Exam EE1P21

Electricity and magnetism

Wednesday 25 May 2016, 9:00–11:00

- This exam consists of 3 pages with 4 exercises.
- A list of formulas is appended after the exercises, starting with page 4.
- The maximum attainable number of points is 90.
- The number of points assigned to each exercise is stated explicitly.
- Start each exercise on a new exam sheet and fill in on each sheet your name, your study number and the number of the current exercise.

The EE1P21 team wishes you a lot of success!

Exercise 1

20 points

Let the configuration in Fig. 1 consisting of 3 point charges $+q$ located at: A $\left((\sqrt{3}/2)d, -(1/2)d\right)$, B $\left(-(\sqrt{3}/2)d, -(1/2)d\right)$ and C $(0, d)$, with $d > 0$ being a constant length.

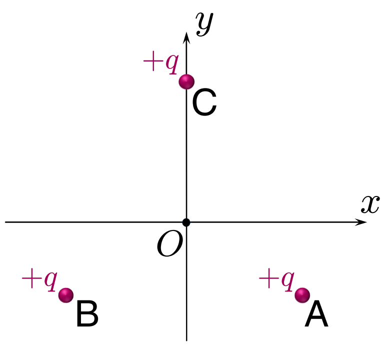

Figure 1: Point charges configuration.

a) Determine the electric field $\vec{E}$ at the origin $(0, 0)$ – *a vectorial quantity is requested.* (5 points)

Solution

$\vec{E}(0,0)$ follows by applying superposition. By making use of elementary geometry, it can directly be written that

$$\vec{E}(0,0) = \vec{E}(0,0)\Big|_{q@\mathrm{A}} + \vec{E}(0,0)\Big|_{q@\mathrm{B}} + \vec{E}(0,0)\Big|_{q@\mathrm{C}}$$
$$= \frac{k\,q}{d^2}\left[\left(-\sqrt{3}/2\,\hat{x} + 1/2\,\hat{y}\right) + \left(\sqrt{3}/2\,\hat{x} + 1/2\,\hat{y}\right) + (-\hat{y})\right] = \vec{0}\,(\mathrm{N/C}) \quad (\mathrm{V/m}). \qquad (1)$$

b) Determine the electric potential V at the origin $(0,0)$. (4 points)

Solution

$V(0,0)$ also follows by applying superposition. By making use of elementary geometry, it can directly be written that

$$V(0,0) = V(0,0)\Big|_{q@\mathrm{A}} + V(0,0)\Big|_{q@\mathrm{B}} + V(0,0)\Big|_{q@\mathrm{C}} = \frac{3\,k\,q}{d}\,(\mathrm{V}). \tag{2}$$

c) Determine the potential energy of the system. (4 points)

Solution

We make use of the expression given in the lecture

$$U_{\mathrm{tot}} = U_{\mathrm{AB}} + U_{\mathrm{BC}} + U_{\mathrm{CA}} \tag{3}$$

with

$$U_{\mathrm{a}} = k\frac{q_{\mathrm{a}}\,q_{\mathrm{b}}}{|\vec{r}_{\mathrm{ab}}|}. \tag{4}$$

For the configuration at hand $|\vec{r}_{\mathrm{ab}}| = \sqrt{3}d$ for all constituents in (3). It then follows that

$$U_{\mathrm{tot}} = 3\,k\,\frac{q^2}{\sqrt{3}d} = \sqrt{3}\,k\,\frac{q^2}{d}\,(\mathrm{J}). \tag{5}$$

d) Determine the work that has to be effectuated for adding to this system another point charge $+q$ at the origin $(0,0)$. (7 points)

Solution

The work effectuated for adding to this system another point charge $+q$ at the origin $(0,0)$ equals the difference between the stored electrostatic energy U'_{tot} in that new state and U_{tot} at point c. The distances between A, B, C and the origin all equal d. It then follows that

$$U'_{\mathrm{tot}} = U_{\mathrm{tot}} + 3\,k\,\frac{q^2}{d} = U_{\mathrm{tot}} + 3\,k\,\frac{q^2}{d}\,(\mathrm{J}). \tag{6}$$

