chain rule for functions of several variables :

chain rule for 1 variable:

Given a chain(function) h of 2 functions of 1 variable:

$$h(x) = f(g(x)).$$

1. question:
   Give a expression for $h'(x)$ in terms $f(x)$and $g(x)$ and their derivatives.
2. answer:
   $h'(x) = f'(g(x))g'(x)$,
3. a more convenient notation: $\frac{dh}{dx} = \frac{df}{dg}\frac{dg}{dx}$.
4. question:
   Find with it the derivative of the chain:

$$\sqrt{\sin(\ln(\arctan(\frac{e^x}{x+1})))}$$

chain rule for functions of several variables :

$$\sqrt{\sin(\ln(\arctan(\frac{e^x}{x+1})))}$$

answer:

$$\frac{1}{2\sqrt{\sin\left(\ln\left(\arctan\left(\frac{e^x}{x+1}\right)\right)\right)}}\cdot\cos\left(\ln\left(\arctan\left(\frac{e^x}{x+1}\right)\right)\right)\cdot\frac{1}{\arctan\left(\frac{e^x}{x+1}\right)}\cdot\frac{1}{1+\left(\frac{e^x}{x+1}\right)^2}\cdot\frac{e^x(x+1)-e^x}{(x+1)^2}$$

chain rule for functions of several variables :

Given the following chain:

$$z(t) = f(x(t), y(t)).$$

question:

Find $\frac{dz}{dt}(t)$.

(express your answer in terms of (partial) derivatives of f, x and y.)

a break: Vectors and Lines in $\mathbb{R}^3$, a repetition

Question:

Given the graph of $f(x, y) = x^2 y^3$ and point $P(1, 1, 1)$ on it.
Find the vector equations of the tangent lines l_1 and l_2 to the graph at P in the plane $x = 1$ and in the plane $y = 1$.

Answer: $l_1 = \langle 1, 1, 1\rangle + \lambda\langle 0, 1, 3\rangle$ and $l_2 = \langle 1, 1, 1\rangle + \lambda\langle 1, 0, 2\rangle$

chain rule for functions of several variables :

Answer:

1. We remark that $z(t)$ is a chain of the function f of 2 variables and 2 common functions $x(t)$ and $y(t)$ of 1 variable.
2. To answer the question we pay attention to the "curve" $\langle x(t), y(t), z(x(t), y(t))\rangle$. We look at this curve as a motion of a particle over the graph $z = f(x, y)$ and which is a function of the variable t (time).
3. So in this view we can associate with the vector $\langle x'(t), y'(t), \frac{dz(x(t),y(t))}{dt}\rangle$ the velocity of the particle which is of course tangent to the "motion curve" and so tangent to graph of f and so a member of the tangent plane γ of graph of f at point $P((x(t), y(t), z(x(t), (y(t))$.
4. We can look at this tangent plane as a 2-dimensional subspace which is a span of two independent vectors of the tangent plane.
5. So $\gamma = span(\langle 1, 0, f_x(x(t), y(t))\rangle, \langle 0, 1, f_y(x(t), y(t))\rangle$
6. So $\langle x'(t), y'(t), \frac{dz(x(t),y(t))}{dt}\rangle = a\langle 1, 0, f_x(x(t), y(t))\rangle + b\langle 0, 1, f_y(x(t), y(t))\rangle$, for some numbers a and b
7. Solving this system of equations we get: $a = x'(t)$ and $b = y'(t)$, reading the last line gives the result:

chain rule for functions of several variables :

Theorem

$$\frac{dz(x(t),y(t))}{dt} = f_x(x(t),y(t))x'(t) + f_y(x(t),y(t))y'(t).$$

From this and the definition of partial derivatives in mind we also get:

- $$\frac{\partial z(x(u,v),y(u,v))}{\partial u} = f_x(x(u,v),y(u,v))x_u(u,v)+f_y(x(u,v),y(u,v))y_u(u,v).$$
- $$\frac{\partial z(x(u,v),y(u,v))}{\partial v} = f_x(x(u,v),y(u,v))x_v(u,v)+f_y(x(u,v),y(u,v))y_v(u,v).$$
- and further more are generalizations for more variables easy to make.

