Probability

Lecture 4.2: Poisson process and stochastic simulation

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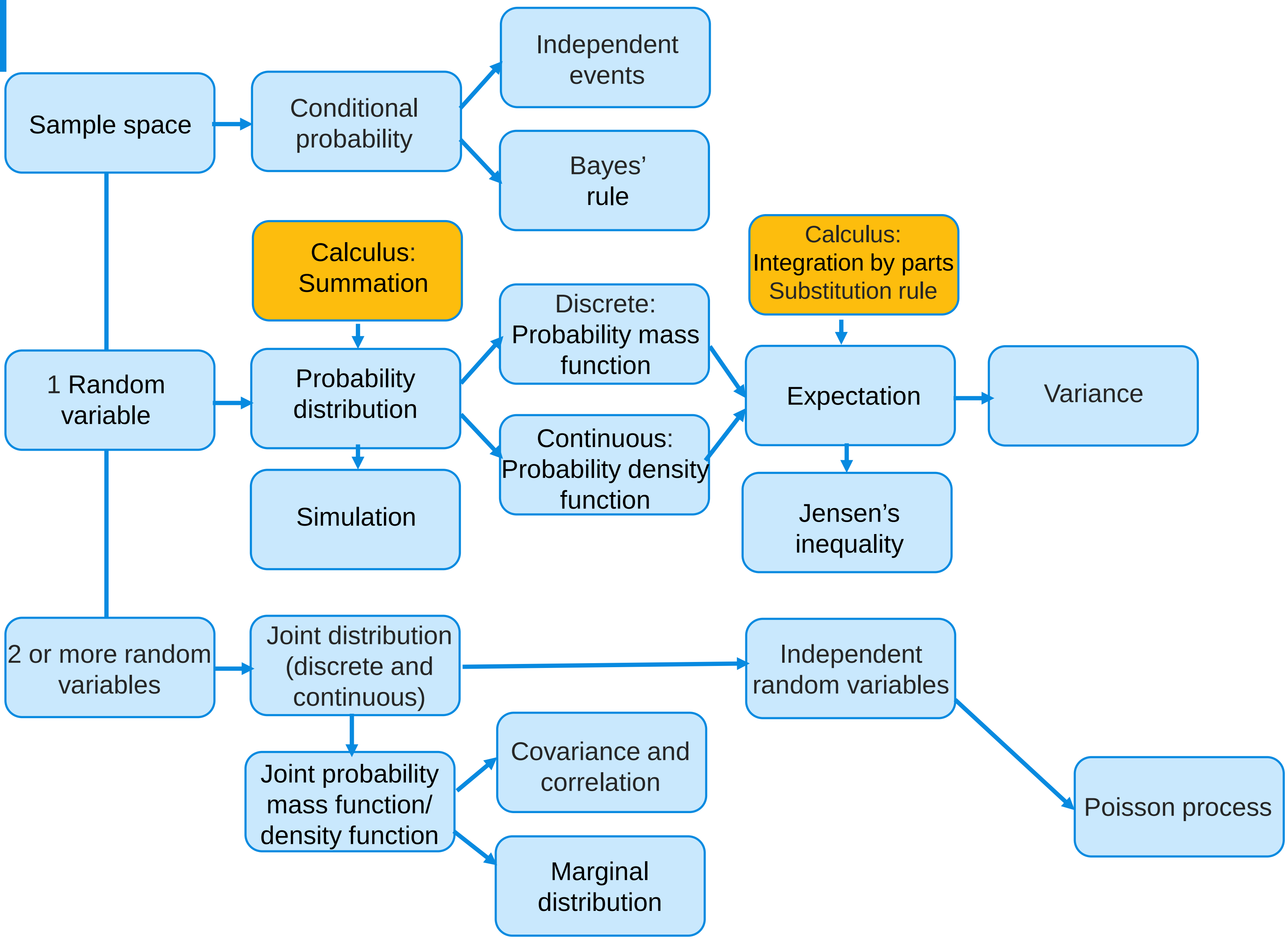
Learning objective

After this class you are able to

- **Know the definition and recognize context of the Poisson process**
- **Compute and derive properties of the Poisson process**
- **Simulate random variables**

Book: Sections 12.1, 12.2, 12.3, 6.1, 6.2

Probability



Before this class (week 3.4 lesson 2):

Watch prelecture *“Poisson Process”*

Book: Sections 12.1

Programme

- The Poisson distribution on nonnegative integers
- Binomial distribution, large *n*, small *p*
- The Poisson *process*
- Modelling with Poisson process
- Simulation of a random variable

From the pre-lecture video: if clients arrive at a shop according to a Poisson process with intensity three clients per ten minutes, find the probability that exactly four clients arrive in fifteen minutes.

