

$$1.(a) \text{ (1) } S = 1 \text{ GeV}, p_1(\sigma | \alpha, \beta) = \frac{1}{\sqrt{2\pi V_1}} e^{-(\sigma_1 - \bar{\sigma}_1)^2 / 2V_1}$$

$$\text{expected mean } \bar{\sigma}_1 = 4\sqrt{1} + 2 = 6 \quad (\sigma_1 = \alpha + \beta)$$

$$\text{variance } V_1 = (0.2 \bar{\sigma}_1)^2 = (1.2)^2 = 1.44$$

$$(2) \text{ energy } S = 2 \text{ GeV}, p_2(\sigma | \alpha, \beta) = \frac{1}{\sqrt{2\pi V_2}} e^{-(\sigma_2 - \bar{\sigma}_2)^2 / 2V_2}$$

$$\text{expected mean } \bar{\sigma}_2 = 4\sqrt{2} + 2 = 7.66 \quad (\sigma_2 = \sqrt{2}\alpha + \beta)$$

$$\text{variance } V_2 = (0.15 \bar{\sigma}_2)^2 = 1.15^2 = 1.32$$

$$(3) \text{ energy } S = 5 \text{ GeV}, p_3(\sigma | \alpha, \beta) = \frac{1}{\sqrt{2\pi V_3}} e^{-(\sigma_3 - \bar{\sigma}_3)^2 / 2V_3}$$

$$\text{expected mean } \bar{\sigma}_3 = 4\sqrt{5} + 2 = 10.94 \quad (\sigma_3 = \sqrt{5}\alpha + \beta)$$

$$\text{variance } V_3 = (0.1 \bar{\sigma}_3)^2 = 1.09^2 = 1.20$$

$$\ln P = \frac{N_1}{V_1} \ln p_1 + \frac{N_2}{V_2} \ln p_2 + \frac{N_3}{V_3} \ln p_3 = -\frac{N_1(\sigma - \bar{\sigma}_1)^2}{2V_1} - \frac{N_2(\sigma - \bar{\sigma}_2)^2}{2V_2} - \frac{N_3(\sigma - \bar{\sigma}_3)^2}{2V_3} + C$$

$$\text{where } \bar{\sigma}_1 = \alpha + \beta, \bar{\sigma}_2 = \sqrt{2}\alpha + \beta, \bar{\sigma}_3 = \sqrt{5}\alpha + \beta$$

$$\Rightarrow I_{ij} = E \left[\sum_{m=1}^3 \frac{N_m}{V_m} \frac{\partial \ln p_m}{\partial \theta_i} \frac{\partial \ln p_m}{\partial \theta_j} \right] \quad \theta_i, \theta_j \in \{\alpha, \beta\}$$

$$= \begin{bmatrix} \frac{N_1}{V_1} + 2\sqrt{\frac{N_2}{V_2}} + 5\sqrt{\frac{N_3}{V_3}} & \frac{N_1}{V_1} + \sqrt{2}\frac{N_2}{V_2} + \sqrt{5}\frac{N_3}{V_3} \\ \frac{N_1}{V_1} + \sqrt{2}\frac{N_2}{V_2} + \sqrt{5}\frac{N_3}{V_3} & \frac{N_1}{V_1} + \frac{N_2}{V_2} + \frac{N_3}{V_3} \end{bmatrix}$$

$$V[\hat{\theta}_{ij}] \geq (I^{-1})_{ij}$$

~~By using Excel, I get~~

A1: when $(N_1, N_2, N_3) = (2, 1, 2)$, we minimize the variance on slope α

$$V = \begin{bmatrix} 0.83 & -1.35 \\ -1.35 & 2.46 \end{bmatrix}$$

A2: when $(N_1, N_2, N_3) = (1, 1, 1)$, we minimize the variance on intercept β

$$V = \begin{bmatrix} 0.95 & -1.21 \\ -1.21 & 1.73 \end{bmatrix}$$

1.6)

$$I_{ij}^{\text{prior}} = \begin{pmatrix} \sigma_{\alpha}^{-2} & 0 \\ 0 & \sigma_{\beta}^{-2} \end{pmatrix} = \begin{pmatrix} (0.1\alpha)^{-2} & 0 \\ 0 & (0.2\beta)^{-2} \end{pmatrix} \quad \cancel{\#}$$

$$I_{ij}^{\text{total}} = I_{ij}^{\text{prior}} + I_{ij}^{\text{measured}}$$

For A1, $V = \begin{pmatrix} 0.073 & -0.045 \\ -0.045 & 0.127 \end{pmatrix}$ minimize α , $(N_1, N_2, N_3) = (2, 1, 2)$

