Sampling and Reconstruction

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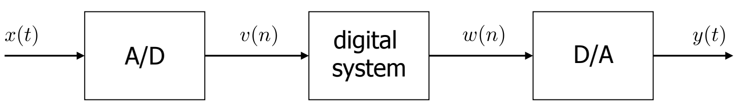
Digital Signal Processing

In its most general form, *digital signal processing* (DSP) refers to processing of analog signals by means of discrete-time (discrete-space) operations implemented on digital hardware.

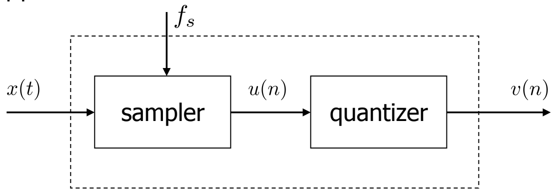
From a system point of view, DSP is concerned with mixed systems:

- the input and output signals are analog
- the processing is done on the equivalent digital signals



Analog-to-Digital Converter

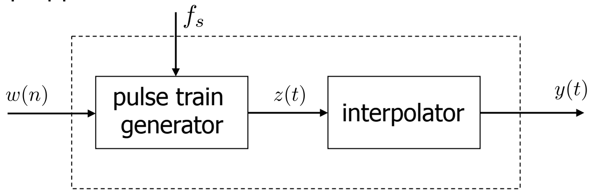
Two-step approach:



- Sampler: $u(n) = x(nT_s)$ where $T_s = 1/f_s$, the sampling period
- Quantizer: $v(n) = (Qu)(n)$, where Q is a (nonlinear) mapping from intervals of the real line (quantization cells) to reproduction levels

Digital-to-Analog Converter

Two-step approach:

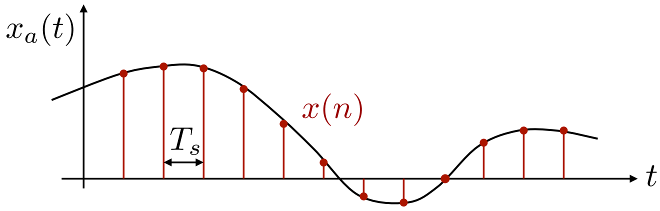


- The pulse-train generator transforms the sequence of numbers $w(n)$ into a sequence of scaled, analog pulses (spaced $T_s = 1/f_s$ seconds apart)
- The interpolator removes high-frequency components in z (via low-pass filtering) to produce a smooth analog output signal

Sampling

To process a continuous-time signal using digital signal processing techniques, it is necessary to convert the signal into a sequence of numbers. This is usually done by *sampling* the analog signal, say $x_a(t)$, periodically every T_s seconds to produce the discrete-time signal $x(n)$ given by

$$x(n) = x_a(nT_s), \quad -\infty < n < \infty$$



Sampling

Recall that the spectrum of the discrete-time signal x is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

where the signal x can be recovered from its spectrum by the inverse Fourier transform

$$x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega$$

Sampling

If x_a is an aperiodic absolutely integrable signal, its Fourier transform (with $\Omega = 2\pi f$) is given by

$$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

where the signal x_a can be recovered from its spectrum by the inverse Fourier transform

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\Omega) e^{j\Omega t} d\Omega$$

The *angular frequency* Ω is expressed in radians per second (rad/s)

Sampling

Key question: What is the relation between $X(\omega)$ and $X_a(\Omega)$, and under what conditions can we recover x_a from $X(\omega)$?

Periodic sampling imposes a relationship between the independent variables t and n in the signals $x_a(t)$ and $x(n)$, respectively

$$t = nT_s = \frac{n}{f_s}$$

and thus between ω and Ω through Fourier transformation.

Sampling

Consider an analog harmonic signal $x_a(t) = e^{j\Omega t}$. What is the relation between the "real" angular frequency Ω and the discrete-time angular frequency ω of the (sampled) discrete-time signal $x(n) = e^{j\omega n}$?

