Our goal this morning is to understand: The potential $\Phi(h)$ above a uniform charged sphere is given by the integral:

$$1/4\,\frac{1}{\pi\,\epsilon_0}\int_{-R}^{R}\int_{0}^{2\,\pi}\int_{0}^{\sqrt{R^2-z^2}}\frac{r\,\sigma}{\sqrt{r^2+(h-z)^2}}\,\mathrm{d}r\,\mathrm{d}\theta\,\mathrm{d}z$$

wherein:

- σ the constant charge density $[\frac{Q}{m^3}]$.
- h is the distance from the center outside the sphere.
- R is the radius of the sphere.
- Maple answer: $1/3\,\frac{\sigma\,R^3}{h\,\epsilon_0}$.
- So the electric field is given by $-\nabla\Phi(x,y,z)=\frac{Q}{4\pi\epsilon_0 z^2}\hat{\rho}$ wherein Q the total charge of the sphere and $\hat{\rho}$ the unity-vector pointing in radial direction w.r.t. the center of the sphere

example 1

An example:
Find the value of

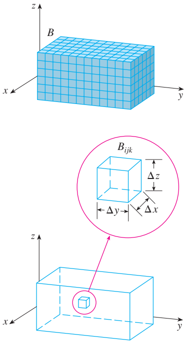
$$\int\int\int_G zdV$$

with G the "solid" domain between the surfaces

- $x = 0, y = 0, z = 0$ and
- $x + y + z = 1$.

Before we do the calculations we need the definition for $\int\int\int_G zdV$.
Of course this definition is similar to the definition of double integral.
Make a start:

The Riemann sum:



With the pictures above we define for triple integrals Riemann sums and the meaning of $\int\int\int_G f(x,y,z)dV$.

The Riemann sum :

Definition

- Given a function $f : G \to \mathbb{R}$. ($G \subset \mathbb{R}^3$
- For $i = 1...n$, let $\triangle B_i$ the size (volume) of B_i, which is a subregion of G, i.e. a member of a partition $\mathcal{P}$ so that $\bigcup_{B_i \in \mathcal{P}} B_i = G$ and almost no overlap between the subregions.We choose in every B_i a point (a_i, b_i, c_i). We write for this choice, $\mathcal{S}$.
- The Riemann sum, denoted with $\mathcal{R}(f, G, \mathcal{P}, \mathcal{S})$, is summation $\Sigma_i f(a_i, b_i, c_i)\triangle B_i$.

- We calculate in some sense the limit of $\mathcal{R}(f, G, \mathcal{P}, \mathcal{S})$ for $\triangle B_i \downarrow 0$.
- Remark: this limit has to be independent of the choices for $\mathcal{P}$ and $\mathcal{S}$.
- This "limit number" is denoted with $\int\int\int_G f(x, y, z)dV$.

example 1:

Back to our example:

Find the value of

$$\int\int\int_G z dV$$

with G the "solid" domain between the surfaces

- $x = 0, y = 0, z = 0$ and
- $x + y + z = 1.$

Question:Can we start for finding its value? What do we need?

Answer: A 3-dim. version of *Fubini*! Give it!

3=dim. Fubini.

Theorem

Given a function $f : G \to \mathbb{R}$. We assume that we can describe the domain G as: $a \le x \le b$, $f_1(x) \le y \le f_2(x)$ and $g_1(x,y) \le z \le g_2(x,y)$ for some numbers a and b , some functions f_1 and f_2 and some functions g_1 and g_2.
Then:

$$\int\int\int_G f(x,y,z)dV = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} \left[\int_{g_1(x,y)}^{g_2(x,y)} f(x,y,z)dz\right] dy\right] dx.$$

Remarks:

- We see on the right side an iterated integral, just a repetition of three "common" integrals; the inside one over z, the middle over y and the outside one over x.
- So the evaluation of the right side is nothing more than finding primitives and doing substitutions.
- Back to our last problem.

example 1:

Back to our example:

Find the value of

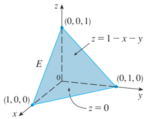
$$\int\int\int_G z dV$$

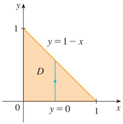
with G the "solid" domain between the surfaces

- $x = 0, y = 0, z = 0$ and
- $x + y + z = 1.$

Question: Find the "Fubini form" for G.

