vector field

The definition of a vector field **F** in space is given by:

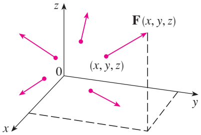
Definition

A vector field $\mathbf{F} : A \to B$, with $A, B \subseteq \mathbb{R}^3$, is map from A to B given by $\mathbf{F}(x, y, z) = P(x, y, z)\mathbf{i} + Q(x, y, z)\mathbf{j} + R(x, y, z)\mathbf{k}$

- Remark: It is easy to find a definition for a vector field in $\mathbb{R}^2$.
- Question: Could you give some examples for vector fields?
- Notation: A briefer notation is $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$.

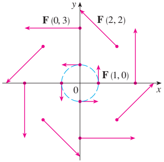
pictures and examples:

- example1:



In the picture above you see a graphical representation of a vector field, it consists of arrows with their origin in a point of the domain.

- example2:



Question: Could you give a formula for it?

pictures and examples:

Some other examples:

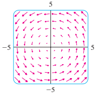
- 

FIGURE 6
$\mathbf{F}(x, y) = \langle -y, x \rangle$

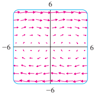


FIGURE 7
$\mathbf{F}(x, y) = \langle y, \sin x \rangle$

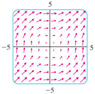


FIGURE 8
$\mathbf{F}(x, y) = \langle \ln(1 + y^2), \ln(1 + x^2) \rangle$

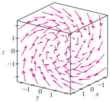
- 

FIGURE 10
$\mathbf{F}(x, y, z) = y\,\mathbf{i} + z\,\mathbf{j} + x\,\mathbf{k}$

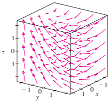


FIGURE 11
$\mathbf{F}(x, y, z) = y\,\mathbf{i} - 2\,\mathbf{j} + x\,\mathbf{k}$

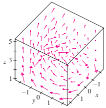
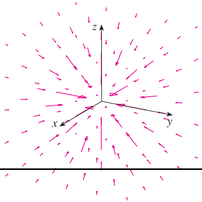
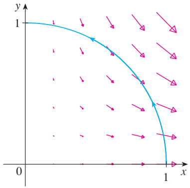


FIGURE 12
$\mathbf{F}(x, y, z) = \frac{y}{z}\,\mathbf{i} - \frac{x}{z}\,\mathbf{j} + \frac{z}{4}\,\mathbf{k}$

- A representation of a gravitational or electric field:



Some Questions:

- Given the circle with equation $x^2 + y^2 = 4$ or the parabola $y = x^2$ with $x \in [-1, 1]$. What are their arc lengths?
- Again given the parabola $y = x^2$ with $x \in [-1, 1]$. Suppose it is charged with a constant charge density ρ_0. Where do you locate its "center of charge"?
- 

What is the work done by the "field force" if a particle moves in a field?

We start with the first problem:

Given the circle with equation $x^2 + y^2 = 4$ or the parabola $y = x^2$ with $x \in [-1, 1]$. What are their arc lengths?

Definition (arc length)

Given a parametrization $\langle x(t), y(t), z(t) \rangle$ of a curve with $t \in [a, b]$, then its arc length is given by the integral:

$$\int_{\mathcal{C}} ds = \int_{\mathcal{C}} \sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \int_a^b \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2} dt.$$

- Why is the definition of this form?
- Pay attention to the different, equivalent integral forms.
- Often is used $ds = \sqrt{x'^2(t) + y'^2(t) + z'^2(t)} dt$.

the answer on

Given the circle with equation $x^2 + y^2 = 4$. What is its arc length?

- parametrization circle: $\langle 2\cos(t), 2\sin(t)\rangle$ with $t \in [0, 2\pi]$,
- so arc length is: $\int_0^{2\pi} 2\sqrt{(-\sin(t))^2 + (\cos(t))^2} dt = 4\pi$

the answer on

Given the parabola $y = x^2$ with $x \in [-1, 1]$. What is its arc length?

- parametrization parabola: $\langle t, t^2\rangle$ with $t \in [-1, 1]$,
- so arc length is: $\int_{-1}^{1} \sqrt{(1 + 4t^2} dt = \sqrt{5} - 1/2 \ln\left(-2 + \sqrt{5}\right)$.

line integral of a scalar - and vector function:

Given the parabola $y = x^2$ with $x \in [-1, 1]$. Suppose it is charged with a constant charge density ρ_0.

Where do you locate its "center of charge"?
A guess: The charge dQ of a small part of the parabola is given by $\rho_0 ds$, so the x-coordinate and y-coordinate of the center of charge will be symbolic given by $\frac{\int_{\mathcal{C}} x\rho_0 ds}{\int_{\mathcal{C}} \rho_0 ds}$ and $\frac{\int_{\mathcal{C}} y\rho_0 ds}{\int_{\mathcal{C}} \rho_0 ds}$

Definition (line integral of a scalar function)

Given a parametrization $\langle x(t), y(t), z(t)\rangle$ of a curve with $t \in [a, b]$ and a scalar function $f(x, y, z)$. Then :

$$\int_{\mathcal{C}} fds = \int_{\mathcal{C}} f\sqrt{(dx)^2 + (dy)^2 + (dz)^2} = \int_a^b f(x(t), y(t), z(t))\sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}dt.$$

- Pay attention to the different, equivalent integral forms.
- In our example is $f(x, y, z) = x\rho_0$, $f(x, y, z) = y\rho_0$ or $f(x, y, z) = \rho_0$.

