EE1P21 Electricity and Magnetism

Electric Field and Dielectric Constant

Topic 2

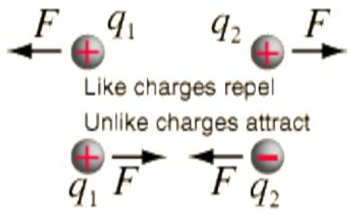
Electric Field
Dielectric Constant

Learning Objectives

Know what the electric field is
Constitutive Relations for simple matter

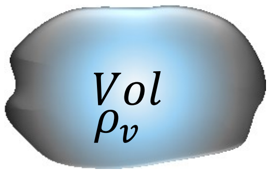
Truly Important from Lecture 1

1)



$$\vec{F}_{12}(\vec{r}, q_1, q_2) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{12}$$

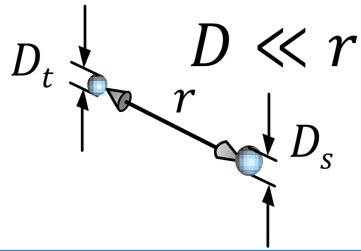
2)



$\vec{r}_t$

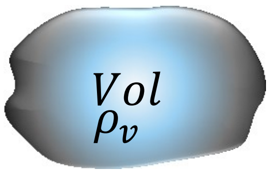
$$\vec{F}(\vec{r}_t) = q_t \iiint\limits_{Vol} \frac{k_e \rho_v(\vec{r}')}{|\vec{r}_t - \vec{r}'|^2}\frac{(\vec{r}_t - \vec{r}')}{|\vec{r}_t - \vec{r}'|}\,d\vec{r}'$$

$D \ll r$



3)

$\vec{r}_t$

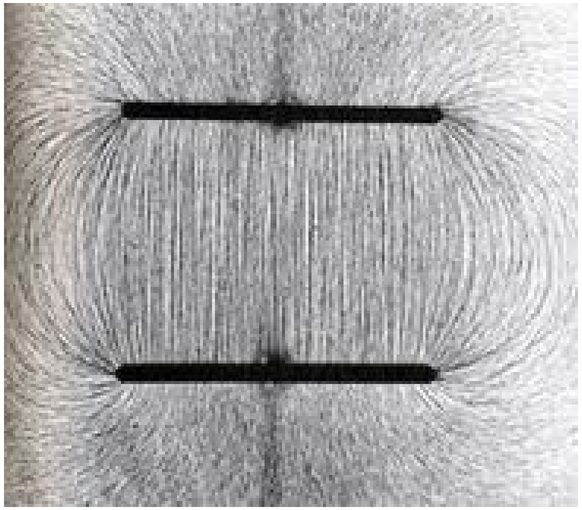


$$\vec{F}(\vec{r}_t) = q_t k_e \frac{\hat{r}_t}{r_t^2} Q$$

The Electric Field

For large systems evaluation of forces and interactions is difficult: immense number of charges, possibly all moving ....

$$\vec{F}(\vec{r}_t) = q_t \iiint_{Vol} \frac{k_e \rho_v(\vec{r}')}{|\vec{r}_t - \vec{r}'|^2} \frac{(\vec{r}_t - \vec{r}')}{|\vec{r}_t - \vec{r}'|} d\vec{r}'$$



Easier to imagine that there is something else called the **electric field**.

The electric field at a point is the force that a unit charge placed at that point would experience:

$$\vec{E}(\vec{r}_t) = \vec{F}(\vec{r}_t) \frac{1}{q_t}$$

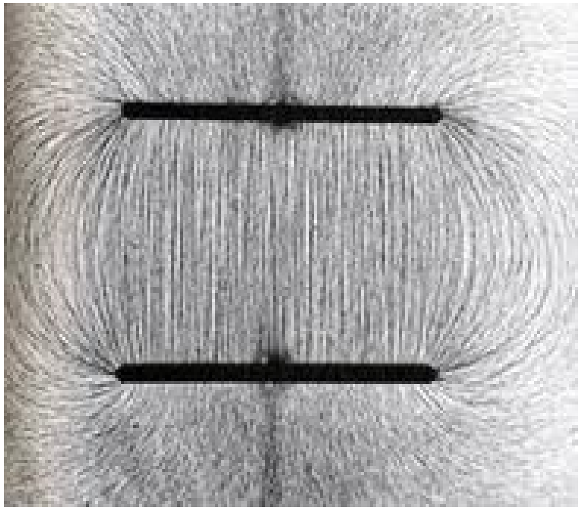
The electric field is generated by all source charges. First calculate this field and than observe the force it exercises.

$$q\vec{E}(\vec{r}) = \vec{F}(\vec{r})$$

The Electric Field

The electric field can be associated to a point in space rather than to a charge. Thus we refer to only to an observation point and we indicate it as $\vec{r}$ rather than $\vec{r}_t$

$$\vec{E}(\vec{r})$$



To create this figure they have distributed some a powder of test charges, but the lines actually represent electric field lines

