Truly Important from Lecture 4

Electric Field is Conservative *1)*

$$\nabla \times \vec{E}(\vec{r}) = 0$$

$$\Updownarrow$$

$$\oint \vec{E}(\vec{l}) \cdot d\vec{l} = 0$$

Potential Energy difference *2)*

$$\Delta U_{AB} = -\int_A^B q\vec{E}(l) \cdot d\vec{l}$$

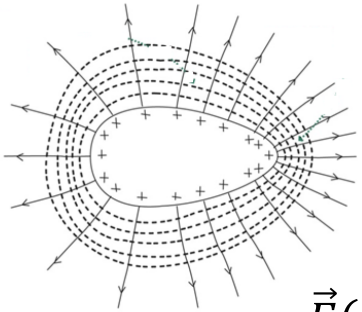
Potential difference *3)*

$$\Delta V_{AB} = -\int_A^B \vec{E}(l) \cdot d\vec{l}$$

Potential of a set of point charges *4)*

$$V(\vec{r}) = \frac{1}{4\pi\epsilon_0\epsilon_r}\sum_{i=1}^{N} \frac{q_i}{|\vec{r} - \vec{r}_i|}$$

5)



$$\vec{E}(\vec{r}) = -\nabla V(\vec{r})$$

EE1P21 Electricity and Magnetism

Electrostatic Energy

Topic 5

Understand the relation between charges, voltage, and energy
Relation between Electric field and stored energy

Learning Objectives

How to calculate the energy in discrete charge ensembles

Energy in charge distributions $$U_{tot} = \frac{1}{2} \iiint\limits_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

Energy in electric field

$$U_{tot} = \frac{1}{2} \iiint\limits_{Vol} \epsilon_0 \epsilon_r \left|\vec{E}(\vec{r}')\right|^2 d\vec{r}'$$

What is Energy?

Physicists: ‘the ability to do work’

‘work’ as shifting energy from one form to another

We do not know what energy is!

It is a quantity that comes in different forms *(we study the Electrostatic form).*

If we count carefully it appears to be always conserved.

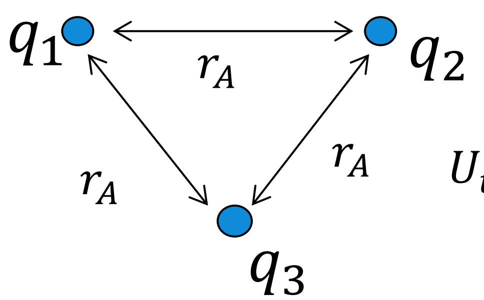
Energy in charge distributions

$$U_{tot} = \frac{1}{2} \iiint\limits_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

This result is also in book of course
but the way we show it will be bit differen

You will understand the meaning of this!

