Truly Important from Lecture 5

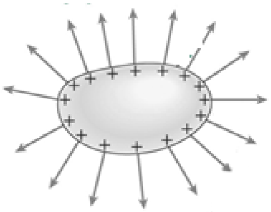
1)

q_1 q_2 q_3 q_N

$$U_{tot} = \frac{1}{2} \sum_{i=1}^{N} q_i V(\vec{r}_i)$$

$$V(\vec{r}_i) = k \sum_{i=1;i\neq j}^{N} \frac{q_i}{r_{ij}}$$

2)



$$U_{tot} = \frac{1}{2} \iiint_{Vol} \rho(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,'$$

3)

$$U_{tot} = \frac{1}{2} \iiint_{Vol} \epsilon_0 \epsilon_r \left|\vec{E}(\vec{r}\,')\right|^2 d\vec{r}\,'$$

EE1P21 Electricity and Magnetism

Capacitors

Topic 6

Storing electrostatic energy

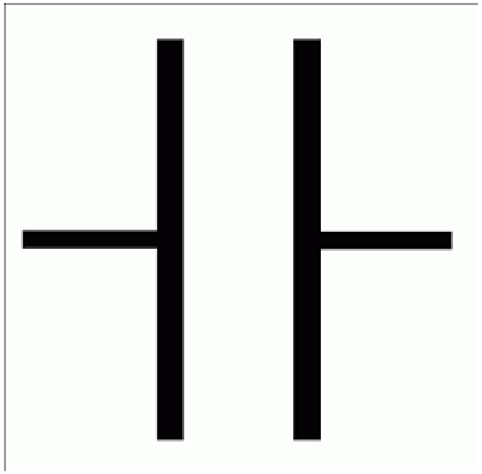
Learning Objectives

What is a capacitor
What is the capacitance
Why they typically involve dielectrics

Capacitors

Devices to store electrostatic energy

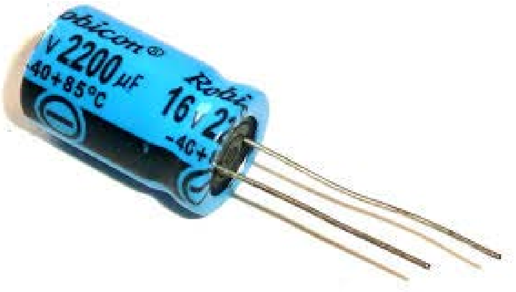
Symbol



- They are composed by a pair of conductors, **insulated** from each other.

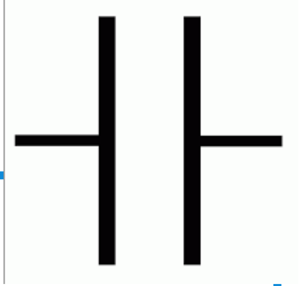
- Opposite and equal charges.

Component



- Work used in separating charge is stored as potential energy in the capacitor.

What we will see today



For all capacitors the **Energy stored** can be expressed as:

$$U_{tot} = \frac{1}{2} Q_1 \Delta V$$

Q_1 *is the charge in each conductor*

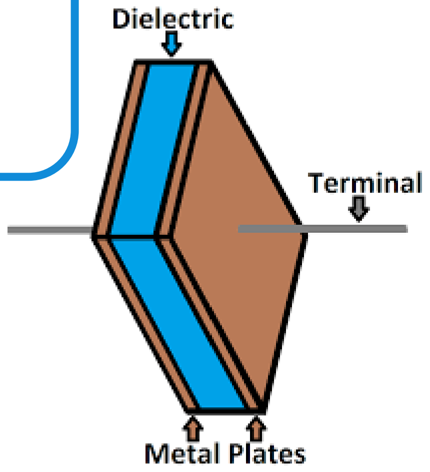
ΔV *is the potential difference between the two conductors*

We will motivate/demonstrate this fact

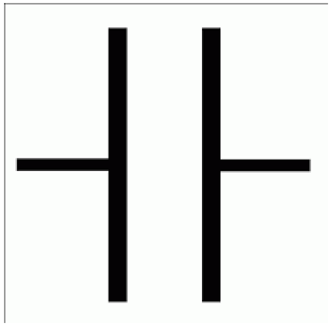
Every capacitor is characterized by ratio of charge and voltage: **Capacitance**

$$C = \frac{Q_1}{\Delta V}$$

We shall see the capacitance for the simplest realization: **Parallel Plate Capacitor**



Capacitor Energy



Energy stored in any system

$$U_{tot} = \frac{1}{2} \iiint\limits_{Vol} \rho(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,'$$

If the charge is superficial

$$\rho(\vec{r}\,') = \sigma(\vec{r}\,')\chi(\vec{r}\,')$$

Existence function

$$\chi(\vec{r}\,') = \begin{cases} 1 & \forall\, \vec{r}\,' \in metal \\ 0 & \forall\, \vec{r}\,' \in (Vol - metal) \end{cases}$$

The Volumetric integral is a surface integral, because only where $\rho(\vec{r}\,') \neq 0$, $\rho(\vec{r}\,')V(\vec{r}\,') \neq 0$

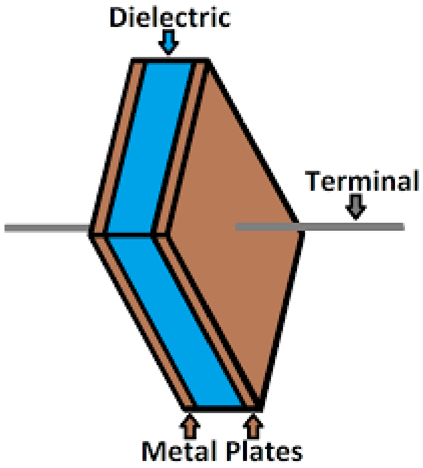
$$U_{tot} = \frac{1}{2} \iint\limits_{A_1 + A_2} \sigma(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,'$$