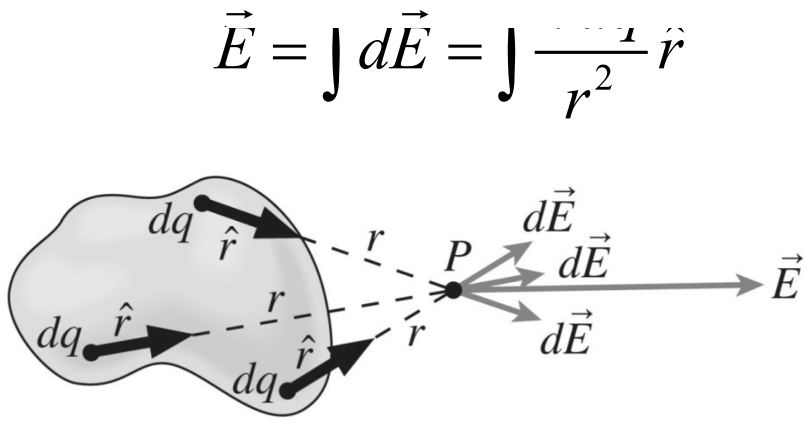
Field due to Charge Distributions

$$\vec{E}(\vec{r}) = \iiint\limits_{Vol} \frac{k_e \rho_v(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\vec{r}\,'$$

$$\vec{E}(\vec{r}) = \iint\limits_{Surf} \frac{k_e \rho_s(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\vec{r}\,'$$

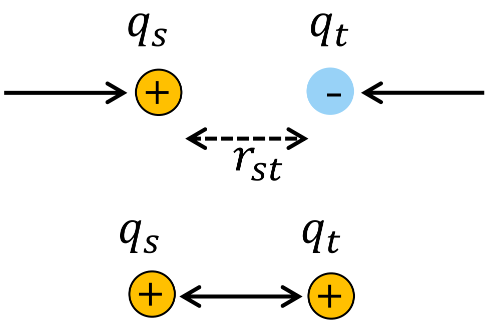
$$\vec{E}(\vec{r}) = \int\limits_{line} \frac{k_e \rho_s(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2} \frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|} d\vec{r}\,'$$

In book you find



Charge distribution

Field of a Point Charge



Attraction

Repulsion

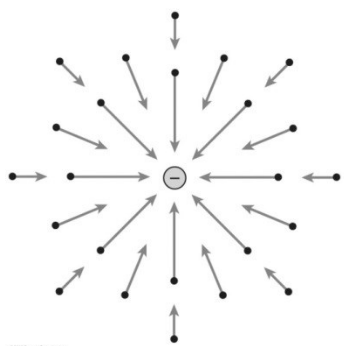
1) Express it with q_s in the origin of the reference system

$$\vec{F}_{st}(\vec{r}_{st}, q_s, q_t) = \frac{k_e q_s q_t}{r_{st}^2}\,\hat{r}_{st}$$

2) Assume that q_s is the source charge and q_t is the test charge

$$\frac{1}{q_t}\vec{F}_{st}(\vec{r}_{st}, q_s, q_t) = \frac{k_e q_s}{r_{st}^2}\hat{r}_{st}$$

The field of a point charge is radial, outward for a positive charge and inward for a negative charge.

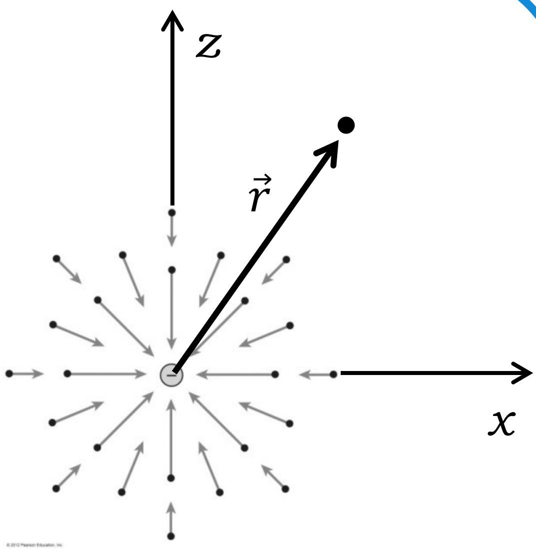


3) Say that this force is a property originated by q_s=q only (forget that q_t exists)

$$\vec{E}(\vec{r} = \vec{r}_{st}, q_s = q) = \frac{k_e q}{r^2}\hat{r}$$

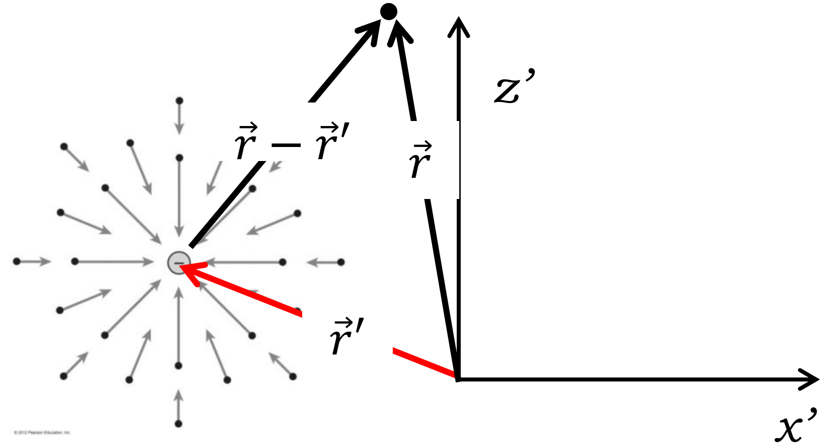
This is the field of a point charge

Field of a Point Charge in generalized reference system



Charge in origin

$$\vec{E}(\vec{r}\,) = \frac{k_e q}{r^2}\hat{r}$$



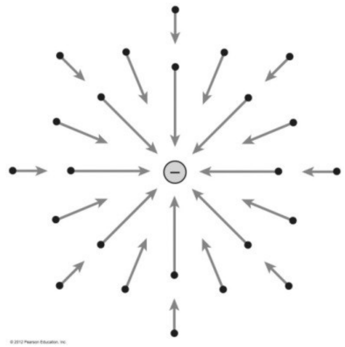
Charge not in origin

$$\vec{E}(\vec{r}\,) = \frac{k_e q}{|\vec{r} - \vec{r}'|^2}\frac{\vec{r} - \vec{r}'}{|\vec{r} - \vec{r}'|}$$

