ELECTRIC POTENTIAL

1. OUTLINE

- ✓ Conservative fields
- ✓ Electric potential energy difference
- ✓ Potential of charge distributions
- ✓ Electric field from the potential
- ✓ Towards the curl operator

2. CONSERVATIVE FIELDS

As it was shown in mechanics for vectors describing forces, one can introduce for a vector field the line integral, C, over a certain length, L, defined as

$$C = \int_L \vec{F} \cdot d\vec{l}$$

Such integral can also be performed over a closed length referred as a loop. The scalar quantity obtained by such operation is typically referred as circulation. When the line integral over a loop for a certain field (i.e., the circulation of the field) is zero, such field is referred to as a conservative field. In other words, when

$$\oint \vec{G}(\vec{l}) \cdot d\vec{l} = 0 \tag{1}$$

$\vec{G}$ is said to be conservative. A well-known example of a conservative field is the gravitational field. It can be shown that the static electric field is also conservative, i.e.,

$$\oint \vec{E}(\vec{l}) \cdot d\vec{l} = 0 \tag{2}$$

3. ELECTRIC POTENTIAL ENERGY DIFFERNCE

By using the definition of the electric field and force, it is trivial that if the electric field is conservative, its force also has such properties (see the definitions in lectures 1 and 2). When a force is conservative, it is useful to define a potential energy difference as the amount of work done against such force for moving the "object" that is experiencing the force from one position to another one. This potential energy can be expressed as

$$\Delta U_{AB} = -W_F = -\int_A^B \vec{F}(\vec{l}) \cdot d\vec{l} \tag{3}$$

where ΔU_{AB} is the amount of work done against the force $\vec{F}$, in order to move from point A to B. Since the electric field is defined as the electric force over a test charge, i.e. $\vec{E} = \frac{\vec{F}}{q}$, one can analogously introduce an electric potential difference as

$$\Delta V_{AB} = \frac{\Delta U_{AB}}{q} = -\int_A^B \vec{E}(\vec{l}) \cdot d\vec{l} \tag{4}$$

It is worth noting that in Eq. (4), ΔV_{AB} does not depend on the integration path. This property can be derived from the definition of conservative field. If two arbitrary integration paths are chosen (see Figure 1), using Eq. (2), it can be shown that

$$\int_{\substack{A \\ path\ 1}}^{B} \vec{E}(\vec{l}) \cdot d\vec{l} + \int_{\substack{B \\ path\ 2}}^{A} \vec{E}(\vec{l}) \cdot d\vec{l} = 0$$

Therefore:

$$\int_{\substack{A \\ path\ 1}}^{B} \vec{E}(\vec{l}) \cdot d\vec{l} = -\int_{\substack{B \\ path\ 2}}^{A} \vec{E}(\vec{l}) \cdot d\vec{l} = \int_{\substack{A \\ path\ 2}}^{B} \vec{E}(\vec{l}) \cdot d\vec{l}$$

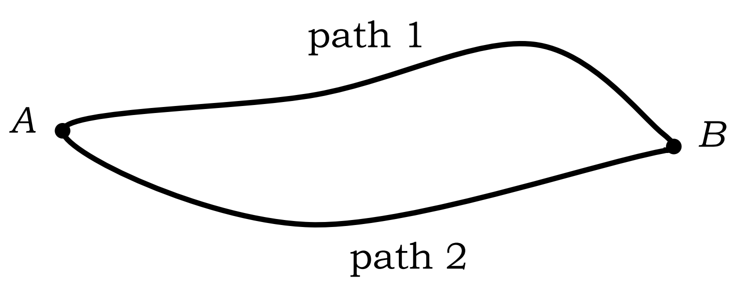

Figure 1. Illustrating two arbitrary line integral paths from the point *A* to the point *B*.

