$\mathcal{Z}$-Transform
(Recap)

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$\mathcal{Z}$-Transform

Why the $\mathcal{Z}$-transform?

- Tool that greatly simplifies the analysis of LTI systems
- Powerfull means of characterizing LTI systems and its response to various input signals by its pole-zero locations
- Generalization of the discrete-time Fourier transform
- A systematic approach for solving linear constant-coeffcient difference equations
- The role played by the $\mathcal{Z}$-transform in the solution of difference equations corresponds to that played by the Laplace transforms in the solution of differential equations

$\mathcal{Z}$-Transform

The $\mathcal{Z}$-transform of a discrete-time signal x is defined as

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

- $X(z)$ is a complex function of a complex variable z
- The $\mathcal{Z}$-transform is an infinite power series (Laurent series expansion of $X(z)$ around $z = 0$)
- The *region of convergence* (ROC) of $X(z)$ is the set of all z-values for which the series converges. That is, for which $X(z)$ is finite.

$\mathcal{Z}$-Transform

Example:

1. $x(n) = \delta(n-3)$

   $X(z) = z^{-3}$, ROC the entire z-plane except $z = 0$

2. $x(n) = \delta(n+2)$

   $X(z) = z^{2}$, ROC entire z-plane except $z = \infty$

3. $x(n) = \{\cdots 0, 2, 0, \underset{\uparrow}{0}, 1, 4, 1, 0, \cdots\}$

   $X(z) = 2z^{2} + z^{-1} + 4z^{-2} + z^{-3}$, ROC the entire z-plane except $z = 0$ and $z = \infty$

$\mathcal{Z}$-Transform

For finite-duration signals, the ROC is the entire z-plane, except possibly the points $z = 0$ and/or $z = \infty$

Example (con't):

4. $x(n) = a^n u(n)$

$$X(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} \left(az^{-1}\right)^n = \frac{1}{1 - az^{-1}} \quad \text{if } |az^{-1}| < 1$$

