Frequency-Domain Analysis of LTI Systems (Recap)

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Response to Complex Exponentials

Recall that the response of a (relaxed) linear time-invariant (LTI) system is given by the convolution sum

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)$$

We excite the system with a complex exponential $x = e^{j\omega_0(\cdot)}$:

$$y(n) = \sum_{k=-\infty}^{\infty} h(k)e^{j\omega_0(n-k)} = \left(\sum_{k=-\infty}^{\infty} h(k)e^{-j\omega_0 k}\right)\underbrace{e^{j\omega_0 n}}_{x(n)}$$

Complex exponentials are eigenfunctions of LTI systems !!

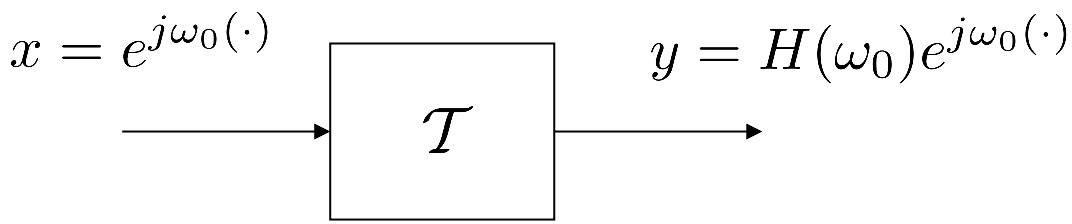
Frequency Response

The function

$$H(\omega) = \sum_{k=-\infty}^{\infty} h(k)e^{-j\omega k}$$

is called the *frequency response* of the system and is the Fourier transform of the impulse response which exists if h is absolutelty summable, that is, if the system is BIBO stable.

The value $H(\omega_0)$ is an *eigenvalue* for the *eigenfunction* $e^{j\omega_0(\cdot)}$:



Periodic Input Signals

Fourier series expansion: $$x(n) = \sum_{m=0}^{N-1} c_m e^{j\frac{2\pi m}{N}n}$$

so that

$$y(n) = \sum_{k=-\infty}^{\infty} h(k) \left(\sum_{m=0}^{N-1} c_m e^{j\frac{2\pi m}{N}(n-k)} \right)$$

$$= \sum_{m=0}^{N-1} c_m \left(\sum_{k=-\infty}^{\infty} h(k) e^{-j\frac{2\pi m}{N}k} \right) e^{j\frac{2\pi m}{N}n}$$

$$= \sum_{m=0}^{N-1} c_m H(\frac{2\pi m}{N}) e^{j\frac{2\pi m}{N}n}$$

Aperiodic Input Signals

Fourier transform: $$x(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega)e^{j\omega n}d\omega$$

so that

$$y = x * h \overset{\mathcal{F}}{\longleftrightarrow} Y(\omega) = H(\omega)X(\omega)$$

$$y(n) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H(\omega)X(\omega)e^{j\omega n}d\omega$$

Conclusions:

- The LTI system modifies the (complex) amplitudes of the frequency components contained in the signal
- The system acts like a *filter*
- Frequencies that are not contained in the input can only be created by *time-variant* and/or *non-linear* systems

Magnitude and Phase Response

In general, $H(\omega)$ is a complex-valued function:

- $H(\omega)$ is 2π-periodic
- Let $H(\omega) = H_R(\omega) + jH_I(\omega) = |H(\omega)|e^{j\angle H(\omega)}$

$$|H(\omega)| = \sqrt{H_R(\omega)^2 + H_I(\omega)^2}$$

and

$$\angle H(\omega) = \arctan\left(\frac{H_I(\omega)}{H_R(\omega)}\right)$$

The function $|H(\omega)|$ is called the *magnitude response* of the system, whereas the function $\angle H(\omega)$ is called the *phase response*.

Properties of Magnitude/Phase Response

If h is real, it follows directly from the properties of Fourier transforms that:

- $H(\omega)$ is conjugate symmetric ($H(\omega) = H^*(-\omega)$)
- $|H(\omega)| = |H(-\omega)|$ (even symmetric)
- $\angle H(\omega) = -\angle H(-\omega)$ (odd symmetric)
- If h symmetric ($h(n) = h(-n)$), then $H(\omega) = H_R(\omega)$
- if h anti-symmetric ($h(n) = -h(-n)$), then $H(\omega) = H_I(\omega)$

Magnitude and Phase Response

Let $x(n) = ce^{j(\omega_0 n + \phi)}$. Then

$$y(n) = cH(\omega_0)e^{j(\omega_0 n+\phi)} = \underbrace{c|H(\omega_0)|}_{c'}e^{j(\omega_0 n + \overbrace{\phi + \angle H(\omega_0)}^{\phi'})}$$

modification of the magnitude — modification of the phase

Let $x(n) = ce^{-j(\omega_0 n + \phi)}$. Then

$$\begin{aligned} y(n) &= cH(-\omega_0)e^{-j(\omega_0 n+\phi)} \\ &= c|H(\omega_0)|e^{-j(\omega_0 n+\phi+\angle H(\omega_0))} \end{aligned}$$

Hence, let $x(n) = c\cos(\omega_0 n + \phi)$. We then have

$$y(n) = c|H(\omega_0)|\cos(\omega_0 n + \phi + \angle H(\omega_0))$$

Magnitude and Phase Response

Example:

$$h(n) = \begin{cases} 1, & \text{for } n = -N, \ldots, N \\ 0, & \text{otherwise} \end{cases}$$

Hence

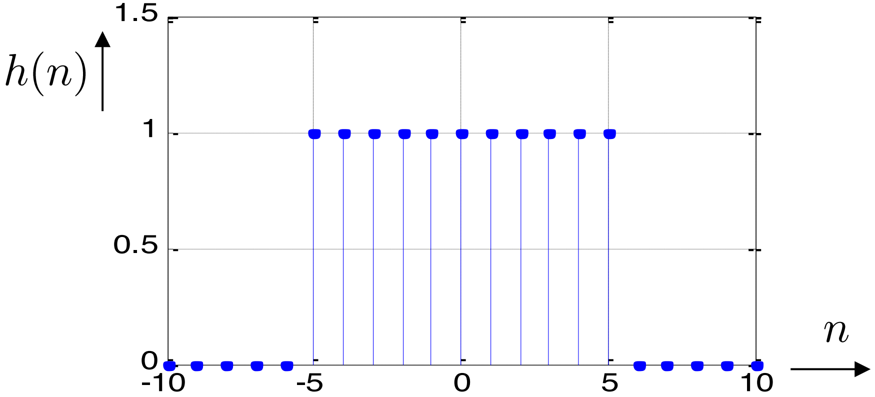
$$H(\omega) = \sum_{k=-N}^{N} e^{-j\omega k} = e^{j\omega N} \sum_{k=0}^{2N} e^{-j\omega k}$$

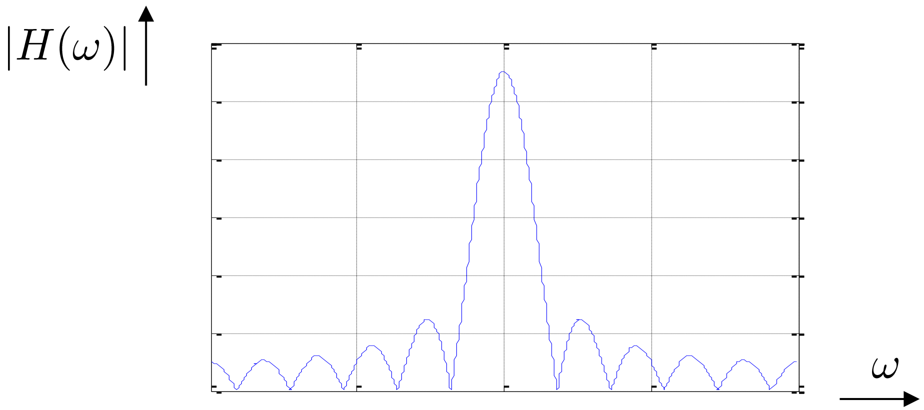
$$= e^{j\omega N} \frac{1 - e^{-j\omega(2N+1)}}{1 - e^{-j\omega}} = \frac{\sin(\omega(N + \frac{1}{2}))}{\sin(\frac{\omega}{2})}$$

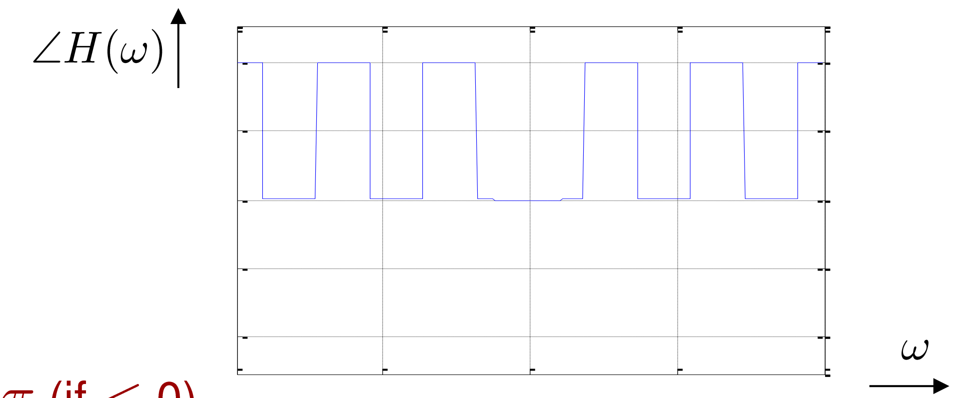
real

Magnitude and Phase Response

Example (con't):







$$|H(\omega)| = \left| \frac{\sin(\omega(N+\frac{1}{2}))}{\sin(\frac{\omega}{2})} \right|$$

$$\angle H(\omega) = \angle \frac{\sin(\omega(N+\frac{1}{2}))}{\sin(\frac{\omega}{2})}$$

0 (if $\geq$ 0) or π (if $<$ 0)

Magnitude and Phase Response

Example:

