Graded homework 1

May 2, 2016

Notes:

Your assignment must be handed as hardcopy, *at the beginning* of the lecture starting at 10:45 on Monday, May 9, 2016. In the case when you are not able to attend the lecture, you are allowed to submit a *good quality* scanned copy of your solution by e-mail, to the address i.e.lager@tudelft.nl.

Solutions submitted after Monday, May 9, 10:45 will not be considered!

Each solution must be handed in on a separate page. Please indicate on each page your name, your study number and the exercise number.

Please indicate in all cases the relevant measure units.

Exercise 1 – 4 points

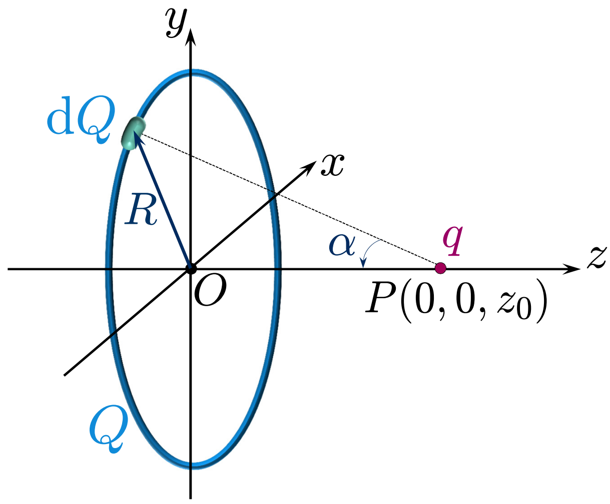

Figure 1: The charged ring – point charge configuration.

Let the configuration in Fig. 1, consisting of:

- a ring of radius R and negligible cross-section, centred at the origin and located in the $z = 0$ plane; it carries a total charge $Q = 1\,\mathrm{mC}$ *uniformly* distributed along the ring;
- a point charge $q = 1\,\mu\mathrm{C}$, located along the z-axis, at $z_0 = 0.1\,\mathrm{m}$.

The configuration is considered in free space.

a) Give an expression for the elementary charge $\mathrm{d}Q$ along an arc of infinitesimal length (see figure) and derive the expression of the *magnitude* of the elementary force $\mathrm{d}F$ exerted by the ring on the charge q. Infer from the figure's symmetry the orientation of the *total* force $\vec{F}$. (2 points)

Solution

The elementary along the arc of infinitesimal length in Fig. 1 amounts to

$$\mathrm{d}Q = \lambda \mathrm{d}L = \frac{Q}{2\pi R}\,\mathrm{d}L = \frac{Q}{2\pi}\,\mathrm{d}\varphi \tag{1}$$

with λ denoting the *constant* linear charge density along the ring and φ being the standard (azimuth) angle in the xOy plane, measured from the positive Ox axis. The elementary Coulomb force exerted by the ring at point z_0 then follows as

$$\mathrm{d}F(z_0) = k\frac{q\,\mathrm{d}Q}{|\vec{r}-\vec{r}'|^2}\frac{\vec{r}-\vec{r}'}{|\vec{r}-\vec{r}'|} \tag{2}$$

in which $\vec{r}'$ is the position vector corresponding to the elementary charge $\mathrm{d}Q$ and $k = 9\cdot 10^9\,\mathrm{Nm^2/C^2}$.

In view of the symmetry of the charge distribution, it can be directly inferred that $F_x = 0$ and $F_y = 0$, while F_z will be non-zero. Moreover, since $Q > 0 \Longrightarrow F_z(z_0) > 0$.

---

b) Determine the radius R such $|\vec{F}| = 1\,\mathrm{N}$. (2 points)

Solution

From the previous point, the only non-zero force component is F_z. By observing that $F_z = F\cos(\alpha)$, it can be written that

$$\mathrm{d}F_z(z_0) = k\,\frac{q\,\mathrm{d}Q}{|\vec{r}-\vec{r}'|^2}\cos(\alpha) = k\,\frac{q\,Q\,z_0}{2\pi\,(z_0^2+R^2)^{3/2}}\,\mathrm{d}\varphi. \tag{3}$$

