HWJ PHY950

1.(a) ① $s = 1\,GeV$, $p_1(\sigma|\alpha,\beta) = \frac{1}{\sqrt{2\pi V_1}} e^{-(\sigma_1-6)^2/2V_1}$

expected mean $\bar{\sigma}_1 = 4\sqrt{1}+2 = 6$ $\quad (\sigma_1 = \alpha+\beta)$

variance $V_1 = (0.2\,\bar{\sigma}_1)^2 = (1.2)^2 = 1.44$

② energy $s = 2\,GeV$, $p_2(\sigma|\alpha,\beta) = \frac{1}{\sqrt{2\pi V_2}} e^{-(\sigma_2-7.66)^2/2V_2}$

expected mean $\bar{\sigma}_2 = 4\sqrt{2}+2 = 7.66$ $\quad (\sigma_2 = \sqrt{2}\alpha+\beta)$

variance $V_2 = (0.15\,\bar{\sigma}_2)^2 = 1.15^2 = 1.32$

③ energy $s = 5\,GeV$, $p_3(\sigma|\alpha,\beta) = \frac{1}{\sqrt{2\pi V_3}} e^{-(\sigma_3-\bar{\sigma}_3)^2/2V_3}$

expected mean $\bar{\sigma}_3 = 4\sqrt{5}+2 = 10.94$ $\quad (\sigma_3 = \sqrt{5}\alpha+\beta)$

variance $V_3 = (0.1\,\bar{\sigma}_3)^2 = 1.09^2 = 1.20$

$$\ln P = \underset{N_1}{\lambda}\ln p_1 + \underset{N_2}{\lambda}\ln p_2 + \underset{N_3}{\lambda}\ln p_3 = -N_1\frac{(\sigma_1-\bar{\sigma}_1)^2}{2V_1} - N_2\frac{(\sigma_2-\bar{\sigma}_2)^2}{2V_2} - N_3\frac{(\sigma_3-\bar{\sigma}_3)^2}{2V_3} + C$$

where $\sigma_1 = \alpha+\beta$, $\sigma_2 = \sqrt{2}\alpha+\beta$, $\sigma_3 = \sqrt{5}\alpha+\beta$

$$\Rightarrow I_{ij} = E\left[\sum_{m=1}^{3} \frac{N_m}{V_m}\frac{\partial \sigma_m}{\partial\theta_i}\frac{\partial \sigma_m}{\partial\theta_j}\right] \quad \theta_i, \theta_j \in \{\alpha,\beta\}$$

$$= \begin{bmatrix} \frac{N_1}{V_1} + 2\frac{N_2}{V_2} + 5\frac{N_3}{V_3} & \frac{N_1}{V_1} + \sqrt{2}\frac{N_2}{V_2} + \sqrt{5}\frac{N_3}{V_3} \\ \frac{N_1}{V_1} + \sqrt{2}\frac{N_2}{V_2} + \sqrt{5}\frac{N_3}{V_3} & \frac{N_1}{V_1} + \frac{N_2}{V_2} + \frac{N_3}{V_3} \end{bmatrix}$$

$V[\hat{\theta}_{ij}] \geq (I^{-1})_{ij}$

By using Excel, I get

A1: when $(N_1, N_2, N_3) = (2, 1, 2)$, we minimize the variance on slope α

$$V = \begin{bmatrix} 0.83 & -1.35 \\ -1.35 & 2.46 \end{bmatrix}$$

A2: when $(N_1, N_2, N_3) = (5, 1, 1)$, we minimize the variance on intersect β

$$V = \begin{bmatrix} 0.95 & -1.21 \\ -1.21 & 1.73 \end{bmatrix}$$

1.(b)

$$I_{ij}^{prior} = \begin{pmatrix} \sigma_\alpha^{-2} & 0 \\ 0 & \sigma_\beta^{-2} \end{pmatrix} = \begin{pmatrix} (0.1\alpha)^{-2} & 0 \\ 0 & (0.2\beta)^{-2} \end{pmatrix}$$

$$I_{ij}^{total} = I_{ij}^{prior} + I_{ij}^{measured}$$

For A1, $V = \begin{pmatrix} 0.073 & -0.045 \\ -0.045 & 0.127 \end{pmatrix}$ minimize α, $(N_1, N_2, N_3) = (2,1,2)$

