Signal Processing EE2S31

Stochastic Processes for EE

Lecture 4

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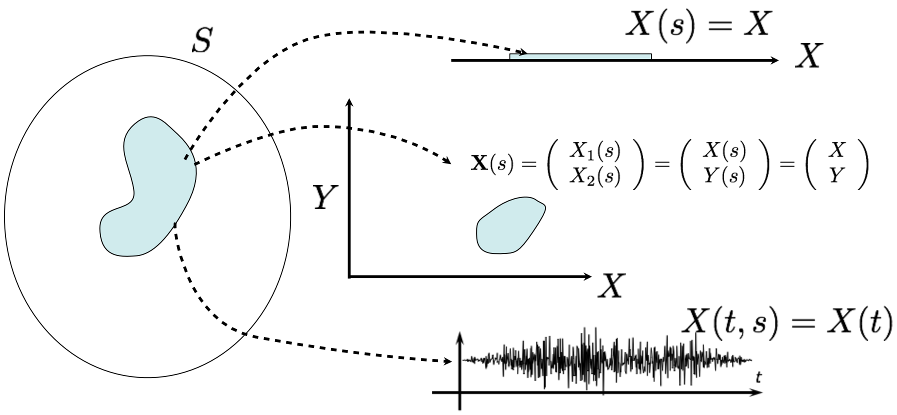
Summarizing



One experiment → one outcome or realization of the stochastic/random process

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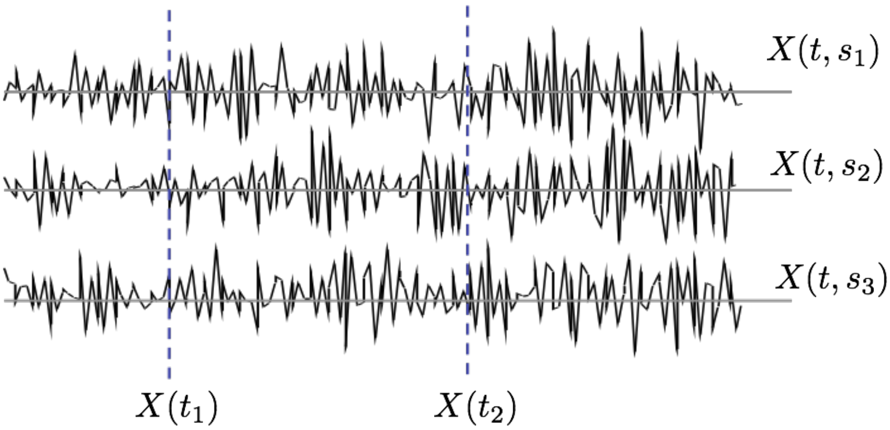
Description random process



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Notice that ...

$$X(t_1)...X(t_k)... \rightarrow f_{X_1,X_2,...,X_k,...}(x_1, x_2, ..., x_k, ...)$$

resembles a vector random variable

- ... but can be of infinite dimensionality
- ... and **ordering** (in time) of $X(t_k)$ is **essential**

With the exception of a few "special cases"

- iid random sequence/process
- Gaussian stochastic process
- Poisson stochastic process

this joint PDF is **very** difficult to get in practice

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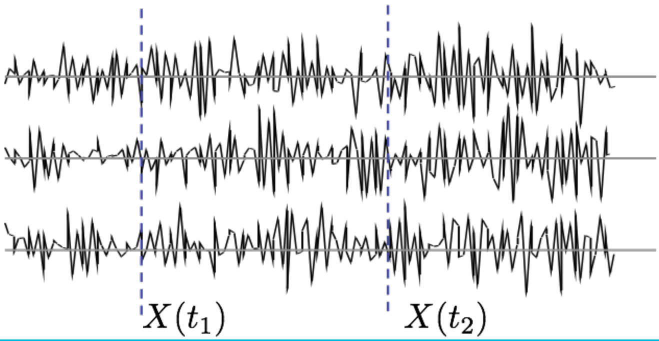
Expected value of X(t)

- Expected value of $X(t_k)$ at time t_k

$$\mu(t_k) = E[X(t_k)] = \int_{-\infty}^{\infty} x f_{X(t_k)}(x) dx$$



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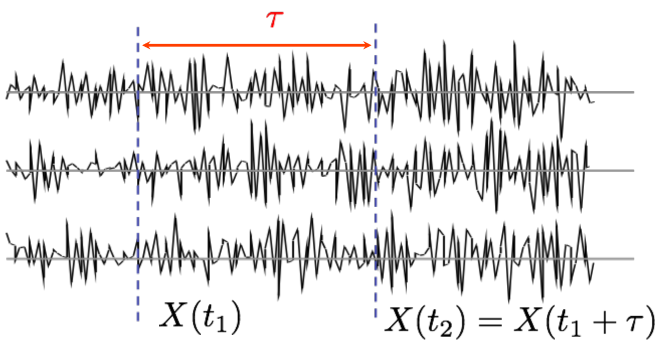
Auto-covariance

- Joint behavior of X(t) at time t_1 and t_2 can be described by the auto-covariance function
  the covariance of $X(t_1)$ and $X(t_2)$ for all t_1 and t_2



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Auto-Covariance function

- The autocovariance function

$$\begin{aligned} C_X(t,\tau) &= Cov[X(t), X(t+\tau)] \\ &= E\left[(X(t)-\mu_X(t))(X(t+\tau)-\mu_X(t+\tau))\right] \\ &= \underbrace{E\left[X(t)X(t+\tau)\right]}_{\text{correlation } R_X(t,\tau)} - E[X(t)]E[X(t+\tau)] \end{aligned}$$

- The autocorrelation

$$R_X(t,\tau) = E[X(t)X(t+\tau)] = \iint xy f_{X(t)X(t+\tau)}(x,y)dxdy$$

- The autocovariance and autocorrelation are functions of time

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Uncorrelated process

- If all pairs $X(t)$, $X(t+\tau)$ are uncorrelated, i.e.

$$C_X(t,\tau) = \begin{cases} var(t) & \text{for all } t \text{ and } \tau = 0 \\ 0 & \text{for all } t \text{ and } \tau \neq 0 \end{cases}$$

then $X(t)$ is called an uncorrelated process

- If all pairs $X(t)$, $X(t+\tau)$ are orthogonal, i.e.

$$R_X(t,\tau) = \begin{cases} E[X^2(t)] & \text{for all } t \text{ and } \tau = 0 \\ 0 & \text{for all } t \text{ and } \tau \neq 0 \end{cases}$$

then $X(t)$ is called an orthogonal process.

