Solution Proef Tentamen #2

Exercise 1.

(a) $\vec{E}_1(0,0,z) = \dfrac{kQ_1}{(x_1^2+y_1^2+z^2)^{3/2}}\left(-x_1\hat{x} - y_1\hat{y} + z\hat{z}\right)$ (V/m) (N/C)

(b) $\vec{F}(0,0,z_2) = \dfrac{kQ_1Q_2}{(x_1^2+y_1^2+z_2^2)^{3/2}}\left(-x_1\hat{x} - y_1\hat{y} + z_2\hat{z}\right)$ (N)

(c) The following symmetry applies to the prescribed surface charge density:

$$\sigma(\varphi) = \sigma(\pi-\varphi) = \sigma(\pi+\varphi) = \sigma(-\varphi) \quad \text{with} \quad 0 \leq \varphi \leq \pi/2$$

Consequently, the configuration will be symmetric with respect to the x=0 plane and y=0 plane

$$dq = \sigma\, r\,dr\,d\varphi = \sin^2(\varphi)\, r\,dr\,d\varphi \quad (C)$$

$$Q = \int_0^R r\,dr \int_0^{2\pi} \sin^2(\varphi)\,d\varphi$$

$$= \frac{r^2}{2}\Big|_0^R \cdot \frac{1}{2}\int_0^{2\pi} [1-\cos(2\varphi)\,d\varphi = \frac{R^2}{4}\left[2\pi - \frac{1}{2}\int_0^{2\pi}\cos(2\varphi)\,d(2\varphi)\right]$$

$$= \frac{R^2}{4}\left[2\pi - \sin(2\varphi)\Big|_0^{2\pi}\right] = \frac{\pi R^2}{2} = 2\pi \; (\mu C)$$

(d) From symmetry, $E_x = E_y = 0$

$$dE_z = \frac{k\,dq\,|z|}{(r^2+z^2)^{3/2}} = \frac{k|z|\sin^2(\varphi)\, r\,dr\,d\varphi}{(r^2+z^2)^{3/2}}$$

$$E_z = k|z| \int_0^R \frac{r\,dr}{(r^2+z^2)^{3/2}} \int_0^{2\pi} \sin^2(\varphi)\,d\varphi$$

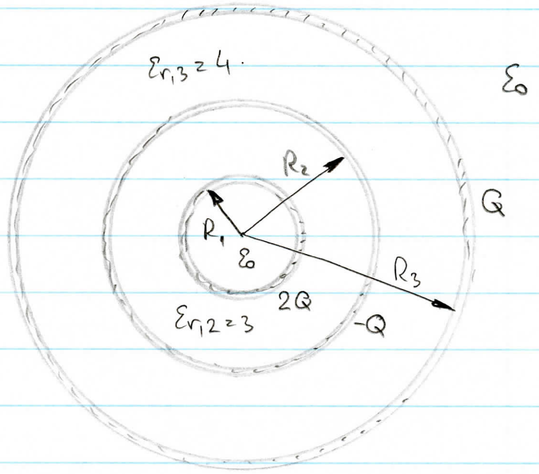
$$= k|z|\left[-\frac{1}{\sqrt{r^2+z^2}}\Big|_0^R\right]\pi = k\pi|z|\left[\frac{1}{|z|} - \frac{1}{\sqrt{R^2+z^2}}\right]$$

By filling in the numerical values and taking into account that σ is in nC/m^2

$$E_z = 9\cdot10^9\,\pi\,2\left[\frac{1}{2} - \frac{1}{2\sqrt{2}}\right]\cdot10^{-9} = 9\pi\left(1-\frac{\sqrt{2}}{2}\right) = 8.281 \;(V/m)\;(N/C)$$

Exercise 2.

(a)



(b) For $r < R_1$: application of Gauss's theorem for a spherical Gaussian surface with $r < R_1$ yields:

$$4\pi r^2 E_1 = 0 \Rightarrow \vec{E}_1 = \vec{0}\ V/m$$

For $r > R_3$: application of Gauss's theorem for a region bounded inwardly by a sperical surface taken inside the outer sheet and outwardly by a spherical surface with $r > R_3$ yields

$$4\pi r^2 E_4 = \frac{q_{enclosed}}{\varepsilon_0} = \frac{Q}{\varepsilon_0} \Rightarrow \vec{E}_4(r) = \frac{Q}{4\pi\varepsilon_0 r^2}\hat{r}\ (V/m).$$

