Probability

TU Delft

Lecture 1.1: Probability spaces and conditional probability

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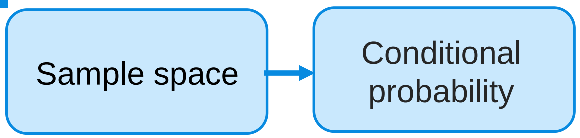
Learning objective

After this class you can

- **Set up a probability model**
- **Compute conditional probabilities**
- **Apply various probability rules**

Book: Sections 2.1, 2.2, 2.3, 2.5, 3.1, 3.2

Probability



Before this class (week 3.1 lesson 1):

Watch prelecture *'Set Theory Part I'* and *'Set Theory Part II'*

Book: Sections 2.1 , 2.2

Programme

Probability function
Addition and complement rule
Infinite sample spaces

Exercises

Conditional probability
Multiplication rule
Law of total probability

Exercises

Give the set $\{0, 1, 2\} \cup \{0, 2, 4\}$

A) $\{0, 0, 1, 2, 2, 4\}$

B) $\{0, 2\}$

C) $\{0, 1, 2, 4\}$

Give the set $Z \cap (0, 2]$

A) $\{1, 2\}$

B) $(0, 2]$

C) $[1, 2]$

Let E and F be two events for which one knows that the probability that at least one of them occurs is $\frac{3}{4}$.
What is the probability that neither E nor F occurs?

A) $\frac{3}{4}$

B) $\frac{3}{2}$

C) $\frac{1}{8}$

D) $\frac{1}{4}$

2.3 Probability function: definition

Definition:

Let Ω be a finite sample space. A *probability function* P assigns to each event A in Ω a number $P(A)$ in $[0, 1]$ such that:

1. $P(\Omega) = 1$.
2. $P(A \cup B) = P(A) + P(B)$, if A and B are disjoint.

The number $P(A)$ is called the *probability* of A.

Example

Experiment: month of birth of random person.
Sample space: $\Omega = \{$ Jan, Feb, ..., Nov, Dec $\}$.

What to choose for probability function? 3 options!

1. All equal:

   $P(\text{Jan}) = 1/12 = P(\text{Feb}) = \ldots = P(\text{Dec}).$

2. Count days:

   $P(\text{Jan}) = 31/365 = \ldots, P(\text{Feb}) = 28/365, P(\text{April}) = 30/365 = \ldots$

3. Count also leap years:

   $P(\text{Jan}) = 124/1461 = \ldots, P(\text{Feb}) = 113/1461, P(\text{April}) = 120/1461 = \ldots$

Example

Experiment: month of birth of random person.
Sample space: $\Omega = \{$ Jan, Feb, …, Nov, Dec $\}$.
Assume equal probabilities for all months.

Let A be the event $A = \{$Born in even month$\}$. What is $P(A)$?

Addition rule

Theorem:

For any two events A and B we have

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

Complement rule

Theorem:

For any event A we have
$$P(A^c) = 1 - P(A)$$

Two exercises

Let $P(A) = 1/3$, $P(B) = 1/2$ and $P(A \cup B) = 3/4$.
Compute $P(A \cap B)$ and $P(A^c \cup B^c)$.

An experiment has only two outcomes. The first has probability p to occur, the second probability p^2. What is p?

2.5 Infinite sample space

Suppose you throw a die until 6 comes up.
Interested in the number of throws needed.
What is the outcome space? $\Omega = \{1, 2, 3, \ldots\}$

Furthermore, $P(n) = \left(\frac{5}{6}\right)^{n-1} \frac{1}{6}$.

Probability function?

Extend property 2 of probability function to infinite unions:

$$P(A_1 \cup A_2 \cup \ldots) = P(A_1) + P(A_2) + \ldots,$$

if $A_1, A_2, \ldots$ are disjoint events.

Check: $P(\Omega) = 1$.

Consider two experiments:
I. Number of earthquakes in Groningen per year.
II: Number of days until next earthquake in Groningen.
Which has/have an infinite sample space?