And since there are always two plates....

$$U_{tot} = \frac{1}{2} \iint\limits_{A_1} \sigma_1(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,' + \frac{1}{2} \iint\limits_{A_2} \sigma_2(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,'$$

Parallel Plate Capacitor Charge

$$U_{tot} = \frac{1}{2}\iint_{A_1} \sigma_1(\vec{r}')V(\vec{r}')d\vec{r}' + \frac{1}{2}\iint_{A_2} \sigma_2(\vec{r}')V(\vec{r}')d\vec{r}'$$



Key approximation

$$\sigma_1(\vec{r}') \sim \frac{Q_1}{A_1}$$

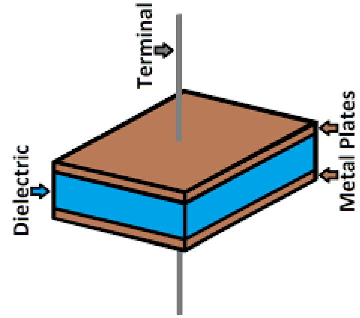
$$\sigma_2(\vec{r}') \sim -\frac{Q_1}{A_2}$$

Charge uniformly distributed on panels

Given this approximation let us calculate the energy

Parallel Plate Capacitor Potential

Conductors
= equipotentials



$$V(\vec{r} \in A_1\ ) = V\left(0,0,\frac{d}{2}\right)$$

$$V(\vec{r} \in A_2\ ) = V\left(0,0,-\frac{d}{2}\right)$$

$$V\left(0,0,\frac{d}{2}\right) = V_1$$

$$V\left(0,0,-\frac{d}{2}\right) = V_2$$

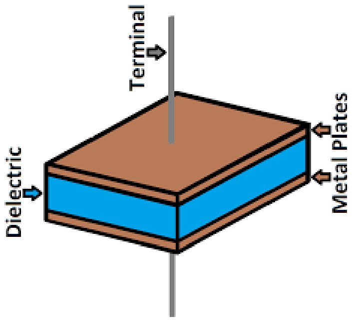
$$U_{tot} = \frac{1}{2}\iint_{A_1} \sigma_1(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,' + \frac{1}{2}\iint_{A_2} \sigma_2(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,'$$

$$V(\vec{r}\,') \neq 0\ \forall\ \vec{r}\,'$$

only needs to be specified in surfaces

Parallel Plate Capacitor Energy

$$U_{tot} = \frac{1}{2}\iint\limits_{A_1} \sigma_1(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,' + \frac{1}{2}\iint\limits_{A_2} \sigma_2(\vec{r}\,')V(\vec{r}\,')d\vec{r}\,'$$



$$\sigma_1(\vec{r}\,') \sim \frac{Q_1}{A_1} \qquad V(\vec{r} \in A_1) = V_1$$

$$\sigma_2(\vec{r}\,') \sim -\frac{Q_1}{A_2} \qquad V(\vec{r} \in A_2) = V_2$$

$$U_{tot} = \frac{1}{2}\iint\limits_{A_1} \frac{Q_1}{A_1} V_1 d\vec{r}\,' - \frac{1}{2}\iint\limits_{A_2} \frac{Q_1}{A_2} V_2 d\vec{r}\,'$$

$$U_{tot} = \frac{1}{2}[Q_1V_1 - Q_1V_2] = \boldsymbol{\frac{1}{2} Q_1 \Delta V}$$

where

$$\Delta V = V_1 - V_2$$

Capacitor Energy

Starting from

$$U_{tot} = \frac{1}{2} \iiint_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

Using

The charge is superficial

Charge uniformly distributed on panels

Conductors = equipotentials

$$U_{tot} = \frac{1}{2} Q_1 \Delta V$$

Applies to all capacitors
not only Parallel Plate

Capacitance

If Q_1 and ΔV are given
the **energy** is calculated $$U_{tot} = \frac{1}{2} Q_1 \Delta V$$

It would be useful to be able to provide the energy once only the charge, or only the voltage is given.

Capacitance is introduced as the charge stored per unit potential difference:

$$C = \frac{Q_1}{\Delta V}$$

Its SI unit is the **farad** (F): 1 F = 1 C/V

Energy expressions with capacitance

$$U_{tot} = \frac{1}{2} \Delta V Q_1$$

$$C = \frac{Q_1}{\Delta V}$$

$$U_{tot} = \frac{1}{2} C (\Delta V)^2$$

The energy can also be expressed in terms of the voltage and capacitance

$$U_{tot} = \frac{1}{2} \frac{{Q_1}^2}{C}$$

Or in terms of the charge and the capacitance

Capacitor design

A classic physics based design problem could be:

select the shape of the conductors
so that you can store a large energy.