It then follows that the effectuated work is

$$W = U'_{\mathrm{tot}} - U_{\mathrm{tot}} = 3\,k\,\frac{q^2}{d}\,(\mathrm{J}). \tag{7}$$

Exercise 2

20 points

A sphere of radius $R = 10\,\text{cm}$ carries electric charge with nonuniform distribution, the volumetric density being $\rho(r) = \rho_0 r$, with ρ_0 being a constant. The total charge on the sphere is $8\,\text{nC}$. Coulomb's constant is $k = 9\cdot 10^9\,\text{N}\cdot\text{m}^2/\text{C}^2$.

a) Determine the value of the constant ρ_0 and specify its measure unit. (8 points)

Solution

By making use of the expression of the elementary volume in spherical coordinates, the total charge on (inside) the charged sphere is

$$
\begin{aligned}
Q_{\text{tot}} &= \int_{\text{Sphere}} \rho(r)\mathrm{d}V = \int_{\varphi=0}^{2\pi}\int_{\vartheta=0}^{\pi}\int_{r=0}^{R} (\rho_0 r)\, r^2 \sin(\vartheta)\,\mathrm{d}\vartheta\mathrm{d}\varphi\mathrm{d}r \\
&= \rho_0 \int_{\varphi=0}^{2\pi} \mathrm{d}\varphi \int_{\vartheta=0}^{\pi} \sin(\vartheta)\,\mathrm{d}\vartheta \int_{r=0}^{R} r^3\,\mathrm{d}r = \rho_0\, \frac{4\pi}{4}\, r^4\Big|_{r=0}^{R} = \rho_0\,\pi\, R^4\,([\rho_0]\text{m}^4).
\end{aligned} \tag{8}
$$

By now equating Q_{tot} to $Q = 8\,\text{nC}$ it follows that

$$
\rho_0 = \frac{Q}{\pi\, R^4} = \frac{8\cdot 10^{-9}}{\pi\cdot 10^{-4}} = \frac{8}{\pi}\cdot 10^{-5} \approx 2.55\cdot 10^{-5}\,(\text{C}/\text{m}^4). \tag{9}
$$

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b) Determine the electric field $\vec{E}(r)$ for $r \geqslant 0$ – *a vectorial quantity is requested.* (6 points)

Solution

The configurations' symmetry implies that the electric field is radial and it has constant magnitude on spheres that are concentric with the charged sphere. Application of Gauss's law on Gaussian spherical surfaces $\mathcal{S}(r)$ of radius r and concentric with the charged sphere then yields

$$
\int_{\mathcal{S}(r)} \vec{E}\cdot\hat{n}\,\mathrm{d}A = \frac{1}{\varepsilon_0}\int_{\mathcal{V}(r)} \rho(r')\mathrm{d}V \;\Longrightarrow\; 4\pi r^2 E(r) = \frac{1}{\varepsilon_0}\int_{\mathcal{V}(r)} \rho(r')\mathrm{d}V \tag{10}
$$

with $\mathcal{V}(r)$ being the volume enclosed by $\mathcal{S}(r)$.

Two cases need being examined:

for $0 < r < R$:

$$
\begin{aligned}
&4\pi r^2 E(r) = \frac{1}{\varepsilon_0}\int_{\mathcal{V}(r')} (\rho_0 r)\mathrm{d}V = \frac{1}{\varepsilon_0}\frac{Q}{\pi\, R^4}\, 4\pi \int_{r'=0}^{r} r'^3\,\mathrm{d}r' = \frac{Q}{\varepsilon_0}\left(\frac{r}{R}\right)^4 \;\Longrightarrow \\
&\vec{E}(r) = \frac{Q}{4\pi\varepsilon_0\, r^2}\left(\frac{r}{R}\right)^4 \hat{r}\,(\text{N}/\text{C})
\end{aligned} \tag{11}
$$

and

for $r > R$:

$$4\pi r^2 E(r) = \frac{Q}{\varepsilon_0} \Longrightarrow \vec{E}(r) = \frac{Q}{4\pi\varepsilon_0\, r^2}\hat{r}\,(\mathrm{N/C}). \tag{12}$$

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c) Determine the potential difference ΔV between the sphere's centre and its surface. (6 points)