Hence, the ROC is $|z| > |a|$

$\mathcal{Z}$-Transform

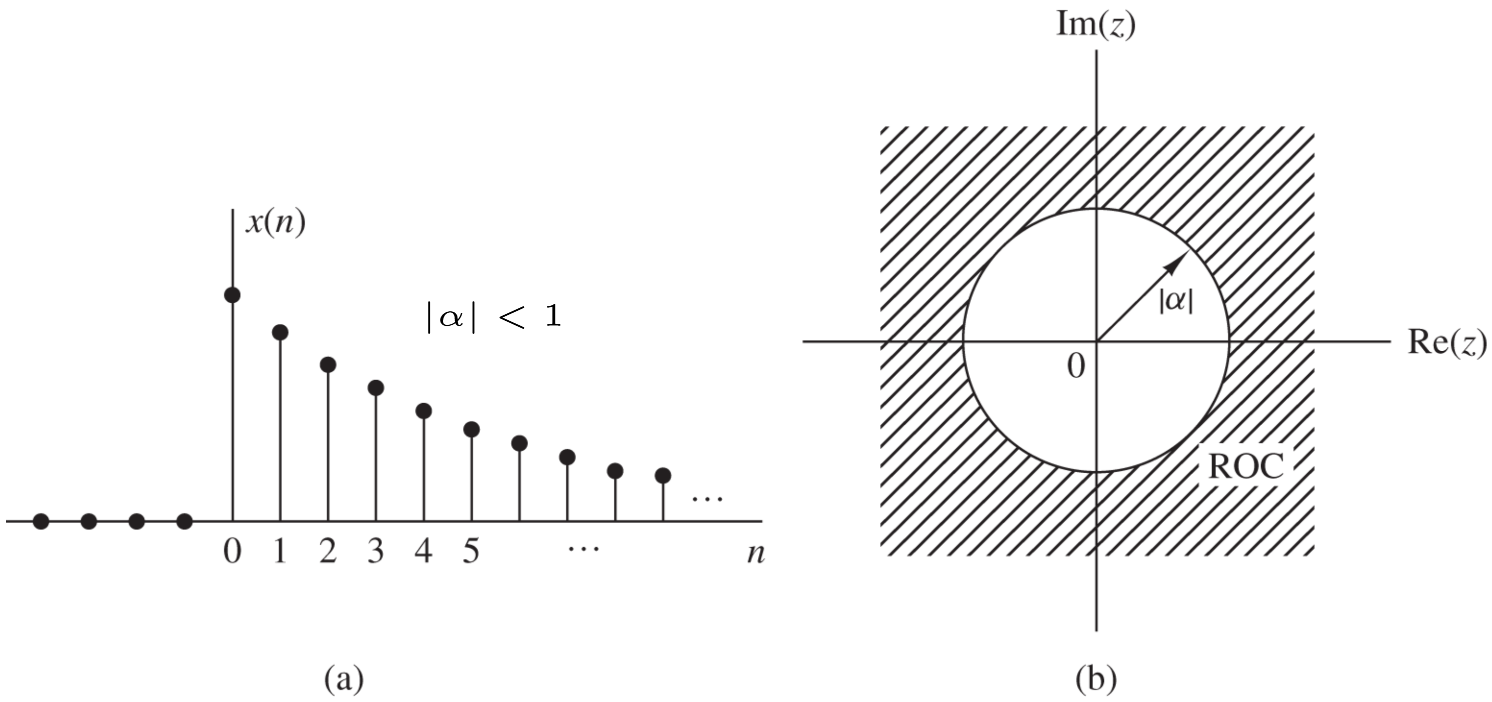
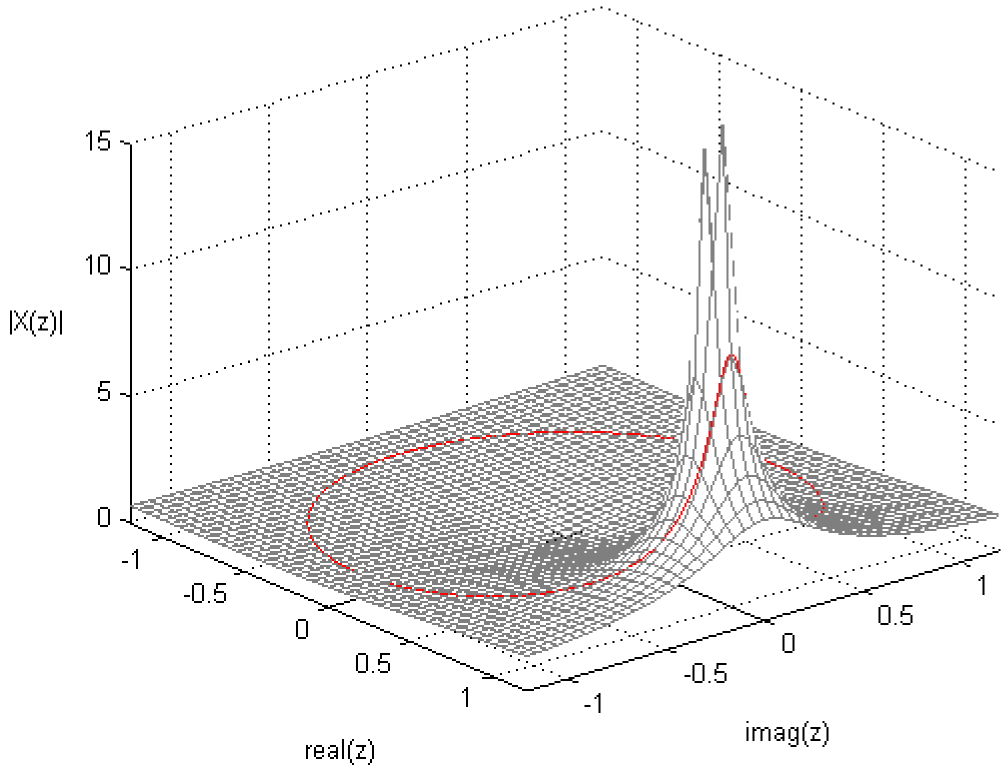


Figure 3.1.2 The exponential signal $x(n) = \alpha^n u(n)$ (a), and the ROC of its z-transform (b).