A) 0.0783

B) 0.1898

C) 0.0627

D) 0.2312

E) 0.1677

Suppose $X_1, X_2, X_3, \ldots, X_n$ are Random variables with Bernoulli(*p*) distributions. The distribution of the sum

$$S = X_1 + X_2 + \cdots + X_n$$

is then exactly

A) *Binomial*

B) *Binomial,* provided the X_i's are <u>independent</u>

C) *Poisson*

D) *Poisson,* provided the X_i's are <u>independent</u>

E) *Other*

Binomial distribution, small *p*, large *n*

Theorem:
Let Y_n be a binomially distributed randomvariable with parameters *n* and $p_n = \mu/n$ (for *n=1,2,...*)
Then, for *k=0,1,2,...*

$$\lim_{n\to\infty} P(Y_n = k) = \lim_{n\to\infty} \binom{n}{k} \left(\frac{\mu}{n}\right)^k \left(1 - \frac{\mu}{n}\right)^{n-k} = \frac{\mu^k}{k!} e^{-\mu}$$

Resulting approximation:

X: binomial with parameters *n*=1000 and *p*=0.01

$$P(X = 8) \approx \frac{10^8}{8!} e^{-10} = 0.1126\ldots$$

$0.1128\ldots$

The Poisson distribution

Definition:
The random variable X has Poisson distribution with parameter $\mu > 0$ if its probability mass function is given by

$$p(k) = P(X = k) = \frac{\mu^k}{k!}e^{-\mu}, \ k = 0, 1, 2, \ldots$$

and zero elsewhere

Siméon Poisson
(1781-1840)

Theorem

If X has a Poisson distribution with paramater μ, then $\mathrm{E}[X] = \mu$ and $\mathrm{Var}(X) = \mu$.

Modelling with the Poisson process

Registering events that happen in time

$N((a,b]) =$ “Number of events in interval $(a,b]$”

Roughly: suppose

- During time interval of length s, there are many potential occurrences, but for each potential occurrence, the probability of actual occurrence is small
- Whether or not a potential occurrence actually occurs, is independent among potential occurrences

Poisson process natural candidate model

Defining properties of the Poisson process with intensity λ on the line

$N(a,b)$: number of events in (a,b).

1. $\mathrm{E}[N(a,b)] = \lambda(b-a)$

2. For disjoint intervals (a,b) and (c,d):
$N(a,b)$ and $N(c,d)$ are **independent** random variables.

Consequence of **1** and **2**:

3. $N(a,b)$ has a Poisson$(\lambda(b-a))$ distribution.

Question: Poisson process natural?

1: The times at which rain drops hit the roof during ten seconds of a rain shower

2: Patients that yearly come to a first aid post with a broken nose.

The Poisson process is a natural model for:

A) None of these processes

B) Only process 1

C) Only process 2

D) Both processes

Two more properties of the Poisson process
4a Distribution of the first 'arrival time'

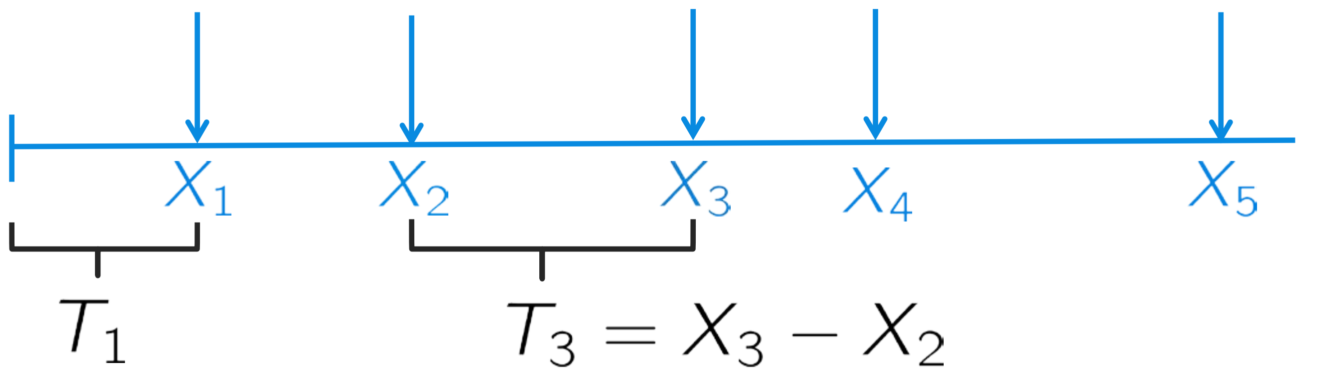
Exercise

Consider a Poisson process $X_1, X_2, X_3, \ldots$ of intensity λ.
Let $T_1 = X_1$ be the time the first event occurs.
Note that $\mathrm{P}(T_1 > t) = \mathrm{P}(N(0,t) = 0)$, where $N(0,t)$ is

the number of events in the interval $(0, t]$.