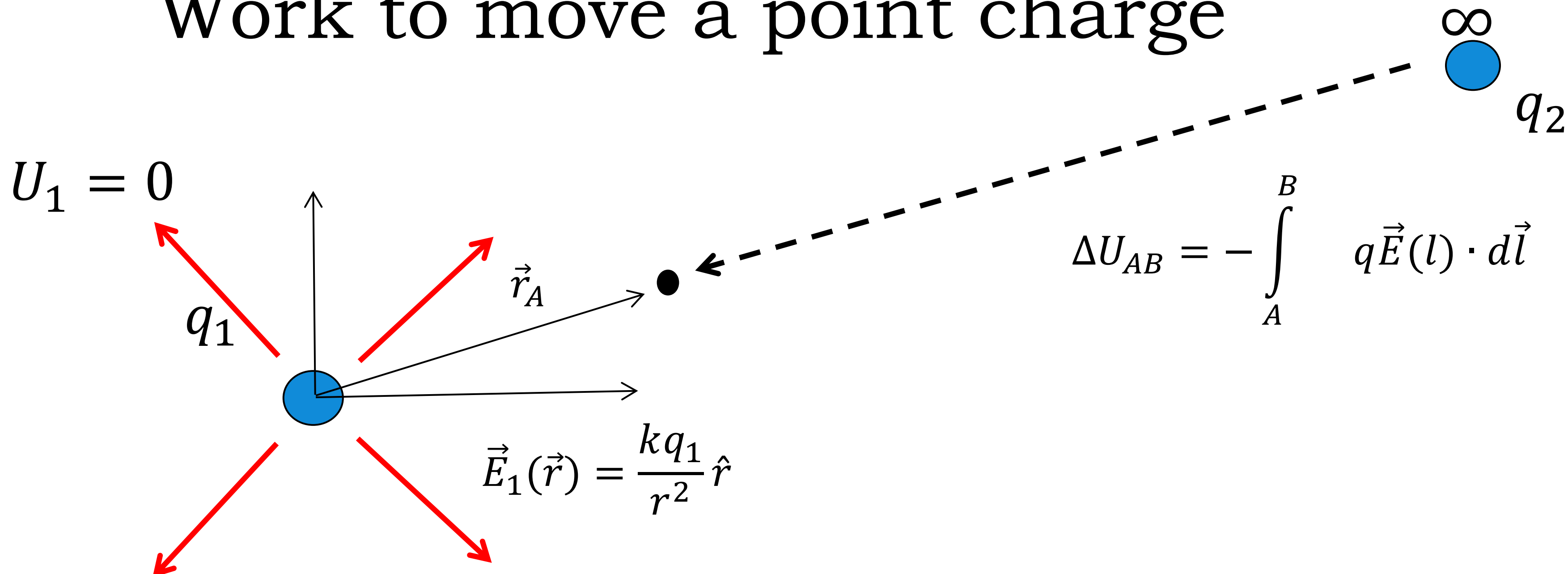
With three charges at same distance



$$U_{tot} = k\frac{q_1q_2}{r_A} + k\frac{q_1q_3}{r_A} + k\frac{q_2q_3}{r_A}$$

This is explained in ten lines in page 390

Work to move a point charge