We have

$$e^{j\omega n} = x(n) = x_a(nT_s) = e^{j\Omega T_s n}$$

and we conclude that

$$\boxed{\omega = \Omega T_s = \frac{\Omega}{f_s} = 2\pi \frac{f}{f_s}}$$

Hence, ω is the normalized angular frequency (dimensionless)

Sampling

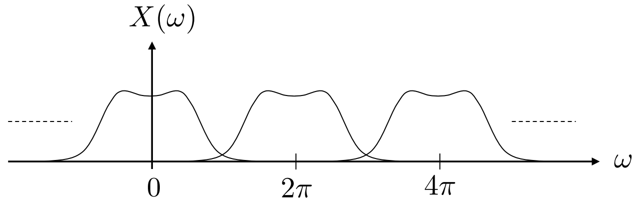
Relation between $X(\omega)$ and $X_a(\Omega)$:

$$\begin{aligned}
\underbrace{\frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega}_{x(n)} &= \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega n T_s}d\Omega}_{x_a(nT_s)} \\
&= \frac{f_s}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)e^{j\omega n}d\omega \\
&= \frac{f_s}{2\pi}\sum_{k=-\infty}^{\infty}\int_{k2\pi}^{(k+1)2\pi} X_a(\omega)e^{j\omega n}d\omega \\
\omega \to \omega + k2\pi \qquad &= \frac{1}{2\pi}\int_0^{2\pi}\left(f_s \sum_{k=-\infty}^{\infty} X_a(\omega + k2\pi)\right)e^{j\omega n}d\omega
\end{aligned}$$

Sampling

Hence,

$$X(\omega) = f_s \sum_{k=-\infty}^{\infty} X_a(\omega + k2\pi)$$



The spectrum of the discrete-time signal consists of shifted copies of the (scaled) analog spectrum

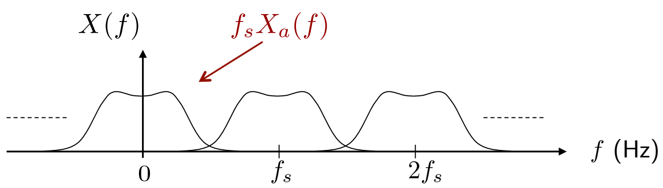
Sampling

Equivalently, we have

$$X(\Omega) = f_s \sum_{k=-\infty}^{\infty} X_a(\Omega + k\Omega_s), \quad \Omega_s = 2\pi f_s$$