Example:

Given the transformation (to polar coordinates):
$x(r,\theta) = r\cos(\theta); y(r,\theta) = r\sin(\theta)$ and a function f two variables x and y, and z defined by $z(r,\theta) = f(x(r,\theta), y(r,\theta))$

- Find z_r.
  Answer:
  $z_r = f_x x_r + f_y y_r$, (in brief notation) so,

$$\frac{\partial z(r,\theta)}{\partial r} = f_x(r\cos(\theta), r\sin(\theta))\cos(\theta) + f_y(r\cos(\theta), r\sin(\theta))\sin(\theta).$$

  (in extended notation)

- Find $z_{r\theta}$.
  Answer:
  In brief notation:

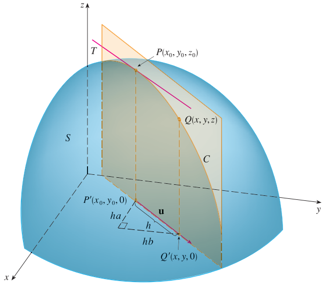
$$z_{r\theta} = (f_x x_r + f_y y_r)_\theta = (f_{xx}x_\theta + f_{xy}y_\theta)x_r + f_x x_{r\theta} + (f_{yx}x_\theta + f_{yy}y_\theta)y_r + f_y y_{r\theta}.$$

  Find for your self the extended version.

Directional Derivative

With partial derivatives we were dealing with tangent lines at a point P in special directions, namely in the x and y direction.

View the next picture:



In the picture you see a tangent line T in a random direction at $P(x_0, y_0, z_0)$, this direction is determined by a unit vector **u** which lies in xy-plane.

Directional Derivative

We want to calculate the slope, which we denote as $D_{\mathbf{u}}f(x_0, y_0)$, of the tangent line T.

Definition

of $D_{\mathbf{u}}f(x_0, y_0)$:

$$\boxed{D_{\mathbf{u}}f(x_0, y_0) = \lim_{h\to 0} \frac{f(x_0+ah, y_0+bh) - f(x_0, y_0)}{h}}$$

hereby is $\mathbf{u} = \langle a, b\rangle$ with $a^2 + b^2 = 1$, *the unit direction*.

Question:
Can we calculate $D_{\mathbf{u}}f(x_0, y_0)$ without the limit?

Directional Derivative

Answer: In 4 steps:

1. Question1:
   Give the vector equation of the tangent line T
   Answer: $T = \langle x_0, y_0, f(x_0, y_0)\rangle + \lambda\langle a, b, D_{\mathbf{u}}f(x_0, y_0)\rangle$
2. Question2:
   Why lie the three space vectors $\mathbf{v} = \langle a, b, D_{\mathbf{u}}f(x_0, y_0)\rangle$, $\mathbf{v}_1 = \langle 1, 0, f_x(x_0, y_0)\rangle$ and $\mathbf{v}_2 = \langle 0, 1, f_y(x_0, y_0)\rangle$ in one plane?
   Answer: They are direction vectors of three tangent lines of the graph of f at point $P(x_0, y_0, f(x_0, y_0))$. So they lie all in the tangent plane
3. Question3:
   Write $\mathbf{v}$ as a (linear) combination of $\mathbf{v}_1$ and $\mathbf{v}_2$.
   Answer: $\mathbf{v} = a\mathbf{v}_1 + b\mathbf{v}_2$
4. Question4:
   Express $D_{\mathbf{u}}f(x_0, y_0)$ in $f_x(x_0, y_0)$ and $f_y(x_0, y_0)$
   Answer: From last answer follows $D_{\mathbf{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0)$
5. So $\boxed{D_{\mathbf{u}}f(x_0, y_0) = af_x(x_0, y_0) + bf_y(x_0, y_0)}$

Directional Derivative: an exercise

Back to our two dimensional distance function of last week:

1. Given $f(x,y) = \sqrt{x^2+y^2}$ and $P(1,1)$. Calculate $D_{\mathbf{u}}f(P)$ for the unit-steps $\mathbf{u} = \langle 1,0\rangle$, $\mathbf{u} = \langle 0,1\rangle$ and $\mathbf{u} = \langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle$

   Answer: $D_{\langle 1,0\rangle} f(P) = \frac{1}{\sqrt{2}}$, $D_{\langle 0,1\rangle} f(P) = \frac{1}{\sqrt{2}}$, $D_{\langle \frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\rangle} f(P) = 1$

2. Calculate $D_{\mathbf{u}}f(P)$ for the general unit step $\mathbf{u} = \langle \cos(\theta), \sin(\theta)\rangle$

   Answer: $D_{\langle \cos\theta, \sin\theta\rangle} f(P) = \frac{1}{\sqrt{2}}(\cos\theta + \sin\theta)$

3. For which θ in the last problem is $D_{\mathbf{u}}f(P)$ a maximum?

   Answer: $\theta = \frac{\pi}{4}$

4. Is the last answer in accordance with your intuition? Why?

Dot product:a Repetition

Given two vectors $\mathbf{a} = \langle a_1, a_2 \rangle$ and $\mathbf{b} = \langle b_1, b_2 \rangle$
Do you remember:

$$\mathbf{a} \cdot \mathbf{b} = a_1 b_1 + a_2 b_2$$

and

$$\mathbf{a} \cdot \mathbf{b} = |\mathbf{a}||\mathbf{b}| \cos(\theta)$$

whereby θ is the angle between the given vectors.
What has this to do with:

$$D_{\mathbf{u}} f(x_0, y_0) = a f_x(x_0, y_0) + b f_y(x_0, y_0)?$$

a convenient formula for $D_{\mathbf{u}}f(x_0, y_0)$

$$D_{\mathbf{u}}f(x_0, y_0) = \langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle \cdot \langle a, b\rangle.$$

and so

$$D_{\mathbf{u}}f(x_0, y_0) = |\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle||\langle a, b\rangle| \cos(\theta).$$