$$\Delta U_{AB} = -\int_A^B q\vec{E}(l) \cdot d\vec{l}$$

$$\vec{E}_1(\vec{r}) = \frac{kq_1}{r^2}\hat{r}$$

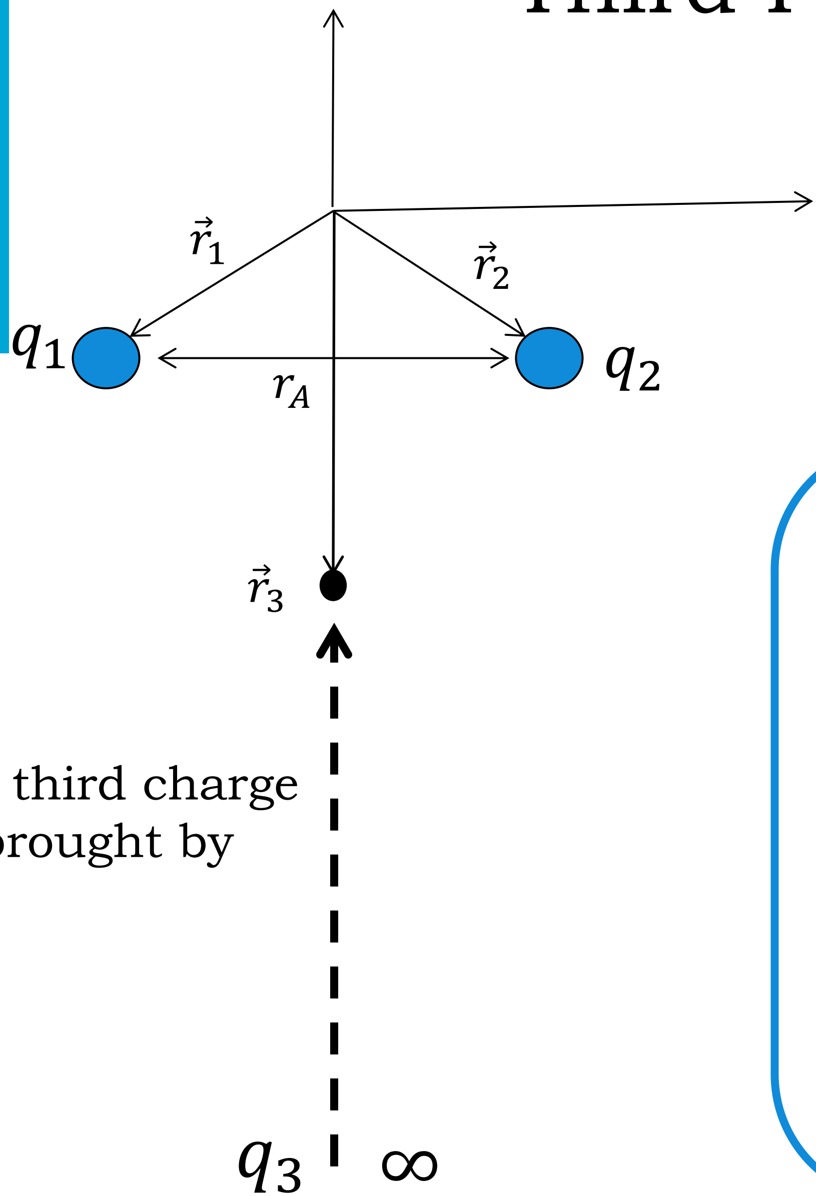
There is some energy internal to the charge. That we don't consider.