ⓒ For $R_1 < r < R_2$: application of Gauss's theorem for a region bounded inwardly by a spherical surface taken inside the inner sheet and outwardly by a spherical surface with $R_1 < r < R_2$ yields:

$$4\pi r^2 E_2 = \frac{q_{enclosed}}{\varepsilon_{r,2}\varepsilon_0} = \frac{2Q}{3\varepsilon_0} \Rightarrow \vec{E}_2(r) = \frac{Q}{6\pi\varepsilon_0 r^2}\hat{r}\ (V/m)$$

For $R_2 < r < R_3$: application of Gauss's theorem for a region bounded inwardly by a sherical surface taken inside the middle sheet and outwardly bounded by a spherical surface with $R_2 < r < R_3$ yields:

$$4\pi r^2 E_3 = \frac{q_{enclosed}}{\varepsilon_{r,3}\varepsilon_0} = \frac{-Q}{4\varepsilon_0} \Rightarrow \vec{E}_3(r) = -\frac{Q}{16\pi\varepsilon_0 r^2}\hat{r}\ (V/m)$$

ⓓ $$\lim_{r \uparrow R_2} \vec{E}_2(r) = \frac{Q}{6\pi\varepsilon_0 R_2^2}\hat{r}$$

$$\lim_{r \downarrow R_2} \vec{E}_3(r) = -\frac{Q}{16\pi\varepsilon_0 R_2^2}\hat{r}$$

Both fields are radial and, hence, the tangential components are zero. The tangential components satisfy then the boundary condition on a perfectly conducting surface.

Note: This is a supplementary material that was not discussed this year. You should not expect such exercises at the exam.

e On the outer sheet

$$\Delta V_{4\infty} = -\int_{\infty}^{R_3} \vec{E}_4(r)\cdot\hat{r}\,dr = -\int_{\infty}^{R} \frac{Q\,dr}{4\pi\varepsilon_0 r^2}$$

$$= \frac{Q}{4\pi\varepsilon_0 r}\Bigg|_{\infty}^{R_3} = \frac{Q}{4\pi\varepsilon_0 R_3} \Rightarrow V_4 = \frac{Q}{4\pi\varepsilon_0 R_3} \quad (V)$$

Between the middle and outer sheets there is a potential difference

$$\Delta V_{34} = -\int_{R_3}^{R_2} \vec{E}\cdot\hat{r}\,dr = \int_{R_3}^{R_2} \frac{Q\,dr}{16\pi\varepsilon_0 r^2} = \frac{Q}{16\pi\varepsilon_0 r}\Bigg|_{R_2}^{R_3}$$

$$= \frac{Q}{16\pi\varepsilon_0}\left(\frac{1}{R_3} - \frac{1}{R_2}\right) \quad (V) \Rightarrow$$

$$V_3 = V_4 + \Delta V_{34} = \frac{Q}{4\pi\varepsilon_0 R_3} + \frac{Q}{16\pi\varepsilon_0 R_3} - \frac{Q}{16\pi\varepsilon_0 R_2}$$

$$= \frac{5Q}{16\pi\varepsilon_0 R_3} - \frac{Q}{16\pi\varepsilon_0 R_2}$$

Between the inner and the middle sheets the is a potential difference

$$\Delta V_{23} = -\int_{R_2}^{R_1} \vec{E}\cdot\hat{r}\,dr = -\int_{R_2}^{R_1} \frac{Q\,dr}{6\pi\varepsilon_0 r^2} = \frac{Q}{6\pi\varepsilon_0 r}\Bigg|_{R_2}^{R_1}$$

$$= \frac{Q}{6\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right) \quad (V) \Rightarrow$$

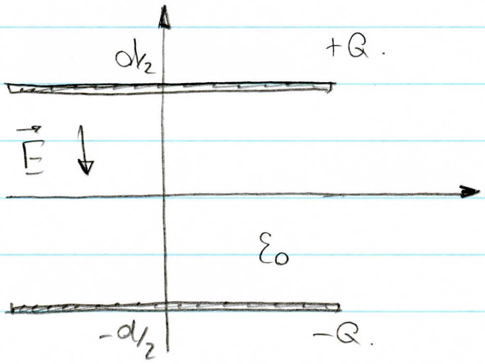
$$V_2 = V_3 + \Delta V_{23} = \frac{5Q}{16\pi\varepsilon_0 R_3} - \frac{Q}{16\pi\varepsilon_0 R_2} + \frac{Q}{6\pi\varepsilon_0 R_1} - \frac{Q}{6\pi\varepsilon_0 R_2}$$

$$= \frac{5Q}{16\pi\varepsilon_0 R_3} - \frac{11 Q}{48\pi\varepsilon_0 R_2} + \frac{Q}{6\pi\varepsilon_0 R_1} \quad (V)$$

f) The capacity corresponds to $2Q$ and ΔV_{23}

$$C = \frac{2Q}{\frac{Q}{6\pi\varepsilon_0}\left(\frac{1}{R_1} - \frac{1}{R_2}\right)} = \frac{12\pi\varepsilon_0}{\frac{1}{R_1} - \frac{1}{R_2}} = \frac{12\pi\varepsilon_0 R_1 R_2}{R_2 - R_1} \quad (F)$$