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Stationary process

- A stochastic process is stationary if and only if every joint-pdf is shift invariant:

$$f_{X(t_1),X(t_2),..,X(t_k)}(x_1,x_2,..,x_k) =$$
$$f_{X(t_1+\Delta t),X(t_2+\Delta t),..,X(t_k+\Delta t)}(x_1,x_2,..,x_k)$$

- **Consequence I**
  - The marginal pdf's are independent of t:

$$f_{X(t)}(x) = f_{X(t+\Delta t)}(x) = f_X(x)$$

  - The marginal pdf's are identical for all t_k !!!

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Stationary process

- Therefore:
  - Expected value is independent of time:

$$\mu_X(t) = E[X(t)] = \mu_X$$

  - Variance is independent of time:

$$Var_X(t) = Var[X(t)] = Var[X] = \sigma_X^2$$

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Stationary process

- Consequence II
  - The 2D joint-pdf is shift invariant

$$f_{X(t_1),X(t_2)}(x_1,x_2) = f_{X(t_1+\Delta t),X(t_2+\Delta t)}(x_1,x_2)$$
$$= f_{X(0),X(t_2-t_1)}(x_1,x_2)$$

  - ... only the 'distance' between t_2 and t_1 matters

- Therefore

$$R_X(t,\tau) = R_X(\tau)$$
$$C_X(t,\tau) = C_X(\tau) = R_X(\tau) - \mu_X^2$$

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Stationary processes

- Example of stationary processes
  - iid process
  - Bernoulli process
  - Poisson process

- Non-stationary processes are difficult to model and to handle in practice

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Wide-Sense stationary processes

- To show that a process is stationary, we need the overall joint-pdf
  - Pretty impossible to get, except for special cases
- We can often estimate the process'
  - expected value
  - correlation function
- If (only) these functions satisfy the property of stationarity, we call this process wide sense stationary (WSS)
  - Don't know anything about other properties of the process!

'zwak stationair'

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WSS Process

- A process is wide-sense stationary, if and only if

$$\begin{aligned} \mu_X(t) &= \mu && \text{for all } t \\ R_X(t,\tau) &= R_X(\tau) && \text{for all } t \end{aligned}$$

$$\begin{aligned} \mu_X(n) &= \mu && \text{for all } n \\ R_X(n,k) &= R_X(k) && \text{for all } n \end{aligned}$$

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Autocorrelation function for WSS

- We will work a lot with the assumption of WSS
- With random time signals, we often (sometimes even implicitly) assume $E[X(k)] = 0$

- The autocorrelation function is the most important property used in random signal processing

$$R_X(0) \geq 0$$
$$R_X(k) = R_X(-k)$$
$$|R_X(k)| \leq R_X(0)$$
if $\lim_{k\to\infty} R_X(k) = C$ then $C = \mu_X^2$

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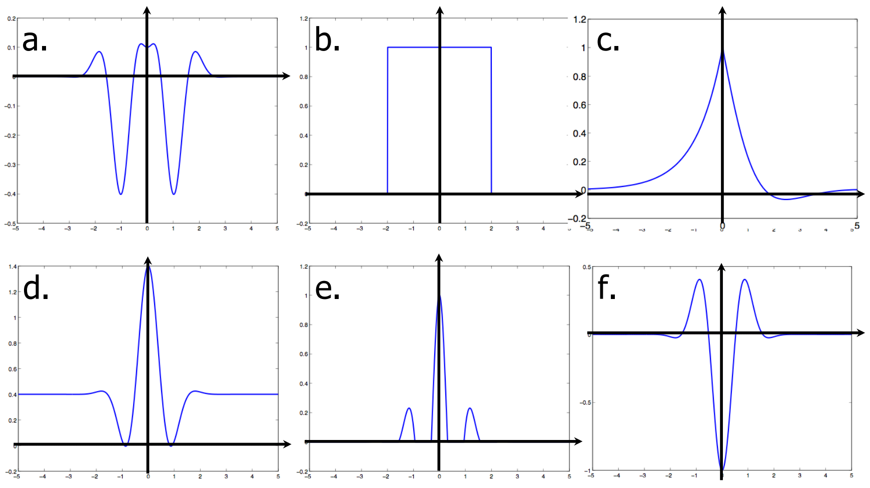
Valid autocorrelation functions?



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Today

- Example: autocorrelation of a random telegraph
- Estimation in real life... ergodicity
- Cross-correlation
- Signal processing of WSS signals

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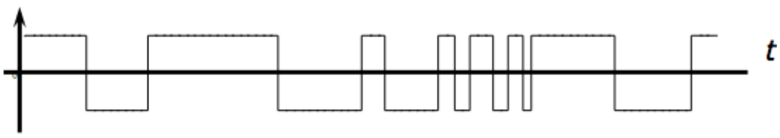
Example: random telegraph

- Send a signal along a telegraph line
- Switching of polarity transmits a bit
- switching is a Poisson process:



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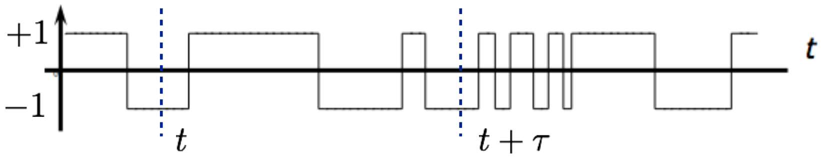
Random telegraph



$$R_X(t,\tau) = \sum_x \sum_y xyP[X(t) = x, X(t+\tau) = y]$$

$$\begin{aligned} R_X(t,\tau) = &(+1)(+1)P[X(t) = 1, X(t+\tau) = 1] \\ &+ (+1)(-1)P[X(t) = 1, X(t+\tau) = -1] \\ &+ (-1)(+1)P[X(t) = -1, X(t+\tau) = +1] \\ &+ (-1)(-1)P[X(t) = -1, X(t+\tau) = -1] \end{aligned}$$

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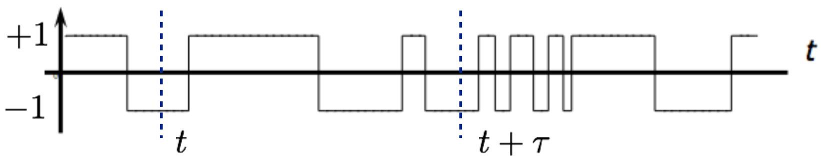
Random telegraph



$$\begin{aligned} R_X(t,\tau) = &(+1)(+1)P[X(t) = 1, X(t+\tau) = 1] \\ &+ (+1)(-1)P[X(t) = 1, X(t+\tau) = -1] \\ &+ (-1)(+1)P[X(t) = -1, X(t+\tau) = +1] \\ &+ (-1)(-1)P[X(t) = -1, X(t+\tau) = -1] \end{aligned}$$

use definition of conditional probability:

$$\begin{aligned} = &+ P[X(t+\tau) = 1 | X(t) = 1]P[X(t) = 1] \\ &- P[X(t+\tau) = -1 | X(t) = 1]P[X(t) = 1] \\ &- P[X(t+\tau) = 1 | X(t) = -1]P[X(t) = -1] \\ &+ P[X(t+\tau) = -1 | X(t) = -1]P[X(t) = -1] \end{aligned}$$

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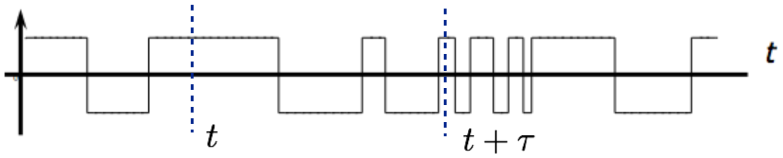
Random telegraph

- Signal changes polarity if an 'arrival' occurs
- Arrivals form a Poisson process, with rate α



$$P[X(t+\tau) = 1 | X(t) = 1] =$$

$$P[\text{even number of arrivals in interval } \tau] =$$

$$P[N(\tau) = \text{even}] = \ldots = \frac{1}{2}\left(1 + e^{-2\alpha\tau}\right)$$

(miracle in step 2)

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Random telegraph

- What is $P[N(\tau) = \text{even}]$?
- We know the Poisson process:

$$P[N(t) = n] = \frac{(\alpha t)^n}{n!}\exp(-\alpha t)$$

so:

$$\begin{aligned} P[N(\tau) = \text{even}] &= P[N(\tau) = 0] + P[N(\tau) = 2] + P[N(\tau) = 4] + \ldots \\ &= \exp(-\alpha\tau) + \frac{(\alpha\tau)^2}{2!}\exp(-\alpha\tau) + \frac{(\alpha\tau)^4}{4!}\exp(-\alpha\tau) \\ &= \left(1 + \frac{(\alpha\tau)^2}{2!} + \frac{(\alpha\tau)^4}{4!} + \ldots\right)\exp(-\alpha\tau) \end{aligned}$$

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Random telegraph

- Now use that: $\cosh(x) = \frac{1}{2}\left(e^{x} + e^{-x}\right) = 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} \ldots$

then

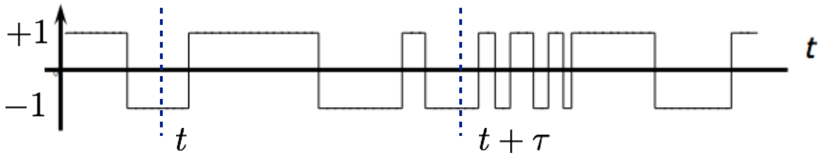
$$\begin{aligned} P[N(t) = \text{even}] &= \left(1 + \frac{(\alpha\tau)^2}{2!} + \frac{(\alpha\tau)^4}{4!} + \frac{(\alpha\tau)^6}{6!} + \ldots\right)\exp(-\alpha\tau) \\ &= \cosh(\alpha\tau)\exp(-\alpha\tau) \\ &= \frac{1}{2}\left(\exp(\alpha\tau) + \exp(-\alpha\tau)\right)\exp(-\alpha\tau) \\ &= \frac{1}{2}(1 + \exp(-2\alpha\tau)) \end{aligned}$$

Random telegraph

- Similarly:

$$\begin{aligned} P[N(\tau) = \text{odd}] &= \left(\frac{\alpha\tau}{1!} + \frac{(\alpha\tau)^3}{3!} + \frac{(\alpha\tau)^5}{5!} + \ldots\right)\exp(-\alpha\tau) \\ &= \sinh(\alpha\tau)\exp(-\alpha\tau) \\ &= \frac{1}{2}\left(e^{\alpha\tau} - e^{-\alpha\tau}\right)\exp(-\alpha\tau) \\ &= \frac{1}{2}(1 - \exp(-2\alpha\tau)) \end{aligned}$$

Random telegraph



- So, we had four terms:

$$\begin{aligned}R_X(t,\tau) = &+P[N(\tau)=\text{even}]P[X(t)=1]\\&-P[N(\tau)=\text{odd}]P[X(t)=1]\\&-P[N(\tau)=\text{odd}]P[X(t)=-1]\\&+P[N(\tau)=\text{even}]P[X(t)=-1]\\=&\exp(-2\alpha|\tau|)\end{aligned}$$