Then there is some energy that is acquired (spent) in bringing two charges together

$$U_2 = -q_2 \int_{\infty}^{\vec{r}_A} \frac{kq_1}{r^2}\hat{r}_{12} \cdot d\vec{r}_{12} = k\frac{q_1 q_2}{r_A}$$

This line integral is along a line to infinity that unites $\vec{r}_A$ and q_1

Third Point Charge



If a third charge is brought by

$$U_2 = k\frac{q_1 q_2}{r_A} = k\frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|}$$

$$U_{3\infty} = -\int_{\infty}^{\vec{r}_3} q_3\,\vec{E}(l)\cdot d\vec{l}$$

Using the many point charges version of Coulomb's law for the electric field

$$\vec{E}(\vec{r}) = \sum_i \frac{kq_i}{|\vec{r}-\vec{r}_i|^2}\frac{\vec{r}-\vec{r}_i}{|\vec{r}-\vec{r}_i|}$$

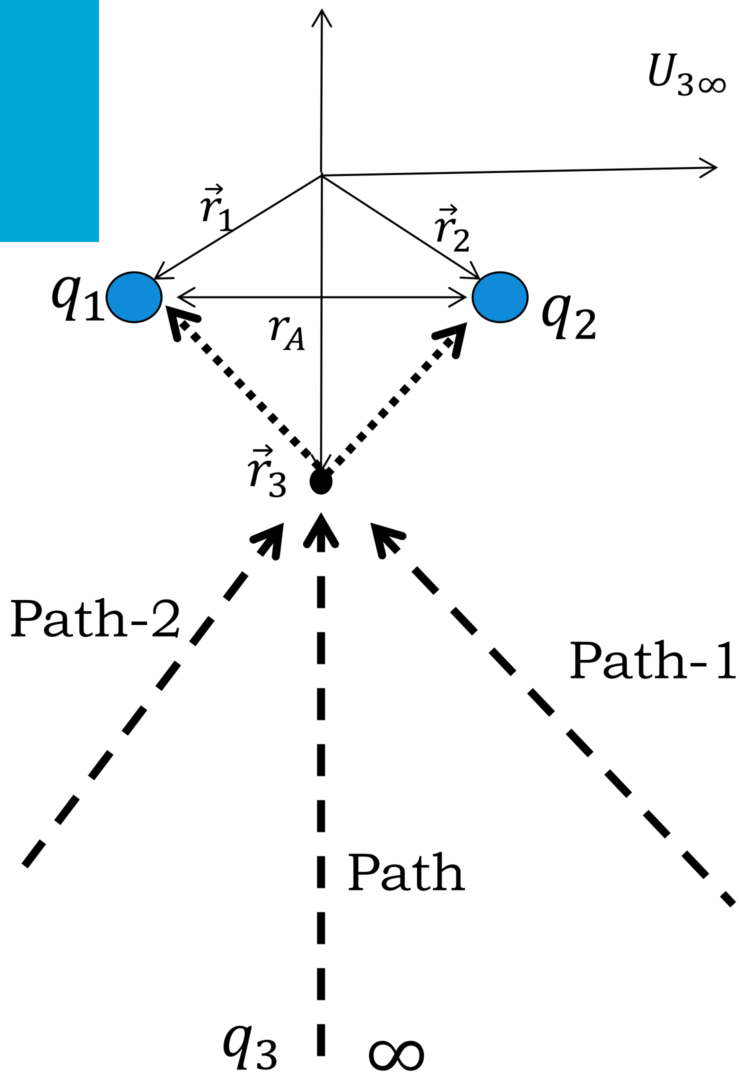
$$\Downarrow$$

$$\vec{E}(\vec{r}) = \vec{E}_1(\vec{r}) + \vec{E}_2(\vec{r})$$

$$\Downarrow$$

$$\vec{E}(\vec{r}) = \frac{kq_1}{|\vec{r}-\vec{r}_1|^2}\frac{\vec{r}-\vec{r}_1}{|\vec{r}-\vec{r}_1|} + \frac{kq_2}{|\vec{r}-\vec{r}_2|^2}\frac{\vec{r}-\vec{r}_2}{|\vec{r}-\vec{r}_2|}$$

Strange Integration paths?



$$U_{3\infty} = -\int_{\infty}^{\vec r_3} q_3\left[\frac{kq_1}{|\vec r-\vec r_1|^2}\frac{\vec r-\vec r_1}{|\vec r-\vec r_1|}+\frac{kq_2}{|\vec r-\vec r_2|^2}\frac{\vec r-\vec r_2}{|\vec r-\vec r_2|}\right]\cdot d\vec l$$

Path

$$U_{3\infty}(q_1) = -\int_{\infty}^{\vec r_3} q_3\frac{kq_1}{|\vec r-\vec r_1|^2}\frac{\vec r-\vec r_1}{|\vec r-\vec r_1|}\cdot d\vec l$$

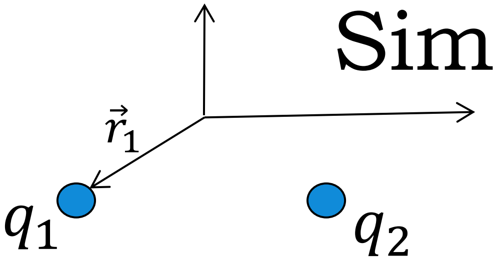
$$U_{3\infty}(q_2) = -\int_{\infty}^{\vec r_3} q_3\frac{kq_2}{|\vec r-\vec r_2|^2}\frac{\vec r-\vec r_2}{|\vec r-\vec r_2|}\cdot d\vec l$$

Do we know how to calculate them?

I would know how to do it on Path-2

I would know how to do it on Path-1

Simple Integration Path!