Geometry → Energy

Parallel Plate Capacitance

Capacitance $C = \dfrac{Q_1}{\Delta V}$

$$V(\vec{r}) = \iiint_{Vol} k \frac{\rho(\vec{r}\,')}{|\vec{r} - \vec{r}\,'|} d\vec{r}\,'$$

Using the fact that charge is uniformly distributed on two surfaces

Key approximation

$$\sigma_1(\vec{r}\,') \sim \frac{Q_1}{A}$$

$$\sigma_2(\vec{r}\,') \sim -\frac{Q_1}{A}$$

Charge uniformly distributed on panels

$$V(\vec{r}) = \frac{Q_1}{A} k \iint_{A_1} \frac{1}{|\vec{r} - \vec{r}_i|} d\vec{r}_i - \frac{Q_1}{A} k \iint_{A_2} \frac{1}{|\vec{r} - \vec{r}_i|} d\vec{r}_i$$

Parallel Plate Capacitance: specifications

$$C = \frac{Q_1}{\Delta V}$$

$$V(\vec{r}_1\ ) = V\left(0,0,\frac{d}{2}\right)$$

$$\Delta V = V(\vec{r}_1) - V(\vec{r}_2)$$

$$V(\vec{r}_2\ ) = V\left(0,0,-\frac{d}{2}\right)$$

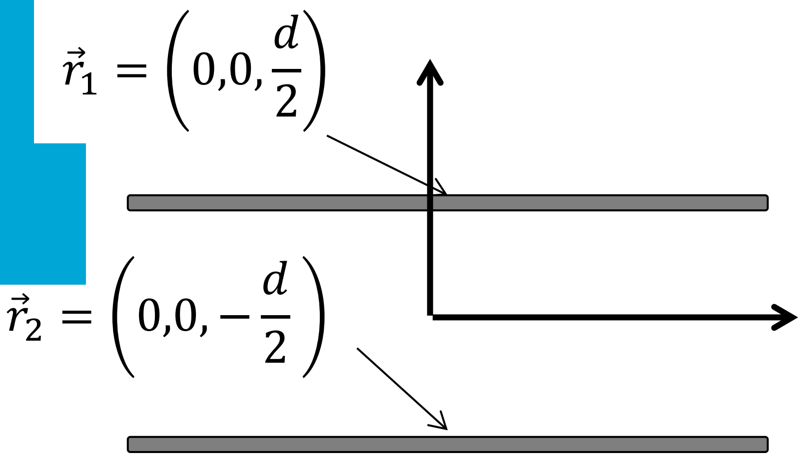
$$V(\vec{r}_i) = \frac{Q_1}{A} k \iint_{A_1} \frac{1}{|\vec{r}_i - \vec{r}'|} d\vec{r}' - \frac{Q_1}{A} k \iint_{A_2} \frac{1}{|\vec{r}_i - \vec{r}'|} d\vec{r}'$$

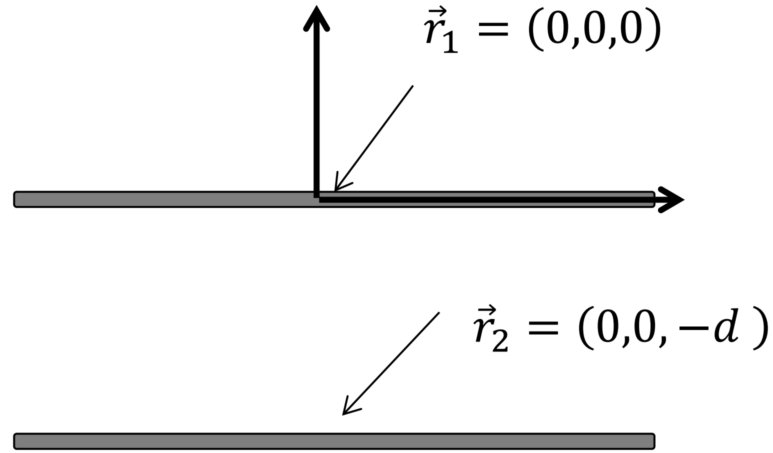
$\downarrow$

i=1,2

$A_1 = A_2 = A_1$

Potential on one plate: representation





$$V(\vec{r}_1) = \frac{Q_1}{A} k \iint_{A_1} \frac{1}{|\vec{r}_1 - \vec{r}\,'|} d\vec{r}\,' - \frac{Q_1}{A} k \iint_{A_2} \frac{1}{|\vec{r}_1 - \vec{r}\,'|} d\vec{r}\,'$$

General expression

$\vec{r}_1 \to \vec{r}_2$

$$V(0,0,0) = \frac{Q_1}{A} k \left[\iint_{A_1} \frac{1}{r'} d\vec{r}\,' - \iint_{A_1} \frac{1}{|\vec{r}_2 - \vec{r}\,'|} d\vec{r}\,' \right]$$

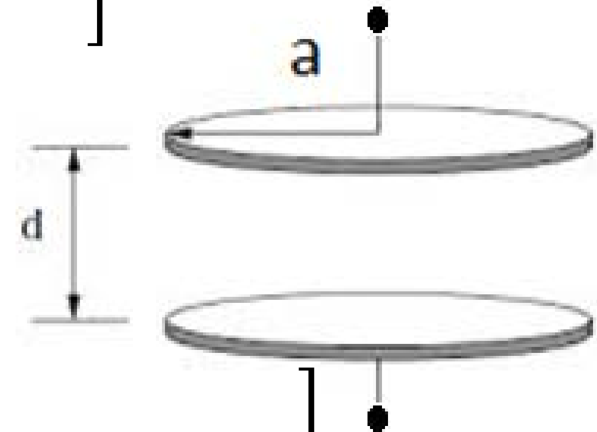
Both integrals in reference system centered in top panel