Find $\mathrm{P}(T_1 \leq t)$, and deduce what the distribution is of T_1.

Two more properties of the Poisson process
4b Distribution of the 'inter-arrival times'



Theorem Let $T_j = X_j - X_{j-1}$ be the time intervals between the events of a Poisson process with intensity λ. (And $T_1 = X_1$.)

Then $T_1, T_2, T_3, \ldots$ are **independent** random variables each with an **Exp(λ) distribution**.

Two more properties of the Poisson process
5 The distribution of the points within an interval

Exercise

Consider a Poisson process $X_1, X_2, X_3, \ldots$ of intensity λ.
Suppose it is **given** that $N(a, b) = n$,
i.e. there are exactly n points in the interval $(a, b]$.
Now let $a < c < b$, and put $p = \dfrac{c-a}{b-a}$.
Show that $\mathrm{P}\left(N(a, c) = k \mid N(a, b) = n\right) = \dbinom{n}{k} p^k (1-p)^{n-k}$.

(If you get dizzy from all the letters take $a = 1, c = 3, b = 6$.)

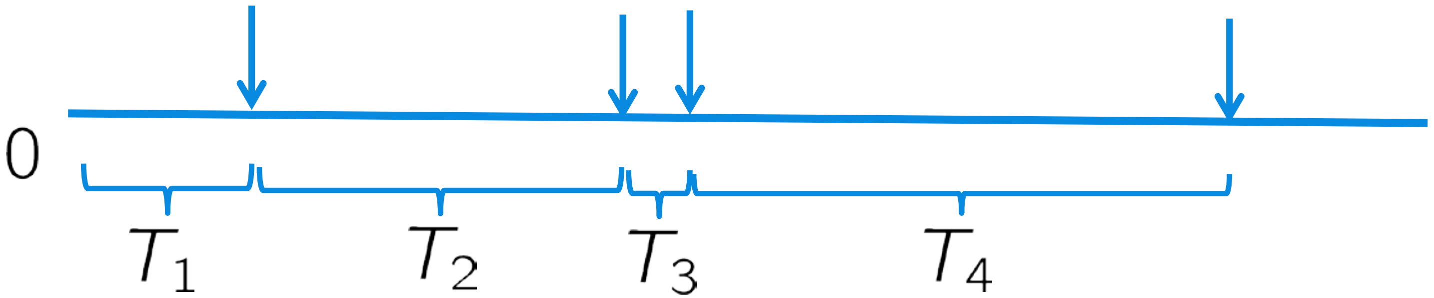
Distribution of inter arrival times

Theorem

Consider a Poisson process N with intensity λ

Let $T_1, T_2, \ldots$ be the inter arrival times

Then $T_1, T_2, \ldots$ are independent and identically distributed random variables, with exponential distribution with parameter λ



Two more properties of the Poisson process

5 The distribution of the points within an interval

Consider a Poisson process $X_1, X_2, X_3, \ldots$ of intensity λ.
Suppose it is **given** that $N(a, b) = n$,
i.e. there are exactly n points in the interval $(a, b]$.

Then the positions of these points in the interval are **independent** random variables **uniformly** distributed over $(a, b]$.

Question How does it follow that given that $N(a, b) = n$, the distribution of $N(a, c)$, for a subinterval $(a, c) \subset (a, b)$ is binomial? (As in the above exercise.)

Suppose $X_1, X_2, X_3, \ldots$ is a Poisson process with intensity λ. If it is given that there are four points in the interval (2,10], what is the probability that three of them lie between 6 and 10?

That is: find P((N(6,10) = 3 | N(2,10)=4).

A) 0.25

B) 0.0625

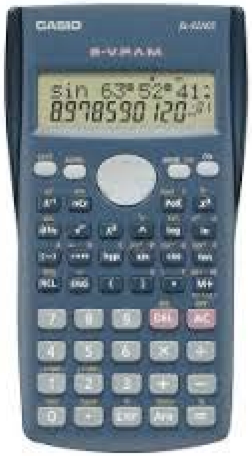
C) 0.1825

D) 0.375

E) 0.5

Stochastic simulation

Starting point: $U \sim \mathrm{Uniform}(0, 1)$




```
> runif(10)
 [1] 0.9597776 0.8796822 0.3489987
 [4] 0.7151867 0.5956950 0.2973708
 [7] 0.8215754 0.4651596 0.9705595
[10] 0.2546833
> 
```

How to simulate a discrete random variable

Exercise

Suppose you have a random number generator that simulates the uniform distribution on the interval $[0, 1]$. How can you use this to simulate

a. A Bernoulli random variable with parameter p?

b. A random variable D that can take on the values $1, 2, 3, 4, 5, 6$, each with probability $1/6$?

c. A RV with a Bin(10, 0.25) distribution?

