Solution Proef Tentamen #1

Exercise 1

(a) $$\vec{E}_1(x,y,z) = \frac{kQ_1}{\left[(x-x_1)^2+(y-y_1)^2+(z-z_1)^2\right]^{3/2}}\left((x-x_1)\hat{x}+(y-y_1)\hat{y}+(z-z_1)\hat{z}\right)$$

(V/m) (N/C).

(b) $$\vec{F}(x_2,y_2,z_2) = \frac{kQ_1Q_2}{\left[(x_2-x_1)^2+(y_2-y_1)^2+(z_2-z_1)^2\right]^{3/2}}\left((x_2-x_1)\hat{x}+(y_2-y_1)\hat{y}+(z_2-z_1)\hat{z}\right)$$

(N)

(c) $dq = \rho_s \, dx\, dy = 2(x^2+y^2+9)^{3/2} dx\, dy$ (C)

(d) $$d\vec{E}(0,0,0) = \frac{dq(\vec{x}')}{\left[x'^2+y'^2+z^2\right]^{3/2}}\left[-x'\hat{x}-y'\hat{y}+|z|\hat{z}\right]$$

$$= \frac{2(x'^2+y'^2+9)^{3/2} dx'dy'}{\left[x'^2+y'^2+9\right]^{3/2}}\left[-x'\hat{x}-y'\hat{y}+3\hat{z}\right]$$

$$= 2\left[-x'\hat{x}-y'\hat{y}+3\hat{z}\right]dx'dy'.$$

$$\vec{E}(0,0,0) = -2\hat{x}\int_{-2}^{2}dy'\int_{-2}^{2}x'dx' - 2\hat{y}\int_{-2}^{2}dx'\int_{-2}^{2}y'dy' + 6\hat{z}\int_{-2}^{2}dx'\int_{-2}^{2}dy'$$

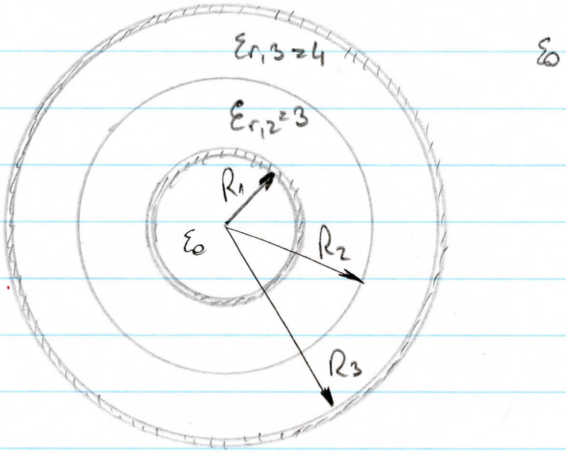
$$= -8\hat{x}\left.\frac{x'^2}{2}\right|_{-2}^{2} - 8\hat{y}\left.\frac{y'^2}{2}\right|_{-2}^{2} + 6\hat{z}\cdot 16$$

$$= 96\hat{z} \ (\text{V/m})$$

Interpret: ρ_s is symmetrical in x and y, hence $E_x = E_y = 0$.

Exercise 2

ⓐ



ⓑ For $r < R_1$: application of Gauss's theorem for a spherical Gaussian surface with $r < R_1$ yields.

$$4\pi r^2 E_1 = 0 \implies \vec{E}_1 = \vec{0} \text{ V/m}$$

For $r > R_3$: application of Gauss's theorem for a region bounded inwardly by a spherical surface taken inside the outer sheet and outwardly bounded by a spherical surface with $R_3 < r$ yields:

$$4\pi r^2 E_4 = \frac{Q_{enclosed}}{\varepsilon_0} = \frac{-Q}{\varepsilon_0} \implies \vec{E}_4(r) = -\frac{Q}{4\pi\varepsilon_0 r^2}\,\hat{r} \text{ (V/m)}$$

ⓒ For $R_1 < r < R_2$: application of Gauss's theorem for a region bounded inwardly by a spherical surface taken inside the inner sheet and outwardly bounded by a spherical surface with $R_1 < r < R_2$ yields:

$$4\pi r^2 E_2 = \frac{Q_{enclosed}}{\varepsilon_{r,2}\varepsilon_0} = \frac{2Q}{3\varepsilon_0} \implies \vec{E}_2(r) = \frac{Q}{6\pi\varepsilon_0 r^2}\,\hat{r} \text{ (V/m)}$$

