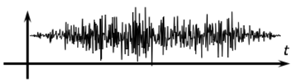
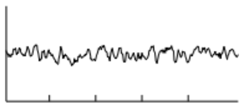
Random signal processing

Stochastic Processes for EE (EE2511)
Lecture 5

Summary Filtered WSS Process





$X(t) \longrightarrow$ linear system (LTI) h(n), h(t) $\longrightarrow Y(t)$

μ_X $\qquad\qquad$ $\mu_Y = \mu_X H_0$

$R_X(k)$ $\qquad\qquad$ $R_Y(k) = h(k) * h(-k) * R_X(k)$

- Time continuous: $R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$

What can we do with Stoch. Proc.?

- System identification
- Modeling of real signals (e.g. speech)
- Prediction (next week)

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Today's Agenda

- Moving average (MA) and autoregressive (AR) processes
- Modeling of speech signals as an AR process
- Fourier Transform of the autocorrelation function: Power Spectral Density

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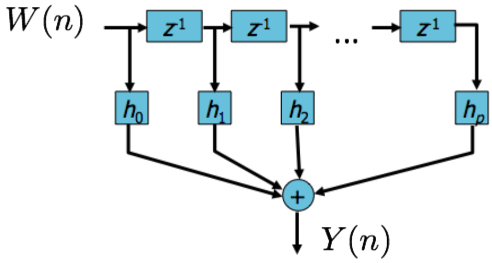
Two typical LTI systems

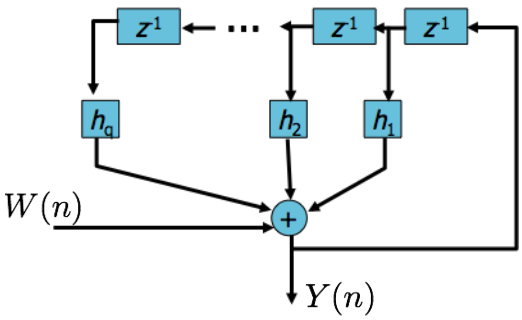
- Tapped delay line or Finite Impulse Response Filter (FIR) system



$$Y(n) = h_0 W(n) + h_1 W(n-1) + \ldots + h_p W(n-p)$$

Two typical LTI systems

- Infinite Impulse Response (IIR) system



$$Y(n) = h_1 Y(n-1) + h_2 Y(n-2) + \ldots + h_q Y(n-q) + W(n)$$

Terminology

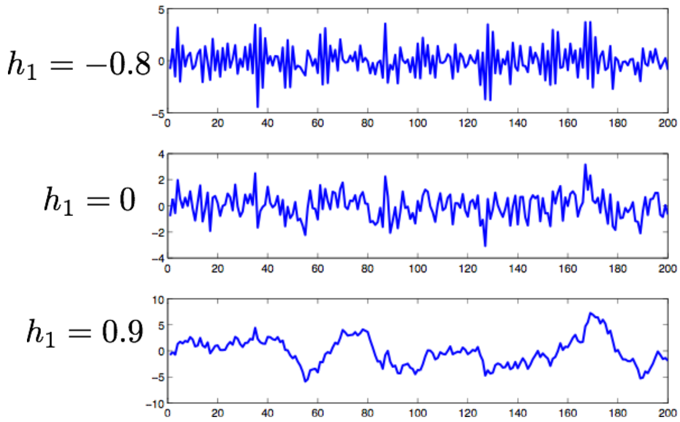
If we input white noise into these LTI systems, then

- output of FIR filter is called a *moving average* (MA) process
- output of IIR filter is called an *autoregressive* (AR) process

- Combination of the two is called ARMA process
- Many speech processing systems (including speech compression in GSM) use AR model

Example of AR realizations

- First order AR process $Y(n) = h_1 Y(n-1) + W(n)$



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Autocorrelation Functions (MA-1)

- First order MA process

$$Y(n) = h_0 W(n) + h_1 W(n-1)$$

- Autocorrelation function (*Method 1*)

$$\begin{aligned} R_Y(k) &= E[Y(n)Y(n+k)] \\ &= E[(h_0 W(n) + h_1 W(n-1))(h_0 W(n+k) + h_1 W(n+k-1))] \\ &= h_0^2 E[W(n)W(n+k)] + h_0 h_1 E[W(n)W(n+k-1)] \\ &\quad + h_0 h_1 E[W(n-1)W(n+k)] + h_1^2 E[W(n-1)W(n+k-1)] \\ &= (h_0^2 + h_1^2) R_W(k) + h_0 h_1 R_W(k+1) + h_0 h_1 R_W(k-1) \end{aligned}$$

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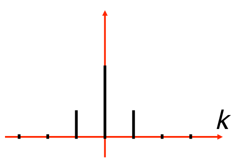
Autocorrelation Functions (MA-1)

- First order MA process

$$Y(n) = h_0 W(n) + h_1 W(n-1)$$

- Autocorrelation function (*Method 1*)

$$\begin{aligned} R_Y(k) &= E[Y(n)Y(n+k)] \\ &= (h_0^2 + h_1^2) R_W(k) + h_0 h_1 R_W(k+1) + h_0 h_1 R_W(k-1) \\ &= \begin{cases} (h_0^2 + h_1^2)\sigma_W^2 & k = 0 \\ h_0 h_1 \sigma_W^2 & k = 1 \\ h_0 h_1 \sigma_W^2 & k = -1 \\ 0 & \text{elsewhere} \end{cases} \end{aligned}$$