FeedbackFruits

In Matlab the command "rand" produces a uniformly distributed random variable . Furthermore, "ceil" rounds upwards, and "floor" rounds downwards. How can you simulate an RV that takes the values 2,4,6,8 with probabilities ¼?

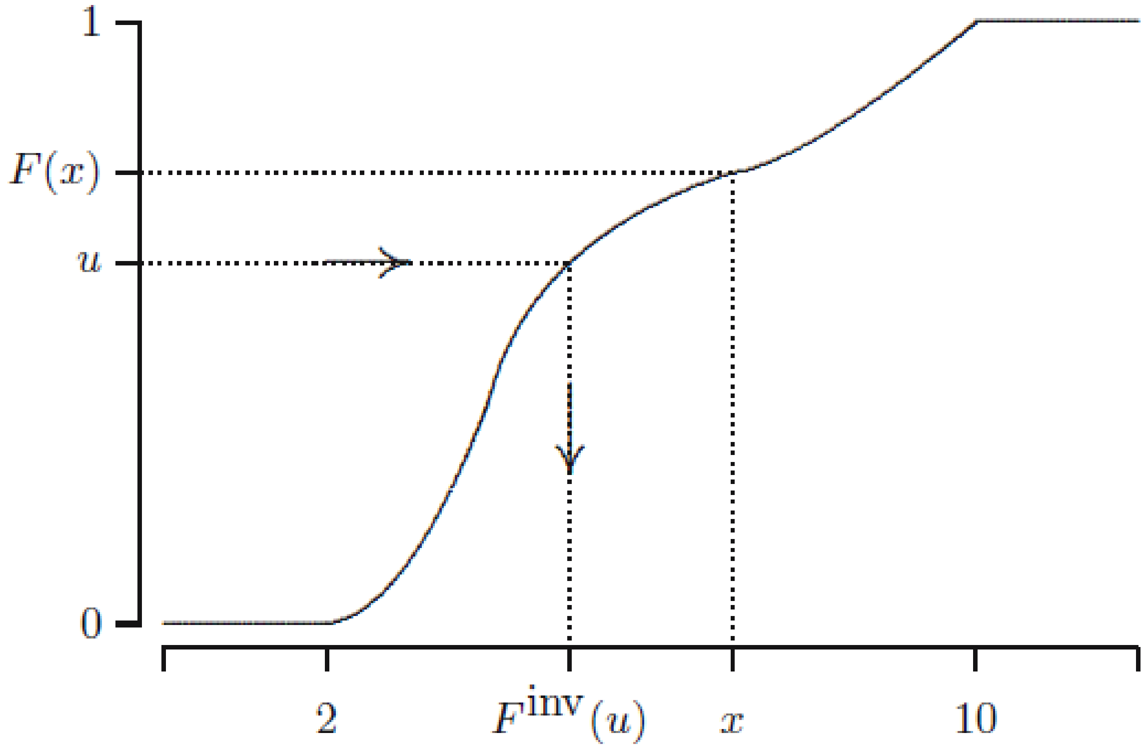
A) `u = rand; x = 4*ceil(2*u);`

B) `u = rand; x = 4*floor(2*u);`

C) `u = rand; x = 2*ceil(4*u);`

D) `u = rand; x = 2*floor(4*u);`

To generate a random variable with cdf F



$U \sim \text{Uniform}(0, 1)$

Define $X = F^{inv}(U)$

Then: X has distribution function F

Given: uniformly distributed random variable U. Needed: random variable with pdf

$$f(x) = \begin{cases} 2x, & \text{if } 0 \leq x \leq 1 \\ 0, & \text{else} \end{cases}$$

The following formula will 'do the trick':

A) $X = U/2$

B) $X = 2U$

C) $X = U^2$

D) $X = \sqrt{U}$

Suppose again that *U* has a uniform distribution on [0,1]. Give a function *g* such that *X = g(U)* has a Pareto(3) distribution, i.e.

$$F(x) = \begin{cases} 0\,, & \text{if } x \leq 1 \\ 1 - 1/x^3, & \text{else} \end{cases}$$

A) $g(u) = 1 - u^{-1/3}$

B) $g(u) = u^{-3}$

C) $g(u) = u^{-1/3}$

D) $g(u) = \frac{1}{3}u^{-2}$

For next class (week 3.5 lesson 1): Prepration for midterm!

Complete MyStatLab assignments and book exercises

Watch prelectures of past weeks

Book: Respective sections of Chapters 1-12

Prepare from Exam June 2014:

- **Multiple choice questions 1-6, 8**
- **Open question 1**

They will be discussed in class!

Probability

Good luck!