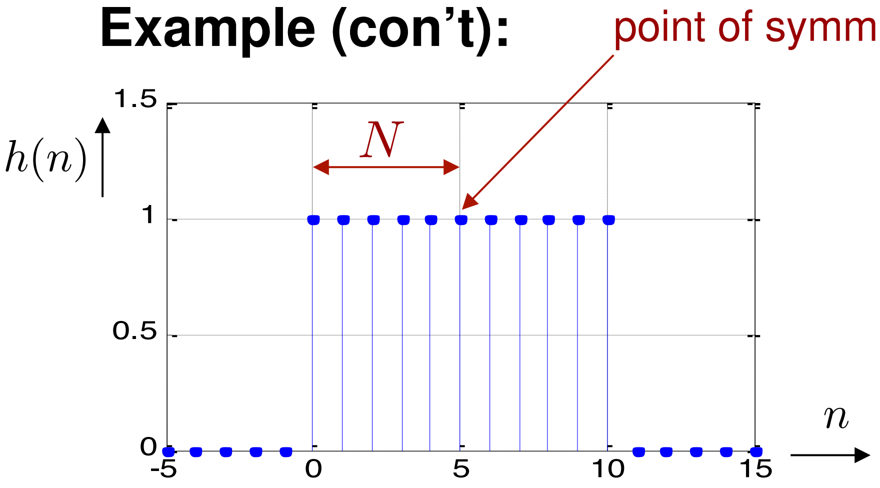
$$h(n) = \begin{cases} 1, & \text{for } n = 0, \ldots, 2N \\ 0, & \text{otherwise} \end{cases}$$

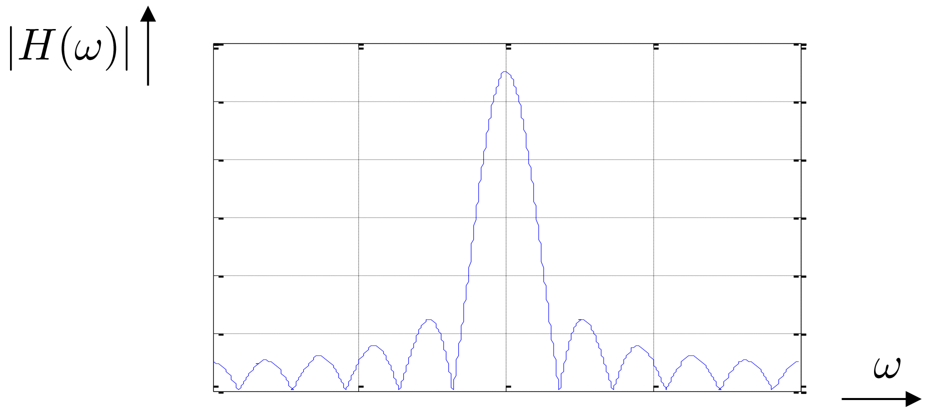
Since $h(n-k) \stackrel{\mathcal{F}}{\longleftrightarrow} e^{-j\omega k} H(\omega)$, we conclude from the previous example that

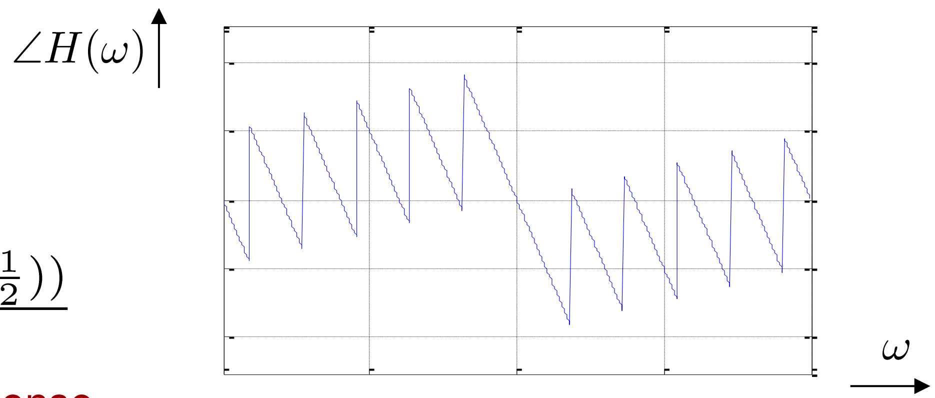
$$H(\omega) = \sum_{k=0}^{2N} e^{-j\omega k} = e^{-j\omega N} \frac{\sin(\omega(N + \frac{1}{2}))}{\sin(\frac{\omega}{2})}$$

Magnitude and Phase Response

Example (con't):







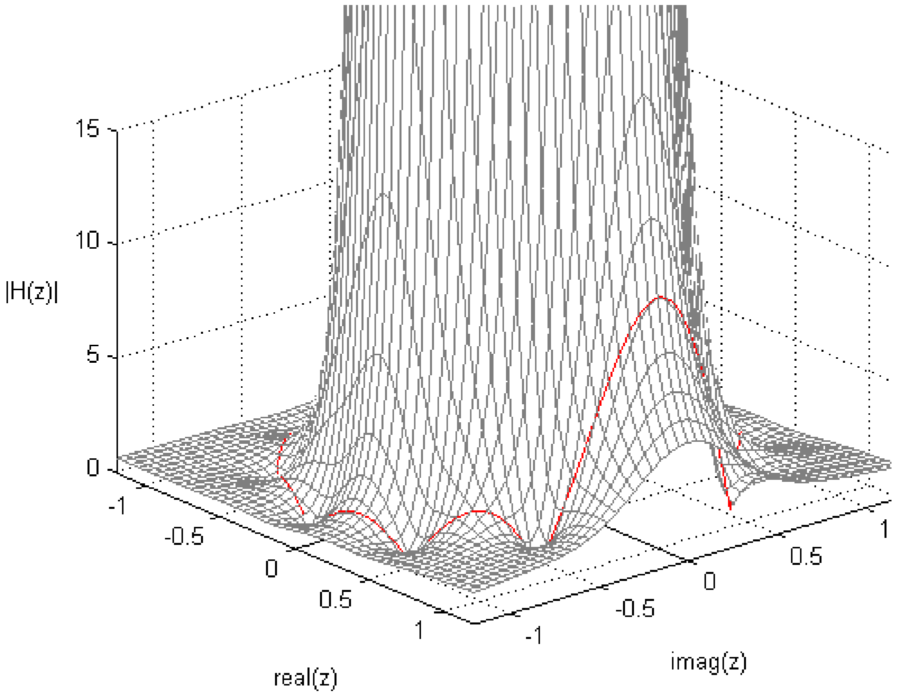
$$|H(\omega)| = \left| \frac{\sin(\omega(N+\frac{1}{2}))}{\sin(\frac{\omega}{2})} \right|$$

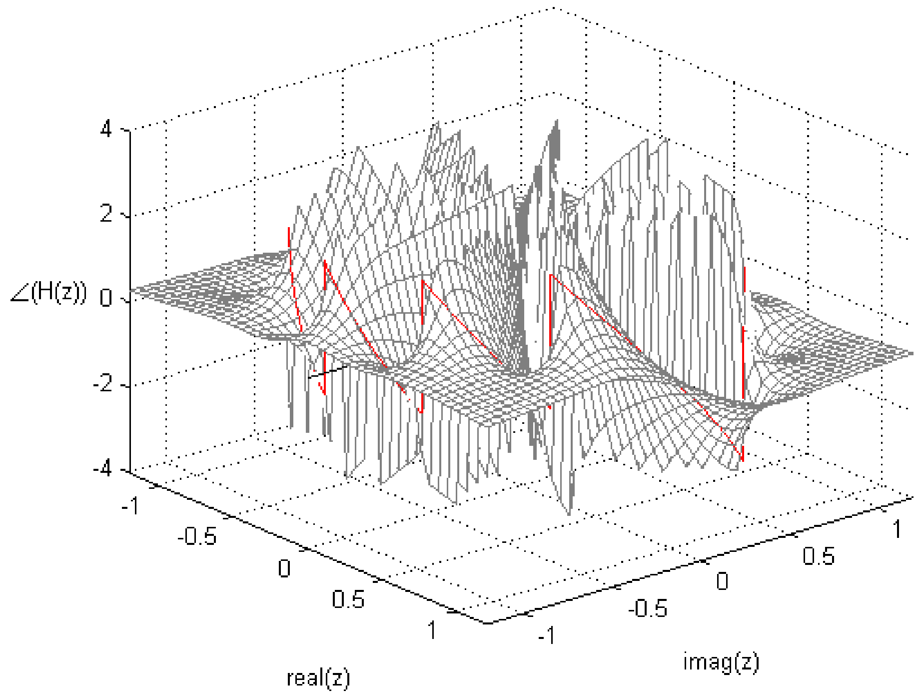
$$\angle H(\omega) = -\omega N + \angle \frac{\sin(\omega(N+\frac{1}{2}))}{\sin(\frac{\omega}{2})}$$

delay of N samples w.r.t. a symmetric response

Magnitude and Phase Response

Note that the Fourier transform of h can also be obtained by evaluating $H(z)$ on the unit circle





Linear Phase Response

Conclusions:

- A (anti)-symmetric impulse response results in a linear phase response: $\angle H(\omega) = -c\omega$
- The slope of the response depends on the point of symmetry
- Linear phase filters have a constant group delay:

$$\tau_{\mathrm{g}}(\omega) = -\frac{d\angle H(\omega)}{d\omega} = c$$

- In that case, all frequency components in the input signal undergo the same time delay:

$$e^{j\omega n} e^{j\angle H(\omega)} = e^{j\omega n} e^{-j\omega c} = e^{j\omega(n-c)}$$

Rational System Functions

If the system function $H(z)$ converges on the unit circle, we can obtain the frequency response $H(\omega)$ by evaluating $H(z)$ on the unit circle

$$
\begin{aligned}
H(\omega) &= H(z)\Big|_{z=e^{j\omega}} \\
&= \frac{b_0}{a_0} z^{N-M} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}\Bigg|_{z=e^{j\omega}} \\
&= \frac{b_0}{a_0} e^{j\omega(N-M)} \frac{(e^{j\omega}-z_1)(e^{j\omega}-z_2)\cdots(e^{j\omega}-z_M)}{(e^{j\omega}-p_1)(e^{j\omega}-p_2)\cdots(e^{j\omega}-p_N)}
\end{aligned}
$$

Rational System Functions

Express the complex-valued factors in polar form as

$$e^{j\omega} - z_k = V_k(\omega) e^{j\Theta_k(\omega)}, \qquad e^{j\omega} - p_k = U_k(\omega) e^{j\Phi_k(\omega)}$$