In conclusion, the potential difference only depends on the starting and ending points of interest. Using Eq. (4), one can calculate the potential difference in various electric scenarios. In the following, the electric potential is derived for the cases of a constant field, the field due to a point charge and, in the next section, for the field due to an electric dipole.

a) ELECTRIC POTENTIAL OF A CONSTANT ELECTRIC FIELD

Referring to Fig. 2, if the electric field is constant in a certain direction (e.g. x-direction), so that it can be expressed as $\vec{E} = E_0\hat{x}$, where E_0 is a constant, the electric potential difference can be calculated using Eq. (4), as

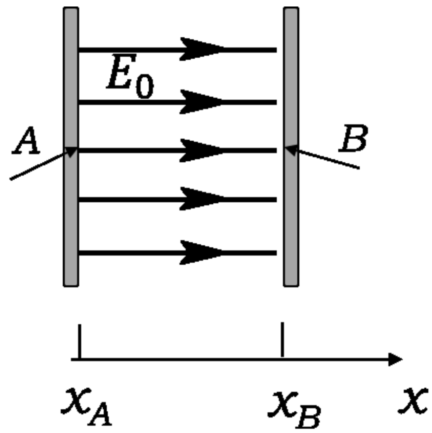

Figure 2. Constant electric field between two infinite parallel plates orthogonal to x-direction.

$$\Delta V_{AB} = -\int_A^B \vec{E}(\vec{l}) \cdot d\vec{l} = -\int_{x_A}^{x_B} E_0\hat{x} \cdot \hat{x}dx = -E_0(x_B - x_A) \qquad (5)$$

b) POTENTIAL OF A POINT CHARGE

The electric field generated by a point charge, q, is expressed as

$$\vec{E} = \frac{k_e q}{r^2}\hat{r} \qquad (6)$$

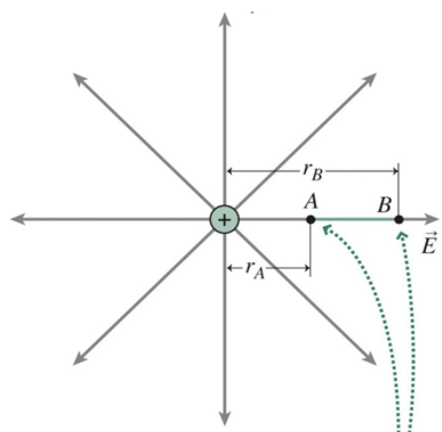

Figure 3. Geometry for the calculation of the electric potential for a point charge.

The potential difference for such field can be calculated by the following radial integral (Fig. 3)

$$\Delta V_{AB} = -\int_A^B \frac{k_e q}{r^2}\hat{r} \cdot d\vec{r} = -k_e q \int_{r_A}^{r_B} \frac{1}{r^2}\hat{r} \cdot \hat{r}dr = k_e q\left(\frac{1}{r_B} - \frac{1}{r_A}\right) \qquad (7)$$

Using the equation above, the potential difference between a point at infinity and the point B can be expressed as

$$\Delta V_{\infty B} = V_B = \frac{k_e q}{r_B} \tag{8}$$

By choosing the reference position at infinity, one can define the potential value V_B at a certain position B.

4. POTENTIAL OF CHARGE DISTRIBUTIONS

As it was discussed in pervious lectures, the electric field generated by a distribution of charges at a certain observation point is equal to the summation of the contribution of each charge. The same superposition principle can be applied for the potential. For a discrete number of charges this leads to the following formula

$$V(\vec{r}) = \sum_{i=1}^{n} \frac{k_e q_i}{|\vec{r} - \vec{r}_i|} \tag{9}$$

where q_i is the point charge placed at the $\vec{r}_i$ position in the global coordinate system, and $\vec{r}$ indicates the position of the observation point. The same discussion can be extended to a continuous charge distribution using the charge density concept, as it was discussed in the first lecture. As an example for a volume charge distribution, one can calculate the potential as

$$V(\vec{r}) = \iiint\limits_{Volume} \frac{k_e \rho(\vec{r}')}{|\vec{r} - \vec{r}'|} \, d\vec{r}' \tag{10}$$

where $\vec{r}'$ indicates the position of an elementary point charge, dq, in a volume with a certain charge density, $\rho(\vec{r}')$.

