GAUSS'S LAW

1. OUTLINE

- ✓ Electric field lines
- ✓ Quantification of the field
- ✓ Flux of the electric field
- ✓ Gauss's law
- ✓ Gauss's law and Coulomb's law
- ✓ Use of Gauss's law
- ✓ Towards the divergence operator
- ✓ Differential form of Gauss's law
- ✓ Gauss's law and conductors

2. ELECTRIC FIELD LINES

We can visualize electric fields by drawing **electric field lines**:

- The field lines are continuous lines whose direction is everywhere the same as that of the electric field; i.e., the electric field is tangent to the line in every point of the line itself. Field lines begin on positive charges and either end on negative charges or extend towards infinity.
- Vectors give the electric field's magnitude and direction at specific points (Fig. 1a).
- The spacing of the field lines describes the magnitude of the field. In Fig. 1b the field is stronger where the lines are closer and weaker where they are farther apart.

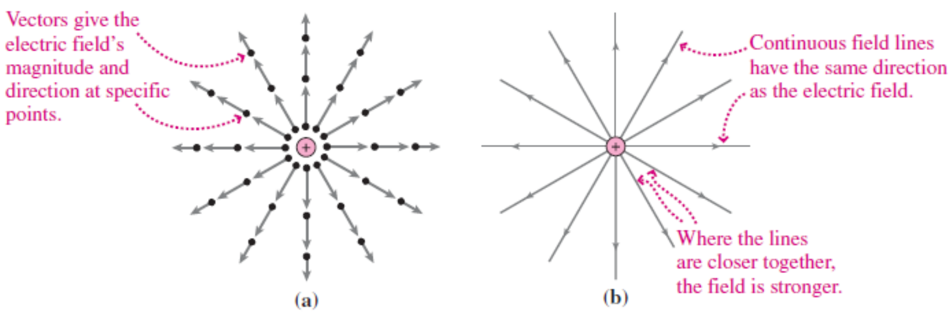

Figure 1. Two different ways to visualize the electric field: (a) vector and (b) field lines

To trace the field lines of a charge distribution one has to follow the direction of the net field, i.e., the vector sum of the field contributions from all charges in the distribution. Usually the field direction varies, so the line is curved. In Fig. 2a the details for one field line of a dipole is shown; whereas Fig. 2b shows a certain number of dipole field lines. It can be seen that the field is stronger near the individual charges and in the region between them.

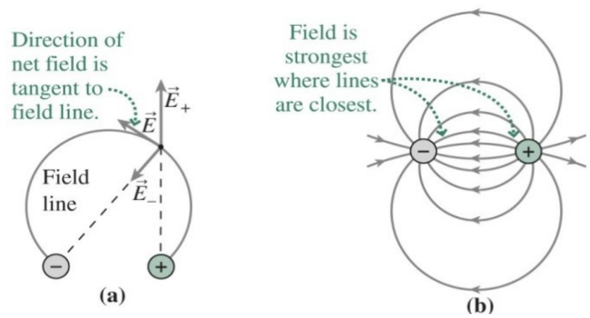

Figure 2. Field of an electric dipole. (a) At each point, the field-line direction is that of the net electric field, $\vec{E} = \vec{E}_+ + \vec{E}_-$. (b) Tracing several field lines shows the overall dipole field.

Notice that the electric field exists everywhere. In theory there is an infinite number of field lines. We obviously can't draw them all. To make field-line pictures, we associate a fixed number of field lines with a charge of a given **magnitude**. In Fig. 3a, 8 field lines correspond to a charge of magnitude q whereas 16 field lines correspond to a charge of magnitude $2q$, as it can be seen in Fig. 3b.