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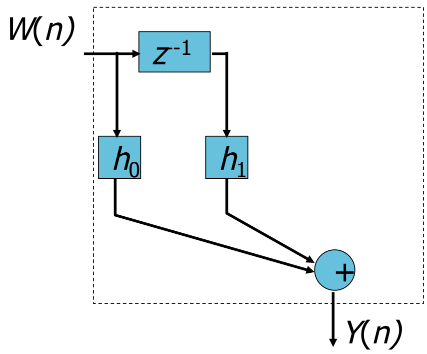
Autocorrelation Functions (MA-1)

- First order MA process

$$Y(n) = h_0 W(n) + h_1 W(n-1)$$

- Autocorrelation function (*Method 2*)



Linear system with impulse response:
…, 0, 0, 0, h_0, h_1, *0, 0, 0, …*

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Autocorrelation Functions (MA-1)

- First order MA process

$$Y(n) = h_0 W(n) + h_1 W(n-1)$$

- Autocorrelation function (*Method 2*)

$$Y(n) = (\ldots,0,0,h_0,h_1,0,0,\ldots) * W(n) = h(n) * W(n)$$

$$\begin{aligned} R_Y(k) &= h(k) * h(-k) * R_W(k) \\ &= (..,0,0,h_0 h_1, h_0^2 + h_1^2, h_0 h_1, 0,0,\ldots) * \sigma_W^2 \delta(k) \end{aligned}$$

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Autocorrelation Functions (MA-1)

Autocorrelation function (*Method 2*)

$$R_Y(k) = h(k) * h(-k) * R_W(k)$$

$$\begin{aligned}
f(k) &= h(k) * h(-k) \\
&= \sum_n h(n)h(-(k-n)) \\
&= h(0)h(-k) + h(1)h(-k+1) \\
R_Y(k) &= \sum_n \left(h(0)h(-n) + h(1)h(-n+1)\right)\sigma_W^2 \delta(k-n) \\
&= \sigma_W^2 h(0)h(-k) + \sigma_W^2 h(1)h(-k+1) \\
&= \begin{cases} (h(0)^2 + h(1)^2)\sigma_W^2 & \text{if } k = 0 \\ h(0)h(1)\sigma_W^2 & \text{if } k = 1 \\ h(0)h(1)\sigma_W^2 & \text{if } k = -1 \end{cases}
\end{aligned}$$

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Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

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Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

- Autocorrelation function (*Method 1*)

$$\begin{aligned} R_Y(k) &= E[Y(n)Y(n+k)] \\ &= E[Y(n)(h_1 Y(n+k-1) + W(n+k))] \\ &= h_1 E[Y(n)Y(n+k-1)] + E[Y(n)W(n+k)] \\ &= h_1 R_Y(k-1) + E[Y(n)]E[W(n+k)] \\ &= h_1 R_Y(k-1) \qquad (k>0) \end{aligned}$$

independent

(Similarly for k<0: $R_Y(k) = h_1 R_Y(k+1) \qquad (k<0)$)

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Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

- Autocorrelation function (*Method 1*)

$$\begin{aligned} R_Y(0) &= E[(h_1 Y(n-1) + W(n))(h_1 Y(n-1) + W(n))] \\ &= h_1^2 E[Y(n-1)^2] + 2h_1 E[Y(n-1)W(n)] + E[W(n)^2] \\ &= h_1^2 R_Y(0) + 2h_1 E[Y(n-1)]E[W(n)] + R_w(0) \\ &= h_1^2 R_Y(0) + \sigma_w^2 \\ &\Downarrow \end{aligned}$$

$$R_Y(0) = \frac{1}{1-h_1^2}\sigma_w^2$$

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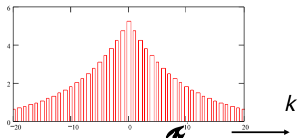
Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

- Autocorrelation function (*Method 1*)

$$\left.\begin{array}{ll} R_Y(k) = h_1 R_Y(k-1) & k > 0 \\ R_Y(k) = h_1 R_Y(k+1) & k < 0 \\ R_Y(0) = \dfrac{1}{1-h_1^2}\sigma_W^2 & \end{array}\right\} \Rightarrow R_Y(k) = \frac{\sigma_W^2}{1-h_1^2} h_1^{|k|}$$



h_1=0.9

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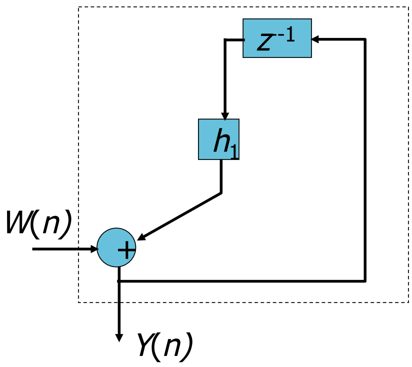
Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

- Autocorrelation function (*Method 2*)



