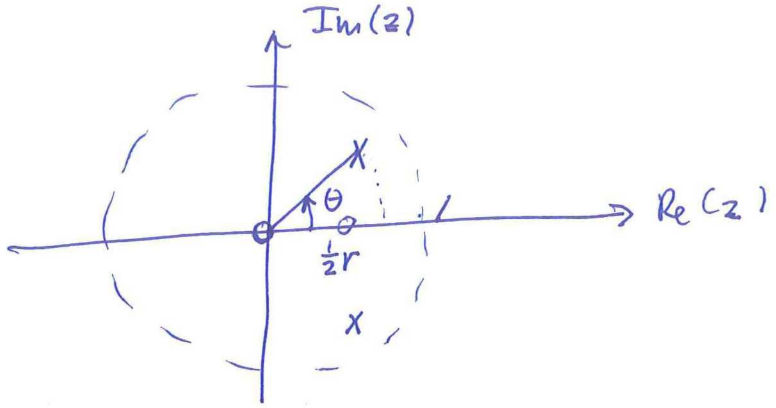
2)

b) $$H(z) = \frac{z\left(z-\frac{1}{2}r\right)}{(z-re^{j\theta})(z-re^{-j\theta})} = \frac{\frac{1}{2}z(2z-r)}{(z-re^{j\theta})(z-re^{-j\theta})}$$

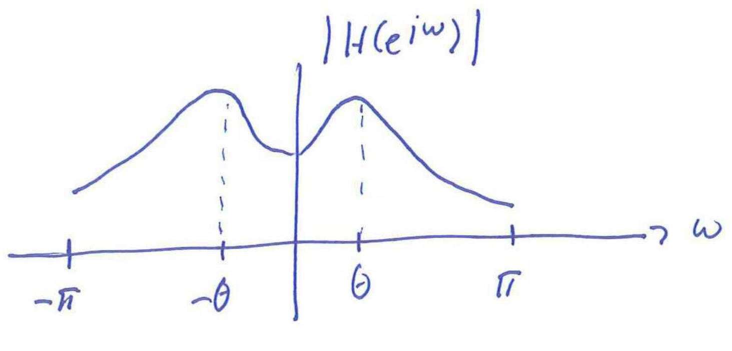
ROC : $|z| > r$, unique upto $c \in \mathbb{C}$.

a)



c) $r < 1$, $\theta \in [0, 2\pi)$ (poles inside unit circle)

d) $$|H(e^{jw})| = \left| \frac{\frac{1}{2}e^{jw}(2e^{jw} - r)}{(e^{jw} - re^{j\theta})(e^{jw} - re^{-j\theta})} \right|$$



e) $$\frac{H(z)}{z} = \frac{\frac{1}{2}(2z-r)}{(z-re^{j\theta})(z-re^{-j\theta})} = \frac{A}{z-re^{j\theta}} + \frac{B}{z-re^{-j\theta}}$$

$$A = \left.\frac{\frac{1}{2}(2z-r)}{z-re^{-j\theta}}\right|_{z=re^{j\theta}} = \frac{re^{j\theta}-\frac{1}{2}r}{re^{j\theta}-re^{-j\theta}}$$

$$B = \left.\frac{\frac{1}{2}(2z-r)}{z-re^{j\theta}}\right|_{z=re^{-j\theta}} = \frac{re^{-j\theta}-\frac{1}{2}r}{re^{-j\theta}-re^{j\theta}} = A^*$$

$$h(n) = A r^n e^{jn\theta} u(n) + B r^n e^{-jn\theta} u(n)$$

$$\left.\begin{aligned} &= r^n \left(A e^{jn\theta} + A^* e^{-jn\theta} \right) u(n) \\ &= |A| r^n \left(e^{j(n\theta + \angle A)} + e^{-j(n\theta + \angle A)} \right) u(n) \\ &= 2|A| r^n \cos(n\theta + \angle A)\, u(n) \end{aligned}\right\} \text{optional}$$