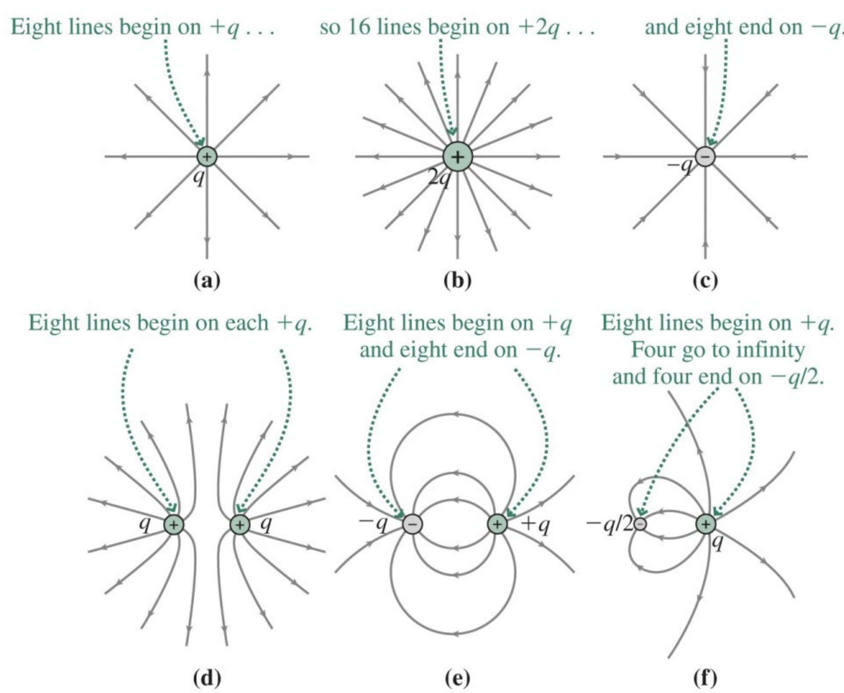

Figure 3. Field lines for six charge distribution.

3. QUANTIFICATION OF THE FIELD

An important property of the fields, in addition to the amplitude, is the polarization. The electric field can be expressed as the sum of its components in the Cartesian coordinates system:

$$\vec{E}(\vec{r}) = E_x(\vec{r})\hat{x} + E_y(\vec{r})\hat{y} + E_z(\vec{r})\hat{z} = E(\vec{r})\hat{e}$$

where $E(\vec{r})$ is the amplitude and $\hat{e}$ the polarization:

$$E(\vec{r}) = \sqrt{E_x^2(\vec{r}) + E_y^2(\vec{r}) + E_z^2(\vec{r})}$$

$$\hat{e} = \frac{\vec{E}(\vec{r})}{E(\vec{r})}$$

The polarization specifies the direction of the electric field.

4. FLUX OF THE ELECTRIC FIELD

The number of field lines crossing a surface depends on three factors: the field strength E, the surface area A, and the orientation of the surface with respect to the field. We get the most field-line crossings when the surface is perpendicular to the field and none when it's parallel (see the example in Fig. 4). If we define a vector $\vec{A}$ normal to the surface, then the number of field-line crossing the surface is proportional to $\cos\theta$, where θ is the angle between the normal vector $\vec{A}$ and the field $\vec{E}$. Therefore, the number of field lines crossing the surface is proportional to $EAcos\theta$. This quantity is called the **electric flux Φ** through the surface. If we set the magnitude of the surface normal vector $\vec{A}$ equal to the surface area A, we can define the flux of the field as the scalar product between the field and the normal vector

$$\Phi = \vec{E} \cdot \vec{A} \tag{1}$$

This equation can be used if the surface is planar and the field is constant. If it is not the case then we have to divide the surface into patches (Fig. 4), each small enough that it is essentially flat and the field is uniform. If each patch has an area dA, where the vector $d\vec{A}$ is normal to the patch, the total flux through the surface is then the sum over all the patches. If we make the patches arbitrarily small, such sum becomes an integral, and the flux is defined as

$$\Phi = \iint_{surface} \vec{E} \cdot d\vec{A} = \iint_{surface} E\, cos\theta\, dA$$

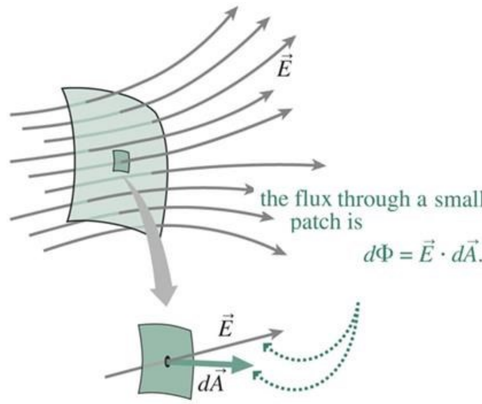

Figure 4. Finding the flux through a small area dA. When it is significantly small it can be considered to be essentially flat.