Linear system with impulse response:

$$\ldots,0,1,h_1,h_1^2,h_1^3,\ldots\ldots$$
$$= \ldots,0,0,0,h_1^n$$

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Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

- Autocorrelation function (*Method 2*)

$$Y(n) = (\ldots,0,0,h_1^k) * W(n) = h(n) * W(n)$$

$$\begin{aligned} R_Y(k) &= h(k) * h(-k) * R_W(k) \\ &= (\ldots, h_1^3, h_1^2, h_1, 1, h_1, h_1^2, h_1^3, \ldots) * \sigma_W^2 \delta(k) \\ &= \frac{\sigma_W^2}{1-h_1^2} h_1^{|k|} \end{aligned}$$

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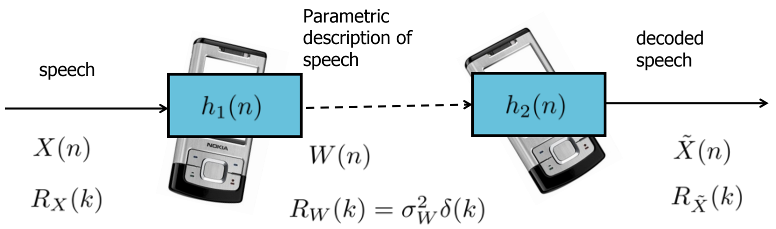
What About Speech?!?



What to transmit?

The statistical description of the speech process

Choose $h_1(n)$ such that the output is uncorrelated with minimum variance.

$W(n) = h_1(n) * X(n)$
Then transmit $h_1(n)$ (= inverse filter of $h_2(n)$) and σ_W^2

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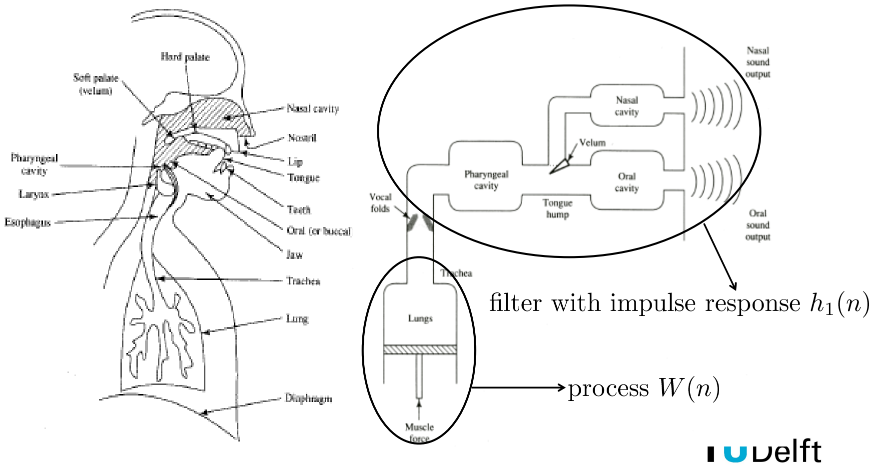
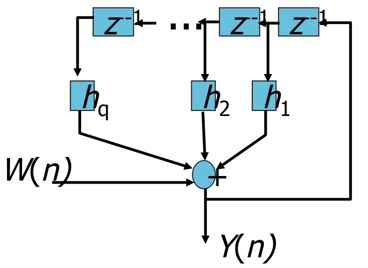
Speech Production Model

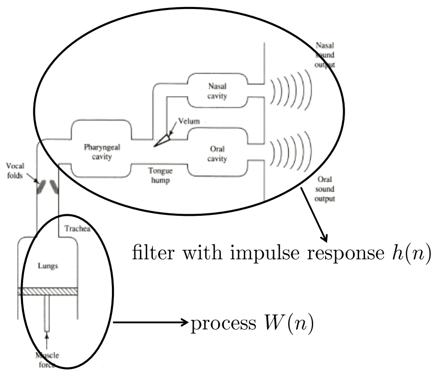
What does $h_1(n)$ model? It is a model of the speech production process.



Speech Production Model

Typically, speech is seen as an AR-process:





Predict current speech sample from the q previous speech samples

$$Y(n) = h_1 Y(n-1) + h_2 Y(n-2) + \ldots + h_q Y(n-q) + W(n)$$

Estimation of the AR Coefficients (1)

- The signal model should match the real data as well as possible
- Real data $Y(n)$
- Modeled data $\hat{Y}(n) = \sum_{k=1}^{q} h_k Y(n-k)$ (*linear prediction*)
- Difference $$Y(n) - \hat{Y}(n) = Y(n) - \sum_{k=1}^{q} h_k Y(n-k) = W(n)$$
- Minimize the difference by choosing the AR coefficients optimally

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Estimation of the AR Coefficients (2)

- Difference is quantified as variance

$$\sigma_W^2 = E[(Y(n) - \hat{Y}(n))^2] = E[(Y(n) - \sum_{k=1}^{q} h_k Y(n-k))^2]$$

- Minimize this difference:

$$\min_{h_i} \sigma_W^2 \qquad i = 1,2,\ldots,q$$

$$\Rightarrow \frac{\partial}{\partial h_i} \sigma_W^2 = 0 \qquad i = 1,2,\ldots,q$$

$$\frac{\partial}{\partial h_i} E[(Y(n) - \sum_{k=1}^{q} h_k Y(n-k))^2] = 0$$

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Estimation of the AR Coefficients (3)

