Solutions of exercises from 'Probability and Stochastic Processes', Yates and Goodman

2013-2014

June 22, 2015

Week 1

Refresher random variables

1.4.1 For the probability $P[H_0]$ that a phone makes no hand-offs, we have to find all (disjoint) events that contain H_0. These are LH_0 and BH_0, so:

$$P[H_0] = P[LH_0] + P[BH_0] = 0.1 + 0.4 = 0.5 \tag{1.1}$$

Similarly, we have for a brief call

$$P[B] = P[BH_0] + P[BH_1] + P[BH_2] = 0.4 + 0.1 + 0.1 = 0.6 \tag{1.2}$$

Finally, the probability that a call is long or makes at least two hand-offs:

$$\begin{aligned} P[L \cup H_2] &= P[LH_0] + P[LH_1] + P[LH_2] + P[BH_2] && (1.3) \\ &= 0.1 + 0.1 + 0.2 + 0.1 = 0.5 && (1.4) \end{aligned}$$

1.5.1 (a) The probability for a brief call is

$$P[B] = P[H_0B] + P[H_1B] + P[H_2B] = 0.6 \tag{1.5}$$

Now the probability that a brief call has no hand-offs is

$$P[H_0|B] = \frac{P[H_0B]}{P[B]} = \frac{0.4}{0.6} = \frac{2}{3} \tag{1.6}$$

(b) The probability of one hand-off is

$$P[H_1] = P[H_1B] + P[H_1L] = 0.2 \tag{1.7}$$

The probability that a call with one handoff is long, is:

$$P[L|H_1] = \frac{P[H_1L]}{P[H_1]} = \frac{0.1}{0.2} = \frac{1}{2} \tag{1.8}$$

(c) The probability of a long call is:

$$P[L] = P[H_0L] + P[H_1L] + P[H_2L] = 0.4 \tag{1.9}$$

The probability that a long call has one or more handoffs is:

$$P[H_1 \cup H_2|L] = \frac{P[H_1L \cup H_2L]}{P[L]} = \frac{0.1 + 0.2}{0.4} = \frac{3}{4} \tag{1.10}$$

1.5.5 First, the sample space is $\{234, 243, 324, 342, 423, 432\}$. Each of the outcomes are equally likely, i.e. 1/6. The events are:

$$E_1 = \{234, 243, 423, 432\} \quad E_2 = \{243, 324, 342, 423\} \quad E_3 = \{234, 324, 342, 432\} \tag{1.11}$$

$$O_1 = \{324, 342\} \quad O_2 = \{234, 432\} \quad O_3 = \{243, 423\} \tag{1.12}$$

Then the rest becomes straighforward:

(a) The cond. prob. that the second card is even, given the first is even:

$$P[E_2|E_1] = \frac{P[E_1E_2]}{P[E_1]} = \frac{P[243, 423]}{P[234, 243, 423, 432]} = \frac{2/6}{4/6} = \frac{1}{2} \tag{1.13}$$

(b) The prob. that the first two cards are even, given the third is even:

$$P[E_1E_2|E_3] = \frac{P[E_1E_2E_3]}{P[E_3]} = \frac{0}{\text{something}} = 0 \tag{1.14}$$

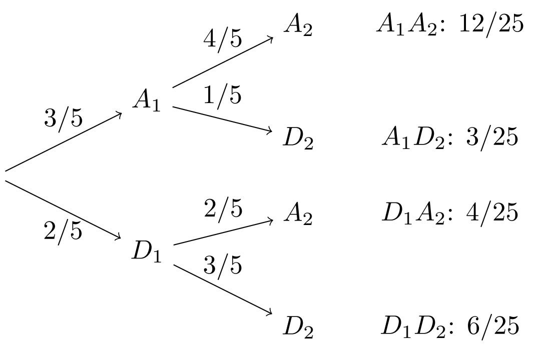
(c) The cond. prob. that the second card is even given the second card is odd:

$$P[E_2|O_1] = \frac{P[O_1E_2]}{P[O_1]} = \frac{P[O_1]}{P[O_1]} = 1 \tag{1.15}$$

(d) The cond. prob. that the second card is odd given the first card is odd:

$$P[O_2|O_1] = \frac{P[O_1O_2]}{P[O_1]} = 0 \tag{1.16}$$

1.7.6 A photodetector can be acceptable (A) or defective (D). Because the two detectors are tested in sequence, we can make the following tree:



(a) To find the probability of one acceptable photodetector, we look at the tree and find:

$$P[E_1] = P[A_1D_2] + P[D_1A_2] = 3/25 + 4/25 = 7/25 \tag{1.17}$$

(b) The probability that both photodetectors are defective is $P[D_1d_2] = 6/25$.

2.4.3 (a) The CDF looks like

(b) The pmf looks like

$$P_X(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise.} \end{cases} \tag{1.18}$$

2.5.5 When we have the PMF

$$P_x(x) = \begin{cases} 0.4 & x = -3 \\ 0.4 & x = 5 \\ 0.2 & x = 7 \\ 0 & \text{otherwise.} \end{cases} \tag{1.19}$$

we can compute the expected value using the definition:

$$E[X] = \sum_x x P_x(x) = -3 \cdot 0.4 + 5 \cdot 0.4 + 7 \cdot 0.2 = 2.2 \tag{1.20}$$

2.8.4 Again, we have the PMF as in (1.18). The expected value was computed in question 2.5.5: $E[X] = 2.2$. The expected value of X^2 is

$$E[X^2] = \sum_x x^2 P_X(x) = (-3)^2 \cdot 0.4 + 5^2 \cdot 0.4 + 7^2 \cdot 0.2 = 23.4 \tag{1.21}$$

Therefore the variance becomes:

$$Var[X] = E[X^2] - E[X]^2 = 23.4 - 2.2^2 = 18.6 \tag{1.22}$$

2.9.3 To compute $E[X|B]$ we need the probability for event B:

$$P[B] = P[X > 0] = P_X(5) + P_X(7) = 0.6 \tag{1.23}$$

Using the definition of the conditional probability:

$$P_{X|B} = \begin{cases} \frac{P_X(x)}{P[B]} & x \in B \\ 0 & \text{otherwise} \end{cases} = \begin{cases} 2/3 & x = 5 \\ 1/3 & x = 7 \\ 0 & \text{otherwise} \end{cases} \tag{1.24}$$