5. GAUSS'S LAW

Gauss's law asserts that the total outward flux of the electric field over any closed surface in a medium is equal to the total charge enclosed in the surface divided by the permittivity of the medium (assuming the medium to be linear, isotropic, and homogeneous)

$$\oiint_{Closed\ Surf.} \vec{E} \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0 \epsilon_r} \tag{2}$$

The left side of this equation is the mathematical description of the electric flux, i.e., the number of electric field lines passing through a closed surface S; whereas the right side is the total amount of charge contained within such surface divided by the permittivity constant, that is the multiplication of the free space permittivity time the relative permittivity of the medium.

The main concept of Gauss's law in integral form is that **electric charge produces an electric field, and the flux of that field passing through any closed surface is proportional to the total charge contained within that surface.**

Gauss's law is one of the four fundamental relations that governs the behaviour of the electromagnetic fields.

6. GAUSS'S LAW AND COULOMB'S LAW

Gauss's law and Coulomb's law look completely different, but they are closely related. As an example, we imagine a charge q_{encl} in $\vec{r}' = 0$. According to the Coulomb's law the field of a point charge is:

$$\vec{E}(\vec{r}, q_{encl}) = \frac{k_e q_{encl}}{r^2}\hat{r} = \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r r^2}\hat{r} \tag{3}$$

If we apply the Gauss's law to such field and we consider a closed spherical surface around the charge q_{encl} we have that:

$$\oiint_{Closed\,Surf.} \vec{E}\cdot d\vec{A} = \oiint_{Closed\,Surf.} \vec{E}\cdot \hat{r}dA = \int_0^{2\pi}\int_0^{\pi} \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r r^2}\hat{r}\cdot\hat{r}\,(r^2 sin\theta d\theta d\phi)$$

where $d\vec{A} = \hat{n}dA = \hat{r}dA$ and, if we perform the integration on the surface in spherical coordinates, the relevant differential can be written as $dA = r^2 sin\theta d\theta d\phi$. One can evaluate the integral in the following way

$$\int_0^{2\pi}\int_0^{\pi} \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r r^2}\hat{r}\cdot\hat{r}\,(r^2 sin\theta d\theta d\phi) = \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r r^2} r^2 \int_0^{2\pi}\int_0^{\pi} sin\theta d\theta d\phi =$$

$$= \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r r^2} r^2 \int_0^{\pi} sin\theta d\theta \int_0^{2\pi} d\phi = \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r}(-[cos\theta]_0^{\pi})2\pi = \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r}\big(-(-1-1)\big)2\pi =$$

$$= \frac{q_{encl}}{4\pi\epsilon_0\epsilon_r}4\pi = \frac{q_{encl}}{\epsilon_0\epsilon_r} \longleftarrow \quad \text{GAUSS'S LAW}$$

In this way one clearly shows that the field of an elementary charge satisfies the Gauss's law.

7. USE OF GAUSS'S LAW

Gauss's law is a universal statement for electric field: it's true for any surface enclosing any charge distribution. For charge distributions with sufficient symmetry, e.g., symmetric with respect to a point, a line, or a plane, Gauss's law also provides a powerful alternative tool to Coulomb's law that makes electric field calculations much easier. In the following, one can find a representation of the discussed symmetries.

a) Spherical symmetry

A charge distribution has spherical symmetry when the charge density depends only of the radial distance r from the centre of the distribution and it is uniform and constant with respect to the polar coordinates θ and ϕ. The only electric field consistent with spherical symmetry is a field that points in the radial direction, either away from or toward the point of symmetry. Fig. 5 shows an example of spherical symmetry.