- Solution of minimization problem:

$$\frac{\partial}{\partial h_i} E[(Y(n) - \sum_{k=1}^{q} h_k Y(n-k))^2] = 0 \qquad i=1,2,\ldots,q$$

$$E[(Y(n) - \sum_{k=1}^{q} h_k Y(n-k))(-2)\left\{\frac{\partial}{\partial h_i} \sum_{k=1}^{q} h_k Y(n-k)\right\}] = 0$$

$$E[(Y(n) - \sum_{k=1}^{q} h_k Y(n-k))Y(n-i)] = 0$$

$$R_Y(i) = \sum_{k=1}^{q} h_k R_Y(i-k)$$

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Estimation of the AR Coefficients (4)

- *Yule Walker* or *normal* equations

$$R_Y(i) = \sum_{k=1}^{q} h_k R_Y(i-k) \qquad i = 1,2,\ldots,q$$

$$\begin{pmatrix} R_Y(1) \\ R_Y(2) \\ \vdots \\ R_Y(q) \end{pmatrix} = \begin{pmatrix} R_Y(0) & R_Y(1) & \cdots & R_Y(q-1) \\ R_Y(1) & R_Y(0) & R_Y(1) & \vdots \\ \vdots & R_Y(1) & \ddots & R_Y(1) \\ R_Y(q-1) & \cdots & R_Y(1) & R_Y(0) \end{pmatrix} \begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_q \end{pmatrix}$$

- Variance of excitation noise:

$$\sigma_W^2 = E[(Y(n) - \sum_{k=1}^{q} h_k Y(n-k))^2] = R_Y(0) - \sum_{k=1}^{q} h_k R_Y(k)$$

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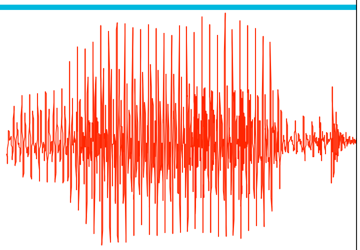
How to Synthesize speech? (1)

- Estimate the correlation function



$$\widetilde{R}_Y(k) = \frac{1}{N-k}\sum_{n=1}^{N-k} Y(n)Y(n+k)$$

- Compute filter coefficients

$$\begin{pmatrix} h_1 \\ h_2 \\ \vdots \\ h_q \end{pmatrix} = \begin{pmatrix} \widetilde{R}_Y(0) & \widetilde{R}_Y(1) & \cdots & \widetilde{R}_Y(q-1) \\ \widetilde{R}_Y(1) & \widetilde{R}_Y(0) & \widetilde{R}_Y(1) & \vdots \\ \vdots & \widetilde{R}_Y(1) & \ddots & \widetilde{R}_Y(1) \\ \widetilde{R}_Y(q-1) & \cdots & \widetilde{R}_Y(1) & \widetilde{R}_Y(0) \end{pmatrix}^{-1} \begin{pmatrix} \widetilde{R}_Y(1) \\ \widetilde{R}_Y(2) \\ \vdots \\ \widetilde{R}_Y(q) \end{pmatrix}$$

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How to Synthesize speech? (2)

- Variance of excitation noise:

$$\sigma_w^2 = \widetilde{R}_Y(0) - \sum_{k=1}^{q} h_k \widetilde{R}_Y(k)$$

- Perform filtering:

$$Y(n) = h_1 Y(n-1) + h_2 Y(n-2) + \ldots + h_q Y(n-q) + W(n)$$

If we apply z-transform: $H(z) = \dfrac{1}{1-\sum_{q=1}^{Q} h_q z^{-q}}$

All-pole!

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Voiced vs. Unvoiced Speech

Does $W(n)$ really get uncorrelated in practice?

This depends on

- the model order used in $H_1(z^{-1})$
- the type of speech sound:
  - voiced - excitation still contains the long term correlation that originates from the vocal cords
  - unvoiced

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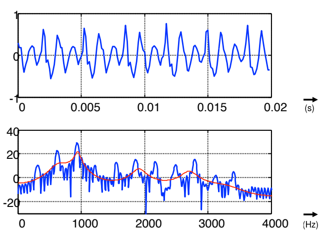
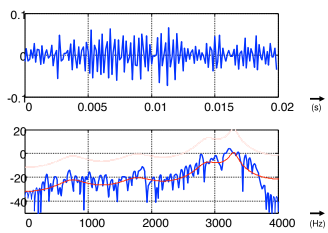
Voiced vs. Unvoiced Speech

- Voiced: Air pushed through the glottis which oscillates, generating quasi-periodic puffs of air (e.g. vowels /a/, /i/, etc.)
- Unvoiced: Air forced through constriction somewhere along the vocal tract (e.g. /s/, /f/).

voiced:



unvoiced:



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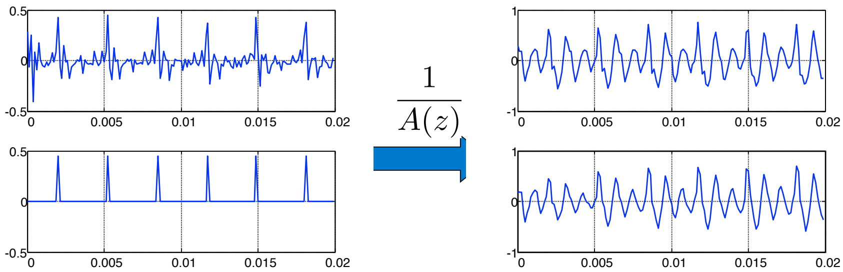
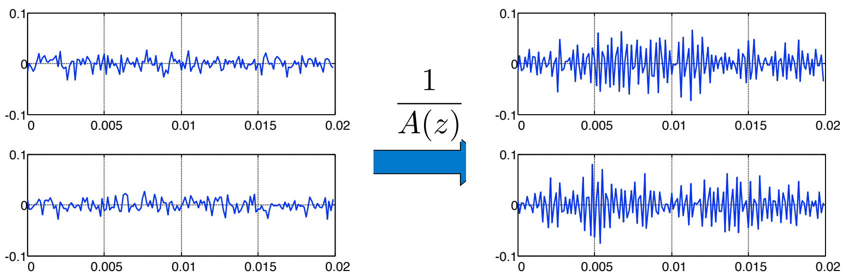
Example: Speech Coding

voiced speech:



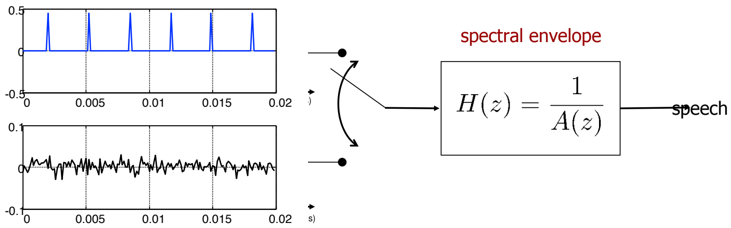
Example: Speech Coding

unvoiced speech:



Vocoder

- Discrete-time linear source-filter model of speech production



Speech can be synthesized if we know:

1. spectral envelope
2. pitch period T_0

(can be computed using the auto-correlation function)

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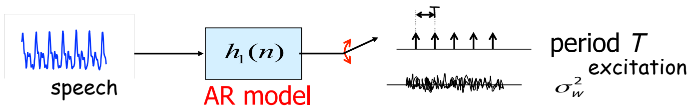
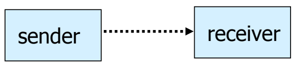
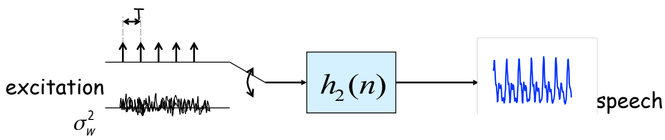
Demonstration - Speech Coding



1. Determine filter h(n) using Yule – Walker equations and error variance
2. Determine whether speech is unvoiced or voiced
3. Transmission/Quantization of *h(n)* and voiced/unvoiced information





4. Synthesis of speech based on received filter *h(n)* and voiced/unvoiced information.

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Demo Vocoder

English female speech, $f_s = 8$ kHz, mono, 8 bit/sample

- original
- modelled
- modelled as unvoiced speech

German male speech, $f_s = 8$ kHz, mono, 8 bit/sample

- original
- modelled
- modelled as unvoiced speech

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Usage of Fourier transforms
The Power Spectral Density

- (needed for system identification)

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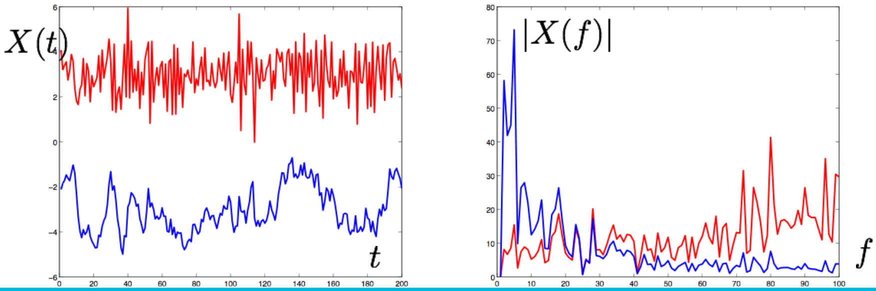
Usual tool in Signal Processing

- Fourier transforms are used to
  - describe deterministic signals (functions) and linear systems
  - analyze and design signals and linear systems



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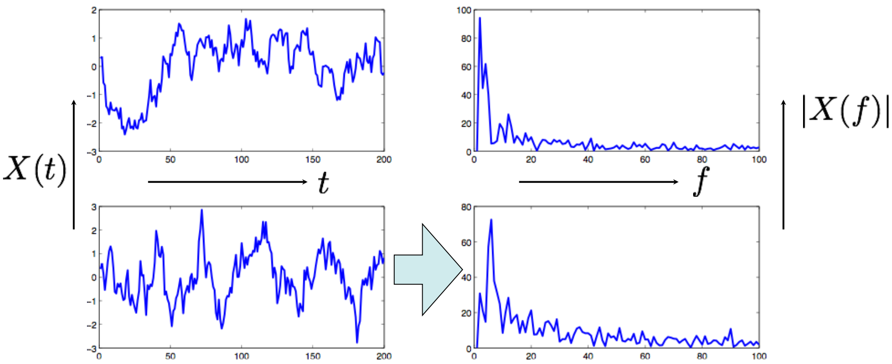
What about random signals?

