ELECTRIC CHARGE AND ELECTROSTATIC FORCE

1. OUTLINE

- ✓ Vector algebra
- ✓ Electric charges and electrostatic force
- ✓ Charge densities

2. VECTOR ALGEBRA

A vector quantity can be represented as

$$\vec{r} = r\hat{r} \tag{1}$$

Which consists of an amplitude (or modulus), r, and a direction, $\hat{r}$. A position in space can be represented by a vector. In 3D space, using the Cartesian coordinates (Fig. 1) one can indicate a position as:

$$\vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \tag{2}$$

where x, y, and z are the coordinates of the generic position, and $\hat{x}$, $\hat{y}$, and $\hat{z}$ are the unit vectors of the Cartesian coordinates. These vectors indicate the direction in the coordinates. Eq. (2) is an alternative representation of Eq. (1); it can also be written as

$$\vec{r} = r\hat{r} = |\vec{r}|\hat{r} = \sqrt{x^2 + y^2 + z^2}\hat{r} \qquad \text{where} \qquad \hat{r} = \frac{\vec{r}}{r}, r = \sqrt{x^2 + y^2 + z^2}$$

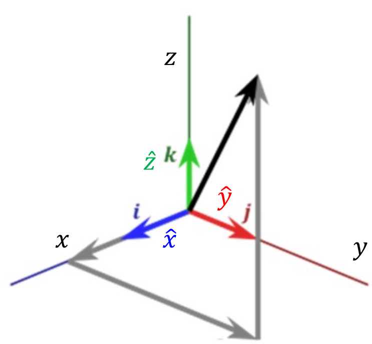

Figure 1. The Cartesian coordinate system.

Summation and difference operations in the vector algebra has to be performed for each vector component separately. In other words:

$$\vec{r}_2 \pm \vec{r}_1 = (x_2 \pm x_1)\hat{x} + (y_2 \pm y_1)\hat{y} + (z_2 \pm z_1)\hat{z} \tag{3}$$

a) SCALAR PRODUCT

The scalar product represents the projection of one vector onto the other. As the name suggests the result of such product is not a vector but a scalar quantity. Formally it is calculated as

$$\vec{a} \cdot \vec{b} = |\vec{a}||\vec{b}|\cos(\gamma) \tag{4}$$

where γ is the angle between the two vectors. The term $|\vec{a}|\cos(\gamma)$ is the projection of $\vec{a}$ onto $\vec{b}$. From Eq. (4) it is apparent that $\vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}$. If $\vec{a}$ and $\vec{b}$ are represented in the Cartesian coordinates as

$$\vec{a} = a_x\hat{x} + a_y\hat{y} + a_z\hat{z}$$
$$\vec{b} = b_x\hat{x} + b_y\hat{y} + b_z\hat{z}$$

the scalar product can be expressed as

$$\vec{a} \cdot \vec{b} = a_x b_x + a_y b_y + a_z b_z \tag{5}$$

b) VECTOR PRODUCT

The vector product of two vectors generates another vector which is perpendicular to the plane which contains the vectors that are vectorially multiplied. The vector product can be represented as

$$\vec{a} \times \vec{b} = |\vec{a}||\vec{b}|\sin(\gamma)\,\hat{n} \tag{6}$$

where γ again is the angle between the two vectors, and $\hat{n}$ indicates a unit vector perpendicular to $\vec{a}$ and $\vec{b}$, which follows the right hand law (Fig. 2).

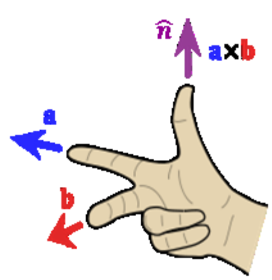

Figure 2. The right hand law indicates the direction ($\hat{n}$) of the vector $\vec{a} \times \vec{b}$.