To find the variance, we first have to compute $E[X^2|B]$:

$$E[X^2|B] = \sum_x x^2 P_{X|B}(x) = 5^2 \cdot 2/3 + 7^2 \cdot 1/3 = 33 \tag{1.25}$$

and $E[X|B]$:

$$E[X|B] = \sum_x x P_{X|B}(x) = 5 \cdot 2/3 + 7 \cdot 1/3 = 17/3 \tag{1.26}$$

therefore

$$Var[X|B] = E[X^2|B] - (E[X|B])^2 = 33 - (17/3)^2 = 8/9 \tag{1.27}$$

3.4.5 (a) Using the definition from Appendix A from the book:

$$f_X(x) = \begin{cases} 1/10 & -5 < x < 5 \\ 0 & \text{otherwise} \end{cases} \tag{1.28}$$

(b) To find the CDF, we have to integrate. For $x \leq -5$ $F_X(x) = 0$, and for $x \geq 5$ we get that $F_X(x) = 1$. For $-5 < x < 5$ we obtain:

$$F_X(x) = \int_{-5}^{x} f_X(u)du = \frac{x+5}{10} \tag{1.29}$$

The CDF is therefore:

$$F_X(x) = \begin{cases} 0 & x \leq -5 \\ (x+5)/10 & -5 < x < 5 \\ 1 & x \geq 5 \end{cases} \tag{1.30}$$

(c) To compute the expected value:

$$E[X] = \int_{-\infty}^{\infty} x f_X(x)dx = \int_{-5}^{5} \frac{x}{10}dx = \left[\frac{x^2}{20}\right]_{-5}^{5} = 0 \tag{1.31}$$

(d) Similarly,

$$E[X] = \int_{-\infty}^{\infty} x^5 f_X(x)dx = \int_{-5}^{5} \frac{x^5}{10}dx = \left[\frac{x^6}{60}\right]_{-5}^{5} = 0 \tag{1.32}$$

(e) Similarly,

$$E[X] = \int_{-\infty}^{\infty} e^x f_X(x)dx = \int_{-5}^{5} \frac{e^x}{10}dx = \left[\frac{e^x}{10}\right]_{-5}^{5} = \frac{e^5 - e^{-5}}{10} = 14.8 \tag{1.33}$$

Week 2

Multivariate distributions

4.1.1 (a) The probability of the event $A = \{X \le 2, Y \le 3\}$ is directly be given by the joint CDF for $x = 2$ and $y = 3$:

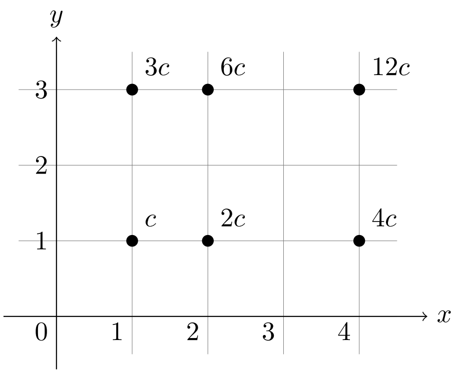
$$P[X \le 2, Y \le 3] = F_{X,Y}(x = 2, y = 3) = (1 - e^{-2})(1 - e^{-3}) \tag{2.1}$$

(b) The marginal pdf can be found by filling in $y = \infty$:

$$F_X(x) = F_{X,Y}(x, y = \infty) = 1 - e^{-x} \tag{2.2}$$

(c) This marginal pdf can be found by filling in $x = \infty$:

$$F_Y(y) = F_{X,Y}(x = \infty, y) = 1 - e^{-y} \tag{2.3}$$



4.2.1 First make a picture:

(a) The c should be chosen such, that the PMF adds up to one:

$$c + 2c + 4c + 3c + 6c + 12c = 28c = 1 \quad \rightarrow \ c = \frac{1}{28} \tag{2.4}$$

(b) In the event $\{Y < X\}$ there are the outcomes $(2, 1)$, $(4, 1)$ and $(4, 3)$, so

$$P[Y < X] = P_{XY}(2, 1) + P_{XY}(4, 1) + P_{XY}(4, 3) = 2c + 4c + 12c = 18/28 \tag{2.5}$$

(c) In the event $\{Y > X\}$ there are the outcomes $(1, 3)$ and $(2, 3)$, so

$$P[Y > X] = P_{XY}(1, 3) + P_{XY}(2, 3) = 3c + 6c = 9/28 \tag{2.6}$$

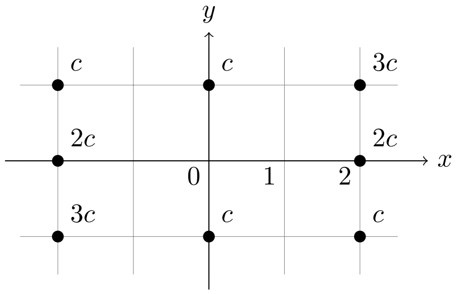
(d) In the event $\{Y = X\}$ there is only the outcome $(1, 1)$ so

$$P[Y = X] = P_{XY}(1, 1) = c = 1/28 \tag{2.7}$$

(e) In the event $\{Y = 3\}$ there are the outcomes $(1, 3)$, $(2, 3)$ and $(4, 3)$, so

$$P[Y = 3] = P_{XY}(1, 3) + P_{XY}(2, 3) + P_{XY}(4, 3) = 3c + 6c + 12c = 21/28 \tag{2.8}$$

4.2.2 It is always a good idea to make a picture:



(a) The constant c is found by summing the PMF over all values for X and Y, and equating it to 1.

$$\sum_x \sum_y P_{X,Y}(x, y) = \sum_{x=-2,0,2} \sum_{y=-1,0,1} c|x + y| = 6c + 2c + 6c = 14c = 1 \tag{2.9}$$

Therefore $c = 1/14$.