- Fourier transforms of random signals are rather useless: the result depends on the particular realization



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Power Spectral Density

- Instead of considering the Fourier transform of one realization, we should look at 'average behavior' in the Fourier domain
- **Power Spectral Density (PSD)**

$$S_X(f) = F\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau$$

$$S_X(f) = F\{R_X(k)\} = \sum_{k=-\infty}^{\infty} R_X(k) \exp(-j2\pi fk)$$

PSD only exists for WSS random processes!

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Actually...

- The original definition of PSD is a bit more ugly:

$$S_X(f) = \lim_{T\to\infty} \frac{1}{2T} E\left[|X_T(f)|^2\right]$$

- It highlights the fact that we are computing the average power in a frequency, but for (in principle) infinitely long signals
- The Wiener-Khintchine theorem shows the equivalence with

$$S_X(f) = \int_{-\infty}^{\infty} R_X(\tau) \exp(-j2\pi f\tau) d\tau$$

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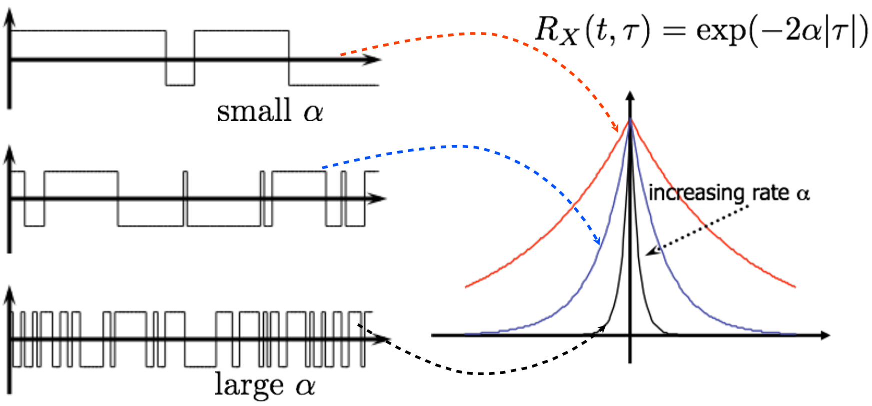
Heuristic interpretation

- (Amplitude component of) Fourier transform of a deterministic signal gives the strength of a particular complex exponential (sine-cosine combination)
- PSD of a random process/signal gives the average power carried by a particular complex exponential
- PSD can be calculated from the autocorrelation function of the WSS random process

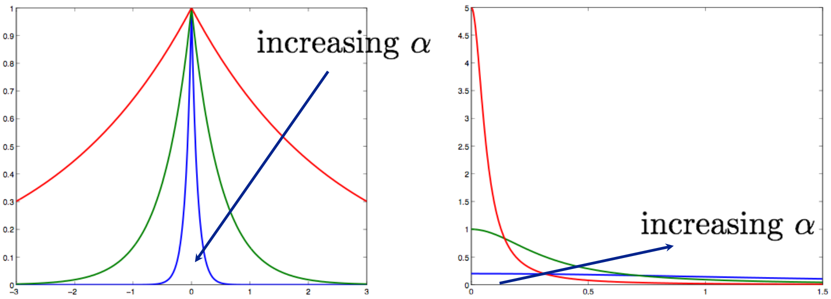
Example Random telegraph



- Realizations of three different random processes

Example Random Telegraph

$$R_X(t,\tau) = \exp(-2\alpha|\tau|)$$
$$S_X(f) = \frac{4\alpha}{4\alpha^2 + 4\pi^2 f^2}$$



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From PSD to autocorrelation funct.

- Continuous time:

$$S_X(f) = F\{R_X(\tau)\} = \int_{-\infty}^{\infty} R_X(\tau)\exp(-j2\pi f\tau)d\tau$$
$$R_X(\tau) = \int_{-\infty}^{\infty} S_X(f)\exp(j2\pi f\tau)df$$

- Discrete time:

$$S_X(f) = F\{R_X(k)\} = \sum_{k=-\infty}^{\infty} R_X(k)\exp(-j2\pi fk)$$
$$R_X(k) = \int_{-1/2}^{1/2} S_X(f)\exp(j2\pi fk)df$$

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Properties of PSD

- Because we are using real-valued signals, the power spectral density function is symmetric around f=0

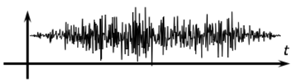
$$S_X(-f) = S_X(f)$$

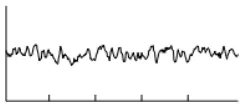
- The function expresses a **density**, and is therefore always non-negative

