Electrostatic energy

1. OUTLINE

- ✓ Electrostatic energy and forces
- ✓ Electrostatic energy in terms of field

2. ELECTROSTATIC ENERGY AND FORCES

In the previous lecture we saw that the electric potential at a point in an electric field is the work required to bring a unit charge from infinity (the zero reference for the potential) to that point. As an example, the amount of work required to bring a charge, q_2, from infinity to a point $\vec{r}_A$ (see Fig. 1), *against* the field generated by a charge q_1 can be expressed as:

$$U_2(\vec{r}_A) = -\int_{\infty}^{\vec{r}_A} q_2 \vec{E}_1(\vec{r}) \cdot d\vec{r} \qquad (1)$$

where $\vec{E}_1(\vec{r})$ is the field generated by q_1 and can be expressed as

$$\vec{E}_1(\vec{r}) = \frac{k_e q_1}{r^2}\hat{r} \qquad (2)$$

Using Eq. (2), Eq. (1) can be expressed as

$$U_2(\vec{r}_A) = -q_2 \int_{\infty}^{\vec{r}_A} \frac{k_e q_1}{r^2}\hat{r}_A \cdot d\vec{r}_A = k_e \frac{q_1 q_2}{r_A} \qquad (3)$$

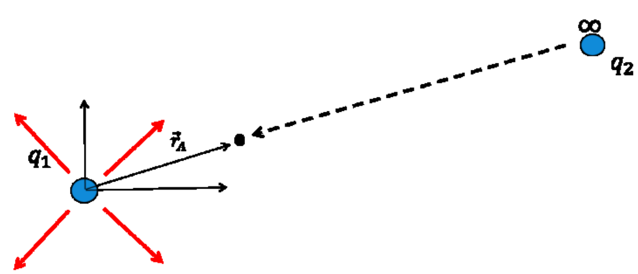

Figure 1. Work to move a point charge from infinity against the field of another point charge.

Since the electrostatic field is conservative, U_2 is independent from the path followed by q_2.

Now, let us consider another slightly different geometry, e.g., two charges q_1 and q_2 located at $\vec{r}_1$ and $\vec{r}_2$, respectively (Fig. 2). In such case, the electrostatic energy of the system can be written as

$$U_2 = k_e \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|} \qquad (4)$$

Let us suppose that another charge q_3 is brought from infinity to a point $\vec{r}_3$ (Fig. 2). The amount of work required to bring q_3 to $\vec{r}_3$ can be expressed as

$$U_{3\infty} = -\int_{\infty}^{\vec{r}_3} q_3 \vec{E}(\vec{r}) \cdot d\vec{r} \tag{5}$$

where $\vec{E}(\vec{r})$ is the field generated by both $q_1$and q_2. Using the discussions in the second lecture, one can express this field as

$$\vec{E}(\vec{r}) = \sum_{i=1}^{2} \frac{kq_i}{|\vec{r}-\vec{r}_i|^2} \frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|} \tag{6}$$

The above equation can be expanded as

$$\vec{E}(\vec{r}) = \frac{kq_1}{|\vec{r}-\vec{r}_1|^2} \frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|} + \frac{kq_2}{|\vec{r}-\vec{r}_2|^2} \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|} \tag{7}$$

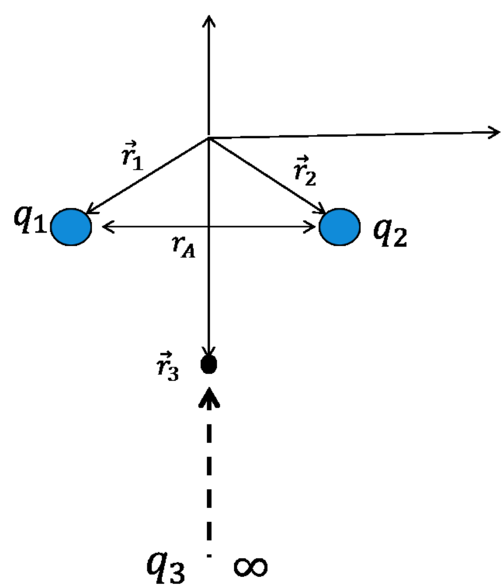

Figure 2. A system of three electric charges.

