Probability

Lecture 3.3.2: Joint distributions

Name teacher

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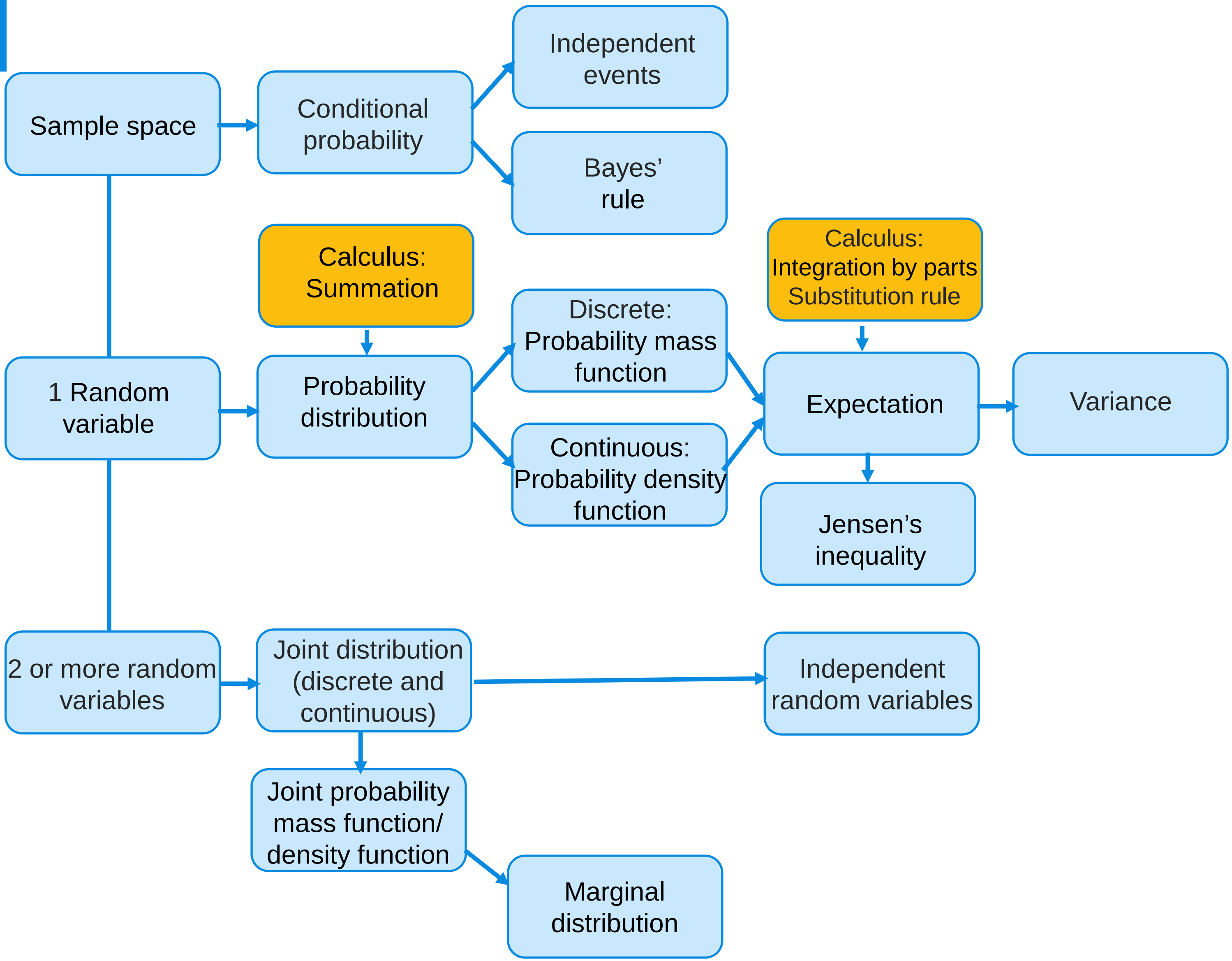
Learning objective

After this class you can

- **Apply Jensen's inequality**
- **Obtain the distribution of a transformed RV**
- **Manage joint distributions of discrete and continuous RVs**
- **Derive marginal distributions from the joint distribution**

Book: Sections 8.1, 8.2, 8.3, 9.1, 9.2

Probability



Before this class (week 3.3 lesson 2):

Watch prelecture *'Jensen's inequality'*

Book: Section 8.3

Programme

Transforming random variables
Transforming a normal RV to a standard normal RV

Exercises

The joint probability of maximum and sum

Joint distributions

Exercises

Let X be a RV with expectation equal to 3.
Compute $\mathrm{E}[3X-6]$.

A) 0

B) 3

C) 9

The function $g(x) = 3x - 6$ is

A) concave

B) convex

C) concave and convex

D) not concave and not convex

Transforming random variables

Let X be a RV and let Y be a function of X, e.g. $Y = \sqrt{X}$.
How to determine distribution of Y?

X **discrete**: via probability mass function of X.

X **continuous**: via distribution function of X.