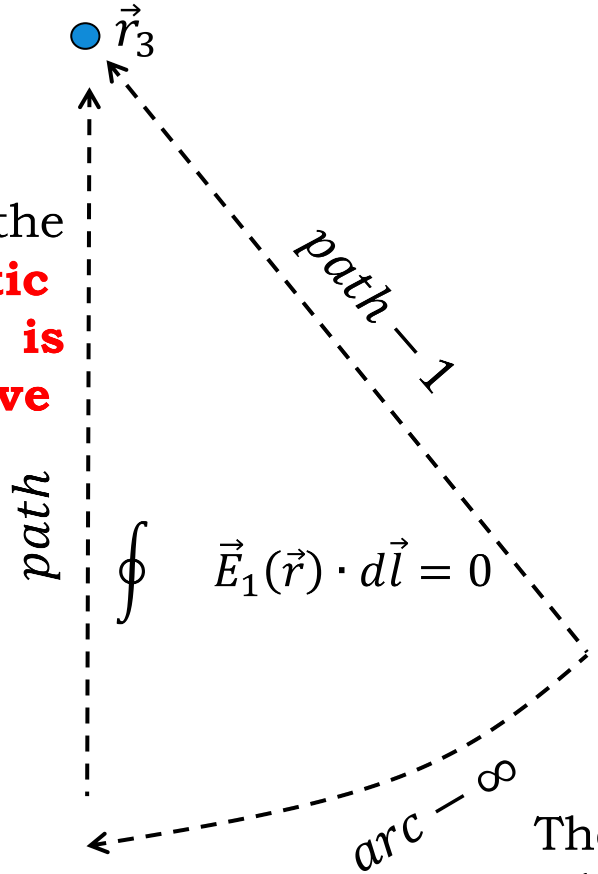


Because the field at infinity is zero

$$0 = \oint \vec{E}_1(\vec{r}) \cdot d\vec{l} = + \int_{arc-\infty} \cancel{\vec{E}_1(\vec{r})} \cdot d\vec{l}_{arc} + \int_{path} \vec{E}_1(\vec{r}) \cdot d\vec{l} - \int_{path-1} \vec{E}_1(\vec{r}) \cdot d\vec{l}_1$$

$$\int_{path} \vec{E}_1(\vec{r}) \cdot d\vec{l} = \int_{path-1} \vec{E}_1(\vec{r}) \cdot d\vec{l}_1$$

Since the **electrostatic field is conservative**



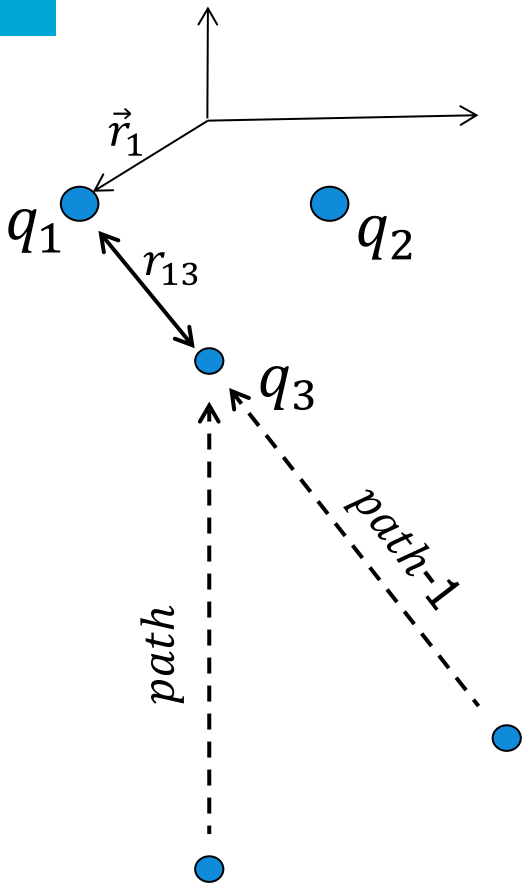
$$\oint \vec{E}_1(\vec{r}) \cdot d\vec{l} = 0$$

Consider the closed path

The line integral can be evaluated along a line to infinity that unites $\vec{r}_3$ and q_1

Energy due to first and third charge

The electrostatic energy accumulated is by definition the work performed against the field to push a charge from infinity to a finite distance position. **It does not matter where at infinity (Because of field is irrotational/conservative)**.



Work done against field of q_1

$$U_{3\infty}(q_1) = -q_3 \int_{path} \vec{E}_1(\vec{r}) \cdot d\vec{l} = -q_3 \int_{path-1} \vec{E}_1(\vec{r}) \cdot d\vec{l}_1$$

$$U_{3\infty}(q_1) = -q_3 \int_{\infty}^{\vec{r}_3} \frac{kq_1}{r^2} \hat{r}_{13} \cdot d\vec{r}_{13} = k\frac{q_1 q_3}{r_{13}}$$