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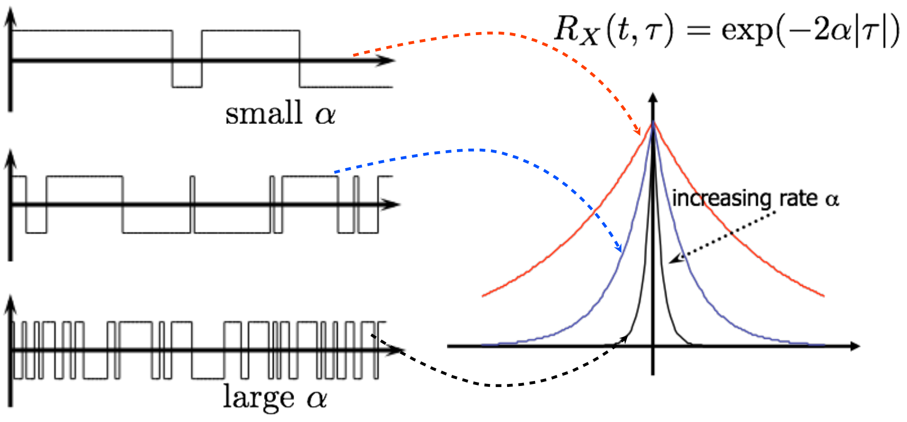
Random telegraph



- Realizations of three different random processes

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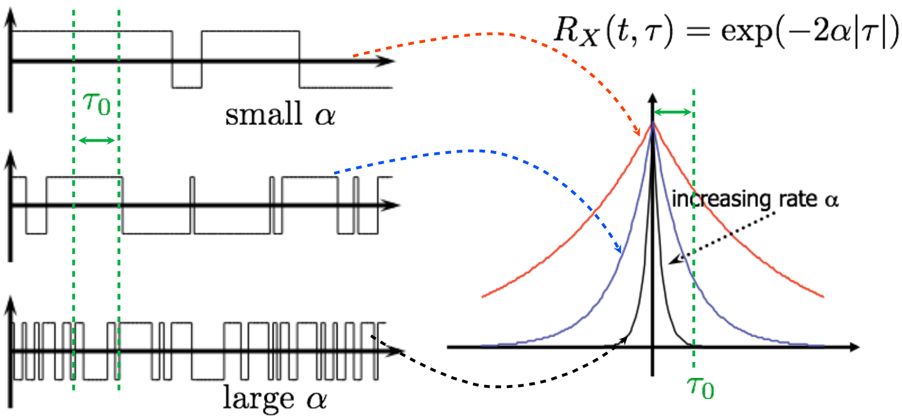
Random telegraph



- Realizations of three different random processes

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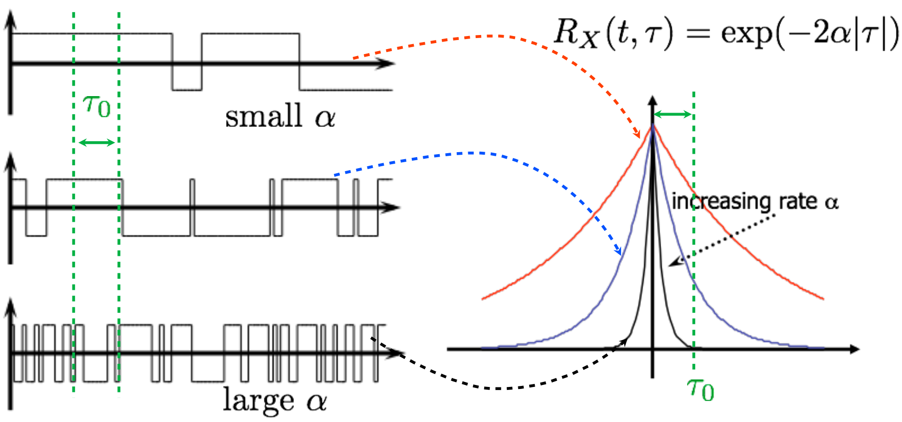
Random telegraph



- Note: autocorrelation function does NOT depend on t!

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Estimated Autocorrelation Function

- Autocorrelation function of a (time discrete) WSS process $X(n)$:

$$R_X(k) = E[X(n)X(n+k)]$$

- How to estimate $R_X(k)$?
  1. Using the j-PDF of $X(n)$ and $X(n+k)$:

$$R_X(k) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} x_1 x_2 f_{X(n),X(n+k)}(x_1, x_2) dx_1 dx_2$$

  2. Using multiple realizations (sample functions)
  3. Using a single realization (for ergodic processes)

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Simpler Case: Estimation of Exp Value

- In practice, properties of random variables are estimated from observations obtained by repeating the experiment
- E.g. Get 100 observations of a random variable X, and estimate the expected value

- Assume $X \sim f_X(x)$
- $E[X] = \mu_X$
- Observe values: 1 4 3 5 1 1 5 6 6 2
- Estimate of μ_X ?

- Sample Mean:

$$M_N = \frac{1}{N}\sum_{i=1}^{N} X_i$$

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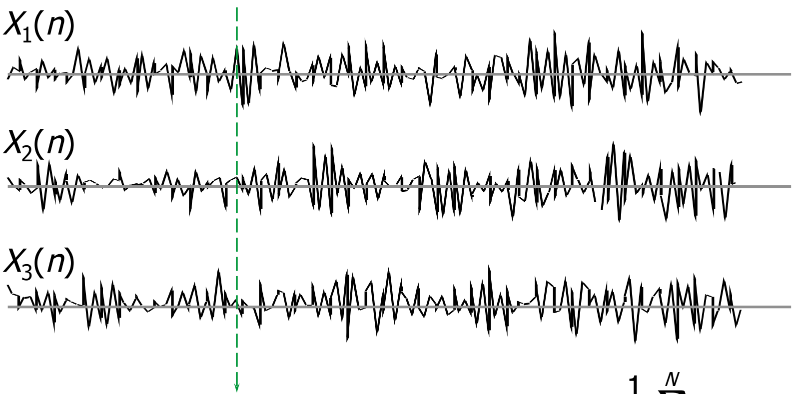
Many Realizations of WSS Process