$r' = |\vec{r}\,'|$

Potential on one plate value

$$V(\vec{r}_1) = \frac{Q_1}{A} k \left[\iint_{A_1} \frac{1}{r'} d\vec{r}' - \iint_{A_1} \frac{1}{|(0,0,-d) - \vec{r}'|} d\vec{r}' \right]$$

$$\frac{1}{|\vec{r}_2 - \vec{r}'|}$$



Cylindrical parametrization $\quad \vec{r}' = (\rho' cos\phi',\ \rho' sin\phi')$

$$V(\vec{r}_1) = \frac{Q_1}{A} k \left[\int_0^a \int_0^{2\pi} \frac{1}{\rho'} \rho' d\rho' d\phi' - \int_0^a \int_0^{2\pi} \frac{1}{\sqrt{\rho'^2 + d^2}} \rho' d\rho' d\phi' \right]$$

$$V(\vec{r}_1) = \frac{Q_1}{A} k \ 2\pi \left\{ a - \sqrt{d^2 + a^2} + d \ \right\}$$

Integral
is an issue in itself

Integral: Mathematical steps I

$$I = \int_0^a \int_0^{2\pi} \frac{1}{\rho}\rho d\rho d\phi - \int_0^a \int_0^{2\pi} \frac{1}{\sqrt{\rho^2 + d^2}}\rho d\rho d\phi$$

$$I = 2\pi\left\{\int_0^a d\rho - \int_0^a \frac{1}{\sqrt{\rho^2 + d^2}}\rho d\rho\right\}$$

$$I = 2\pi\left\{a - \int_0^a \frac{1}{\sqrt{\rho^2 + d^2}}\rho d\rho\right\} = 2\pi\left\{a - \sqrt{d^2 + a^2} + d\right\}$$

Demonstration in the next slides

Integral: Mathematical steps II

$$\int_0^a \frac{1}{\sqrt{\rho^2 + d^2}} \rho d\rho \qquad \sqrt{\rho^2 + d^2} = d\sqrt{1 + \frac{\rho^2}{d^2}}$$

$$\int_0^a \frac{1}{d\sqrt{1 + \frac{\rho^2}{d^2}}} \rho d\rho = d \int_0^{a/d} \frac{y}{\sqrt{1 + y^2}} dy$$

$$\frac{\rho^2}{d^2} = y^2$$

Next slide

$$d \int_0^{a/d} \frac{y}{\sqrt{1 + y^2}} dy = \sqrt{d^2 + a^2} - d$$

Integral: Mathematical steps III

$$d\int_0^{a/d} \frac{y}{\sqrt{1+y^2}}\,dy = \sqrt{d^2+a^2} - d$$

$$\frac{d}{dy}\sinh^{-1} y = \frac{1}{\sqrt{1+y^2}}$$

Suggests to use integration per parts

$$d\int_0^{a/d} \frac{y}{\sqrt{1+y^2}}\,dy = d\, y\sinh^{-1} y\Big]_0^{a/d} - d\underbrace{\int_0^{a/d} \sinh^{-1} y\, dy}$$

Known integral

$$\int_0^{a/d} \sinh^{-1} y\, dy = \left[y\sinh^{-1} y - \sqrt{1+y^2}\right]_0^{a/d}$$

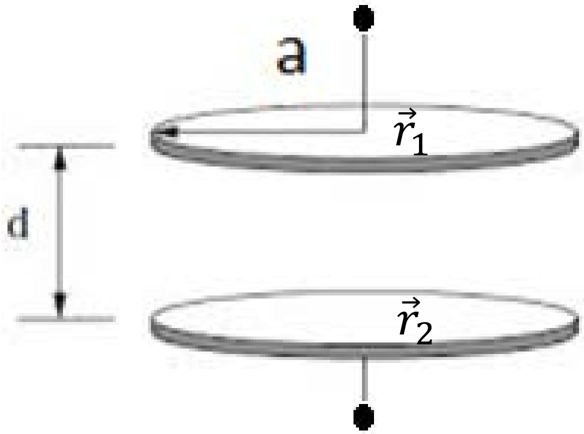
$$d\int_0^{a/d} \frac{y}{\sqrt{1+y^2}}\,dy = \cancel{d\, y\sinh^{-1} y\Big]_0^{a/d}} - d\left[\cancel{y\sinh^{-1} y} - \sqrt{1+y^2}\right]_0^{\frac{a}{d}}$$

$$= d\sqrt{1+\left(\frac{a}{d}\right)^2} - d = \sqrt{d^2+a^2} - d$$

Voltages

$$V(\vec{r}_1) = \frac{Q_1}{A} k \iint\limits_{A_1} \frac{1}{|\vec{r}_1 - \vec{r}'|} d\vec{r}' - \frac{Q_1}{A} k \iint\limits_{A_2} \frac{1}{|\vec{r}_1 - \vec{r}'|} d\vec{r}'$$

$$V(\vec{r}_1) = \frac{Q_1}{A} k 2\pi \left\{ a - \sqrt{d^2 + a^2} + d \right\}$$



$$V(\vec{r}_2) = -\frac{Q_1}{A} k \iint\limits_{A_2} \frac{1}{|\vec{r}_2 - \vec{r}'|} d\vec{r}' + \frac{Q_1}{A} k \iint\limits_{A_1} \frac{1}{|\vec{r}_2 - \vec{r}'|} d\vec{r}'$$

$$V(\vec{r}_2) = -\frac{Q_1}{A} k 2\pi \left\{ a - \sqrt{d^2 + a^2} + d \right\}$$

$$\Delta V = V(\vec{r}_1) - V(\vec{r}_2) = \frac{Q_1}{A} k 4\pi \left\{ a - \sqrt{d^2 + a^2} + d \right\}$$

$$\Delta V = \frac{Q_1}{A} \frac{1}{\epsilon_0 \epsilon_r} \left\{ a - \sqrt{d^2 + a^2} + d \right\}$$

$$k = \frac{1}{4\pi\epsilon_0\epsilon_r}$$ *Coulomb's constant*

Energy stored in Parallel Plate Capacitor

$$U_{tot} = \frac{1}{2}\iiint_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

$$U_{tot} = \frac{1}{2}Q_1\Delta V$$

$$U_{tot} = \frac{1}{2}\frac{Q_1^2}{A}\frac{1}{\epsilon_0\epsilon_r}\left\{a - \sqrt{d^2 + a^2} + d\right\}$$