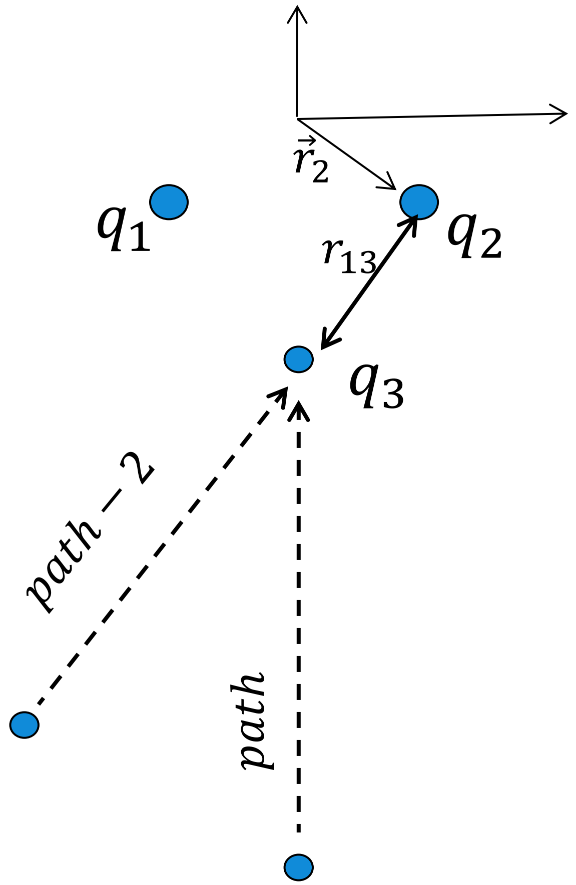
Energy due to second and third charge

Work performed against the field to push a charge from infinity to a finite distance position. **It does not matter where at infinity**.

Work done against field of q_2

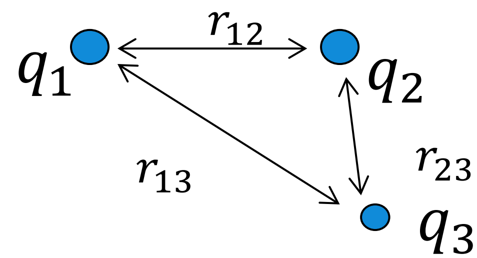
$$U_{3\infty}(q_2) = -q_3 \int_{path} \vec{E}_2(\vec{r}) \cdot d\vec{l} = -q_3 \int_{path-2} \vec{E}_2(\vec{r}) \cdot d\vec{l}_2$$

$$U_{3\infty}(q_2) = -q_3 \int_{\infty}^{\vec{r}_3} \frac{kq_2}{r^2} \hat{r}_{23} \cdot d\vec{r}_{23} = k\frac{q_2 q_3}{r_{23}}$$



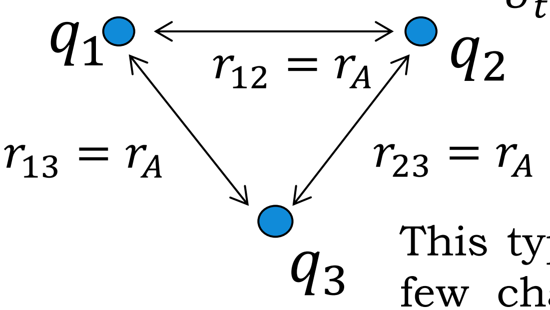
Potential Energy of three charges

$$U_{tot} = U_{12} + U_{13} + U_{23} = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}}$$



With three charges at same distance

$$U_{tot} = k\frac{q_1 q_2}{r_A} + k\frac{q_1 q_3}{r_A} + k\frac{q_2 q_3}{r_A}$$

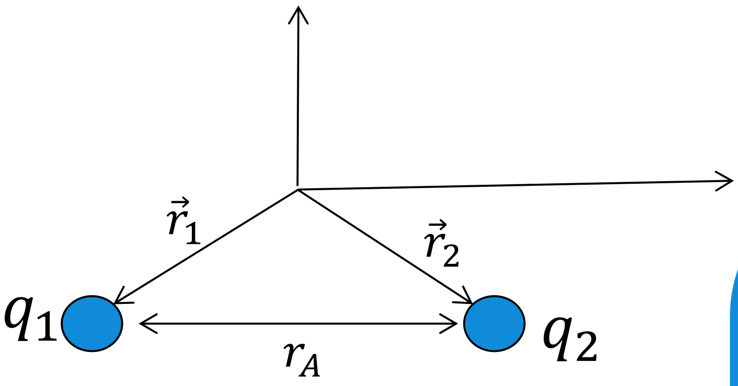


This type of configurations with few charges are representative for the energy in molecules: unions of charged particles

Three charges look like a special case…. there is a very simple way to extend it to many charges. Using the potential.

