Sums of Random Variables

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PDF of the Sum of Two Random Variables

Consider two random variables X and Y. What is the pdf of $W = X + Y$?

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Remember Lecture 1 for scalar RVs: Derived Random Variables

- The PDf of an arbitrary derived random variable $Y = g(X)$ is often difficult to calculate. A general procedure is to
  1. Find the CDF of $F_Y(y) = P[Y \leq y]$
  2. Compute the PDF by calculating the $f_Y(y) = \frac{dF_Y(y)}{dy}$
- Special case is a linear transformation:

$$Y = aX + b \Leftrightarrow f_Y(y) = \frac{1}{|a|} f_X\left(\frac{y-b}{a}\right)$$

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$$f_W(w) = \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)dx = \int_{-\infty}^{\infty} f_{X,Y}(w-y, y)dy$$

Proof: Make use of knowledge on how to derive pdf of derived RVs!

$$F_W(w) = P[X + Y \leq w] = \int_{-\infty}^{\infty} \left(\int_{-\infty}^{w-x} f_{X,Y}(x,y)dy\right) dx$$

PDF is given by the derivative of the CDF:

$$\begin{aligned} f_W(w) &= \frac{dF_W(w)}{dw} = \int_{-\infty}^{\infty} \left(\frac{d}{dw}\left(\int_{-\infty}^{w-x} f_{X,Y}(x,y)dy\right)\right) dx \\ &= \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)dx \end{aligned}$$

Use differentiation under the integral sign:

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)dt = f(x,b(x))b'(x) - f(x,a(x))a'(x) + \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt$$

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PDF of the Sum of Two Independent Random Variables

For independent RVs: $f_{X,Y} = f_X f_Y$

So, for two independent RVs X and Y we get

$$\begin{aligned} f_W(w) &= \int_{-\infty}^{\infty} f_{X,Y}(x, w-x)dx \\ &= \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx, \end{aligned}$$

In words: the PDF of the sum of two independent RVs is the convolution of the two PDFs. (Equivalent for discrete RVs).

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Expected value of sums of Random Variables

Consider the sum $W_n = X_1 + W_2 + \cdots + X_n$. The expected value $E[W_n]$ is given by

$$E[W_n] = E[X_1] + E[W_2] + \cdots + E[X_n]$$

The variance of W_n is given by

$$Var[W_n] = \sum_{i=1}^{n}\sum_{j=n}^{n} Cov[X_i, X_j] = \sum_{i=1}^{n} Var[X_i] + 2\sum_{i=1}^{n-1}\sum_{j=i+1}^{n} Cov[X_i, X_j]$$

For uncorrelated variables we thus get $Var[W_n] = \sum_{i=1}^{n} Var[X_i]$

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PDF of the Sum of n Independent Random Variables

What about the pdf of the sum of n independent variables?
For two independent RVs X and Y we have

$$f_W(w) = \int_{-\infty}^{\infty} f_X(x) f_Y(w-x)dx.$$

Calculating such convolutional integrals for large n is tricky...

However, from system theory we know that convolutions in time domain correspond to multiplications in the frequency domain.

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PDF of the Sum of n Independent Random Variables

In probability theory we can also use transforms to replace the convolution between pdfs by the multiplication of their frequency transforms.

The transform of a PDF or PMF is called: the moment generating function (if Laplace transform is used) or, the characteristic function (if Fourier transform is used).

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Moment generating and characteristic function

The characteristic function is the (inverse) Fourier transform of the pdf:

$$\phi_X(u) = \int_{-\infty}^{\infty} f_X(x)e^{jux}dx = E[e^{jux}].$$

- $u \in \mathbb{R}$ is thus the "frequency".

Equivalently we can use the moment generating function (MGF), which is defined as the (inverse) Laplace transform of the pdf for real s:

$$\phi_X(s) = \int_{-\infty}^{\infty} f_X(x)e^{sx}dx = E[e^{sx}].$$

- In this lecture we will follow the book and use the MGF.

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Moment generating function: Properties

For continues RVs:

$$\phi_X(s) = \int_{-\infty}^{\infty} f_X(x)e^{sx}dx = E[e^{sX}].$$

For discrete RVs

$$\phi_X(s) = \sum_{x_i \in S_x} P_X(x_i)e^{sx_i} = E[e^{sX_i}].$$

- $\phi_X(0) = E[e^0] = 1$
- The MGF is extremely useful to calculate the moments $E[X^n]$ of a RV: $E[X^n] = \frac{d^n\phi_X(s)}{ds^n}\Big|_{s=0}$.

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Moment generating function: Properties

Proof: $E[X^n] = \frac{d^n \phi_X(s)}{ds^n}\Big|_{s=0}$.

$$\begin{aligned}
\frac{d^n \phi_X(s)}{ds^n}\Big|_{s=0} &= \frac{d^n}{ds^n} \int_{\infty}^{\infty} e^{sx} f_X(x) dx \Big|_{s=0} \\
&= \int_{\infty}^{\infty} x^n e^{sx} f_X(x) dx \Big|_{s=0} \\
&= \int_{\infty}^{\infty} x^n f_X(x) dx \\
&= E[X^n]
\end{aligned}$$

Example

Let X be exponentially distributed with MGF $\phi(s) = \frac{\lambda}{\lambda - s}$

- $E[X] = \frac{d\phi_X(s)}{ds}\Big|_{s=0} = \frac{\lambda}{(\lambda-s)^2}\Big|_{s=0} = \frac{1}{\lambda}$
- $E[X^2] = \frac{d^2\phi_X(s)}{ds^2}\Big|_{s=0} = \frac{2\lambda}{(\lambda-s)^3}\Big|_{s=0} = \frac{2}{\lambda^2}$
- $E[X^n] = \frac{d^n\phi_X(s)}{ds^n}\Big|_{s=0} = \frac{n!\lambda}{(\lambda-s)^n+1}\Big|_{s=0} = \frac{n!}{\lambda^n}$

MGF of Linearly transformed RVs

The MGF of $Y = aX + b$ is $\phi_Y(s) = e^{sb}\phi_X(as)$.

