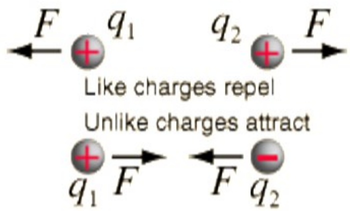
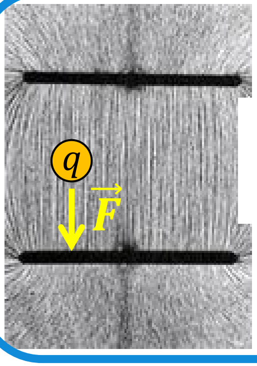
Truly Important from Lecture 1-2

1)



$$\vec{F}_{12}(\vec{r}, q_1, q_2) = \frac{1}{4\pi\epsilon_0}\frac{q_1 q_2}{r_{12}^2}\,\hat{r}_{12}$$

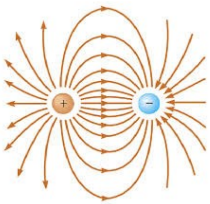
2)



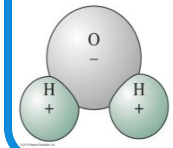
$$\vec{E}(\vec{r}) \equiv \frac{\vec{F}(\vec{r})}{q}$$

3)

$$\vec{E}(\vec{r}) = \sum_{i=1}^{N} \frac{k_e q_i}{|\vec{r}-\vec{r}_i|^2}\frac{(\vec{r}-\vec{r}_i)}{|\vec{r}-\vec{r}_i|}$$



4)



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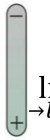
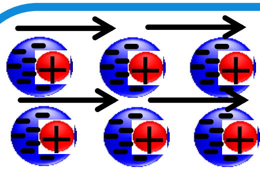


$$\lim_{\to large} \vec{E}(\vec{r}) \approx \frac{k_e}{r^3}\left((-\vec{p}) + \frac{3r\hat{r}\cdot\vec{p}}{r^2}r\hat{r}\right)$$

5)



ϵ_0

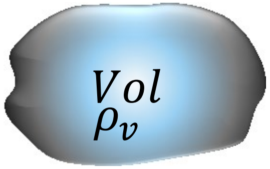
$\boldsymbol{E}_{\text{ext}}$

$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_p^{ave}}$$

$\epsilon_0\epsilon_r$

$\boldsymbol{E}_{\text{tot}}^{\boldsymbol{ave}}$

6)



$$\vec{E}(\vec{r}) = \iiint_{Vol} \frac{k_e \rho_v(\vec{r}\,')}{|\vec{r}-\vec{r}\,'|^2}\frac{(\vec{r}-\vec{r}\,')}{|\vec{r}-\vec{r}\,'|}\,d\vec{r}\,'$$

EE1P21 Electricity and Magnetism

Gauss Law

Learning Objectives

Use of gauss Law as a tool to evaluate the electric fields

Topic 3

Gauss Law in Integral Form
Use of Gauss Law to evaluate the electric fields
Divergence
Gauss Law in Local Form

Quantification of the field

Besides the amplitude an important property of the electric field is its polarization

$$\vec{E}(\vec{r}) = E_x(\vec{r})\hat{x} + E_y(\vec{r})\hat{y} + E_z(\vec{r})\hat{z}$$

$$\vec{E}(\vec{r}) = E(\vec{r})\hat{e}$$

$$\hat{e} = \frac{\vec{E}(\vec{r})}{E(\vec{r})}$$

Polarization

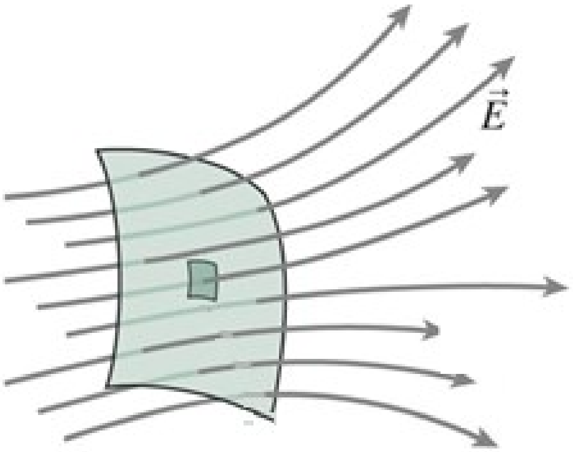
$$E(\vec{r}) = \sqrt{E_x^2(\vec{r}) + E_y^2(\vec{r}) + E_z^2(\vec{r})}$$

Amplitude

Amplitude $\equiv$ *magnitude in book*

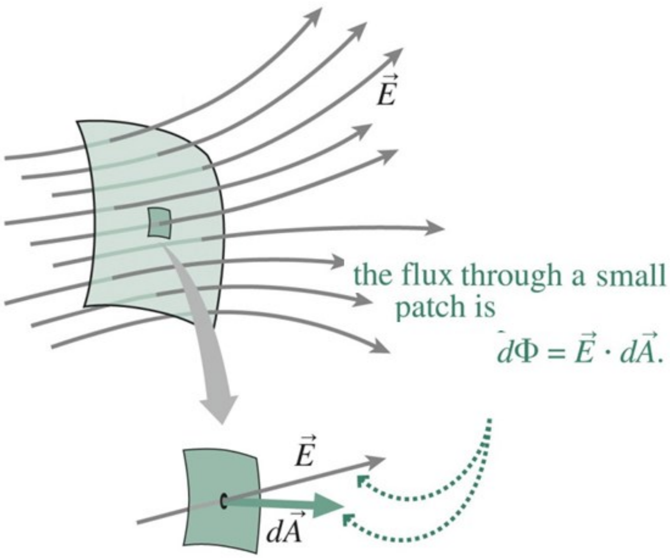
Polarization $\equiv$ *direction in book*

The quantification of the field typically implies knowing the amplitude and the polarization with respect to surfaces



Flux of the Electric Field

Dealing with electric fields (vectors) …
…. you get a scalar after a scalar product