We then have

$$|H(\omega)| = \left|\frac{b_0}{a_0}\right| \frac{\prod_{k=1}^{M} V_k(\omega)}{\prod_{k=1}^{N} U_k(\omega)}$$

and

0 or π

$$\angle H(\omega) = \angle \frac{b_0}{a_0} + \omega(N - M) + \sum_{k=1}^{M} \Theta_k(\omega) - \sum_{k=1}^{N} \Phi_k(\omega)$$

Rational System Functions

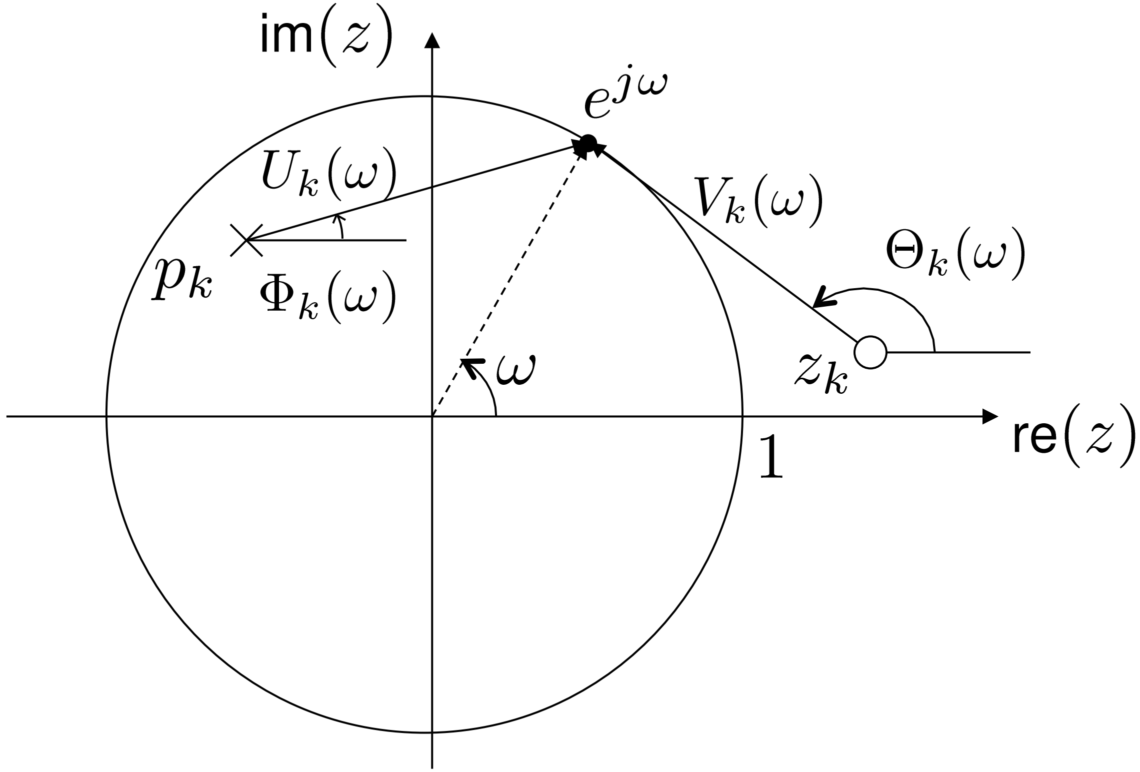
Geometrical interpretation of the contribution of a pole and zero to the magnitude and phase response

- Contribution to the magnitude response:

$$\frac{V_k(\omega)}{U_k(\omega)}$$

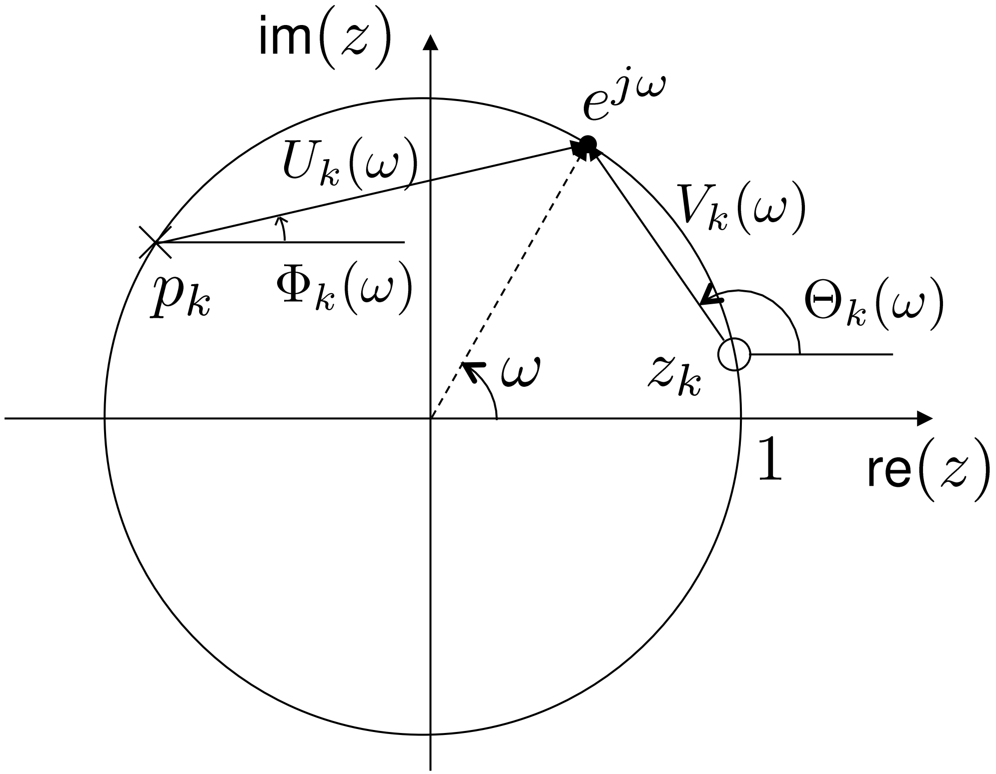
- Contribution to the phase response:

$$\Theta_k(\omega) - \Phi_k(\omega)$$



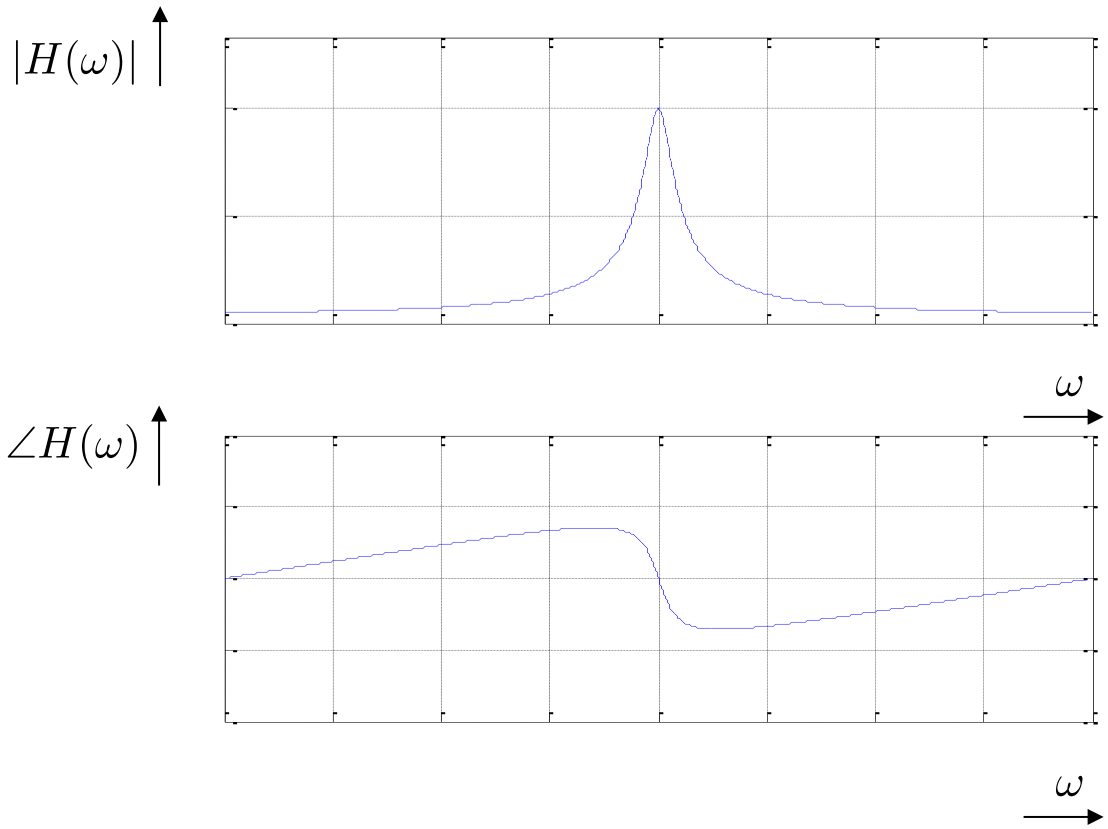
Rational System Functions

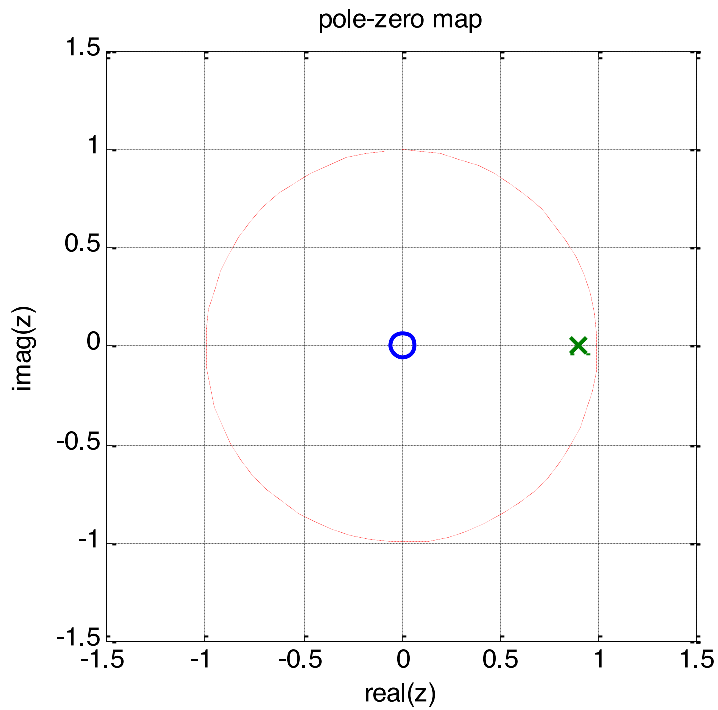
Poles/zeros on the unit circle



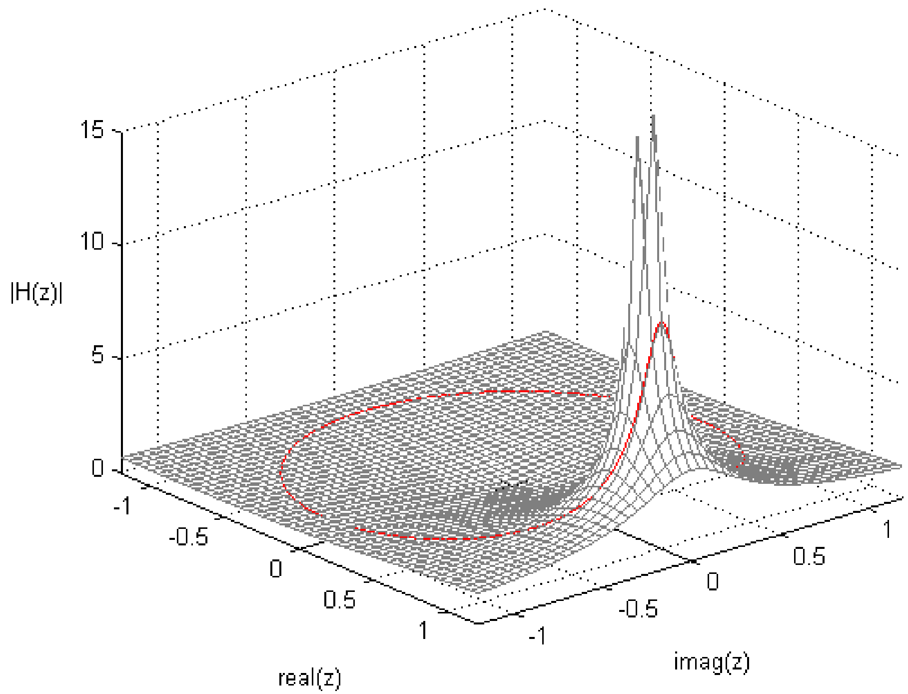
- a zero on the unit circle causes $|H(\omega)| = 0$ at $\omega = \angle z_k$
- a pole on the unit cirle causes $|H(\omega)| = \infty$ at $\omega = \angle p_k$

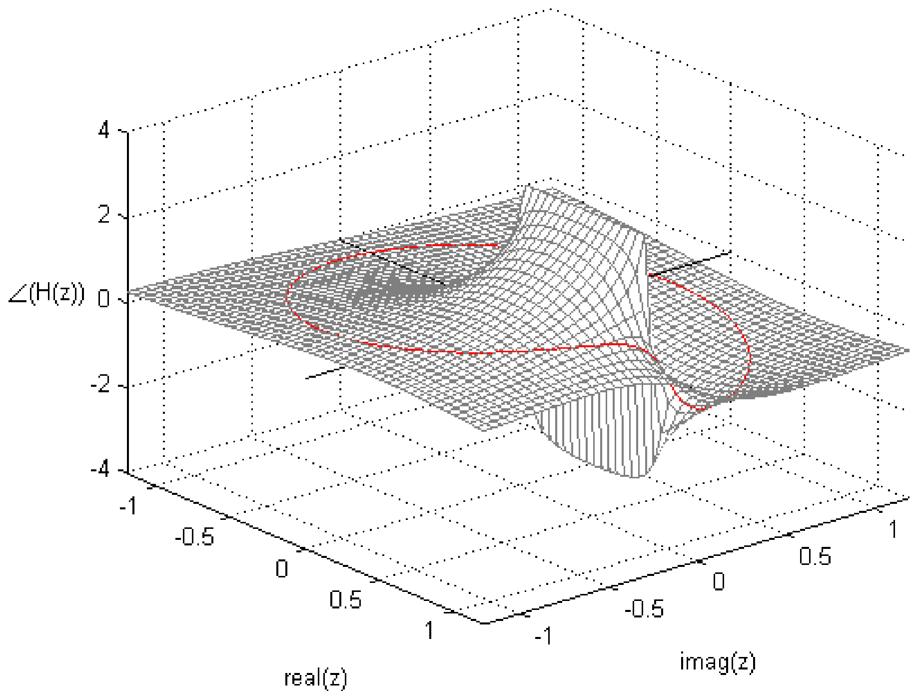
Rational System Functions





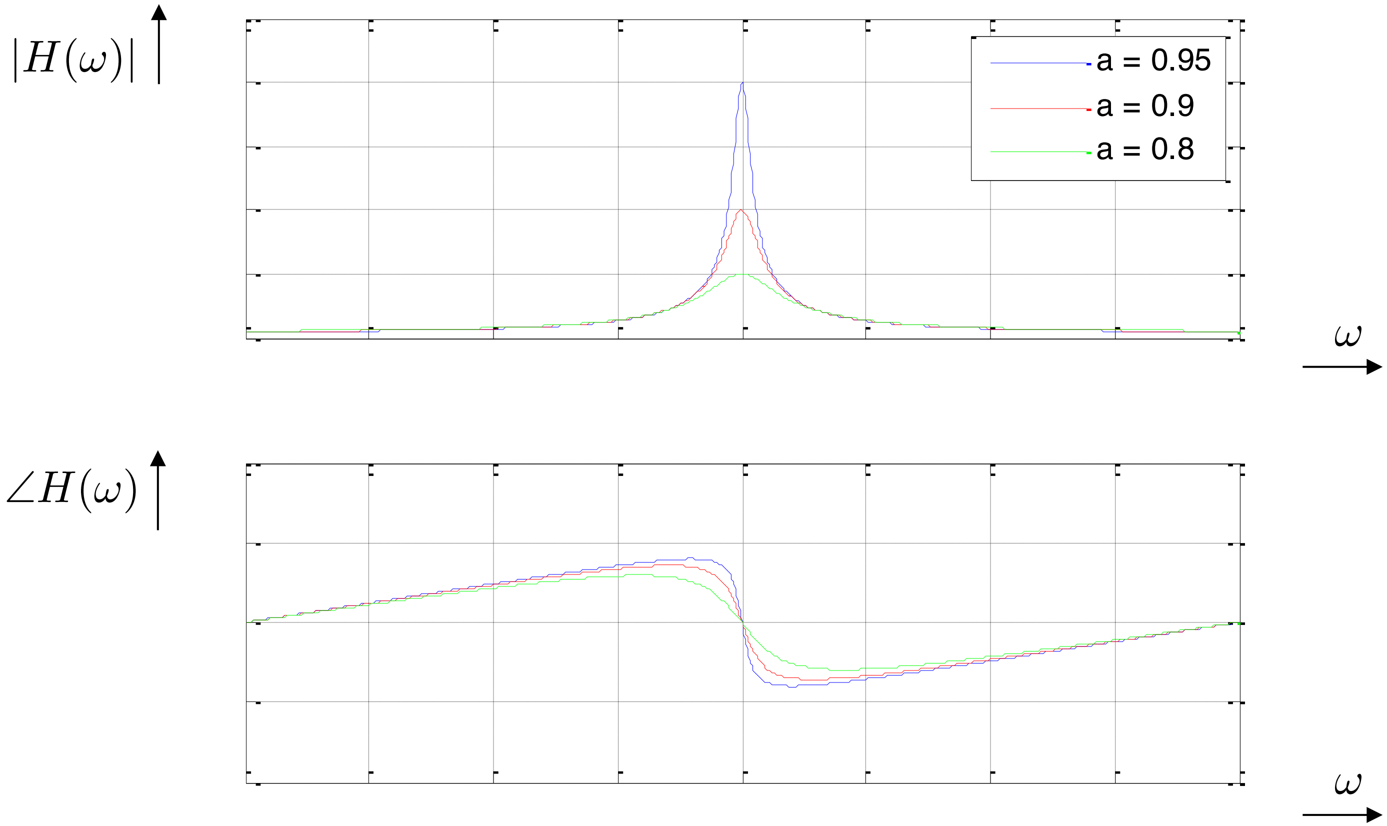
Rational System Functions





Rational System Functions

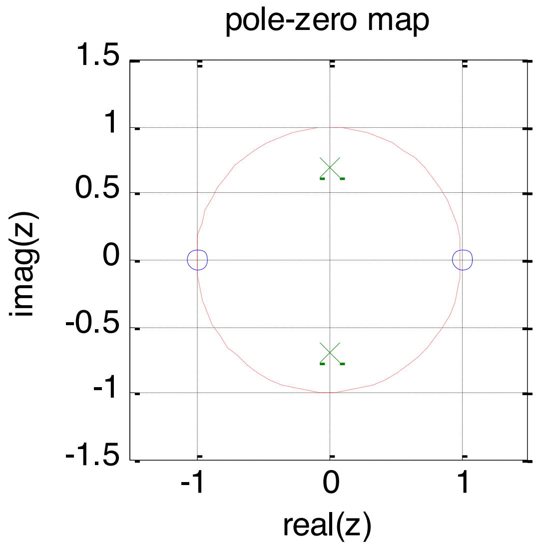
Effect of pole location:

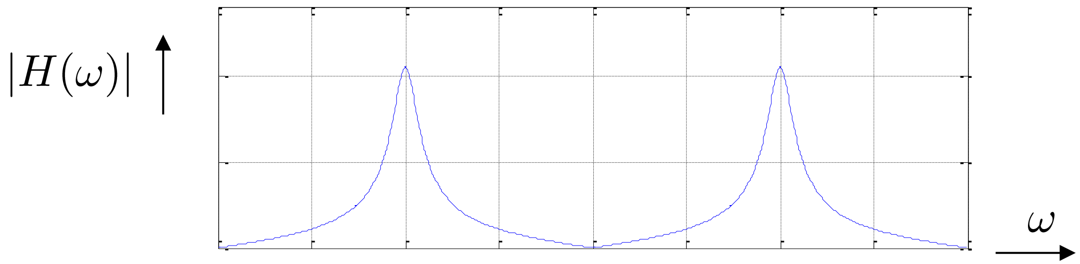


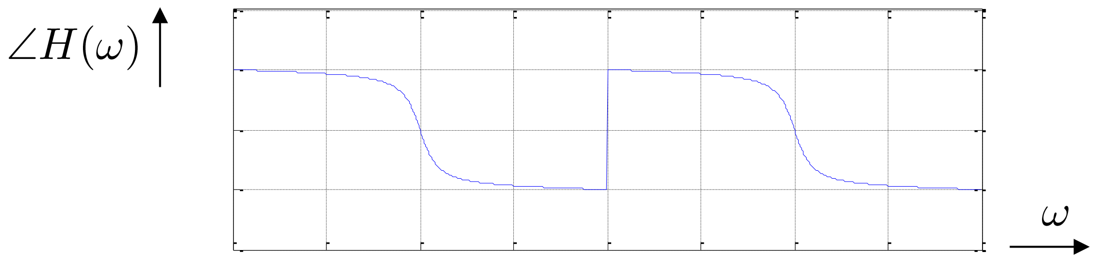
Rational System Functions

Example: bandpass filter

$$H(z) = \frac{(z-1)(z+1)}{(z-ja)(z+ja)} \qquad (a = 0.9)$$

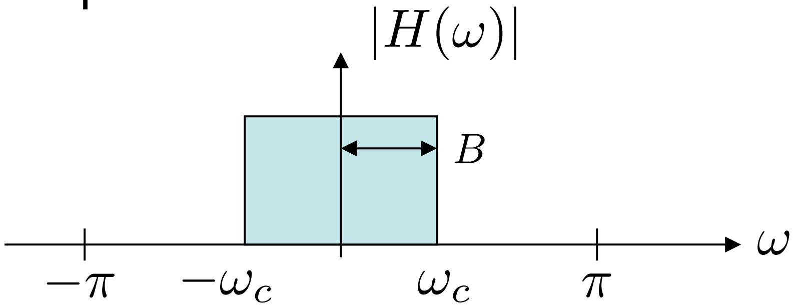




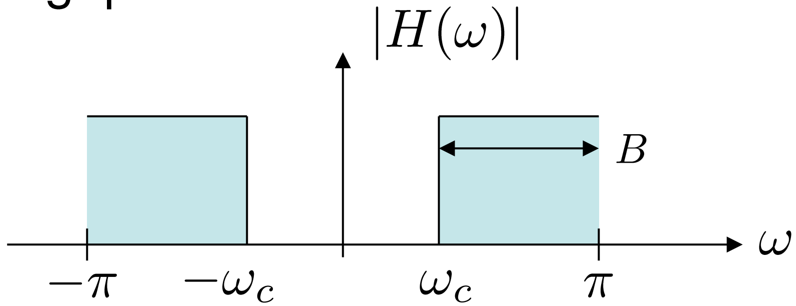


Frequency Selective Filters

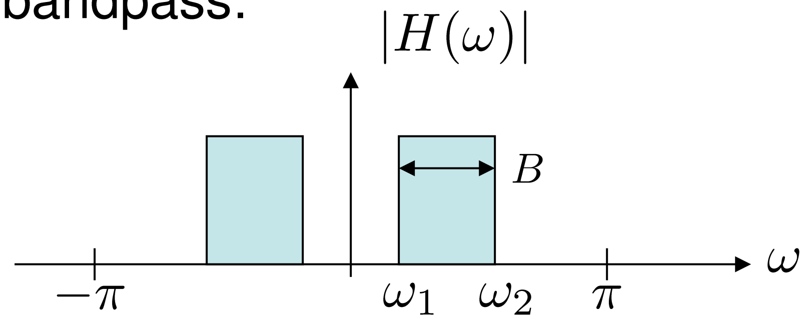
lowpass:



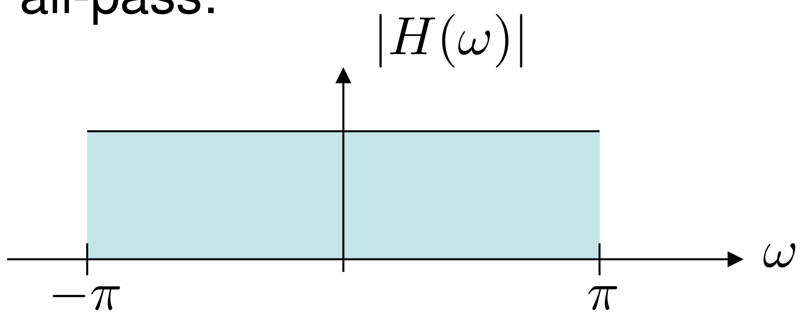
highpass:



bandpass:



all-pass:



Frequency Selective Filters

- ideal magnitude response: brickwall filter
- ideal phase response: linear phase response

Indeed, let

$$H(\omega) = \begin{cases} ce^{-j\omega n_0}, & \omega_1 < \omega < \omega_2 \\ 0, & \text{otherwise} \end{cases}$$

We then have

$$Y(\omega) = H(\omega)X(\omega) = cX(\omega)e^{-j\omega n_0}, \quad \omega_1 < \omega < \omega_2$$

and thus

$$y(n) = cx(n - n_0)$$

Frequency Selective Filters

Ideal brickwall filters cannot be realized

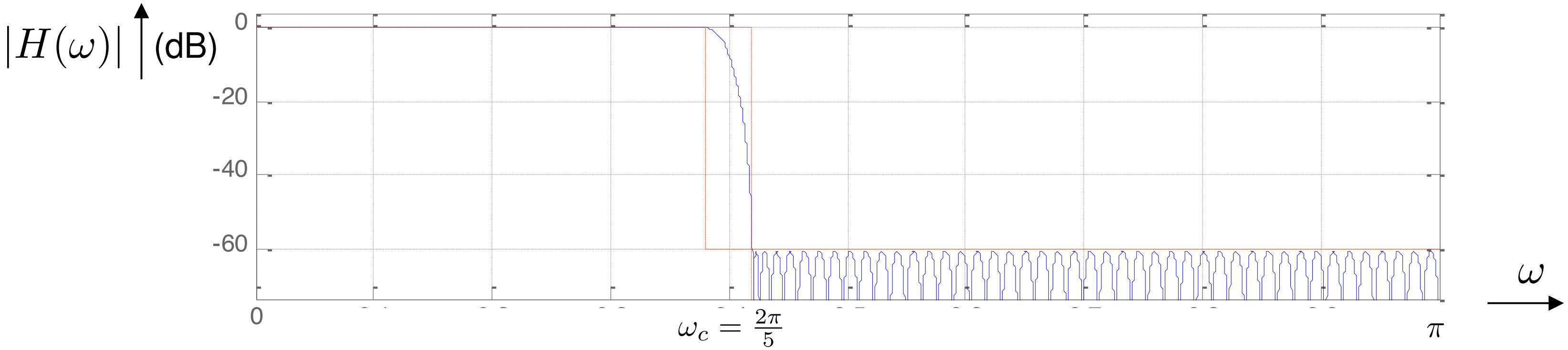
$$H(\omega) = \begin{cases} 1, & |\omega| < \omega_c \\ 0, & \text{otherwise} \end{cases} \quad \overset{\mathcal{F}}{\longleftrightarrow} \quad h(n) = \frac{\sin(\omega_c n)}{\pi n}$$

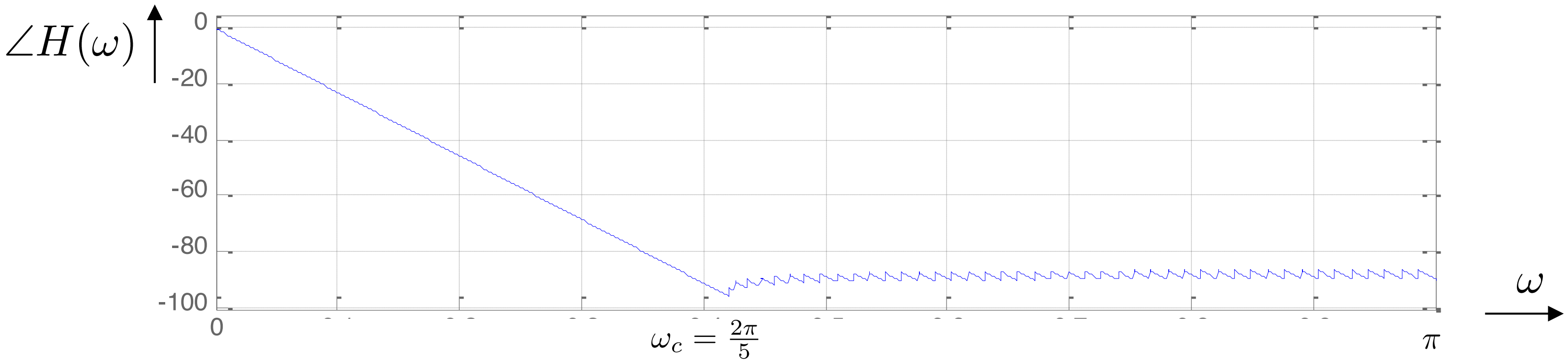
Since h is *not* absolutely summable, the filter is unstable; it can produce an unbounded output for a bounded input signal. However, we can approximate the ideal frequency response very closely

