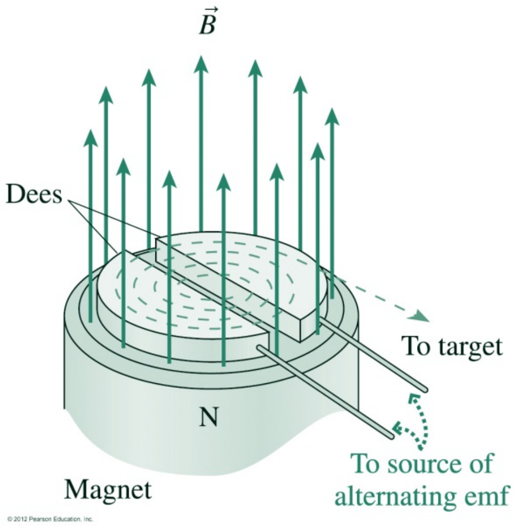
Partial exam magnetism (Electricity and Magnetism, EE1P21)
Exam magnetism (Electricity and Magnetism, EE1210 part B)
1st of July 2015 from 9.00 to 11.00

This exam consists of 4 questions.
Answer every question on a new sheet of paper.
An A4 with equations can be used.
A simple calculator may be used.
Smartphones, tablets, programmable calculators and other devices that may contain old exams or other information may not be used.

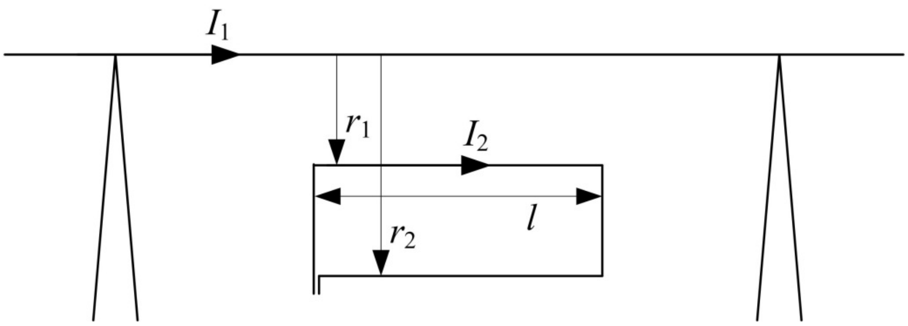
Question 1 (4+3+3+5+5+5=25 points)

A cyclotron with a diameter of 2 m is used to accelerate deuterium nuclei (consisting of one proton and one neutron). The mass of a deuterium nucleus is $m = 3.32\times10^{-27}$ kg and its charge is $e = 1.6\cdot10^{-19}$ C.
The magnetic flux density B is 1 T and the direction is upwards: $\vec{B} = [0,1,0]$ T.



a) Calculate the force $\vec{F}$ on a deuterium nucleus with a velocity $\vec{v} = [1,0,0]\cdot10^{6}$ m/s.
b) Why does the deuterium nucleus follow a circular path in this cyclotron?
c) What is the rotation direction of the deuterium nuclei seen from above, clockwise or counter clockwise?
d) At what frequency should the dee voltage be alternated to accelerate the deuterium nuclei?
e) What is the maximum kinetic energy of the deuterium nuclei?
f) If the amplitude of the potential difference between the dees is 1000 V, how may orbits do the deuterium nuclei complete before reaching maximum energy?

Question 2 (4+4+8+3+3+3+3+3+3=34 points)
This question deals with a simplified power line. At first, a DC current of I_1=10000 A is flowing through the line. We assume the line is exactly horizontal.



a) Give an expression for the magnetic flux density outside the line as a function of the distance r from the line.

A second conductor forms a rectangular loop. This rectangular loop is situated in the same vertical plane as the power line.
The length of the loop is l=10 m.
The top conductor of the rectangular loop is at a distance of r_1=5 m from the line.
The bottom conductor of the rectangular loop is at a distance of r_2=10 m from the line.
At first, the current in the loop is zero.

b) Calculate the magnetic flux through the rectangular loop.

The loop is then connected to an ideal DC current source generating a current I_2=10A.

c) Calculate the magnitudes and the directions of the forces on the 4 sides of the loop.

Next, the loop is closed via a resistance R, and the loop is moved down, away from the line, while it stays in the vertical plane through the line.

d) What is the direction of the current induced in the loop, clockwise or counter clockwise?
e) What is the direction of the total force on the loop?