The electric flux on a surface is integral
of the scalar product between the field and a surface

$$\Phi = \iint_{Surface} \vec{E} \cdot d\vec{A} = \iint_{Surface} E \, \cos\theta \, dA$$

$$d\vec{A} = dA \, \hat{\boldsymbol{n}}$$

$$\vec{E} \cdot d\vec{A} = E \, dA \cos\theta$$

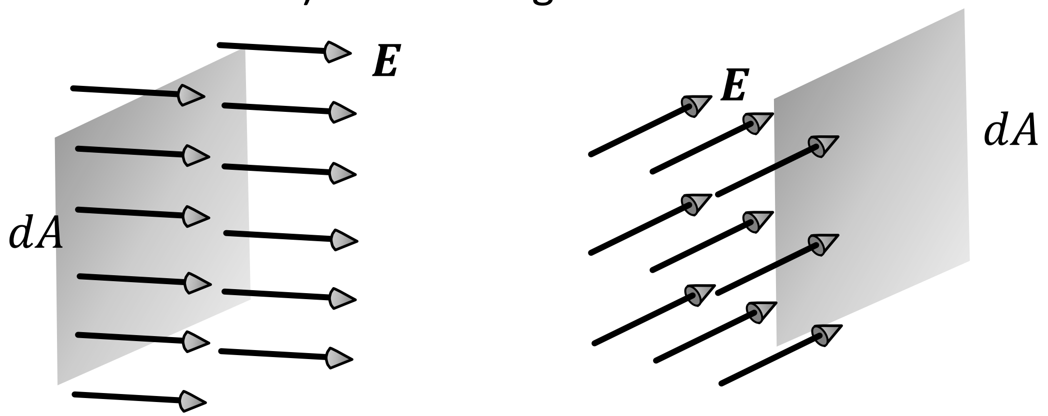
If the surface is planar and E is constant :

$$\Phi = \iint_{Surface} E \, \cos\theta \, dA = E cos\theta \; A$$

Flux of the Electric Field(2)

$$\Phi = \iint_A \vec{E} \cdot d\vec{A}$$

Flux of a vector field (idea arises from the flow of a fluid): How many field lines, (how much electric field) are coming out of a surface?



A surface allows maximum 'flow' when it is normal to the field lines and minimum (zero) when it is parallel to them.

Gauss's Law

The electric flux through any closed surface is proportional to the charge enclosed.

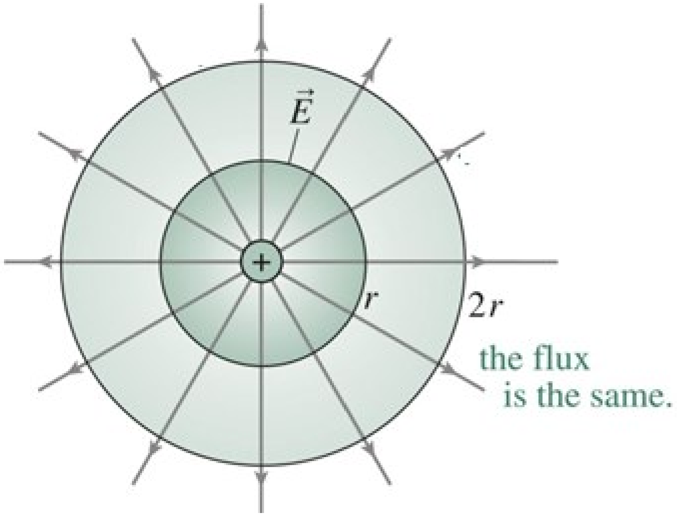
$$\oint\!\!\!\oint_{Closed\,Surf.} \vec{E}\cdot d\vec{A} = \frac{q_{enclosed}}{\epsilon_0 \epsilon_r}$$

Warning: only if it makes sense to define a dielectric constant

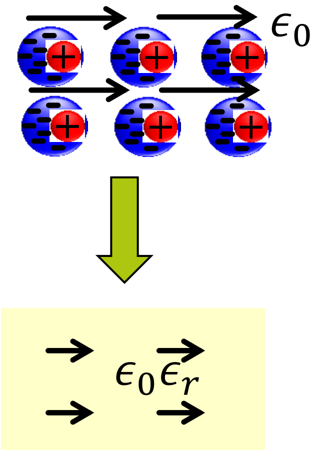
This statement is **Gauss's law.**

Gauss's law is one of the four fundamental laws of electromagnetism.

The flux can be calculated in any surface



Dielectric Constant



$$\vec{E}_{tot}^{ave} = \vec{E}_{ext} + \vec{E}_{p}^{ave}$$

If $\vec{E}_{p}^{ave}$ is parallel to $\vec{E}_{ext}$

It makes sense to define a dielectric constant

$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_{p}^{ave}}$$

If material

- is uniform in all space
- responds linearly
- and responds uniformly in all directions (isotropy)

It makes sense to associate a dielectric constant to the medium

<u>*Warning:*</u> In many practical applications one cannot simply apply the dielectric constant concept. However the deviations are too many, too different and also simple to understand when you need them

Gauss's Law and Coulomb's Law

They are *equivalent*:

both describe the inverse square dependence of the point-charge field.

Coulombs Law

Imagine a charge, q_{encll} in $\vec{r}'=0$

$$\vec{E}(\vec{r},q_{encl}) = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{q_{encl}}{r^2}\hat{r}$$

Gauss Law

$$\oiint_{Closed\ Surf.} \vec{E}\cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0\epsilon_r}$$

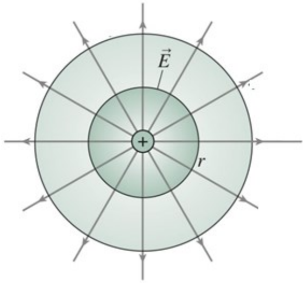
$$\oiint_{Sphere\ of\ radius\ r}$$

$$\int_0^{2\pi}\int_0^{\pi}\frac{q_{encl}}{4\pi\epsilon_0\epsilon_r}\frac{1}{r^2}\hat{r}\cdot\overbrace{\hat{r}\,(r^2 sin\theta d\theta d\phi)}^{d\vec{A}=\hat{n}dA} = \frac{q_{encl}}{\epsilon_0\epsilon_r}$$

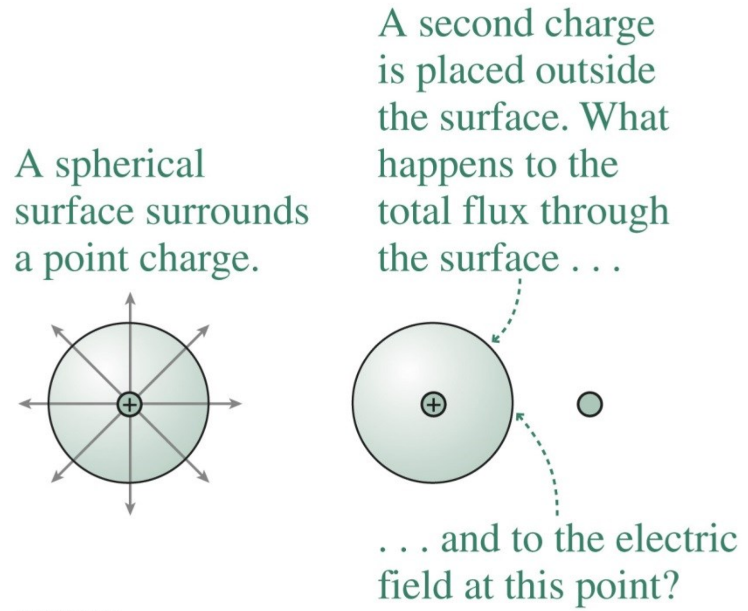
$$\frac{q_{encl}}{4\pi\epsilon_0\epsilon_r}\int_0^{2\pi}\int_0^{\pi} sin\theta d\theta d\phi = \frac{q_{encl}}{\epsilon_0\epsilon_r}$$

