Frequency Analysis: Fourier Transform (Recap)

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Frequency Decomposition

Decomposition of signals in terms of sinusoidal (or complex exponential) components

- For periodic signals, this decomposition is called a *Fourier series*
- For the class of finite energy signals, this decomposition is called the *Fourier transform*

Why important?

- Complex exponentials are eigenfunctions of linear time-invariant systems
- The response of an LTI system to a sinusoidal input signal is a sinusoid of the same frequency but of different amplitude and phase

Fourier Series

Theorem: Let $\{e_k\}_{k=1}^{\infty}$ denote a complete orthonormal system in $L^2(E)$ and let $x \in L^2(E)$. Then

$$x = \sum_{k=1}^{\infty} (x, e_k) e_k$$

Moreover, we have (Parseval's identity)

$$\|x\|^2 = \sum_{k=1}^{\infty} |(x, e_k)|^2$$

The series is called the *Fourier series* w.r.t. the system $\{e_k\}$; the coefficients (x, e_k) are called the *Fourier coefficients*.

Complex Fourier series

The *Fourier series representation* of a periodic signal $x(t)$ of period T_0 is given by

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t}, \quad \Omega_0 = \frac{2\pi}{T_0}$$

with *Fourier coefficients* X_k

$$X_k = \frac{1}{T_0} \int_{t_0}^{t_0+T_0} x(t) e^{-jk\Omega_0 t} dt, \quad k \in \mathbb{Z}$$

Moreover, we have *(Parseval's identity)*: $\|x\|^2 = T_0 \sum_{k=-\infty}^{\infty} |X_k|^2$

Trigonometric Fourier series

The *Fourier series representation* of a periodic signal $x(t)$ of period T_0 is given by

$$x(t) = c_0 + 2\sum_{k=0}^{\infty}\left(c_k\cos(k\Omega_0 t) + d_k\sin(k\Omega_0 t)\right), \quad \Omega_0 = \frac{2\pi}{T_0}$$

with *Fourier coefficients* c_k and d_k

$$c_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\cos(k\Omega_0 t)dt, \quad k = 0, 1, 2, \cdots$$
$$d_k = \frac{1}{T_0}\int_{t_0}^{t_0+T_0} x(t)\sin(k\Omega_0 t)dt, \quad k = 1, 2, \cdots$$

Convergence of the Fourier Series

$\Omega_0 = 1$

Example: $x(t) = t, t \in [\pi, \pi]$. Since x is odd, $c_k = 0$ for all k and

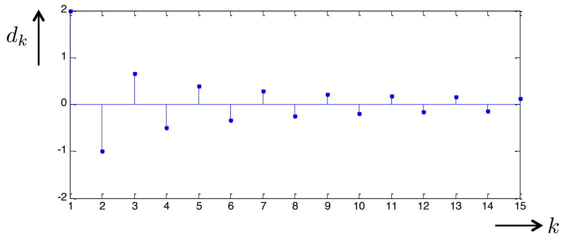
$$d_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} t\sin(kt)dt$$
$$= \frac{-1}{2\pi k} t\cos(kt)\Big|_{-\pi}^{\pi} + \frac{1}{2\pi k}\underbrace{\int_{-\pi}^{\pi}\cos(kt)dt}_{=0} = \frac{1}{k}(-1)^{k+1}$$

Hence, the Fourier series becomes

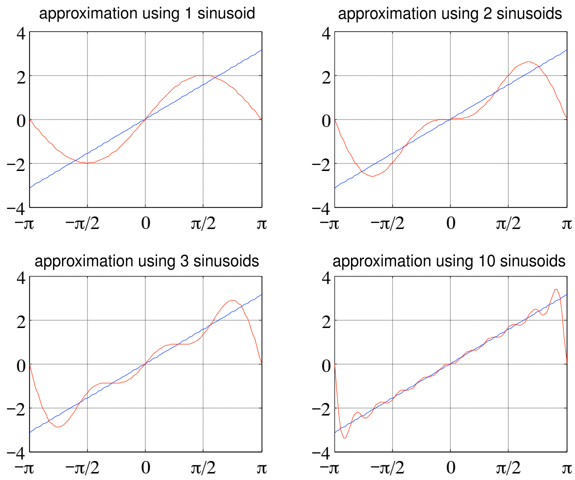
$$\frac{2}{1}\sin(t) - \frac{2}{2}\sin(2t) + \frac{2}{3}\sin(3t) - \frac{2}{4}\sin(4t) + \cdots$$