Work in terms of potentials

Two charges



$$U_{12} = k\frac{q_1 q_2}{r_A}$$

An artificial representation is

$$U_2 = k\frac{q_1 q_2}{r_A} = \frac{1}{2} q_1 k \frac{q_2}{|\vec{r}_1 - \vec{r}_2|} + \frac{1}{2} q_2 k \frac{q_1}{|\vec{r}_2 - \vec{r}_1|}$$

Which can also be expressed as

$$U_2 = \frac{1}{2}\sum_{i=1}^{2} q_i V(\vec{r}_i) \qquad V(\vec{r}_i) = k \sum_{j=1;j\neq i}^{2} \frac{q_j}{|\vec{r}_i - \vec{r}_j|};$$

It is the same expression where energy is associated to each charge and their Potentials.

These expressions can be generalized!!

Work to form charges ensembles

Two charges

$$U_2 = k\frac{q_1 q_2}{r_A}$$

$$U_2 = \frac{1}{2}\sum_{i=1}^{2} q_i V(\vec{r}_i) \qquad V(\vec{r}_i) = k\sum_{j=1;j\neq i}^{2} \frac{q_j}{|\vec{r}_i - \vec{r}_j|};$$

Three charges

$$U_3 = U_{12} + U_{13} + U_{23} = k\frac{q_1 q_2}{r_{12}} + k\frac{q_1 q_3}{r_{13}} + k\frac{q_2 q_3}{r_{23}}$$

$$U_3 = \frac{1}{2}\left[q_1\left(k\frac{q_2}{r_{12}} + k\frac{q_3}{r_{13}}\right) + q_2\left(k\frac{q_3}{r_{23}} + k\frac{q_1}{r_{12}}\right) + q_3\left(k\frac{q_1}{r_{13}} + k\frac{q_2}{r_{23}}\right)\right]$$

$$U_3 = \frac{1}{2}\sum_{i=1}^{3} q_i V(\vec{r}_i) \qquad V(\vec{r}_i) = k\sum_{j=1;j\neq i}^{3} \frac{q_j}{|\vec{r}_i - \vec{r}_j|};$$

Work to form charges ensembles

Three charges

$$U_3 = \frac{1}{2}\sum_{i=1}^{3} q_i V(\vec{r}_i) \qquad V(\vec{r}_i) = k \sum_{j=1; j\neq i}^{3} \frac{q_j}{|\vec{r}_i - \vec{r}_j|};$$

N charges

It only makes sense for N>1

$$U_{tot} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i,) \qquad V(\vec{r}_i) = k \sum_{j=1; j\neq i}^{N} \frac{q_j}{|\vec{r}_i - \vec{r}_j|};$$

Energy Associated to Charge and Potential

$$U_{tot} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)$$

$$V(\vec{r}_i) = k \sum_{j=1;i\neq j}^{N} \frac{q_j}{r_{ij}}$$

Only intuitive here, no dem.

$$U_{tot} = \frac{1}{2}\iiint_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

The total potential energy is the integral superposition of the products between the charges and the potential they generate

In the book you have equations 23.7 and 23.8 which are derived for a parallel plate capacitor only

Electrostatic Energy in terms of Field

$$U_{tot} = \frac{1}{2} \iiint\limits_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

$$\Downarrow$$

Electrostatic Energy
can also be expressed as

$$U_{tot} = \frac{1}{2} \iiint\limits_{Vol} \epsilon_0 \epsilon_r \left|\vec{E}(\vec{r}')\right|^2 d\vec{r}'$$

