Probability

Lecture 3.1: Expectation and variance

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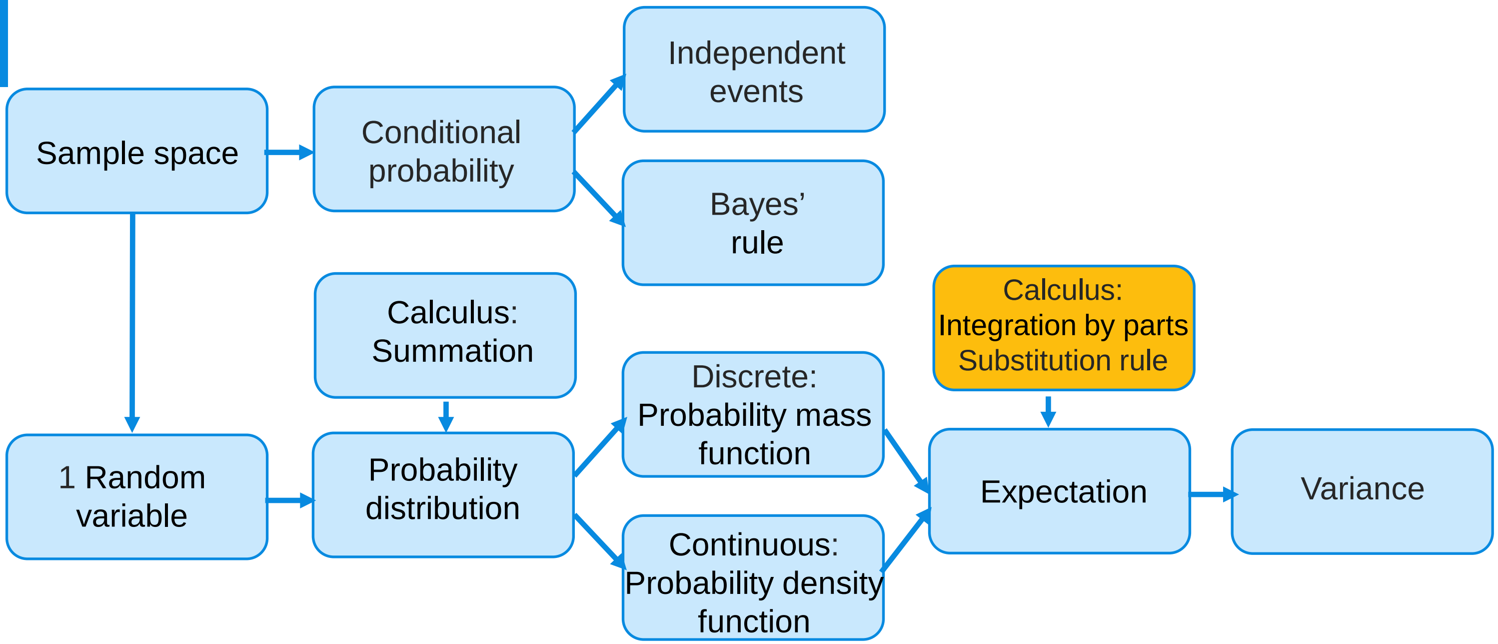
Learning objective

After this class you can

- **Compute the expectation and variance of discrete and continuous RVs**
- **Apply the change-of-variables formula**
- **Apply the change-of-units formula**

Book: Sections 7.1, 7.2, 7.3, 7.4

Probability



Before this class (week 3.3 lesson 1):

Watch prelecture ‘Expectation’

Book: Section 7.1

After this class you can

- **Compute the expectation and variance of discrete and continuous RVs**
- **Apply the change-of-variables formula**
- **Apply the change-of-units formula**

Programme

- Expectation and Examples
- Exercises
- Change-of-variable formula
- Variance and Standard deviation
- Exercises
- Change-of-units formula

Let X be a discrete RV with probability mass function p(-1)=1/5, p(0)=2/5 and p(1)=2/5. Let Y be X^2. Compute E[X] and E[Y].

A) E[X]=0, E[Y]=0

B) E[X]=1/5, E[Y]=1/25

C) E[X]=1, E[Y]=1

D) E[X]=1/5, E[Y]=3/5

Expectation: center of mass

Definition:

The *expectation* of a discrete RV X is defined as

$$\mathrm{E}[X] = \sum_i a_i \mathrm{P}(X = a_i)$$

Interpretation: weighted average/center of mass.

The *expectation* of a continuous RV X with pdf f is given by

$$\mathrm{E}[X] = \int_{-\infty}^{+\infty} x f(x)\, \mathrm{d}x$$

Examples

Let $X \sim Bin(n, p)$, then $\mathrm{E}[X] = \sum_{k=0}^{n} k \binom{n}{k} p^k (1-p)^{n-k} = np$.

Let $X \sim Geo(p)$, then $\mathrm{E}[X] = \sum_{k=1}^{\infty} kp(1-p)^{k-1} = \frac{1}{p}$.