Convergence of the Fourier Series

- Notice that $\lim_{k\to\infty} d_k = 0$ and that the decay is $\mathcal{O}(k^{-1})$
- $x(\pi) = \pi$ but substituting $t = \pi$ in the Fourier series yields 0



Convergence of the Fourier Series



Convergence of the Fourier Series

Example: $x(t) = |t|, t \in [-\pi, \pi]$. Since x is even, $d_k = 0$ for all k

$$c_k = \frac{1}{\pi}\int_0^{\pi} t\cos(kt)dt$$
$$\stackrel{(k\neq 0)}{=} \underbrace{\frac{1}{k\pi} t\sin(kt)\Big|_0^{\pi}}_{=0} - \frac{1}{k\pi}\int_0^{\pi}\sin(kt)dt = \begin{cases} 0, & k = 2, 4, \ldots \\ \frac{-2}{\pi k^2}, & k \text{ odd} \end{cases}$$

and

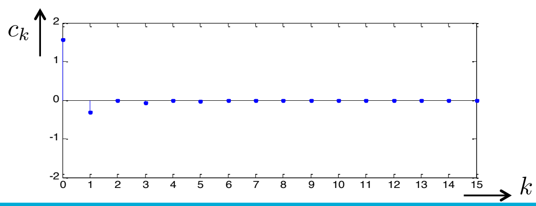
$$c_0 = \frac{1}{\pi}\int_0^{\pi} tdt = \frac{\pi}{2}$$

Convergence of the Fourier Series

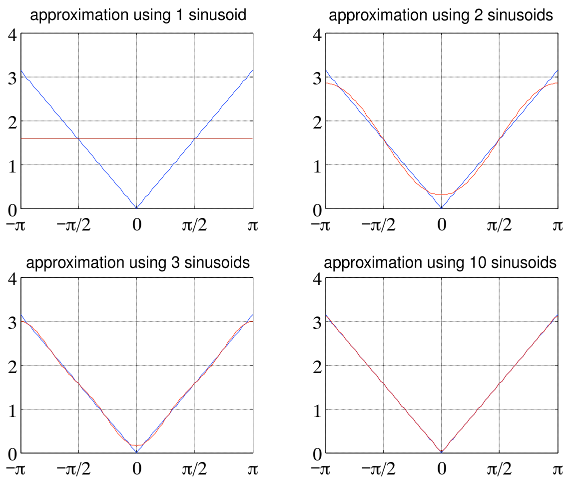
The Fourier series becomes

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos(t)}{1^2} + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right)$$

Notice that $\lim_{k\to\infty} c_k = 0$ and that the decay is of order $\mathcal{O}(k^{-2})$



Convergence of the Fourier Series



Convergence of the Fourier Series

- If x is p times differentiable and all derivatives are in $L^1(E)$, then

$$x^{(p)}(t) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad (jk\Omega_0)^p X_k$$

