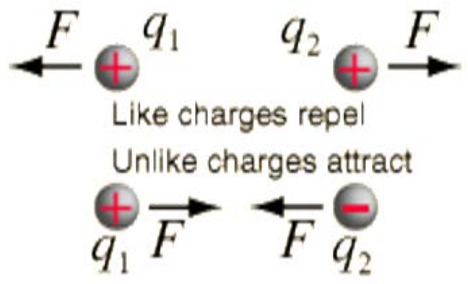
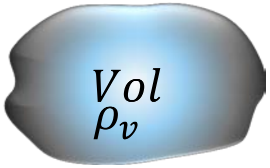
Truly Important from Lectures 1-2-3

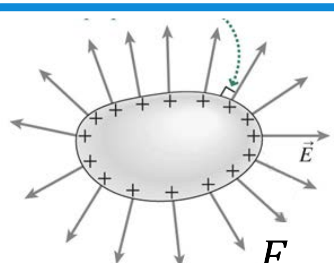
1)



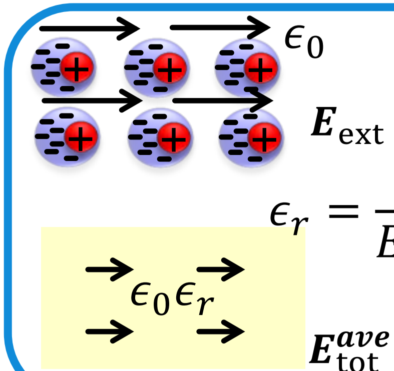
$$\vec{F}_{12}(\vec{r}, q_1, q_2) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{12}$$



$$\vec{E}(\vec{r}) = \frac{1}{4\pi\epsilon_0\epsilon_r}\iiint_{Vol}\frac{\rho_v(\vec{r}')}{|\vec{r}-\vec{r}'|^2}\frac{(\vec{r}-\vec{r}')}{|\vec{r}-\vec{r}'|}\,d\vec{r}'$$



$$E_n = \frac{\rho_s}{\epsilon_0\epsilon_r}$$



$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_p^{ave}}$$

$$\nabla \cdot \vec{E} = \frac{\rho_V}{\epsilon_0\epsilon_r}$$

Gauss Law

Electric Potential

Learning Objectives

- Understand the relation between forces, electric field and potential energy
- Learn how to calculate the potential
- Learn how to calculate the field from the potential

Topic 4

- Conservative and irrotational fields
- Electric Potential Energy Difference
- Potential of charge distributions
- Electric Field from the Potential

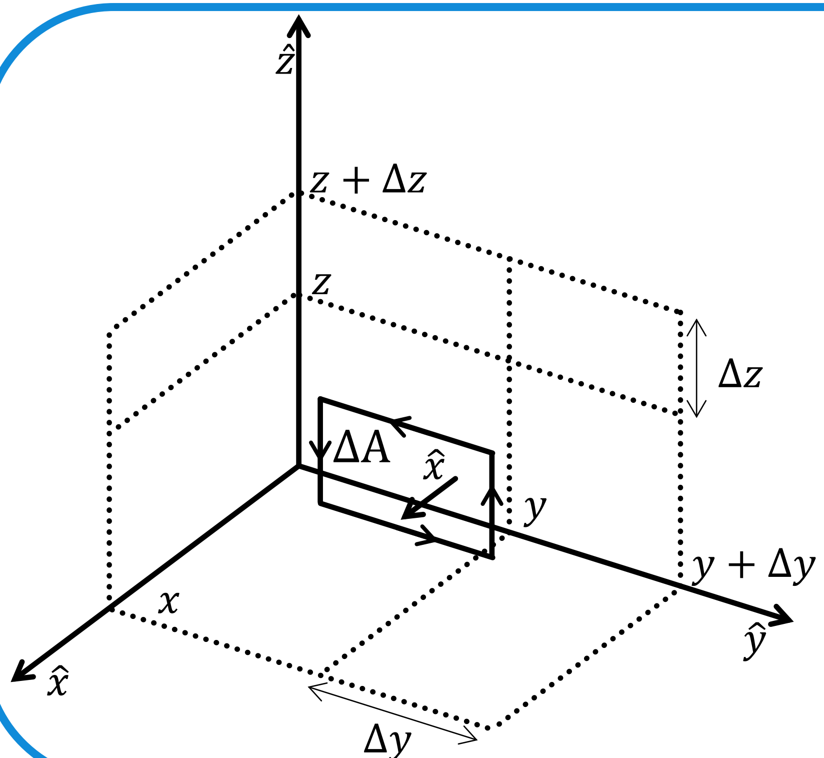
Line integrals

Line integral C of a vector field (e.g., the electric field) on a line L

$$C = \int_L \vec{F} \cdot d\vec{l}$$

$$\vec{F} \equiv (F_x, F_y, F_z)$$

$d\vec{l}$ incremental vector tangent to the line L



Line integral C of a vector field on **a closed line** L_x

$$C = \oint_{L_x} \vec{F} \cdot d\vec{l}$$

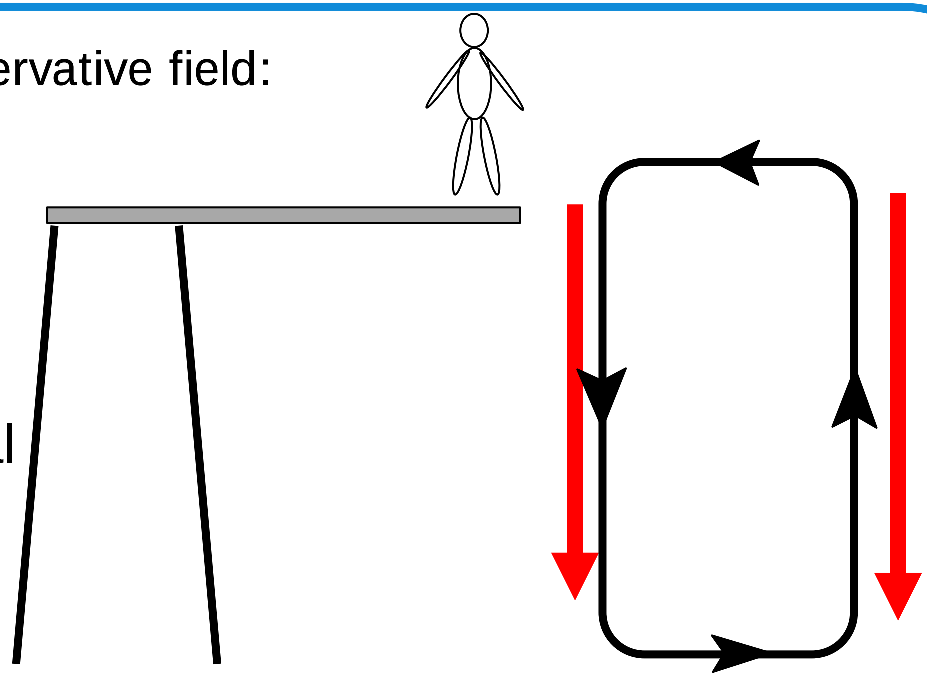
Conservative fields

Consider a field $\vec{G}(\vec{l})$

And the line integration of the field $\int_{line} \vec{G}(\vec{l}) \cdot d\vec{l}$