(b) Simply:

$$P[Y < X] = P_{X,Y}(0, -1) + P_{X,Y}(2, -1) + P_{X,Y}(2, 0) + P_{X,Y}(2, 1) = 1/2 \tag{2.10}$$

(c) Surprisingly:

$$P[Y > X] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(-2, 1) + P_{X,Y}(0, 1) = 1/2 \tag{2.11}$$

(d) There is no outcome with $X = Y$ so $P[X = Y] = 0$.

(e)

$$P[X < 1] = P_{X,Y}(-2, -1) + P_{X,Y}(-2, 0) + P_{X,Y}(0, -1) + P_{X,Y}(0, 1) = 8/14 \tag{2.12}$$

4.3.2 (a) Use the definition of the marginal probability:

$$P_X(x) = \sum_y P_{X,Y}(x, y) = \sum_{y=-1,0,1} P_{X,Y}(x, y) = \begin{cases} 6/14 & x = -2, 2 \\ 2/14 & x = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.13}$$

Similarly for $P_Y(y)$:

$$P_Y(y) = \sum_x P_{X,Y}(x, y) = \sum_{x=-2,0,2} P_{X,Y}(x, y) = \begin{cases} 5/14 & y = -1, 1 \\ 4/14 & y = 0 \\ 0 & \text{otherwise} \end{cases} \tag{2.14}$$

(b) Given the marginal pdf's it is easy to compute $E[X]$:

$$E[X] = \sum_x xP_X(x) = \sum_{x=-2,0,2} xP_X(x) = -2 \cdot 6/14 + 2 \cdot 6/14 = 0 \tag{2.15}$$

$$E[Y] = \sum_y yP_Y(y) = \sum_{y=-1,0,1} yP_Y(y) = -1 \cdot 5/14 + 1 \cdot 5/14 = 0 \tag{2.16}$$

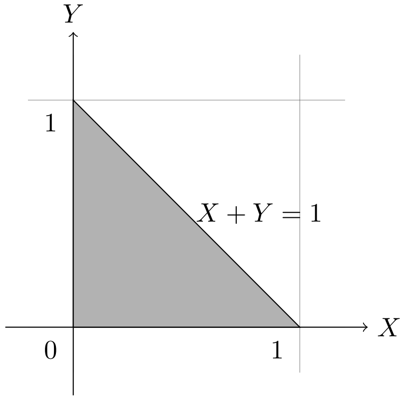
(c) For both variances $Var(X) = E[X^2] - (E[X])^2 = E[X^2]$:

$$Var(X) = E[X^2] = \sum_x x^2 P_X(x) = (-2)^2 \cdot 6/14 + 2^2 \cdot 6/14 = 24/7 \quad (2.17)$$

$$Var(Y) = E[Y^2] = \sum_y y^2 P_Y(y) = (-1)^2 \cdot 5/14 + 1^2 \cdot 5/14 = 5/7 \quad (2.18)$$

So $\sigma_X = \sqrt{24/7}$ and $\sigma_Y = \sqrt{5/7}$.

4.4.1 Let's first make a picture:



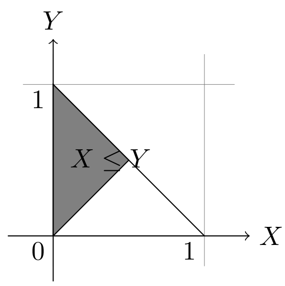
With: $f_{X,Y} = \begin{cases} c & x+y \leq 1,\ x \leq 0,\ y \leq 0 \\ 0 & \text{otherwise} \end{cases}$

(a) The integral over the shaded area should become one:

$$\int f_{X,Y}(x,y)dxdy = \int_0^1 \int_0^{1-x} cdydx = \int_0^1 [cy]_0^{1-x} dx = c \int_0^1 (1-x)dx = c\left[x - \frac{1}{2}x^2\right]_0^1 = c/2 = 1 \quad (2.19)$$

and therefore $c = 2$.

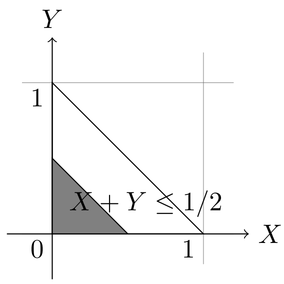
(b) For $P[X \leq Y]$ we have to integrate over the following area:



That means (first integrating over y, then over x):

$$P[X \leq Y] = \int_0^{1/2} \int_x^{1-x} cdydx = c\int_0^{1/2} [cy]_x^{1-x} dx = c\int_0^{1/2} (1-2x)dx = c\left[x - x^2\right]_0^{1/2} = 1/2$$

(c) For $P[X+Y \leq 1/2]$ we have to integrate over the following area:



That means (first integrating over y, then over x):

$$P[X \leq Y] = \int_0^{1/2} \int_0^{1/2-x} cdydx = c\int_0^{1/2} [cy]_0^{1/2-x} dx = c\int_0^{1/2} (1/2-x)dx = c\left[x/2 - x^2/2\right]_0^{1/2} = 1/4$$

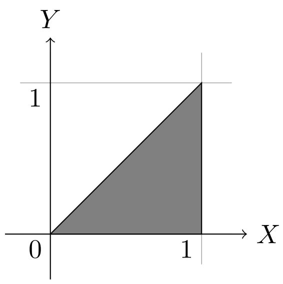
4.5.2 (a) This joint pdf is the same as in question 4.4.1. The marginals are computed using the definition:

$$f_X(x) = \int_y f_{X,Y}(x,y)dy = \int_0^{1-x} 2dy = \begin{cases} 2(1-x) & 0 \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.20}$$

(b) And...

$$f_Y(y) = \int_x f_{X,Y}(x,y)dx = \int_0^{1-y} 2dx = \begin{cases} 2(1-y) & 0 \leq y \leq 1 \\ 0 & \text{otherwise} \end{cases} \tag{2.21}$$

4.5.6 (a) As always, first make a picture:

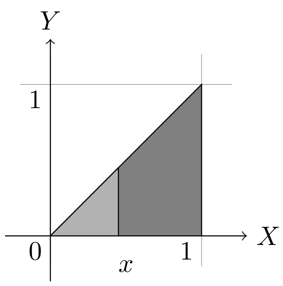


(b) The constant c is determined by the fact that the pdf should integrate to 1. I decide to first integrate over y for a given value of x. For a given value of x variable y runs between 0 and x. The second integral is then over x, between 0 and 1:

$$\int_0^1 \int_0^x cydydx = \int_0^1 \frac{1}{2}cx^2dx = \left[\frac{1}{6}cx^3\right]_0^1 = \frac{1}{6}c = 1 \tag{2.22}$$

And therefore $c = 6$.