Solution

From the definition of the potential difference ΔV, (11) yields

$$\Delta V = -\int_0^R \vec{E}\cdot\hat{r}\mathrm{d}r = -\frac{Q}{4\pi\varepsilon_0\, R^4}\int_0^R r^2\mathrm{d}r = -\frac{Q}{12\pi\varepsilon_0\, R^4}r^3\Big|_0^R = -\frac{Q}{12\pi\varepsilon_0\, R} \approx -240\,(\mathrm{V}). \tag{13}$$

Exercise 3

25 points

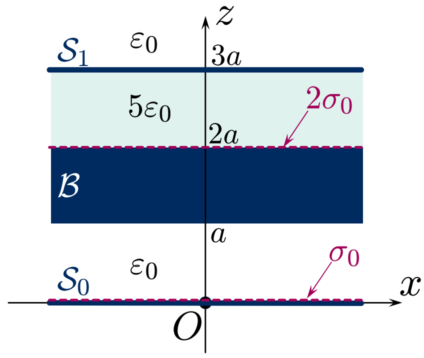

Figure 2: Layered configuration – cross-section for $y = 0$.

Let the configuration in Fig. 2 consisting of two perfectly conducting sheets $\mathcal{S}_0$ and $\mathcal{S}_1$ of very small, yet non-vanishing thickness $t \ll a$, and a thick, perfectly conducting block $\mathcal{B}$. The locations of these elements are given in the figure. All elements are taken to be infinitely extended in the x and y directions. The space between $\mathcal{S}_0$ and $\mathcal{B}$ is empty (the permittivity is ε_0) and the space between $\mathcal{B}$ and $\mathcal{S}_1$ is filled with a dielectric with permittivity $5\varepsilon_0$. On the *upper* face of $\mathcal{S}_0$ there is electric charge with *uniform* surface charge density $\sigma_0\,\mathrm{C/m^2}$. On the *upper* face of $\mathcal{B}$ there is electric charge with *uniform* surface charge density $2\sigma_0\,\mathrm{C/m^2}$. The configuration is at electrostatic equilibrium. The potential at $z \to \infty$ is taken to be 0 V.

a) Determine the electric field $\vec{E}(z)$ in the regions $z < 0$, $a < z < 2a$ and $z > 3a$. (3 points)

Solution

Since the configuration is at electrostatic equilibrium, if there is a surface charge density σ_0 on the upper side of $\mathcal{S}_0$ there has to be a surface charge density σ_0 on its lower side (see Fig. 21.26 in the book). By accounting for the fact that the configuration is infinite in the x and y directions, application of Gauss's law on a cylindrical Gaussian surface of area A and lateral surface orthogonal to $\mathcal{S}_0$, with one of its bases inside the perfectly conducting sheet $\mathcal{S}_0$ (t is small, but non-vanishing) and one at $z < 0$ yields

$$\text{for } z < a: \quad \vec{E}(z) \cdot A(-\hat{z}) = \frac{\sigma_0 A}{\varepsilon_0} \Rightarrow \vec{E}(z) = -\frac{\sigma_0}{\varepsilon_0}\hat{z}\,(\mathrm{N/C}) \text{ or } (\mathrm{V/m}). \tag{14}$$

The space $a < z < 2a$ is inside the perfectly conducting block $\mathcal{B}$ where $\vec{E}(z) = \vec{0}\,(\mathrm{N/C})$.

For $z > 3a$, application of Gauss's law on a cylindrical Gaussian surface of area A and lateral surface orthogonal to $\mathcal{S}_1$, with one of its bases inside the perfectly conducting sheet $\mathcal{S}_1$ (t is small, but non-vanishing) and one at $z > 3a$ yields

$$\text{for } z > 3a: \quad \vec{E}(z) \cdot A\hat{z} = \frac{\sigma_{\mathcal{S}_1,\mathrm{upp}}}{\varepsilon_0} \tag{15}$$