In the last formula is θ the angle between vectors $\mathbf{u} = \langle a, b\rangle$ and $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$
Because $\mathbf{u} = \langle a, b\rangle = 1$ (unit vector) we get

$$D_{\mathbf{u}}f(x_0, y_0) = |\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle| \cos(\theta).$$

We denote $\langle f_x(x_0, y_0), f_y(x_0, y_0)\rangle$ as $\nabla f(x_0, y_0)$ and it is called the gradient of f at point (x_0, y_0).

An important interpretation of the gradient.

Look at

$$D_{\mathbf{u}} f(x_0, y_0) = |\nabla f(x_0, y_0)| \cos(\theta)$$

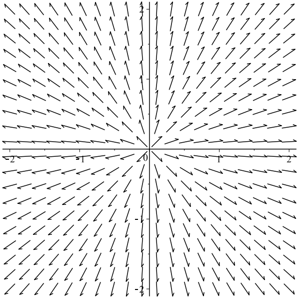
Again the question: In which direction $\mathbf{u}$ is $D_{\mathbf{u}} f(x_0, y_0)$ a maximum?
answer: "the gradient direction"

Some other questions:

1. In what direction do you have to take your (unit) step for minimizing $D_{\mathbf{u}} f(x_0, y_0)$?
2. and in what direction do you have to take your (unit) step(s) such that value $f(x_0, y_0)$ is "unaffected"?
3. Make a walk in the xy-plane which minimize in every step the value of $\sqrt{x^2 + y^2}$!

A gradient-plot.

In next picture you see a with Maple constructed gradient plot for the function $f(x,y) = \sqrt{x^2+y^2}$:



If you follow the "opposite gradient vectors" starting at a random point you "always" end at $(0,0)$ which belongs to the minimum value of f.

Generalizations for directional derivatives:

- In $\mathbf{R}^n$ (the n input variables are $x_1, x_2 ... x_n$) the directional derivative is calculated as:

$$D_{\mathbf{u}} f(\mathbf{p}) = \nabla f(\mathbf{p}) \cdot \mathbf{u}$$

- with $\mathbf{u} = \langle u_1, \cdots u_n \rangle$ a (step-)vector in $\mathbb{R}^n$,
- with $\mathbf{p} = \langle p_1, \cdots p_n \rangle$ a (position-)vector in $\mathbb{R}^n$,
- with $\nabla f(\mathbf{p}) = \langle f_{x_1}(\mathbf{p}), f_{x_2}(\mathbf{p}), \cdots f_{x_n}(\mathbf{p}) \rangle$,
- with dot product $\mathbf{a} \cdot \mathbf{b}$ defined as $a_1 b_1 + a_2 b_2 + \cdots + a_n b_n$,
- Interpretation for gradient $\nabla f(\mathbf{p})$ is:

  If we choose $\mathbf{u}$ equal to $\frac{\nabla f(\mathbf{p})}{|\nabla f(\mathbf{p})|}$ then for given $\mathbf{p}$, $D_{\mathbf{u}} f(\mathbf{p})$ is approximately the maximal "increment" of the function value if unit steps are taken in several directions from $\mathbf{p}$.
- Remark: The the gradient vector lives in the "input space" (domain).

An Example.

Question

Given a rectangular box with sizes 8[m] for the length l, 4[m] for width w and 2[m] for height h.
We change l, w and h such that $\triangle l$, $\triangle w$, $\triangle h$ fulfill the condition:

$$\sqrt{(\triangle l)^2 + (\triangle w)^2 + (\triangle h)^2} = 1.$$

Estimate which $\langle \triangle l, \triangle w, \triangle h \rangle$ -*step direction* gives largest volume increment of the box?

An Example.

Answer:

- We consider the volume V as function:$V(l, w, h) = l \cdot w \cdot h$.
- We are "sitting" in $P(8, 4, 2)$.
- $\nabla V = \langle wh, lh, lw \rangle$ so in P this equals to $\langle 8, 16, 32 \rangle$
- So "the unit step increment" of the volume of the box is maximal for "step" $\mathbf{u} = \langle \frac{1}{\sqrt{21}}, \frac{2}{\sqrt{21}}, \frac{4\rangle}{\sqrt{21}} \rangle$
  A good estimate is $\triangle l = \frac{1}{\sqrt{21}}, \triangle w = \frac{2}{\sqrt{21}}, \triangle l = \frac{4}{\sqrt{21}}$.