Electrostatic Energy in terms of Field I

$$U_{tot} = \frac{1}{2}\iiint\limits_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

$$\nabla \cdot \vec{E}(\vec{r}') = \frac{\rho(\vec{r}')}{\epsilon_0\epsilon_r}$$

Gauss Law
in Local form

$$\epsilon_0\epsilon_r\nabla \cdot \vec{E}(\vec{r}') = \rho(\vec{r}')$$

$$U_{tot} = \frac{1}{2}\iiint\limits_{Vol} \epsilon_0\epsilon_r\nabla \cdot \vec{E}(\vec{r}')V(\vec{r}')d\vec{r}'$$

$$\nabla \cdot \left[\vec{E}(\vec{r}')V(\vec{r}')\right] = \nabla \cdot \vec{E}(\vec{r}')\ V(\vec{r}') + \vec{E}(\vec{r}') \cdot \nabla V(\vec{r}')$$

$$\nabla \cdot \vec{E}(\vec{r}')\ V(\vec{r}') = \nabla \cdot \left[\vec{E}(\vec{r}')V(\vec{r}')\right] - \vec{E}(\vec{r}') \cdot \nabla V(\vec{r}')$$

$$U_{tot} = \frac{1}{2}\iiint\limits_{Vol} \epsilon_0\epsilon_r\nabla \cdot \left[\vec{E}(\vec{r}')V(\vec{r}')\right]d\vec{r}' - \frac{1}{2}\iiint\limits_{Vol} \epsilon_0\epsilon_r\vec{E}(\vec{r}') \cdot \nabla V(\vec{r}')d\vec{r}'$$

Electrostatic Energy in terms of Field II

$$U_{tot} = \frac{1}{2}\iiint\limits_{Vol} \epsilon_0\epsilon_r \nabla\cdot\left[\vec{E}(\vec{r}\,')V(\vec{r}\,')\right]d\vec{r}\,' - \frac{1}{2}\iiint\limits_{Vol} \epsilon_0\epsilon_r \vec{E}(\vec{r}\,')\cdot\nabla V(\vec{r}\,')d\vec{r}\,'$$

Divergence Theorem

$$U_{tot} = \underbrace{\frac{1}{2}\oiint\limits_{S} \epsilon_0\epsilon_r V(\vec{r}\,')\vec{E}(\vec{r}\,')\cdot\hat{n}d\vec{r}\,'}_{} - \frac{1}{2}\iiint\limits_{Vol} \epsilon_0\epsilon_r \vec{E}(\vec{r}\,')\cdot\nabla V(\vec{r}\,')d\vec{r}\,'$$

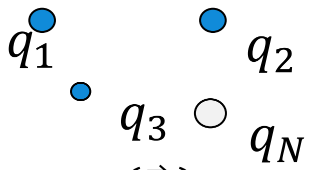
$$\vec{E}(\vec{r}\,') = -\nabla V(\vec{r}\,')$$

$$For\ S \rightarrow \infty \oiint\limits_{S} \epsilon_0\epsilon_r \underset{\frac{1}{r}}{V(\vec{r}\,')}\underset{\frac{1}{r^2}}{\vec{E}(\vec{r}\,')}\cdot\hat{n}\, r^2\, sin\theta d\theta d\phi \rightarrow 0$$

$$\boxed{U_{tot} = \frac{1}{2}\iiint\limits_{Vol} \epsilon_0\epsilon_r \left|\vec{E}(\vec{r}\,')\right|^2 d\vec{r}\,'}$$

Truly Important from Lecture 5

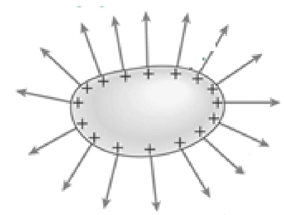
1)



$$U_{tot} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)$$

$$V(\vec{r}_i) = k \sum_{i=1;i\neq j}^{N} \frac{q_i}{r_{ij}}$$

2)



$$U_{tot} = \frac{1}{2}\iiint_{Vol} \rho(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,'$$

3)

$$U_{tot} = \frac{1}{2}\iiint_{Vol} \epsilon_0\epsilon_r\left|\vec{E}(\vec{r}\,')\right|^2 d\vec{r}\,'$$