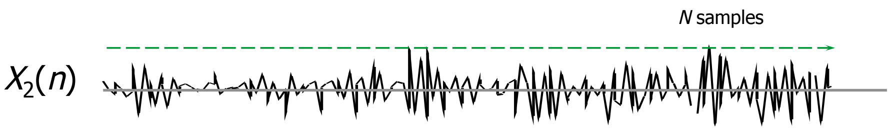
Estimate expected value via sample mean: *Ensemble mean* (*ensemble average*)

$$M_N(n) = \frac{1}{N}\sum_{i=1}^{N} X_i(n)$$

= constant for all n

Ergodic Process

- But we can also average over time ("time average")



$$\overline{M}_N = \frac{1}{N}\sum_{n=1}^{N} X(n)$$

- Only for *ergodic processes* time and ensemble averages are the same
  - In practice, we typically assume so

Notice the Differences

- Average over N different realizations:

$$M_N(n) = \frac{1}{N}\sum_{i=1}^{N} X_i(n)$$

- Average over N time samples of a *single* realization:

$$\overline{M}_N = \frac{1}{N}\sum_{n=1}^{N} X(n)$$

- Identical if $X(n)$ is a (wide sense) ergodic process

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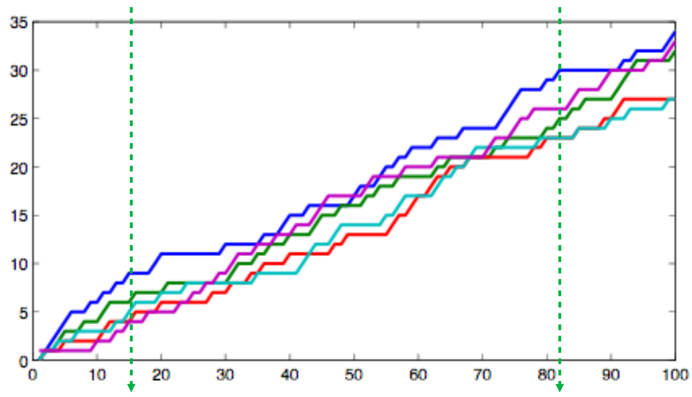
Note that not all processes are WSS!



- Sum/counting process is NOT WSS.

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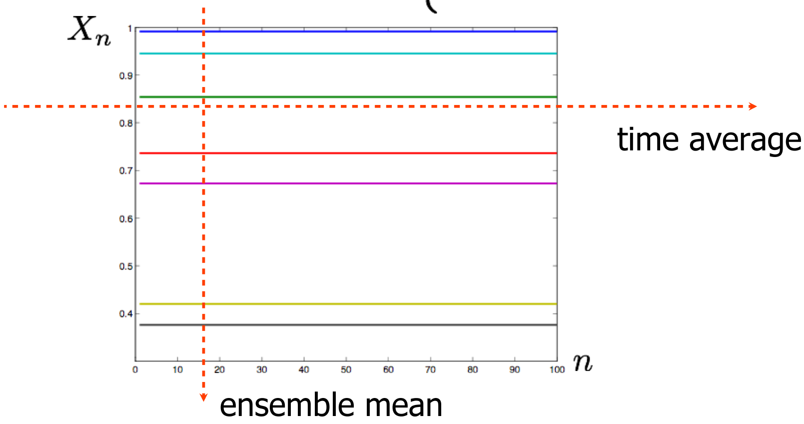
Not all processes are ergodic!

$$X_n = A \qquad f_A(a) = \begin{cases} 1 & 0 \le a \le 1 \\ 0 & \text{otherwise} \end{cases}$$



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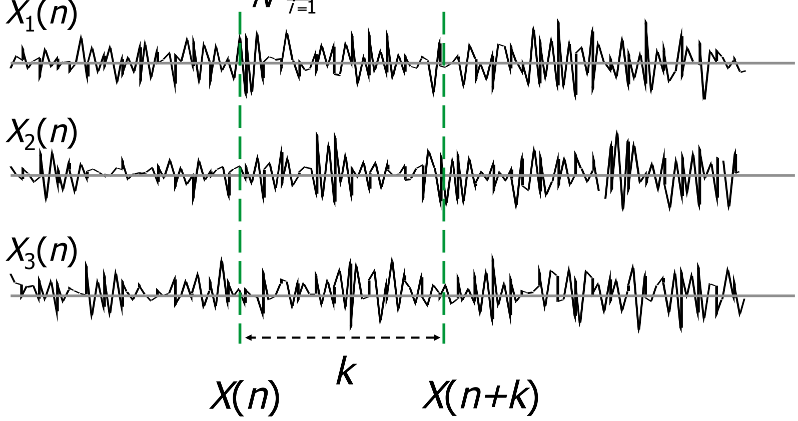
Similar for Autocorrelation

$$R_X(k) = \frac{1}{N}\sum_{i=1}^{N} X_i(n)X_i(n+k)$$



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Estimated Autocorrelation Function (2)

- Instead of autocorrelation function based on ensembles

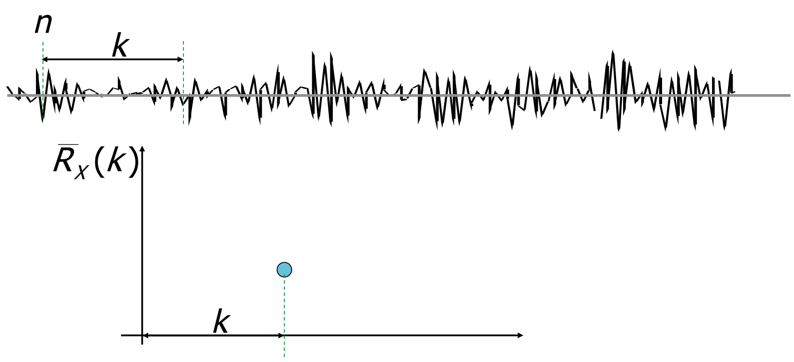
$$R_X(k) = \frac{1}{N}\sum_{i=1}^{N} X_i(n)X_i(n+k)$$