Linear phase filters can be obtained by using symmetric (FIR) filters

Frequency Selective Filters

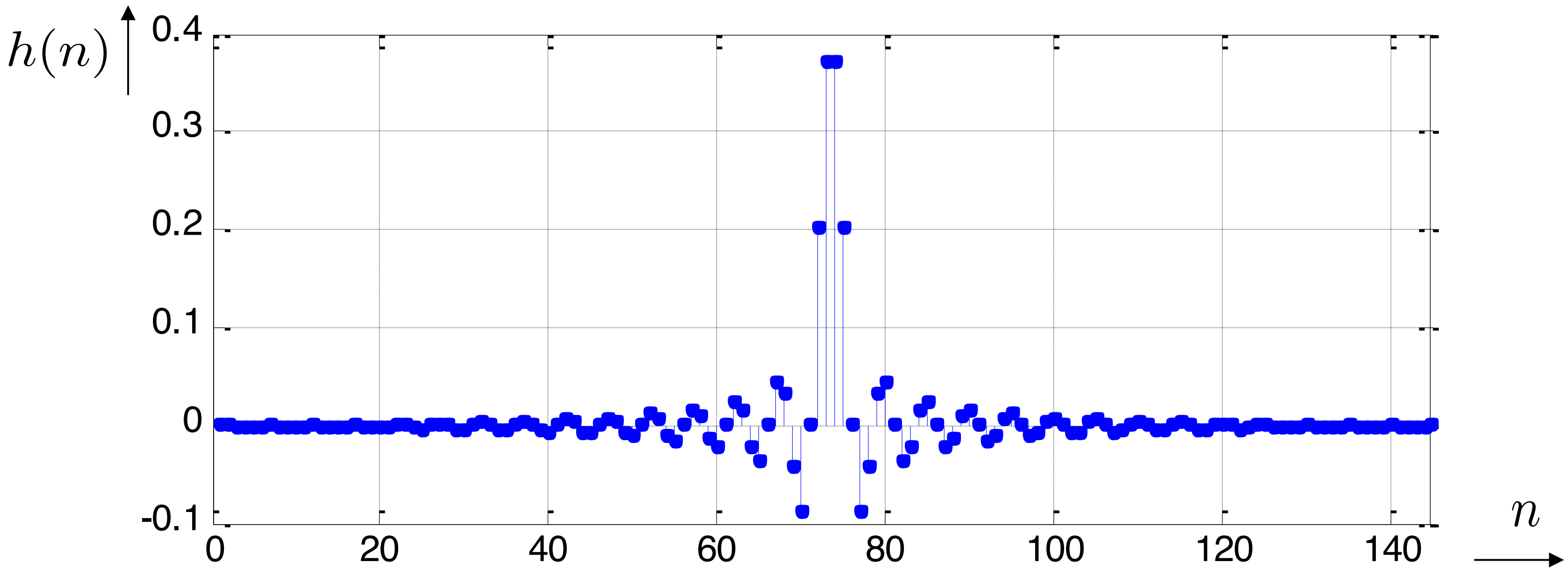
Example: equiripple FIR design ($\omega_c = \frac{2\pi}{5}$, transition width 0.04π, stopband attentuation 60 dB)





Frequency Selective Filters

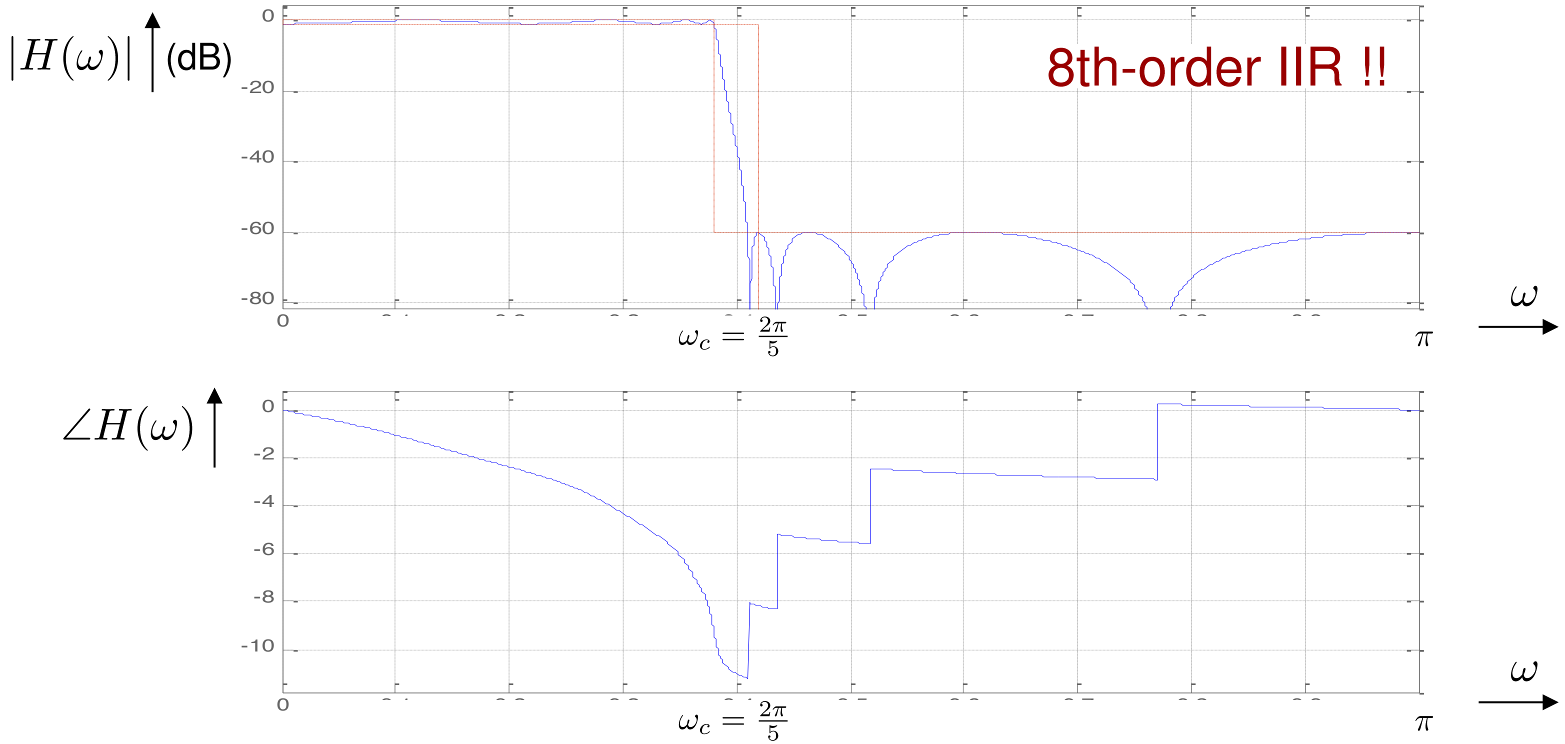
146-taps symmetric FIR impulse response:



Because of the symmetry of h around $n = 72\frac{1}{2}$, the phase response is linear: $\angle H(\omega) = -72\frac{1}{2}\omega$

Frequency Selective Filters

Example: elliptic IIR design ($\omega_c = \frac{2\pi}{5}$, transition width 0.04π, stop-band attentuation 60 dB)



Frequency Selective Filters

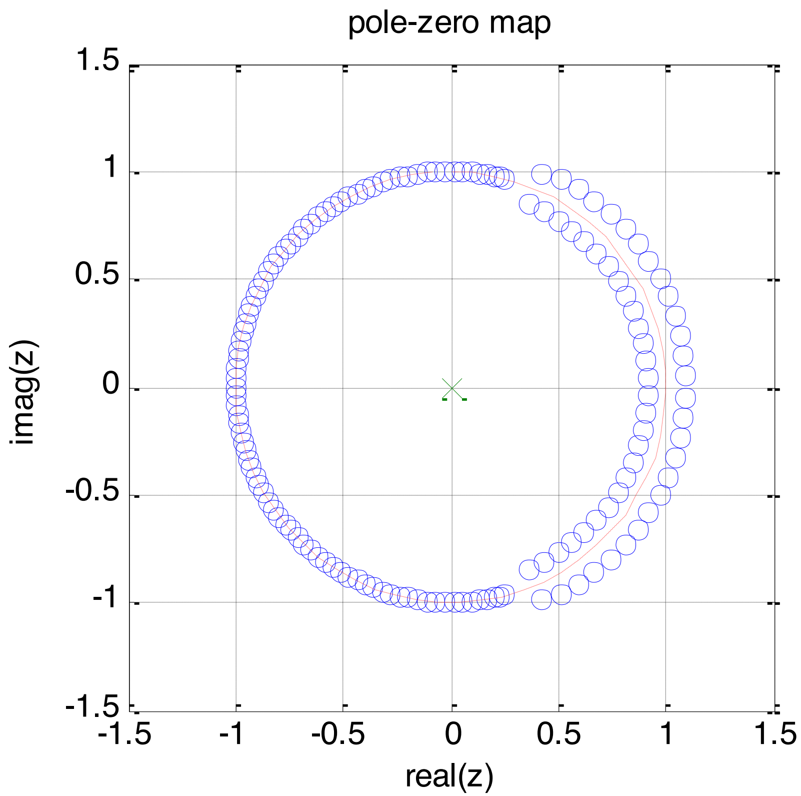
Lowpass filter design:

- poles should be placed near the unit circle at points corresponding to low frequencies (near $\omega = 0$)
- zeros should be placed near or on the unit circle at points corresponding to high frequencies (near $\omega = \pi$).

The opposite holds for highpass filters.

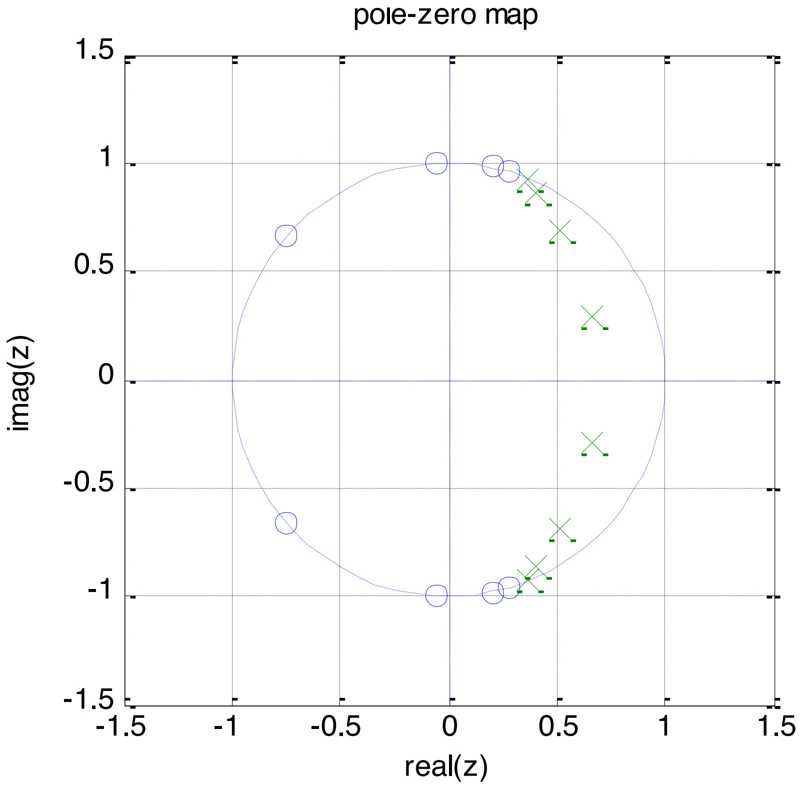
Frequency Selective Filters

Example: 146-taps equiripple FIR filter



Frequency Selective Filters

Example: 8th-order elliptic IIR filter



Notch Filters

A *notch filter* is a filter having one or more deep notches, or ideally, perfect nulls in the frequency response

Notch filters are useful when specific frequencies have to be eliminated, for example the 50 HZ power line frequency.