The field is conservative if the line integral over a closed loop is zero $\oint \vec{G}(\vec{l}) \cdot d\vec{l} = 0$

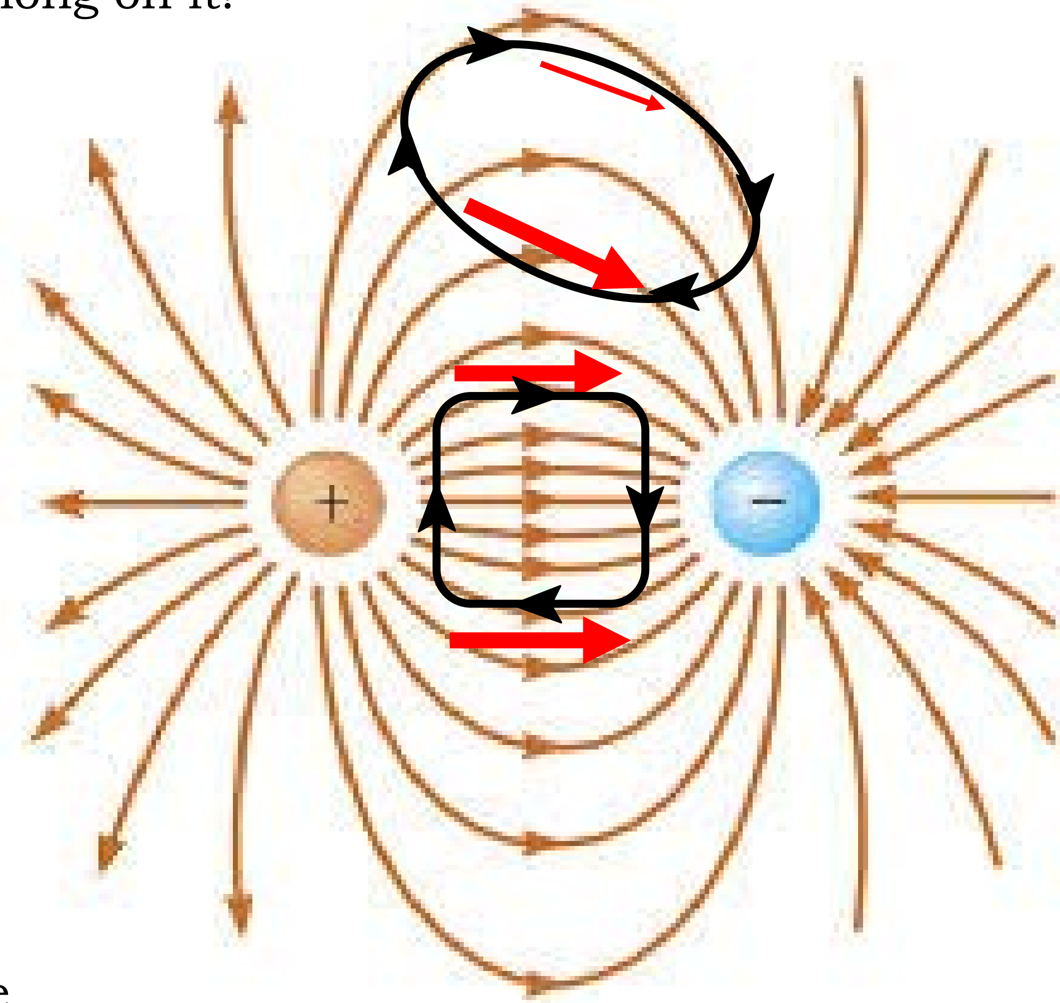
You have already seen a conservative field:
the gravitational field



Imagine any loop and try
to integrate the gravitational
field along it!
It's going to give zero

Static electric field is conservative

Imagine any loop and
try to integrate electric field along on it!



Electric field due to dipole

Since integral of electric field is null for any loop

$$\oint \quad \vec{E}(\vec{l}) \cdot d\vec{l} = 0$$

Static electric field is conservative

Potential Energy Difference

If a force $\vec{F}$ is conservative

Irrotational $\nabla \times \vec{F} = 0$

It makes sense to define a **potential energy difference** associated to this force $\vec{F}$

- In fact the work done by a force is its spatial integral

$$W = \int_{line} \vec{F}(l) \cdot d\vec{l}$$

- Definition: The **potential energy difference** associated to a conservative force is the work done against thus force (hence the minus).

$$\Delta U_{AB} = -\int_{line} \vec{F}(l) \cdot d\vec{l}$$

Electric Potential Energy Difference

Since the **electric field acting on a charge provides a force,** $\vec{F} = q\vec{E}$

and $\vec{E}$ is conservative, also $\vec{F}$
Is conservative

It makes sense to define an **Electric** potential energy difference

In fact the work done by the electric field is it's the spatial integral of the field times the charge

$$W_e = \int_A^B q\vec{E}(l) \cdot d\vec{l}$$

So by definition: The **Electric potential energy difference** is the work done **against ("-")** the electric field

$$\Delta U_{AB} = -q\int_A^B \vec{E}(l) \cdot d\vec{l}$$

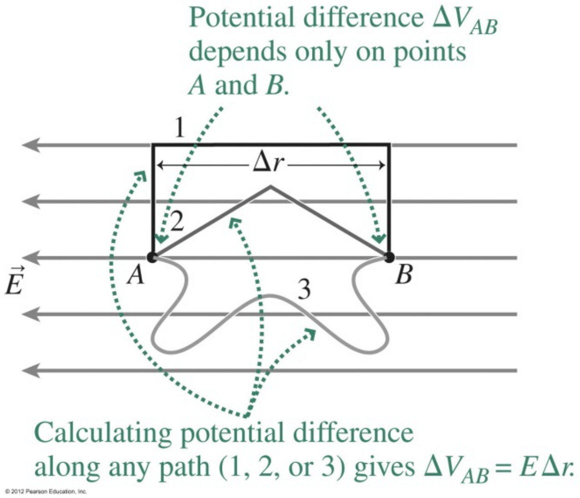
We will stop specifying "electric"

Potential Energy Difference vs. Potential Difference

$$\Delta U_{AB} = -\int_A^B q\vec{E}(l)\cdot d\vec{l}$$ Potential ~~Energy~~ difference

$$\Delta V_{AB} = \frac{\Delta U_{AB}}{q} = -\int_A^B \vec{E}(l)\cdot d\vec{l}$$ Potential difference

Potential difference is a property *of two points*



The Unit of the Potential Difference

- The unit of (electric) potential difference is the **volt** (V).
  - 1 volt is 1 joule per coulomb (1 V = 1 J/C).