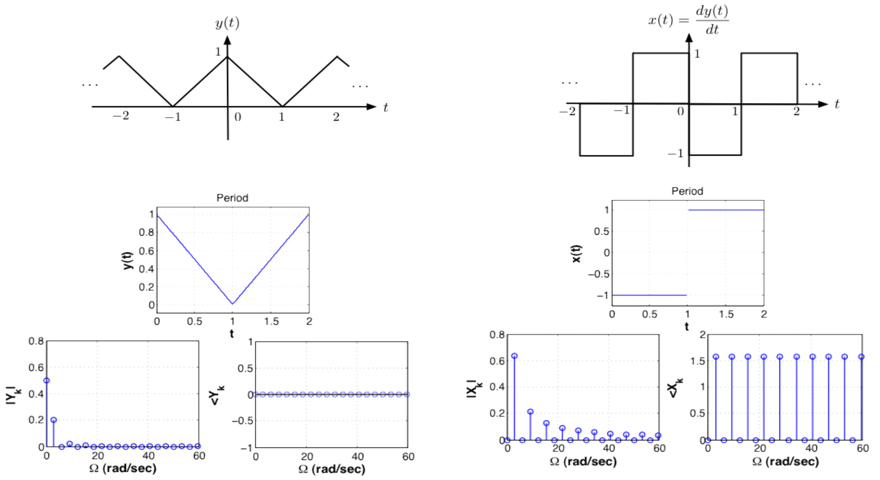
- Applying the Riemann-Lebesgue lemma on $x^{(p)}$, we conclude that

$$\lim_{k\to\pm\infty} (k\Omega_0)^p X_k = 0$$

  so that regularity of x translates to rapid decay of X_k

This explains the faster decay of the Fourier coefficients of the function $x(t) = |t|$ as compared to those of $x(t) = t$.

Convergence of the Fourier Series



Fourier Transform

Some observations:

- Fourier series to represent periodic signals
- Periodic signals have a line spectrum
- The line spacing is equal to the fundamental frequency
- What happens if the fundamental period becomes infinite (fundamental frequency zero)? $\longrightarrow$ continuous spectrum?

From Fourier Series to Fourier Intergral

Fourier transform

We have

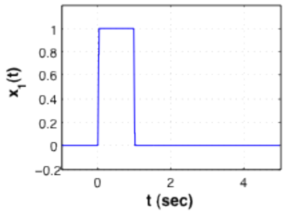
$$X(\Omega) = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt, \quad \Omega \in \mathbb{R}$$

$$x(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} X(\Omega)e^{j\Omega t}d\Omega, \quad t \in \mathbb{R}$$

is called the *Fourier transform* pair.

Fourier transform

Example: $x(t) = u(t) - u(t-1)$



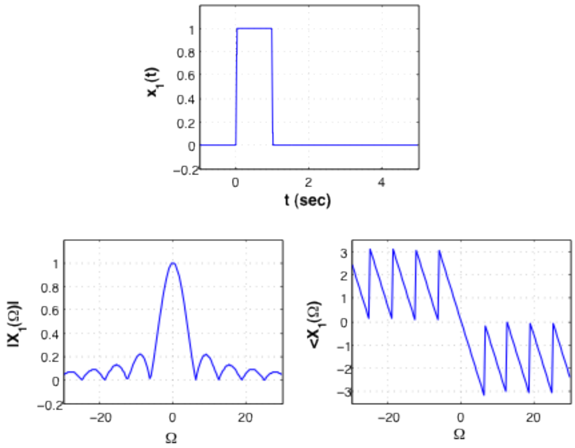
Solution:

$$X(\Omega) = \int_0^1 e^{-j\Omega t}dt = \frac{1}{j\Omega}\left(1 - e^{-j\Omega}\right) = \frac{\sin(\Omega/2)}{\Omega/2}e^{-j\Omega/2}$$

$$|X(\Omega)| = \left|\frac{\sin(\Omega/2)}{\Omega/2}\right|, \qquad \angle X(\Omega) = \angle\frac{\sin(\Omega/2)}{\Omega/2} - \Omega/2$$

Fourier transform

Solution:



Relation to Laplace transform

If the region of convergence (ROC) of $X(s) = \mathcal{L}\{x(t)\}$ contains the j-axis, then

$$X(\Omega) = \mathcal{F}\{x(t)\} = \int_{-\infty}^{\infty} x(t)e^{-j\Omega t}dt = \mathcal{L}\{x(t)\}\Big|_{s=j\Omega}$$