The energy is proportional to the square of the charge in each plate

If d ≪ a

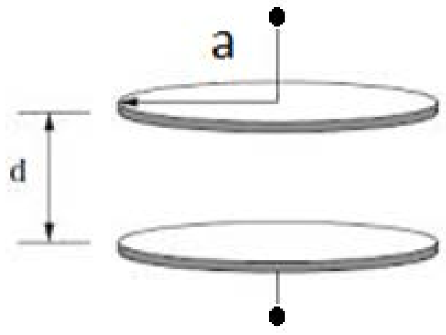
$$V\left(\vec{r} = (0,0,\frac{d}{2})\right) \sim \frac{Q_1}{A}k2\pi d$$

$$V\left(\vec{r} = (0,0,-\frac{d}{2})\right) \sim -\frac{Q_1}{A}k2\pi d$$

$$U_{tot} \sim \frac{1}{2}Q_1^2\frac{1}{\epsilon_0\epsilon_r}\frac{d}{A}$$

Capacitance of Parallel Plate Capacitor

$$\Delta V = \frac{Q_1}{A}\frac{1}{\epsilon_0\epsilon_r}\left\{a - \sqrt{d^2 + a^2} + d\right\}$$

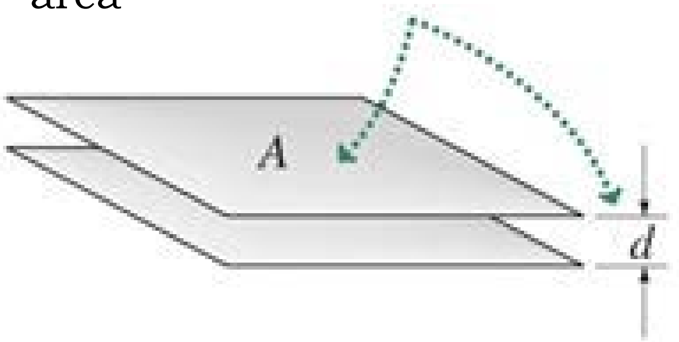


$$C = \frac{Q_1}{\Delta V} = \frac{\epsilon_0\epsilon_r A}{\left\{a-\sqrt{d^2+a^2}+d\right\}}$$

Whatever is the shape the capacitance *approximately* depends only on the facing area

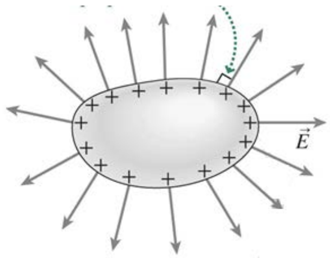
If d ≪ a

$$C = \frac{Q_1}{\Delta V} \sim \frac{\epsilon_0\epsilon_r A}{d}$$



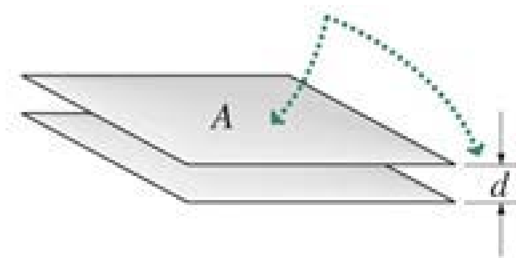
Alternative Procedure for Parallel Plate Capacitance

A charged metal generates field:



$$E_n = \frac{\sigma}{\epsilon_0 \epsilon_r}$$

Where σ is the charge density

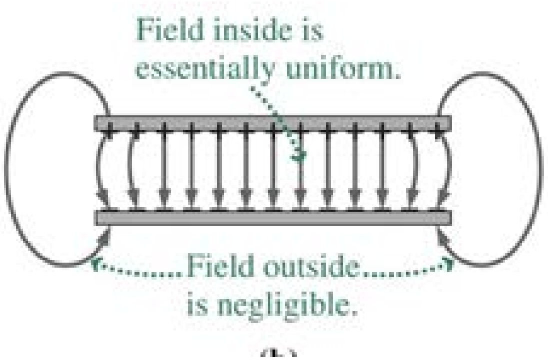


If Q is the total charge $\sigma \approx \dfrac{Q_1}{A}$

$$E_{normal} \sim \frac{Q}{A}\frac{1}{\epsilon_0 \epsilon_r} \quad \Rightarrow \quad E_{normal} d = \Delta V$$