$$\frac{q_{encl}}{4\pi\epsilon_0\epsilon_r}4\pi = \frac{q_{encl}}{\epsilon_0\epsilon_r}$$

$$\frac{q_{encl}}{\epsilon_0\epsilon_r} = \frac{q_{encl}}{\epsilon_0\epsilon_r}$$



Use of Gauss's Law



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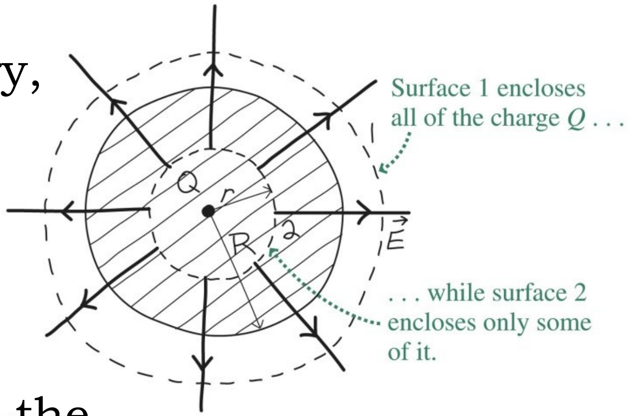
- Gauss's law is useful for calculating the electric field in situations with sufficient symmetry:
  - Spherical symmetry
  - Line symmetry
  - Plane symmetry

- Gauss's law is always true, so it holds in both situations shown.
- Both surfaces surround the same net charge, so the flux through each is the same.
- However, only the left-hand situation has enough symmetry to allow the use of Gauss's law to calculate the field. .

Example, Electric Field of Uniformly Charged Sphere

- **Field:** The field is a vector function of the space coordinates $\vec{E}(\vec{r}) = \vec{E}(r, \theta, \phi)$
- **However:** The situation has spherical symmetry, so field is radial and dependent only from distance.

$$\vec{E}(\vec{r}) = E_r(r)\,\hat{r}$$



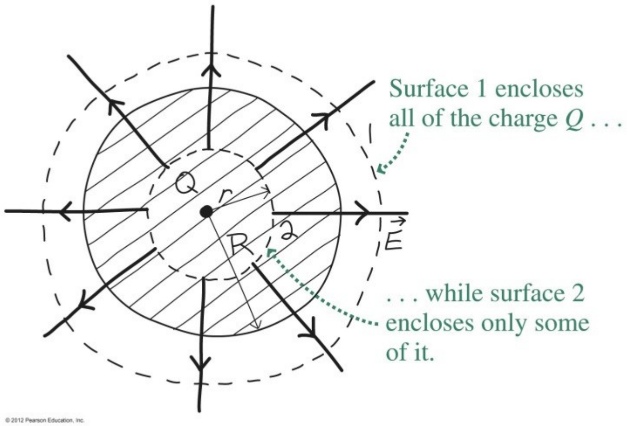
- **Inside/outside:** It's going to be different inside the sphere and outside the sphere

First, the entire charge:

$$\int_0^{2\pi}\int_0^{\pi}\int_0^{R} \rho(\vec{r})dv = \rho_{uni}\int_0^{2\pi}\int_0^{\pi}\int_0^{R} dv = \rho_{uni}\,\frac{4}{3}\pi R^3 = Q$$

The Field **Outside** a Uniformly Charged Sphere

$$\vec{E}(\vec{r}) = E_r \hat{r}(r)$$



$$\oiint_{sphere-R} \vec{E}(\vec{r}) \cdot d\vec{A} = \int_0^{2\pi} \int_0^{\pi} E_r(R)\, \hat{r} \cdot \hat{r}\, R^2 sin\theta d\theta d\phi$$

$$= E_r\ (R) R^2 \int_0^{2\pi} \int_0^{\pi} sin\theta d\theta d\phi$$

$$= E_r(R) R^2 4\pi$$

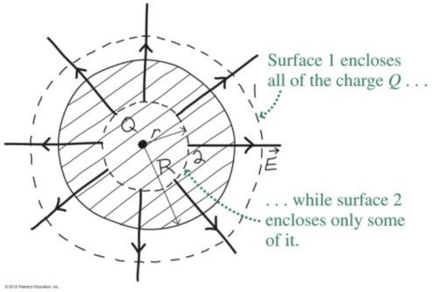
$$\oiint_{sphere-R} \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q}{\epsilon_0 \epsilon_r}$$

$$E_r(R) = \frac{1}{4\pi R^2} \frac{Q}{\epsilon_0 \epsilon_r}$$

If we evaluate the Gauss integral over a larger sphere, for *r>R*,

$$\boxed{E_r(r) = \frac{1}{4\pi r^2} \frac{Q}{\epsilon_0 \epsilon_r}}$$

The Field **Inside** a Uniformly Charged Sphere



$$\vec{E}(\vec{r}) = E_r \hat{r}(r)$$

$$\oiint\limits_{sphere-r<R} \vec{E}(\vec{r}) \cdot d\vec{A} = \int_0^{2\pi}\int_0^{\pi} E_r(r)\, \hat{r} \cdot \hat{r}\, r^2 sin\theta d\theta d\phi$$

$$= E_r\ (r) r^2 4\pi$$

$$\oiint\limits_{sphere-r<R} \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{1}{\epsilon_0 \epsilon_r} \iiint\limits_{sphere-r<R} \rho_{uni} dv$$