- Integral over power spectral density is **average power**

$$\int_{-\infty}^{\infty} S_X(f)df = E[X(t)^2] = R_X(0)$$

Filtered WSS Process





$R_X(t)$ → linear system (LTI) H(f) → $R_Y(t)$

$S_X(f)$ $S_Y(f)$

- The power spectral density of Y(t) is given by

$$S_Y(f) = |H(f)|^2 S_X(f)$$

Result can easily be understood

- Since

$$R_Y(\tau) = h(\tau) * h(-\tau) * R_X(\tau)$$

$$\Updownarrow \quad \text{Fourier transform}$$

- then

$$\begin{aligned} S_Y(f) &= H(f)H^*(f)S_X(f) \\ &= |H(f)|^2 S_X(f) \end{aligned}$$

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Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

- Autocorrelation function

① $$\begin{aligned} R_Y(k) &= E[Y(n)Y(n+k)] \\ &= E[Y(n)(h_1 Y(n+k-1) + W(n+k))] \\ &= \ldots \end{aligned}$$

② $$Y(n) = (\ldots,0,0,h_1^k) * W(n) = h(n) * W(n)$$

$$\begin{aligned} R_Y(k) &= h(k) * h(-k) * R_W(k) \\ &= (\ldots, h_1^3, h_1^2, h_1, 1, h_1, h_1^2, h_1^3, \ldots) * \sigma_W^2 \delta(k) = \frac{\sigma_W^2}{1 - h_1^2} h_1^{|k|} \end{aligned}$$

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Autocorrelation Functions (AR-1)

- First order AR process

$$Y(n) = h_1 Y(n-1) + W(n)$$

- Impulse response

$$h(n) = \begin{cases} 0 & n < 0 \\ h_1^n & n \geq 0 \end{cases}$$

- Frequency response

$$H(f) = \frac{1}{1 - h_1 \exp(-j2\pi f)}$$

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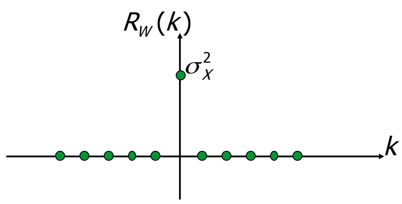
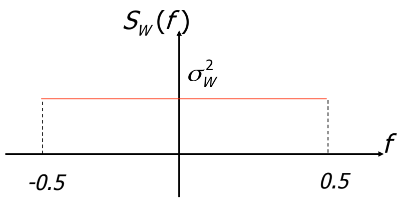
Autocorrelation Functions (AR-1)

- Power Spectral Density of Output $Y(k)$

$$S_Y(f) = |H(f)|^2 \, S_W(f)$$

- Power Spectral Density of Input $W(k)$





$$S_Y(f) = |H(f)|^2 \, \sigma_W^2 = \frac{\sigma_W^2}{|1 - h_1 \exp(-j2\pi f)|^2} \quad \Rightarrow R_Y(k)$$

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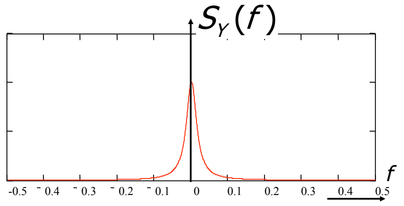
Autocorrelation Functions (AR-1)

$$W(n) \longrightarrow \boxed{h(n) = \begin{cases} 0 & k < 0 \\ h_1^k & k \geq 0 \end{cases}} \longrightarrow Y(n)$$

•Power Spectral Density

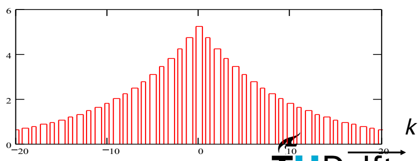
$$S_Y(f) = |H(f)|^2 \sigma_W^2$$

$$= \frac{\sigma_W^2}{|1 - h_1 \exp(-j2\pi f)|^2}$$



•Autocorrelation Function

$$R_Y(k) = \frac{\sigma_W^2}{1 - h_1^2} h_1^{|k|}$$



h_1=0.9

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Example (time continuous)

$$X(t) \longrightarrow \boxed{h(t) = \begin{cases} 3\exp(-t) & t \geq 0 \\ 0 & t < 0 \end{cases}} \longrightarrow Y(t)$$

$R_X(\tau) = 4 + 3\delta(\tau)$

- What is $R_Y(\tau) = ?$

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Example (time continuous)

$$X(t) \longrightarrow \boxed{h(t) = \begin{cases} 3\exp(-t) & t \geq 0 \\ 0 & t < 0 \end{cases}} \longrightarrow Y(t)$$

$R_X(\tau) = 4 + 3\delta(\tau)$

- then $S_X(f) = 4\delta(f) + 3$ and $H(f) = \dfrac{3}{1 + j(2\pi f)}$ (pg 413. book)
- and
$$S_Y(f) = |H(f)|^2 S_X(f) = \frac{9}{1+(2\pi f)^2}(4\delta(f)+3)$$
$$= 36\delta(f) + \frac{27}{1+(2\pi f)^2}$$
Inverse Fourier transform
- so $R_Y(\tau) = 36 + \dfrac{27}{2}\exp(-|\tau|)$

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Using the PSD

- White noise process (again)
- Cross-correlation (again)
- System identification