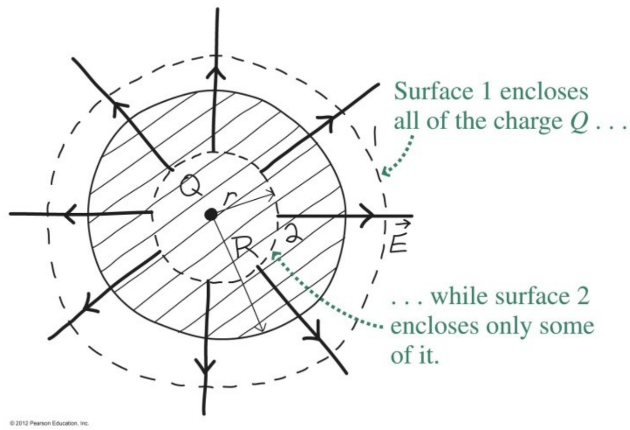

Figure 5. A charge Q is spread uniformly throughout a sphere of radius R.

b) Line Symmetry

A charge distribution has line symmetry when its charge density depends only on the distance r perpendicular to an infinite line. Such symmetry requires that the field point radially and that the magnitude depend only on the distance from the axis. It also requires that the charge distribution has to be infinitely long, so there is no variation parallel to the line. This is impossible, of course, but nevertheless the infinite line is a reasonable approximation for elongated structures like wires. Fig. 6 shows an example of line symmetry.

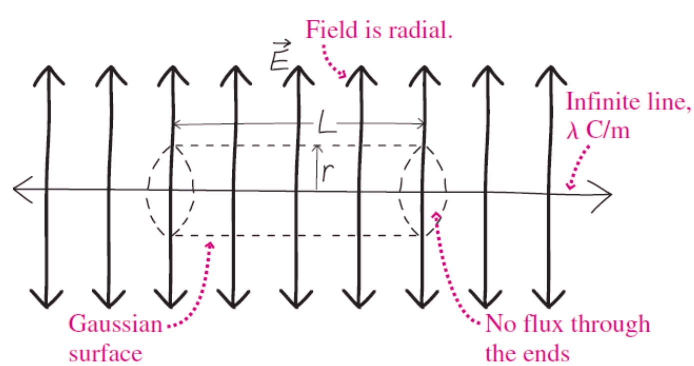

Figure 6. Infinite line charge with charge density λ in coulombs per meter.

c) Plane symmetry

A charge distribution has plane symmetry when its charge density depends only on the perpendicular distance from a plane. The only electric field direction consistent with this symmetry is perpendicular to the plane. As for the line symmetry, true plane symmetry implies charge distributions infinitely extent. Again, this is not feasible, but plane symmetry remains a good approximation when charge is spread uniformly over large, flat surfaces, or slabs. Fig. 7 shows an example of plane symmetry.

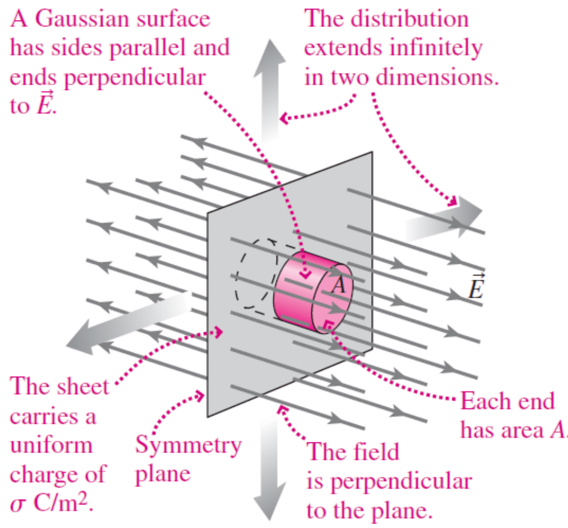

Figure 7. An infinite sheet of charge with uniform surface charge density σ in coulombs per square meter.

8. TOWARDS THE DIVERGENCE OPERATOR

We discussed the concept "flux of a vector" in terms of integral operations. In this section, we introduce a concept which corresponds to that of the flux in terms of differential operations.