If the source charge is not in the origin its location must be explicitly specified

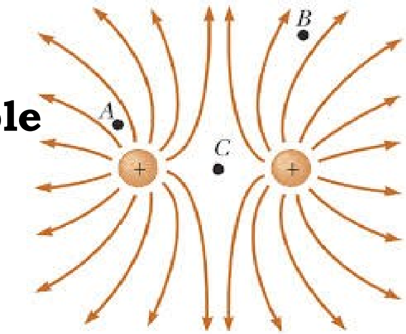
Charge Distributions

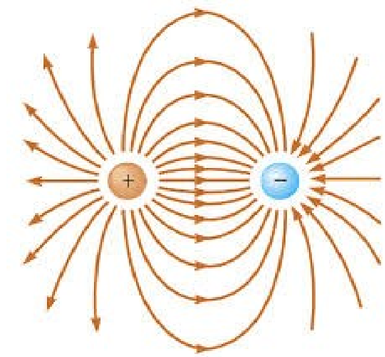
If a point charge generates field:



$$\vec{E}(\vec{r},q) = \frac{k_e q}{r^2}\hat{r}$$

Superposition principle for the electric field





Mathematically the superposition principle for N sources becomes becomes

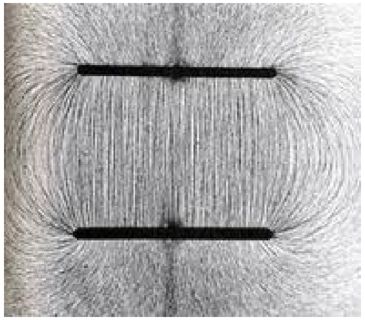
$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{k_e q_i}{|\vec{r}-\vec{r}_i|^2}\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|}$$

The reference system can be taken everywhere since also the location of the sources is specified

Electric Field Lines

- **Electric field lines** provide a convenient and insightful way to represent electric fields.
  - A field line is a curve whose direction at each point is the direction of the electric field at that point.

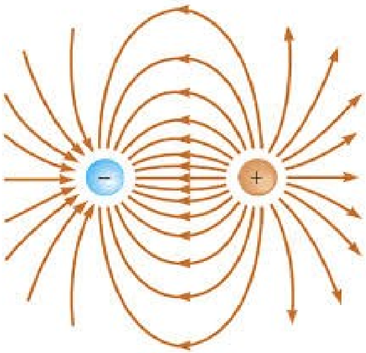
Powder

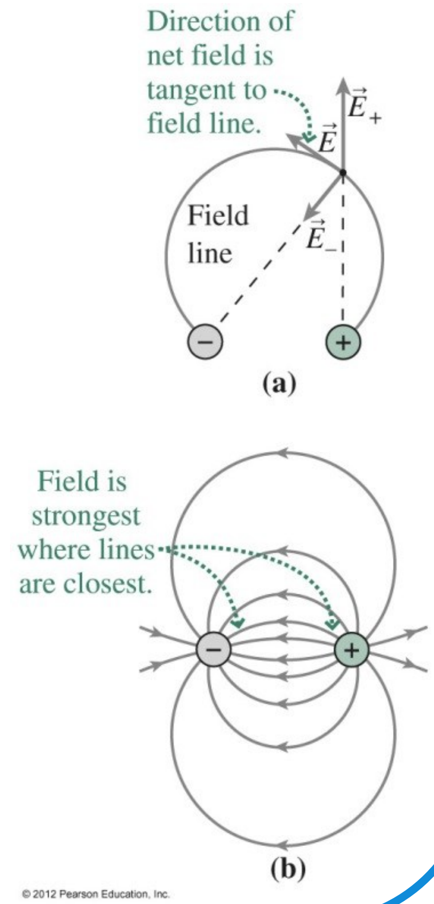


- The spacing of field lines describes the magnitude of the field; where lines are closer, the field is stronger.

Extremely intuitive!