$$\frac{Q}{A}\frac{1}{\epsilon_0 \epsilon_r} = \Delta V / d$$

$$\frac{Q_1}{\Delta V} = \frac{\epsilon_0 \epsilon_r A}{d}$$

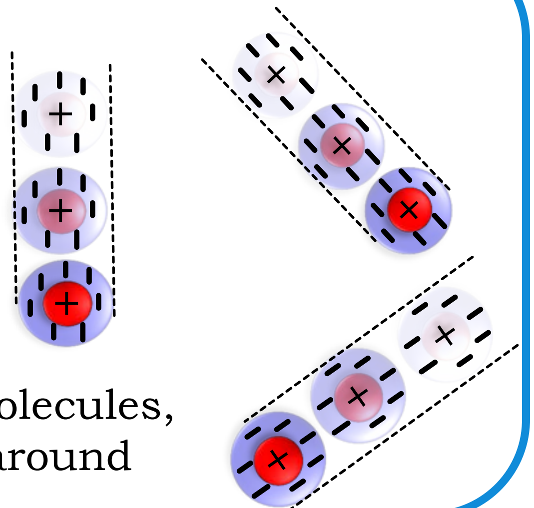


And thus for a parallel plate capacitor

$$C = \frac{\epsilon_0 \epsilon_r A}{d}$$

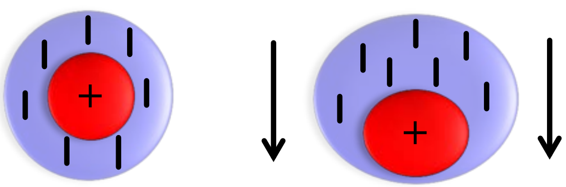
Break- down in Gases

A)



Gas molecules, move around

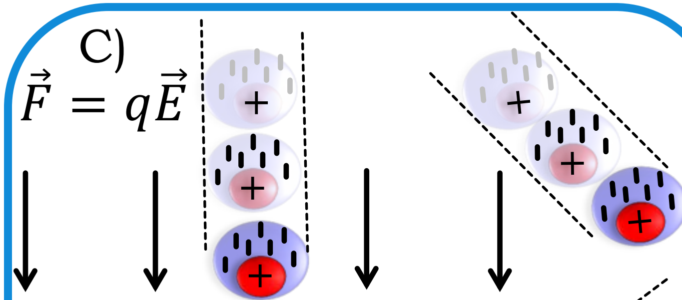
B) **Low Electric** fields can polarize molecules



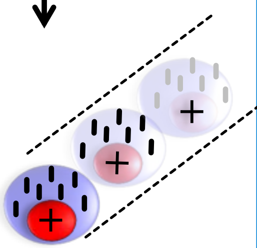
Charges within molecules go as far as chemical bonds allow

C)

$\vec{F} = q\vec{E}$



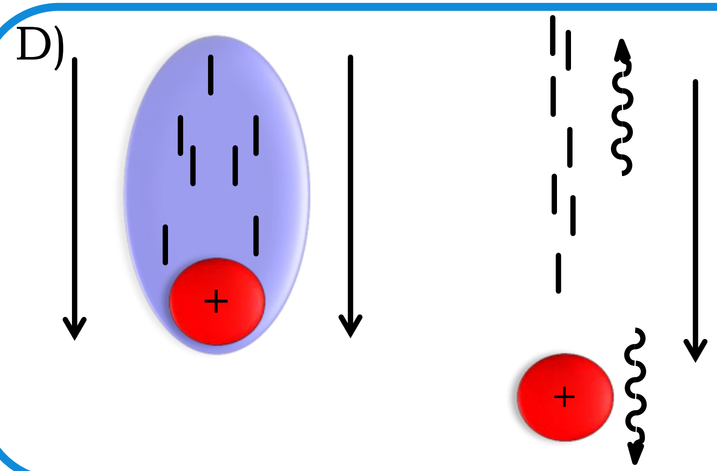
Low Electric fields do not accelerate molecules.



Because each molecule has $q = 0$

Break- down in Gases (2)

D)



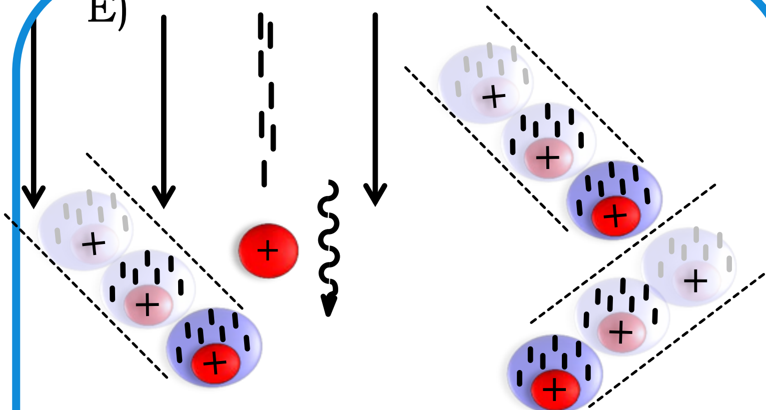
Strong electric fields can break molecules

Molecules aligned with field break !

Energy:
Potential Electrostatic

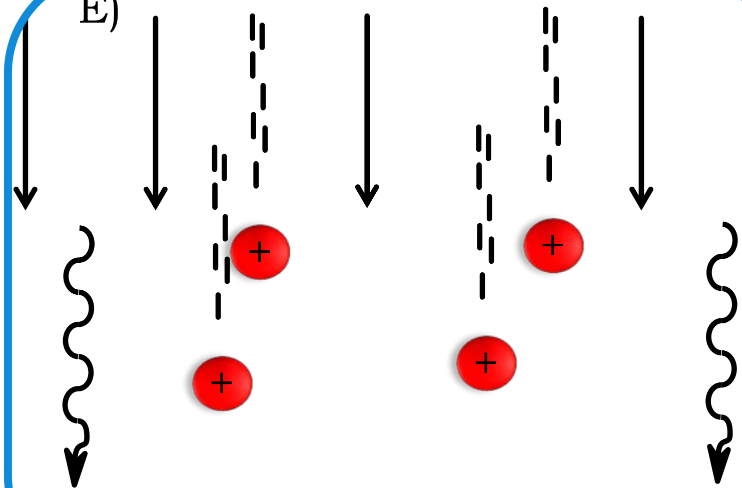
Kinetic energy

E)