The flux of a vector $\vec{F}$ across a surface A is

$$\Phi = \iint_A \vec{F} \cdot d\vec{A}$$

where $\vec{F} = (F_x, F_y, F_z)$, $d\vec{A} = \hat{n}\, dA$, and $\hat{n}$ is the unit vector normal to the incremental surface element dA. From previous courses we know that the Scalar Product (also Dot or Inner Product) of two vectors $\vec{a} = (a_x, a_y, a_z)$ and $\vec{b} = (b_x, b_y, b_z)$ is

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z$$

If we take a cube-like structure as the one in Fig. 8 and we calculate the flux across the closed surface of the cube, i.e., the flux across all the six surfaces of the cube we have:

$$\Phi = \oiint_A \vec{F} \cdot d\vec{A}$$

$$= \int_y^{y+\Delta y} \int_z^{z+\Delta z} \vec{F}(x+\Delta x) \cdot \hat{x}\, dydz + \int_x^{x+\Delta x} \int_z^{z+\Delta z} \vec{F}(y+\Delta y) \cdot \hat{y}\, dxdz +$$

$$\int_{x}^{x+\Delta x}\int_{y}^{y+\Delta y}\vec{F}(z+\Delta z)\cdot\hat{z}\,dxdy+\int_{y}^{y+\Delta y}\int_{z}^{z+\Delta z}\vec{F}(x)\cdot(-\hat{x})\,dydz+$$

$$+\int_{x}^{x+\Delta x}\int_{z}^{z+\Delta z}\vec{F}(y)\cdot(-\hat{y})dxdz+\int_{x}^{x+\Delta x}\int_{y}^{y+\Delta y}\vec{F}(z)\cdot(-\hat{z})dxdy$$

$$=F_x(x+\Delta x)\Delta y\Delta z+F_y(y+\Delta y)\Delta x\Delta z+F_z(z+\Delta z)\Delta x\Delta y-F_x(x)\Delta y\Delta z-F_y(y)\Delta x\Delta z-F_z(z)\Delta x\Delta y$$

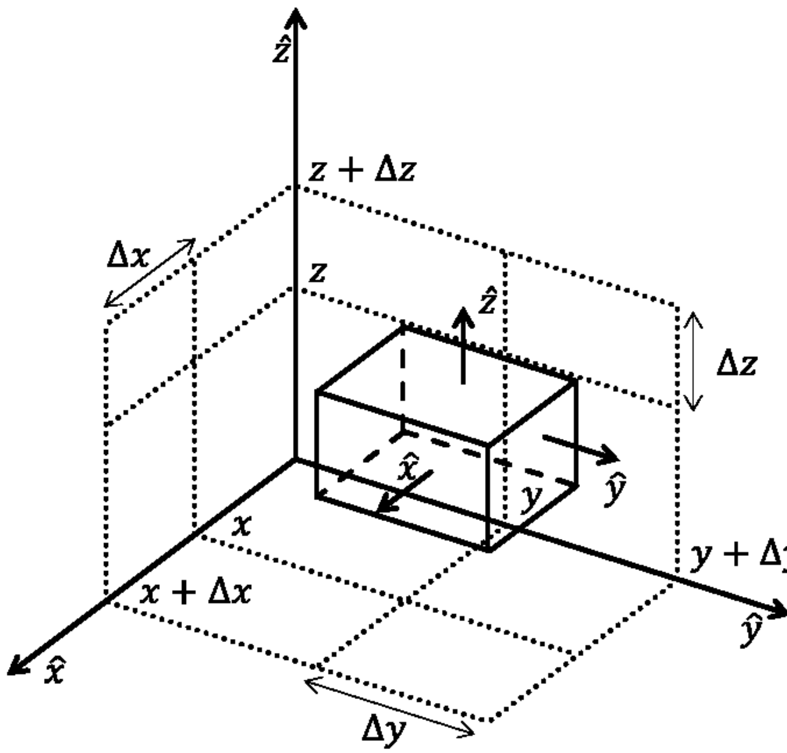

Figure 8. Example of a cube structure with its differential volume.