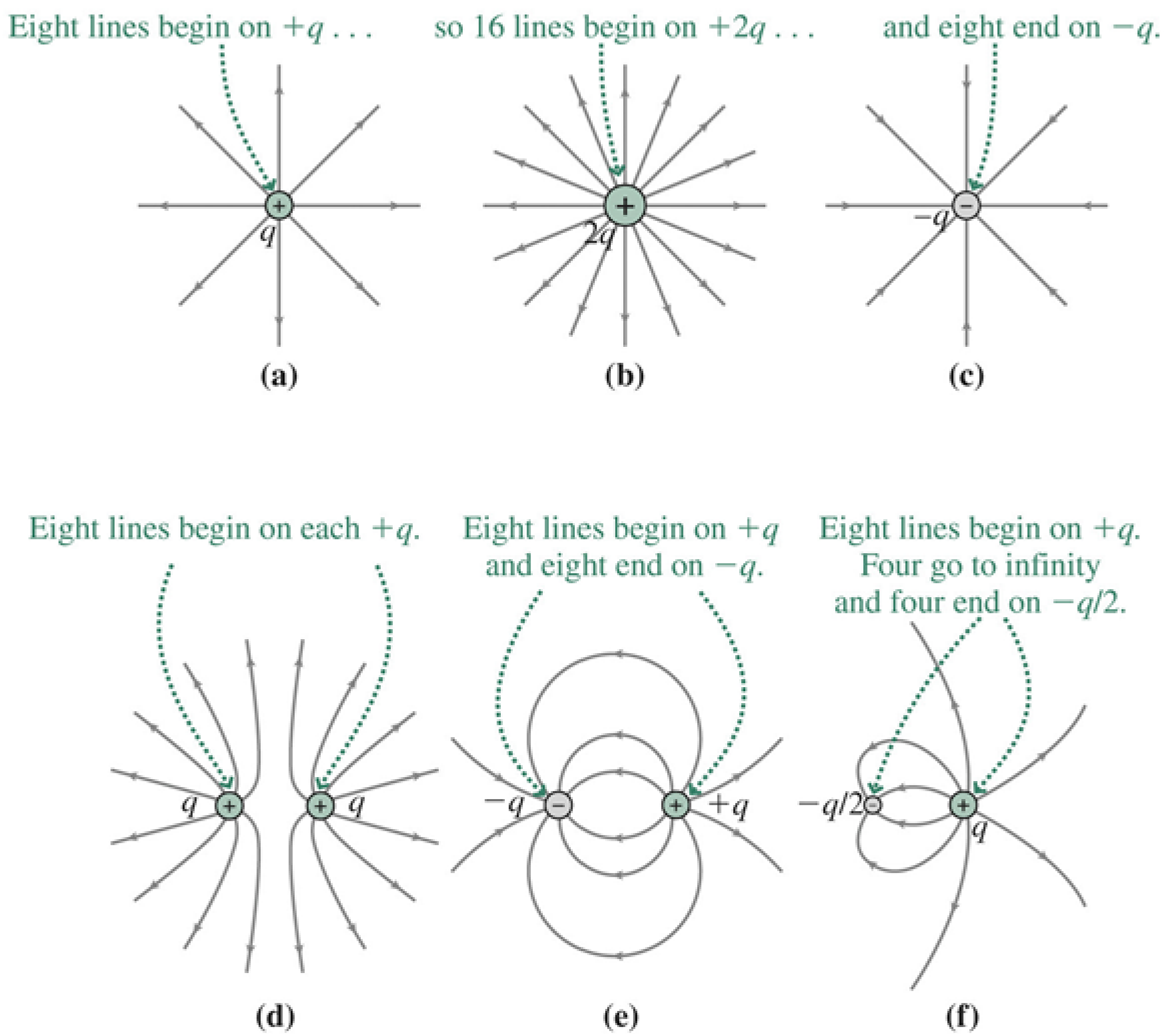
Tracing the field
of two sources of opposite sign





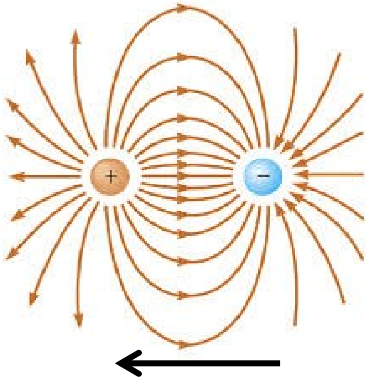
Electric Field Lines (2)

- In drawing field-line diagrams, we associate a certain finite number of field lines with a charge of a given **magnitude**.
- In the diagrams shown, 8 lines are associated with a charge of magnitude q.
- Note that field lines of static charge distributions always begin and end on charges, or extend to infinity.



The Dipole: an Important Charge Distribution

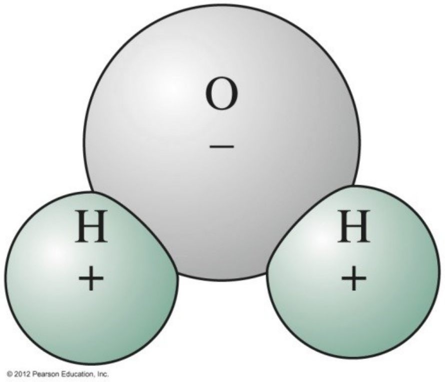
- An **electric dipole** consists of two point charges of equal magnitude but opposite signs, held a short distance apart.



d

The product of the charge and separation is the **dipole moment**: $\vec{p} = \vec{d}q$.

$$\vec{E}(\vec{r}) = \sum_{i=1}^{2} \frac{k_e q_i}{|\vec{r} - \vec{r}_i|^2} \frac{(\vec{r} - \vec{r}_i)}{|\vec{r} - \vec{r}_i|}$$