$\mathcal{Z}$-Transform

Example: $a = 0.9$



$\mathcal{Z}$-Transform

Example (con't):

5. $x(n) = -a^n u(-n-1)$

$$X(z) = \sum_{n=-\infty}^{-1} -a^n z^{-n} = -\sum_{n=1}^{\infty} (a^{-1}z)^n$$

$$= 1 - \sum_{n=0}^{\infty} (a^{-1}z)^n = \frac{1}{1-az^{-1}} \quad \text{if } |a^{-1}z| < 1$$

same as in Example 4!

Hence, the ROC is $|z| < |a|$

but different ROC!

$\mathcal{Z}$-Transform

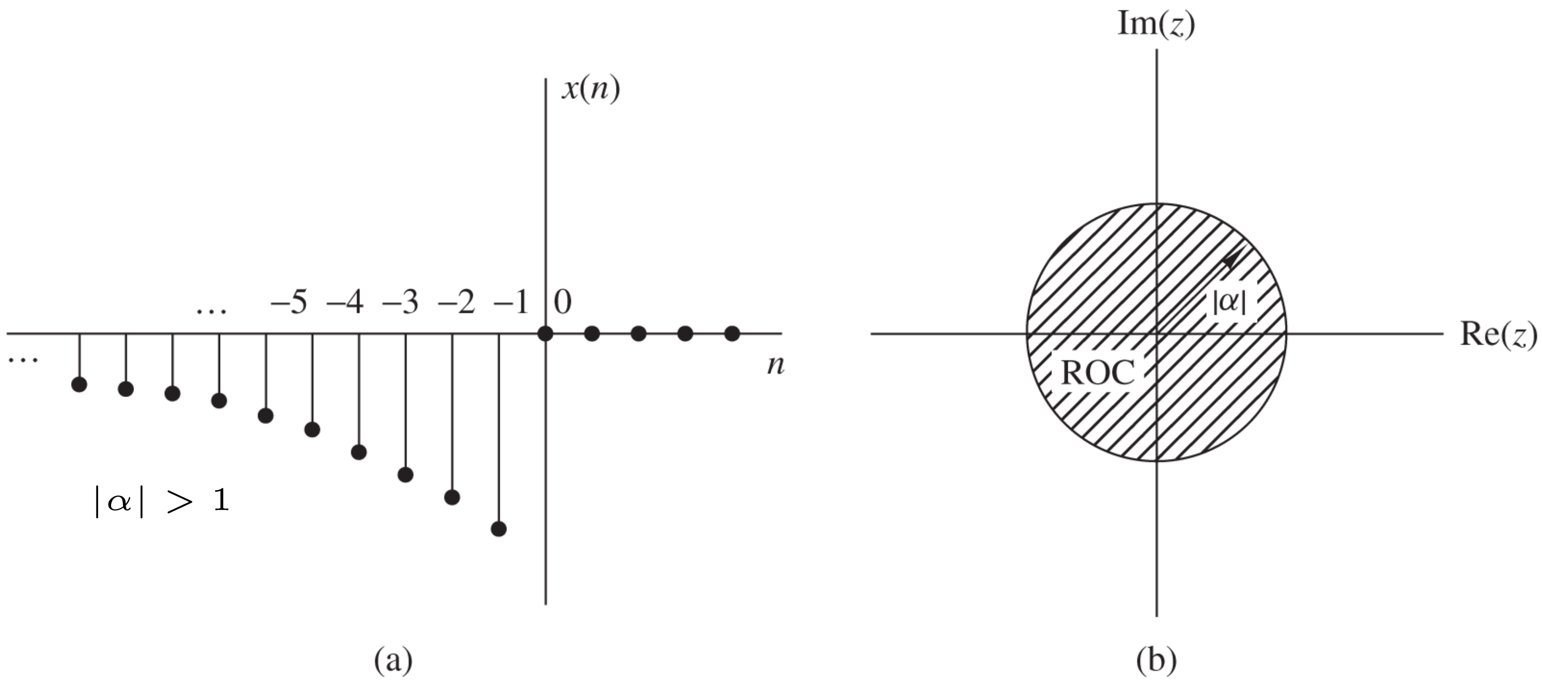


Figure 3.1.3 Anticausal signal $x(n) = -\alpha^n u(-n-1)$ (a), and the ROC of its z-transform (b).

$\mathcal{Z}$-Transform

Let $z = re^{j\theta}$. We then have

$$
\begin{aligned}
|X(z)| &= \left| \sum_{n=-\infty}^{\infty} x(n) r^{-n} e^{-j\theta n} \right| \\
&\leq \sum_{n=-\infty}^{\infty} |x(n) r^{-n}| \\