This holds for causal, anti-causal, and non-causal signals

Regularity and Fourier transform

- The decay of $X(\Omega)$ depends on the worst singular behaviour of x
- If x is p times differentiable and all derivatives are in $L^1(\mathbb{R})$, then

$$\lim_{\Omega \to \pm\infty} |\Omega|^p X(\Omega) = 0$$

so that regularity of x translates to rapid decay of $X(\Omega)$

Fourier transform of periodic signal

Representing a T_0-periodic signal $x(t)$ by its Fourier series, we have the following Fourier transform pair

$$x(t) = \sum_{k=-\infty}^{\infty} X_k e^{jk\Omega_0 t} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad X(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k \delta(\Omega - k\Omega_0)$$

Note that x has a line spectrum!

$\Delta\Omega_k = \Omega_0$

Example:

$$\mathcal{F}\{\cos(\Omega_0 t)\} = \mathcal{F}\left\{\frac{1}{2}e^{j\Omega_0 t} + \frac{1}{2}e^{-j\Omega_0 t}\right\} = \pi\delta(\Omega - \Omega_0) + \pi\delta(\Omega + \Omega_0)$$

Fourier transform of distributions

The Fourier transform is well defined for distributions as well. We have

- $\mathcal{F}\{\delta(t)\} = \mathcal{L}\{\delta(t)\}\Big|_{s=j\Omega} = 1\Big|_{s=j\Omega} = 1$
- $\mathcal{F}\{\delta'(t)\} = \mathcal{L}\{\delta'(t)\}\Big|_{s=j\Omega} = s\Big|_{s=j\Omega} = j\Omega$

Time-frequency relation

The support of $X(\Omega)$ is inversely proportional to the support of $x(t)$. If the Fourier transform $X(\Omega)$ of $x(t)$ exists and $a \neq 0 \in \mathbb{R}$, then $x(at)$

- is contracted ($|a| > 1$)
- is expanded ($-1 < a < 1$)
- reflected ($a < 0$)

We have

$$x(at) \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad \frac{1}{|a|} X\left(\frac{\Omega}{a}\right), \quad a \neq 0$$

Discrete-Time Fourier Transform

We have

$$\boxed{\begin{aligned} X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}, \quad \omega \in [0, 2\pi) \\ x(n) &= \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega, \quad n \in \mathbb{Z} \end{aligned}}$$

is called the *discrete-time Fourier transform* pair.

Relation Different Transforms

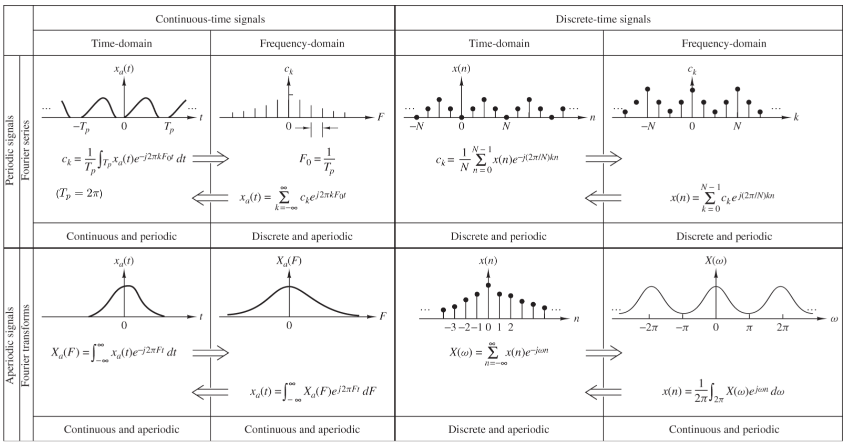


Figure 4.3.1 Summary of analysis and synthesis formulas.

Relation to the $\mathcal{Z}$-Transform

Recall that the z-transform of a sequence x was given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where $r_2 < |z| < r_1$ is the region of convergence. If $X(z)$ converges for $|z| = 1$, we have on the unit circle ($z = e^{j\omega}, \; -\pi \leq \omega < \pi$):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

In conclusion, if the unit circle is contained in the ROC, the Fourier transform of x is obtained by evaluating the z-transform $X(z)$ on the unit circle!