Consequently,

$$F_z(z_0) = k\frac{q\,Q\,z_0}{2\pi\,(z_0^2+R^2)^{3/2}}\int_{\varphi=0}^{2\pi}\mathrm{d}\varphi = k\frac{q\,Q\,z_0}{(z_0^2+R^2)^{3/2}} \tag{4}$$

implying that

$$R^2 = \left(k\frac{q\,Q\,z_0}{|\vec{F}|}\right)^{2/3} - z_0^2 \tag{5}$$

and, hence

$$R = \sqrt{\left(k\frac{q\,Q\,z_0}{|\vec{F}|}\right)^{2/3} - z_0^2} = 0.96\,(\mathrm{m}). \tag{6}$$

Exercise 2 – 2 points

Let the case of an electric field in free space having the expression

$$\vec{E}(\vec{r}) = E_0 \frac{1}{r^4} \cos(\vartheta)\, \vec{r}, \text{ for } r > 0$$

in which ϑ is the standard (elevation) angular coordinate in a spherical reference frame (see Fig. 2).

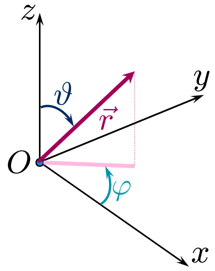

Figure 2: Spherical reference frame.

a) Determine the flux of the relevant electric field $\vec{E}(\vec{r})$ through a spherical surface of arbitrary radius $R > 0$, centred at the origin. (1 point)

Solution

We firstly observe that, for the configuration at hand, the outwardly oriented normal to the sphere of radius R is $\hat{n} = \vec{r}/|\vec{r}|$, with $\vec{r}$ being the position vector of a point on the sphere and $|\vec{r}| = R$. Consequently, from the flux's definition it follows that

$$\begin{aligned}\Phi &= \oint\!\!\oint_{\mathcal{S}} \vec{E} \cdot \hat{n}\, \mathrm{d}A = \int_{\vartheta=0}^{\pi} \int_{\varphi=0}^{2\pi} E_0 \frac{1}{R^4} \cos(\vartheta) \hat{r} \cdot \hat{r} \left[R^2 \sin(\vartheta) \mathrm{d}\vartheta \mathrm{d}\varphi\right] \\ &= E_0 \frac{2\pi}{R^2} \int_{\vartheta=0}^{\pi} \sin(\vartheta) \mathrm{d}\vartheta = -E_0 \frac{2\pi}{R^2} \cos(\vartheta)\Big|_{\vartheta=0}^{\vartheta=\pi} = 0\,(\mathrm{V/m}). \end{aligned} \tag{1}$$

b) Determine the total charge enclosed by the sphere. (1 point)

Solution

From Gauss's law, (1) immediately implies that $Q_{\text{enclosed}} = 0\,\mathrm{C}$.

Exercise 3 – 4 points

A thick spherical shell of inner radius R and outer radius $2R$ contains electric charge with *non-uniform* volume density

$$\rho_\mathrm{v}(r) = \rho_{\mathrm{v},0}\left(\frac{r}{R} - 1\right) \quad \text{for } R \leqslant r \leqslant 2R.$$

The shell is centred at the origin and is located in free space.

a) Determine the expression of the electric field strength $\vec{E}(\vec{r})$ at arbitrary locations. (3 points)

Hint: Pay attention to the symmetry of the configuration that may allow an alternative to direct integration.

Solution

The configuration has spherical symmetry and the evaluation of $\vec{E}(\vec{r})$ can be done by applying Gauss's law on spherical Gaussian surfaces of radius r and centred at the origin.

I. For $0 \leqslant r < R$:

In this region, $Q_\mathrm{enclosed} = 0 \Rightarrow \vec{E}(\vec{r}) = \vec{0}$.

II. For $R \leqslant r < 2R$:

In this region

$$\begin{aligned}
Q_\mathrm{enclosed}(r) &= \rho_{\mathrm{v},0}\int_{\varphi=0}^{2\pi}\mathrm{d}\varphi\int_{\vartheta=0}^{\pi}\sin(\vartheta)\mathrm{d}\theta\int_{\rho=R}^{r}\left(\frac{\rho}{R}-1\right)\rho^2\,\mathrm{d}\rho \\
&= 4\pi R^3\rho_{\mathrm{v},0}\int_{\rho=R}^{r}\left(\frac{\rho}{R}-1\right)\left(\frac{\rho}{R}\right)^2\mathrm{d}\left(\frac{\rho}{R}\right) \\
&= 4\pi R^3\rho_{\mathrm{v},0}\int_{u=1}^{r/R}(u-1)u^2\,\mathrm{d}u = 4\pi R^3\rho_{\mathrm{v},0}\left(u^4/4 - u^3/3\right)\Big|_{u=1}^{u=r/R} \\
&= 4\pi R^3\rho_{\mathrm{v},0}\left[\frac{1}{4}\left(\frac{r}{R}\right)^4 - \frac{1}{3}\left(\frac{r}{R}\right)^3 + \frac{1}{12}\right]\text{(C)}. \qquad (1)
\end{aligned}$$