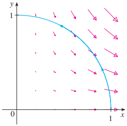
the answer on

Given the parabola $y = x^2$ with $x \in [-1, 1]$. Suppose it is charged with a constant charge density ρ_0.

Where do you locate its "center of charge"?

- parametrization parabola: $\langle t, t^2 \rangle$ with $t \in [-1, 1]$,
- so the x-location is: $\dfrac{\int_{\mathcal{C}} x\rho_0 ds}{\int_{\mathcal{C}} \rho_0 ds} = \dfrac{\int_{-1}^{1} \rho_0 t\sqrt{(1+4t^2)}dt}{\int_{-1}^{1} \rho_0 \sqrt{(1+4t^2)}dt}$
- so the y-location is: $\dfrac{\int_{\mathcal{C}} y\rho_0 ds}{\int_{\mathcal{C}} \rho_0 ds} = \dfrac{\int_{-1}^{1} \rho_0 t^2\sqrt{(1+4t^2)}dt}{\int_{-1}^{1} \rho_0 \sqrt{(1+4t^2)}dt}$
- after calculations we get 0 for the x-location and $\dfrac{\frac{9}{16}\sqrt{5}+1/32\ \ln\left(-2+\sqrt{5}\right)}{\sqrt{5}-1/2\ \ln\left(-2+\sqrt{5}\right)}$ for the y-location.

line integral of a scalar - and vector function:



What is the work done by the "field force" if a particle moves in a field? The answer:

- From secondary education we know from physics: $W = Fs$, all this under the assumption that we are dealing with a constant force in the direction of the displacement.
- A more convenient formula, also known from high school is $W = Fs\cos(\alpha)$, now the force is still constant but it makes a constant angle with the displacement.
- The last in vector form: $W = \mathbf{F}\cdot\mathbf{s}$.
- Let $\mathcal{C}$ be some "motion-curve" in space in a (force-) field $\mathbf{F}(x, y, z)$ then the work dW done during a infinite displacement $d\mathbf{s}$ is given by $dW = \mathbf{F}\cdot d\mathbf{s}$.
- So $W = \int_{\mathcal{C}} dW = \int_{\mathcal{C}} \mathbf{F}\cdot d\mathbf{s}$.
- Suppose the field is defined by $\mathbf{F} = \langle P, Q, R\rangle$ with P, Q and R the x, y an z-components of the force, with each of them a function of x, y and z, then we write $W = \int_{\mathcal{C}} Pdx + Qdy + Rdz$, wherein we have calculated the dot product between $\mathbf{F} = \langle P, Q, R\rangle$ and $d\mathbf{s} = \langle dx, dy, dz\rangle$.
- Finally, if $\mathcal{C}$ is given by $\langle x(t), y(t), z(t)\rangle$, then $d\mathbf{s} = \langle dx, dy, dz\rangle = \langle x'(t)dt, y'(t)dt, z'(t)dt\rangle$,for $t \in [a, b]$,
- and so $W = \int_a^b \left(P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t)\right) dt$.

We make things formal by the following definition:

Definition (line integral of a vector function)

Given a parametrization $\langle x(t), y(t), z(t)\rangle$ of a curve with $t \in [a, b]$ and a vector function $\mathbf{F}(x, y, z) = \langle P(x, y, z), Q(x, y, z), R(x, y, z)\rangle$. Then :

$$\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} =$$

$$\int_{\mathcal{C}} P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz =$$

$$\int_a^b \Big(P(x(t), y(t), z(t))x'(t) + Q(x(t), y(t), z(t))y'(t) + R(x(t), y(t), z(t))z'(t) \Big)\, dt$$

An example: A very familiar problem: Determine the work done by the gravity-force on a particle with mass m and a "motion curve" which is given by $\langle \cos(\phi), \sin(\phi), \phi$, with $\phi \in [0, 2\pi]$

- $\mathbf{F}(x, y, x) = \langle 0, 0, -mg \rangle$.
- $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} =$
- $\int_{\mathcal{C}} P(x, y, z)dx + Q(x, y, z)dy + R(x, y, z)dz =$
- $\int_0^{2\pi} -mg \, d\phi$
- So the work done by the gravity force is $-2mg\pi$, which is a familiar result!

Definition (conservative vector field)

A vector field is conservative if there is a scalar function with $\nabla f = \mathbf{F}$

- Given $f(x, y, z) = -mgx$, then $\mathbf{F}(x, y, z) = \langle 0, 0, -mg \rangle$.
- Given $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$, then $\mathbf{F}(x, y, z) = \frac{-1}{\left(\sqrt{(}x^2+y^2+z^2)\right)^3} \langle x, y, z \rangle$.
- Probably you recognize in this something of a potential function!

path independence

Theorem (path independence)

Given a conservative vector field $\mathbf{F} = \nabla f$ *in space and given some curve* $\mathcal{C}$ *given by* $\mathbf{r}(t)$ *for* $t \in [a, b]$

then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

- So if a path is closed then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 0$ for every conservative field.
- Why does a swing work? (also in the outer space)
- Examine the force $\mathbf{F}(x, y) = \langle -2x, -2y \rangle$. (a two dimensional spring)
- Could you reverse the theorem i.e. holds: "path independence implies conservative force"?
  Not in general!