Example: A 9-V battery supplies 9 joules of energy to every **coulomb** of charge that passes through an external circuit connected between its two terminals.

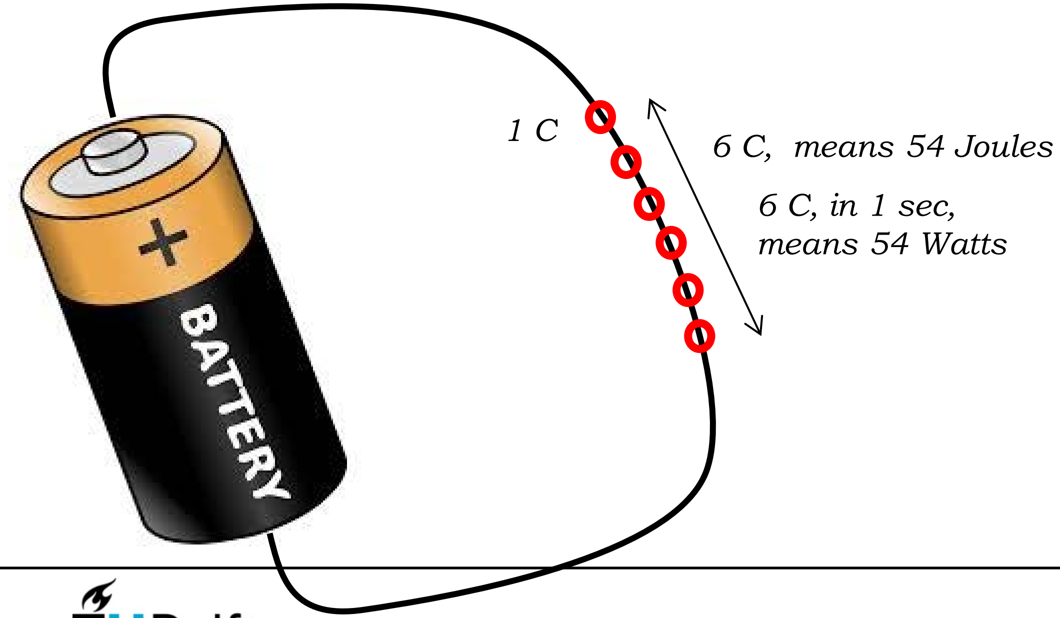
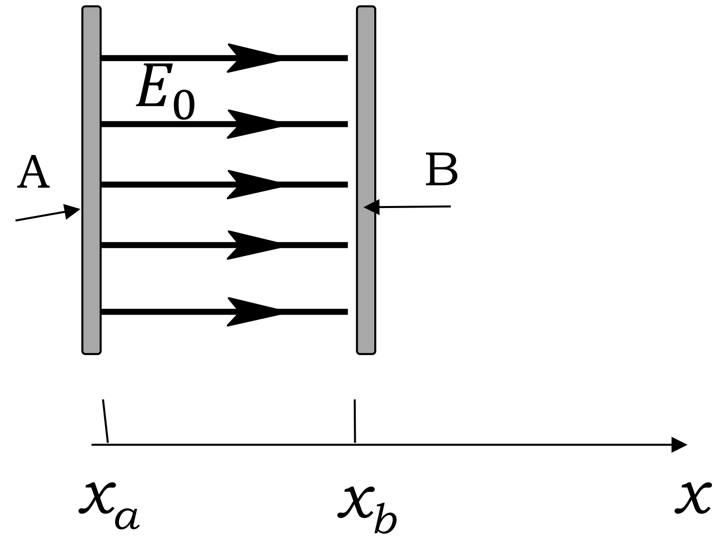


Table 22.2 Typical Potential Differences

Between human arm and leg due to heart's electrical activity	1 mV
Across biological cell membrane	80 mV
Between terminals of flashlight battery	1.5 V
Car battery	12 V
Electric outlet (depends on country)	100–240 V
Between long-distance electric transmission line and ground	365 kV
Between base of thunderstorm cloud and ground	100 MV

Potential Difference for a Constant Field

If the electric field is constant, E_0, for every position the **potential differences** between two points (A and B) in the field can be found simply



$$\Delta V_{AB} = -\int_A^B \vec{E}(l) \cdot d\vec{l}$$

$$\Delta V_{AB} = -\int_{x_a}^{x_b} E_0 \hat{x} \cdot \hat{x} dx$$

$$= -E_0 \int_{x_a}^{x_b} dx$$

$$= -E_0 (x_b - x_a)$$

Potential Difference for the field of a Point Charge

- If the electric field changes with position, the **potential differences** between two points in the field must be found by integration.

$$\Delta V_{AB} = -\int_{A}^{B} \vec{E}(l) \cdot d\vec{l}$$

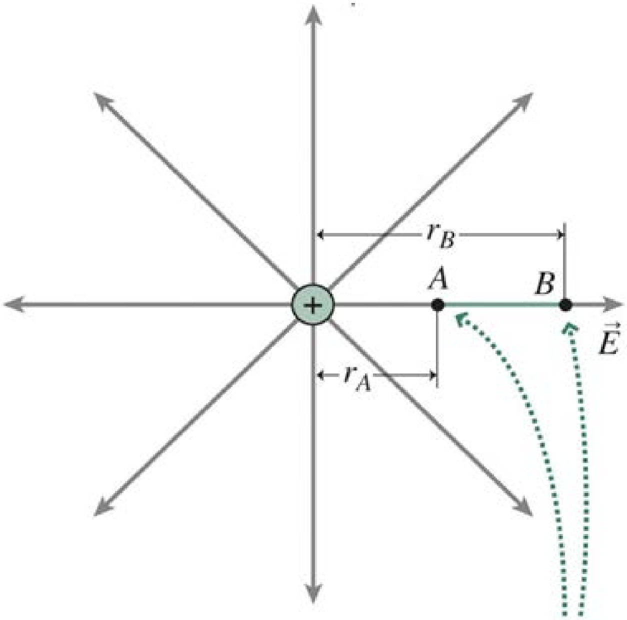
$$\Delta V_{AB} = -\int_{A}^{B} \frac{kq}{r^2}\hat{r} \cdot d\vec{r}$$

$$= -\int_{A}^{B} \frac{kq}{r^2}\hat{r} \cdot d\vec{r}$$

$$= -kq \int_{r_A}^{r_B} \frac{1}{r^2} dr$$

$$= kq \left.\frac{1}{r}\right]_{r_A}^{r_B}$$

$$= kq \left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$



If the electric field source is a **point charge** it generates a field

$$\vec{E}(\vec{r}) = \frac{kq}{r^2}\hat{r}$$

Potential for the field due to a Point Charge

- Since the **potential differences** between two points in the field is found by integration.

