Signal Processing EE2S31

July 2015

Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (13 points)

A joint probability density function of the random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c & \text{for } 0 \leq x \leq 2, \text{ and } 0 \leq y \leq 2-x \\ 0 & \text{elsewhere.} \end{cases}$$

(2 p) (a) Calculate constant c.

(2 p) (b) Compute the marginal pdfs of X and Y and show that they are given by

$$f_X(x) = 2c - cx \text{ for } 0 \leq x \leq 2,$$

and

$$f_Y(y) = 2c - cy \text{ for } 0 \leq y \leq 2.$$

(1 p) (c) Explain whether X and Y are independent.

(1 p) (d) Compute the conditional pdf $f_{X|Y}(X|Y)$.

(2 p) (e) Compute the expected value $E[X]$.

In an experiment we observe realizations of random variable Y, while we want to make an estimate of X. To do so, we can make use of linear or non-linear estimators.

(2 p) (f) Proof that the MMSE estimate for X is given by $E[X|y]$.

(2 p) (g) Calculate $E[X|y]$.

(1 p) (h) As an alternative, one could also calculate the linear MMSE estimator. Given an expression for the linear MMSE estimator in terms of expected values (there is no need to calculate them) and argue which of the two estimators (the linear or the non-linear) leads to a smaller mean-squared error.

Question 2 (14 points)

Let $Y[n]$ be an auto-regressive process that is given by the following input-output relation:

$$Y[n] = aY[n-1] + X[n].$$

The input process $X[n]$ has a variance σ_X^2.

(1 p) (a) Give an expression for the autocorrelation function $R_X[k]$ of the input.

(1 p) (b) Explain for both $Y[n]$ and $X[n]$ whether they form an IID process.

(2 p) (c) Determine the system function $H(z)$ and calculate the impulse response.

(2 p) (d) Calculate the value for a in the above input-output relation, such that the cross-correlation between input and output becomes $R_{XY}[k] = \sigma_X^2(-\frac{1}{4})^n u[k]$.

For the following questions, assume that the impulse response is given by

$$h[n] = \left(\frac{1}{3}\right)^n u[n].$$

The auto-correlation $R_Y[k]$ of the output $Y[n]$ is given by

$$R_Y[k] = h[k] * h[-k] * R_X[k].$$

The convolution $f[k] = h[k] * h[-k]$ can also be seen as the concatenation of two filters. One with impulse response $h[k]$ and one with impulse response $h[-k]$. The autocorrelation $R_X[k]$ and $R_Y[k]$ are then related to each other by a convolution of $R_X[k]$ with an overall filter that has impulse response $f[k]$.

(3 p) (e) Show by explicitly calculating the convolutions that $f[k] = \frac{9}{8}(\frac{1}{3})^{|n|}$.
Hint: To show this, you might want to use the generalized expression for the geometric series, given by

$$\sum_{k=a}^{b} r^k = \frac{r^a - r^{b+1}}{1-r}.$$

(2 p) (f) Give the system function $F(z)$ of the overall filter and plot its pole-zero diagram.

(1 p) (g) Argue whether or not this is a stable filter.

(2 p) (h) Calculate the autocorrelation function $R_Y[k]$.

Question 3 (13 points)

Consider the following pole-zero map of a causal linear time-invariant system having two zeros at the origin and two poles at $z = a$.

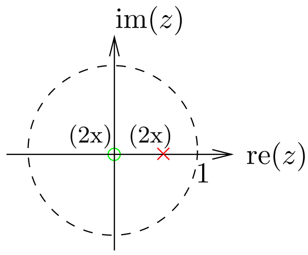

Figure 1: Pole-zero map.

(2 p) (a) Determine the corresponding system function. Is this function unique?

(2 p) b) Is this system BIBO stable? Motivate your answer.

(3 p) (c) Sketch the magnitude and phase response of the system and compute the values at $\omega = 0$ and $\omega = \pm\pi$.

(3 p) (d) Compute the inverse $\mathcal{Z}$-transform of the system function $H(z)$ in case the region of convergence is $|z| > |a|$.

(3 p) (e) Compute the step response $y(n)$ of the system, with $y(n) = y_{\text{tr}}(n) + y_{\text{ss}}(n)$, where $y_{\text{tr}}(n)$ and $y_{\text{ss}}(n)$ are the *transient* and *steady-state* response of the system, respectively.

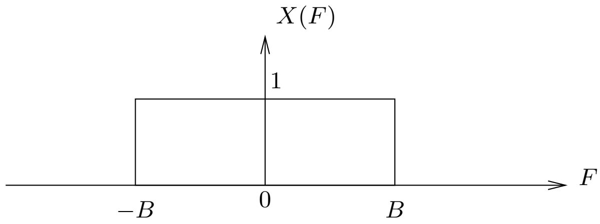

Figure 2: Spectrum $X(F)$.

Question 4 (16 points)

Consider a signal x of which its spectrum is given as depicted in Figure 2.

(1 p) (a) Is this signal x a discrete-time or continuous-time signal? Motivate your answer.

(1 p) (b) Is the signal x a periodic or a non-periodic signal? Motivate your answer.

Suppose we sample the signal $x(t)$ with sampling frequency $F_s = 3B$. Afterwards we can reconstruct the (analog) signal out of its samples, say $x(n)$, by a proper interpolation scheme.

(1 p) (c) What is the relation between $x(t)$ and $x(n)$?

(2 p) (d) Sketch the spectrum of $x(n)$.

(2 p) (e) What is the minimum sampling frequency such that we can perfectly reconstruct $x(t)$ out of its samples $x(n)$ and give the corresponding reconstruction formula.

Assume we want to reconstruct the continuous-time signal using the interpolation function as depicted in Figure 3.

(2 p) (f) How does the continuous-time reconstructed signal look like if we reconstruct using the above mentioned interpolation function?

The reconstruction formula can be expressed in the Fourier domain and is given by

$$\tilde{X}(F) = X(F)G(F),$$

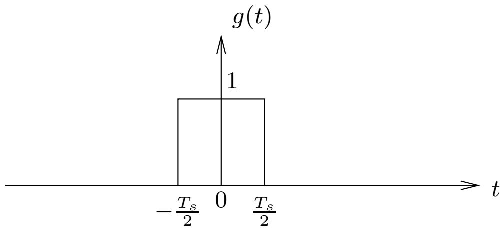

Figure 3: Interpolation function.

where $\tilde{X}$ is the spectrum of the reconstructed signal, X is the spectrum of the discrete-time signal $x(n)$ and G is the Fourier transform of the interpolation function $g(t)$.

(3 p) (g) Compute $G(F)$.

(2 p) (h) Sketch the spectrum of $\tilde{X}(F)$

(2 p) (i) Can we obtain a perfect reconstruction of $x(t)$ by using the interpolation function depicted in Figure 3? Motivate your answer.