Note that $h(0) = \lim_{z \to \infty} H(z) = 1$

$$h(0) = A + A^* = \frac{re^{j\theta}-\frac{1}{2}r}{re^{j\theta}-re^{-j\theta}} + \frac{re^{-j\theta}-\frac{1}{2}r}{re^{-j\theta}-re^{j\theta}} = \frac{re^{j\theta}-re^{-j\theta}}{re^{j\theta}-re^{-j\theta}}$$

$$= 1$$

Using residue calculus:

$n \geq 0$:

$$\operatorname*{Res}_{z=re^{j\theta}} \frac{(z-\frac{1}{2}r)\, z^n}{(z-re^{j\theta})(z-re^{-j\theta})} = \left.\frac{(z-\frac{1}{2}r)\, z^n}{z-re^{-j\theta}}\right|_{z=re^{j\theta}} = A r^n e^{jn\theta}$$

$$\operatorname*{Res}_{z=re^{-j\theta}} \frac{(z-\frac{1}{2}r)\, z^n}{(z-re^{j\theta})(z-re^{-j\theta})} = \left.\frac{(z-\frac{1}{2}r)\, z^n}{(z-re^{j\theta})}\right|_{z=re^{-j\theta}} = A^* r^n e^{-jn\theta}$$

$$h(n) = A r^n e^{jn\theta} + A^* r^n e^{-jn\theta}, \quad n \geq 0$$

$h(n) = 0$ for $n<0$ (h causal)

f) $X(z) = \dfrac{z}{z - e^{j\pi/3}}$

$$Y(z) = H(z)X(z) = \frac{z^2\left(z - \frac{1}{2}r\right)}{\left(z - re^{j\theta}\right)\left(z - re^{-j\theta}\right)\left(z - e^{j\pi/3}\right)}$$

$$\operatorname*{Res}_{z = re^{j\theta}} \frac{\left(z - \frac{1}{2}r\right) z^{n+1}}{\left(z - re^{j\theta}\right)\left(z - re^{-j\theta}\right)\left(z - e^{j\pi/3}\right)} = \underbrace{\frac{\left(re^{j\theta} - \frac{1}{2}r\right) re^{j\theta}}{\left(re^{j\theta} - re^{-j\theta}\right)\left(re^{j\theta} - e^{j\pi/3}\right)}}_{A} r^n e^{jn\theta}$$

$$\operatorname*{Res}_{z = re^{-j\theta}} \left(\cdots \right) = \underbrace{\frac{\left(re^{-j\theta} - \frac{1}{2}r\right) re^{-j\theta}}{\left(re^{-j\theta} - re^{j\theta}\right)\left(re^{-j\theta} - e^{j\pi/3}\right)}}_{B} r^n e^{-jn\theta}$$

$$\operatorname*{Res}_{z = e^{j\pi/3}} \left(\cdots \right) = \underbrace{\frac{\left(e^{j\pi/3} - \frac{1}{2}r\right) e^{j\pi/3}}{\left(e^{j\pi/3} - re^{j\theta}\right)\left(e^{j\pi/3} - re^{-j\theta}\right)}}_{C} e^{j\frac{\pi}{3}n}$$

$$y(n) = A r^n e^{jn\theta} + B r^n e^{-jn\theta} + C e^{j\frac{\pi}{3}n}, \quad n \geq 0$$

$$y_{ss}(n) = \lim_{n \to \infty} y(n) = C e^{j\frac{\pi}{3}n}$$

$$y_{tr}(n) = A r^n e^{jn\theta} + B r^n e^{-jn\theta} \to 0 \text{ when } n \to \infty \ (r < 1)$$