$$\Delta V_{AB} = -\int_A^B \frac{kq}{r^2}\hat{r} \cdot d\vec{r} = kq\left[\frac{1}{r_B} - \frac{1}{r_A}\right]$$

Taking the zero of potential at infinity gives

$$V_A \equiv \Delta V_{A\infty} = -\int_\infty^r \vec{E}(l) \cdot d\vec{l} = \frac{kq}{r}$$

By definition the "potential" of a point rather than the potential difference

This potential is a field,
it's a **simpler** expression
than the electric field!

Charge distributions

If there is a distribution of point charges, the potential can be found by summing point-charge potentials:

$$V(\vec{r}) = \sum_{i=1}^{N} \frac{kq_i}{|\vec{r} - \vec{r}_i|}$$

Notation in book

$$V = \sum_{i=1}^{N} \frac{kq_i}{r_i}$$

$$r_i = |\vec{r} - \vec{r}_i|$$

For a continuous charge distribution

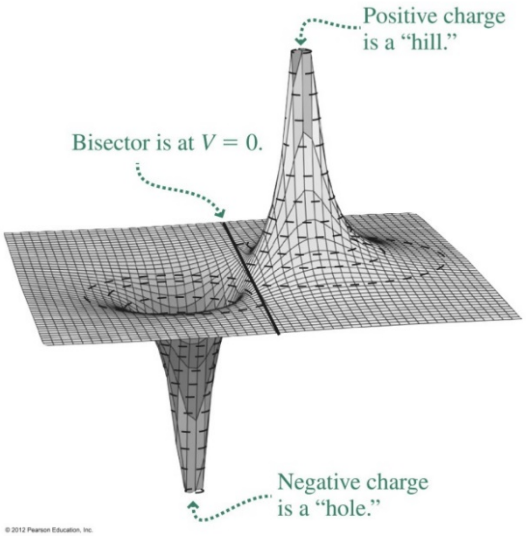
$$V(\vec{r}) = \iiint_{Volume} \frac{k\rho(\vec{r}')d\vec{r}'}{|\vec{r} - \vec{r}'|} = \int \frac{kdq}{r}$$

Potential of a dipole

$$V(\vec{r}) = \sum_{i=1}^{2} \frac{kq_i}{|\vec{r} - \vec{r}_i|} \qquad q_2 = -q_1$$

$$\Downarrow$$

$$V(\vec{r}) = \frac{kq}{|\vec{r} - \vec{r}_1|} - \frac{kq}{|\vec{r} - \vec{r}_2|}$$



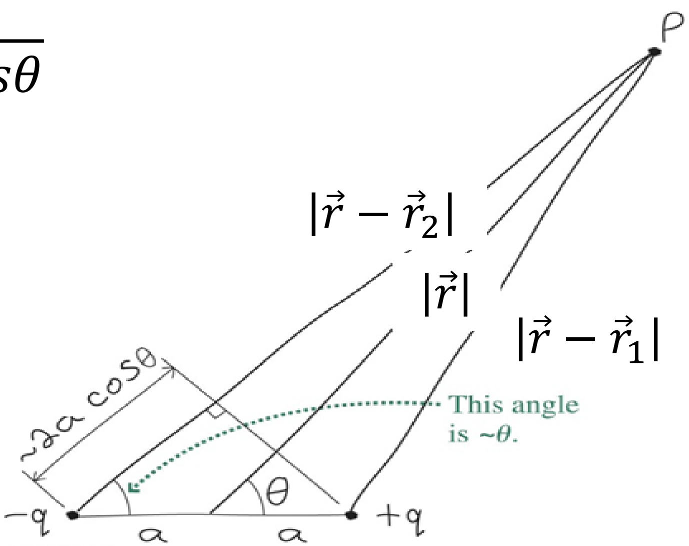
For observation points far away…..

$$V(\vec{r}) \sim \frac{kq}{|\vec{r} - \vec{r}_1|} - \frac{kq}{|\vec{r} - \vec{r}_1| + 2acos\theta}$$

$$V(\vec{r}) \sim \frac{kq\ (|\vec{r} - \vec{r}_1| + 2acos\theta - |}{|\vec{r} - \vec{r}_1|^2 + |\vec{r} - \vec{r}_1|2ac}$$

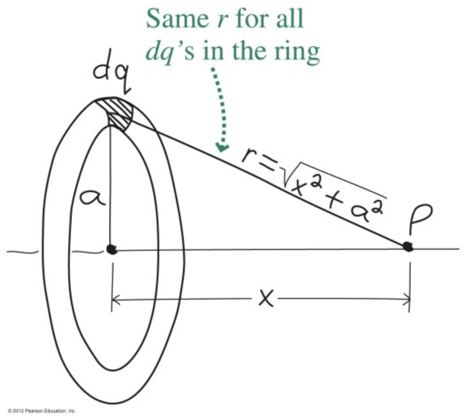
$$V(\vec{r}) \sim \frac{kq2acos\theta}{|\vec{r} - \vec{r}_1|^2} \sim \frac{kpcos\theta}{|\vec{r}|^2}$$

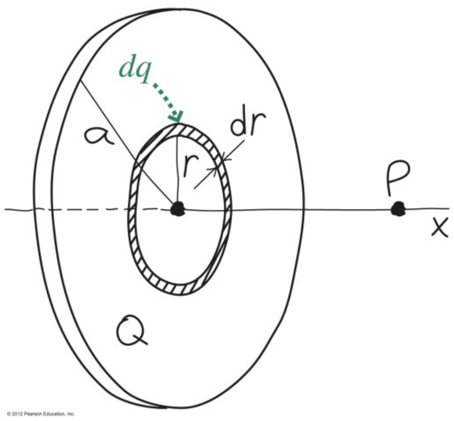
where $p = 2a\ q$ is
the ***dipole moment amplitude***



Continuous Distributions: A Ring and a Disk

- You will do these exercises on the next days





Nabla, Divergence and Gradient

You have seen
and understood

Divergence (scalar)

$$\frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z = \nabla \cdot \vec{F}$$