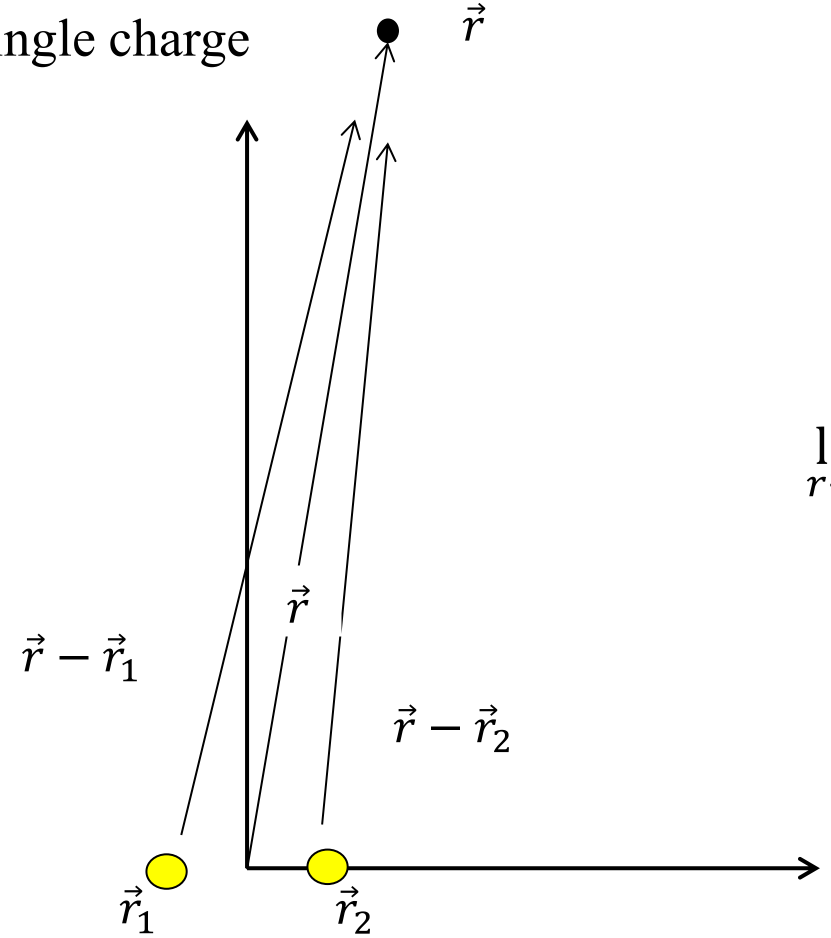


$\simeq$

– Many charge distributions, especially molecules, behave like electric dipoles.

The Dipole: Field at Infinity

The dipole at large distance generates much less field than a single charge



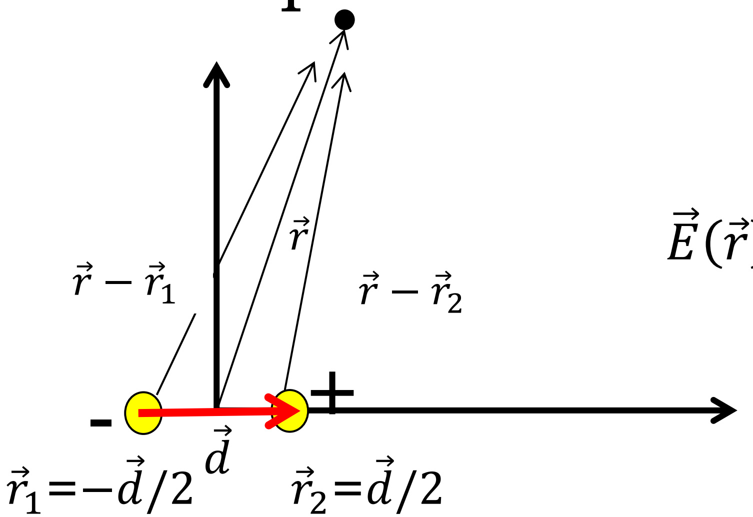
$$\vec{E}(\vec{r}) = \sum_{i=1}^{2} \frac{k_e q_i}{|\vec{r}-\vec{r}_i|^2} \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|}$$

$$\lim_{r\to\infty} \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|} \to \frac{\vec{r}}{|\vec{r}|} = \hat{r}$$

$$\lim_{r\to\infty} \vec{E}(\vec{r}) = \frac{k_e q_1}{r^2}\hat{r} + \frac{k_e q_2}{r^2}\hat{r} = 0$$

Because $q_2 = -q_1$

The Dipole: Field at Finite Large Distance



$$\vec{E}(\vec{r}) = \sum_{i=1}^{2} \frac{k_e q_i}{|\vec{r}-\vec{r}_i|^2} \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|}$$

$$\vec{E}(\vec{r}) = \sum_{i=1}^{2} \frac{k_e q_i}{|\vec{r}-\vec{r}_i|^3} (\vec{r}-\vec{r}_i)$$

$$\vec{E}(\vec{r}) = k_e q_1 \left(\frac{(\vec{r}-\vec{r}_1)}{|\vec{r}-\vec{r}_1|^3} - \frac{(\vec{r}-\vec{r}_2)}{|\vec{r}-\vec{r}_2|^3} \right) = k_e(-q) \left(\frac{(\vec{r}+\vec{d}/2)}{\left|\vec{r}+\vec{d}/2\right|^3} - \frac{(\vec{r}-\vec{d}/2)}{\left|\vec{r}-\vec{d}/2\right|^3} \right)$$

$$= k_e q \left(\frac{(\vec{r}-\vec{d}/2)}{\left|\vec{r}-\vec{d}/2\right|^3} - \frac{(\vec{r}+\vec{d}/2)}{\left|\vec{r}+\vec{d}/2\right|^3} \right)$$

The Dipole in intermediate distance (d<<r)

$$\frac{1}{|\vec{r}-\vec{d}/2|^3} = |\vec{r}-\vec{d}/2|^{-3} = [(\vec{r}-\vec{d}/2)\cdot(\vec{r}-\vec{d}/2)]^{-3/2}$$

$$= \left[\left(r^2 - \vec{r}\cdot\vec{d} + \frac{d^2}{4}\right)\right]^{-3/2}$$

$$= \left[r^2\left(1 - \frac{\vec{r}\cdot\vec{d}}{r^2} + \frac{d^2}{r^2 4}\right)\right]^{-3/2} = r^{-3}\left[\left(1 - \frac{\vec{r}\cdot\vec{d}}{r^2} + \frac{d^2}{r^2 4}\right)\right]^{-3/2}$$

$$\approx r^{-3}\left[\left(1 - \frac{\vec{r}\cdot\vec{d}}{r^2}\right)\right]^{-3/2}$$

$$\approx r^{-3}\left(1 + \frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2}\right)$$

