Some Questions:

Our questions for this lecture:

- What is the value, in a decimal representation, for $\sin(1)$, $\ln(2)$?
- Find $\lim_{x \to 0} \frac{e^x - 1}{x}$
- How can we find an approximation for $\int_0^1 e^{-x^2}\,dx$?

Taylor series:

Definition (Taylor series)

Given a function f for which all derivatives $f^{(n)}(a)$ exist in point a. We define

$$T(x) = \sum_{k=0}^{\infty} \frac{f^{(k)(a)}}{n!}(x-a)^k.$$

T is called the Taylor series of f around point $a \in \mathbb{R}$.

Remarks:

- $f^{(k)}(a)$ denotes the k^{th} derivative of f in a.
- Given the function f we can find a power series which probably converges to the function on some interval.
- $T_n(x) = \sum_{k=0}^{n} \frac{f^{(k)(a)}}{n!}(x-a)^k$ is the called the Taylor polynomial of the index n.
- The following holds: $f^{(k)}(a) = T_n^{(k)}(a)$.

Taylor series examples:

Find the Taylor series for e^x around 0.

- $f^{(k)}(0) = 1$ for all $k = 0, 1, 2...$
- So $T(x) = \sum_{k=0}^{\infty} \frac{1}{k!} x^k$. We have found this result earlier.

Find the Taylor series for $\sin(x)$ around 0.

- $f^{(2p+1)}(0) = 1$ and $f^{(2p)}(0) = 0$ for all $p = 0, 1, 2...$
- So $T(x) = \sum_{k=0}^{\infty} (-1)^k \frac{1}{(2k+1)!)} x^{(2k+1)}$ which is convergent on $\mathbb{R}$.
- We connect this series with $\sin(x)$.
- The answer of the last starting question "what is sin(1)?"
  answer: $\sin(1) = 1 - \frac{1}{3!} + \frac{1}{5!} + ...$

  The same calculations holds for $\ln 2$

Definition (Powerserie)

An expression like $\sum_{n=0}^{\infty} c_n(x-a)^n$ with c_n and a real numbers is called a power series around a.

Remarks:

- So Taylorseries are special powerseries.
- We know a familiar power series which can identify with a formula : $1+x+x^2+x^3+\ldots=\frac{1}{1-x}$ on $(-1,1)$. The geometric series.
- In the definition you find the variable x which is in $\mathbb{R}$. Our question is for what values of x is our power series convergent? The power series is at least convergent for $x=a$, the geometric is convergent for $x\in(-1,1)$.
- In general is a power series convergent in a point, on intervals like $(-R,R)$, $[-R,R)$, $(-R,R]$, $[-R,R]$ or $\mathbb{R}$. (During this course we don't explain how to find this these domains of convergence.) On these sets we can associate with the power series a function on these sets.
- Suppose $f(x)=\sum_{n=0}^{\infty} c_n(x-a)^n$then
  $f'(x)=\sum_{n=1}^{\infty} c_n n(x-a)^{n-1}$ on the *interior* of the domain of f.
  $\int f(x)\,dx=\sum_{n=0}^{\infty} c_n\frac{1}{n+1}(x-a)^{n+1}+C$ on the *interior* of the domain of f

Theorem

Under some conditions the following holds:

$$f(x) = T_n(x) + R_n(x)$$

with $R_n(x) \leq \frac{M}{n+1)!}(x-a)^{(n+1)}$ *and* M *the maximum value of* $|f^{(n+1)}(c)|$ *with* c *between* x *and* a.