- the *autocorrelation function* is *estimated* on the basis of a single realization

$$\bar{R}_X(k) = \frac{1}{N}\sum_{n=1}^{N} X(n)X(n+k)$$

Estimated Autocorrelation Function (3)

$$\bar{R}_X(k) = \frac{1}{N}\sum_{n=1}^{N} X(n)X(n+k)$$



Estimated Autocorrelation Function (4)

- The basic estimator form

$$\bar{R}_X(k) = \frac{1}{N}\sum_{n=1}^{N} X(n)X(n+k)$$

To estimate N values of $R_X(k)$ i.e., $R_X(0)\ldots R_X(N-1)$ we need 2N-1 data-samples.

Example for $k=0$, $k=1$ and,$k=2$ and $N=3$

$$\begin{aligned} R_X(0) &= \frac{1}{3}\{x(1)^2+x(2)^2+x(3)^2\} \\ R_X(1) &= \frac{1}{3}\{x(1)x(2)+x(2)x(3)+x(3)x(4)\} \\ R_X(2) &= \frac{1}{3}\{x(1)x(3)+x(2)x(4)+x(3)x(5)\} \end{aligned}$$

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Estimated Autocorrelation Function (5)

- Modified estimator form:

$$\hat{R}_X(k) = \frac{1}{N}\sum_{n=1}^{N-k} X(n)X(n+k)$$

$N=3$

$$\begin{aligned} R_X(0) &= \frac{1}{3}\{x(1)^2+x(2)^2+x(3)^2\} \\ R_X(1) &= \frac{1}{3}\{x(1)x(2)+x(2)x(3)\} \\ R_X(2) &= \frac{1}{3}\{x(1)x(3)\} \end{aligned}$$

- This estimator is biased: $E[\hat{R}_X(k)] = \frac{N-k}{N}R_X(k)$
- Usual estimator form:

$$\tilde{R}_X(k) = \frac{1}{N-k}\sum_{n=1}^{N-k} X(n)X(n+k)$$

Advantage: This estimator can be efficiently implementated using FFTs

Disadvantage: Less accurate for large “k”

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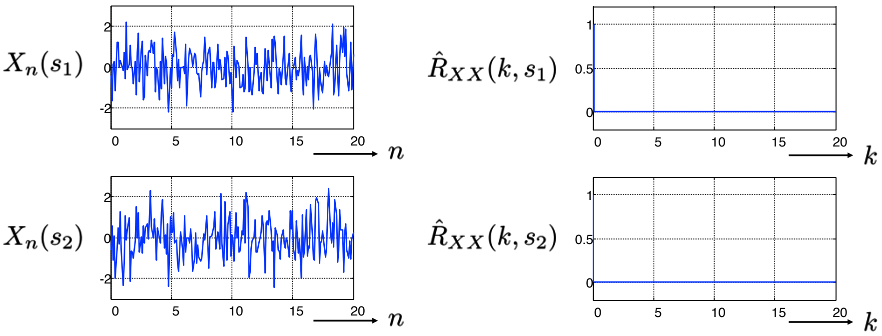
Estimated Autocorrelation Function

Example uncorrelated, zero mean, unit variance noise

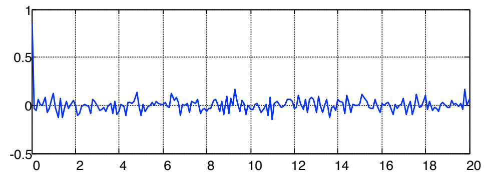
$\{X_n, n \in \mathbb{Z}\}$

$$\Rightarrow \quad R_{XX}(k) = \delta(k)$$

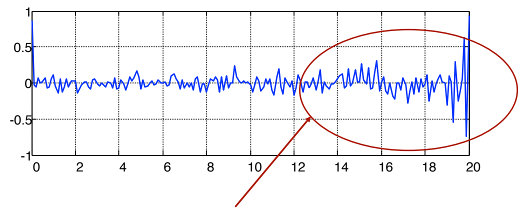


Estimated Autocorrelation Function

- $\hat{R}_{XX}(k,s) = \dfrac{1}{N} \sum_{n=0}^{N-1} X_{n+k}(s) X_n(s), \quad k = 0, \ldots, N-1$



- $\hat{R}_{XX}(k,s) = \dfrac{1}{N-k} \sum_{n=0}^{N-k-1} X_{n+k}(s) X_n(s), \quad k = 0, \ldots, N-1$

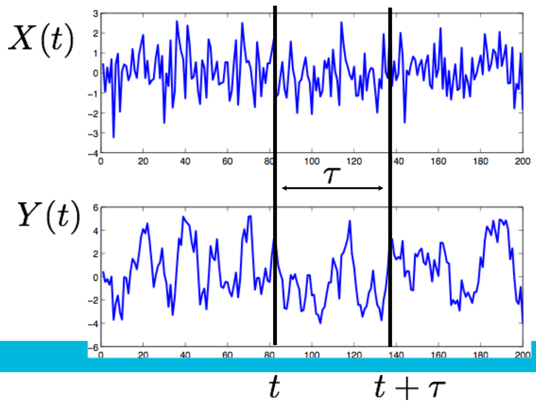


Inaccuracy due to limited number of data samples

Cross-correlation function

- Cross-correlation:

$$R_{XY}(t,\tau) = E[X(t)Y(t+\tau)]$$



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Cross correlation function for Jointly WSS signals

- Two signals are jointly WSS if the signals are both WSS and the cross correlation only depends on the time difference.
- Note the order of the two processes in the definition.
- Changing the order of the processes, changes the sign of the cross correlation:

$$\begin{aligned} R_{XY}(\tau) &= E[X(t)Y(t+\tau)] \\ &= E[Y(t+\tau)X(t)] \\ &= E[Y(t)X(t-\tau)] \\ &= R_{YX}(-\tau) \end{aligned}$$

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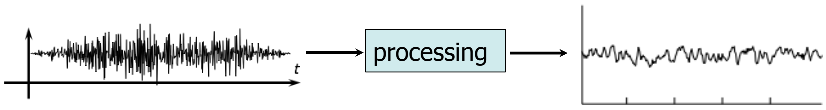
Random signal processing

- Signals (speech, music, images) are often processed:
  - noise removal
  - equalization
  - modulation
  - compression
- Consider signals are realizations of random processes



- Relation between stochastic properties of original and processed signal?