where we assumed that the volume is **small enough** such that $\vec{F}$ components can be approximated as constant within the surfaces which delimit the volume. Then, the flux can be written as

$$\Phi=[F_x(x+\Delta x)-F_x(x)]\Delta y\Delta z+[F_y(y+\Delta y)-F_y(y)]\Delta x\Delta z+[F_z(z+\Delta z)-F_z(z)]\Delta x\Delta y$$

$$=\left[\frac{F_x(x+\Delta x)-F_x(x)}{\Delta x}\right]\Delta x\Delta y\Delta z+\left[\frac{F_y(y+\Delta y)-F_y(y)}{\Delta y}\right]\Delta x\Delta y\Delta z+\left[\frac{F_z(z+\Delta z)-F_z(z)}{\Delta z}\right]\Delta x\Delta y$$

$$=\left[\frac{F_x(x+\Delta x)-F_x(x)}{\Delta x}\right]\Delta V+\left[\frac{F_y(y+\Delta y)-F_y(y)}{\Delta y}\right]\Delta V+\left[\frac{F_z(z+\Delta z)-F_z(z)}{\Delta z}\right]\Delta V$$

Dividing the flux by the volume and performing the limit for small volume, one gets

$$\lim_{\Delta V \to 0} \frac{\Phi}{\Delta V} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_A \vec{F} \cdot d\vec{A}$$

$$= \lim_{\Delta x \to 0} \left[\frac{F_x(x + \Delta x) - F_x(x)}{\Delta x} \right] + \lim_{\Delta y \to 0} \left[\frac{F_y(y + \Delta y) - F_y(y)}{\Delta y} \right] + \lim_{\Delta z \to 0} \left[\frac{F_z(z + \Delta z) - F_z(z)}{\Delta z} \right]$$

Remembering the definition of the derivative of a function

$$\frac{d}{dx} f(x) = \lim_{h \to 0} \left[\frac{f(x + h) - f(x)}{h} \right]$$

One can write

$$\lim_{\Delta V \to 0} \frac{\Phi}{\Delta V} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_A \vec{F} \cdot d\vec{A} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z$$

If one introduces the operator vector "*nabla*", $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z} \right)$, one can write

$$\lim_{\Delta V \to 0} \frac{\Phi}{\Delta V} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_A \vec{F} \cdot d\vec{A} = \frac{\partial}{\partial x} F_x + \frac{\partial}{\partial y} F_y + \frac{\partial}{\partial z} F_z = \nabla \cdot \vec{F}$$

i.e.,

$$\nabla \cdot \vec{F} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_A \vec{F} \cdot d\vec{A}$$

which defines the differential "*divergence operator*", when the limit exists. This concept will be used in the next section where the differential form of the Gauss's law will be introduced.

9. DIFFERENTIAL FORM OF GAUSS'S LAW

The integral form of Gauss's law for electric fields relates the electric flux over a closed surface to the charge enclosed by that surface. Gauss's law can also be written in *differential* form. For linear, homogeneous, isotropic media, the differential form can be written as

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0 \epsilon_r} \tag{8}$$

The left side of this equation is the divergence of the electric field, whereas the right side is the electric charge density divided by the permittivity of the dielectric medium.

The main idea of Gauss's law in differential form is that **the electric field produced by electric charge densities diverges from positive charge densities and converges upon negative charge densities.** In other words, the only places at which the divergence of the electric field is not zero are those locations at which charge is present. If a positive charge is present, the divergence is positive, meaning that the electric field tends to "flow" away from that location. If a negative charge is present, the divergence is negative, and the field lines tends to "flow" towards that point.