(c) To compute the cdf $F_X(x) = P[X \leq x]$ we have to integrate over the nonzero probability region right of the vertical line at x:



When we choose $x < 0$ this integral is 0, and if we choose $x > 1$ then this integral is 1. We now only have to worry for situations where $0 \leq x \leq 1$.

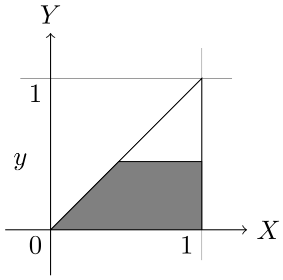
I introduce two integration variables u (over X) and v (over Y), and the integral becomes:

$$\begin{aligned} F_X(x) &= \int_0^x \int_0^u cv\,dv\,du & (2.23) \\ &= \int_0^x \frac{1}{2}cu^2du & (2.24) \\ &= \frac{1}{6}cx^3 = x^3 & (2.25) \end{aligned}$$

So, in total the cdf becomes:

$$F_X(x) = \begin{cases} 0 & x < 0 \\ x^3 & 0 \leq x \leq 1 \\ 1 & x > 1 \end{cases} \tag{2.26}$$

(d) To compute the cdf $F_Y(y)$, we have to integrate over the nonzero probability region below the horizontal line at y. When we choose $y < 0$ this integration becomes 0, and when we choose $y > 1$, this integration becomes 1. When we have $0 \le y \le 1$ we have to make the integration over the following region:



Again, I introduce two integration variables u (over X) and v (over Y). I decide to integrate first in the Y direction, then over the X direction:

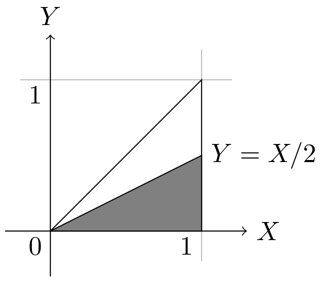
$$F_Y(y) = \int_0^y \int_v^1 cv\, du\, dv \tag{2.27}$$

$$= \int_0^y [cvu]_v^1\, dv \tag{2.28}$$

$$= \int_0^y cv(1-v)dv \tag{2.29}$$

$$= c\left[\frac{1}{2}v^2 - \frac{1}{3}v^3\right]_0^y = c\left(\frac{1}{2}y^2 - \frac{1}{3}y^3\right) \tag{2.30}$$

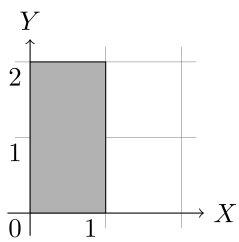
(e) To compute $P(Y \le X/2)$ we first have to find the event. It is indicated in the figure:



I integrate first over Y, then over X:

$$P[Y \le X/2] = \int_0^1 \int_0^{x/2} cy\, dy\, dx = \int_0^1 \left[\frac{1}{2}cy^2\right]_0^{x/2} dx = \int_0^1 \frac{1}{8}cx^2\, dx = \left[\frac{1}{24}cx^3\right]_0^1 = \frac{1}{24}c \tag{2.31}$$

4.7.8 Here we go again. First a picture:



(a) For the computation of $E[X] = \int x f_X(x)dx$ and $E[Y]$ we need the marginal distributions, so let's do that first:

$$f_X(x) = \int f_{X,Y}(x,y)dy = \int_0^2 (x+y)/3dy = \left[xy/3 + y^2/6\right]_0^2 = \frac{2x+2}{3} \quad \text{for } 0 \le x \le 1$$

and

$$f_Y(y) = \int f_{X,Y}(x,y)dx = \int_0^1 (x+y)/3dx = \left[x^2/6 + xy/3\right]_0^1 = \frac{2y+1}{6} \quad \text{for } 0 \le y \le 2$$

The expected value for X is

$$E[X] = \int x f_X(x)dx = \int_0^1 x\frac{2x+2}{3}dx = \left[\frac{2x^3}{9} + \frac{x^2}{3}\right]_0^1 = \frac{5}{9} \tag{2.32}$$

and

$$E[X^2] = \int x^2 f_X(x)dx = \int_0^1 x^2\frac{2x+2}{3}dx = \left[\frac{x^4}{6} + \frac{2x^3}{9}\right]_0^1 = \frac{7}{18} \tag{2.33}$$

The variance therefore becomes

$$Var[X] = E[X^2] - (E[X])^2 = \frac{7}{18} - \left(\frac{5}{9}\right)^2 = \frac{13}{162} \tag{2.34}$$

(b) The expected value for Y is

$$E[Y] = \int y f_Y(y)dy = \int_0^2 y\frac{2y+1}{6}dy = \left[\frac{y^2}{12} + \frac{x^3}{9}\right]_0^2 = \frac{11}{9} \tag{2.35}$$

and

$$E[Y^2] = \int y^2 f_Y(y)dy = \int_0^2 y^2\frac{2y+1}{6}dx = \left[\frac{y^3}{18} + \frac{x^4}{12}\right]_0^2 = \frac{16}{9} \tag{2.36}$$

The variance therefore becomes

$$Var[Y] = E[Y^2] - (E[Y])^2 = \frac{16}{9} - \left(\frac{11}{9}\right)^2 = \frac{23}{81} \tag{2.37}$$

(c) The covariance is defined as $Cov[X,Y] = E[XY] - E[X]E[Y]$, so we need the correlation first:

$$\begin{aligned} E[XY] &= \int xyf_{X,Y}(x,y)dxdy = \int_0^1\int_0^2 xy\left(\frac{x+y}{3}\right)dydx && (2.38)\\ &= \int_0^1 \left[\frac{x^2y^2}{6} + \frac{xy^3}{9}\right]_0^2 dx && (2.39)\\ &= \left(\frac{2x^2}{3} + \frac{8x}{9}\right)dx = \left[\frac{2x^3}{9} + \frac{4x^2}{9}\right]_0^1 = \frac{2}{3} && (2.40)\end{aligned}$$

so the covariance:

$$Cov[X,Y] = \frac{2}{3} - \frac{5}{9}\cdot\frac{11}{9} = -\frac{1}{81} \tag{2.41}$$

(d) Well, this is simply $E[X+Y] = E[X] + E[Y] = 16/9$.