Watch:





$$0 \leq x \leq 1, 0 \leq y \leq 1 - x, 0 \leq z \leq 1 - x - y$$

example 1:

All this yields the iterated integral:

$$\int_0^1 \int_0^{1-x} \int_0^{1-x-y} z\,dz\,dy\,dx$$

Its value is: $\frac{1}{24}$.

an example and cylinder coordinates

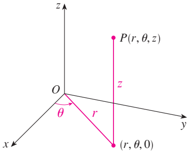
Find the value of

$$\int\int\int_G \sqrt{x^2+y^2}dV$$

with G the "solid" domain between the surfaces

- $x=0, y=0, z=0, z=1$ and
- $x^2+y^2=4.$

For finding its value it is convenient to describe integral in cylinder coordinates.
Look at the following picture:



From this we get $\boxed{x=r\cos(\theta), y=r\sin(\theta), z=z}$.

We make our "translation" of

Find the value of

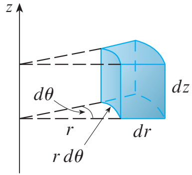
$$\int\int\int_G \sqrt{x^2+y^2}dV$$

with G the "solid" domain between the surfaces

- $x=0, y=0, z=0, z=1$ and
- $x^2+y^2=4$.

Answer:

- Domain G is described by: $0 \le r \le 2, 0 \le \theta \le \frac{\pi}{2}, 0 \le z \ge 1$
- The integrand as: r,
- dV as: $rdrd\theta dz$. (Look at picture below.)



We get:

$\int_0^1 \int_0^{1/2\,\pi} \int_0^2 r^2 dr\, d\theta\, dz$ with answer: $4/3\,\pi$.

an example and spherical coordinates

Find the value of

$$\int\int\int_G \sqrt{x^2+y^2+z^2}\,dV$$

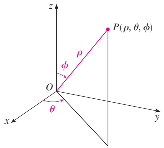
with G the "solid" domain between the surfaces

- $x=0, y=0, z=0$ and
- $x^2+y^2+z^2=4.$

For finding its value it is convenient to describe integral in spherical coordinates.

Look at the following picture:

$\rho \geqslant 0 \qquad 0 \leqslant \phi \leqslant \pi$



From this we get

$\boxed{x=\rho\sin(\phi)\cos(\theta), y=\rho\sin(\phi)\sin(\theta), z=\rho\cos(\phi)}.$

an example and spherical coordinates

We make our "translation" of Find the value of

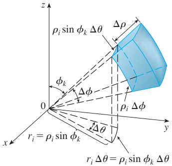
$$\int\int\int_G \sqrt{x^2+y^2+z^2}dV$$

with G the "solid" domain between the surfaces

- $x = 0, y = 0, z = 0$ and
- $x^2 + y^2 + z^2 = 4.$

Answer:

- Domain G is described by: $0 \leq \rho \leq 2, 0 \leq \theta \leq \frac{\pi}{2}, 0 \leq \phi \leq \frac{\pi}{2}$
- The integrand as: ρ,
- dV as: $\rho^2 \sin(\phi) d\rho d\theta d\phi$. (Look at picture below.)



We get: $\int_0^2 \int_0^{1/2\,\pi} \int_0^{1/2\,\pi} \rho^3 \sin(\phi)\, d\phi\, d\theta\, d\rho$, calculation gives $2\Pi\ \pi$

dA and dV for general transformations

- For polar coordinates we get $dA = rdrd\theta$.
- For cylinder coordinates we get $dV = rdrd\theta dz$.
- For spherical coordinates we get $dV = \rho^2 \sin(\phi) d\rho d\theta d\phi$.
- Question: What do we get for dA for a transformation $x = x(u, v)$ and $y = y(u, v)$and for
- for dV for a transformation $x = x(u, v, w)$, $y = y(u, v, w)$ and $z = z(u, v, w)$?
- Answer: Crystal clear is: $\boxed{dA = \left| det \begin{bmatrix} \frac{\partial}{\partial u} x(u, v) & \frac{\partial}{\partial v} x(u, v) \\ \frac{\partial}{\partial u} y(u, v) & \frac{\partial}{\partial v} y(u, v) \end{bmatrix} \right| dudv}$
- For yourself $\boxed{dV = \left| det \begin{bmatrix} \frac{\partial}{\partial u} x(u, v, w) & \frac{\partial}{\partial v} x(u, v, w) & \frac{\partial}{\partial w} x(u, v, w) \\ \frac{\partial}{\partial u} y(u, v, w) & \frac{\partial}{\partial v} y(u, v, w) & \frac{\partial}{\partial w} y(u, v, w) \\ \frac{\partial}{\partial u} z(u, v, w) & \frac{\partial}{\partial v} z(u, v, w) & \frac{\partial}{\partial w} z(u, v, w) \end{bmatrix} \right| dudvdw}$
- With it check former results for pole, cylinder and spherical coordinates.