Nabla vector $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$

$\vec{F} = \left(F_x, F_y, F_z\right)$

Nabla vector

Can be multiplied with a scalar

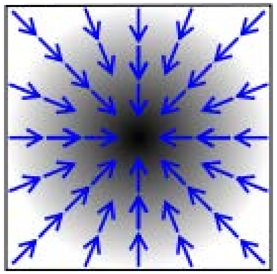
Gradient (vector)

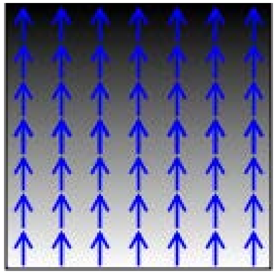
$$\frac{\partial S}{\partial x}\hat{x} + \frac{\partial S}{\partial y}\hat{y} + \frac{\partial S}{\partial z}\hat{z} = \nabla S$$

$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$ $\quad S$

Nabla vector

Darkness indicates S **lines** indicate the **Gradient**





- **vector** indicates direction of the greatest variation
- the amplitude quantifies it

Potential and the Electric Field

Potential difference involves an integral over the electric field.
$$V(B) = -\int_{\infty}^{B} \vec{E}(l) \cdot \hat{l}\, dl$$

Specifically, the component of the electric field in a given direction is the negative of the rate of change (the derivative) of potential in that direction.

$$\left\{ \begin{aligned} dV &= -\vec{E}(l) \cdot \hat{l} dl \\ \frac{dV}{dl} &= -\vec{E}(l) \cdot \hat{l} \end{aligned} \right.$$

$$\frac{dV}{dx} = -E_x; \frac{dV}{dy} = -E_y; \frac{dV}{dz} = -E_z$$

Applied to Cartesian coordinates

$$\vec{E} = -\frac{dV}{dx}\hat{x} - \frac{dV}{dy}\hat{y} - \frac{dV}{dz}\hat{z}$$

Which means

- This approach may be used to find the field from the potential.
- Potential is **easier** to calculate, since it's a **scalar rather than a vector**.

$$\boxed{\vec{E} = -\nabla V}$$

Vector algebra form is that the electric field is the **gradient** of the potential

Equipotentials

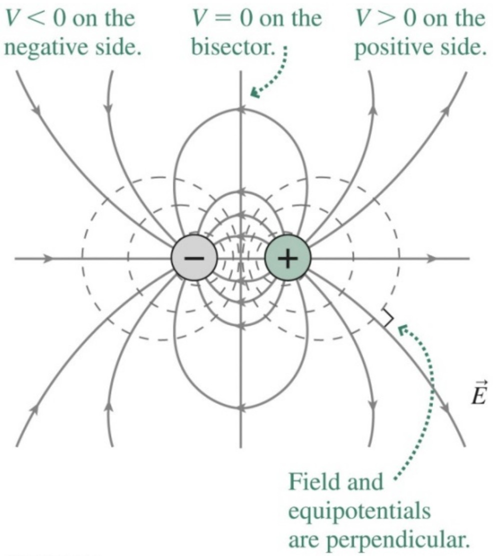
An **equipotential** is a surface on which the potential is constant.

The electric field is oriented in the direction of maximum variation of V

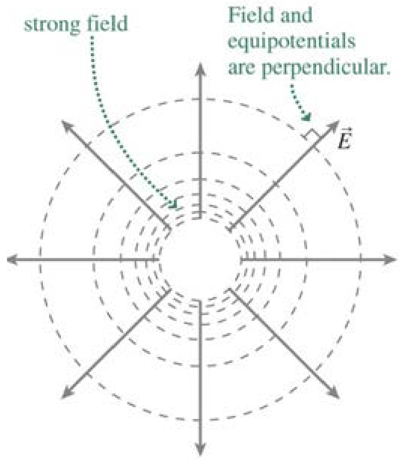
$$\vec{E} = -\nabla V$$

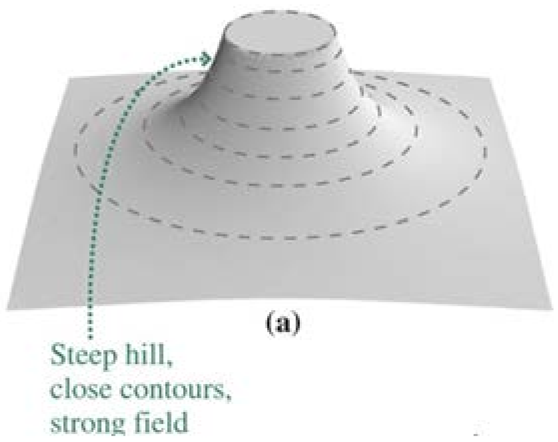
In the direction orthogonal to where the field is oriented the field is zero, so potential is constant

The electric field is perpendicular to the equipotentials.



Equipotentials for a dipole:





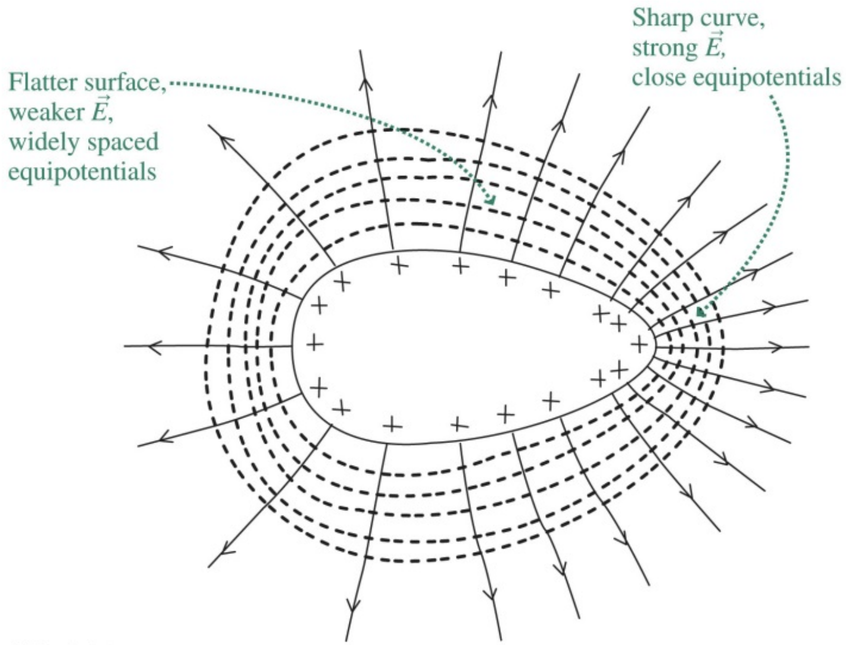
Equipotentials for a charge:

Charged Conductors

- The electric field is orthogonal to at the boundaries of a conductor (last lecture)
- So the surface of a conductor in electrostatic equilibrium is an equipotential.
- Actually the entire conductor is equipotential because there is no field inside

Since $$\vec{E} = -\nabla V$$

the electric field stronger and the charge density higher, where the conductor curves more sharply.