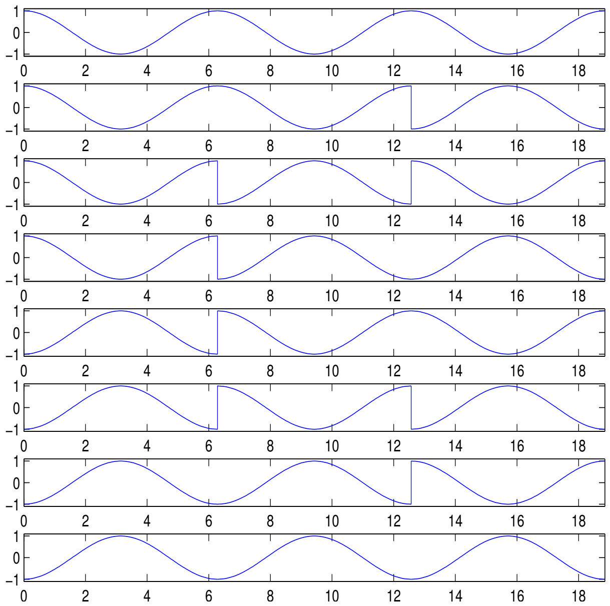
(e) Use Theorem 4.15,

$$Var[X+Y] = Var[X] + Var[Y] + 2Cov[X,Y] = \frac{13}{162} + \frac{23}{81} - \frac{2}{81} = \frac{55}{162} \tag{2.42}$$

Week 3

Random processes

10.2.3 Each of the sample functions should encode one of the sequences $\{(0,0,0),(1,0,0),...,(1,1,1)\}$.



10.10.3 To find the autocorrelation function of the random process $W(t) = X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t)$ (with uncorrelated RV's X and Y) we have to compute:

$$\begin{aligned}
R_W(t,\tau) &= E[W(t)W(t+\tau)] \qquad (3.1)\\
&= E[(X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t))(X\cos(2\pi f_0(t+\tau)) + Y\sin(2\pi f_0(t+\tau)))]\\
&= E[X\cos(2\pi f_0 t)X\cos(2\pi f_0(t+\tau)) + X\cos(2\pi f_0 t)Y\sin(2\pi f_0(t+\tau)) +\\
&\qquad Y\sin(2\pi f_0 t)X\cos(2\pi f_0(t+\tau))) + Y\sin(2\pi f_0 t)Y\sin(2\pi f_0(t+\tau)))]\\
&= E[X^2]\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) + E[XY]\cos(2\pi f_0 t)\sin(2\pi f_0(t+\tau)) +\\
&\qquad E[XY]\sin(2\pi f_0 t)\cos(2\pi f_0(t+\tau))) + E[Y^2]\sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau)))]\\
&\qquad (3.2)
\end{aligned}$$

Because X and Y are uncorrelated $Cov[X,Y] = E[XY] - E[X]E[Y] = 0$, and because the expected value $E[X] = E[Y] = 0$, we know $E[XY] = 0$. Furthermore is given that $Var[X] = E[X^2] - E[X]^2 = \sigma^2$, and therefore $E[X^2] = E[Y^2] = \sigma^2$. Combining gives:

$$R_W(t,\tau) = \sigma^2\left(\cos(2\pi f_0 t)\cos(2\pi f_0(t+\tau)) + \sin(2\pi f_0 t)\sin(2\pi f_0(t+\tau))\right)$$

Now using Math Fact B.2, we can derive:

$$\begin{aligned}\cos(A)\cos(B)+\sin(A)\sin(B) &= \frac{1}{2}\left[\cos(A-B)+\cos(A+B)+\cos(A-B)-\cos(A+B)\right] \\ &= \cos(A-B) \qquad (3.3)\end{aligned}$$

Substituting $A = 2\pi f_0 t$ and $B = 2\pi f_0(t+\tau)$ we obtain:

$$R_W(t,\tau) = \sigma^2 \cos(2\pi f_0 \tau) \qquad (3.4)$$

and we see that the autocovariance function is independent of t.

To check if $W(t)$ is wide sense stationary, we also have to check if the expected value $E[W]$ is independent of t:

$$E[W(t)] = E\left[X\cos(2\pi f_0 t) + Y\sin(2\pi f_0 t)\right] = E[X]\cos(2\pi f_0 t) + E[Y]\sin(2\pi f_0 t) = 0+0 = 0 \qquad (3.5)$$

So, $W(t)$ is WSS.

10.4.1 For Y_k to be iid, it should have identical distribution for different k, and it should be independent. Each Y_k is the sum of two identical independent Gaussian random variables, so each Y_k has the same pdf. Next, Y_k is independent of Y_l when $l \neq k$ because they do not share any samples of X_k.

10.4.2 Again, each W_k is the sum of two identical independent Gaussian random variables, so they have the same pdf. But variables W_k and W_{k-1} share the sample X_{k-1}, so W_k and W_{k-1} are *not* independent.

4.11.1 Looking at the joint pdf:

$$F_{X,Y}(x,y) = ce^{-x^2/8-y^2/18} \qquad (3.6)$$

it looks suspiciously like a Gaussian distribution. Looking at Definition 4.17, pg 191 of the book, we see that this definition contains μ_X, μ_Y, σ_1, σ_2 and ρ.