Relation to the $\mathcal{Z}$-Transform

The inverse z-transform, where C is taken to be the unit circle, is given by

$$\begin{aligned} x(n) &= \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz \\ &= \frac{1}{2\pi j} \oint_C X(\omega) e^{j\omega(n-1)} \underbrace{d\left(e^{j\omega}\right)}_{je^{j\omega} d\omega} \\ &= \frac{1}{2\pi} \int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega \end{aligned}$$

which is consistent with the results obtained before.

Relation to the $\mathcal{Z}$-Transform

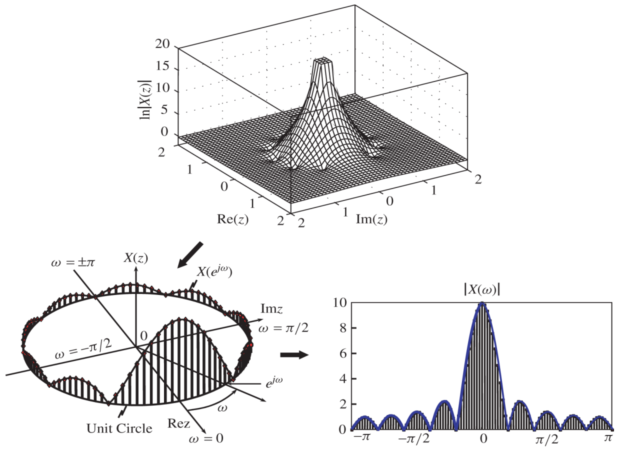


Figure 4.2.9 relationship between $X(z)$ and $X(\omega)$ for the sequence in Example 4.2.4, with $A = 1$ and $L = 10$

Application: Filtering

If $x(t)$ (periodic or not) is input to a BIBO stable LTI system with impulse response $h(t)$ (and transfer function $H(s)$), then the output of the system, say $y(t)$, is given by the convolution $y(t) = (x * h)(t)$ with Fourier transform

$$Y(\Omega) = X(\Omega)H(j\Omega)$$

If $x(t)$ is T_0-periodic, the Fourier transform of the output is given by

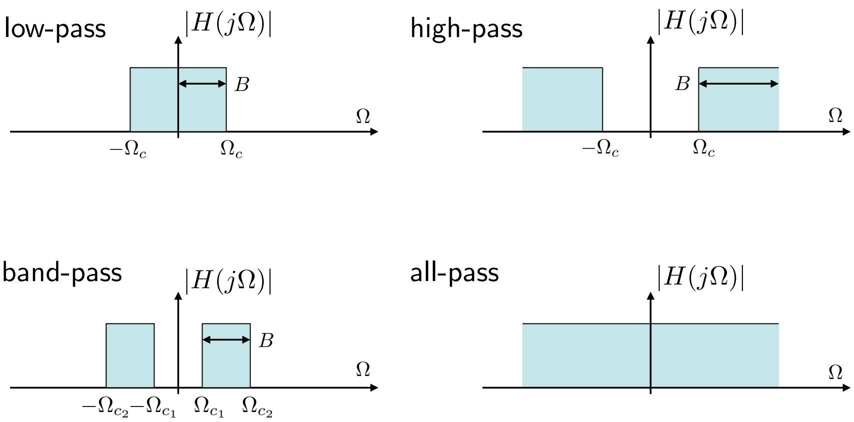
$$Y(\Omega) = \sum_{k=-\infty}^{\infty} 2\pi X_k H(jk\Omega_0)\delta(\Omega - k\Omega_0)$$

with X_k de Fourier series coefficients of $x(t)$ and $\Omega_0 = \dfrac{2\pi}{T_0}$

