Our goal this morning is to understand:

The strength of an electrical field above the center of a uniform charged disk is given by the integral:

$$1/4\,\frac{1}{\pi\,\epsilon}\int_0^{2\,\pi}\int_0^R \frac{h\,\sigma\,r}{\left(h^2+r^2\right)^{3/2}}\,\mathrm{d}r\,\mathrm{d}\theta$$

wherein:

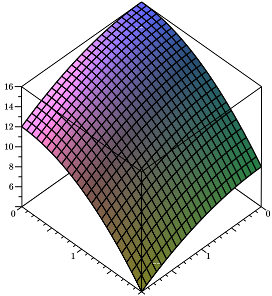
- σ the charge density $[\frac{Q}{m^2}]$.
- h, R are the height above - and radius of the disk.

an integral problem:

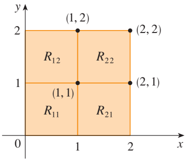
Given the function f of several variables with domain D $0 \leq x \leq 2$ and $0 \leq y \leq 2$ defined by:

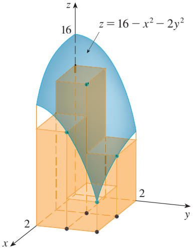
$$f(x,y) = 16 - x^2 - 2y^2$$

We want to calculate the volume between D and the graph of f which see in the next picture:



a integral problem:



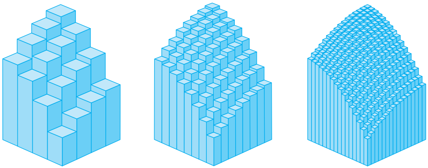


What is happening here?

a integral problem:

The last pictures suggest that you can make an approximation of the answer by adding the volumes of the blocks (calculations of these volumes have you learned on your grammar school).

You can get better approximations by the following refinements made in pictures:



(a) $m = n = 4$, $V \approx 41.5$ (b) $m = n = 8$, $V \approx 44.875$ (c) $m = n = 16$, $V \approx 46.46875$

We call these approximations Riemann Sums.

Riemann Sum:

We give a more formal definition of a Riemann sum:

Definition

- Given a function $f : G \to \mathbb{R}$.
- For $i = 1...n$, let $\triangle G_i$ the size (area) of G_i, which is a subregion of G, i.e. a member of a partition $\mathcal{P}$ so that $\bigcup_{G_i \in \mathcal{P}} G_i = G$.We choose in every G_i a point (a_i, b_i). We write for this choice, $\mathcal{S}$.
- The Riemann sum, denoted with $\mathcal{R}(f, G, \mathcal{P}, \mathcal{S})$, is summation $\Sigma_i f(a_i, b_i) \triangle G_i$.

- We calculate in some sense the limit of $\mathcal{R}(f, G, \mathcal{P}, \mathcal{S})$ for $\triangle G_i \downarrow 0$.
- Remark: this limit has to be independent of the choices for $\mathcal{P}$ and $\mathcal{S}$.
- This "limit number" is denoted with $\int\int_G f(x, y) dA$.

An example:

Question 1:

Determine $\int\int_G (1 - \sqrt{x^2 + y^2}) dA$ for
$G = \{(x, y) | 0 \leq x^2 + y^2 \leq 1\}$.

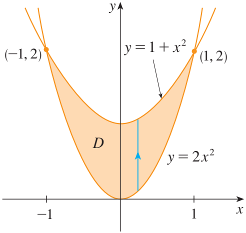
Answer 1:

The graph of f is a cone , the surface (base) is a circular disc with surface area π, the height is 1, so its volume is $\frac{1}{3}\pi$ and so $\int\int_G (1 - \sqrt{x^2 + y^2}) dA = \frac{1}{3}\pi$

Another example:

Question 2:

Can you answer(?):
Determine $\int\int_D (x + 2y)dA$ for D given by:



Answer2: no

Fubini:

The theorem of Fubini is the key for solving the last question:

Theorem

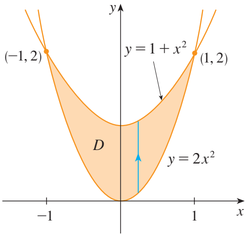
Given a function $f : G \to \mathbb{R}$. We assume that we can describe the domain G as: $a \leq x \leq b$ and $f_1(x) \leq y \leq f_2(x)$ for some numbers a and b and some functions f_1 and f_2. Then:

$$\int \int_G f(x,y)dA = \int_a^b \left[\int_{f_1(x)}^{f_2(x)} f(x,y)dy \right] dx.$$

Remarks:

- We see on the right side an iterated integral, just a repetition of two "common" integrals; the inside one over y, the outside one over x.
- So the evaluation of the right side is nothing more than finding primitives and doing substitutions.
- Back to our last problem.

