Definition (conservative vector field)

A vector field is conservative if there is a scalar function with $\nabla f = \mathbf{F}$

- Given $f(x, y, z) = -mgz$, then $\mathbf{F}(x, y, z) = \langle 0, 0, -mg\rangle$.
- Given $f(x, y, z) = \frac{1}{\sqrt{x^2+y^2+z^2}}$ (gravity law), then $\mathbf{F}(x, y, z) = \frac{-1}{\left(\sqrt{(x^2+y^2+z^2)}\right)^3}\langle x, y, z\rangle$.
- Probably you recognize in this something of a potential function!
- Is a force of 1 variable always conservative?
- The reverse problem: Is force $\mathbf{F}(x, y, z) = \langle yz, xz, xy\rangle$ conservative?
- And is force $\mathbf{F}(x, y) = \langle yx, y^2\rangle$ conservative?

path independence

Theorem (path independence)

Given a conservative vector field $\mathbf{F} = \nabla f$ *in space and given some curve* $\mathcal{C}$ *given by* $\mathbf{r}(t)$ *for* $t \in [a, b]$

then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = f(\mathbf{r}(b)) - f(\mathbf{r}(a))$.

- So if a path is closed then $\int_{\mathcal{C}} \mathbf{F} \cdot d\mathbf{s} = 0$ for every conservative field.
- Why does a swing work? (also in the outer space)
- Examine the force $\mathbf{F}(x, y) = \langle -2x, -2y \rangle$. (a two dimensional spring)

The of conservation of mechanical energy and work:

What is the relation between kinetic energy, work and conservation of "energy" a "start":

Suppose a particle with a mass m is moving under influence of a force $\mathbf{F}$ along a curve $\mathcal{C}$ from point A to B.

We denote force , position, velocity, acceleration and work as $\mathbf{F}$, $\mathbf{r}$, $\mathbf{v}$, $\mathbf{a}$ and W.

- Why is the kinetic energy E_k for a particle defined as $E_k = \frac{1}{2}m\mathbf{v}\cdot\mathbf{v} = \frac{1}{2}mv^2$?
- Examine: Work done by a force $\mathbf{F}$ is $W = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{dr} = \int_{\mathcal{C}} m\mathbf{a}\cdot\mathbf{dr} =$
- $\int_{\mathcal{C}} m\frac{d\mathbf{v}}{dt}\cdot\mathbf{v}dt = \int_{\mathcal{C}} m\mathbf{v}\cdot d\mathbf{v} =$
- $\frac{1}{2}mv^2(B) - \frac{1}{2}mv^2(A)$.
- So work W done by a force is equal to the difference of the kinetic energy in the initial - and end point.
- We have heard of the law of conservation of potential - and kinetic energy. Where is it coming from?
- The final answer: Suppose that our force $\mathbf{F}$ is a conservative one. Then $W = \int_{\mathcal{C}} \mathbf{F}\cdot\mathbf{dr} =$
- $\nabla\phi(B) - \nabla\phi(A)$, with $\mathbf{F} = \nabla\phi$.
- So we get: $\frac{1}{2}mv^2(B) - \frac{1}{2}mv^2(A) = \phi(B) - \phi(A)$, which we rewrite as
- $\frac{1}{2}mv^2(B) - \phi(B) = \frac{1}{2}mv^2(A) - \phi(A)$.
- We define potential energy $V(x,y,z)$ as $-\phi(x,y,z)$.
- Finally we get our familiar law: $E_k + V$ is conserved.

Given a vector field $\mathbf{F}(x,y) = \langle P(x,y), Q(x,y)\rangle$, what are the conditions for P and Q so that $\mathbf{F}$ is a conservative field?