Heavy mass protons, accelerated by external field, break other molecules

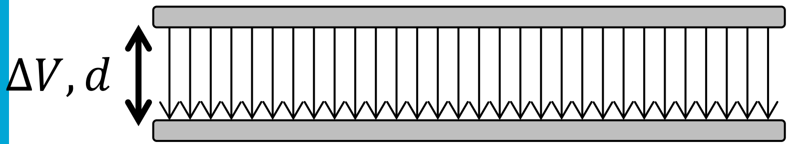
E)



All of them Break (discharge)

Strong Voltages

Increasing ΔV

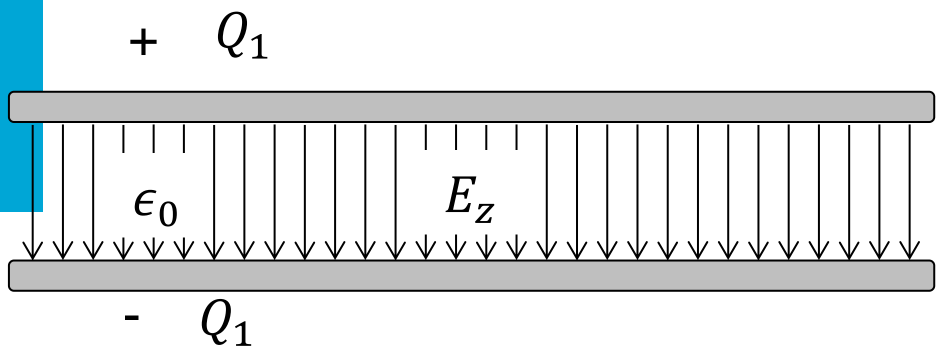


In pumped vacuum no particles, **you could pump up the voltage!**

In air High Electric Fields (3MV/m) can break molecular ties of gas molecules

Capacitors and dielectrics

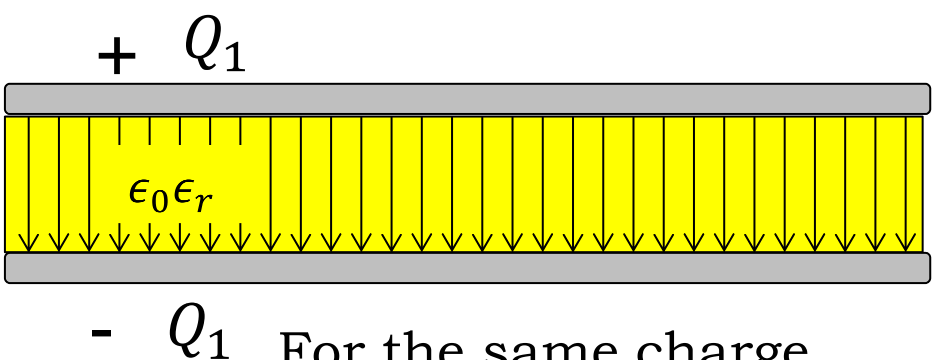
$$\boxed{C = \frac{Q_1}{\Delta V}}$$



$$E_{ext} = \frac{Q_1}{A\epsilon_0}$$

$$\Delta V = E_{ext} d = \frac{Q_1}{A\epsilon_0} d$$

$$C = \frac{\epsilon_0 A}{d}$$



For the same charge
Electric field becomes
smaller

$$\epsilon_r = \frac{E_{ext}}{E_{ext} + E_p}$$

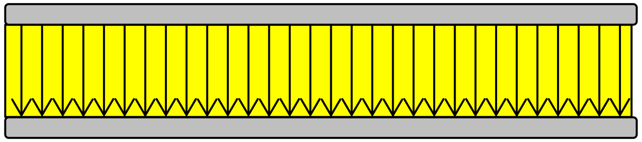
$$E_{ext} = \frac{Q_1}{A\epsilon_0\epsilon_r}$$

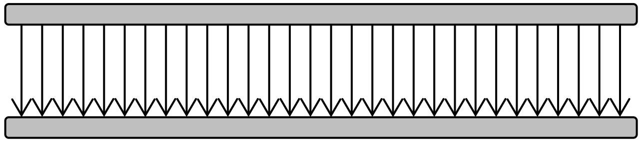
$$\Delta V = E_{ext} d$$

$$C = \frac{\epsilon_0 \epsilon_r A}{d}$$

What does it mean

Given a battery of voltage ΔV,





The capacitor accumulates $q=\varepsilon_r C_0\ V$ charges

compared to $q_0=C_0\ V$ charges

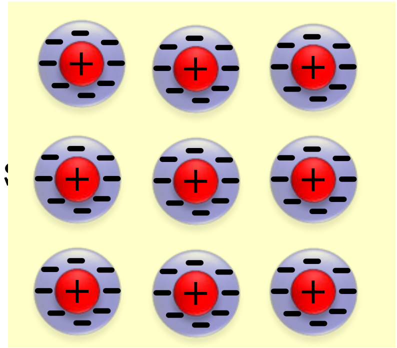
$$\frac{1}{2}\epsilon_r C_0 V^2 > \frac{1}{2} C V^2$$

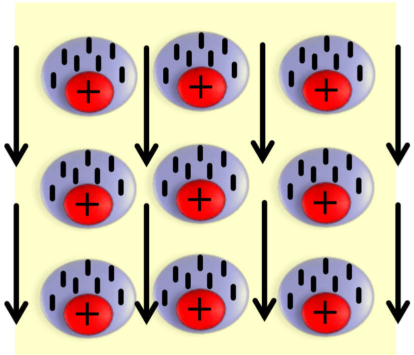
The total average electric field eventually is the same

$$E_{tot}^{ave} = E_{\text{ext}} = -\Delta V/d$$

Where is the energy accumulated?