Proof:

$$\phi_Y(s) = E[e^{s(aX+b)}] = e^{sb}E[e^{saX}] = e^{sb}\phi_X(as)$$

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The MGF for sums of RVs

Consider the sum of independent RVs $X_1, ...x_n$. The moment generating function of

$$W = \sum_{i=1}^{n} X_i$$

is then given by

$$\phi_W(s) = E[e^{sW}] = E[e^{s\sum_{i=1}^{n} X_i}] = E\left[\prod_{i=1}^{n} e^{sX_i}\right] = \prod_{i=1}^{n} \phi_{X_i}(s)$$

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Example

Let $K_1, K_2, ...$ denote a sequence of iid Bernoulli Rvs.
Let $M = K_1 + \cdots + K_n$.

- Find MGF $\phi_K(s)$.
- Find MGF $\phi_M(s)$.
- $E[M]$ and $E[M^2]$

Example

Let $K_1, K_2, ...$ denote a sequence of iid Bernoulli Rvs.
Let $M = K_1 + \cdots + K_n$.

- Find MGF $\phi_K(s)$:
  $\phi_K(s) = E[e^{Ks}] = (1-p)e^0 + pe^s = 1 - p + pe^s$.
- Find MGF $\phi_M(s)$:
  $\phi_M(s) = \prod_{i=1}^{n} \phi_{K-i}(s) = (1 - p + pe^s)^n$
- $E[M]$ and $E[M^2]$:
  $E[M] = \frac{d}{ds} (1 - p + pe^s)^n \mid_{s=0} = npe^s (1 - p + pe^s)^{n-1} \mid_{s=0} =$
  np
  $E[M^2] = \frac{d^2}{ds^2} npe^s (1 - p + pe^s)^{n-1} \mid_{s=0} = n(n-1)(pe^s)^2 (1 - p + pe^s)^{n-2} +$
  $npe^s(1 - p + pe^s)^{n-1} \mid_{s=0} = n(n-1)p^2 + np$

The sum of Gaussian Rvs

Let $X_1, X_2, ...X_n$ denote a sequence of Gaussian Rvs.
What is the distribution of $W = X_1 + X_2 + ... + X_n$?

$$\begin{aligned} \phi_W(s) &= \phi_{X_1}(s)\phi_{X_2}(s)\dots\phi_{X_n}(s) \\ &= e^{s\mu_1+\sigma_1^2 s^2/2} e^{s\mu_2+\sigma_2^2 s^2} \dots e^{s\mu_n+\sigma_n^2 s^2} \\ &= e^{s(\mu_1+\mu_2+\cdots+\mu_n)+(\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2)s^2/2}. \end{aligned}$$

Distribution of W is thus again Gaussian with mean $\mu_1+\mu_2+\cdots+\mu_n$ and variance $\sigma_1^2 + \sigma_2^2 + \cdots + \sigma_n^2$.

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The Central Limit Theorem

Given a sequence of iid random variables $X_1, X_2, \dots, X_n$, each with expected value μ_X and variance σ_X^2.

The CDF of $Z_n = (\sum_{i=1}^n X_i - n\mu_X)/\sqrt{n\sigma_X^2}$ then has the property:

$$\lim_{n\to\infty} F_n = \Phi(z).$$

In other words, if n becomes "large", the distribution of the sum of iid random variables approaches a Gaussian distribution.

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The Central Limit Theorem

- Although the theorem states that $\lim_{n\to\infty}$, it already holds for a relatively small number of random variables.
- There do exist "weaker" versions of the central limit theorem where the sum should consist not even of IID Rvs.

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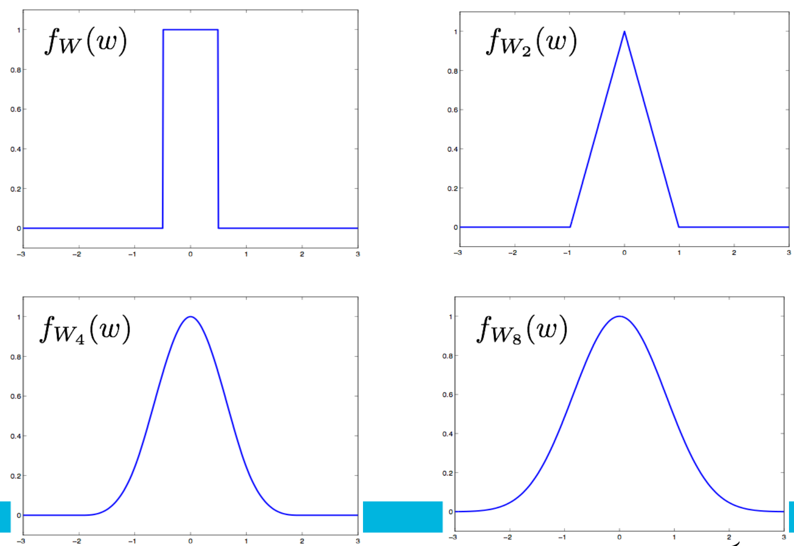
The Central Limit Theorem: illustration



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