&= \underbrace{\sum_{n=-\infty}^{-1} |x(n) r^{-n}|}_{\text{anti-causal}} + \underbrace{\sum_{n=0}^{\infty} |x(n) r^{-n}|}_{\text{causal}}
\end{aligned}
$$

Convergence if both series converge.

$\mathcal{Z}$-Transform

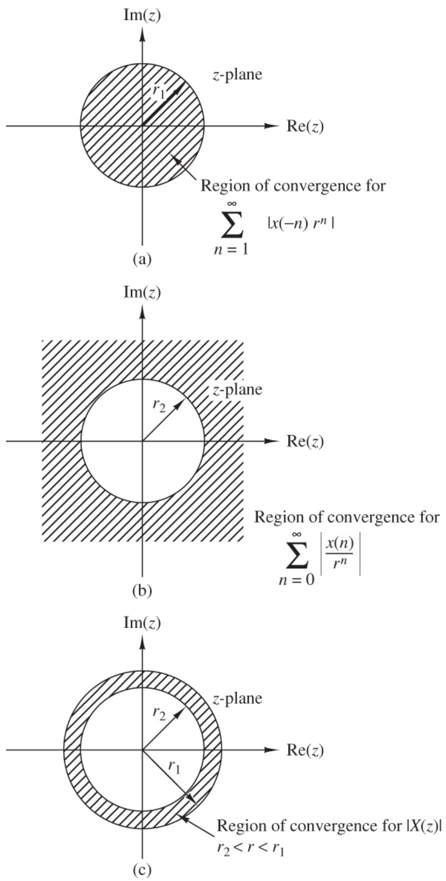


Figure 3.1.1 Region of convergence for $X(z)$ and its corresponding causal and anticausal components.

$\mathcal{Z}$-Transform

Conclusions so far:

- The closed-form expression of the $\mathcal{Z}$-transform does not uniquely specify the signal in the time domain
- The ambiguity can be resolved only if, in addition to the closed-form expression, the ROC is specified
- The ROC of a causal signal is the exterior of a circle of some radius r_2, while the ROC of an anti-causal signal is the interior of a circle of some radius r_1
- In general, the ROC is given by $r_2 < |z| < r_1$.