$$= \frac{1}{\epsilon_0 \epsilon_r} \rho_{uni} \frac{4}{3} \pi r^3$$

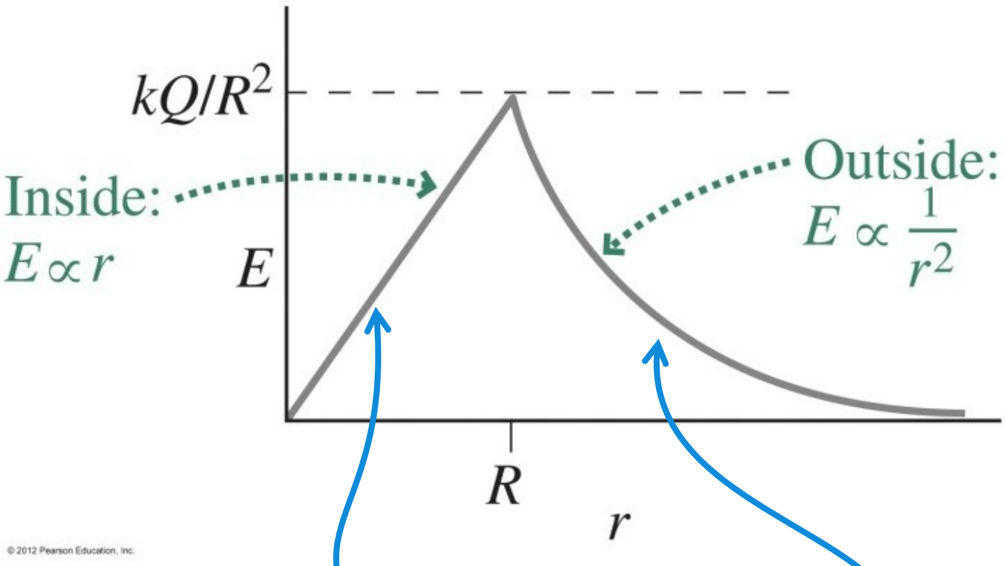
$$= \frac{1}{\epsilon_0 \epsilon_r} \frac{Q}{\frac{4}{3}\pi R^3} \frac{4}{3} \pi r^3$$

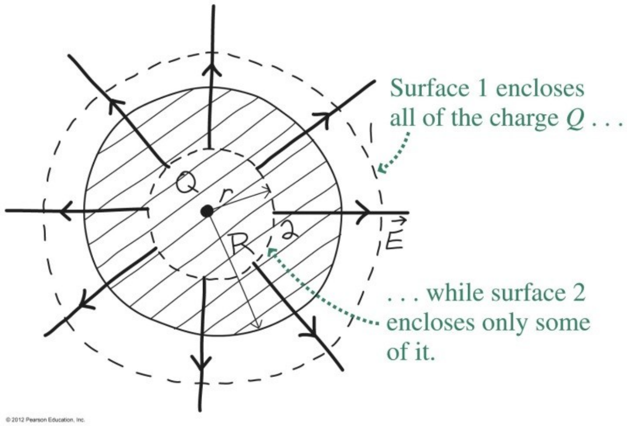
$$= Q \frac{r^3}{R^3} \frac{1}{\epsilon_0 \epsilon_r}$$

$$E_r(r) r^2 4\pi = Q \frac{r^3}{R^3} \frac{1}{\epsilon_0 \epsilon_r}$$

$$\boxed{E_r(r) = \frac{Q}{4\pi} \frac{1}{\epsilon_0 \epsilon_r} \frac{r}{R^3}}$$

The Field of a Uniformly Charged Sphere





$$E_r(r) = \frac{Q}{4\pi\epsilon_0\epsilon_r}\frac{r}{R^3}$$

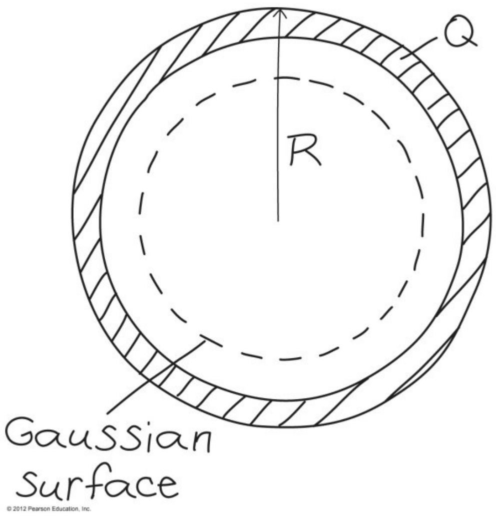
$$E_r(r) = \frac{Q}{4\pi\epsilon_0\epsilon_r}\frac{1}{r^2}$$

Shielding

- Applying Gauss's law to a hollow spherical shell is similar to that for a spherical charge, but now the enclosed charge is zero.

$$4\pi r^2 \epsilon_0 E(r) = q_{enclosed} = 0$$

  - Therefore the field inside the shell is zero.



Differential form of Gauss Law

$$\oiint_{Surface} \vec{E}(\vec{r}) \cdot d\vec{A} = \frac{Q}{\epsilon_0 \epsilon_r}$$

This is a law that lends itself to experiments (only form of Gauss Law in book)

However there is a form of Gauss Law that is equivalent but much more used in advanced EM

Provides a punctual relation between the electric field and the charges

The ones of you that want to have a shot at engineering should follow carefully

Towards the Divergence

Flux of a vector (e.g., flux of the electric field)
across a closed surface A

$$\vec{F} \equiv (F_x, F_y, F_z) \qquad d\vec{A} = \hat{n} dA$$

$$\Phi = \oiint_A \vec{F} \cdot d\vec{A}$$

$\hat{n}$ unit vector normal to the incremental surface element dA

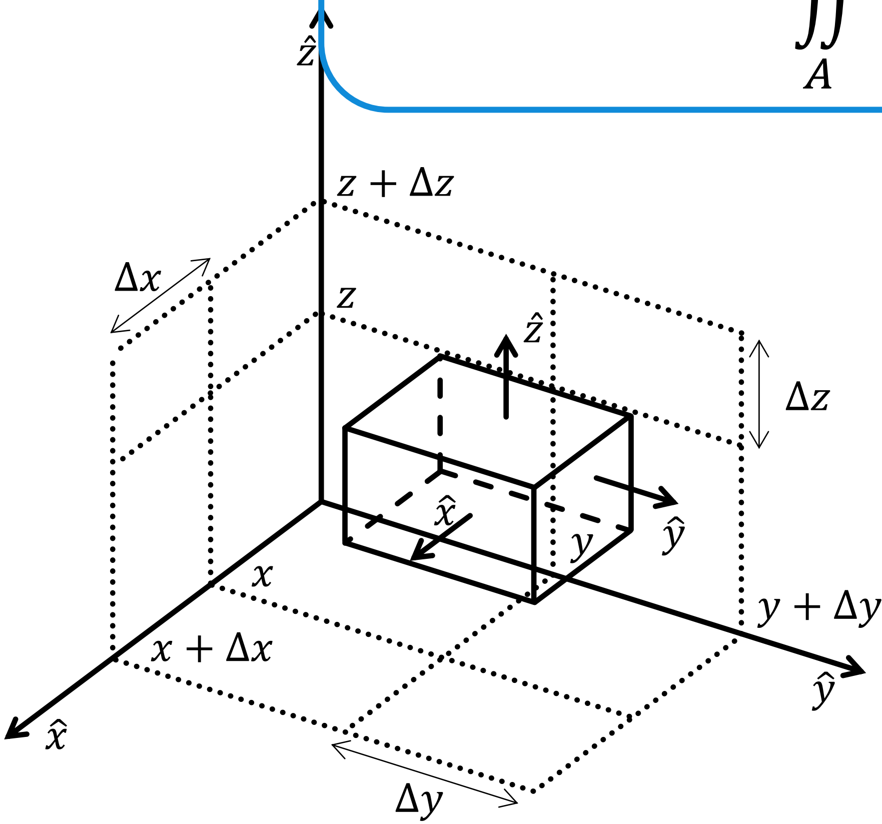
From previous courses:

Scalar Product (also Dot or Inner Product)

$\vec{a} \equiv (a_x, a_y, a_z)$

$\vec{b} \equiv (b_x, b_y, b_z)$

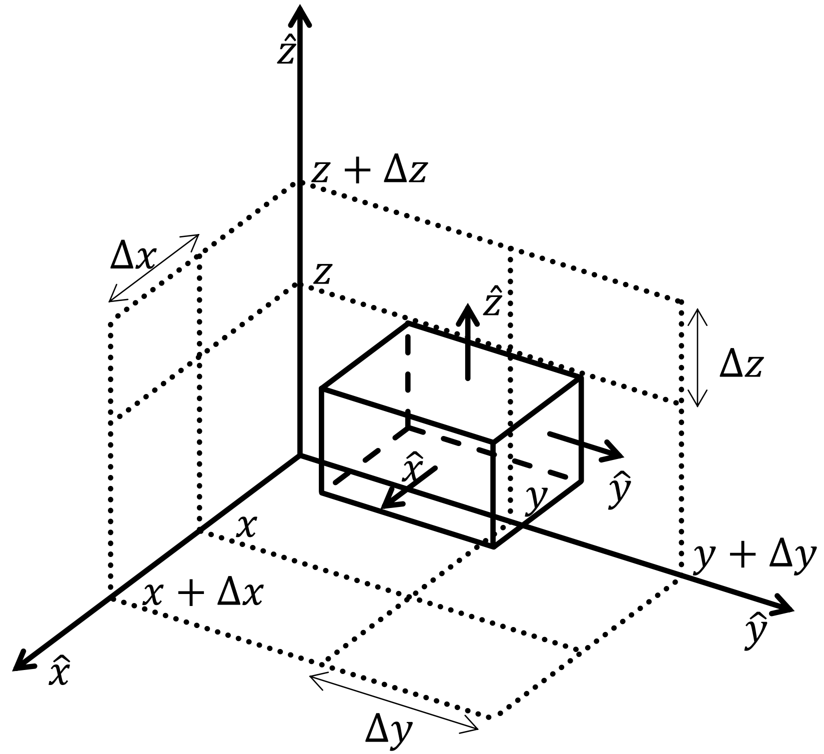
$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$



Let us take a cube like structure

Towards the Divergence (2)

ΔV **small enough** such that $\vec{F}$ components can be approximated as **constant** within the **surfaces** which delimit the volume.
I.e. there can be a variation, but if we assume a constant field equal to the average we do not get significant errors



$$\Phi = \oiint_A \vec{F} \cdot d\vec{A}$$

$$= \int_y^{y+\Delta y} \int_z^{z+\Delta z} \vec{F}(x+\Delta x)\cdot\hat{x}\, dydz + \int_x^{x+\Delta x} \int_z^{z+\Delta z} \vec{F}(y+\Delta y)\cdot\hat{y}\, dxdz + \int_x^{x+\Delta x} \int_y^{y+\Delta y} \vec{F}(z+\Delta z)\cdot\hat{z}\, dxdy$$
$$+ \int_y^{y+\Delta y} \int_z^{z+\Delta z} \vec{F}(x)\cdot(-\hat{x})\, dydz + \int_x^{x+\Delta x} \int_z^{z+\Delta z} \vec{F}(y)\cdot(-\hat{y}) dxdz + \int_x^{x+\Delta x} \int_y^{y+\Delta y} \vec{F}(z)\cdot(-\hat{z}) dxdy$$

$$= F_x(x+\Delta x)\Delta y\Delta z + F_y(y+\Delta y)\Delta x\Delta z + F_z(z+\Delta z)\Delta x\Delta y - F_x(x)\Delta y\Delta z - F_y(y)\Delta x\Delta z - F_z(z)\Delta x\Delta y$$

Towards the Divergence (3)

$$\Phi = [F_x(x+\Delta x) - F_x(x)]\Delta y \Delta z + [F_y(y+\Delta y) - F_y(y)]\Delta x \Delta z + [F_z(z+\Delta z) - F_z(z)]\Delta x \Delta y$$

$$= \left[\frac{F_x(x+\Delta x) - F_x(x)}{\Delta x}\right]\Delta x \Delta y \Delta z + \left[\frac{F_y(y+\Delta y) - F_y(y)}{\Delta y}\right]\Delta x \Delta y \Delta z + \left[\frac{F_z(z+\Delta z) - F_z(z)}{\Delta z}\right]\Delta x \Delta y \Delta z$$

$$= \left[\frac{F_x(x+\Delta x) - F_x(x)}{\Delta x}\right]\Delta V + \left[\frac{F_y(y+\Delta y) - F_y(y)}{\Delta y}\right]\Delta V + \left[\frac{F_z(z+\Delta z) - F_z(z)}{\Delta z}\right]\Delta V$$

Dividing by the volume and doing the limit for small volume

$$\lim_{\Delta V \to 0} \frac{\Phi}{\Delta V} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oint\!\!\!\oint_A \vec{F} \cdot d\vec{A}$$

$$= \lim_{\Delta x \to 0} \left[\frac{F_x(x+\Delta x) - F_x(x)}{\Delta x}\right] + \lim_{\Delta y \to 0} \left[\frac{F_y(y+\Delta y) - F_y(y)}{\Delta y}\right] + \lim_{\Delta z \to 0} \left[\frac{F_z(z+\Delta z) - F_z(z)}{\Delta z}\right]$$