Taylor/Maclaurin Series

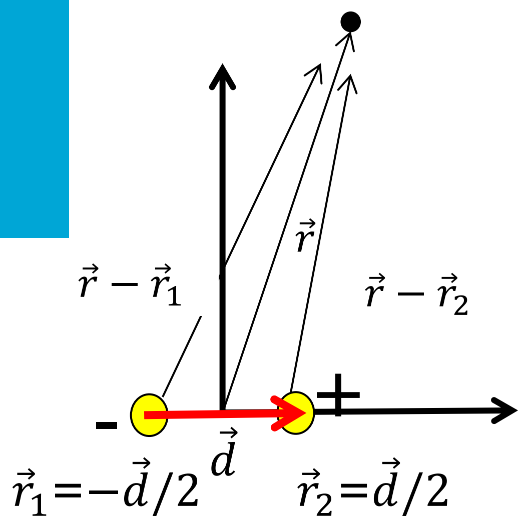
$$\left[\left(1 - \frac{\vec{r}\cdot\vec{d}}{r^2}\right)\right]^{-3/2} = \left(1 + \frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2} + \ \ldots.\right)$$

$$\frac{1}{|\vec{r}-\vec{d}/2|^3} \approx r^{-3}\left(1 + \frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2}\right)$$

similarly

$$\frac{1}{|\vec{r}+\vec{d}/2|^3} \approx r^{-3}\left(1 - \frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2}\right)$$

The Dipole: Field at Finite Large Distance



$$\vec{E}(\vec{r}) = k_e q\left(\frac{(\vec{r}-\vec{d}/2)}{|\vec{r}-\vec{d}/2|^3} - \frac{(\vec{r}+\vec{d}/2)}{|\vec{r}+\vec{d}/2|^3}\right)$$

$$\frac{1}{|\vec{r}-\vec{d}/2|^3} \approx r^{-3}\left(1+\frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2}\right)$$

$$\frac{1}{|\vec{r}+\vec{d}/2|^3} \approx r^{-3}\left(1-\frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2}\right)$$

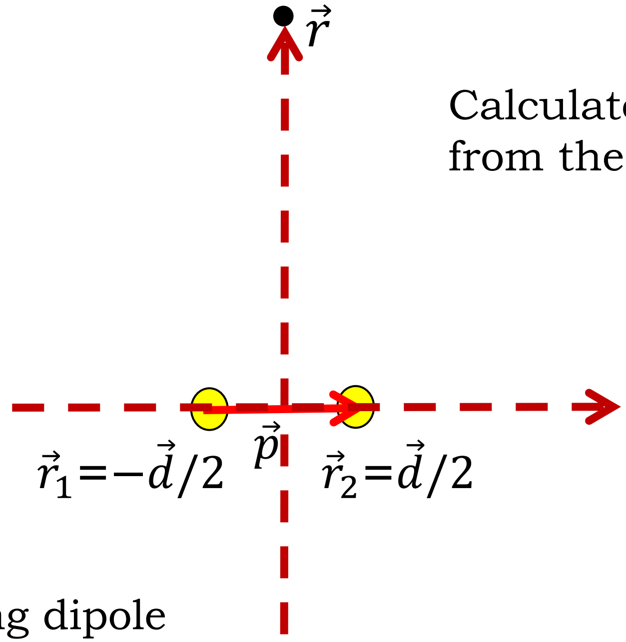
$$\vec{E}(\vec{r}) \approx \frac{k_e q}{r^3}\left(\left(1+\frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2}\right)(\vec{r}-\vec{d}/2) - \left(1-\frac{3\,\vec{r}\cdot\vec{d}}{2\,r^2}\right)(\vec{r}+\vec{d}/2)\right)$$

$$= \frac{k_e q}{r^3}\left((-\vec{d}) + \frac{3\,\vec{r}\cdot\vec{d}}{r^2}\vec{r}\right)$$

$$= \frac{k_e}{r^3}\left((-q\vec{d}) + \frac{3\,\vec{r}\cdot q\vec{d}}{r^2}\vec{r}\right)$$

$$\boxed{\vec{E}(\vec{r}) \approx \frac{k_e}{r^3}\left((-\vec{p}) + \frac{3\,\vec{r}\cdot\vec{p}}{r^2}\vec{r}\right) \qquad r \gg d}$$

Exercise



Calculate electric field at large distance from the dipole in the two scanning lines

$$\vec{E}(\vec{r}) \approx \frac{k_e}{r^3}\left((-\vec{p}) + \frac{3\,\vec{r}\cdot\vec{p}}{r^2}\vec{r}\right) \qquad r \gg d$$

Along dipole

$$\vec{E}(\vec{r}) \approx \frac{k_e}{r^3}\left((-\vec{p}) + \frac{3r\hat{r}\cdot\vec{p}}{r^2}r\hat{r}\right)$$

$$\vec{E}(\vec{r}) \approx \frac{k_e}{r^3}\left((-\vec{p}) + 3\hat{r}\cdot\vec{p}\hat{r}\right) =$$
$$\frac{k_e}{r^3}\left((-\vec{p}) + 3p\hat{r}\right) = \frac{k_e}{r^3}\left((-p\hat{p}) + 3p\hat{p}\right) = \frac{k_e}{r^3}2\vec{p}$$

$$\vec{E}(\vec{r}) \approx = \frac{k_e}{r^3}2\vec{p}$$

Orthogonal to the dipole

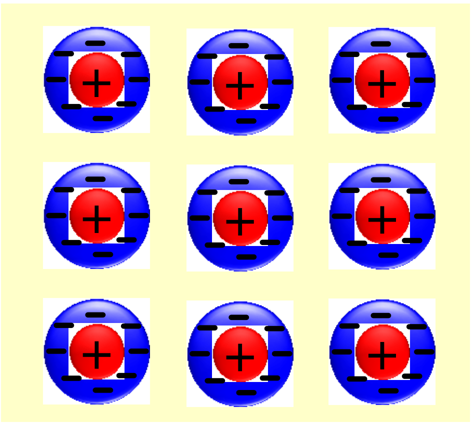
$$\vec{E}(\vec{r}) \approx -\frac{k_e}{r^3}\vec{p}$$