Basics of filtering

- Filtering consists in getting rid of undesirable components of a signal, and/or emphasizing desired components
- Filter design: find $H(s) = B(s)/A(s)$ satisfying certain specifications, e.g., flat frequency response up to a certain frequency Ω_p, and sufficient suppression for $|\Omega| > \Omega_s$.
- Frequency discriminating filters keep the frequency components of interest and attenuate the rest
- Filter design is a *rational approximation* problem

Frequency selective filters



Frequency selective filters

- Ideal filters cannot be realised:

$$H(j\Omega) = \begin{cases} c\,e^{-j\Omega t_0}, & |\Omega| < \Omega_c \\ 0, & \text{otherwise} \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad h(t) = \frac{\sin(\Omega_c(t-t_0))}{\pi(t-t_0)}$$

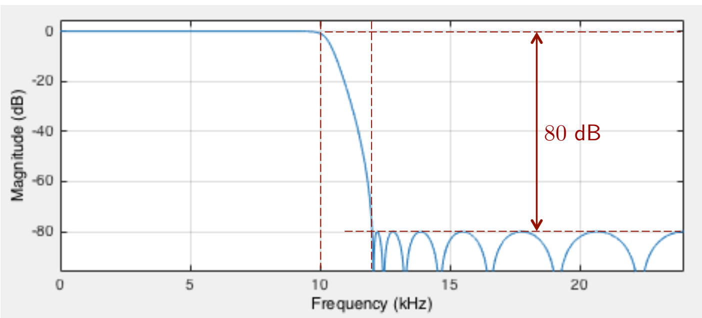
  Since $h \notin L^1(\mathbb{R})$, the filter is unstable; it can produce an unbounded output for a bounded input signal.

- Ideal filters are non-causal. If we approximate h by a causal filter, the frequency response cannot be zero in *any* frequency band
- ... but we can approximate the ideal filter closely

Frequency selective filters

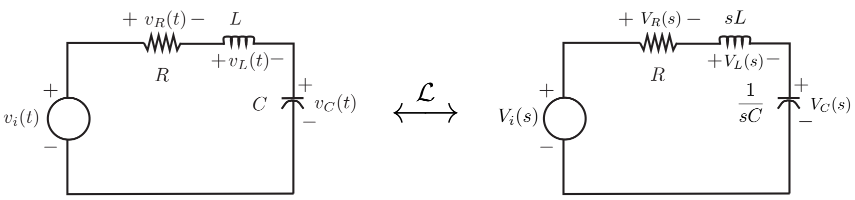
Design specs:

- $f_p = 10$ kHz, $f_s = 12$ kHz, stop-band attenuation 80 dB



Frequency selective filters

Example: (initial conditions zero)



$$V_i(s) = V_R(s) + V_L(s) + V_C(s)$$

Frequency selective filters

High-pass filter: voltage across the inductor

$$V_c(s) = \frac{s^2 LCV_i(s)}{s^2LC + sRC + 1}$$

so that the transfer function is given by

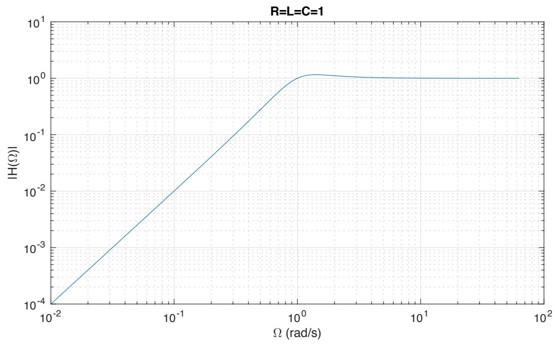
$$H(s) = \frac{s^2LC}{s^2LC + sRC + 1}$$

and thus

$$|H(j\Omega)| = \frac{\Omega^2 LC}{\sqrt{(1 - \Omega^2 LC)^2 + (\Omega RC)^2}}$$

