PHY 950 HW2

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2.(a) $f(z) = \int_{-\infty}^{\infty} g(x)\, h(\frac{z}{x}) \frac{dx}{|x|}$

$g(x) = \begin{cases} 1 & 0<x<1 \\ 0 & \text{other} \end{cases}$; $h(\frac{z}{x}) = \begin{cases} 1 & 0<\frac{z}{x}<1 \\ 0 & \text{other} \end{cases}$

$= \int_0^1 h(\frac{z}{x}) \frac{dx}{x}$ $\quad 0<x<1$

$= \int_0^{1/z} h(\frac{z}{x}) \frac{d(\frac{x}{z})}{x/z}$ $\quad t = \frac{x}{z}$ $\quad \underline{z = xy}$

$= \int_0^{1/z} h(t) \frac{dt}{t}$ $\quad 0<\frac{z}{x}<1 \Rightarrow$

$= \begin{cases} -\ln z & \rightarrow 0<z<1 \\ 0 & \text{otherwise} \end{cases}$

(b) $z = xy$, $u = x$

$g(u) = f(z)|J| \Rightarrow J = \frac{\partial z}{\partial u} = y$

$\int_a^b f(z)dz = \int_\alpha^\beta f(z(u)) \frac{\partial z}{\partial u} du$

$\int_\alpha^\beta f(u)du = \int_a^b f(u(z)) \frac{\partial u}{\partial z} dz = \int_a^b f[u(z)] \frac{1}{y} dz$

$= \int_0^1 g(x)dx = x$

$\Rightarrow \int_0^1 f[x(z)] \frac{1}{y} dz = x$

$\int_0^1 f(\frac{z}{y}) \frac{1}{y} dz = x = \frac{z}{y}$

$\Rightarrow f(z) = \begin{cases} -\ln z & 0<z<1 \\ 0 & \text{otherwise} \end{cases}$

(b) $z = xy$, $u = x$, pdf $f(x,y) = g(x)h(y)$

$g(z,u) = f(x,y)|J|$

$J^{-1} = \begin{vmatrix} \frac{\partial z}{\partial x} & \frac{\partial z}{\partial y} \\ \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \end{vmatrix} = \begin{vmatrix} y & x \\ 1 & 0 \end{vmatrix} \Rightarrow |J| = \frac{1}{|J^{-1}|} = \frac{1}{x}$

$= -x$

$\Rightarrow g(z,u) = g(x)h(y) \frac{1}{x}$

$f(z) = \int_{-\infty}^{\infty} g(z,u)du = \int_0^1 g(x) h(\frac{z}{x}) \frac{1}{x} dx = \begin{cases} -\ln z & 0<z<1 \\ 0 & \text{otherwise} \end{cases}$

5. $V[\frac{X_1^2}{X_2}] = E^2[X_1^2]\, V[\frac{1}{X_2}] + E^2[\frac{1}{X_2}]\, V[X_1^2] + V[X_1^2]\, V[\frac{1}{X_2}]$

Because X_1, X_2 are independent $\Rightarrow$ $E(X_1 X_2) = E(X_1)\, E(X_2)$

$$\Rightarrow V(\frac{X_1^2}{X_2}) = E[(\frac{X_1^2}{X_2})^2] - E^2(\frac{X_1^2}{X_2})$$

$$= E(X_1^4)\, E(\frac{1}{X_2^2}) - E^2(X_1^2)\, E^2(\frac{1}{X_2})$$

$E(X_1^4) = \sigma_1^2(X_1^2) + E^2(X_1^2)$

$E(X_1^2) = \sigma_1^2(X_1) + E^2(X_1) = \sigma_1^2 + \mu_1^2 = 101$

$E(\frac{1}{X_2}) = \frac{1}{N_2} \sum^{N_2} \frac{1}{X_i} = \frac{1}{\prod_{N_2} X_i} E(X_2)$