Our questions of this week:

- The area of a sphere is $4\pi R^2$, why?
- What is the center of mass of a hemisphere with uniform density and radius R?
- What is the (radiation) flux through a sphere from the field coming from a point charge placed in the origin?

For answering the questions we needed the definition of a parametric surface:

Definition (parametric surface)

A parametric surface $\mathcal{S}$ is a map from a domain D of $\mathbb{R}^2$ to $\mathbb{R}^3$ given by

$$\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$$

wherein $x(u,v), y(u,v)$ and $z(u,v)$ are proper functions.

example:
Give a parametrization for the hemisphere $x^2 + y^2 + z^2 = 4$ with $z \geq 0$.

solution:
Inspired by spherical coordinates we get:
$\mathbf{r}(\theta, \phi) = \langle 2\sin(\phi)\cos(\theta), 2\sin(\phi)\sin(\theta), 2\cos(\phi)\rangle$, with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$.

parametric surfaces and their integrals

Our first question: The area of a sphere is $4\pi R^2$, why?

- Given in domain D the parametrization of $\mathcal{S}$: $\mathbf{r}(u,v) = x(u,v)\mathbf{i} + y(u,v)\mathbf{j} + z(u,v)\mathbf{k}$.
- What is the meaning of $\mathbf{r}_u(u,v)$? and of $\mathbf{r}_v(u,v)$?
- And what is this: $|\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)|dudv$?
- And so $\int\int_D |\mathbf{r}_u(u,v) \times \mathbf{r}_v(u,v)|dudv$ .....the area of $\mathcal{S}$.
- So for our problem we finally get

$$\int_0^{2\pi} \left[\int_0^{\pi} R^2 \sin\phi \, d\phi \right] d\theta.$$

- And so the area of a sphere is $4\pi R^2$.

Our second question: What is the center of mass of a hemisphere with uniform density and radius R?

For answering this we need a new definition:

Definition (surface integrals for scalar functions)

Given a proper scalar function $f(x, y, z)$ and a surface $\mathcal{S}$ then (par definition)

$$\int\int_{\mathcal{S}} f dS = \int\int_{R} f(\mathbf{r}(u, v))|\mathbf{r}_u \times \mathbf{r}_v| dudv,$$

wherein $\mathbf{r}(u, v)$ a parametrization is for $\mathcal{S}$ with domain R.

scalar-function integrals over a surface

Our second question: What is the center of mass of a hemisphere with uniform density ρ_0 and radius R?

The answer:

- Because of symmetry the center of mass lies on the z-axis.
- We denote z-coordinate with $\overline{z}$.
- Now: $\overline{z} = \frac{\int\int_{\mathcal{S}} z\rho_0 dS}{\int\int_{\mathcal{S}} \rho_0 dS}$.
- A parametrization for $\mathcal{S}$ is $\mathbf{r}(\theta,\phi) = \langle R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)\rangle$, with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \frac{\pi}{2}$.
- So $dS = |\mathbf{r}_\theta \times \mathbf{r}_\phi| d\theta d\phi = R^2\sin(\phi) d\theta d\phi$.
- The denominator: $\int\int_{\mathcal{S}} \rho_0 dS = 2\pi R^2 \rho_0$.
- The numerator: $\int\int_{\mathcal{S}} z\rho_0 dS = \rho_0 \int_0^{2\pi} \left[\int_0^{\frac{\pi}{2}} R\cos(\phi) R^2 \sin(\phi) d\phi\right] d\theta = \pi R^3 \rho_0$
- So $\overline{z} = \frac{\pi R^3 \rho_0}{2\pi R^2 \rho_0} = \frac{R}{2}$.

vector-function integrals over a surface

Our third question: What is the (radiation) flux through a sphere from the field coming from a point charge placed in the origin?
For answering this we need a new definition:

Definition (surface integrals for vector functions)

Given a proper vector function $\mathbf{f}(x, y, z)$ and a *oriented* surface $\mathcal{S}$ then (par definition)

$$\int\int_{\mathcal{S}} \mathbf{f} \cdot \mathbf{dS} = \int\int_{\mathcal{S}} \mathbf{f} \cdot \mathbf{n} dS = \pm \int\int_{R} \mathbf{f}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) du dv$$

wherein $\mathbf{r}(u, v)$ a parametrization is for $\mathcal{S}$ with domain R.