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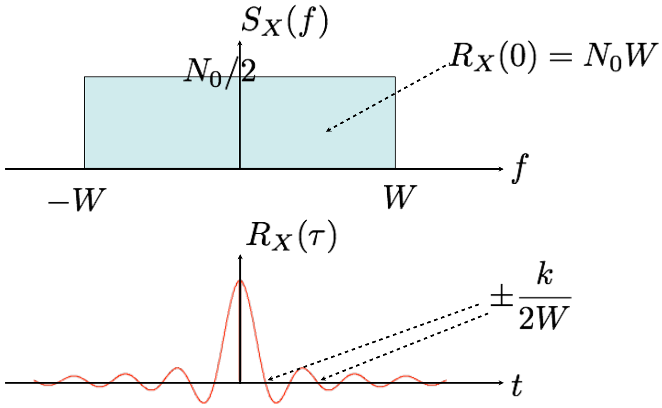
White noise signal/process

- Bandlimited white noise process with $\mu_X = 0$



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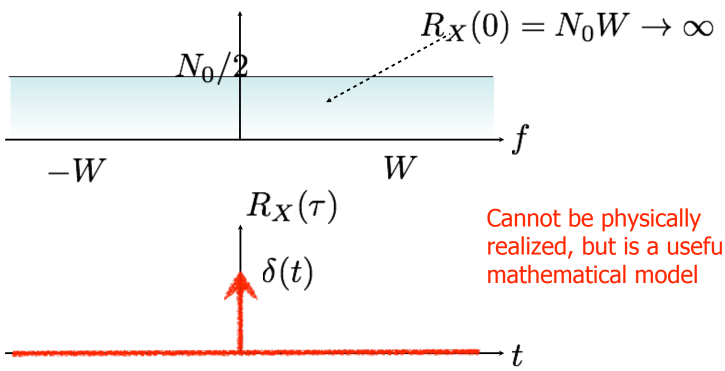
White noise signal/process

- Let $W \to \infty$
  White noise or uncorrelated noise process



Cannot be physically realized, but is a useful mathematical model

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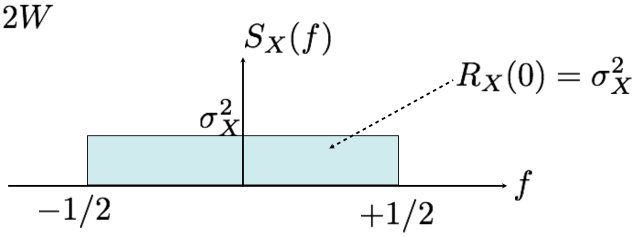
White noise signal/process

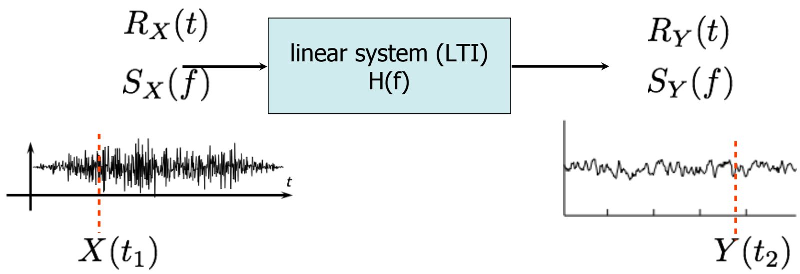
- If we sample the time continuous white noise process at $f_s = 2W$



- This time discrete process **can** be realized:

$$R_X(k) = \begin{cases} \sigma_X^2 & k = 0 \\ 0 & k \neq 0 \end{cases}$$

Cross correlation function



- What is the stochastic relation between $X(t_1)$ and $Y(t_2)$?

Cross correlation function

- Cross correlation function for a jointly WSS random process is:

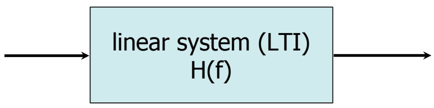
$$R_{XY}(\tau) = E[X(t)Y(t+\tau)]$$
$$= R_{YX}(-\tau)$$

- We get in frequency domain:

$$R_{XY}(\tau) = h(\tau) * R_X(\tau)$$
$$S_{XY}(f) = H(f)S_X(f)$$

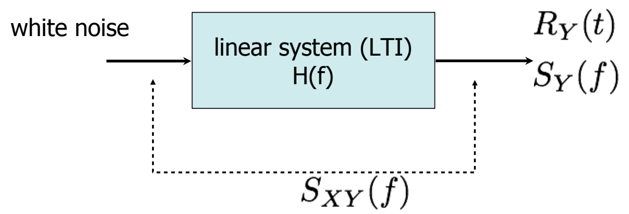
cross power spectral density

Application: system identification



- Find the impulse (or frequency) response H(f) of the system

Application: system identification



- When feeding white noise into a linear system, we obtain

$$S_{XY}(f) = H(f)S_X(f) = \frac{N_0}{2}H(f)$$

Covered Today

- Chapter 11
- Key terms
  - Moving average, autoregressive processes
  - Autocorrelation for infinite impulse response
  - Power spectral density
  - Average power
  - White Gaussian noise process
  - Cross power spectral density
  - System identification