Nabla and Divergence

From previous courses:
Derivative Definition

$$\frac{d}{dx}f(x) = \lim_{h\to 0}\left[\frac{f(x+h)-f(x)}{h}\right]$$

$$\lim_{\Delta V\to 0}\frac{1}{\Delta V}\oiint_A \vec{F}\cdot d\vec{A}$$

$$= \lim_{\Delta x\to 0}\left[\frac{F_x(x+\Delta x)-F_x(x)}{\Delta x}\right] + \lim_{\Delta y\to 0}\left[\frac{F_y(y+\Delta y)-F_y(y)}{\Delta y}\right] + \lim_{\Delta z\to 0}\left[\frac{F_z(z+\Delta z)-F_z(z)}{\Delta z}\right]$$

$$= \frac{\partial}{\partial x}F_x + \frac{\partial}{\partial y}F_y + \frac{\partial}{\partial z}F_z \quad = \nabla\cdot\vec{F}$$

$$\nabla \equiv \left(\frac{\partial}{\partial x},\frac{\partial}{\partial y},\frac{\partial}{\partial z}\right)$$

Nabla vector

Divergence Operator Definition:

$$\nabla\cdot\vec{F} = \lim_{\Delta V\to 0}\frac{\oiint_A \vec{F}\cdot d\vec{A}}{\Delta V} \qquad (\textit{when the limit exhists})$$

Gauss Theorem in Local Form

$$\Phi = \oiint_A \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0 \varepsilon_r}$$

Gauss Theorem

$$\lim_{\Delta V \to 0} \frac{\Phi}{\Delta V} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_A \vec{E} \cdot d\vec{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{Q}{\varepsilon_0 \varepsilon_r}$$

Gauss Theorem for small volumes

$$\nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{\rho \Delta V}{\varepsilon_0 \varepsilon_r} = \lim_{\Delta V \to 0} \frac{\rho}{\varepsilon_0 \varepsilon_r} = \frac{\rho}{\varepsilon_0 \varepsilon_r} \quad \rightarrow \quad \nabla \cdot \vec{E} = \frac{\rho}{\varepsilon_0 \varepsilon_r}$$

Gauss Theorem in Local (differential) form

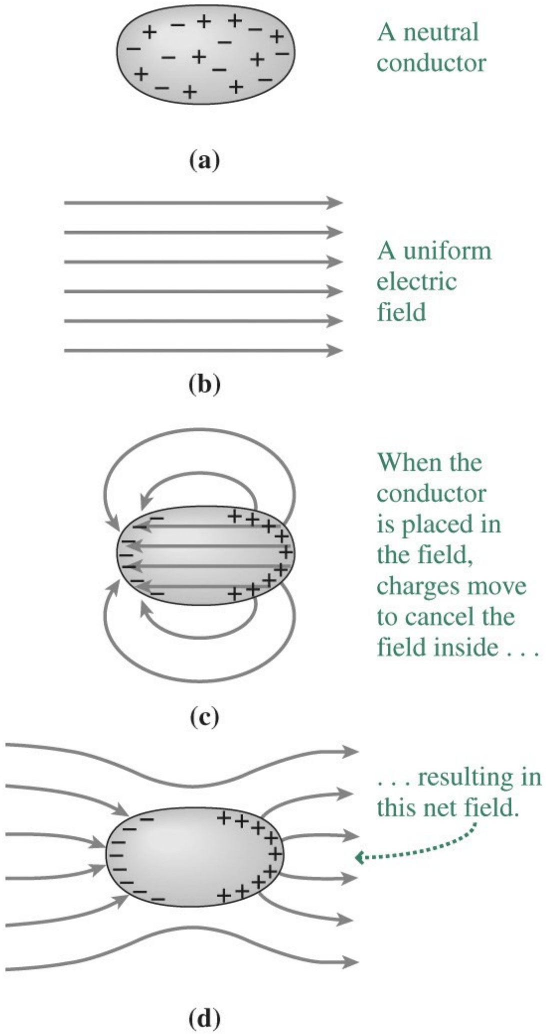
$$\boxed{\oiint_A \vec{E} \cdot d\vec{A} = \frac{Q}{\varepsilon_0 \varepsilon_r} \quad \xRightarrow{\lim_{\Delta V \to 0}} \quad \nabla \cdot \vec{E}(\vec{r}) = \frac{\rho(\vec{r})}{\varepsilon_0 \varepsilon_r}}$$

Electrostatic Field Inside Conductors=0

Charges in conductors are free to move, and they do so in response to an applied electric field.

- If a conductor is allowed
  to reach **electrostatic equilibrium**, a condition in which there is no net charge motion, then charges redistribute themselves to cancel the applied field inside the conductor.

- Therefore **the electric field is zero inside a conductor in electrostatic equilibrium.**



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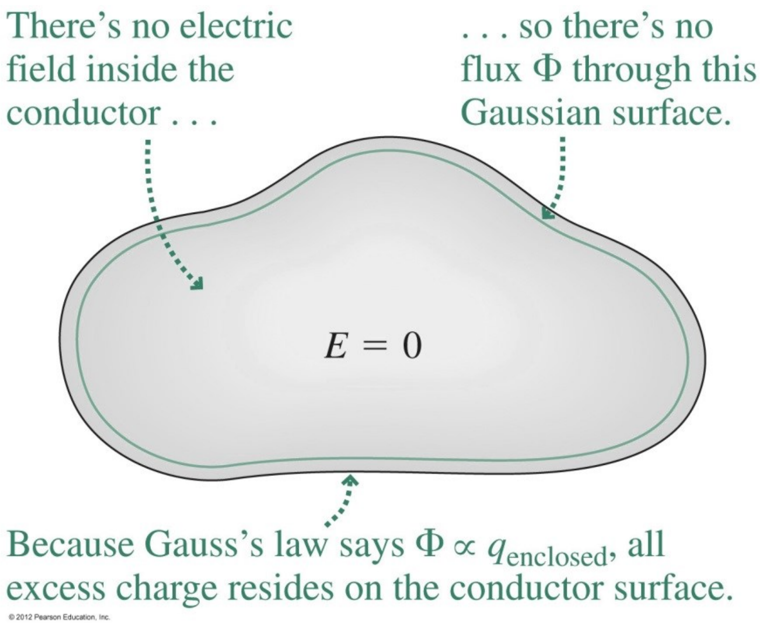
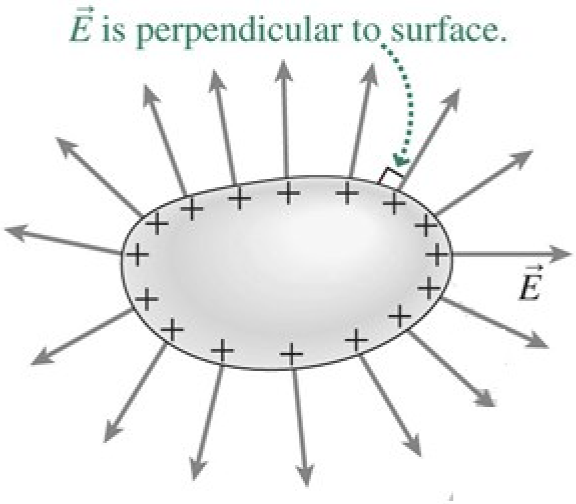
Charged Conductors

- Gauss's law requires that any free charge on a conductor reside on the conductor surface.