For $R_2 < r < R_3$: similarly

$$4\pi r^2 E_3 = \frac{Q_{enclosed}}{\varepsilon_{r,3}\varepsilon_0} = \frac{2Q}{4\varepsilon_0} \implies \vec{E}_3(r) = \frac{Q}{8\pi\varepsilon_0 r^2}\,\hat{r} \text{ (V/m)}$$

d) At $r=R_2$ the electric field has a jump discontinuity

$$\frac{E_2}{E_3} = \frac{4}{3} = \frac{\varepsilon_{r,3}}{\varepsilon_{r,2}}$$

This is in accordance with the corresponding boundary interface conditions.

Note: This is a supplementary material that was not discussed this year. You should not expect such exercises at the exam.

e) On the outer sheet

$$\Delta V_{4\infty} = -\int_{\infty}^{R_3} \vec{E}_4(r)\cdot\hat{r}\,dr = +\int_{\infty}^{R_3} \frac{Q\,dr}{4\pi\varepsilon_0 r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 r}\Big|_{R_3}^{\infty} = -\frac{Q}{4\pi\varepsilon_0 R_3} \Rightarrow V_4 = -\frac{Q}{4\pi\varepsilon_0 R_3}\ (V)$$

Between the two sheets there is a potential difference

$$\Delta V_{24} = -\int_{R_3}^{R_1} \vec{E}\cdot\hat{r}\,dr = -\int_{R_3}^{R_2} E_3\,dr - \int_{R_2}^{R_1} E_2\,dr$$

$$= -\int_{R_3}^{R_2} \frac{Q\,dr}{8\pi\varepsilon_0 r^2} - \int_{R_2}^{R_1} \frac{Q\,dr}{6\pi\varepsilon_0 r^2} =$$

$$= \frac{Q}{\pi\varepsilon_0}\left[\frac{1}{8R_2} - \frac{1}{8R_3} + \frac{1}{6R_1} - \frac{1}{6R_2}\right]$$

$$= \frac{Q}{24\pi\varepsilon_0}\left[\frac{4}{R_1} - \frac{1}{R_2} - \frac{3}{R_3}\right]\ (V) \Rightarrow$$

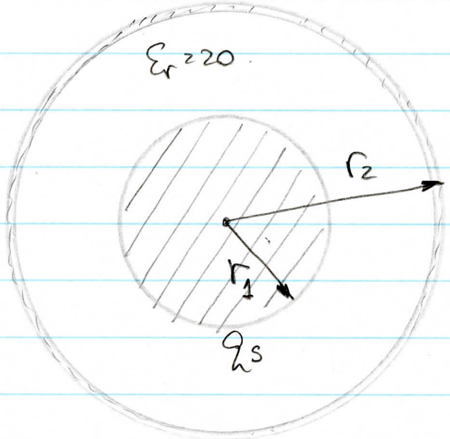
On the inner shell: $V_1 = V_4 + \Delta V_{24}$

(f) The capacity corresponds to a capacitor with +/- 2Q and voltage V_{24}

$$C = \frac{2Q}{\Delta V_{24}} = \frac{2Q}{\frac{Q}{24\pi\varepsilon_0}\left[\frac{4}{R_1} - \frac{1}{R_2} - \frac{3}{R_3}\right]}$$

$$= \frac{48\pi\varepsilon_0}{\frac{4}{R_1} - \frac{1}{R_2} - \frac{3}{R_3}} \quad (F)$$

Exercise 3.



(a) $\lambda = \rho_s \cdot 2\pi r_1 =$
$= 2\pi \cdot 10^{-8} \cdot 5 \cdot 10^{-4}$
$= \pi \cdot 10^{-11}$ (C/m)

(b) Application of Gauss's theorem on a Gaussian cylindrical surface of unit length yields for $r_1 < r < r_2$

$$2\pi r L E = \frac{Q_{enclosed}}{\varepsilon_r \varepsilon_0} = \frac{\lambda L}{20\varepsilon_0} \Rightarrow$$

$$\vec{E} = \frac{\pi \cdot 10^{-11}}{40\pi\varepsilon_0 r}\hat{r} = \frac{10^{-12}}{4\varepsilon_0 r}\hat{r} \quad V/m$$