We consider the same configuration, but now, the current trough the line is an alternating current: $I_1 = I_{1p}\sin(\omega t)$ with $\omega = 100\pi\,\mathrm{rad/s}$. The amplitude of the current through the line has such a value, that the amplitude of the magnetic flux through the loop is given by $\Phi_{mp} = 30\,\mathrm{mWb}$.

f) Calculate the voltage induced in the loop as a function of time.

The loop is now closed via a large resistance. The resistance is so large that the effect of the current in the loop on the flux through the loop is negligible.

g) Is there a net average force on the loop? Justify your answer.

The loop is now short-circuited and the resistance of the loop is zero.

h) Calculate the flux through the loop.
i) Is there a net average force on the loop? Justify your answer.

Question 3 (3+2+4+4=13 points)

A square current loop has its 4 corners at [a,a,0], [a,-a,0], [-a,-a,0] and [-a,a,0]. The current in the loop is *I* and flows clockwise.

a) Calculate the magnetic dipole moment $\vec{\mu}$ of the loop.

Next, an expression for the magnetic flux density at the origin $\vec{B}(0,0,0)$ has to be derived by applying Biot-Savart to this current loop. This derivation should be done in the following derivation steps.

b) Give an expression for $d\vec{B}$ according to Biot-Savart.

c) Show how Biot-Savart is applied to the current loop.

d) Integrate to obtain $\vec{B}(0,0,0)$.

Hint: $\int \frac{dx}{(x^2 \pm a^2)^{3/2}} = \frac{\pm x}{a^2\sqrt{x^2 \pm a^2}}$

Question 4 (3+3+3+3+3+3=18 points)

Sunlight comes to earth as an electromagnetic wave, and can be considered a plane electromagnetic wave. We assume the wave direction being perpendicular to the earth's surface. In this question, the average intensity is $\bar{S} = 1\,\mathrm{kW/m^2}$.

a) Calculate the amplitude of the magnetic flux density of this electromagnetic wave.

b) Calculate the amplitude of the electric field of this electromagnetic wave.

c) Calculate the average energy density of the electric field.

d) Calculate the average energy density of the magnetic field.

e) Calculate the radiation pressure.

f) Calculate the frequency for a wavelength of 550 nm.

Answer question 1

DEVELOP Particles in a cyclotron get a boost in velocity each time they pass from one dee to the other. The magnetic field holds them in a circular orbit, so they make multiple passes. In order to always be accelerating the particles, the voltage has to be alternated every time they make a half circle of their orbit. In other words, the voltage needs to cycle at the same rate as the particles revolve in the magnetic field, which is just the cyclotron frequenqcy: $f = qB/2\pi m$ (Equation 26.4). In this case, the particles are deuterium nuclei, which have atomic mass 2 and charge $+e$:

$$m = 2(1.66\times10^{-27}\,\text{kg}) = 3.32\times10^{-27}\,\text{kg}$$

$$q = +1.60\times10^{-19}\,\text{C}$$

The frequency does not depend on the speed (energy) of the nuclei, but the radius of their orbit does: $r = mv/eB$ (Equation 26.3). The maximum energy is achieved when the nuclei reach the outer rim of the cyclotron. We can figure out how many orbits it takes to reach this maximum by dividing by the kinetic energy gain of each orbit. We'll assume the nuclei have negligible kinetic energy to begin with.

a) $\vec{F} = q\vec{v}\times\vec{B} = 1.6\cdot10^{-13}\begin{bmatrix}0\\0\\1\end{bmatrix}\text{N}$

b) The force remains perpendicular to the speed.

c) Clockwise.

d) The electromagnetic force on a deuteron is $\vec{F} = q\vec{v}\times\vec{B}$. In this case: $F = e\omega rB$ because the speed and the flux density are perpendicular. The force is perpendicular to the speed, so the motion is rotational.
Therefore, the electromagnetic force must be equal to the centrifugal force $F = ma = m\omega^2 r$.
Therefore, $F = m\omega^2 r = e\omega rB$ and $\omega = \dfrac{eB}{m}$.
The frequency at which the voltage should be alternated is

$$f = \frac{\omega}{2\pi} = \frac{eB}{2\pi m} = \frac{(1.60\times10^{-19}\,\text{C})(1.0\text{ T})}{2\pi(3.32\times10^{-27}\,\text{kg})} = 7.67\,\text{MHz}.$$

e) The maximum kinetic energy can be derived from the speed at the cyclotron's radius. The radius follows from the same force balans: $\omega = \dfrac{v}{r} = \dfrac{eB}{m}$. Therefore, $v = \dfrac{reB}{m}$.
The maximum kinetic energy is

$$K = \frac{1}{2}mv^2 = \frac{r^2e^2B^2}{2m} = \frac{\left[\left(\frac{1}{2}\cdot2\text{m}\right)\left(1.60\times10^{-19}\,\text{C}\right)\left(1.0\text{ T}\right)\right]^2}{2(3.32\times10^{-27}\,\text{kg})} = 3.86\cdot10^{-12}\text{ J} = 24.1\,\text{MeV}.$$