POTENTIAL OF AN ELECTRIC DIPOLE

An electric dipole with dipole moment of $\vec{p} = q\vec{d}$ is illustrated in the Figure 5. In the following, the formula for calculating the potential at a certain observation point is derived. Let $|\vec{r}| = r$ indicates the distance between the observation point, P, and the center of the dipole; and θ is the angle between $\vec{d}$ and $\vec{r}$. Moreover, referring to Fig. 4, $\vec{r}_1$ and $\vec{r}_2$ are the vectors between the dipole charges and the observation point; and θ_1 and θ_2 are the angles between $\vec{d}$ and $\vec{r}_1$ and $\vec{d}$ and $\vec{r}_2$, respectively. If one assumes that the observation point is far away from the dipole, the following approximations are valid

$$\theta_1 \simeq \theta_2 \simeq \theta$$

$$|\vec{r}_2| \simeq |\vec{r}| - \frac{|\vec{d}|}{2}\cos\theta = r - \frac{d}{2}\cos\theta$$

$$|\vec{r}_1| \simeq |\vec{r}| + \frac{|\vec{d}|}{2}\cos\theta = r + \frac{d}{2}\cos\theta$$

Using the above approximations the potential can be expressed as

$$V(\vec{r}) \simeq \frac{k_e q}{|\vec{r}_2|} - \frac{k_e q}{|\vec{r}_1|} = k_e q\left(\frac{1}{r - \frac{d}{2}\cos\theta} - \frac{1}{r + \frac{d}{2}\cos\theta}\right) \simeq \frac{k_e q}{r}\left(1 + \frac{d}{2r}\cos\theta - \left(1 - \frac{d}{2r}\cos\theta\right)\right) \rightarrow$$

$$V(\vec{r}) \simeq \frac{k_e q d}{r^2}\cos\theta = \frac{k_e|\vec{p}|}{r^2}\cos\theta = \frac{k_e\vec{p}\cdot\hat{r}}{r^2} \tag{11}$$

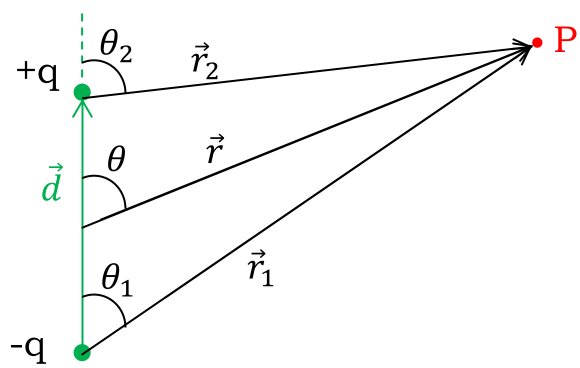

Figure 4. Illustrating the geometry for calculating the potential of an electric dipole.

5. ELECTRIC FIELD FROM THE POTENTIAL

As it was discussed, it is possible to define an electric potential at a certain position using the electric field, as a line integral (Eq. (4)). The potential is a scalar value which is easier to calculate in comparison to the electric field itself. Using the definition of the potential, one can derive an expression for the electric field as below

$$V(B) = -\int_{\infty}^{B}\vec{E}(r)\cdot\hat{r}\,dr \rightarrow dV = -\vec{E}(r)\cdot\hat{r}\,dr \rightarrow \frac{dV}{dr} = -\vec{E}(r)\cdot\hat{r}$$

For each component of the field one can write

$$\begin{cases} \frac{dV}{dx} = -E_x \\ \frac{dV}{dy} = -E_y \\ \frac{dV}{dz} = -E_z \end{cases}$$

In a vectorial notation one has

$$\vec{E} = -\frac{dV}{dx}\hat{x} - \frac{dV}{dy}\hat{y} - \frac{dV}{dz}\hat{z}$$

or, more compactly, by introducing the symbolic vector operator *nabla*, $\nabla \equiv \left(\frac{\partial}{\partial x}, \frac{\partial}{\partial y}, \frac{\partial}{\partial z}\right)$,

$$\vec{E} = -\nabla V \tag{12}$$

From the above equation, one can see that the electric field is oriented toward the maximum spatial variation of the potential. The surfaces where the potential is constant are referred to as equipotential surfaces. An equipotential surface is perpendicular to the direction of the electric field. From the previous lecture, we know that the electric field is orthogonal to the surface of a conductor. Therefore, the surface of a conductor is an equipotential surface. Moreover, since the field is zero everywhere inside the conductor, the whole conductor has the same potential of its surface.