Transforming normal RVs

Theorem:

Suppose $X \sim N(\mu, \sigma^2)$, then the random variable $rX + s$ also has a normal distribution:

$$rX + s \sim N(r\mu + s, r^2\sigma^2)$$

Corollary:

Every normally distributed RV can be transformed into a standard normal RV.

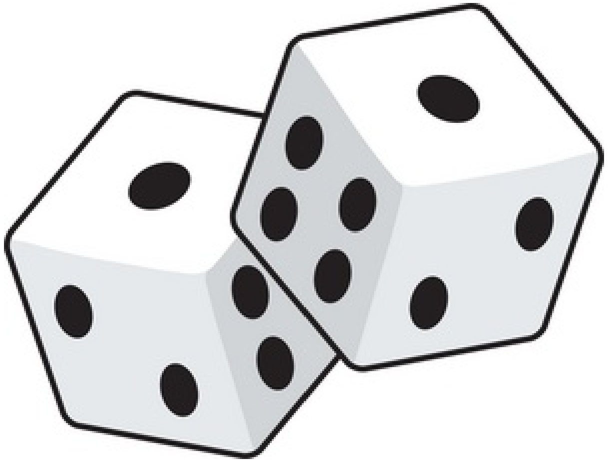
Exercises

Let X have a $Par(\alpha)$ distribution. Define the RV $Y = \ln X$. What is the distribution of Y?

Compute the probabilities $\mathrm{P}(X \leq 5)$ and $\mathrm{P}(X \geq 2)$ for X with a $N(4, 25)$ distribution.

Joint distribution function

Maximum and sum of two dice



Definition:
Let X and Y be two RVs.
The joint distribution function F of X and Y is the function $F : R^2 \to [0, 1]$ defined by

$$F(a, b) = \mathrm{P}(X \leq a, Y \leq b) \text{ for } -\infty < a, b < \infty$$

Compute $F(5, 3)$ for the joint distribution function of the pair (S, M).

The joint probability mass function

Definition:
Let X and Y be two discrete RVs.
The <u>joint probability mass function</u> p of X and Y is the function $p : R^2 \to [0, 1]$ defined by

$$p(a, b) = \mathrm{P}(X = a, Y = b) \quad \text{for} \quad -\infty < a, b < \infty$$

<u>From joint to marginal:</u>
Take sum of rows and colums

Exercise

Let X and Y be two RVs with joint distribution function the *Melencolia* distribution given by the table on page 128 in the book.

Compute

(a) $\mathrm{P}(X = Y)$

(b) $\mathrm{P}(X + Y = 5)$

(c) Give the marginal probability mass functions of X and Y

The joint density function

Definition:
Let X and Y be two continuous RVs.
The joint density function f of X and Y is the function $f : R^2 \to R$ such that

$$P(a_1 \leq X \leq b_1, a_2 \leq Y \leq b_2) = \int_{a_1}^{b_1} \int_{a_2}^{b_2} f(x, y)\, dy dx$$

The density function has to satisfy

1. $f(x, y) \geq 0$
2. $\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x, y) dx dy = 1$

Exercise

The joint probability density function of the pair (X, Y) is given by

$$f(x, y) = K(3x^2 + 8xy) \text{ for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2$$
$$f(x, y) = 0 \quad \text{for all other values of x and y}$$

(a) Find K
(b) Compute $P(2X \leq Y)$

From joint to marginal density function

Let f be the joint density function of X and Y.
Then the *marginal densities* of X and Y can be found as

$$f_X(x) = \int_{-\infty}^{\infty} f(x,y)dy \quad \text{and} \quad f_Y(y) = \int_{-\infty}^{\infty} f(x,y)dx$$

From joint density function to joint distribution function and vice versa

Let f be the joint density function of continuous variables X and Y.
Then the joint *distribution* function of X and Y can be found as

$$F(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f(x,y)\,dx\,dy$$

Let F be the joint distribution function of X and Y.
Then the joint *density* function of X and Y can be found as

$$f(x,y) = \frac{\partial^2}{\partial x \partial y} F(x,y)$$

Exercise

The joint probability density function of the pair (X, Y) is given by

$$f(x, y) = \frac{1}{10}(3x^2 + 8xy) \text{ for } 0 \leq x \leq 1 \text{ and } 0 \leq y \leq 2$$

$$f(x, y) = 0 \qquad \text{for all other values of x and y}$$

(a) Determine the marginal density of X
(b) Determine the joint distribution function of X and Y.
(c) Determine the marginal distribution function of Y

For next class (week 3.4 lesson 1):

Complete MyStatLab assignments and book exercises:
8.8, 8.12, 8.13, 9.10, 9.11, 9.14 (a)

Watch prelectures *'Independence of RVs'*

Book: Section 9.4

After this class you can

- **Check and use independence of RVs**
- **Apply the change-of-variable formula for two RVs**
- **Compute the covariance and correlation between two RVs**
- **Determine the distribution of the sum of two independent RVs**

Probability

Good luck!