To create a null in the frequency response at a frequency ω_0, we simply produce a pair of complex-conjugated zeros on the unit circle at angle ω_0. That is,

$$z_{1,2} = e^{\pm j\omega_0}$$

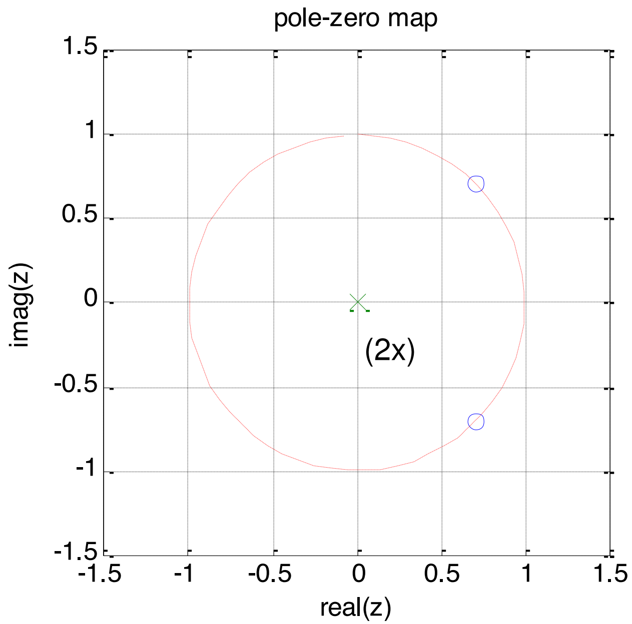
so that

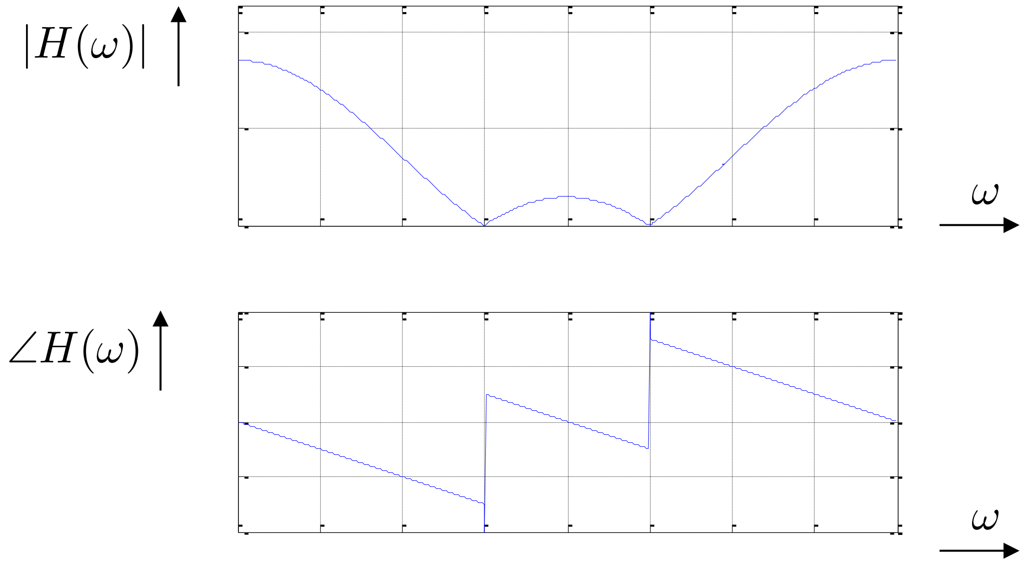
FIR notch filter

$$H(z) = (1 - z_1 z^{-1})(1 - z_2 z^{-1}) = 1 - 2\cos\omega_0 z^{-1} + z^{-2}$$

Notch Filters

$$H(z) = \frac{(z - e^{j\omega_0})(z - e^{-j\omega_0})}{z^2}$$



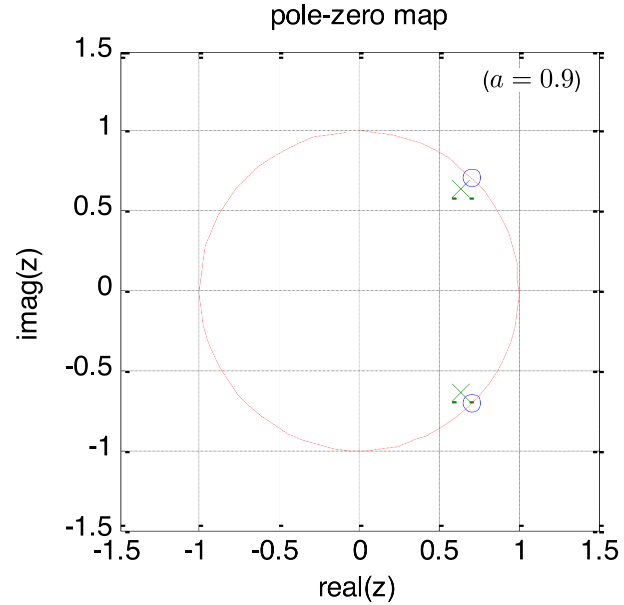


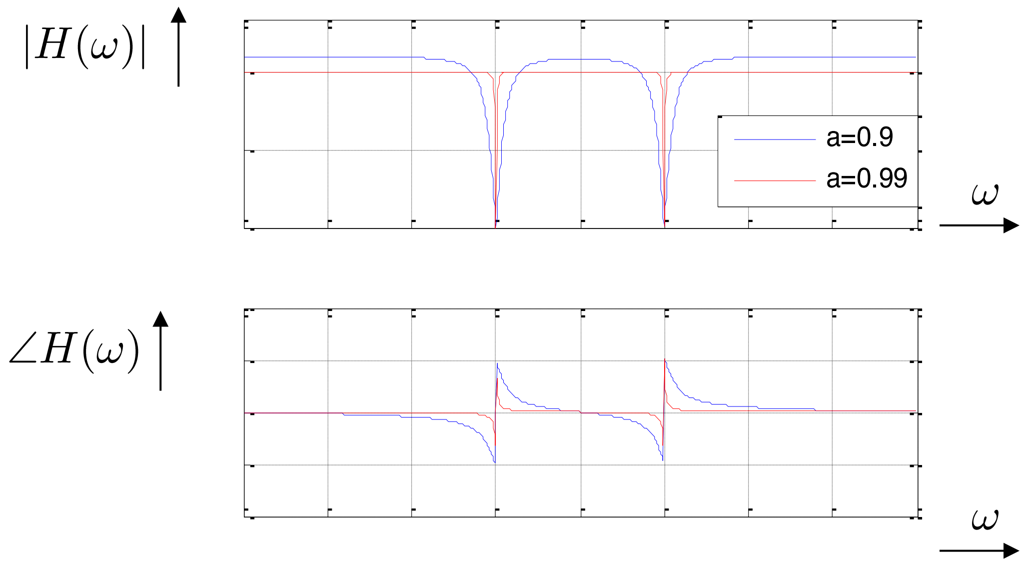
With the FIR notch filter, the bandwidth of the notch is relatively large

Notch Filters

To improve the frequency response characteristics, we can introduce (non-trivial) poles in the system

$$H(z) = \frac{(z - e^{j\omega_0})(z - e^{-j\omega_0})}{(z - ae^{j\omega_0})(z - ae^{-j\omega_0})}$$





All-Pass Filters

An all-pass filter is defined as a system that has a constant magnitude response for all frequencies, that is,

$$|H(\omega)| = 1, \quad 0 \leq \omega \leq \pi$$

All-pass filters find application as phase equalizers; when placed in cascade with a system having an undesired phase response, a phase equalizer is designed to compensate for the poor phase characteristics of the system

General form:

$$H(z) = \frac{a_N^* + a_{N-1}^* z^{-1} + \cdots + a_1^* z^{-N+1} + z^{-N}}{1 + a_1 z^{-1} + \cdots + a_N z^{-N}} = z^{-N} \frac{A_*(z^{-1})}{A(z)}$$

All-Pass Filters

We have

$$|H(\omega)|^2 = H(z)H_*(z^{-1})\Big|_{z=e^{j\omega}} = 1$$

Example: first-order all-pass filter

$$H(z) = z^{-1}\frac{1+p^*z}{1+pz^{-1}} = p^*\frac{z+1/p^*}{z+p}$$

Note that if z_0 is a zero of $H(z)$, $1/z_0^*$ is a pole of $H(z)$. Hence, if $z_0 = re^{j\theta}$, then $1/z_0^* = r^{-1}e^{j\theta}$. The pole and zero sit at the same angle but have reciprocal magnitudes (i.e., they are reflections of each other across the boundary of the unit circle).