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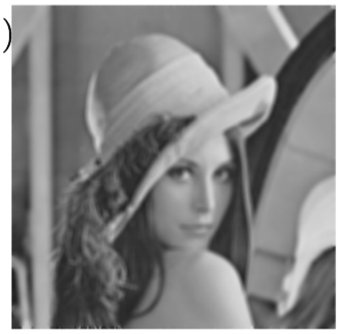
Image example

$X(i,j)$

$Y(i,j)$



- In theory: find relation between pdf’s of X(t) and Y(t)
- In practice: assume signals to be WSS, and consider relations between means and autocorrelation functions

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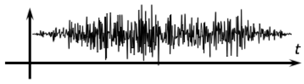
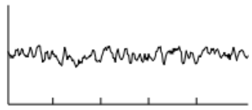
Consider only LTI systems





$X(t)$ → linear system (LTI) h(n), h(t) → $Y(t)$

- Describe Linear Time Invariant system with an impulse response function h(n) (or h(t))

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Response to Linear system (LTI)

$x(n)$ → h(n) → $y(n)$

- Linear time invariant system is described by its impulse response:
  response (output) of the system when the input is an impulse signal at n=0
- Output and input are related to each other via the convolution of $x(n)$ and $h(n)$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = h(n) * x(n)$$

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Response to Linear system (LTI)

$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = h(n) * x(n)$$

- Example: $(h(0), h(1), h(2), h(3), ...) = (1, 2, -1, 0, 0, ...)$

$$y(0) = ?$$

Response to Linear system (LTI)

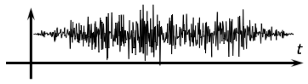
$$x(n) \longrightarrow \boxed{h(n)} \longrightarrow y(n)$$

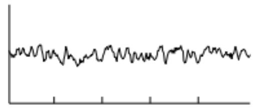
$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k) = h(n) * x(n)$$

- Example: $(h(0), h(1), h(2), h(3), ...) = (1, 2, -1, 0, 0, ...)$

$$\begin{aligned} y(0) &= h(0)x(0) + h(1)x(-1) + h(2)x(-2) \\ &= x(0) + 2x(-1) - x(-2) \\ y(1) &= x(1) + 2x(0) - x(-1) \\ &\dots \\ y(k) &= 1 \cdot x(k) + 2x(k-1) - 1 \cdot x(k-2) \end{aligned}$$

Filtered WSS Process





$X(t) \longrightarrow$ linear system (LTI) h(n), h(t) $\longrightarrow Y(t)$

- If X(t) is a WSS process, what can we say about the properties of the filtered signal Y(t)?
  - Is Y(t) also WSS?
  - What are μ_Y and $R_Y(k)$?

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E[] of filtered WSS process (1)

$x(n) \longrightarrow$ h(n) $\longrightarrow y(n)$

- Expected value

$$E[Y(n)] = E[h(n) * X(n)] = E\left[\sum_{k=-\infty}^{\infty} h(k)X(n-k)\right]$$

$$= \sum_{k=-\infty}^{\infty} h(k)E[X(n-k)] = E[X(n)] \cdot \sum_{k=-\infty}^{\infty} h(k) = E[X(n)]H_0$$

$$\mu_Y = E[X(n)]H_0 = \mu_X H_0$$

- Expected value for continuous time:

$$\mu_Y = E[Y(t)] = E[X(t)]H_0 = \mu_X \int_{-\infty}^{\infty} h(t)dt$$

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E[] of filtered WSS process (2)

- We assume $h(k) = 2,\, 1,\, 0,\, 0,\, 0, \ldots \quad E[X(n)] = -2$

$$\begin{aligned} E[Y(n)] &= E[2X(n) + X(n-1)] \\ &= 2E[X(n)] + \underbrace{E[X(n-1)]}_{=E[X(n)] \text{ because WSS}} \\ &= 2 \cdot -2 + 1(-2) = -6 \end{aligned}$$

- alternative:

$$E[Y(n)] = E[X(n)] \underbrace{\sum_k h(k)}_{H(0)} = -2 \cdot (2+1) = -6$$

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R(k) for filtered WSS processes

- Specific case:

$$h(k) = 2,\, 1,\, 0,\, 0, \ldots \qquad R_X(k) = \begin{cases} 3 & k = 0 \\ 1 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\begin{aligned} R_Y(k) &= E[Y(n)Y(n+k)] \\ &= E\left[(h(n) * X(n))(h(n) * X(n+k))\right] \\ &= E\left[(2X(n) + X(n-1))(2X(n+k) + X(n+k-1))\right] \\ &= 4E[X(n)X(n+k)] + 2E[X(n-1)X(n+k)] \\ &\quad + 2E[X(n)X(n+k-1)] + E[X(n-1)X(n+k-1)] \\ &= 5R_X(k) + 2R_X(k-1) + 2R_X(k+1) \end{aligned}$$

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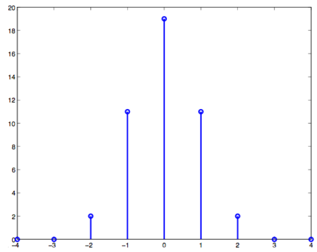
R(k) for filtered WSS processes

$$h(k) = 2,\ 1,\ 0,\ 0, \ldots \qquad R_X(k) = \begin{cases} 3 & k = 0 \\ 1 & k = \pm 1 \\ 0 & \text{otherwise} \end{cases}$$