We've written the answer in eV, as this unit is easier to work with for particles.

f) Each orbit in the cyclotron accounts for two passes across the potential difference between the dees. Therefore, the kinetic energy gain in each orbit is $\Delta K = 2q\Delta V$, and the number of orbits needed to reach the maximum energy is
Every revolution, the energy of the deuteron nucleus increases twice with 1000 eV.
Therefore, $\dfrac{K}{\Delta K} = \dfrac{24.1\text{ MeV}}{2(e)(1000\text{ V})} = \dfrac{24.1\text{ MeV}}{(2000\text{ eV})} = 12050$.

Answer question 2

a) Applying Ampere's law $\oint \vec{B}\cdot \mathrm{d}\vec{r} = \mu_0 I$ to a circular path around the line gives

$2\pi r B = \mu_0 I$. Therefore $B = \dfrac{\mu_0 I}{2\pi r}$.

b) The flux density $\vec{B}$ has the same direction as $d\vec{A}$. Therefore,

$$\Phi_m = \int \vec{B}\cdot d\vec{A} = \int_0^l \int_{r_1}^{r_2} B dr dx = l\int_{r_1}^{r_2} \frac{\mu_0 I}{2\pi r} dr = l\,\frac{\mu_0 I}{2\pi}(\ln(r_2)-\ln(r_1)) = 13.9\,\mathrm{mWb}$$

c) The Lorenz force can be calculated as $\vec{F} = I\vec{L}\times\vec{B}$.

The forces on the sides of the loop caused by fields due to the currents in the loop are negligible compared to the forces on the sides of the loop caused by the field of the power line because the current in the power line is three orders of magnitude larger than the current in the loop.

The flux density is always perpendicular to the current in the conductor.

For the top side, the force is $F = ILB(r_1) = 40\,\mathrm{mN}$ upwards.

For the right side, the force is

$F = I_2\int_{r_1}^{r_2} B dr = I_2\int_{r_1}^{r_2}\dfrac{\mu_0 I_1}{2\pi r}dr = \mu_0\dfrac{I_2 I_1}{2\pi}(\ln(r_2)-\ln(r_1)) = 13.9\,\mathrm{mN}$ to the right.

For the top side, the force is $F = ILB(r_1) = 20\,\mathrm{mN}$ down.

For the left side, the force is

$F = I_2\int_{r_1}^{r_2} B dr = I_2\int_{r_1}^{r_2}\dfrac{\mu_0 I_1}{2\pi r}dr = \mu_0\dfrac{I_2 I_1}{2\pi}(\ln(r_2)-\ln(r_1)) = 13.9\,\mathrm{mN}$ to the left.

The total force is 20 mN upwards.

d) Faraday's law says: $\oint \vec{E}\cdot \mathrm{d}\vec{r} = -\dfrac{\mathrm{d}\Phi_m}{\mathrm{d}t}$. Therefore, a current is induced that tries to oppose the change of the field. If the loop is moved away from the line, the flux is reducing. To oppose that, the induced current must flow clockwise.

e) The electromagnetic force on the loop opposes the change of the flux, so the force is upwards, towards the power line.

f) Because the current is sinusoidal, the magnetic flux is also sinusoidal:

$\Phi_m = \Phi_{mp}\sin(\omega t)$.

Therefore, the induced voltage is $E = -\dfrac{d\Phi_m}{dt} = -\omega\Phi_{mp}\cos(\omega t) = -9.42\cos(100\pi t)\,\mathrm{V}$

g) The average force on the loop is zero, because the flux density is sinusoidal, while the current is cosinusoidal.

h) Faradays law says that the induced current opposes the change of the magnetic field. If we have a perfect conductor (with zero resistance):

$$\frac{d\Phi_m}{dt} = 0$$

Therefore, the current has such a value that it completely cancels the change of the magnetic field of the line. It must be in phase with the current in the line.

i) The average force on the loop has a value and pushes the loop away, because the direction of the current in the top side of the loop is opposite to the direction of the current in the line and in phase with that current.