(c) $$\Delta V_{12} = -\int_{r_2}^{r_1} \vec{E}\cdot\hat{r}\,dr = -\int_{r_2}^{r_1} \frac{10^{-12}}{4\varepsilon_0 r}\,dr = \frac{10^{-12}}{4\varepsilon_0}\ln\frac{r_2}{r_1}$$

$$= \frac{10^{-12}}{4\varepsilon_0}\ln 4 \quad (V)$$

(d) $$C = \frac{\lambda}{\Delta V_{12}} = \frac{\pi \cdot 10^{-11}}{\frac{10^{-12}}{4\varepsilon_0}\ln 4} = \frac{40\pi\varepsilon_0}{\ln 4} \quad (F/m)$$

(e) From Ohm's law it follows that

$$E = \frac{J}{\sigma} \quad \text{with} \quad J = \frac{I}{2\pi r}$$ (recall that I is per unit length!)

$$E = \frac{I}{2\pi r \sigma}$$

$$\Delta V_{12} = -\int_{r_2}^{r_1} \vec{E} \cdot \hat{r}\, dr = -\int_{r_2}^{r_1} \frac{I\, dr}{2\pi\sigma r} = \frac{I}{2\pi\sigma} \ln \frac{r_2}{r_1}$$

$$= \frac{I}{2\pi\sigma} \ln 4 \quad (V)$$

The resistance per unit length is then:

$$R = \frac{\Delta V_{12}}{I} = \frac{\ln 4}{2\pi\sigma} \quad (\Omega/m)$$

Exercise 4.

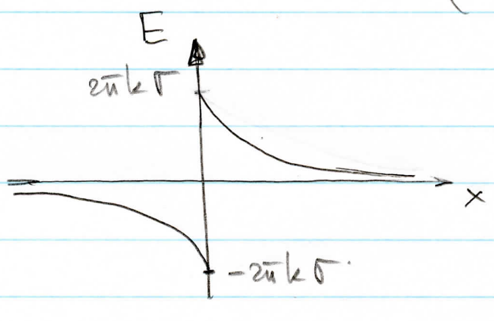
(a) $\vec{E} = -\nabla V = -\partial_x V \hat{x}$

for $x \geq 0$ $\vec{E} = -\partial_x 2\pi k\sigma \left(\sqrt{x^2+R^2} - x\right) \hat{x}$

$$= 2\pi k\sigma \left(1 - \frac{x}{\sqrt{x^2+R^2}}\right) \hat{x} \quad (V/m)$$

for $x < 0$ $\vec{E} = -\partial_x 2\pi k\sigma \left(\sqrt{x^2+R^2} + x\right) \hat{x}$

$$= -2\pi k\sigma \left(1 + \frac{x}{\sqrt{x^2+R^2}}\right) \hat{x}$$



(b) For $x > 0$ $\quad \vec{E}(x) = 2\pi k \sigma \left(1 - \frac{x}{\sqrt{x^2+R^2}}\right)\hat{x}$

$$\lim_{R \to \infty} E(x) = 2\pi k \sigma \quad (V/m)$$

(c) The plate is perfectly conducting, hence the potential on the plate is constant V_0

(d) The electric field is along $\hat{x}$, hence at $x=0$ it automatically satisfies the tangential continuity (the tangential field is 0)

By knowing that $\bar{D} = \varepsilon_0 \vec{E}$

$$\lim_{x \uparrow 0} D = \lim_{x \uparrow 0} \varepsilon_0 E = \lim_{x \uparrow 0} -\varepsilon_0 \frac{2\pi\sigma}{4\pi\varepsilon_0}\left(1 + \frac{x}{\sqrt{x^2+R^2}}\right) = -\frac{\sigma}{2}$$

$$\lim_{x \downarrow 0} D = \lim_{x \downarrow 0} \varepsilon_0 E = \lim_{x \downarrow 0} \varepsilon_0 \frac{2\pi\sigma}{4\pi\varepsilon_0}\left(1 - \frac{x}{\sqrt{x^2+R^2}}\right) = \frac{\sigma}{2}$$

$\lim_{x \downarrow 0} D - \lim_{x \uparrow 0} D = \sigma$ as required by the interface conditions.

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