6. TOWARDS THE CURL OPERATOR

The curl operation on a vector field is a vector product between the nabla operator, $\nabla = \frac{\partial}{\partial x}\hat{x} + \frac{\partial}{\partial y}\hat{y} + \frac{\partial}{\partial z}\hat{z}$, and the field. By using the vector product properties, discussed in the first lecture, it can be calculated symbolically as the determinant of the matrix in the following equation

$$\nabla \times \vec{F} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ F_x & F_y & F_z \end{vmatrix} \tag{13}$$

In the Cartesian coordinates frame, Eq. (13) can be explicitly expressed as

$$\nabla \times \vec{F} = \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{x} - \left(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x\right)\hat{y} + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{z} \tag{14}$$

In the following we will see that the meaning of the curl operator is strictly related to the concept of the circulation of a vector field. Without loss of generality, let us consider the circulation of a vector field $\vec{F}$ on a small rectangular path in a plane parallel to the yz-plane (Fig. 5). The path has to be small enough such that the field can be considered constant on each segment. The circulation can be expressed as

$$C = \oint_{L_x} \vec{F} \cdot d\vec{l} =$$

$$\int_y^{y+\Delta y} \vec{F}(z) \cdot \hat{y} dy + \int_z^{z+\Delta z} \vec{F}(y+\Delta y) \cdot \hat{z} dz + \int_y^{y+\Delta y} \vec{F}(z+\Delta z) \cdot (-\hat{y}) dy + \int_z^{z+\Delta z} \vec{F}(y) \cdot (-\hat{z}) dz \tag{15}$$

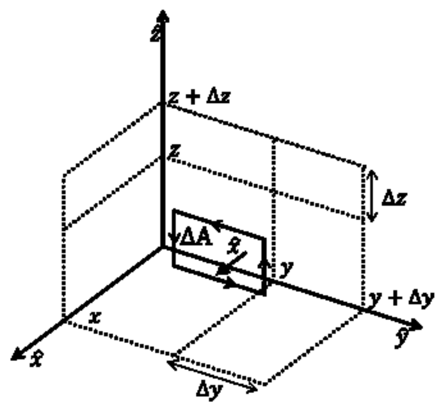

Figure 5: Small rectangular path where the circulation integral is performed.

Since the field is constant over each segment, Eq. (15) can be expressed as

$$C = F_y(z)\Delta y + F_z(y+\Delta y)\Delta z - F_y(z+\Delta z)\Delta y - F_z(y)\Delta z \tag{16}$$

or

$$C = \oint_{L_x} \vec{F}\cdot d\vec{l} = \left[\frac{F_y(z)-F_y(z+\Delta z)}{\Delta z}\right]\Delta y\Delta z + \left[\frac{F_z(y+\Delta y)-F_z(y)}{\Delta y}\right]\Delta y\Delta z \tag{17}$$

The product $\Delta y\Delta z$ give the area ΔA of the rectangle bounded by the path (Fig. 5). If one considers the limit, for the area that tends to zero, of the ratio between the circulation and the area of the rectangle itself, one gets the following expression

$$\lim_{\Delta A\to 0}\frac{1}{\Delta A}\oint_{L_x} \vec{F}\cdot d\vec{l} = \lim_{\Delta y\to 0}\left[\frac{F_z(y+\Delta y)-F_z(y)}{\Delta y}\right] - \lim_{\Delta z\to 0}\left[\frac{F_y(z+\Delta z)-F_y(z)}{\Delta z}\right] = \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right) \tag{18}$$

One can multiply the unit normal vector of the discussed surface, $\hat{x}$, on both side of Eq. (18), thus obtaining

$$\lim_{\Delta A\to 0}\frac{\hat{x}}{\Delta A}\oint_{L_x} \vec{F}\cdot d\vec{l}=\left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{x} \tag{19}$$

If one generalizes the above discussion to a rectangle, arbitrarily oriented in the space, with unit normal vector $\hat{n}$ (Fig. 6), one obtains the following expression

$$\lim_{\Delta A\to 0}\frac{\hat{n}}{\Delta A}\oint_{L} \vec{F}\cdot d\vec{l} = \left(\frac{\partial}{\partial y}F_z - \frac{\partial}{\partial z}F_y\right)\hat{x} - \left(\frac{\partial}{\partial x}F_z - \frac{\partial}{\partial z}F_x\right)\hat{y} + \left(\frac{\partial}{\partial x}F_y - \frac{\partial}{\partial y}F_x\right)\hat{z} \tag{20}$$