Answer question 3

a) The magnetic dipole moment of the loop is given by $\vec{\mu} = NI\vec{A} = [0, 0, -4a^2 I]$

b) Biot-Savard says: $\vec{B} = \int \mathrm{d}\vec{B} = \int \frac{\mu_0}{4\pi} \frac{I\,\mathrm{d}\vec{L} \times \hat{r}}{r^2}$ or $d\vec{B} = \frac{\mu_0}{4\pi} \frac{I\,\mathrm{d}\vec{L} \times \hat{r}}{r^2}$

c) Each of the 4 sides of the current loop has the same contribution to the flux density. If we take the top side where *y*=*a*, we can integrate from *x*=-*a* to *x*=*a*.

The distance is $r = \sqrt{x^2 + a^2}$.

The cross product of the vectors $\mathrm{d}\vec{L} \times \hat{r}$ becomes $\mathrm{d}\vec{L} \times \hat{r} = \frac{a}{\sqrt{x^2 + a^2}} dx$.

Or $d\vec{B} = \frac{\mu_0 I \mathrm{d}\vec{L} \times \vec{r}}{4\pi r^3} = \frac{\mu_0 I}{4\pi r^3} \begin{bmatrix} dx \\ 0 \\ 0 \end{bmatrix} \times \begin{bmatrix} -x \\ -y \\ 0 \end{bmatrix} = \frac{\mu_0 I y}{4\pi (x^2 + y^2)^{3/2}} dx \begin{bmatrix} 0 \\ 0 \\ -1 \end{bmatrix}$

The flux density is in the negative *z*-direction.

d) Integrate to obtain the result:

$$B_z = -4 \int_{-a}^{a} \frac{\mu_0 I}{4\pi} \frac{1}{(x^2 + a^2)} \frac{a}{\sqrt{x^2 + a^2}} dx$$

$$= -\frac{\mu_0 I a}{\pi} \int_{-a}^{a} \frac{1}{(x^2 + a^2)^{3/2}} dx = -\frac{\mu_0 I a}{\pi} \frac{x}{a^2 \sqrt{x^2 + a^2}} \Bigg|_{-a}^{a} = -\frac{\sqrt{2}\mu_0 I}{\pi a}$$

$$\vec{B} = \left[0, 0, -\frac{\sqrt{2}\mu_0 I}{\pi a} \right]$$

Answer question 4

a) $\overline{S}$ is the average intensity. The magnetic flux density and the electric field intensity vary sinusoidally as a function of time (and space). In the question, you are asked to calculate the amplitude of the magnetic flux density.

$\overline{S} = \frac{\overline{E}\overline{B}}{\mu_0} = \frac{E_p B_p}{2\mu_0} = \frac{c B_p^2}{2\mu_0}$ Therefore,

$$B_p = \sqrt{\frac{2\mu_0 \overline{S}}{c}} = \sqrt{\frac{2 \cdot 4 \cdot \pi \cdot 10^{-7} \cdot 1000}{3 \cdot 10^8}}\ \mathrm{T} = 2.89 \cdot 10^{-6}\ \mathrm{T} = 2.89 \mu\,\mathrm{T}$$

b) $E_p = c B_p = 3 \cdot 10^8 \cdot 2.89 \cdot 10^{-6}\ \mathrm{V/m} = 868\,\mathrm{V/m}$

c) The instantaneous local energy density of the electric field is $u_E = \frac{1}{2} \varepsilon_0 \varepsilon_r E^2$

The average energy density of the electric field is given by

$$u_E = \frac{1}{2} \varepsilon_0 \varepsilon_r \overline{E^2} = \frac{1}{4} \varepsilon_0 E_p^2 = 1.67 \mu\,\mathrm{J/m^3}$$

d) The instantaneous local energy density of the electric field is $u_B = \frac{B^2}{2\mu_0 \mu_r}$

The average energy density of the electric field is given by

$$u_B = \frac{\overline{B^2}}{2\mu_0 \mu_r} = \frac{B_p^2}{4\mu_0} = 1.67 \mu\,\mathrm{J/m^3}$$

Adding average the energy densities of the electric field and the magnetic field and multiplying them by the speed of light again gives the average intensity.

e) $P_{\text{rad}} = \dfrac{\bar{S}}{c} = 3.33\mu\,\text{N/m}^2$

f) $f = \dfrac{c}{\lambda} = \dfrac{3\cdot 10^8}{550\cdot 10^{-9}}\cdot 10^{-9}\ \text{Hz} = 545\cdot 10^{12}\ \text{Hz}$