In Cartesian coordinates the vector product can be represented as

$$\vec{a} \times \vec{b} = \begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix} \tag{7}$$

where $\begin{vmatrix} \hat{x} & \hat{y} & \hat{z} \\ a_x & a_y & a_z \\ b_x & b_y & b_z \end{vmatrix}$ is symbolically the determinant of a 3x3 matrix. Therefore, Eq. (7) is simplified as

$$\begin{aligned} \vec{a} \times \vec{b} &= \hat{x} \begin{vmatrix} a_y & a_z \\ b_y & b_z \end{vmatrix} - \hat{y} \begin{vmatrix} a_x & a_z \\ b_x & b_z \end{vmatrix} + \hat{z} \begin{vmatrix} a_x & a_y \\ b_x & b_y \end{vmatrix} \\ &= \left(a_y b_z - a_z b_y\right)\hat{x} - \left(a_x b_z - a_z b_x\right)\hat{y} + \left(a_x b_y - a_y b_x\right)\hat{z} \end{aligned} \tag{8}$$

c) VECTOR FIELD

A field is a function with several variables, e.g., $f(\vec{r}) = f(x, y)$ is a scalar field that depends on x and y. In general, the field can also be a vector. Formally, in three-dimensional Cartesian coordinates a vectorial field can be expressed as

$$\vec{f}(\vec{r}) = f_x(x, y, z)\hat{x} + f_y(x, y, z)\hat{y} + f_z(x, y, z)\hat{z}$$

Such function depends on three variables (x, y, z), and it is a vector.

3. ELECTRIC CHARGE AND ELECTROSTATIC FORCE

The electric charge is a fundamental property of the matter. It can be either a positive or a negative value. The smallest electric charge is typically expressed as e and has value is 1.6×10^{-19}C, where C is Coulomb (according to the International System of Units). An electron has “negative” $-e$ charge whereas a proton has “positive” $+e$ charge. The charge in a closed system is conserved.

a) COULOMB’S LAW

The force that a certain particle with charge q_1 induces on another particle with charge q_2 can be expressed as

$$\vec{F}_{12}(\vec{r}, q_1, q_2) = \frac{k_e q_1 q_2}{r_{12}^2} \hat{r}_{12} \tag{9}$$

where

$$k_e = \frac{1}{4\pi\epsilon_0} \tag{10}$$

is the Coulomb's constant, and ϵ_0 is the permittivity constant of the free space ($\epsilon_0 = 8.854 \cdot 10^{-12}\text{F/m}$). Moreover, $\vec{r}_{12} = r_{12}\hat{r}_{12}$ is a vector that represents the distance between the two charges from q_1 to q_2 (Fig. 3).

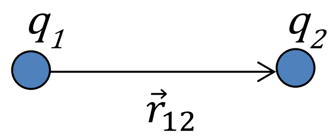

Figure 3. Two electric charges placed at a distance r_{12} one from each other. $\vec{r}_{12}$ is the relevant vector.

As it can be seen in Eq. (9), the direction of the force is either $\hat{r}_{12}$ or $-\hat{r}_{12}$, depending on the charges sign. When both charges have the same sign, $\hat{F}_{12} = \hat{r}_{12}$, i.e. the charges repel each other. Conversely, when the charges have different signs, $\hat{F}_{12} = -\hat{r}_{12}$, i.e. the charges attract each other.

The superposition principle can be used to calculated the force induced by two or more other charges on another charge (Fig. 4).

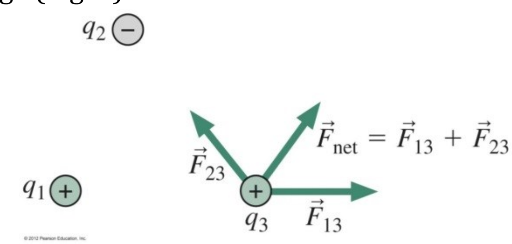

Figure 4. A system of three point charges. $\vec{F}_{\text{net}}$ is the force induced on the charge q_3 by the charges q_1 and q_2.