Theorem (criterium for a conservative field)

Let $\mathbf{F}(x,y) = P(x,y)\mathbf{i} + Q(x,y)\mathbf{j}$ *with P and Q proper functions of 2 variables.*
Let D a simply connected domain for the given functions, then:
$\mathbf{F}$ *is a conservative field only and only then if* $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$ *on the domain D.*

Some remarks

- Roughly said: "Simple connected Region" means "one piece no holes".
- The proof for one "direction" is simple (which one?) for the other we need "theorem of Green".
- Our former example: Is force $\mathbf{F}(x,y) = \langle yx, y^2\rangle$ conservative?

An important exercise 16.3 Ex.15

- Why the restriction simple connected?
- Given the field

$$\mathbf{F}(x, y) = \frac{-y\mathbf{i} + x\mathbf{j}}{x^2 + y^2}$$

- Show $\frac{\partial P}{\partial y} = \frac{\partial Q}{\partial x}$.
- Still $\mathbf{F}(x, y)$ is not conservative, why?
- Is there a contradiction?

Theorem of Green

Theorem (Green)

Given a region R with a boundary C which is positively orientated. Then

$$\int_{\mathcal{C}} Pdx + Qdy = \int\int_G \left(\frac{\partial Q}{\partial x} - \frac{\partial P}{\partial y} \right) dA$$

Remarks:

- Concerning ..."region R with a boundary C which is positively orientated...":
  For instance take for R the domain between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.
  The boundary consists of the 2 circles, the "outside" one is oriented anticlockwise and the inside one clockwise.
  "If you are walking along a curve you will find the domain (the interior) on your left side".
- Back to exercise 16.3:
  Explore the field $\mathbf{F}(x, y) = \frac{-y\mathbf{i}+x\mathbf{j}}{x^2+y^2}$:
  Again we choose the domain as the region between the circles $x^2 + y^2 = 1$ and $x^2 + y^2 = 9$.
  What do we know about the line integrals $\int_{C_1} \mathbf{F}{\cdot}d\mathbf{r}$ and $\int_{C_2} \mathbf{F}{\cdot}d\mathbf{r}$ with C_1 the circle $x^2 + y^2 = 1$,and C_2 the circle $x^2 + y^2 = 9$and for $\int_{\mathcal{C}} \mathbf{F}{\cdot}d\mathbf{r}$ with $\mathcal{C}$ any square around the origin?
- The proof: *"counting" total work in 2 ways*.
- Why is our "criterium for a conservative field" true?

∇f, $\nabla \times \mathbf{f}$, $\nabla \cdot \mathbf{f}$

Some definitions of important differential operators:

- Gradient of a scalar-function f: $\nabla f = \langle f_x, f_y, f_z \rangle$.
- Divergence of a vector-function $\mathbf{f} = \langle P, Q, R \rangle$: $\nabla \cdot \mathbf{f} = P_x + Q_y + R_z$
- Rotation of a vector-function $\mathbf{f} = \langle P, Q, R \rangle$:$\nabla \times \mathbf{f} = \langle R_y - Q_z, P_z - R_x, Q_x - P_y \rangle$

Remarks:

- We know interpretations for the gradient.
- For interpretation of the Rotation: Calculate $\nabla \times \mathbf{f}$ for $\mathbf{f} = \langle P(x, y), Q(x, y), 0 \rangle$, then use Green to come to the equation $\int_{\mathcal{C}} \mathbf{f} \cdot d\mathbf{s} = \int \int_G (\nabla \times \mathbf{f}) \cdot \mathbf{k} dA$ with $\mathcal{C}$ a closed curve (anti-clockwise) which is circumference of domain G. So the interpretation for rotation is....
- For interpretation of the Divergence: Again for a closed curve around a domain G, calculate $\int_{\mathcal{C}} \mathbf{f} \cdot \mathbf{n} ds$ wherein $\mathbf{n}$ unit normal vector is of the curve $\mathcal{C}$ which is outward chosen, we get $\int_{\mathcal{C}} Pdy - Qdx$, use Green to see that integral equals to $\int \int_G \left(P_x + Q_y \right) dA$.pause So the interpretation for divergence is....