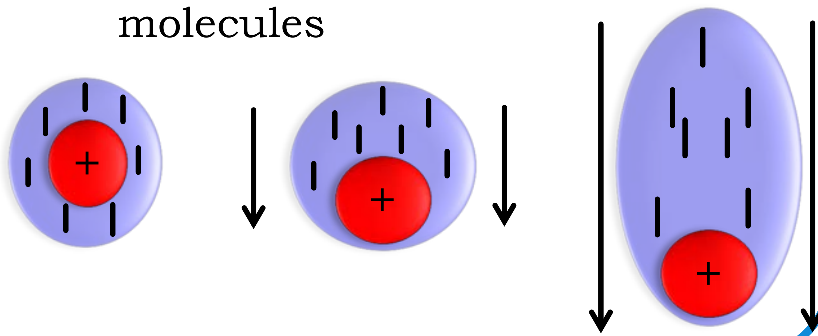
Inside the polarized molecules



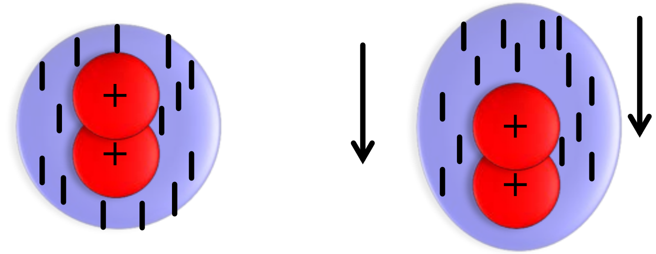


Break- down in Dielectrics

A) **Gases:** strong electric fields can break molecules



B) **Dielectrics:** strong Electric often only polarize bigger molecules

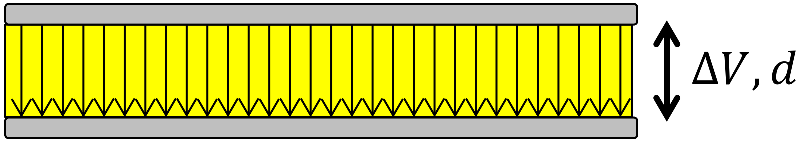


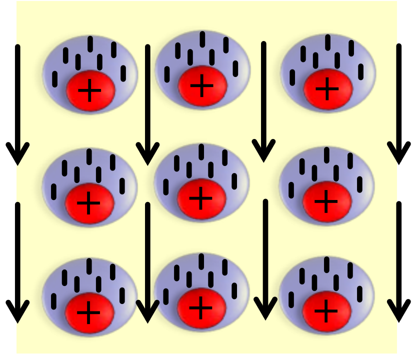
Charges within molecules go as far as stronger ordered chemical bonds allow

Same mechanism as gases; only stronger chemical bonds typically need to be broken than in gases.

Strong Voltages in dielectrics

Increasing ΔV





Much stronger electric fields are compatible with dielectrics with respect to gases

Table 23.1 Properties of Some Common Dielectrics

Dielectric Material	Dielectric Constant	Breakdown Field (MV/m)
Air	1.0006	3
Aluminum oxide	8.4	670
Glass (Pyrex)	5.6	14
Paper	3.5	14
Plexiglas	3.4	40
Polyethylene	2.3	50
Polystyrene	2.6	25
Quartz	3.8	8
Tantalum oxide	26	500
Teflon	2.1	60
Water	80	depends on time and purity

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Truly Important from Lecture 6

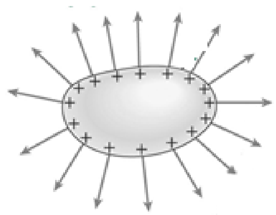
1)

q_1 q_2 q_3 q_N

$$U_{tot} = \frac{1}{2}\sum_{i=1}^{N} q_i V(\vec{r}_i)$$

$$V(\vec{r}_i) = k \sum_{i=1;i\neq j}^{N} \frac{q_i}{r_{ij}}$$

2)



$$U_{tot} = \frac{1}{2}\iiint_{Vol} \rho(\vec{r}')V(\vec{r}')d\vec{r}'$$

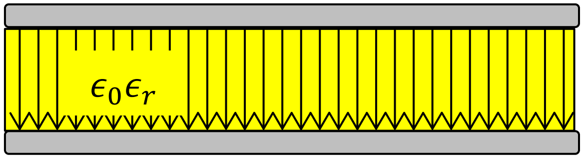
3)

$$U_{tot} = \frac{1}{2}Q_1\Delta V$$

$$C = \frac{Q_1}{\Delta V} \qquad U_{tot} = \frac{1}{2}C(\Delta V)^2$$

4)

$$C = \frac{\epsilon_0\epsilon_r A}{d}$$



5)

$$U_{tot} = \frac{1}{2}\iiint_{Vol} \epsilon_0\epsilon_r\left|\vec{E}(\vec{r}')\right|^2 d\vec{r}'$$