Book page 340, formulas 20.6a, 20.6b

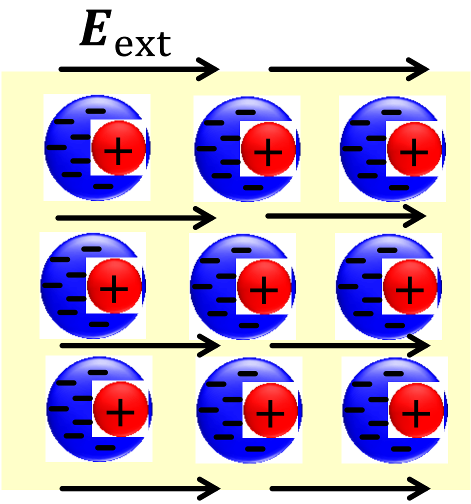
Dielectrics

A dielectric is composed by atoms with electrons strongly bound to them

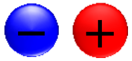
In absence of an E-field, the material is electrically neutral



An external electric field cannot move electrons like in a conductor but can distort them: polarization

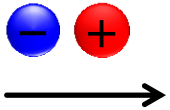


Equivalent to dipoles



Dipoles align themselves with the field

The dipole produces a small electric field that opposes the external field: polarization field $\boldsymbol{E}_p$

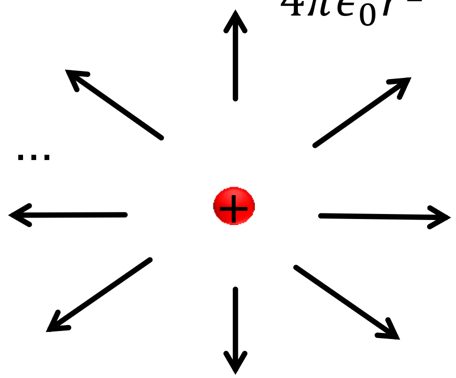


$$\boldsymbol{E}_{tot} = \boldsymbol{E}_{ext} + \boldsymbol{E}_p$$

Charges embedded in dielectrics

$$\vec{E}_{ext}(\vec{r},q) = \frac{q}{4\pi\epsilon_0 r^2}\hat{r}$$

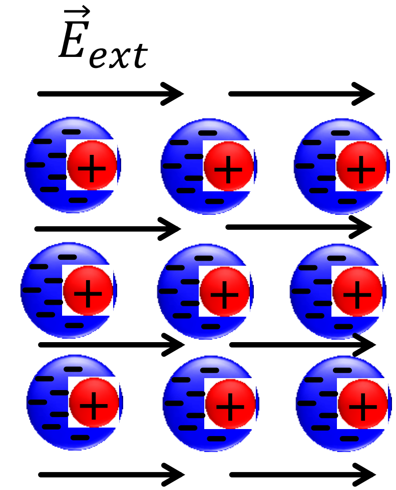
Let us look at …



$r \rightarrow large$

$$\vec{E}_{ext}(\vec{r},q) \sim E_{ext}(q)\hat{x}$$

$$E_{ext}(q) = Cost\frac{1}{\epsilon_0}$$



The dipoles radiate the polarization field

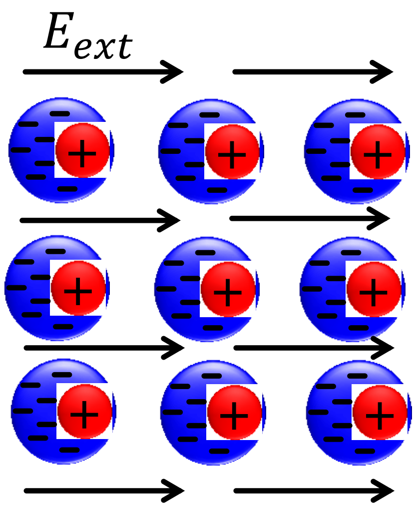
$$\vec{E}_{tot} = E_{ext}\hat{x} + \vec{E}_p(\vec{r})$$

Total field changes dramatically as function of location and has crazy orientation (polarization)

Instead of looking at variable real field we look at average field in the dielectric. This turn out to be parallel to $\hat{x}$ as well and essentially constant

$$\vec{E}_{tot}^{ave} = E_{ext}\hat{x} + E_p^{ave}\hat{x}$$

Charges embedded in dielectrics



$$\vec{E}_{tot}^{ave} = E_{tot}^{ave}\hat{x} = \left(E_{ext} + E_p^{ave}\right)\hat{x}$$

$$E_{ext}(q) = \frac{Cost}{\epsilon_0}$$

$$E_{tot}^{ave} = E_{ext}\left(1 + \frac{E_p^{ave}}{E_{ext}}\right) = \frac{Cost}{\epsilon_0}\left(\frac{E_{ext} + E_p^{ave}}{E_{ext}}\right)$$