Using Eq. (7), $U_{3\infty}$ can be expressed as

$$U_{3\infty} = -\int_{\infty}^{\vec{r}_3} q_3 \left[\frac{kq_1}{|\vec{r}-\vec{r}_1|^2} \frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|} + \frac{kq_2}{|\vec{r}-\vec{r}_2|^2} \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|}\right] \cdot d\vec{r} = U_{3\infty}(q_1) + U_{3\infty}(q_2) \tag{8}$$

where $U_{3\infty}(q_1)$ and $U_{3\infty}(q_2)$ are the energy due to the presence of q_1 and q_2, respectively, and can be expressed as

$$U_{3\infty}(q_1) = -\int_{\infty}^{\vec{r}_3} q_3 \frac{kq_1}{|\vec{r}-\vec{r}_1|^2} \frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|} \cdot d\vec{r} \tag{9}$$

$$U_{3\infty}(q_2) = -\int_{\infty}^{\vec{r}_3} q_3 \frac{kq_2}{|\vec{r}-\vec{r}_2|^2} \frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|} \cdot d\vec{r} \tag{10}$$

The electrostatic energy accumulated is by definition the work performed against the field to push a charge from infinity to a finite distance position. Since the field is conservative, again it does not matter where at infinity the charge is coming from. Therefore, the integration path for Eq. (9) and (10) can be chosen in such a way to simplify the calculation (Fig. 3).

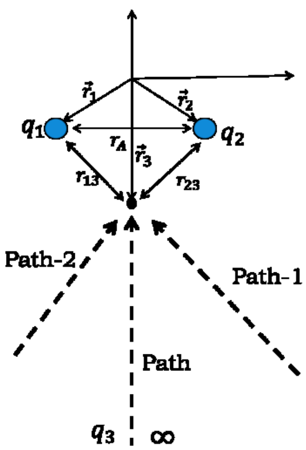

Figure 3. The integration paths used to calculate the amount of energy stored in a configuration of three charges.

Since in Eq. (9) and (10), the terms $\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|}$ and $\frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|}$ represent a direction parallel to path 1 and 2, respectively, it is easier to calculate the relevant integrals in such paths. As it is shown in Fig. 3, using the discussed paths, Eq. (9) and (10) are simplified as

$$U_{3\infty}(q_1) = -q_3 \int_{path-1} \vec{E}_1(\vec{r}) \cdot d\vec{r} = -q_3 \int_{\infty}^{\vec{r}_3} \frac{k_e q_1}{r^2} \hat{r}_{13} \cdot d\vec{r}_{13} = k_e \frac{q_1 q_3}{r_{13}} \tag{11}$$

$$U_{3\infty}(q_2) = -q_3 \int_{path-2} \vec{E}_2(\vec{r}) \cdot d\vec{r} = -q_3 \int_{\infty}^{\vec{r}_3} \frac{k_e q_2}{r^2} \hat{r}_{23} \cdot d\vec{r}_{23} = k_e \frac{q_2 q_3}{r_{23}} \tag{12}$$

Therefore, the total amount of energy stored in the 3 charges configuration is the summation of $U_{3\infty}(q_1)$, $U_{3\infty}(q_2)$, and U_2, that can be expressed as

$$U_3 = k_e \frac{q_1 q_2}{r_{12}} + k_e \frac{q_1 q_3}{r_{13}} + k_e \frac{q_2 q_3}{r_{23}} \tag{13}$$

Work in terms of potential

Work can be artificially represented in terms of potential. As an example Eq. (4) can be expressed as

$$U_2 = k \frac{q_1 q_2}{|\vec{r}_1-\vec{r}_2|} = \frac{1}{2} q_1 k \frac{q_2}{|\vec{r}_1-\vec{r}_2|} + \frac{1}{2} q_2 k \frac{q_1}{|\vec{r}_2-\vec{r}_1|} \tag{14}$$

The above representation can be also reformulate as

$$U_2 = \frac{1}{2}\sum_{i=1}^{2} q_i V(\vec{r}_i) \tag{15}$$

where

$$V(\vec{r}_i) = k\sum_{j=1;j\neq i}^{2} \frac{q_j}{|\vec{r}_i-\vec{r}_j|} \tag{16}$$

The same representation can be used for Eq. (13) as

$$U_3 = \frac{1}{2}\left[q_1\left(k\frac{q_2}{r_{12}} + k\frac{q_3}{r_{13}}\right) + q_2\left(k\frac{q_3}{r_{23}} + k\frac{q_1}{r_{12}}\right) + q_3\left(k\frac{q_1}{r_{13}} + k\frac{q_2}{r_{23}}\right)\right] \tag{17}$$

or

$$U_3 = \frac{1}{2}\sum_{i=1}^{3} q_i V(\vec{r}_i) \tag{18}$$

where

$$V(\vec{r}_i) = k\sum_{j=1;j\neq i}^{3} \frac{q_j}{|\vec{r}_i-\vec{r}_j|} \tag{19}$$

