Probability

Lecture 1.2: Bayes, Independence & Discrete Random Variables

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Learning objective

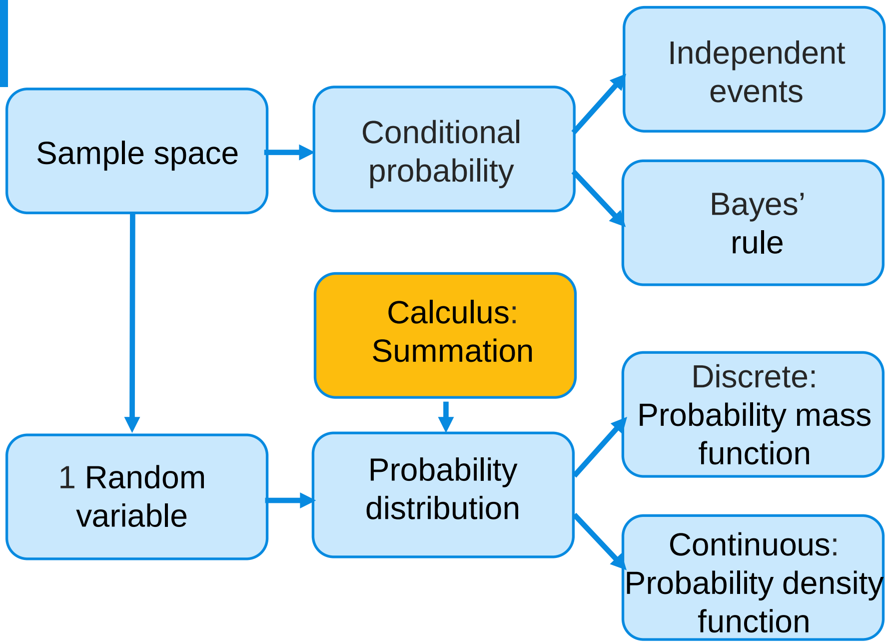
After this class you can
- **Compute conditional probabilities with Bayes' rule**
- **Check whether events are independent**
- **Manage discrete RVs via the probability mass function and the distribution function**

Book: Sections 3.3, 3.4, 4.1, 4.2

Before this class (week 3.1 lesson 2):

- Watch the prelecture '*Bayes' Rule*'
- Watch the prelecture '*Independence of events*'
- Book: Sections 3.3 and 3.4

Probability



Programme

- General Bayes' rule
- Exercises
- Independence of *n* events
- Exercises
- Discrete Random Variables
- Conclusion

Question from prelecture about Bayes' rule:

Suppose $P(B)$ equals 0.000013 instead of 0.02. Compute $P(B|T)$.

A) 0.000013

B) 0.125

C) 0.70

D) 0.000091

Question from prelecture about independent events:
Are A={Queen} and B={Hearts} independent when 2 jokers are added to the deck?

A) Yes

B) No

C) Sometimes

Bayes' Rule

Theorem:
Suppose the events $C_1, C_2, \ldots, C_m$ are disjoint and fill up the sample space Ω. Then the conditional probability of C_i given some event *A* is given by

$$\mathrm{P}(C_i \mid A) = \frac{\mathrm{P}(A \mid C_i)\mathrm{P}(C_i)}{\mathrm{P}(A \mid C_1)\mathrm{P}(C_1) + \ldots + \mathrm{P}(A \mid C_m)\mathrm{P}(C_m)}$$

Very often (as in the prelecture)

$$m = 2 \text{ and } C_1 = B,\; C_2 = B^c.$$

Exercise

Suppose a country has three provinces R_1, R_2, R_3 with respectively 50%, 20% and 30% of the whole population. Suppose further that in these regions respectively 40%, 60% and 30% will vote for the largest political party.
If a ‘random’ inhabitant confirms that she votes for this party, find the (conditional) probability she is from region R_3.

Two dice are thrown, and we consider three events:
A: the sum of the outcomes equals 7
B: the first outcome equals 3
C: the second outcome is bigger than the first
Which sets of events are independent? (It's a bit tricky!)

A) A and B, A and C, B and C.

B) A and B, A and C

C) A and B, B and C

D) A and C, B and C

E) A and B

F) A and C

G) B and C.

Independence of more than two events

Definition:
The events $A_1, A_2, \ldots, A_m$ are called independent if for each collection $A_{i_1}, A_{i_2}, \ldots, A_{i_k}$ it holds that

$$\mathrm{P}(A_{i_1} \cap A_{i_2} \cap \ldots \cap A_{i_k}) = \mathrm{P}(A_{i_1})\,\mathrm{P}(A_{i_2}) \cdots \mathrm{P}(A_{i_k})$$

Remark: there is again an equivalent characterization using conditional probabilities. See the next exercise.

Exercise

1	2	3	4
5	6	7	8
9	10	11	12
13	14	15	16
17	18	19	20
21	22	23	24
25	26	27	28
29	39	31	32
33	34	35	36

Consider the fictitious roulette table on the left. By the roulette wheel a random number is drawn. Define the events *E*: even, *R*: red, and *L*: low (i.e. ≤ 18).

1. Show that each set of two of these events is independent, but that the three together are dependent.
2. Also, compute $\mathrm{P}(E \mid R \cap L)$.
   Note that $\mathrm{P}(E \mid R \cap L) \neq \mathrm{P}(E)$

Discrete random variables

Definition: Let Ω be a sample space. A *discrete random variable* is a function $X : \Omega \to \mathbb{R}$ that takes on a finite number of values $a_1, a_2, \ldots, a_n$
or an infinite number of values $a_1, a_2, a_3, \ldots$

Remark: this definition is rather formal. See the next slide for an example.