Truly Important from Lecture 3

1) Electric Field is Conservative

$$\nabla \times \vec{E}(\vec{r}) = 0$$

$$\Updownarrow$$

$$\oint \vec{E}(\vec{l}) \cdot d\vec{l} = 0$$

2) Potential Energy difference

$$\Delta U_{AB} = -\int_A^B q\vec{E}(l) \cdot d\vec{l}$$

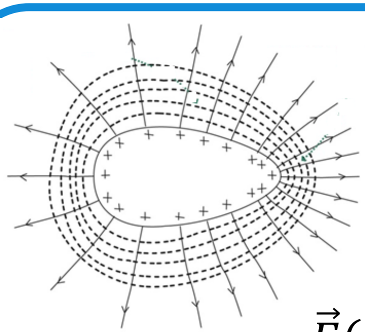
3) Potential difference

$$\Delta V_{AB} = -\int_A^B \vec{E}(l) \cdot d\vec{l}$$

4) Potential of a set of point charges

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0\epsilon_r}\sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

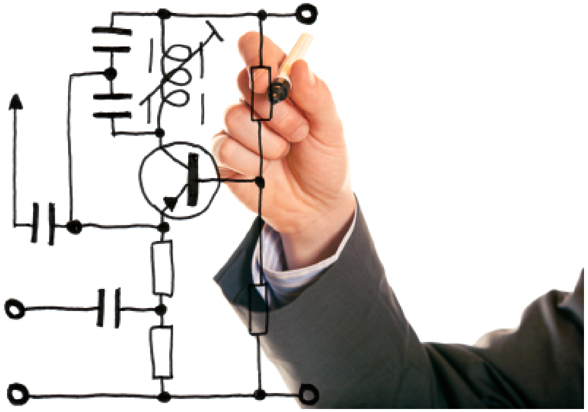
5)



$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

Voltages in circuits

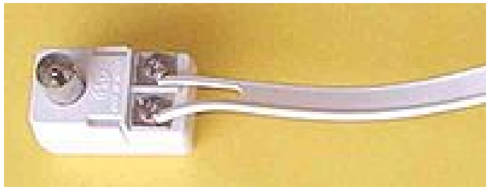
Circuit theory



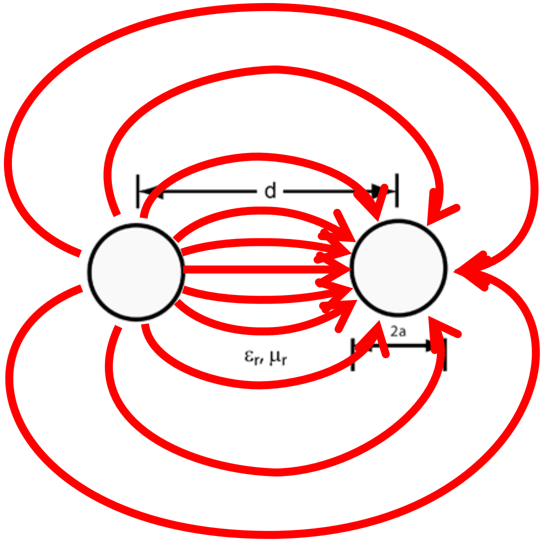
You have familiarized with voltages and currents. You should realize that **voltage is only a convenient representation for** Electric field

$$\vec{E} = -\nabla V$$

Field is a vector and would be more difficult to handle than voltage (scalar)



$\boldsymbol{E}(x,y)=V\,\boldsymbol{e}(x,y)$



***Nabla vector is your anchor of savatage*!**

Differential Axioms

$$\nabla \cdot \vec{E} = \frac{\rho_V}{\epsilon_0 \epsilon_r}$$

Super compact representation
For Coulomb's and Gauss Law
(third lecture)

$$\nabla \times \vec{E} = 0$$

Electric Field is Conservative
(now)

All there is to electrostatics

Conservative = Irrotational

$$\oint \vec{E}(\vec{l}) \cdot d\vec{l} = 0$$

Static electric field is conservative,
If this integral is zero for every loop

I showed you a couple of loops
and you believed me

Conservative property of a field can also be expressed for every observation point as

$$\nabla \times \vec{E} = 0$$

Curl of Static Electric field is zero
or, equivalently said,

Static electric field is irrotational

The curl statement is equivalent but more powerful.
Not only for any chosen path.

What is the Curl?

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \mathrm{F} = \left(F_x, F_y, F_z\right)$$

Mnemonic rule

$$\nabla \times \vec{F}(\vec{r}) = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix}$$

$$\nabla \times \vec{F}$$

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{x} - \left(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x\right)\hat{y} + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{z}$$

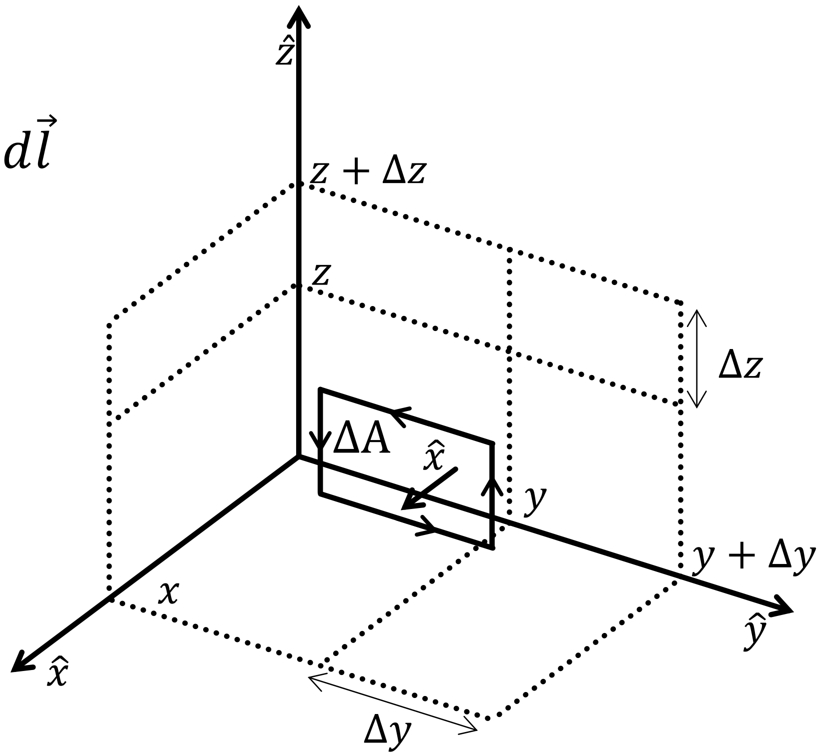
What does it mean ?