When we can identify σ_1, σ_2 and ρ, the constant c can be computed as:

$$c = \frac{1}{2\pi\sigma_1\sigma_2\sqrt{1-\rho^2}} \qquad (3.7)$$

To find σ_1 and $E[X] = \mu_1$, we have to solve:

$$\frac{1}{2}\left(\frac{x-E[X]}{\sigma_1}\right)^2 = \frac{x^2}{8} \quad \rightarrow E[X] = 0, \quad \text{and} \quad \sigma_1 = \sqrt{4} = 2 \qquad (3.8)$$

$$\frac{1}{2}\left(\frac{y-E[Y]}{\sigma_2}\right)^2 = \frac{y^2}{18} \quad \rightarrow E[Y] = 0, \quad \text{and} \quad \sigma_2 = \sqrt{9} = 3 \qquad (3.9)$$

Because there is no cross term with $x \cdot y$, we have to conclude that $\rho = 0$. Solving c gives:

$$c = \frac{1}{2\pi\sqrt{8}\sqrt{18}\sqrt{1-0}} = \frac{1}{12\pi} \qquad (3.10)$$

And because $\rho = 0$ the variables X and Y are uncorrelated, which means for a Gaussian that they are also independent.

4.11.4 (a) When the two random variables X and Y are iid continuous uniform between -50 and 50, it can actually happen that the archer misses the circular target completely! (For instance, when $x = 49$ and $y = 49$.) Because X and Y are independent, we can easily give the joint pdf:

$$f_{X,Y}(x,y) = f_X(x)f_Y(y) = \begin{cases} 10^{-4} & -50 \le x \le 50,\ -50 \le y \le 50 \\ 0 & \text{otherwise} \end{cases} \tag{3.11}$$

(both X and Y are uniformly distributed between -50 and 50). Therefore the probability of bullseye is:

$$P[A] = P[X^2 + Y^2 \le 2^2] = 10^{-4} \cdot \pi 2^2 = 0.0013 \tag{3.12}$$

(b) When $f_{X,Y}(x,y)$ is uniform over the circular area (and X and Y are not independent anymore!), the density becomes the inverse of the area:

$$f_{X,Y}(x,y) = \begin{cases} \frac{1}{\pi 50^2} & x^2 + y^2 \le 50^2 \\ 0 & \text{otherwise} \end{cases} \tag{3.13}$$

Then the probability of bullseye becomes:

$$P[A] = P[X^2 + Y^2 \le 2^2] = \frac{\pi 2^2}{\pi 50^2} = 0.0016 \tag{3.14}$$

(c) When X and Y are independent Gaussian distributions with mean 0 and variance σ^2, we have

$$f_X(x) = \frac{1}{\sqrt{2\pi}\sigma} e^{-x^2/2\sigma^2}, \quad f_Y(y) = \frac{1}{\sqrt{2\pi}\sigma} e^{-y^2/2\sigma^2}, \tag{3.15}$$

and the joint probability density becomes:

$$f_{X,Y}(x,y) = \frac{1}{2\pi\sigma^2} e^{-(x^2+y^2)/2\sigma^2}, \tag{3.16}$$

For the probability of bullseye we have to compute an interesting integral:

$$\begin{aligned} P[A] &= P[X^2 + Y^2 \le 2^2] && (3.17) \\ &= \int_{x^2+y^2 \le 2^2} f_{X,Y}(x,y)\,dxdy && (3.18) \\ &= \frac{1}{2\pi\sigma^2} \int_{x^2+y^2 \le 2^2} e^{-(x^2+y^2)/2\sigma^2} dxdy && (3.19) \end{aligned}$$

Now we have to do a trick, a coordinate transform in polar coordinates: $r^2 = x^2 + y^2$ and $dxdy = rdrd\theta$, and we integrate:

$$\begin{aligned} P[A] &= \frac{1}{2\pi\sigma} \int_0^2 \int_0^{2\pi} e^{-r^2/2\sigma^2} rdrd\theta && (3.20) \\ &= \frac{1}{\sigma^2} \int_0^2 re^{-r^2/2\sigma^2} dr && (3.21) \\ &= \left[-e^{-r^2/2\sigma^2}\right]_0^2 = 1 - e^{-4/200} = 0.020 && (3.22) \end{aligned}$$

10.5.1 The arrivals of new telephone calls can be modelled by a Poisson process. The rate $\lambda = 4$ is given, and therefore our PRM is defined:

$$P_{N(T)}(n) = \begin{cases} (4T)^n e^{-4T}/n! & n = 0, 1, 2, 3, ... \\ 0 & \text{otherwise} \end{cases} \tag{3.23}$$

(a) Now we only have to fill in:

$$P_{N(1)}(0) = (41)^0 e^{-4\cdot 1}/0! = e^{-4} \tag{3.24}$$

(b) and (c) Similarly:

$$P_{N(1)}(4) = (4)^4 e^{-4}/4! = 10.67e^{-4} \tag{3.25}$$
$$P_{N(2)}(2) = (8)^2 e^{-8}/2! = 32e^{-8} \tag{3.26}$$

10.5.6 It is given that the response time T is an exponential random variable with mean 8. That means that the pdf is:

$$f_T(t) = \begin{cases} \frac{1}{8}e^{-t/8} & t \geq 0 \\ 0 & \text{otherwise} \end{cases} \tag{3.27}$$

(a) The probability of a response time larger than 4:

$$\begin{aligned} P[T \geq 4] &= 1 - P[T < 4] = 1 - \int_{-\infty}^{4} f_T(t)dt & (3.28)\\ &= 1 - \int_0^4 \frac{1}{8}e^{-t/8}dt = 1 + \left[e^{-t/8}\right]_0^4 & (3.29)\\ &= 1 + e^{-4/8} - 1 = e^{-1/2} & (3.30) \end{aligned}$$

(b) The conditional probability is (because when $T \geq 13$ then T is also always larger than 5):

$$P[T \geq 13 | T \geq 5] = \frac{P[T \geq 13, T \geq 5]}{P[T \geq 5]} = \frac{P[T \geq 13]}{P[T \geq 5]} \tag{3.31}$$