For A2, $V = \begin{pmatrix} 0.096 & -0.051 \\ -0.051 & 0.1155 \end{pmatrix}$ minimize β, $(N_1, N_2, N_3) = (4,3,0)$

① The prior measurement dramatically increase the accuracy because the variance of the measurement dreased with a factor around 10.

② The prior measurement somehow changed the conclusion. From ~~(5,1,1)~~ $(N_1, N_2, N_3) = (5,1,1)$ to $(4,3,0)$ to minimize β

HW5 PHY950

2.(a) $\log L(\nu_0, k) = \sum_{i=1}^{4} (n_i \log \nu_i - \nu_i)$

$\nu_i(z) = \nu_0 \exp\left(-\frac{4\pi r^3 \Delta\rho g}{3T} \cdot \frac{z}{k}\right)$

$= \sum_{i=1}^{4} \left[n_i \left(\ln \nu_0 - A\frac{z}{k}\right) - \nu_0 e^{-Az/k}\right]$

$= \nu_0 \exp\left(-A\frac{z}{k}\right)$

Let $A = 4\pi r^3 \Delta\rho g / 3T$

$\frac{\partial \ln L(\nu_0, k)}{\partial \nu_0} = \sum_{i=1}^{N} \left(\frac{n_i}{\nu_0} - e^{-Az/k}\right) = 0$ ①

$\frac{\partial \ln L(\nu_0, k)}{\partial k} = \sum_{i=1}^{N} \left(n_i \frac{Az_i}{k^2} + \nu_0 \frac{Az_i}{k^2} e^{-Az/k}\right) = 0$ ②

① $\rightarrow$ $\sum_{i=1}^{N} n_i - \nu_0 \sum_{i=1}^{N} e^{-Az_i/k} = 0$ ③

② $\rightarrow$ $\frac{A}{k^2} \sum_{i=1}^{N} n_i z_i + \frac{A\nu_0}{k^2} \sum_{i=1}^{N} z_i e^{-Az_i/k} = 0$

$\sum_{i=1}^{N} n_i z_i + \nu_0 \sum_{i=1}^{N} z_i e^{-Az_i/k} = 0$ ④

By coding, I get $\begin{cases} \nu_0 = 1845 \\ k = 1.1987 \times 10^{-23} \end{cases}$

(b) $N_A = R/k = 6.02899 \times 10^{23}$ correct ✓ $(R = 8.32\ \mathrm{J/mol \cdot k})$

(c) $\sigma_i = \sqrt{\nu_i}$, $\sigma_i^2 = \nu_i$

$\chi^2 = \sum_{i=1}^{N} \frac{(n_i - \nu_0 e^{-Az_i/k})^2}{\nu_0 e^{-Az_i/k}} = \sum_{i=1}^{N} \left(\frac{n_i^2}{\nu_i} - 2n_i + \nu_i\right)$

$\frac{\partial \chi^2}{\partial \nu_0} = \sum_{i=1}^{N} \left(-\frac{n_i^2}{\nu_0^2} e^{Az_i/k} + e^{-Az_i/k}\right) = 0$ ⑤

$\frac{\partial \chi^2}{\partial k} = \sum_{i=1}^{N} \left(-\frac{n_i^2}{\nu_0} \frac{Az_i}{k^2} e^{Az_i/k} + \frac{\nu_0 A z_i}{k^2} e^{-Az_i/k}\right) = 0$

$\Rightarrow \sum_{i=1}^{N} \left(-n_i^2 e^{Az_i/k} + \nu_0^2 e^{-Az_i/k}\right) = 0$ ⑥ = ⑤

$\Rightarrow \sum_{i=1}^{N} \left(-n_i^2 + \nu_i^2\right) = 0$