For A2, $V = \begin{pmatrix} 0.096 & -0.051 \\ -0.051 & 0.1155 \end{pmatrix}$ minimize β , $(N_1, N_2, N_3) = (4, 3, 0)$

① The prior measurement dramatically increase the accuracy because the variance of the measurement decreased with a factor around 10.

② The prior measurement somehow changed the conclusion.

From ~~(5, 1, 1)~~ $(N_1, N_2, N_3) = (5, 1, 1)$ to $(4, 3, 0)$ to minimize β

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$$2.(a) \log L(v_0, k) = \sum_{i=1}^N (n_i \ln v_i - v_i) \quad | \quad v_i(z) = v_0 \exp\left(-\frac{4\pi r^3 \Delta \rho g}{3T} \cdot \frac{z}{k}\right)$$

$$= \sum_{i=1}^N [n_i (\ln v_0 - A \frac{z}{k}) - v_0 e^{-Az/k}] \quad | \quad = v_0 \exp(-A \frac{z}{k})$$

$$\frac{\partial \ln L(v_0, k)}{\partial v_0} = \sum_{i=1}^N \left(\frac{n_i}{v_0} - e^{-Az/k} \right) = 0 \quad (1) \quad \text{Let } A = 4\pi r^3 \Delta \rho g / 3T$$

$$\frac{\partial \ln L(v_0, k)}{\partial k} = \sum_{i=1}^N \left(n_i \frac{Az_i}{k^2} + v_0 \frac{Az_i}{k^2} e^{-Az_i/k} \right) = 0 \quad (2)$$

$$(1) \rightarrow \sum_{i=1}^N n_i - v_0 \sum_{i=1}^N e^{-Az_i/k} = 0 \quad (3)$$

$$(2) \rightarrow \frac{A}{k^2} \sum_{i=1}^N n_i z_i + \frac{A v_0}{k^2} \sum_{i=1}^N z_i e^{-Az_i/k} = 0$$

$$\sum_{i=1}^N n_i z_i + v_0 \sum_{i=1}^N z_i e^{-Az_i/k} = 0 \quad (4)$$

By solving, I get $\begin{cases} v_0 = 1845 \\ k = 1.1987 \times 10^{-23} \end{cases}$

$$(b) N_A = R/k = 6.02899 \times 10^{30} \text{ correct } \checkmark$$

$$(c) \sigma_i = \sqrt{v_i}, \sigma_i^2 = v_i$$

$$\chi^2 = \sum_{i=1}^N \frac{(n_i - v_0 e^{-Az_i/k})^2}{v_0 e^{-Az_i/k}} = \sum_{i=1}^N \left(\frac{n_i^2}{v_i} - 2n_i + v_i \right)$$

$$\frac{\partial \chi^2}{\partial v_0} = \sum_{i=1}^N \left(-\frac{n_i^2}{v_0^2} e^{Az_i/k} + e^{-Az_i/k} \right) = 0 \quad (5)$$

$$\frac{\partial \chi^2}{\partial k} = \sum_{i=1}^N \left(-\frac{n_i^2}{v_0} \frac{Az_i}{k^2} e^{Az_i/k} + \frac{v_0 Az_i}{k^2} e^{-Az_i/k} \right) = 0$$

$$\Rightarrow \sum_{i=1}^N (-n_i^2 e^{Az_i/k} + v_0^2 e^{-Az_i/k}) = 0 \quad (6) = (5)$$

$$\Rightarrow \sum_{i=1}^N (-n_i^2 + v_i^2) = 0$$