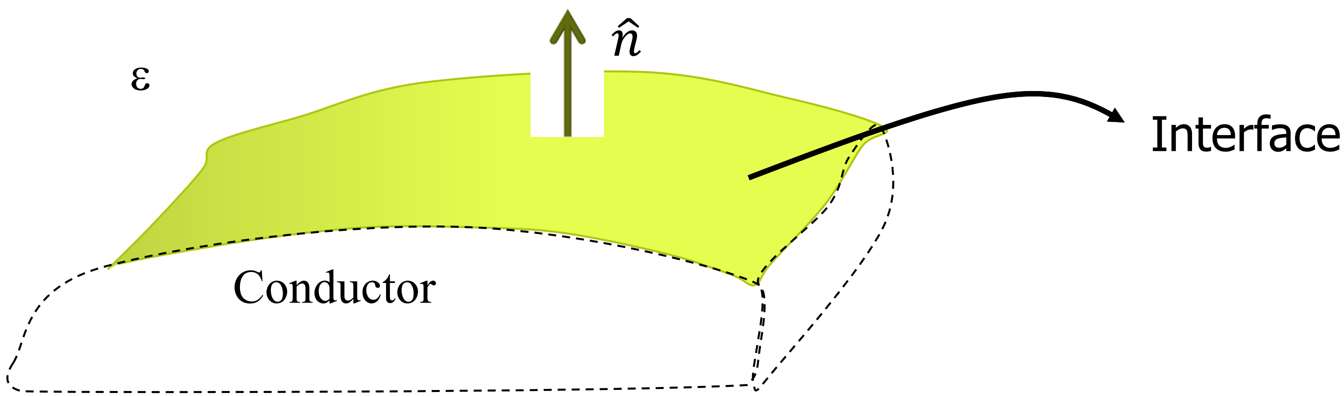
The polarization of Electric Field at a Conductor Surface

- The electric field at the surface of a charged conductor in electrostatic equilibrium is perpendicular to the surface.
    - **If it weren't, charge would move along the surface until equilibrium was reached.**



The intensity of the Electric Field at a Conductor Surface

Let us consider two media (1) and (2), characterized by ε, and perfect conductivity respectively and separated by a certain interface.

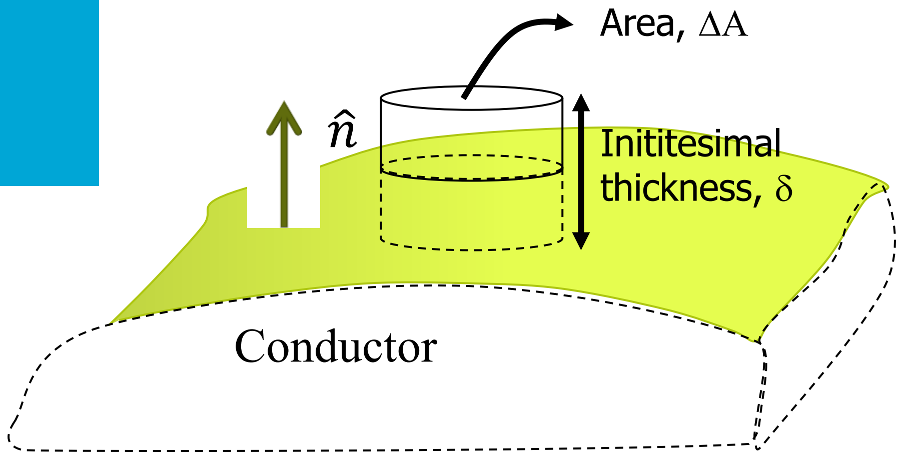


Let us indicate with $\hat{n}$ the normal outside the metal

What happens to the electric fields at the boundaries?

We will establish relations for electrostatic field using Gauss law in integral form

The intensity of the Electric Field at a Conductor Surface



Gauss' law: the electrical charges are the sources of the electrical field

$$\iiint_V \nabla \cdot \vec{E}\, dV = \iiint_V \frac{\rho_V}{\epsilon_0 \epsilon_r} dV$$

Integrating over the volume LHS and RHS

Using the divergence theorem on the LHS

S=entire cylinder

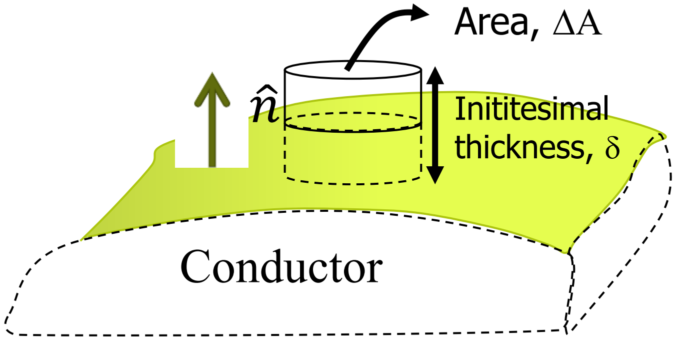
$$\oiint_S \vec{E} \cdot d\vec{A} = \iiint_V \frac{\rho_V}{\epsilon_0 \epsilon_r} dV$$

Surface conductivity

Let us imagine, that the following decomposition is valid

$$\rho_V(\rho,\phi,z) = \rho_V^{\rho,\phi}(\rho,\phi)\rho_V^z(z)$$

It is certainly valid when we investigate small thickness



$$\int_0^a \int_0^{2\pi} \int_{-\delta/2}^{\delta/2} \rho_v(\rho,\phi,z)\rho d\rho d\phi dz$$

Considering, ΔA so small that charge is constant w.r.t. ρ and ϕ

$$\sim \rho_v^{\rho,\phi}(\rho,\phi) \int_0^a \int_0^{2\pi} \rho d\rho d\phi \int_{-\delta/2}^{\delta/2} \rho_v^z(z)dz$$