Finding $\int\int_D (x+2y)dA$:



- From the picture we get:
  D is given by $-1 \leq x \leq 1$ and $2x^2 \leq y \leq 1 + x^2$.
- From Fubini we get $\int\int_D (x+2y)dA = \int_{-1}^{1} \left[\int_{2x^2}^{1+x^2} x + 2y dy\right] dx$.
- This equals to $\int_{-1}^{1} \left[xy + y^2\right]_{2x^2}^{1+x^2} dx$.
- Going on: $\int_{-1}^{1}(x(1+x^2) + (1+x^2)^2 - x2x^2 - (2x^2)^2)dx$.
- And going on: $\int_{-1}^{1}(-3x^4 - x^3 + 2x^2 + x + 1)dx = \frac{32}{15}$.

Fubini:order of integration reversed:

Another not surprising form of Fubini is:

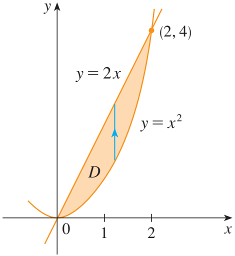
Theorem

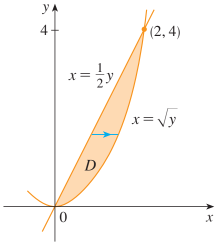
Given a function $f : G \to \mathbb{R}$. We assume that we can describe the domain G as: $c \leq y \leq d$ and $g_1(y) \leq x \leq g_2(y)$ for some numbers c and d and some functions g_1 and g_2.
Then:

$$\int\int_G f(x,y)dA = \int_c^d \left[\int_{g_1(y)}^{g_2(y)} f(x,y)dx\right] dy.$$

Finding $\int\int_D (x^2+y^2)dA$:

Two descriptions of the same region D:





We answer the question by using both Fubini's iterated integrals:

- The first: $\int_0^2 \int_{x^2}^{2x} x^2 + y^2 dy\, dx$.
- And second: $\int_0^4 \int_{1/2y}^{\sqrt{y}} x^2 + y^2 dx\, dy$
- Both calculations yield: $\frac{216}{35}$.

Reversing order: Are you a kwakkwak?

After some examination of the iterated integral

$$\int_0^1 \int_x^1 \sin\left(y^2\right)\, dy\, dx$$

you will see that direct calculation fails.

Reversing order: a little story.

$$\int_0^1 \int_x^1 \sin\left(y^2\right) dy\, dx$$

Teacher:What can we do?

Student:Reverse the order

Teacher:Why?

Student:To find the primitive function of $\sin\left(y^2\right)$ w.r.t. x is easy

Teacher:Marvelous, I think you are a really smart fellow

Student:Yes, I am!

Teacher:OK. show it!

Student: Here it is: $\int_x^1 \int_0^1 \sin\left(y^2\right) dx\, dy$

Teacher:Indeed you are a smart !!

Are you listeners smart kwakkwaks as well ? or not, show it!

Polar coordinates:

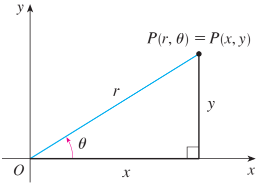
Given the double integral:

$$\int\int_R e^{x^2+y^2} dA$$

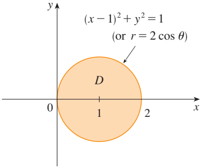
with Region R the circular disc $x^2 + y^2 \leq 4$.

Find its value.
After a closer examination you see that both iterated integrals won't help you, why not?
You will find its value if you use the substitution
$x = r\cos(\theta), y = r\sin(\theta)$.
You try to translate the integral and domain in terms of r and θ and if you succeed, you will find a much simpler integral to calculate.

Polar coordinates:



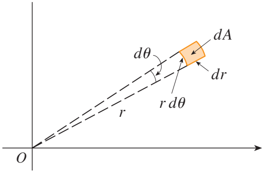
From this picture you immediately deduce the relations:
$x = r\cos(\theta), y = r\sin(\theta)$
and do you understand the result in picture :



?

Polar coordinates:

The solution in steps:

- R in polar coordinates: $0 \leq r \leq 2$, $0 \leq \theta \leq 2\pi$.
- 

  From this picture we deduce $dA = rdrd\theta$
- And so we have full filled our duty:
  The full translation in polar coordinates of our integral in iterated form is:
  $\int_0^{2\pi} \left[\int_0^2 re^{r^2} dr \right] d\theta.$
  and is calculation an easy job to do.