An alternative notation for the Coulomb's Law between two charges is the following

$$\vec{F}_{so}(\vec{r}_o - \vec{r}_s, q_s, q_o) = \frac{k_e q_s q_o}{r_{so}^2}\hat{r}_{so} \tag{11}$$

where

$$\hat{r}_{so} = \frac{\vec{r}_o - \vec{r}_s}{r_{so}} \qquad \text{and} \qquad r_{so} = |\vec{r}_o - \vec{r}_s|$$

This notation explicitly takes into account for the position of the source $\vec{r}_s$, which is inducing the force, and the position of the observing charge $\vec{r}_o$, which is experiencing the force.

b) MATHEMATICS OF SUPERPOSITION PRINCIPLE

Let us consider N active charges q_i and their location $\vec{r}_i$ with $i = 1, \dots, N$. Let us also consider a test charge q_t located at $\vec{r}_t$. This system of charges is illustrated in Figure 5. In virtue of the superposition principle, the electrostatic force experienced by q_t can be expressed as

$$\vec{F}(\vec{r}_t) = \sum_{i=1}^{N} \vec{F}_{it}(\vec{r}_t, \vec{r}_i) \tag{12}$$

where

$$\vec{F}_{it}(\vec{r}_t,\vec{r}_i) = q_t \frac{k_e q_i}{|\vec{r}_t - \vec{r}_i|^2} \frac{\vec{r}_t - \vec{r}_i}{|\vec{r}_t - \vec{r}_i|} \tag{13}$$

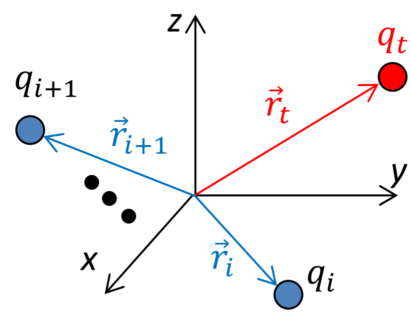

Figure 5. A system of several charges inducing an electrostatic force on the test charge q_t.

4. CHARGE DENSITIES

Until now the discussions were focused on point charges. However, when the space that charges occupy is larger or comparable to the distance between them, it is useful to characterise them together as a whole. In order to do so, one can introduce a charge density distributed in such space. Three type of charge densities are discussed here: line, surface, and volume charge densities. The charge density is defined as the limit of the ratio between the amount of charges contained in a small space and the small space itself (Fig. 6). Formally,

$$\rho_l = \lim_{\Delta l \to 0} \frac{\Delta q}{\Delta l} = \frac{dq}{dl} \to dq = \rho_l dl \tag{14}$$

$$\rho_s = \lim_{\Delta S \to 0} \frac{\Delta q}{\Delta S} = \frac{dq}{dS} \to dq = \rho_s dS \tag{15}$$

$$\rho_v = \lim_{\Delta V \to 0} \frac{\Delta q}{\Delta V} = \frac{dq}{dV} \to dq = \rho_v dV \tag{16}$$

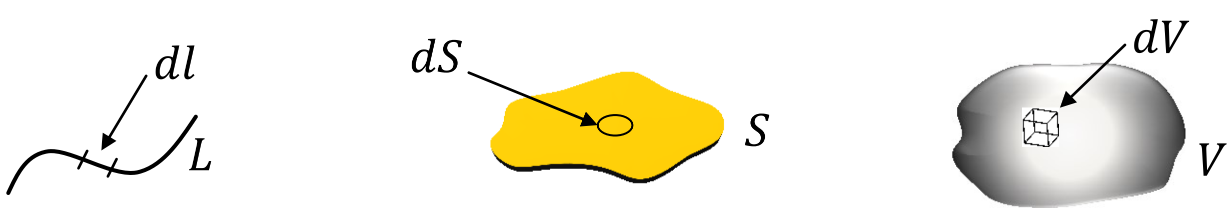

Figure 6. The different spaces (line, surface, and volume) for defining charge densities.