By accounting for the applicable symmetry, the flux through the same surface amounts to

$$\Phi(r) = 4\pi r^2 E(r) \qquad (2)$$

and, hence

$$\begin{aligned}
\vec{E}(\vec{r}) &= \frac{Q_\mathrm{enclosed}(r)}{4\pi r^2\varepsilon_0}\frac{\vec{r}}{r} = \frac{\rho_{\mathrm{v},0}R}{\varepsilon_0}\left(\frac{R}{r}\right)^2\left[\frac{1}{4}\left(\frac{r}{R}\right)^4 - \frac{1}{3}\left(\frac{r}{R}\right)^3 + \frac{1}{12}\right]\frac{\vec{r}}{r} \\
&= \frac{\rho_{\mathrm{v},0}R}{\varepsilon_0}\left[\frac{1}{4}\left(\frac{r}{R}\right)^2 - \frac{1}{3}\left(\frac{r}{R}\right) + \frac{1}{12}\left(\frac{r}{R}\right)^{-2}\right]\frac{\vec{r}}{r}\,\text{(V/m)}. \qquad (3)
\end{aligned}$$

III. For $2R \leqslant r$:

In this region

$$\begin{aligned}
Q_\mathrm{enclosed}(r) &= 4\pi R^3\rho_{\mathrm{v},0}\int_{\rho=R}^{2R}\left(\frac{\rho}{R}-1\right)\left(\frac{\rho}{R}\right)^2\mathrm{d}\left(\frac{\rho}{R}\right) \\
&= 4\pi R^3\rho_{\mathrm{v},0}\left[\frac{1}{4}2^4 - \frac{1}{3}2^3 + \frac{1}{12}\right] = 4\pi R^3\rho_{\mathrm{v},0}\frac{17}{12}\,\text{(C)}. \qquad (4)
\end{aligned}$$

By using (2), it directly follows that

$$\vec{E}(\vec{r}) = \frac{Q_{\text{enclosed}}(r)}{4\pi r^2 \varepsilon_0} \frac{\vec{r}}{r} = \frac{\rho_{\text{v},0}}{\varepsilon_0} \frac{17}{12} \frac{R^3}{r^2} \frac{\vec{r}}{r} \, (\text{V/m}) \tag{5}$$

---

b) Show that the determined expression satisfies the continuity of $\vec{E}(\vec{r})$ at the spherical surfaces $r = R$ and $r = 2R$ (spherical coordinates definitions). (1 point)

Solution

I. At $r = R$:

- from the solution for $0 \leqslant r < R$ $\Longrightarrow$ $\vec{E}(\vec{r}) = \vec{0}\,(\text{V/m})$;
- from the solution for $R \leqslant r < 2R$, (3) $\Longrightarrow$ $\vec{E}(\vec{r}) = \vec{0}\,(\text{V/m})$.

The continuity of $\vec{E}(\vec{r})$ is thus satisfied.

II. At $r = 2R$:

- from the solution for $R \leqslant r < 2R$, (3) $\Longrightarrow$

$$\vec{E}(\vec{r}) = \frac{17\, \rho_{\text{v},0}\, R}{48\, \varepsilon_0} \frac{\vec{r}}{r} \, (\text{V/m}); \tag{6}$$

- from the solution for $2R \leqslant r$, (5) $\Longrightarrow$

$$\vec{E}(\vec{r}) = \frac{17\, \rho_{\text{v},0}\, R}{48\, \varepsilon_0} \frac{\vec{r}}{r} \, (\text{V/m}). \tag{7}$$

The continuity of $\vec{E}(\vec{r})$ is thus satisfied.

Note that the continuity of $\vec{E}(\vec{r})$ is a direct consequence of the fact there is no surface charge at the two spherical surfaces $\Longrightarrow$ the radial component of $\vec{E}(\vec{r})$ must be continuous (lecture notes) that, corroborated with the fact that the field has no tangential components $\Longrightarrow$ $\vec{E}(\vec{r})$ must be continuous.