Frequency selective filters

High-pass filter: voltage across the inductor



Frequency response and poles/zeros

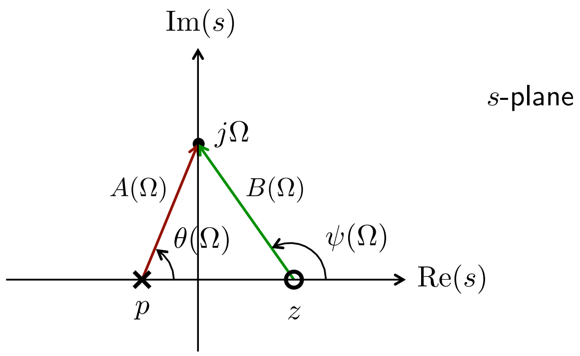
Consider a rational transfer function

$$H(s) = \frac{s-z}{s-p}$$

Frequency response:

$$H(j\Omega) = \frac{j\Omega - z}{j\Omega - p} = \underbrace{\frac{|j\Omega - z|}{|j\Omega - p|}}_{|H(j\Omega)|} \underbrace{e^{j(\angle(j\Omega - z) - \angle(j\Omega - p))}}_{e^{j\angle H(j\Omega)}}$$

Frequency response and poles/zeros



$$|H(j\Omega)| = \frac{B(\Omega)}{A(\Omega)}, \qquad \angle H(j\Omega) = \psi(\Omega) - \theta(\Omega)$$

Frequency response and poles/zeros

More general, for a Nth-order LTI system we have

$$\begin{aligned} H(j\Omega) &= \left.\frac{b_0 + b_1 s + \cdots + b_M s^M}{a_0 + a_1 s \cdots + a_N s^N}\right|_{s=j\Omega} \\ &= \left.\frac{b_M}{a_N} \frac{(s-z_1)(s-z_2)\cdots(s-z_M)}{(s-p_1)(s-p_2)\cdots(s-p_N)}\right|_{s=j\Omega} \\ &= \frac{b_M}{a_N} \frac{(j\Omega-z_1)(j\Omega-z_2)\cdots(j\Omega-z_M)}{(j\Omega-p_1)(j\Omega-p_2)\cdots(j\Omega-p_N)} \\ &= \frac{b_M}{a_N} \frac{\prod_{k=1}^{M}(j\Omega-z_k)}{\prod_{k=1}^{N}(j\Omega-p_k)} \end{aligned}$$

Frequency response and poles/zeros

Express the complex-valued factors in polar form as

$$j\Omega - z_k = B_k(\Omega)e^{j\psi_k(\Omega)}, \quad j\Omega - p_k = A_k(\Omega)e^{j\theta_k(\Omega)}$$

Hence, we have

$$|H(j\Omega)| = \left|\frac{b_M}{a_N}\right| \frac{\prod_{k=1}^{M} B_k(\Omega)}{\prod_{k=1}^{N} A_k(\Omega)}$$

and

$$\angle H(j\Omega) = \angle\frac{b_M}{a_N} + \sum_{k=1}^{M} \psi_k(\Omega) - \sum_{k=1}^{N} \theta_k(\Omega)$$

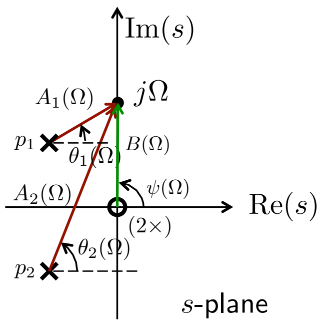
Frequency selective filters

Low-pass filter: voltage across the inductor

Transfer function:

$$H(s) = \frac{s^2}{s^2+s+1} = \frac{s^2}{(s+p_1)(s+p_2)}$$

$$p_1 = p_2^* = \frac{1}{2}(1+j\sqrt{3})$$



Frequency response and poles/zeros

Hence

- $|H(j0)| = 0$
- $\lim\limits_{\Omega \pm \infty} |H(j\Omega)| = 1$
- $\angle H(j0^{+}) = \pi$
- $\angle H(j0^{-}) = -\pi$
- $\lim\limits_{\Omega \pm \infty} \angle H(j\Omega) = 0$