Now we can compute:

$$\begin{aligned} P[T \geq 13] &= 1 - \int_0^{13} \frac{1}{8}e^{-t/8}dt = e^{-13/8} & (3.32)\\ P[T \geq 5] &= 1 - \int_0^{5} \frac{1}{8}e^{-t/8}dt = e^{-5/8} & (3.33)\\ & & (3.34) \end{aligned}$$

so therefore:

$$P[T \geq 13 | T \geq 5] = \frac{e^{-13/8}}{e^{-5/8}} = e^{-1} \tag{3.35}$$

(c) This seems simple: we have a sequence of arrivals, and their interarrival time is exponential. So the $N(t)$ should be a Poisson process:

$$P[N(t) = n] = \begin{cases} \left(\frac{t}{8}\right)^n e^{-t/8}/n! & n = 0, 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{3.36}$$

But there is one tricky detail here: it is given that the first query is made at time zero. So the Poisson process does not start with $N(t) = 0$ counts, but with $N(t) = 1$ counts. We have to shift the Poisson distribution by one count:

$$P[N(t) = n] = \begin{cases} \left(\frac{t}{8}\right)^{n-1} e^{-t/8}/(n-1)! & n = 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{3.37}$$

Note that for $t > 0$ we will always have that $N(t) \geq 1$.

Week 4

The autocorrelation function

10.8.1 First, write the definition:

$$\begin{aligned} C_X[m,k] &= E[(X_m - \mu_X)(X_{m+k} - \mu_X)] && (4.1)\\ &= E[X_m X_{m+k} - X_m \mu_X - \mu_X X_{m+k} + \mu_X^2] && (4.2)\\ &= E[X_m X_{m+k}] - E[X_m]\mu_X - \mu_X E[X_{m+k}] + \mu_X^2 && (4.3) \end{aligned}$$

Now is given that X_n is iid, with mean μ_X and variance σ_X^2, we can simplify it further. For $m \neq k$:

$$\begin{aligned} C_X[m,k] &= E[X_m X_{m+k}] - E[X_m]\mu_X - \mu_X E[X_{m+k}] + \mu_X^2 && (4.4)\\ &= E[X_m]E[X_{m+k}] - \mu_X\mu_X - \mu_X\mu_X + \mu_X^2 = 0 && (4.5) \end{aligned}$$

For $k = 0$ we get

$$\begin{aligned} C_X[m,0] &= E[X_m X_m] - E[X_m]\mu_X - \mu_X E[X_m] + \mu_X^2 && (4.6)\\ &= E[X_m^2] - \mu_X^2 = \sigma^2 && (4.7) \end{aligned}$$

So, in total:

$$C_X[m,k] = \begin{cases} \sigma^2 & k = 0 \\ 0 & \text{otherwise} \end{cases} \tag{4.8}$$

10.8.3 (a) To compute the expected value, use the definition, and the fact that X_n is iid with zero mean:

$$E[C_n] = E\left[16\left(1 - \cos\frac{2\pi n}{365}\right) + 4X_n\right] = 16E\left[1 - \cos\frac{2\pi n}{365}\right] + 4E[X_n] = 16(1 - \cos\frac{2\pi n}{365}) \tag{4.9}$$

(b) Again, fill in the definition:

$$\begin{aligned} C_C[m,k] &= E[(C_m - \mu_X(m))(C_{m+k} - \mu_X(m+k))] && (4.10)\\ &= E[C_m C_{m+k}] - E[C_m]E[C_{m+k}] && (4.11)\\ &= E[\left(16\left(1 - \cos\frac{2\pi m}{365}\right) + 4X_m\right)\left(16\left(1 - \cos\frac{2\pi(m+k)}{365}\right) + 4X_{m+k}\right)] \\ &\quad -16^2(1 - \cos\frac{2\pi m}{365})(1 - \cos\frac{2\pi(m+k)}{365}) && (4.12)\\ &= E[4X_m \cdot 4X_{m+k}] = 16E[X_m X_{m+k}] && (4.13) \end{aligned}$$

Now is given that X_n is iid, and using the result of question 10.8.1, we get:

$$C_C[m,k] = \begin{cases} 16 & k=0 \\ 0 & \text{otherwise} \end{cases} \tag{4.14}$$

(c) It captures the overall seasonal changes, but it does not capture the correlations between consecutive days.

10.9.1 So, we assume that for an arbitrary a, we have $Y(t) = X(t+a)$. Further it is given that $X(t)$ is stationary. If X is stationary, then holds that:

$$f_{X(t_1+\Delta t),\ldots,X(t_k+\Delta t)} = f_{X(t_1),\ldots,X(t_k)} \tag{4.15}$$

If we want to check if process Y is also stationary, we have to check if this holds:

$$f_{Y(t_1+\Delta t),\ldots,Y(t_k+\Delta t)} = f_{Y(t_1),\ldots,Y(t_k)} \tag{4.16}$$

So, we fill in for Y:

$$\begin{aligned} f_{Y(t_1+\Delta t),\ldots,Y(t_k+\Delta t)} &= f_{X(t_1+a+\Delta t),\ldots,X(t_k+a+\Delta t)} && (4.17) \\ &= f_{X(t_1+a),\ldots,X(t_k+a)} = f_{Y(t_1),\ldots,Y(t_k)} && (4.18) \end{aligned}$$

So, Y should also be stationary.

10.10.4 (a) When $X(t)$ is WSS with average power 1, then by definition $E[X^2(t)] = 1$.