Properties of the $\mathcal{Z}$-Transform

- Linearity:

$$x(n) = a_1 x_1(n) + a_2 x_2(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z) = a_1 X_1(z) + a_2 X_2(z)$$

- Time shifting:

$$x(n-k) \stackrel{\mathcal{Z}}{\longleftrightarrow} z^{-k} X(z)$$

- Time reversal:

$$x(-n) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z^{-1})$$

- Differentiation:

$$n x(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} -z \frac{dX(z)}{dz}$$

Properties of the $\mathcal{Z}$-Transform

- Convolution:

$$\sum_{k=-\infty}^{\infty} x(k)y(n-k) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z)Y(z)$$

- Correlation:

$$\sum_{k=-\infty}^{\infty} x(k)y(n+k) \stackrel{\mathcal{Z}}{\longleftrightarrow} X(z^{-1})Y(z)$$

- Multiplication:

$$x(n)y(n) \stackrel{\mathcal{Z}}{\longleftrightarrow} \frac{1}{2\pi j} \oint_C X(v)Y\left(\frac{z}{v}\right) v^{-1} dv$$

Rational $\mathcal{Z}$-Transforms

Rational function (ratio of two polynomials in z (or z^{-1}))

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}}$$

- The *zeros* of a z-transform $X(z)$ are the values of z for which $X(z) = 0$.
- The *poles* of a z-transform $X(z)$ are the values of z for which $X(z) = \infty$.

Rational $\mathcal{Z}$-Transforms

If $a_0 \neq 0$ and $b_0 \neq 0$:

$$X(z) = \frac{b_0 z^{-M}\left(z^M + (b_1/b_0)z^{M-1} + \cdots + b_M/b_0\right)}{a_0 z^{-N}\left(z^N + (a_1/a_0)z^{N-1} + \cdots + a_N/a_0\right)}$$

$$= \frac{b_0}{a_0} z^{N-M} \frac{(z-z_1)(z-z_2)\cdots(z-z_M)}{(z-p_1)(z-p_2)\cdots(z-p_N)}$$

- $X(z)$ has M finite zeros $z_1, \ldots, z_M$
- $X(z)$ has N finite poles $p_1, \ldots, p_N$
- $X(z)$ has $|N-M|$ zeros ($N > M$) or poles ($N < M$) at the origin

Rational $\mathcal{Z}$-Transforms

Example:

$$x(n) = a^n u(n) \quad \Rightarrow \quad X(z) = \frac{1}{1 - az^{-1}} = \frac{z}{z - a}$$

$X(z)$ has one zero at $z = 0$ and one pole at $z = a$. Note that the pole is not included in the ROC.

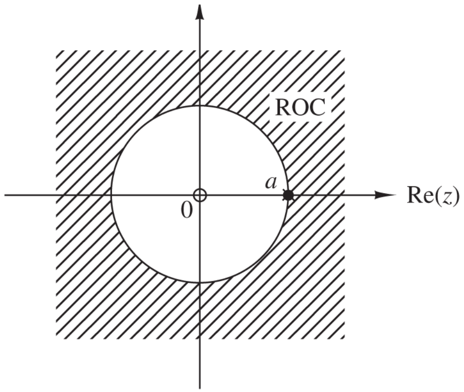


Figure 3.3.1 Pole–zero plot for the causal exponential signal $x(n) = a^n u(n)$.

Stability

BIBO stable: $\sum_n |h(n)| < \infty$

We have

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)z^{-n}| = \sum_{n=-\infty}^{\infty} |h(n)||z^{-n}|$$

When evaluated on the unit circle, that is $|z| = 1$:

$$|H(z)| \leq \sum_{n=-\infty}^{\infty} |h(n)|$$

BIBO stable if the unit circle is contained in the ROC!

Stability: Causal Systems

- For BIBO stability, the ROC includes the unit circle, whether h is causal or not.

Causal systems:

- For causal systems, the ROC is the exterior of some circle of radius r
- Causal and stable systems: $|z| > r, \ r < 1$
- Since the ROC cannot contain any poles of $H(z)$, it follows that a *causal linear time-invariant system* is BIBO stable if and only if all poles of $H(z)$ are inside the unit circle!