Example of a discrete random variable

Throw two dice, and let M be the maximum.

$$\Omega = \{(1,1),\ (1,2),\ (2,1),\ (1,3),\ \ldots\ ,\ (6,6)\}$$

$$M((2,2)) = 2$$

$$\xrightarrow{M} \{1, 2, 3, 4, 5, 6\}$$

The probability mass function

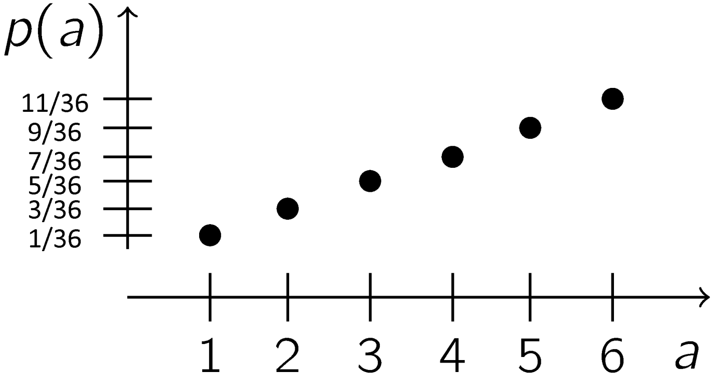
Definition: *the probability mass function* p of a discrete random variable X is the function $p : \mathbb{R} \to [0, 1]$ defined by

$$p(a) = \mathrm{P}(X = a) \quad \text{for} \quad -\infty < a < \infty$$

Note that $p(a) = 0$ for most values of a.

Example:
Probability mass function of maximum of 2 dice

a	$p(a)$
1	$1/36$
2	$3/36$
3	$5/36$
4	$7/36$
5	$9/36$
6	$11/36$



Exercise

Let D be the difference of the outcomes of two throws with a dice. $D((1,3)) = D((3,1)) = 2$, $D((4,4)) = 0$, etc.

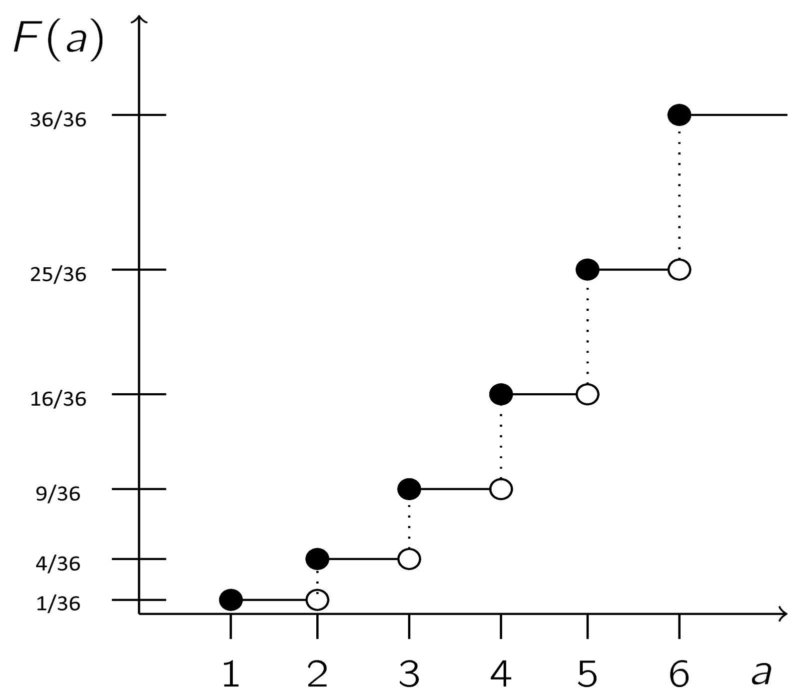
Give the probability mass function of D in the form of a table, as at the bottom of page 43.

The distribution function

Definition: the *distribution function* F of a discrete random variable X is the function $F : \mathbb{R} \to [0,1]$ defined by

$$F(a) = \mathrm{P}(X \leq a) \quad \text{for} \quad -\infty < a < \infty$$

Example: Distribution function of maximum of 2 dice



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Remark: both the probability mass function and the distribution function contain the 'complete' information of the random variable X. The distribution function will also play an important role for continuous variables later.

Two exercises

Consider a discrete RV X with the following p.m.f.:

a	-1	0	1	2
$p(a)$	$1/4$	$1/8$	$1/8$	$1/2$

Let Y be defined by $Y=X^2$. Calculate the p.m.f of Y.
Secondly, compute the value of the distribution function of both X and Y in $a=1$, $a=3/4$ and $a=\pi-3$.

Suppose that the distribution function of a discrete RV X is given by

$$F(a) = \begin{cases} 0, & a < 0 \\ 1/3, & 0 \leq a < 2 \\ 1/2, & 2 \leq a < 3 \\ 1, & a \geq 3. \end{cases}$$

Compute the p.m.f. of X.

For next class (week 3.2 lesson 1):

Complete MyStatLab assignments and book exercises

Watch prelectures *‘Binomial coefficients’*

Book: Section 4.3

After that class you

- **know five standard discrete distributions**
- **recognize the contexts in which they occur**

Probability

Good luck!