Fairly Criptic

Closed Line Integrals on Small Domain

ΔA small enough such that $\vec{F}$ components can be considered constant along each straight line path which delimits the surface A

$$C = \oint_{L_x} \vec{F} \cdot d\vec{l}$$



$$C = \int_y^{y+\Delta y} \vec{F}(z) \cdot \hat{y}\, dy$$

$$+ \int_z^{z+\Delta z} \vec{F}(y + \Delta y) \cdot \hat{z}\, dz$$

$$+ \int_y^{y+\Delta y} \vec{F}(z + \Delta z) \cdot (-\hat{y}) dy$$

$$+ \int_z^{z+\Delta z} \vec{F}(y) \cdot (-\hat{z}) dz$$

$$C = F_y(z)\Delta y + F_z(y + \Delta y)\Delta z - F_y(z + \Delta z)\Delta y - F_z(y)\Delta z$$

Closed Line Integrals in terms of derivatives

$$C = \oint_{L_x} \vec{F} \cdot d\vec{l}$$

$$C = \left[F_y(z) - F_y(z + \Delta z)\right]\Delta y + \left[F_z(y + \Delta y) - F_z(y)\right]\Delta z$$

$$C = \left[\frac{F_y(z) - F_y(z + \Delta z)}{\Delta z}\right]\Delta y \Delta z + \left[\frac{F_z(y + \Delta y) - F_z(y)}{\Delta y}\right]\Delta y \Delta z$$

$$C = \left[\frac{F_z(y + \Delta y) - F_z(y)}{\Delta y}\right]\Delta A - \left[\frac{F_y(z + \Delta z) - F_y(z)}{\Delta z}\right]\Delta A$$

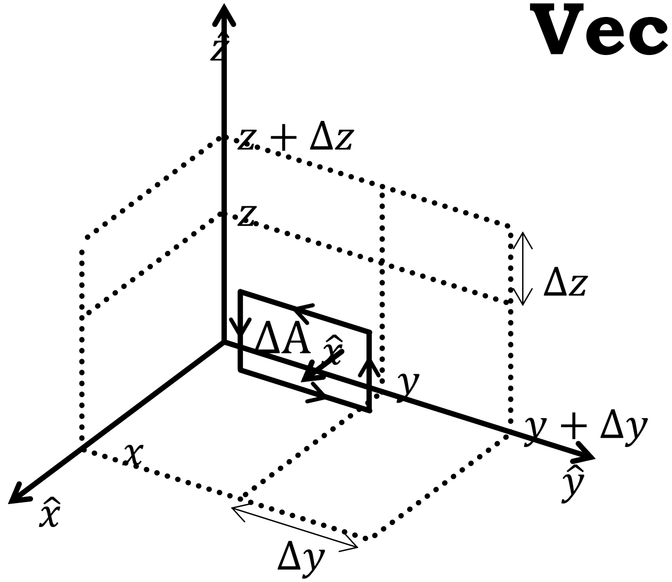
$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{L_x} \vec{F} \cdot d\vec{l} = \lim_{\Delta y \to 0} \left[\frac{F_z(y + \Delta y) - F_z(y)}{\Delta y}\right] - \lim_{\Delta z \to 0} \left[\frac{F_y(z + \Delta z) - F_y(z)}{\Delta z}\right]$$

From previous courses:
Derivative Definition

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h}\right]$$

$$\lim_{\Delta A \to 0} \frac{1}{\Delta A} \oint_{L_x} \vec{F} \cdot d\vec{l} = \left(\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y\right)$$

Vector identity

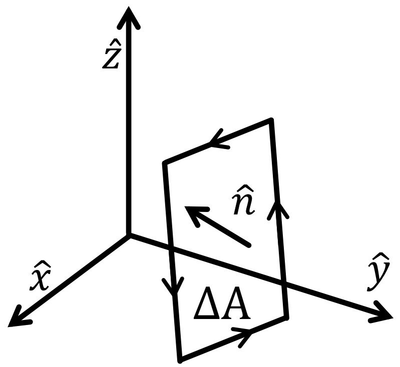


Multiplying LHS and RHS per $\hat{x}$
we still have an equation

$$\lim_{\Delta A \to 0} \frac{\hat{x}}{\Delta A} \oint_{L_x} \vec{F} \cdot d\vec{l} = \left(\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y \right) \hat{x}$$

The same steps can be done for paths contained in planes orthogonal to the unit vectors $\hat{y}$ and $\hat{z}$

Combining the three components, for a path contained in a plane orthogonal to an arbitrary oriented unit vector $\hat{n}$, one obtains



$$\lim_{\Delta A \to 0} \frac{\hat{n}}{\Delta A} \oint_L \vec{F} \cdot d\vec{l}$$

$$= \left(\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y \right) \hat{x} - \left(\frac{\partial}{\partial x} F_z - \frac{\partial}{\partial z} F_x \right) \hat{y} + \left(\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x \right) \hat{z}$$

We have already seen this operator

Recognizing a Vector Product

$$\lim_{\Delta A \to 0} \frac{\hat{n}}{\Delta A} \oint_L \vec{F} \cdot d\vec{l}$$

$$= \left(\frac{\partial}{\partial y} F_z - \frac{\partial}{\partial z} F_y\right)\hat{x} - \left(\frac{\partial}{\partial x} F_z - \frac{\partial}{\partial z} F_x\right)\hat{y} + \left(\frac{\partial}{\partial x} F_y - \frac{\partial}{\partial y} F_x\right)\hat{z}$$