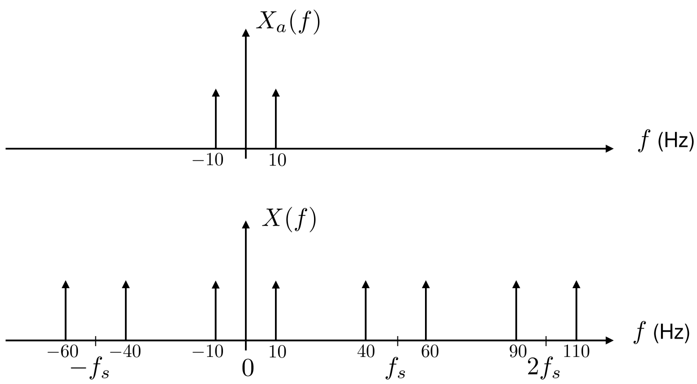
or

$$X(f) = f_s \sum_{k=-\infty}^{\infty} X_a(f + kf_s)$$

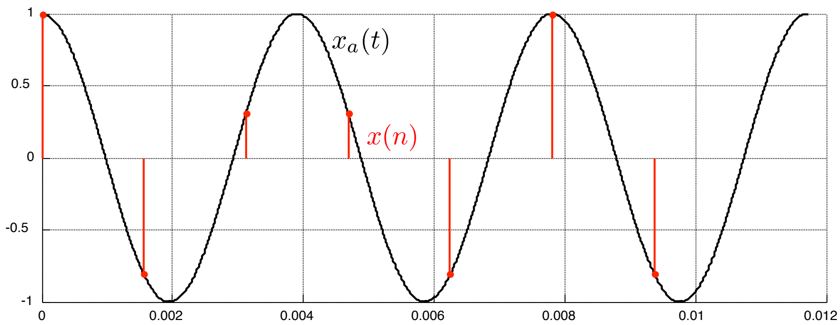


Sampling

Example: $x_a(t) = \cos(2\pi 10 t), \quad f_s = 50$ Hz

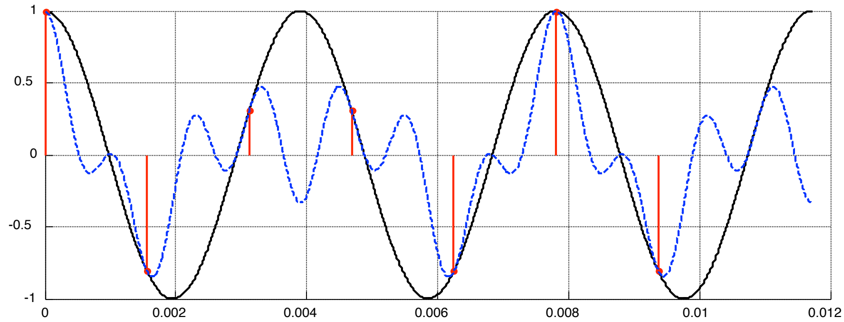


Spectrum of harmonic signals



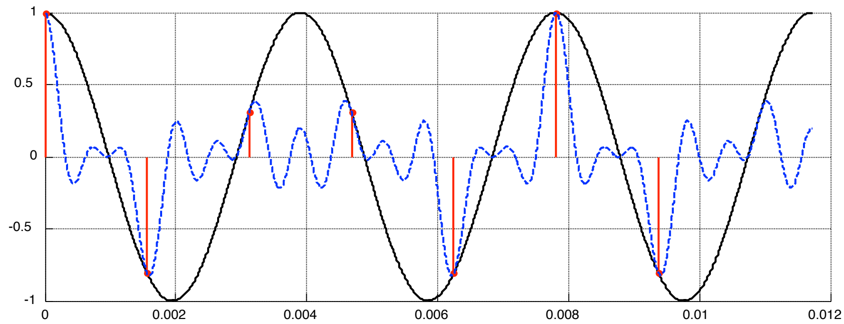
Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, three cosines with frequencies $(f_0, f_s - f_0, f_s + f_0)$



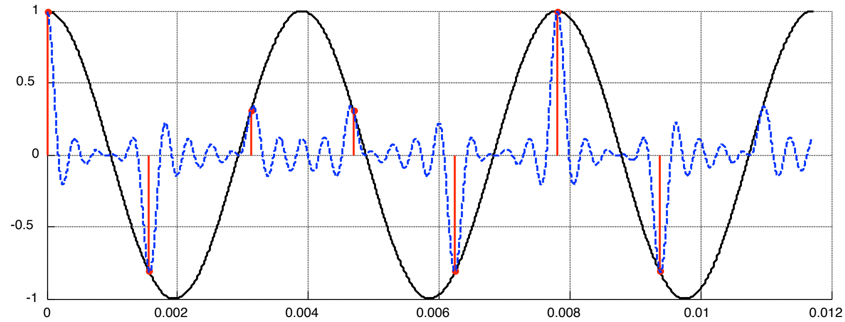
Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, five cosines with frequencies $(f_0, f_s - f_0, f_s + f_0, 2f_s - f_0, 2f_s + f_0)$



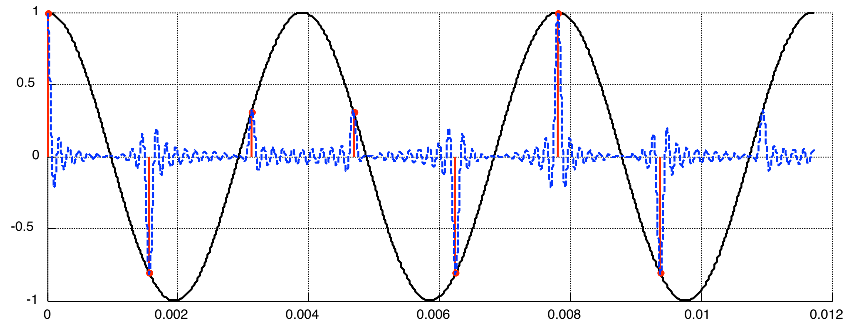
Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, nine cosine terms



Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, 21 cosine terms



Sampling

If the spectrum of the analog signal is band limited to, say B Hz, and the sampling frequency satisfies $f_s > 2B$, we have

$$X(f) = f_s X_a(f) \text{ for } |f| \leq \frac{f_s}{2}$$

since the periodic repetition of the spectrum of X_a does not introduce spectral overlap

In this case, we can perfectly reconstruct the original analog signal by scaling the input spectrum by f_s^{-1} and removing all spectral components $|f| > \frac{f_s}{2}$

$$X_a(f) = \begin{cases} f_s^{-1} X(f), & |f| \leq \frac{f_s}{2} \\ 0, & |f| > \frac{f_s}{2} \end{cases}$$

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Reconstruction

$$\begin{aligned} x_a(t) &= \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X_a(f) e^{j2\pi ft} df \\ &= T_s \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X(f) e^{j2\pi ft} df \\ &= T_s \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} \left(\sum_{n=-\infty}^{\infty} x(n) e^{-\overbrace{j2\pi f T_s}^{\omega = \Omega T_s = 2\pi f T_s} n} \right) e^{j2\pi ft} df \\ &= T_s \sum_{n=-\infty}^{\infty} x(n) \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} e^{j2\pi f(t - nT_s)} df \\ &= \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\frac{\pi}{T_s}(t - nT_s)\right)}{\frac{\pi}{T_s}(t - nT_s)} = \sum_{n=-\infty}^{\infty} x(n) g(t - nT_s) \end{aligned}$$

no spectral overlap

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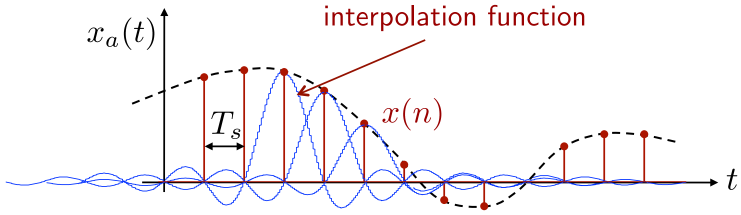
Reconstruction



Hence, $x_a(t)$ is a linear combination of time-shifted signals $g(t-nT_s)$ weighted by the sample values $x(n)$

The family $\{g(t-nT_s)\}_{n=-\infty}^{\infty}$ forms an orthonormal basis for band-limited signals, assuming that $f_s > 2B$

Reconstruction

Since the Fourier transform of g is given by

$$g(t) = \frac{\sin(\frac{\pi}{T_s}t)}{\frac{\pi}{T_s}t} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad G(f) = \begin{cases} f_s^{-1}, & |f| \leq \frac{f_s}{2} \\ 0, & |f| > \frac{f_s}{2} \end{cases}$$

we conclude that the ideal interpolator scales the input spectrum by a factor f_s^{-1} and removes frequencies components for $|f| > \frac{f_s}{2}$. Basically, the ideal interpolator acts as a frequency window that removes the discrete-time spectral periodicity to generate an aperiodic continuous-time signal spectrum

Aliasing

Aliasing occurs when the signal is sampled at a rate which is too low. For real signals, the effect can be described by folding of the frequency axis