The total charge in a medium with a specific charge density can be computed as

$$Q = \int dq = \begin{cases} \int_L \rho_l \, dl & For\ line\ charge\ density \\ \iint_S \rho_s dS & For\ surface\ \ charge\ density \\ \iiint_V \rho_v dV & For\ volume\ charge\ density \end{cases}$$

Using the above concepts, for continuously distributed charge densities, one can rewrite Eq. (12)–(13) as a continuous summation integral, i.e.,

$$\vec{F}(\vec{r}_t) = \iiint_V q_t \frac{k_e \rho_v(\vec{r}\,')}{|\vec{r}_t - \vec{r}\,'|^2} \frac{\vec{r}_t - \vec{r}\,'}{|\vec{r}_t - \vec{r}\,'|} \, d\vec{r}\,' \tag{17}$$

$$\vec{F}(\vec{r}_t) = \iint_S q_t \frac{k_e \rho_s(\vec{r}\,')}{|\vec{r}_t - \vec{r}\,'|^2} \frac{\vec{r}_t - \vec{r}\,'}{|\vec{r}_t - \vec{r}\,'|} \, d\vec{r}\,' \tag{18}$$

$$\vec{F}(\vec{r}_t) = \int_L q_t \frac{k_e \rho_l(\vec{r}\,')}{|\vec{r}_t - \vec{r}\,'|^2} \frac{\vec{r}_t - \vec{r}\,'}{|\vec{r}_t - \vec{r}\,'|} \, d\vec{r}\,' \tag{19}$$

for volume, surface, and line charge distributions. Note that, referring to Fig. 7 the integration variable is denoted by $\vec{r}\,'$. The above integrals compute the electrostatic force experienced by the test charge q_t at $\vec{r}_t$, taking into account for all the contributions of the distributed charges.

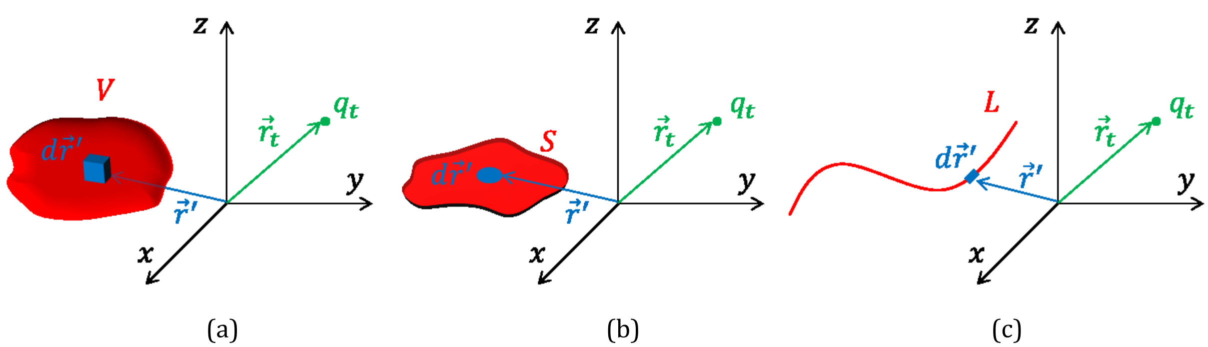

Figure 7. A generic volume (a), surface (b), and line (c) charge distribution in a coordinate system along with the test charge q_t.

One can also explicitly express the components of the force. For example, by considering a volume charge distribution, starting from Eq. (17) the components of the force are

$$F_x(\vec{r}_t) = \iiint\limits_V q_t \frac{k_e \rho_v(\vec{r}')}{|\vec{r}_t - \vec{r}'|^2} \frac{(x_t - x')}{|\vec{r}_t - \vec{r}'|} \, d\vec{r}'$$

$$F_y(\vec{r}_t) = \iiint\limits_V q_t \frac{k_e \rho_v(\vec{r}')}{|\vec{r}_t - \vec{r}'|^2} \frac{(y_t - y')}{|\vec{r}_t - \vec{r}'|} \, d\vec{r}'$$

$$F_z(\vec{r}_t) = \iiint\limits_V q_t \frac{k_e \rho_v(\vec{r}')}{|\vec{r}_t - \vec{r}'|^2} \frac{(z_t - z')}{|\vec{r}_t - \vec{r}'|} \, d\vec{r}'$$