $\mathscr{L}_{int} = \frac{g}{n!}\phi^n$ convention).

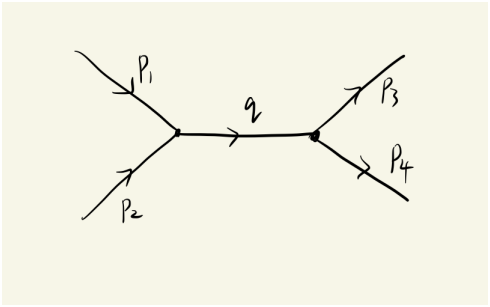
3.For each vertex,insert a factor $(2\pi)^4\delta^4(\sum \pm q)$. In the delta function,incoming momentum is q_j,outgoing momentum is $-q_j$.All the momentum,regardless whether it is external or internal, should be included.

4.Integrate internal momentum $\frac{d^4q_j}{(2\pi)^4}$

5.Cancel the factor $(2\pi)^4\delta(\sum p_i - \sum p_f)$,what is left is contribution to $i\mathscr{M}$ from this particular diagram.

6.The additional rule,which doesn't concern marix elements,but concern cross sections, is that if there are n identical particles in the final state,divide the cross section by n!. This is to avoid double-counting the phase space.

6.4 A Simple Example(ϕ^3)



As an illustration of these rules. Let us calculate the matrix element of this diagram.

$$\begin{aligned}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)i\mathscr{M} &= \int \frac{d^4q}{(2\pi)^4}\frac{i}{q^2-m^2+i\epsilon}(-ig)(-ig)(2\pi)^4\delta^4(p_1+p_2-q)(2\pi^4)\delta^4(p_3+p_4-q)\\ &= \frac{-ig^2}{(p_1+p_2)^2-m^2+i\epsilon}(2\pi)^4\delta^4(p_1+p_2-p_3-p_4)\end{aligned} \tag{44}$$

7 Tie the loose ends

As closing words of this short project, we will fix a few problems we shy away from in our previous derivation.

It is a trivial generalization of section 6 if we have two or more real scalar fields. We just need to introduce two sets of independent creation/annihilation operators. The recipe of deriving corresponding Feynman rules are exactly the same.

What would happen when the $\mathscr{L}_{int}$ depends upon derivative? First, derivative coupling is rare. For example in QED and QCD we don't see derivative coupling. But what if there are? The answer is **there are none**. Lorentz invariance allows $\mathscr{L}_{int} = \phi^2(\partial\phi)^2$,but it turns our that this term is **non-renormalizable**,and thus shouldn't appear.Renormalizability places a heavy constraint on

the types of interaction we can construct. A quick rule is that every ϕ or ∂ contributes a factor of m,so $\mathscr{L}_{int} = \phi^2(\partial\phi)^2$ contributes m^6. The rule is: if the power of m is larger than 4, the theory is non-renormalizable[8]. [13] Suppose we have two real scalar fields, then at most one can have coupling $\partial_\mu\phi_1\partial^\mu\phi_2$,which will not render our previous work invalid. Since π_1 will gain an additional term $\partial_t\phi_2$, but ϕ_1 certainly commutes with ϕ_2.

Ostrogradsky theorem[9] also limits the order of the derivative that can appear in a Lagrangian. All these constraints would render the procedure we describe in the previous part of this article very general.

The author would also like to mention that there is a better, and a more rigorous approach of introducing second quantization[6].It starts by postulating only the principle of indistinguishability and the fact that fermions are antisymmetric/bosons are symmetric, and arrive at the commutation relations among operators. However, strictly following that will take us too far away from the derivation of Feynman rules.

We also ask ourselves:is every particle in our universe described by real scalar fields? We certainly would hope so,but the answer is no. Different particles are described by different fields characterized by the **spin**,which is strictly speaking, a mathematical concept in Representation Theory[10][5]. Deriving those results will require much more knowledge in mathematics. A long journey still awaits.

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