Linear Time-Invariant System

Convolution property:

$$Y(z) = H(z)X(z)$$

so that

$$H(z) = \frac{Y(z)}{X(z)}$$

In particular useful when the system is described by a linear constant-coefficient difference equation:

$$y(n) = \sum_{k=1}^{N} a_k y(n-k) + \sum_{k=0}^{M} b_k x(n-k)$$

Linear Time-Invariant System

Example: step response of first-order recursive system (initially in rest):

$$y(n) = \left(-\frac{a^{n+1}}{1-a} + \frac{1}{1-a} \right) u(n)$$

We then have

$$Y(z) = \frac{-a}{1-a}\frac{z}{z-a} + \frac{1}{1-a}\frac{z}{z-1} = \frac{z^2}{(z-a)(z-1)}$$

Since $X(z) = \frac{z}{z-1}$, we conclude that $H(z) = \frac{Y(z)}{X(z)} = \frac{z}{z-a}$ and thus $h(n) = a^n u(n)$.

Inversion of the $\mathcal{Z}$-Transform

Inverse $\mathcal{Z}$-transform:

$$x(n) = \frac{1}{2\pi j} \oint_C X(z) z^{n-1} dz$$

Three methods for evaluation of the inverse Z-transform:

1. Evaluation by contour integration
2. Partial-fraction expansion
3. Power-series expansion (long division)

Inversion by Contour Integration

Inverse $\mathcal{Z}$-transform:

$$x(n) = \frac{1}{2\pi j}\oint_C \overbrace{X(z)z^{n-1}}^{f(z)} dz = \sum_{k=1}^{N} \operatorname*{Res}_{z=z_k} \overbrace{X(z)z^{n-1}}^{f(z)}$$

Example:

$$X(z) = \frac{1}{1-az^{-1}}, \quad |z| > |a|$$

We have

$$x(n) = \frac{1}{2\pi j}\oint_C \frac{z^{n-1}}{1-az^{-1}} dz = \frac{1}{2\pi j}\oint_C \frac{z^n}{z-a} dz$$

Inversion by Contour Integration

Two cases:

$n \geq 0$: z^n has no poles inside C.

$$x(n) = \operatorname*{Res}_{z=a} \frac{z^n}{z-a} = (z-a)\frac{z^n}{z-a}\bigg|_{z=a} = a^n$$

$n < 0$: $x(n) = 0$ since the sequence is causal (ROC is the exterior of a circle)

Hence

$$x(n) = a^n u(n)$$

Inversion by Partial-Fraction Expansion

If $X(z)$ is a *proper* rational function, that is, $a_N \neq 1$ and $M < N$, we can decompose $X(z)$ into terms which can be inversely transformed by table lookup.

In case of distinct poles, we decompose $X(z)$ into

$$X(z) = \sum_{k=1}^{N} \frac{zA_k}{(z - z_k)}$$

where the constants A_k are found by

$$A_k = (z - z_k)\frac{X(z)}{z}\bigg|_{z=z_k}$$

Inversion by Partial-Fraction Expansion

Example:

$$X(z) = \frac{z}{(z+1)(z-\frac{1}{2})}$$

We decompose $X(z)$ as

$$X(z) = A\frac{z}{z+1} + B\frac{z}{z-\frac{1}{2}},$$

where the individual contributions are found by table lookup:

$$\frac{z}{z+1} \overset{\mathcal{Z}}{\longleftrightarrow} (-1)^n u(n), \qquad \frac{z}{z-\frac{1}{2}} \overset{\mathcal{Z}}{\longleftrightarrow} \left(\frac{1}{2}\right)^n u(n)$$

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Inversion by Partial-Fraction Expansion

Example (cont'): The constants A and B are found as

$$A = (z+1)\frac{X(z)}{z}\bigg|_{z=-1} = -\frac{2}{3}$$

and

$$B = (z-\frac{1}{2})\frac{X(z)}{z}\bigg|_{z=\frac{1}{2}} = \frac{2}{3}$$