with $\sigma_{\mathcal{S}_1,\mathrm{upp}}$ being the surface charge density on the upper side of $\mathcal{S}_1$. Now, should $\sigma_{\mathcal{S}_1,\mathrm{upp}} \neq 0$, this would yield a *constant* field for $z > 3a$ that is incompatible with the fact that the potential at $z \to \infty$ is 0 V: the line integral is infinite, this yielding an infinite potential on $\mathcal{S}_1$, that is not physical; recall the explanations during the instruction concerning specifying reference conditions for V in the case of (in)finite charge distributions! Consequently, the only physical solution is that $\sigma_{\mathcal{S}_1,\mathrm{upp}} = 0$, this implying that

$$\text{for } z > 3a: \quad \vec{E}(z) \cdot A\hat{z} = 0 \Rightarrow \vec{E}(z) = \vec{0}\,(\mathrm{N/C}). \tag{16}$$

b) Determine the electric field $\vec{E}(z)$ in the regions $0 < z < a$ and $2a < z < 3a$. (7 points)

Solution

Similar applications of Gauss's law yield

$$\text{for } 0 < z < a: \quad \vec{E}(z) \cdot A\hat{z} = \frac{\sigma_0 A}{\varepsilon_0} \Rightarrow \vec{E}(z) = \frac{\sigma_0}{\varepsilon_0}\hat{z}\,(\mathrm{N/C}) \tag{17}$$

$$\text{for } 2a < z < 3a: \quad \vec{E}(z) \cdot A\hat{z} = \frac{2\sigma_0 A}{5\varepsilon_0} \Rightarrow \vec{E}(z) = \frac{2\sigma_0}{5\varepsilon_0}\hat{z}\,(\mathrm{N/C}) \tag{18}$$

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c) Determine the electric charge density on the *lower* side of the block $\mathcal{B}$. (5 points)

Solution

Application of Gauss's law on a cylindrical Gaussian surface of area A and lateral surface orthogonal to $\mathcal{S}_0$, with one of its bases inside the perfectly conducting sheet $\mathcal{S}_0$ (t is small, but non-vanishing) and one inside $\mathcal{B}$ yields

$$0 = \frac{\sigma_0 A}{\varepsilon_0} + \frac{\sigma_{\mathcal{B},\mathrm{low}} A}{\varepsilon_0} \Rightarrow \sigma_{\mathcal{B},\mathrm{low}} = -\sigma_0\,(\mathrm{C/m^2}). \tag{19}$$

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d) Determine the electric potential on the sheet $\mathcal{S}_1$. (5 points)

Solution

At point (a) it was established that $\vec{E}(z) = \vec{0}$ for $z > 3a$. Consequently

$$-\int_{z=\infty}^{3a} \vec{E}(z) \cdot (-\hat{z})\mathrm{d}z = V(3a) - V\Big|_{z=\infty} = 0 \Rightarrow V(3a) = V\Big|_{z=\infty} = 0\,(\mathrm{V}). \tag{20}$$

Interpret:
The fact that $V(3a) = 0\,\mathrm{V}$ indicates that the physical modality to ensure that $\sigma_{\mathcal{S}_1,\mathrm{upp}} = 0$ is by grounding the $\mathcal{S}_1$ plate. This is the only modality to ensure that the potential at $z \to \infty$ is $0\,\mathrm{V}$, as required by the exercise.

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e) Determine the electric potential on the sheet $\mathcal{S}_0$. (5 points)

Solution

For $2a < z < 3a$ it holds that

$$-\int_{z=2a}^{3a} \vec{E}(z) \cdot \hat{z}\mathrm{d}z = V(3a) - V(2a) = -V(2a) \Rightarrow V(2a) = \int_{z=2a}^{3a} E(z)\mathrm{d}z = \frac{2\sigma_0 a}{5\varepsilon_0}\,(\mathrm{V}). \quad (21)$$

The block $\mathcal{B}$ is perfectly conducting, hence it's faces are at the same potential. For $0 < z < a$ it holds that

$$-\int_{z=0}^{a} \vec{E}(z) \cdot \hat{z}\mathrm{d}z = V(a) - V(0) = V(2a) - V(0) \Rightarrow \quad (22)$$

$$V(0) = \int_{z=0}^{a} E(z)\mathrm{d}z + V(2a) = \frac{\sigma_0 a}{\varepsilon_0} + \frac{2\sigma_0 a}{5\varepsilon_0} = \frac{7\sigma_0 a}{5\varepsilon_0}\,(\mathrm{V}). \quad (23)$$

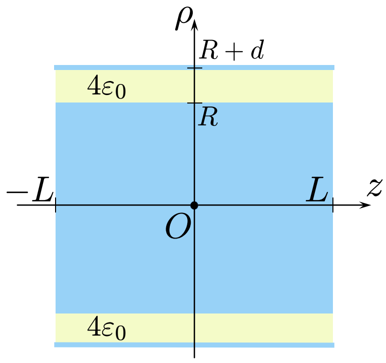

Figure 3: Capacitor configuration – axial cross-section.