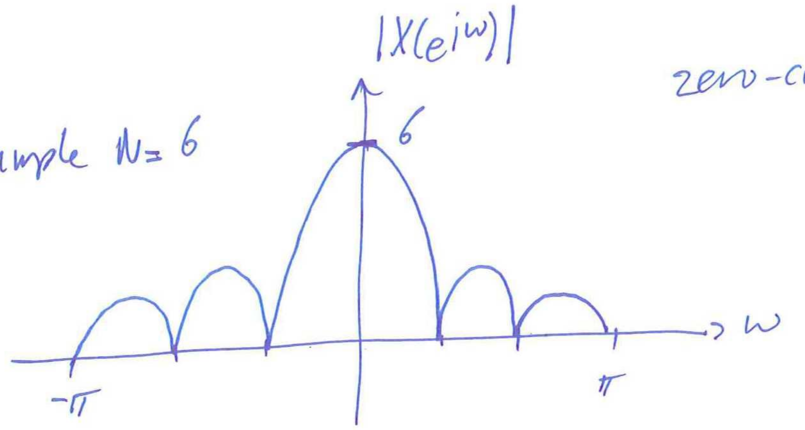
3)

a) $$X(e^{jw}) = \sum_{n=0}^{N-1} e^{-jwn} = \frac{1-e^{-jwN}}{1-e^{-jw}} = \frac{e^{-jwN/2}\left(e^{jwN/2} - e^{-jwN/2}\right)}{e^{-jw/2}\left(e^{jw/2} - e^{-jw/2}\right)}$$

$$= e^{-jw\left(\frac{N-1}{2}\right)} \frac{\sin(wN/2)}{\sin(w/2)}$$

* $X(e^{jw})$ periodic since x is discrete-time
* $X(e^{jw})$ continuous since x is non-periodic

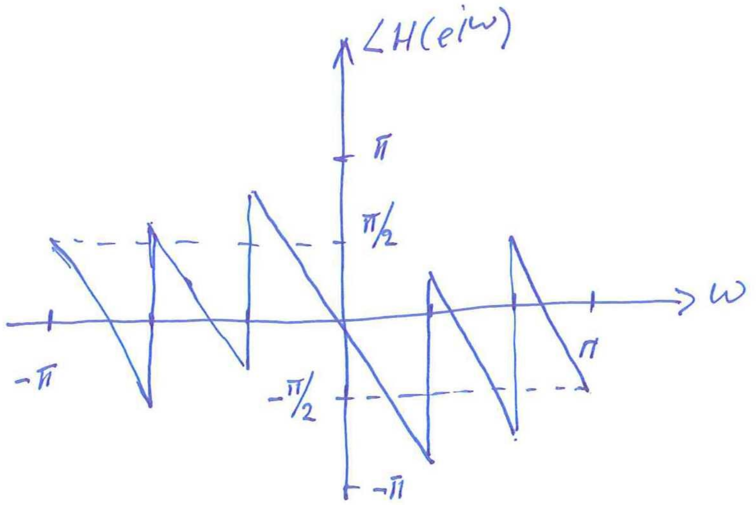
example $N=6$



zero-crossings $wN/2 = \pi \mod \pi$

$\Rightarrow w = \frac{2\pi}{N} \mod \frac{2\pi}{N}$

$w \neq 0$



b) $M \geq N$: sampling of $X(e^{jw})$ results in a periodic repitition of $x(n)$. (periodicity M). if $M < N$, the repetitions have overlaps which results in time-domain aliasing

c) $$x_p(n) = \sum_{k=-\infty}^{\infty} x(n + \overset{M}{3kN})$$

* periodic (see equation above, but also from the fact that the spectrum is discrete.
* discrete-time since the spectrum is still periodic.

d) reconstruction:

$$X(e^{jw}) = \frac{1}{M} \sum_{k=0}^{M-1} X(e^{j\frac{2\pi}{M}k}) \, G(e^{j(w - \frac{2\pi}{M}k)})$$

where

$$G(e^{jw}) = e^{-jw\frac{M-1}{2}} \frac{\sin(wM/2)}{\sin(w/2)}$$

and

$$g(n) = \frac{1}{2\pi} \int_0^{2\pi} G(e^{jw}) e^{jw} dw = \begin{cases} 1, & n = 0, \dots, M-1 \\ 0, & \text{otherwise} \end{cases}$$

$\Rightarrow$ time-domain windowing with rectangular window