Proof:

Let us derive the differential form of the Gauss's law for the electric field from its relevant integral form. The integral form of Gauss's law for the electric field is

$$\Phi = \oiint_{Closed\,Surf.} \vec{E} \cdot d\vec{A} = \frac{q_{encl}}{\epsilon_0 \epsilon_r}$$

If one considers the limit of the ratio between the flux and the volume enveloped by the surface, for the volume which tends to zero, one gets

$$\lim_{\Delta V \to 0} \frac{\Phi}{\Delta V} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \oiint_{A} \vec{E} \cdot d\vec{A} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{q_{encl}}{\epsilon_0 \epsilon_r}$$

Remembering that the charge q_{encl} can be expressed in terms of the volumetric charge density ρ_v as $q_{encl} = \rho_v \Delta \mathrm{V}$, one can write

$$\lim_{\Delta V \to 0} \frac{\Phi}{\Delta V} = \nabla \cdot \vec{E} = \lim_{\Delta V \to 0} \frac{1}{\Delta V} \frac{\rho_v \Delta V}{\epsilon_0 \epsilon_r} = \lim_{\Delta V \to 0} \frac{\rho_v}{\epsilon_0 \epsilon_r} = \frac{\rho_v}{\epsilon_0 \epsilon_r}$$

thus obtaining,

$$\nabla \cdot \vec{E} = \frac{\rho_v}{\epsilon_0 \epsilon_r}$$

which is the **differential (local) form of Gauss's law** for the electric field.

10. GAUSS'S LAW AND CONDUCTORS

Electrostatic Equilibrium

We define conductors as materials that contain free charges, like the free electrons in metal. Free charges respond to the electric force qE by moving (in the directions of the field if they are positive, or opposite to the field if they are negative). The resulting charge separation gives rise to an electric field within the material that is opposite to the applied field. As more charge moves, the internal field becomes stronger until its magnitude eventually equals that of the applied field. At that point free charges within the conductor experience zero net force, and the conductor is in **electrostatic**

equilibrium. Although individual charges continue to move randomly for thermal motion, there is no longer any net charge motion. Once equilibrium is reached, the internal and applied field are equal but opposite, and therefore **the total electric field is zero inside a conductor in electrostatic equilibrium.**

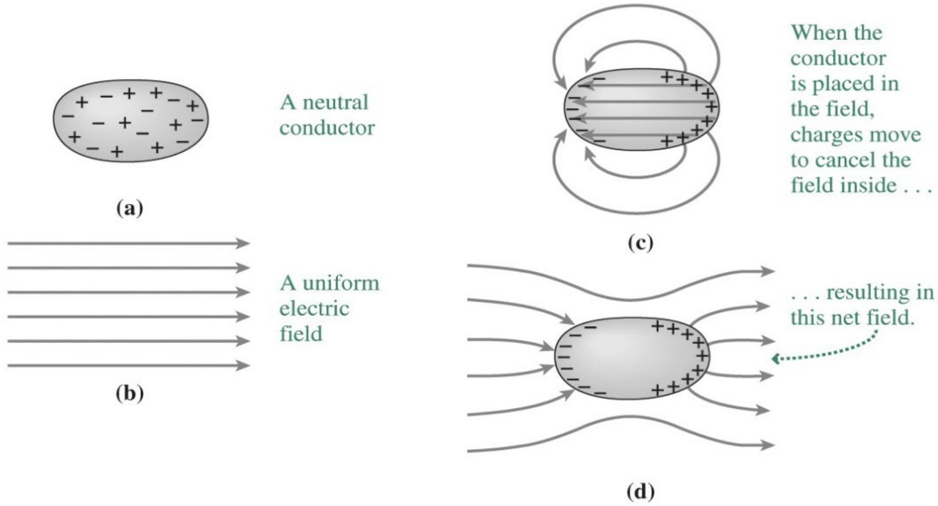

Figure 9. A conductor in a uniform electric field.

Charged Conductors

Although they contain free charges, conductors are normally electrically neutral because they include an equal number of electrons and protons. But suppose we give to a conductor a nonzero net charge, for example, by injecting an excess of electrons into its interior. There is a mutual repulsion among the electrons and, because these are excess electrons, there is no compensating attraction from positive charges. We might expect, therefore, that the electrons will move as far apart as possible, namely, to the surface of the conductor.