$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_p^{ave}}$$

$$E_{tot}^{ave} = \frac{Cost}{\epsilon_0 \epsilon_r}$$

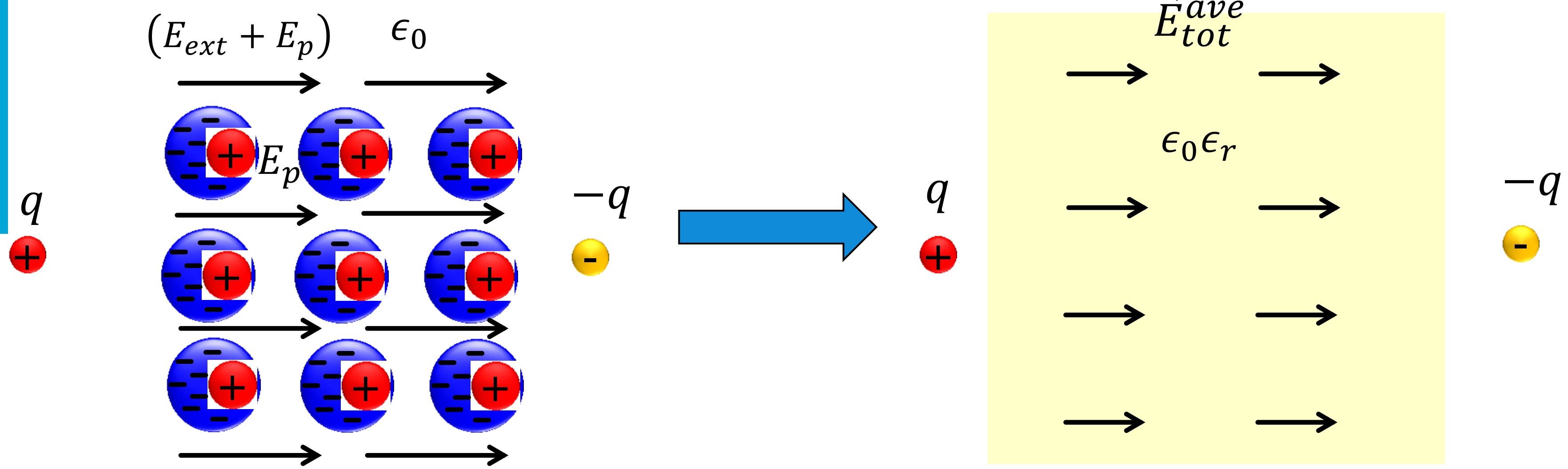
So, if we are not interested in the real fields but only in average fields in a dielectric material we can just assume that Coulomb's law for dielectrics is different:

$$\vec{E}_{tot}^{ave}(\vec{r}, q) = \frac{q}{4\pi(\epsilon_0\epsilon_r)r^2}\hat{r}$$

Materials are then characterized once one knows the dielectric constant $\epsilon_0\epsilon_r$ or the relative dielectric constant ϵ_r

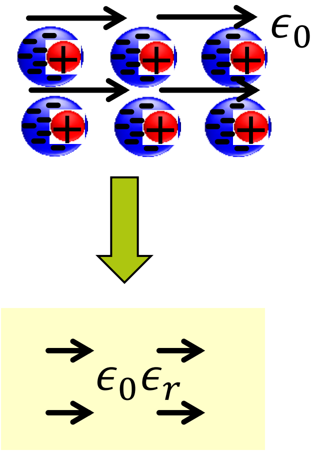
$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_p^{ave}} > 1$$

Homogenization



$$\vec{E}_{tot}^{ave}(\vec{r}, q) = \frac{q}{4\pi(\epsilon_0\epsilon_r)r^2}\hat{r}$$

Dielectric Constant



$$\vec{E}_{tot}^{ave} = \vec{E}_{ext} + \vec{E}_{p}^{ave}$$

If $\vec{E}_{p}^{ave}$ is parallel to $\vec{E}_{ext}$

It makes sense to define a dielectric constant

$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_{p}^{ave}}$$

If material

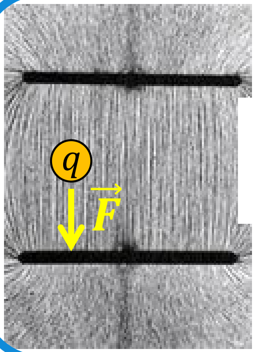
- is uniform in all space
- responds linearly
- and responds uniformly in all directions (isotropy)

It makes sense to associate a dielectric constant to the medium

<u>Warning:</u> In many practical applications one cannot simply apply the dielectric constant concept. However the deviations are too many, too different and also simple to understand when you need them

Truly Important

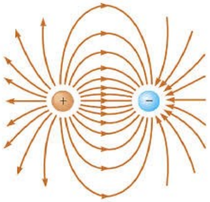
1)



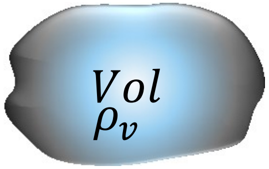
$$\vec{E}(\vec{r}) \equiv \frac{\vec{F}(\vec{r})}{q}$$

2)

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{k_e q_i}{|\vec{r}-\vec{r}_i|^2} \frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|}$$

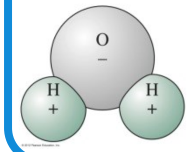


3)



$$\vec{E}(\vec{r}) = \iiint_{Vol} \frac{k_e \rho_v(\vec{r}')}{|\vec{r}-\vec{r}'|^2} \frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|} d\vec{r}'$$

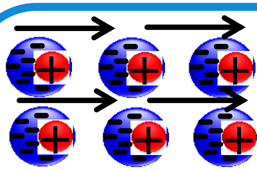
4)



$\simeq$

$$\lim_{r\to large} \vec{E}(\vec{r}) \approx \frac{k_e}{r^3}\left((-\vec{p}) + \frac{3r\hat{r}\cdot\vec{p}}{r^2} r\hat{r}\right)$$

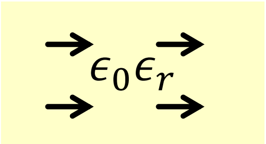
5)



ϵ_0

$\boldsymbol{E}_{\text{ext}}$

$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_p^{ave}}$$



$\boldsymbol{E}_{\text{tot}}^{\boldsymbol{ave}}$