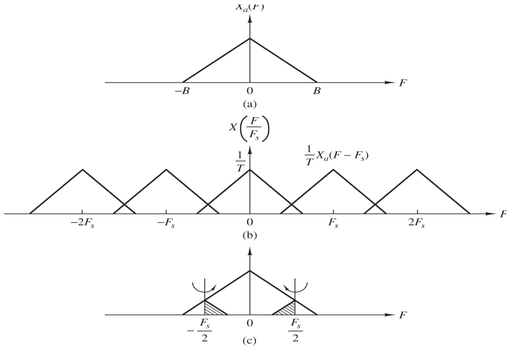
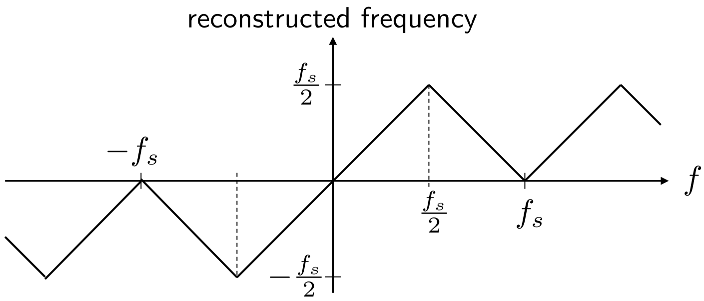


Figure 6.1.3 Illustration of aliasing around the folding frequency.

Aliasing



Example: frequency sweep 0 - 4 kHz

$f_s = 8$ kHz
$f_s = 2$ kHz

Anti-Aliasing Filter

In order to avoid aliasing effects, a prefilter (or anti-aliasing filter) is used

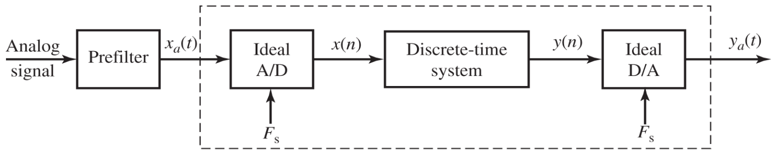


Figure 6.2.1 System for the discrete-time processing of continuous-time signals.

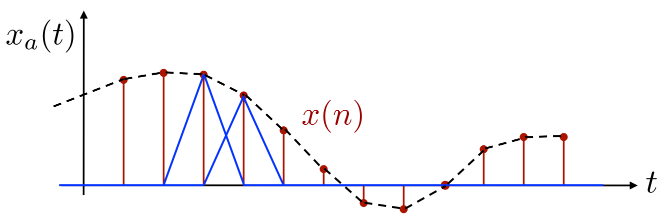
original	original
with prefilter	with prefilter
without prefilter	without prefilter

Practical Interpolation

The ideal interpolator is not a practical interpolator. Why?

Alternatively, we can use a linear interpolator scheme:

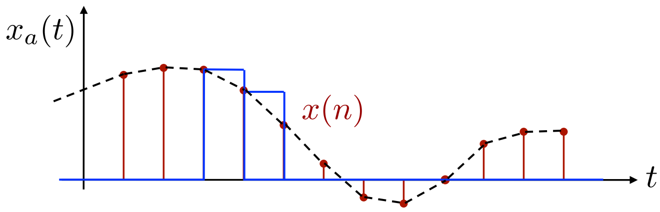
$$\hat{x}_a(t) = \sum_{n=-\infty}^{\infty} x(n) g_{\text{lin}}(t - nT_s), \quad g_{\text{lin}}(t) = \begin{cases} 1 - \frac{|t|}{T_s}, & |t| < T_s \\ 0, & \text{otherwise} \end{cases}$$



Practical Interpolation

Even simpler: 0th-order sample-and-hold

$$\hat{x}_a(t) = \sum_{n=-\infty}^{\infty} x(n) g_{\text{sh}}(t - nT_s), \quad g_{\text{sh}}(t) = \begin{cases} 1, & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases}$$



Practical Interpolation

Again, similar to what we derived for the ideal interpolator, we now have that the Fourier transform of the output of the (practical) interpolator is given by $\hat{X}_a(f) = X(f)G_{(\cdot)}(f)$, where

$$G_{\text{ideal}}(f) = \begin{cases} f_s^{-1}, & |f| \le \frac{f_s}{2} \\ 0, & |f| > \frac{f_s}{2} \end{cases}$$

$$G_{\text{lin}}(f) = T_s \left(\frac{\sin(\pi f T_s)}{\pi f T_s} \right)^2$$

$$G_{\text{sh}}(f) = T_s \frac{\sin(\pi f T_s)}{\pi f T_s} e^{-i\pi f T_s}$$

Practical Interpolation