We now use Gauss's law to prove that the excess charge must be at the surface of a conductor in electrostatic equilibrium. Fig. 10 shows a conducting material with a Gaussian surface drawn just below the material surface. In equilibrium there is no electric field inside the conductor, and thus the field is zero everywhere on the Gaussian surface. The flux, $\oiint_{Closed\ Surf.} \vec{E} \cdot d\vec{A}$, through the Gaussian surface is therefore also zero. But Gauss's law says that the flux trough a closed surface is proportional to the net charge enclosed, and therefore the net charge inside our Gaussian surface must be zero. This is true no matter where the Gaussian surface is as long as it is inside the conductor. We can move it arbitrarily close to the conductor surface and it still encloses no net charge. If there is a net charge on the conductor, it lies outside the Gaussian surface, and therefore we conclude: **If a conductor in electrostatic equilibrium carries a net charge, it must reside on the conductor surface.**

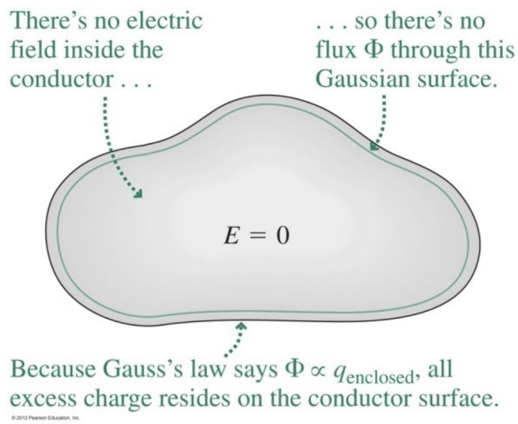

Figure 10. Gauss's law implies that any net charge resides on the surface of a conductor in electrostatic equilibrium.

Surface Charge Distribution on Charged Conductors

There cannot be an electric field within a conductor in electrostatic equilibrium, but there may be a field right at the conductor surface (Fig. 11a). Such a field must be perpendicular to the surface; otherwise, charge would move along the surface in response to the field's parallel component, and we wouldn't have equilibrium.

We can compute the strength of this surface field by considering a small Gaussian surface that straddles the conductor surface, as shown in Fig. 11b. There is no flux through the sides (lateral cylindrical surface in the figure) since the field is orthogonal to the normal vector to the cylindrical surface, and because the field is zero inside the conductor, there is no flux through the inner circular flat surface. Therefore, the only present flux is the one through the outer circular flat surface, with area A. Since such surface is perpendicular to the field, the flux is EA. For an infinitesimal surface element, the net charge on the conductor can be written in terms of its surface charge distribution, ρ_s, as $q_{encl} = \rho_s A$. From the Gauss's law one can equate the flux to $\frac{q_{encl}}{\epsilon_0 \epsilon_r}$ thus obtaining $EA = \rho_s A/\epsilon_0 \epsilon_r$, or

$$E = \frac{\rho_s}{\epsilon_0 \epsilon_r} \tag{9}$$

which is the magnitude of the electric field at the conductor surface. This result shows that a large field is present where the charge density on a conductor is high.

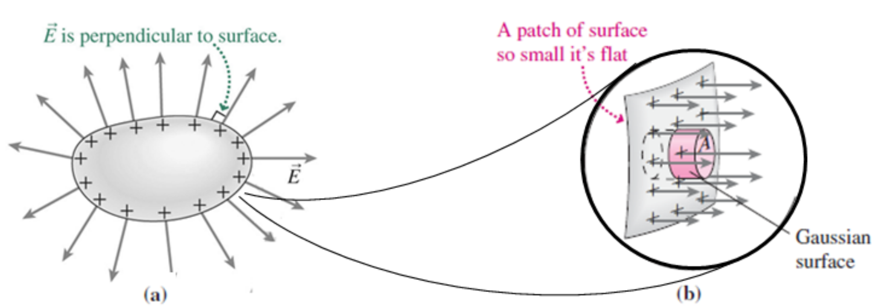

Figure 11. (a) The electric field at the surface of a charged conductor is perpendicular to the conductor surface. (b) The cylindrical Gaussian surface that straddles the conductor surface.