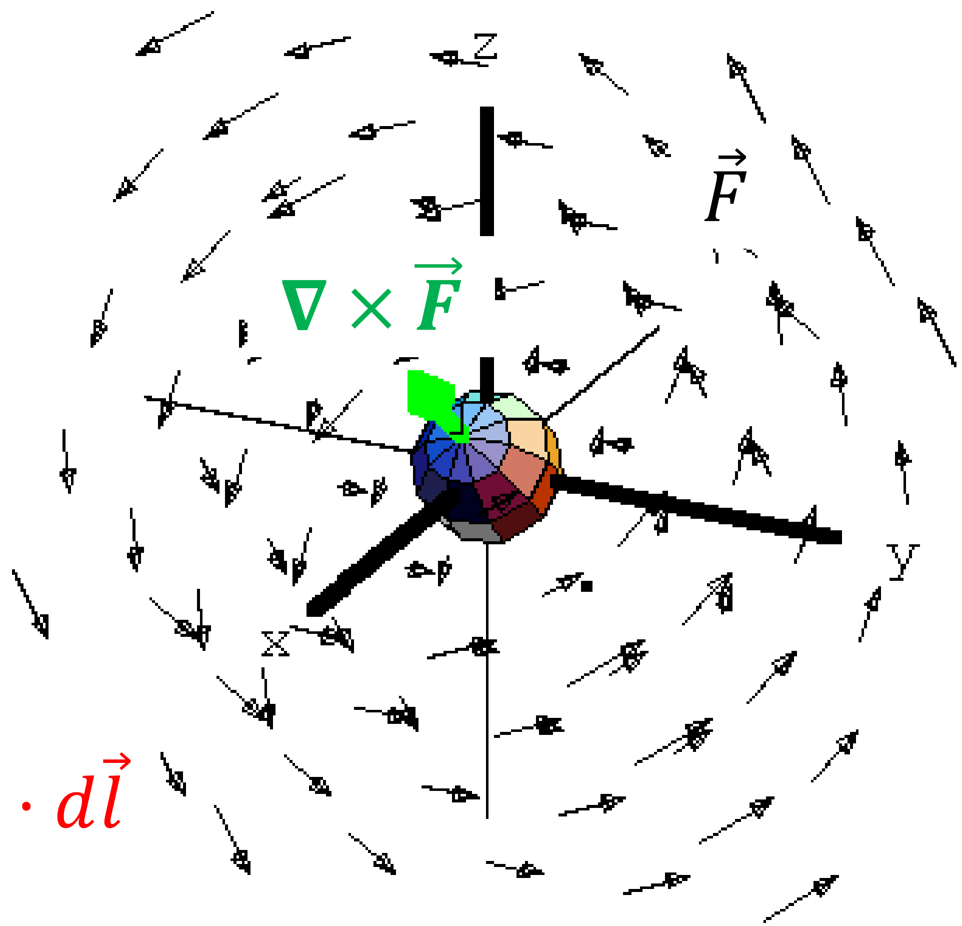
$$= \nabla \times \vec{F}$$

$$\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right) \qquad \mathrm{F} = \left(F_x, F_y, F_z\right)$$

$$\mathit{Curl\ Operator\ Definition:}\quad \nabla \times \vec{F} = \lim_{\Delta A \to 0} \frac{\hat{n}}{\Delta A} \oint_L \vec{F} \cdot d\vec{l} \quad (\mathit{when\ the\ limit\ exhists})$$

Meaning of the curl

The curl provide the **microscopic circulation** of a vector: how much it rotates around itself and which direction it defines



$$\nabla \times \vec{F} = \lim_{\Delta A \to 0} \frac{\hat{n}}{\Delta A} \oint_L \vec{F} \cdot d\vec{l}$$

Relationship Between currents and Magnetic Fields

Ampere law:
allows to express currents in terms of circulation of magnetic fields

$$\oint_C \vec{h} \cdot d\vec{l} = \iint_S \vec{j} \cdot d\vec{s}$$

Can be expressed in local form only resorting to **Stokes theorem**

$$\oint_C \vec{h} \cdot d\vec{l} = \iint_S \left(\nabla \times \vec{h}\right) \cdot d\vec{s}$$

$$\iint_S \left(\nabla \times \vec{h}\right) \cdot d\vec{s} = \iint_S \vec{j} \cdot d\vec{s}$$

If this is valid for every S the only possibility is that

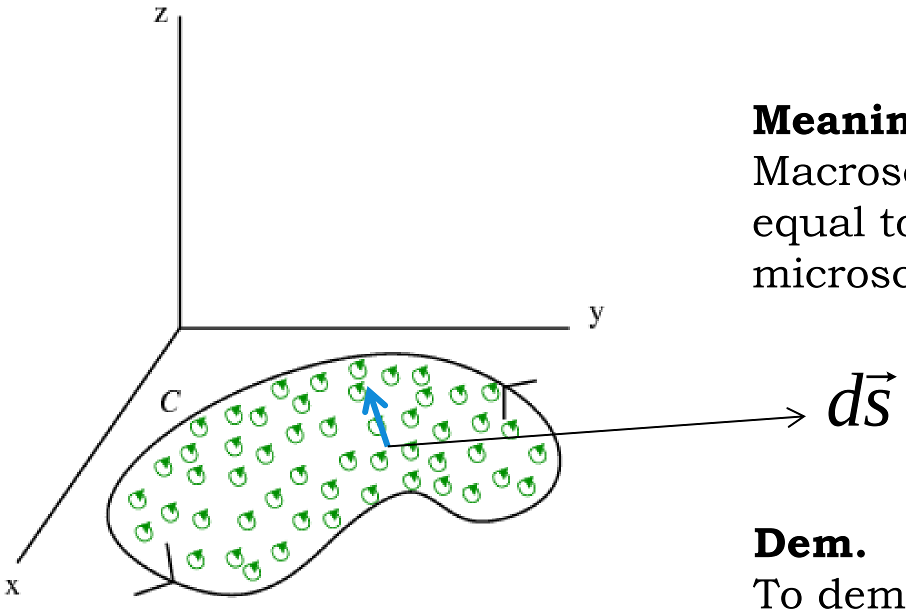
$$\nabla \times \vec{h} = \vec{j}$$

local form

Stokes Theorem

$$\oint_L \vec{F} \cdot d\vec{l} = \iint_A \nabla \times \vec{F} \cdot d\vec{s}$$

Is valid for every A



Meaning:
Macroscopic circulation is equal to flux over surface of microscopic circulation

Dem.
To demonstrate it for any size it is sufficient to demonstrate it for infinitesimally small surface. Because then all neighbouring contributions cancel out.

Stokes Theorem (2)

$$\oint_L \vec{F} \cdot d\vec{l} = \iint_A \nabla \times \vec{F} \cdot d\vec{s}$$

Is this a valid identity for infinitesimally small Area A?

$$\lim_{\Delta A \to 0} \oint_{\Delta L} \vec{F} \cdot d\vec{l} = \iint_{\Delta A} \nabla \times \vec{F} \cdot d\vec{s}$$

$$\oint_{\Delta L} \vec{F} \cdot d\vec{l} = \Delta A \, \nabla \times \vec{F}(\vec{r}_c) \cdot \hat{n}(\vec{r}_c)$$

$$\oint_{\Delta L} \vec{F} \cdot d\vec{l} = \oint_{\Delta L} \vec{F} \cdot d\vec{l}$$

However we can demonstrate that

$$\lim_{\Delta A \to 0} \nabla \times \vec{F}(\vec{r}_c) = \frac{\hat{n}(\vec{r}_c)}{\Delta A} \oint_{\Delta L} \vec{F} \cdot d\vec{l}$$

And accordingly substituting it in