Exercise 3



(a) By applying Gauss's theorem for a Gaussian surface enclosing the upper plate it follows that

$$E \cdot a^2 = \frac{q_{enclosed}}{\varepsilon_0} \Rightarrow$$

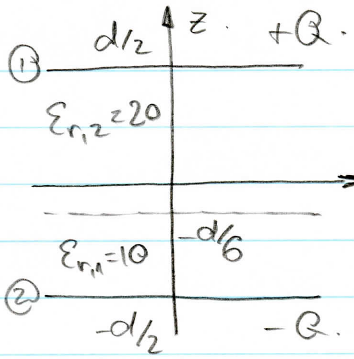
$$\vec{E} = -\frac{Q}{\varepsilon_0 a^2} \hat{z} \quad (V/m)$$

$$\Delta V_{12} = -\int_2^1 \vec{E}\,\hat{z}\,dz = \int_2^1 \frac{Q}{\varepsilon_0 a^2} dz = \frac{Qd}{\varepsilon_0 a^2} \quad [V]$$

(b) $$C = \frac{Q}{\Delta V_{12}} = \frac{\varepsilon_0 a^2}{d} \quad (F)$$

$$U = \frac{Q^2}{2C} = \frac{Q^2 \cdot d}{2\varepsilon_0 a^2} \quad (J)$$

(c) By using the same reasoning as at a it can be derived that



for $-d/2 < z < -d/6$

$$\vec{E}_1 = -\frac{Q}{10\varepsilon_0 a^2}\hat{z} \ (V/m)$$

$$\vec{E}_2 = -\frac{Q}{20\varepsilon_0 a^2}\hat{z} \ (V/m)$$

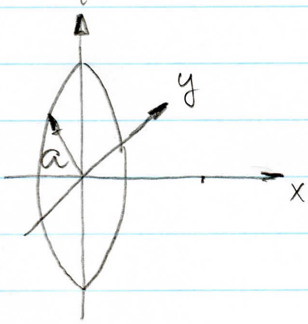
$$\Delta V_{12} = -\int_2^1 \vec{E}\,\hat{z}\,dz = \int_{-d/2}^{-d/6} \frac{Q\,dz}{10\varepsilon_0 a^2} + \int_{-d/6}^{d/2} \frac{Q\,dz}{20\varepsilon_0 a^2}$$

$$= \frac{Q}{10\varepsilon_0 a^2}\left(\frac{d}{3} + \frac{d}{3}\right) = \frac{Qd}{15\varepsilon_0 a^2} \ (V)$$

(d) $$C = \frac{Q}{\Delta V_{12}} = \frac{15\varepsilon_0 a^2}{d} \ (F)$$

$$U = \frac{Q^2}{2C} = \frac{Q^2 d}{30\varepsilon_0 a^2} \ (J)$$

Exercise 4



(a) $$dV = \frac{k\,dq}{\sqrt{x^2+a^2}} = \frac{kQa\,d\varphi}{2\pi a\sqrt{x^2+a^2}}$$

$$= \frac{kQ\,d\varphi}{2\pi\sqrt{x^2+a^2}}$$

$$V = \int_0^{2\pi} \frac{kQ\,d\varphi}{2\pi\sqrt{x^2+a^2}} = \frac{kQ}{\sqrt{x^2+a^2}} \ (V)$$

(b) $\vec{E} = -\nabla V = -\partial_x V \hat{x} = \frac{kQx}{(x^2+a^2)^{3/2}} \hat{x} \; (V/m)$

(c) The disc is decomposed in elementary rings for which it holds that:

$$dV = \frac{k\sigma\, 2\pi r\, dr}{\sqrt{x^2+r^2}} \quad \text{with} \quad \sigma = \frac{Q}{\pi R^2} \Rightarrow$$

$$dV = \frac{kQ2}{R^2} \frac{r\,dr}{\sqrt{x^2+r^2}} \quad (V)$$

$$V = \frac{2kQ}{R^2} \int_0^R \frac{r\,dr}{\sqrt{x^2+r^2}} = \frac{2kQ}{R^2} \sqrt{x^2+r^2}\Big|_0^R$$

$$= 2kQ \frac{\sqrt{x^2+R^2} - |x|}{R^2} \quad (V)$$