- we got:

$$R_Y(k) = 5R_X(k) + 2R_X(k-1) + 2R_X(k+1)$$

$$R_Y(k) = \begin{cases} 19 & k = 0 \\ 11 & k = \pm 1 \\ 2 & k = \pm 2 \\ 0 & \text{otherwise} \end{cases}$$



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R(k) of filtered WSS processes

- Autocorrelation function

$$x(n) \longrightarrow \boxed{\text{h(n)}} \longrightarrow y(n)$$

$$\begin{aligned} R_Y(k) &= E[Y(n)Y(n+k)] \\ &= E[(h(n) * X(n))(h(n) * X(n+k)] \end{aligned}$$

$$\begin{aligned} R_Y(k) &= E[\sum_m h(m)X(n-m) \sum_p h(p)X(n+k-p)] \\ &= \sum_m h(m) \sum_p h(p) E[X(n-m)X(n+k-p)] \\ &= \sum_m h(m) \sum_p h(p) R_X(k-p+m) \\ &= h(k) * h(-k) * R_X(k) \end{aligned}$$

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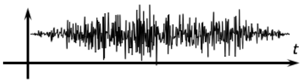
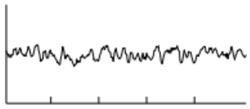
Summary Filtered WSS Process





$X(t)$ → linear system (LTI) h(n), h(t) → $Y(t)$

μ_X $\qquad\qquad$ $\mu_Y = \mu_X H_0$

$R_X(k)$ $\qquad\qquad$ $R_Y(k) = h(k) * h(-k) * R_X(k)$

- Time continuous: $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$

Example (time continuous)

$X(t)$ → $h(t) = \begin{cases} 3\exp(-t) & t \geq 0 \\ 0 & t < 0 \end{cases}$ → $Y(t)$

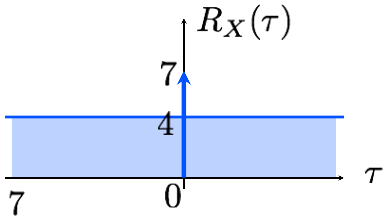
$R_X(\tau) = 4 + 3\delta(\tau)$

- what is $\mu_X, \ \sigma_X^2$?

Example (time continuous)

$$X(t) \longrightarrow \boxed{h(t) = \begin{cases} 3\exp(-t) & t \geq 0 \\ 0 & t < 0 \end{cases}} \longrightarrow Y(t)$$

$R_X(\tau) = 4 + 3\delta(\tau)$



- what is $\mu_X,\ \sigma_X^2$?

$$\lim_{k \to \infty} R_X(k) = 4 = \mu_X^2$$

$$R_X(0) = E[X^2(t)] = 7$$

$$\sigma_X^2 = Var(X) = E[X^2(t)] - E[X(t)]^2 = 3$$

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Example (time continuous)

$$X(t) \longrightarrow \boxed{h(t) = \begin{cases} 3\exp(-t) & t \geq 0 \\ 0 & t < 0 \end{cases}} \longrightarrow Y(t)$$

$R_X(\tau) = 4 + 3\delta(\tau) \qquad \mu_X = 2,\ \sigma_X^2 = 3$

- what is expected value of the output?

$$\mu_Y = H(0)\mu_X = 3 \cdot 2 = 6$$

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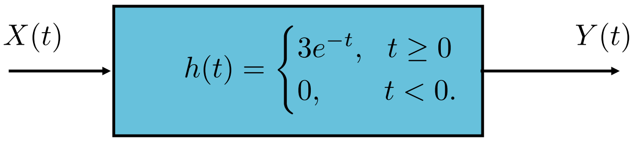
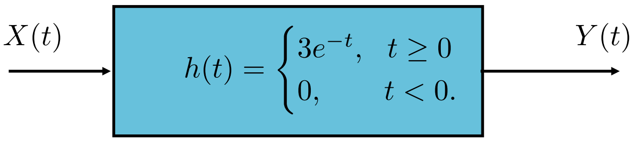
Example (Time Continuous)



$$
\begin{aligned}
R_Y(\tau) &= h(\tau) * h(-\tau) * R_X(\tau) \\
&= f(\tau) * R_X(\tau)
\end{aligned}
$$

$$
\begin{aligned}
f(\tau) &= h(\tau) * h(-\tau) \\
&= \int_{-\infty}^{\infty} 3e^{-t}u(t)3e^{-t+\tau}u(-\tau+t)dt \\
&= \begin{cases} 9e^{\tau}\int_{\tau}^{\infty} e^{-2t}dt = \frac{9}{2}e^{-\tau} & \text{if } \tau \geq 0 \\ 9e^{\tau}\int_{0}^{\infty} e^{-2t}dt = \frac{9}{2}e^{\tau} & \text{if } \tau < 0 \end{cases}
\end{aligned}
$$

Example (Time Continuous)



$$
\begin{aligned}
f(\tau) * R(\tau) &= \left(\frac{9}{2}e^{-\tau}u(\tau) + \frac{9}{2}e^{\tau}u(-\tau)\right) * (4 + 3\delta(\tau)) \\
&= \int_{-\infty}^{+\infty} \frac{9}{2}\left(e^{-t}u(t) + e^{t}u(-t)\right)(4 + 3\delta(\tau - t))dt \\
&= \frac{36}{2}\int_{0}^{\infty} e^{-t}dt + \frac{36}{2}\int_{-\infty}^{0} e^{t}dt + \frac{27}{2}e^{-\tau}u(\tau) + \frac{27}{2}e^{\tau}u(-\tau) \\
&= 36 + \frac{27}{2}e^{-|\tau|}
\end{aligned}
$$

Covered this lecture

- Chapter 11
- Key terms
  - Uncorrelated process
  - Random telegraph example
  - Ergodic processes
  - Random signal processing
  - Filtered WSS processes