25 *points*

Exercise 4

Let the configuration in Fig. 3 consisting of a solid, perfectly conducting kernel of radius R and a perfectly conducting mantle of radius $R+d$, with d being significantly smaller than the overall length $2L$. Between the kernel and the mantle there is a cylindrical slider consisting of a perfect dielectric with permittivity $4\varepsilon_0$ that can frictionlessly slide in the $\hat{z}$ direction. Fringing effects are neglected throughout this exercise.

a) Determine the capacitance of the capacitor. (5 points)

Solution

Let Q being the total *positive* charge on the kernel. Since the configuration is assumed to be at electrostatic equilibrium, this charge must be uniformly distributed on the perfectly conductive kernel.

By applying Gauss's law in the usual manner it can be written that

$$2\pi\rho\, 2L\, E = \frac{Q}{4\varepsilon_0} \Rightarrow \tag{24}$$

$$\vec{E}(\rho) = \frac{Q}{16\pi\varepsilon_0\, L}\,\frac{\hat{\rho}}{\rho}\ \text{(N/C) or (V/m)}. \tag{25}$$

The voltage between the kernel (k) and the mantle (m) is then

$$\Delta V_{\mathrm{km}} = V_{\mathrm{m}} - V_{\mathrm{k}} = -\int_{\mathrm{k}}^{\mathrm{m}} E\mathrm{d}\rho = \frac{Q}{16\pi\varepsilon_0\, L}\ln\left(\frac{R+d}{R}\right)\ \text{(V)}. \tag{26}$$

From the definition of the capacitance we then infer that

$$C = \frac{Q}{\Delta V_{\mathrm{km}}} = \frac{16\pi\varepsilon_0\, L}{\ln\left(\frac{R+d}{R}\right)}\ \text{(F)}. \tag{27}$$

b) This capacitor is connected to a DC voltage source generating the voltage V_0 V, with the $\oplus$ being connected to the kernel. After electrostatic equilibrium was established, the DC source is disconnected. Determine the charge Q_m stored on the mantle. *Both the value and the sign of charge are required!* (3 points)

Solution

The absolute value of the charge on the mantle is

$$|Q_\mathrm{m}| = CV_0 = \frac{16\pi\varepsilon_0\, L\, V_0}{\ln\left(\frac{R+d}{R}\right)} \text{ (C)}. \tag{28}$$

Since the DC voltage was applied with the $\oplus$ on the kernel it follows that

$$Q_\mathrm{m} = -\frac{16\pi\varepsilon_0\, L\, V_0}{\ln\left(\frac{R+d}{R}\right)} \text{ (C)}. \tag{29}$$

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c) Determine the electrostatic energy W_1 stored in the capacitor. (3 points)

Solution

The stored electrostatic energy is

$$W_1 = \frac{C\, V_0^2}{2} = \frac{Q_\mathrm{m}^2}{2C} = \frac{8\pi\varepsilon_0\, L\, V_0^2}{\ln\left(\frac{R+d}{R}\right)} \text{ (J)}. \tag{30}$$

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d) *With the source disconnected*, the slider is slowly moved towards $-\hat{z}$ such that it only protrudes between the kernel and the mantle until z_d, with $-L < z_\mathrm{d} < L$ (see Fig. 4). The remainder of the space between the kernel and the mantle is empty. Calculate the electrostatic energy $W_2(z_\mathrm{d})$ stored in the resulting capacitor. (6 points)

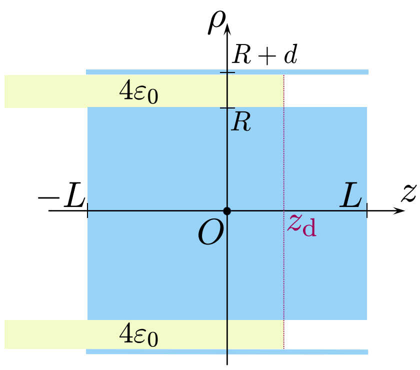

Figure 4: Capacitor configuration / displaced slider – axial cross-section.