Extending this procedure to N discrete point charges, one gets the following general expression for the potential energy

$$U_{tot} = \frac{1}{2}\sum_{i=1}^{N} q_i V_i \tag{20}$$

where V_i is the electric potential at $\vec{r}_i$, and is due to all the charges excluding the i-th charge, q_i. It has the following expression:

$$V_i = k\sum_{\substack{j=1\\(j\neq i)}}^{N} \frac{q_j}{r_{ij}} = k\sum_{\substack{j=1\\(j\neq i)}}^{N} \frac{q_j}{|\vec{r_i} - \vec{r_j}|} \tag{21}$$

For a continuous volumetric charge distribution of density ρ the formula for U_{tot} in Eq. (20) has to be modified. Without going through a separate proof we replace q_j by ρdv and the summation by an integration, thus obtaining

$$U_{tot} = \frac{1}{2}\iiint_{V'} \rho V dv = \frac{1}{2}\iiint_{Volume} \rho(\vec{r}')V(\vec{r}')d\vec{r}' \tag{22}$$

In Eq. (22), V is the potential at the point where the volume charge density is ρ, and V' is the volume of the region where ρ exists.

3. ELECTROSTATIC ENERGY IN TERM OF FIELD

In the previous section, we saw that the energy stored in any system can be expressed as

$$U_{tot} = \frac{1}{2} \iiint_{Volume} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

By using Gauss's law in the local differential form

$$\nabla \cdot \vec{E}(\vec{r}') = \frac{\rho(\vec{r}')}{\epsilon_0 \epsilon_r} \rightarrow \epsilon_0 \epsilon_r \nabla \cdot \vec{E}(\vec{r}') = \rho(\vec{r}')$$

we can write

$$U_{tot} = \frac{1}{2} \iiint_{Volume} \epsilon_0 \epsilon_r \nabla \cdot \vec{E}(\vec{r}')\, V(\vec{r}')d\vec{r}' \tag{23}$$

Using the following property for the differential operator *nabla,*

$$\nabla \cdot (\vec{A}B) = (\nabla \cdot \vec{A})B + \vec{A} \cdot (\nabla B)$$

one has that

$$\begin{gathered} \nabla \cdot \left[\vec{E}(\vec{r}')V(\vec{r}')\right] = \nabla \cdot \vec{E}(\vec{r}')V(\vec{r}') + \vec{E}(\vec{r}') \cdot \nabla V(\vec{r}') \rightarrow \\ \nabla \cdot \vec{E}(\vec{r}')V(\vec{r}') = \nabla \cdot \left[\vec{E}(\vec{r}')V(\vec{r}')\right] - \vec{E}(\vec{r}') \cdot \nabla V(\vec{r}') \end{gathered} \tag{24}$$

Using Eq.(24) in Eq.(23) we have

$$U_{tot} = \frac{1}{2} \iiint_{Volume} \epsilon_0 \epsilon_r \nabla \cdot \left[\vec{E}(\vec{r}')V(\vec{r}')\right]d\vec{r}' - \frac{1}{2} \iiint_{Volume} \epsilon_0 \epsilon_r \vec{E}(\vec{r}') \cdot \nabla V(\vec{r}')d\vec{r}' \tag{25}$$

By applying the divergence Gauss theorem, which states that

$$\iiint_{Volume} (\nabla \cdot \vec{A})d\vec{r}' = \oiint_{Surface} \vec{A} \cdot d\vec{S}$$

where the surface is the one which encloses the volume, in Eq. (25), and by using the expression $\vec{E}(\vec{r}') = -\nabla V(\vec{r}')$, which links the electric field to the potential, we can write

$$U_{tot} = \frac{1}{2} \oiint_S \epsilon_0 \epsilon_r \vec{E}(\vec{r}\,') V(\vec{r}\,') \cdot \hat{n}\, d\vec{r}\,' - \frac{1}{2} \iiint_{Volume} \epsilon_0 \epsilon_r \vec{E}(\vec{r}\,') \cdot \left(-\vec{E}(\vec{r}\,')\right) d\vec{r}\,' \tag{26}$$

Since the volume (and the relevant surface) can be any volume that contains all the charges, we may choose it to be large sphere with radius R. For R which tends to infinity the integrand in the surface integral tends to zero with a faster rate with respect to the one by which the surface grows. In detail, $\vec{E}(\vec{r}\,') \propto 1/R^2$ and $V(\vec{r}\,') \propto 1/R$. The area of the bounding surface grows as R^2. Therefore, the first integral tends to zero. The energy stored in any system can then be expressed in terms of the electric field as

$$U_{tot} = \frac{1}{2} \iiint_{Volume} \epsilon_0 \epsilon_r \left|\vec{E}(\vec{r}\,')\right|^2 d\vec{r}\,' \tag{27}$$