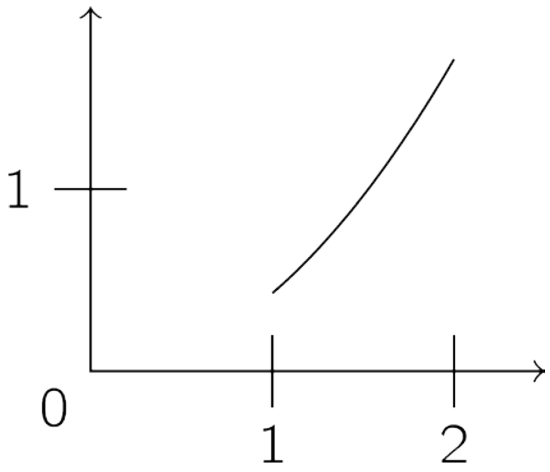
Let $X \sim Pois(\lambda)$, then $\mathrm{E}[X] = \sum_{k=0}^{\infty} k \frac{\lambda^k}{k!} e^{-\lambda} = \lambda$.

Let $X \sim Exp(\lambda)$, then $\mathrm{E}[X] = \int_0^{\infty} x \lambda e^{-\lambda x} \, \mathrm{d}x = \frac{1}{\lambda}$.

Let $X \sim N(\mu, \sigma^2)$, then $\mathrm{E}[X] = \int_{-\infty}^{+\infty} x \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, \mathrm{d}x = \mu$.

Three exercises

The RV X has pdf $f(x) = \frac{3x^2}{7}$
for x in [1,2] and 0 elsewhere.
First: do you think E[X] is closer to 1 or to 2?
Secondly, compute E[X].



Let X have a Pareto distribution with parameter α, i.e.

$$f(x) = \frac{\alpha}{x^{\alpha+1}}, \text{ for } x \geq 1.$$

For which α does E[X] exist? Compute E[X] for these α.

Show that the expectation of a Poisson distributed RV with parameter λ `surprisingly’ equals λ.

Change-of-variable formula

Theorem:

Let X be a RV and $g : \mathbb{R} \to \mathbb{R}$ a function.

If X is discrete, then $\mathrm{E}[g(X)] = \sum_i g(a_i)\mathrm{P}(X = a_i)$.

If X is continuous, then $\mathrm{E}[g(X)] = \int_{-\infty}^{+\infty} g(x)f(x)\,\mathrm{d}x$.

Two exercises

Recall the following exercise: Let X be a discrete RV with p.m.f. $p(-1)=1/5$, $p(0)=2/5$ and $p(1)=2/5$. Let Y be X^2. Now, compute $E[Y]$ using the change-of-variable formula.

Let X be an exponentially distributed RV with parameter λ. Compute $E[X^2]$.

Variance

Definition:

The *variance* of a RV *X* is defined as

$$\operatorname{Var}(X) = \operatorname{E}\left[(X - \operatorname{E}[X])^2\right]$$

The *standard deviation* of a RV *X* is defined as

$$\operatorname{SD}(X) = \sqrt{\operatorname{Var}(X)}$$

For any RV X the variance can be computed by

$$\operatorname{Var}(X) = \operatorname{E}\left[X^2\right] - (\operatorname{E}[X])^2$$

For a certain RV X it is known that $E[X]=3$ and $Var(X) = 2$. What is $E[X^2]$?

A) -7

B) 11

C) 9

D) 1

Example

Let $X \sim N(\mu, \sigma^2)$, then

$$\begin{aligned}\mathrm{Var}(X) &= \mathrm{E}\left[(X - \mathrm{E}[X])^2\right] \\ &= \int_{-\infty}^{+\infty} (x-\mu)^2 \frac{1}{\sigma\sqrt{2\pi}} e^{-\frac{1}{2}\left(\frac{x-\mu}{\sigma}\right)^2} \, \mathrm{d}x \\ &= \ldots = \sigma^2.\end{aligned}$$

Two exercises

Compute the variance of an exponentially distributed RV with parameter λ.

For which α is the variance of a Pareto-distributed RV with parameter α finite? Determine the variance for these α.

Change-of-units formula

Theorem:
For any RV X and any real values r and s it holds that

$$\mathrm{E}[rX + s] = r\mathrm{E}[X] + s, \text{ and } \mathrm{Var}(rX + s) = r^2\mathrm{Var}(X).$$

Remark: This follows from the change-of-variable formula.
Do you see how?

Let X be a RV with $E[X]=2$ and $E[X^2]=9$. Let $Y=3-2X$. Compute $E[Y]$ and $Var[Y]$.

A) E[Y]= 1, Var[Y] = -10

B) E[Y]= -1, Var[Y] = -10

C) E[Y]= -1, Var[Y] = 20

D) E[Y]= 1, Var[Y] = 20

For next class (week 3.3 lesson 2):

Complete MyStatLab assignments and book exercises

Watch prelectures *'Preparation to Jensen'*

Book: Section 8.3

After this class you can

- **Apply Jensen's inequality**
- **Manage joint distributions of discrete and continuous RVs**
- **Derive marginal distributions from the joint distribution**

Probability

Good luck!

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