Solution

The new capacitor consists of 2 capacitors in parallel

$$C_2(z_\mathrm{d}) = C_{2,4\varepsilon_0}(z_\mathrm{d}) + C_{2,\varepsilon_0}(z_\mathrm{d}) = \frac{4\pi\varepsilon_0}{\ln\left(\frac{R+d}{R}\right)}\left[4(L+z_\mathrm{d}) + L - z_\mathrm{d}\right] = \frac{4\pi\varepsilon_0}{\ln\left(\frac{R+d}{R}\right)}(5L+3z_\mathrm{d})\ (\mathrm{C}). \tag{31}$$

Since the DC voltage source is disconnected, $|Q_\mathrm{m}|$ remains constant, implying that the stored electrostatic energy becomes

$$W_2(z_\mathrm{d}) = \frac{Q_\mathrm{m}^2}{2C_2(z_\mathrm{d})} = \frac{16^2\pi^2\varepsilon_0^2L^2V_0^2}{\left[\ln\left(\frac{R+d}{R}\right)\right]^2}\frac{\ln\left(\frac{R+d}{R}\right)}{2\cdot 4\pi\varepsilon_0\,(5L+3z_\mathrm{d})} = \frac{32\pi\varepsilon_0L^2V_0^2}{\ln\left(\frac{R+d}{R}\right)}\frac{1}{5L+3z_\mathrm{d}}\ (\mathrm{J}). \tag{32}$$

---

e) Determine the force $\vec{F}(z_\mathrm{d})$ exerted on the slider in the positive $\hat{z}$ direction – *a vectorial quantity is requested.* (5 points)

Solution

The force in the positive $\hat{z}$ direction is calculated as

$$F(z_\mathrm{d}) = -\frac{\mathrm{d}W_2(z_\mathrm{d})}{\mathrm{d}z_\mathrm{d}} = \frac{32\pi\varepsilon_0L^2V_0^2}{\ln\left(\frac{R+d}{R}\right)}\frac{3}{(5L+3z_\mathrm{d})^2} \tag{33}$$

implying that

$$\vec{F}(z_\mathrm{d}) = \frac{96\pi\varepsilon_0L^2V_0^2}{\ln\left(\frac{R+d}{R}\right)}\frac{1}{(5L+3z_\mathrm{d})^2}\,\hat{z}\ (\mathrm{N}). \tag{34}$$

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f) Calculate the value of this force for $z_\mathrm{d} \downarrow -L$ and $z_\mathrm{d} \uparrow L$. Give a physical interpretation of these results. (3 points)

Solution

By taking the limits it is found that

$$\lim_{z_\mathrm{d}\downarrow -L}\vec{F}(z_\mathrm{d}) = \frac{24\pi\varepsilon_0V_0^2}{\ln\left(\frac{R+d}{R}\right)}\,\hat{z}\ (\mathrm{N}) \tag{35}$$

and

$$\lim_{z_\mathrm{d}\uparrow L}\vec{F}(z_\mathrm{d}) = \frac{3\pi\varepsilon_0V_0^2}{2\ln\left(\frac{R+d}{R}\right)}\,\hat{z}\ (\mathrm{N}). \tag{36}$$

The capacitance is minimum when the slider is practically pulled out and maximum when it is fully plugged in. Since $|Q_\mathrm{m}|$ is constant, the energy decreases as the slider gets in, and so does the force exerted on that slider. Note that the force is not zero when $z_\mathrm{d} \uparrow L$.

Interpret: The fact that the force is not zero when $z_\mathrm{d} \uparrow L$ does not imply that the slider will move, since this is a *virtual* force that were exerted *if the slider were displaced*! For the given system, the stored electrostatic energy is minimum when the slider is fully plugged in, that is then the stable state of this system.