A) Only I.

B) Only II.

C) Both I and II.

D) None.

3.1 Conditional probability: motivation

A die is thrown twice. Suppose you are told that the sum equals 9, what is now the probability that the first throw is 5?

A={first throw is 5}={(5,1),…(5,6)}. Clearly, P(A)=1/6.

C={sum is 9} = {(3,6),(4,5), (5,4), (6,3)}.

(5,4) only one that is in A.

Therefore, *given that C occurs*,

the “conditional” probability of A is 1/4.

Information changed the probability!

Conditional probability: definition

Definition:

The *conditional probability* of A given C is defined as:

$$P(A|C) = \frac{P(A \cap C)}{P(C)},$$

provided $P(C) > 0$.

Interpretation:
Suppose you know C occurs. Then conditional probability of A is the fraction of C that lies also in A.

Exercise

Consider the month of birth of a randomly selected person.

Let L be the event:

$$L = \{\text{Born in ''long'' month}\}$$
$$= \{\text{Jan, Mar, May, Jul, Aug, Oct, Dec}\}.$$

Let S be the event:

$$S = \{\text{Born in summer month}\}$$
$$= \{\text{Jun, Jul, Aug}\}$$

Compute $P(S|L)$ and $P(L|S)$.

Consider a family with two children. You are told they have at least one boy. What is the probability of two boys?

A) 1/4

B) 1/3

C) 1/2

D) 2/3

Two exercises

Prove that $P(A|C) + P(A^c|C) = 1$.

Draw two cards from a regular deck of 52.
Let S_1 be the event that the first one is a spade and
let S_2 be the event the second one is a spade.

Compute $P(S_2|S_1)$ and $P(S_2|S_1^c)$.

3.2 Multiplication rule

You travel from Sydney via Bangkok to Amsterdam.
Your suitcase gets lost with probability 0.02 in Sydney and with probability 0.05 in Bangkok.
What is the probability your suitcase arrives in Amsterdam?

Theorem:
For any events A and C it holds that

$$P(A \cap C) = P(A|C)P(C)$$

Multiplication rule: example

Consider a group of 25 people. What is the probability at least two people have the same birthday?

3.3 Law of total probability

Theorem (Version I) :

Let A and C be two events. We have:

$$P(A) = P(A|C)P(C) + P(A|C^c)P(C^c).$$

Theorem (Version II) :

Suppose $C_1, C_2, \ldots, C_m$ are disjoint events such that $C_1 \cup C_2 \cup \ldots \cup C_m = \Omega$. For any event A we have

$$P(A) = P(A|C_1)P(C_1) + P(A|C_2)P(C_2) + \ldots + P(A|C_m)P(C_m)$$

Law of total probability: example

In 2001 the European Commission tested cows for the "Mad Cow Disease" (BSE).
This test was not perfect: a cow having BSE was tested positive with probability 70%. While a cow **not** having BSE was tested negative with probability 90%.
Suppose that the probability of a cow having BSE was 2%.

What is the probability that an **arbitrary** cow tests positive?

Consider a vase with 9 balls: 3 red and 6 blue. Draw a ball and replace with two extra balls of the same colour. Then draw a ball again: What is the probability this one is red?

A) 3/11

B) 1/3

C) 15/99

D) 1/9

Two exercises

Let $P(A) = 1/3$ and $P(B|A^c) = 1/4$. Compute $P(A \cup B)$.

Recall the following experiment:
Draw two cards from a regular deck of 52.
Let S_1 be the event that the first one is a spade and
let S_2 be the event the second one is a spade.

Compute $P(S_2)$.

For next class (week 3.1 lesson 2):

Complete MyStatLab assignments and book exercises

Watch prelectures *'Bayes' rule'* and '*Independence of events*'

Book: Section 3.3 and 3.4

After this class you know the general version of Bayes' rule, and you know when more than two events are independent.
Furthermore, you know how to manage discrete random variables via the probability mass function and the distribution function.

Probability

Good luck!