Remarks:

- *Oriented* surface $\mathcal{S}$: With $\mathcal{S}$ you choose a proper vector field of unit normals $\mathbf{n}$ on your surface . Mostly (not in general) you can do this in two ways. (why?)
- The $\pm$-sign: Given de parametrization $\mathbf{r}(u, v)$, then $\mathbf{n}$ equals to $\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$ or $-\frac{\mathbf{r}_u \times \mathbf{r}_v}{|\mathbf{r}_u \times \mathbf{r}_v|}$.
- Why $\int\int_R \mathbf{f}(\mathbf{r}(u, v)) \cdot (\mathbf{r}_u \times \mathbf{r}_v) dA$:
  Remember that dS equals to $|\mathbf{r}_u \times \mathbf{r}_v)| du dv$, So $\mathbf{f} \cdot \mathbf{n} dS = \cdots$.

vector-function integrals over a surface

Our third question: What is the (radiation) (outward) flux through a sphere with radius R from the field coming from a point charge q placed in the origin?
Answer:

- A parametrization of the sphere: $\mathbf{r}(\theta, \phi) = \langle R\sin(\phi)\cos(\theta), R\sin(\phi)\sin(\theta), R\cos(\phi)\rangle$, with $0 \leq \theta \leq 2\pi$ and $0 \leq \phi \leq \pi$.
- $\mathbf{r}_u(u, v) \times \mathbf{r}_v(u, v) = \langle R^2\sin^2(\phi)\cos(\theta), R^2\sin^2(\phi)\sin(\theta), R^2\sin(\phi)\cos(\phi)\rangle$.
- The field $\mathbf{E}$ for a point charge q is in general given by

$$\frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\mathbf{r}$$

.

- In our situation we get: $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}\langle\sin(\phi)\cos(\theta), \sin(\phi)\sin(\theta), \cos(\phi)\rangle$.
- So the flux $\int\int_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}dS = \frac{q}{\epsilon_0}$.

The calculation in a more convenient way:

1. Again $\mathbf{E} = \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\mathbf{r}$.
2. The outward normal: $\mathbf{n} = \frac{\mathbf{r}}{|\mathbf{r}|} = \frac{\mathbf{r}}{r}$.
3. So $\int\int_{\mathcal{S}} \mathbf{E}\cdot\mathbf{n}dS = \int\int_{\mathcal{S}} \frac{1}{4\pi\epsilon_0}\frac{q}{r^3}\mathbf{r}\cdot\frac{\mathbf{r}}{r}dS = \int\int_{\mathcal{S}} \frac{1}{4\pi\epsilon_0}\frac{q}{r^2}dS = \int\int_{\mathcal{S}} \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}dS = \frac{1}{4\pi\epsilon_0}\frac{q}{R^2}4\pi R^2 = \frac{q}{\epsilon_0}$.

Gauß and Stokes'

There exist two Green like theorems in space: The first one:

Theorem (Gauß or Divergence theorem)

$$\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}dS = \int\int\int_{R} \nabla\cdot\mathbf{F}dV.$$

wherein **F** *a proper vector field, R a region in* $\mathbb{R}^3$, $\mathcal{S}$ *the (closed) boundary of R and* **n** *the outward unit normal on* $\mathcal{S}$.

Question:
Consider the vector field $\mathbf{F}(x, y, z) = \langle x, y, z\rangle$ and some closed surface $\mathcal{S}$.
Find the outward flux of **F** through $\mathcal{S}$.
Answer:

- Observe that $div\mathbf{F} = 3$.
- So the outward flux is $\int\int_{\mathcal{S}} \mathbf{F}\cdot\mathbf{n}dS = 3\int\int\int_R dV$ =*three times volume region enclosed by* $\mathcal{S}$

Question:
Why holds Gauß?

Answer: Ask me!

The second one:

Theorem (Stokes')

$$\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{ds} = \int \int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS.$$

wherein **F** *a proper vector field, $\mathcal{S}$ some surface with boundary $\mathcal{C}$, the orientations for $\mathcal{C}$ and $\mathcal{S}$ corresponds i.e. they full fill the "right hand rule".*

Question:
Consider again the vector field $\mathbf{F}(x, y, z) = \langle x, y, z \rangle$ and some closed curve $\mathcal{C}$.
Find $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{ds}$.
Answer:

- Observe that $rot\mathbf{F} = \mathbf{0}$.
- So $\int_{\mathcal{C}} \mathbf{F} \cdot \mathbf{ds} = \int \int_{\mathcal{S}} (\nabla \times \mathbf{F}) \cdot \mathbf{n} dS = 0$. wherein $\mathcal{S}$ is any surface with $\mathcal{C}$ as boundary

Why holds Stokes'?

Answer: Ask me!