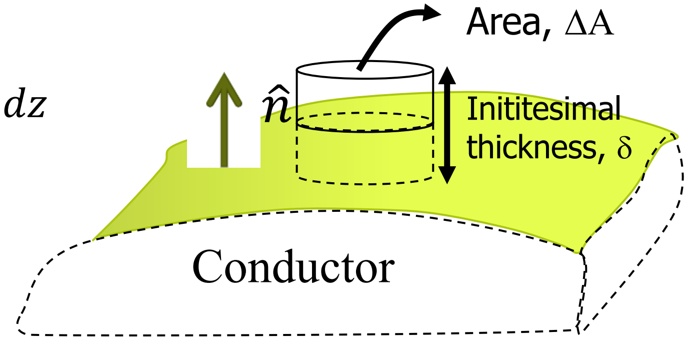
$$\sim \rho_v^{\rho,\phi}(\rho,\phi)\pi a^2 \int_{-\delta/2}^{\delta/2} \rho_v^z(z)dz$$

If charge distribution is volumetric the integral is zero

$$\lim_{\delta\to 0} \iiint_V \rho_v \, dv = 0$$

Surface conductivity (2)

$$\sim\rho_v^{\rho,\phi}(\rho,\phi)\pi a^2 \int_{-\delta/2}^{\delta/2} \rho_v^z(z)dz$$



$$\lim_{\delta\to 0}\int_{-\delta/2}^{\delta/2} \rho^z(z)dz \neq 0$$

C is a constant

$$\rho^z(z) = C\delta(z)$$

$$\rho_V(\rho,\phi,z) = \rho_V^{\rho,\phi}(\rho,\phi)C\delta(z) = \rho_S^{\rho,\phi}(\rho,\phi)\,\delta(z)$$

Surface charge distribution

$$\int_{-\delta/2}^{\delta/2} \delta(z)dz = 1$$

DELTA FUNCTION

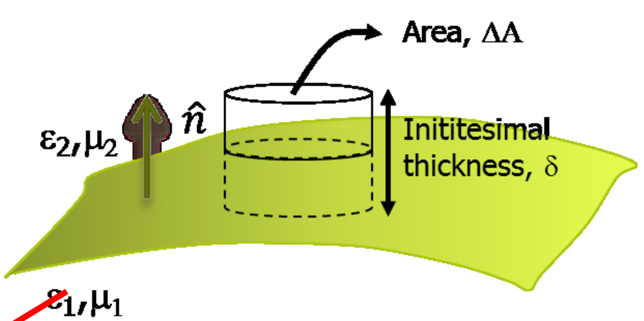
So having indicated

$$\pi a^2 = \Delta A;\ \rho_S^{\rho,\phi}(\rho = 0,\phi) = \rho_s$$

$$\lim_{\delta\to 0}\iiint_V \rho_v\, dv = \rho_s \Delta A$$

Boundary Conditions

$$\oiint_S \vec{E} \cdot d\vec{s} = \frac{\rho_s \Delta A}{\epsilon_0 \epsilon_r} \qquad \iiint_V \frac{\rho_V}{\epsilon_0 \epsilon_r} dV$$



$$\oiint_S \vec{E} \cdot d\vec{s} = \iint_{\Delta A} \vec{E} \cdot \hat{n} ds + \cancel{\iint_{\Delta A} \vec{E} \cdot -\hat{n} ds} + \cancel{\iint_{\Delta lat} \vec{E} \cdot d\hat{s}}$$

$$\lim_{\delta \to 0} \iint_{\Delta lat} \vec{E} \cdot d\hat{s} = 0$$

Because the lateral surface goes to zero,

$$\iint_{\Delta A} \vec{E} \cdot -\hat{n} ds = 0$$

Because field in conductor is zero

$$\iint_{\Delta A} \vec{E} \cdot \hat{n} ds = \iint_{\Delta A} E_n \hat{n} \cdot \hat{n} ds \sim E_n(\rho = 0, \frac{\delta}{2}) \Delta A$$

$$E_n(\rho = 0, \frac{\delta}{2}) \Delta A = \frac{\rho_s \Delta A}{\epsilon_0 \epsilon_r}$$

$$\boxed{\boldsymbol{E_n\left(z = \frac{\delta}{2}\right) = \frac{\rho_s}{\epsilon_0 \epsilon_r}}}$$

Truly Important from Lecture 3

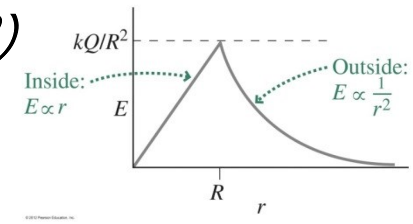
1)

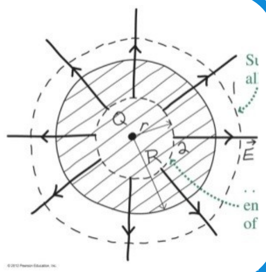
$$\oiint_{Closed\ Surf.} \vec{E} \cdot d\vec{A} = \frac{q_{enclosed}}{\epsilon_0 \epsilon_r}$$

$$\Updownarrow$$

$$\vec{F}_{12} = \frac{1}{4\pi\epsilon_0\epsilon_r}\frac{q_1 q_2}{r_{12}^2}\hat{r}_{12}$$

2)





3)

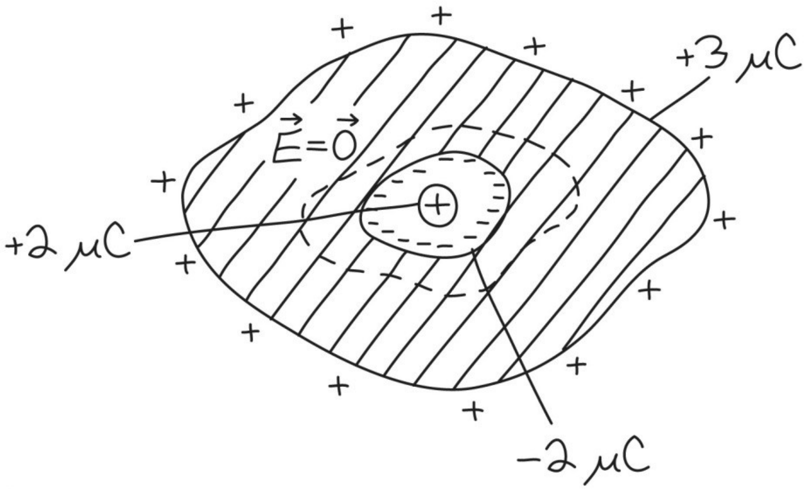
$$\nabla \cdot \vec{E} = \frac{\rho_V}{\epsilon_0 \epsilon_r}$$

5)

6)

$$E_n = \frac{\rho_s}{\epsilon_0 \epsilon_r}$$

Charged hollow Conductors



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- When charge resides inside a hollow, charged conductor, then there may be charge on the inside surface of the conductor.

This charged conductor (shaded) carries a net charge of 1 μC. There's a 2-μC point charge within a hollow cavity in the conductor. Notice how the charge redistributes itself to be consistent with Gauss's law.