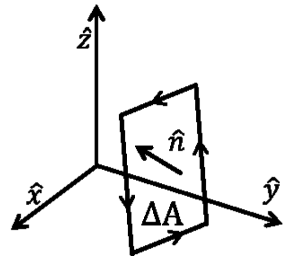

Figure 6: Arbitrarily oriented small rectangular path where the circulation integral is performed..

By comparing Eq. (14) to the right hand side of Eq. (20), it is evident that

$$\nabla \times \vec{F} = \lim_{\Delta A \to 0} \frac{\hat{n}}{\Delta A} \oint_L \vec{F} \cdot d\vec{l} \tag{21}$$

when the limit exists. From Eq. (21) it also is evident how the curl operator is connected to the circulation of a vectorial field $\vec{F}$. The curl provides the microscopic circulation of a vector and quantifies how much the vector field rotates around a point and in which direction (Fig. 7).

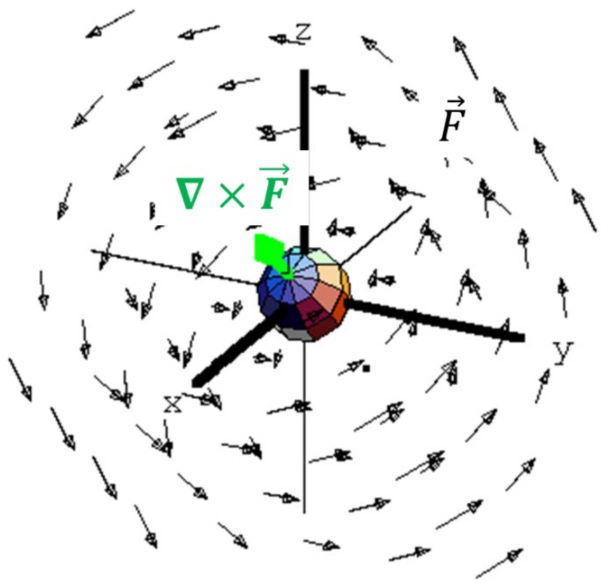

Figure 7: Qualitative representation of the curl operation of a vector field $\vec{F}$.

It is worth also noting that the macroscopic circulation of a field on a closed path is related to the flux of the curl operator vector through the surface bounded by the circulation path, i.e.,

$$\oint_C \vec{F}(\vec{l}) \cdot d\vec{l} = \iint_S (\nabla \times \vec{F}) \cdot d\vec{S} \tag{22}$$

This result is known as Stokes Theorem.

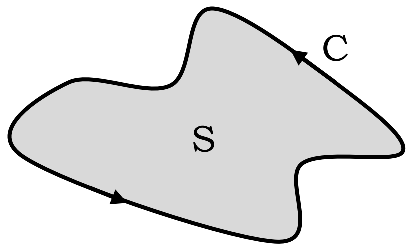

Figure 8: Integration path and surface in the Stokes Theorem.

It is worth noting that, in the electrostatic case, the fact that the circulation of the electric field is zero implies that the curl of the electrostatic field has to be zero, since the equality (in the Stokes Theorem) has to be satisfied by every surface bounded by the same path C, namely

$$\oint_C \vec{E}(\vec{l}) \cdot d\vec{l} = 0 \Rightarrow \iint_S (\nabla \times \vec{E}) \cdot d\vec{S} = 0 \Rightarrow \nabla \times \vec{E} = 0 \tag{23}$$

When the curl of a vector is zero, the vector is said to be irrotational. The electrostatic field is irrotational.

By combining the results from the previous lecture on the Gauss's law and the equation above, one obtains the compact differential axioms which characterize the electrostatic field, i.e.,

$$\begin{cases} \nabla \cdot \vec{E} = \dfrac{\rho_v}{\epsilon_0 \epsilon_r} \\ \nabla \times \vec{E} = 0 \end{cases} \tag{24}$$