All-Pass Filters

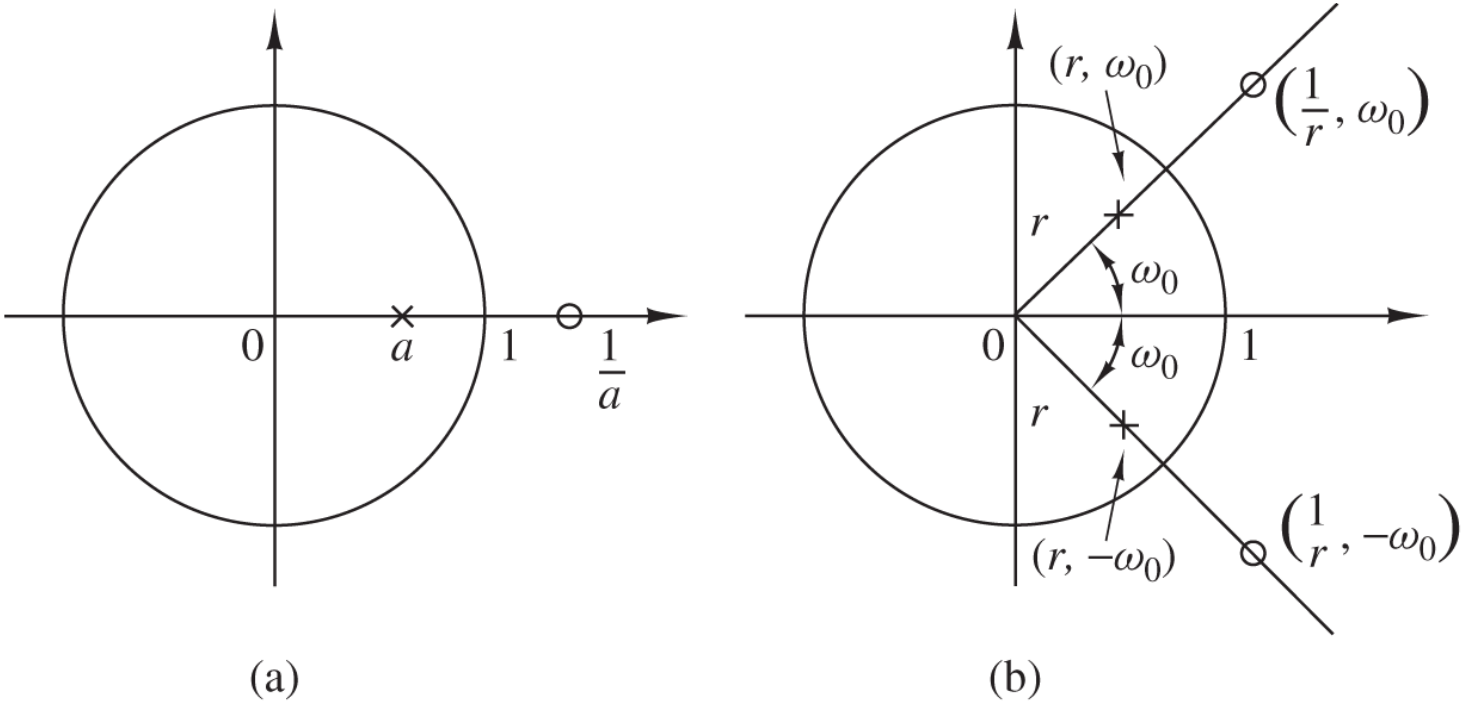


Figure 5.4.16 Pole–zero patterns of (a) a first-order and (b) a second-order all-pass filter.

Minimum-Phase Systems

Consider the two FIR systems:

$$H_1(z) = 1 + \frac{1}{2}z^{-1} = \frac{z + \frac{1}{2}}{z}$$

$$H_2(z) = \frac{1}{2} + z^{-1} = \frac{\frac{1}{2}z + 1}{z}$$

System 1 has a zero at $z = -\frac{1}{2}$ and impulse response $h(0) = 1$ and $h(1) = \frac{1}{2}$. System 2 has a zero at $z = 2$ and impulse response $h(0) = \frac{1}{2}$ and $h(1) = 1$, which is the reverse of that of system 1.

$$H_2(z) = z^{-1}H_1(z^{-1}) \quad \Rightarrow \quad h_2(n) = h_1(-n+1)$$

Minimum-Phase Systems

The frequency responses are given by

$$H_1(\omega) = 1 + \frac{1}{2}e^{-j\omega}, \qquad H_2(\omega) = \frac{1}{2} + e^{-j\omega}$$

Hence,

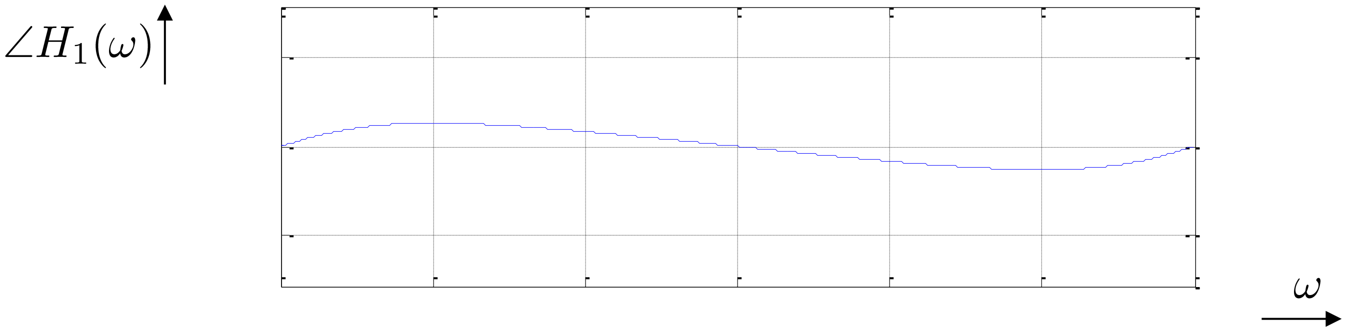
$$|H_1(\omega)| = |H_2(\omega)| = \sqrt{\frac{5}{4} + \cos(\omega)}$$

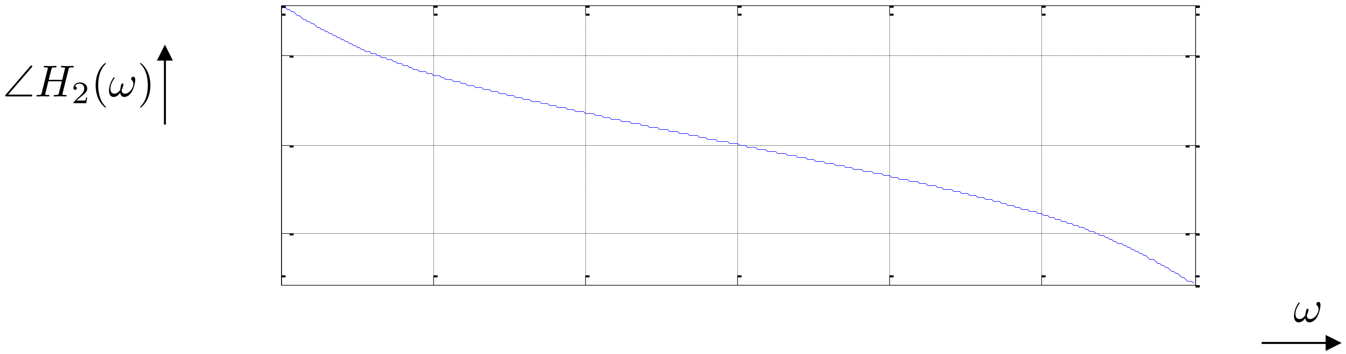
and

$$\angle H_1(\omega) = -\arctan\left(\frac{\sin(\omega)}{2 + \cos(\omega)}\right)$$

$$\angle H_2(\omega) = -\arctan\left(\frac{\sin(\omega)}{\frac{1}{2} + \cos(\omega)}\right)$$

Minimum-Phase Systems





TU Delft

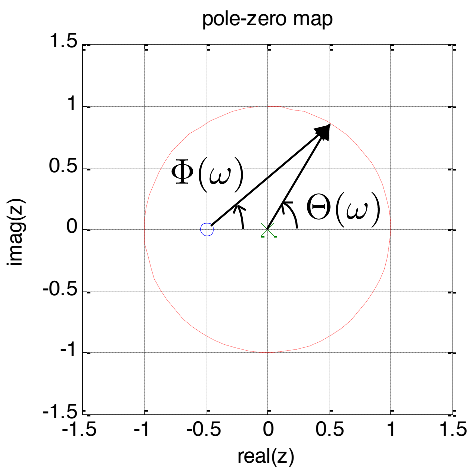
Minimum-Phase Systems

$$\angle H(\omega) = \Theta(\omega) - \Phi(\omega) :$$

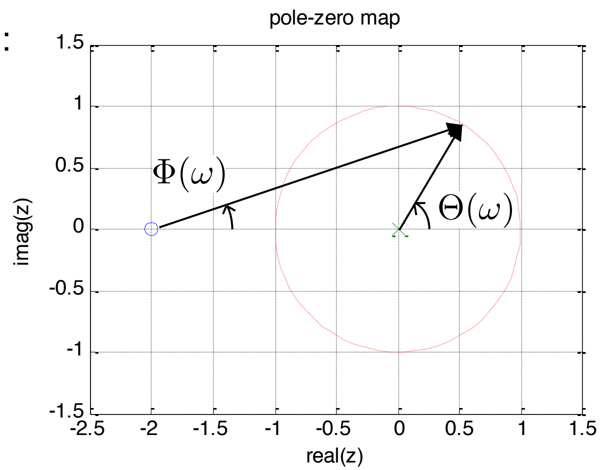
System 1: $\angle H(\pi) - H(0) = 0$

System 2: $\angle H(\pi) - H(0) = \pi$

system 1:



system 2:



Minimum-Phase Systems

- System 1 is called a *minimum phase* system
- System 2 is called a *maximum phase* system

More general, a FIR system of order $M+1$ has M zeros:

$$H(\omega) = b_0(1 - z_1 e^{-j\omega})(1 - z_2 e^{-j\omega}) \cdots (1 - z_M e^{-j\omega})$$

When all zeros are inside the unit circle, the net phase change between $\omega = 0$ and $\omega = \pi$ is zero. On the other hand, when all zeros are outside the unit circle, the net phase change will be $M\pi$, which is the largest possible phase change for a FIR system of order M. When some zeros are outside and some are inside the unit circle, the system is called *mixed phase*.

Minimum-Phase Systems

A stable and causal pole-zero system that is minimum phase has a stable causal inverse which is also minimum phase

$$H(z) = \frac{A(z)}{B(z)} \quad \Rightarrow \quad H^{-1}(z) = \frac{B(z)}{A(z)}$$

Hence, minimum phase ensures the stability of the inverse system $H^{-1}(z)$ and the stability of $H(z)$ implies the minimum-phase property of $H^{-1}(z)$. Mixed-phase and maximum-phase systems result in unstable causal inverse systems.