(b) Given that Θ is a uniform random variable over $[0, 2\pi]$, we can compute:

$$\begin{aligned} E[\cos(2\pi f_c t + \Theta)] &= = \int \cos(2\pi f_c t + u) f_\Theta(u) du && (4.19) \\ &= \int_0^{2\pi} \cos(2\pi f_c t + u) \frac{1}{2\pi} du && (4.20) \\ &= \frac{1}{2\pi} \left[\sin(2\pi f_c t + u)\right]_0^{2\pi} = 0 && (4.21) \end{aligned}$$

(c) We are saved by the fact that X and Θ are independent, because then:

$$E[Y(t)] = E[X(t)\cos(2\pi f_c t + \Theta)] = E[X(t)]E[\cos(2\pi f_c t + \Theta)] = 0 \tag{4.22}$$

(d) Finally, the average power becomes:

$$\begin{aligned} E[Y^2(t)] &= E[X^2(t)\cos^2(2\pi f_c t + \Theta))] = E[X^2(t)]E[\cos^2(2\pi f_c t + \Theta))] && (4.23) \\ &= E[\cos^2(2\pi f_c t + \Theta))] && (4.24) \end{aligned}$$

Now we have to be creative to integrate the $\cos^2()$. We can use Math Fact B.2 and derive:

$$\cos^2\theta = \frac{1}{2}(1 + \cos(2\theta)) \tag{4.25}$$

so now we can integrate:

$$\begin{aligned} E[Y^2(t)] &= E[\cos^2(2\pi f_c t + \Theta))] && (4.26) \\ &= E[\frac{1}{2}(1 + \cos(4\pi f_c t + 2\Theta))] && (4.27) \\ &= E[\frac{1}{2}] + E[\cos(4\pi f_c t + 2\Theta))] && (4.28) \\ &= \frac{1}{2} + 0 = \frac{1}{2} && (4.29) \end{aligned}$$

10.10.1 R_1 and R_2 are valid autocorrelation functions, because it is symmetric around $\tau = 0$ and $R_X(0) \geq R_X(\tau)$. R_3 is not valid, because it is not symmetric around 0, and R_4 is not valid because for $\tau \to \infty$ the autocorrelation function does not converge to μ^2 (it converges to a negative number!).

11.1.1 To compute $R_Y(t,\tau)$ we rewrite the definition:

$$\begin{aligned} R_Y(t,\tau) = &= E[Y(t)Y(t+\tau)] = E[(2+X(t))(2+X(t+\tau))] \\ &= E[4+2X(t)+2X(t+\tau)+X(t)X(t+\tau)] \\ &= 4+2E[X(t)]+2E[X(t+\tau)]+E[X(t)X(t+\tau)] \\ &= 4+0+0+R_X(t,\tau) = R_X(t,\tau)+4 \qquad (4.30) \end{aligned}$$

Because X is WSS, $E[X]$ and R_X do not depend on t and therefore also Y is WSS.

11.1.3 This is very simple, when we use Theorem 11.2 on page 396:

$$\begin{aligned} \mu_Y = 1 &= \mu_X \int h(u)du \\ &= 4\int_0^\infty e^{-u/a}du = \left[-4ae^{-t/a}\right]_0^\infty = 4a \qquad (4.31) \end{aligned}$$

Solving gives $a = 1/4$.

11.1.3 We know that the expected value of the incoming signal is $\mu_X = 4$, while the outgoing signal is 1. In a time-continuous LTI we can relate the input and output mean by

$$\mu_Y = \mu_X \int_{-\infty}^{\infty} h(t)dt \qquad (4.32)$$

Because

$$\int_{-\infty}^{\infty} h(t)dt = \int_0^\infty e^{-t/a}dt = \left[-ae^{-t/a}\right]_0^\infty = a \qquad (4.33)$$

Therefore $a = 1/4$.

E1 The expected value can directly be computed using the definition (and using that the expected value of X_n is 0):

$$E[W_n] = E[\frac{1}{2}(X_n + X_{n+1})] = \frac{1}{2}(E[X_n] + E[X_{n+1}]) = 0 \qquad (4.34)$$

Next, the variance (using that the variance of X_n is 1):

$$\begin{aligned} Var[W_n] &= E[W_n^2] - E[W_n]^2 = E[W_n^2] \qquad (4.35) \\ &= \frac{1}{4}E\left[X_n^2 + 2X_nX_{n-1} + X_{n-1}^2\right] \qquad (4.36) \\ &= \frac{1}{4}E[X_n^2] + \frac{1}{2}E[X_nX_{n-1}] + \frac{1}{4}E[X_{n-1}^2] \qquad (4.37) \\ &= 1/4 + 0 + 1/4 = 1/2 \qquad (4.38) \end{aligned}$$

Then the covariance:

$$\begin{aligned} Cov[W_{n+1}, W_n] &= E[W_{n+1}W_n] - E[W_{n-1}]E[W_n] = E[W_{n+1}W_n] \qquad (4.39) \\ &= \frac{1}{4}E[(X_{n+1}+X_{n+2})(X_n+X_{n+1})] \qquad (4.40) \\ &= \frac{1}{4}E[X_{n+1}X_n + X_{n+2}X_n + X_{n+1}X_{n+1} + X_{n+2}X_{n+1}] \qquad (4.41) \\ &= \frac{1}{4}(R_X(1) + R_X(-2) + R_X(0) + R_X(-1)) \qquad (4.42) \end{aligned}$$

Now use that X_n is iid:

$$Cov[W_{n+1}, W_n] = \frac{1}{4}\,(0 + 0 + 1 + 0) = \frac{1}{4} \tag{4.43}$$

Finally,

$$\rho_{W_{n+1},W_n} = \frac{Cov[W_{n+1}W_n]}{\sqrt{Var[W_{n+1}]Var[W_n]}} = \frac{1/4}{1/2} = \frac{1}{2} \tag{4.44}$$

Week 5

Using the autocorrelation function

11.2.1 (a) The impuls response becomes obvious when you rewrite it:

$$Y_n = \frac{X_{n+1} + X_n + X_{n-1}}{3} = \frac{X_{n+1}}{3} + \frac{X_n}{3} + \frac{X_{n-1}}{3} \tag{5.1}$$

So:

$$h_n = \begin{cases} \frac{1}{3} & n = -1, 0, 1, \\ 0 & \text{otherwise} \end{cases} \tag{5.2}$$

(b) Be strong!:

$$\begin{aligned}
R_Y[n,k] &= E[Y(n)Y(n+k)] && (5.3)\\
&= E[(\frac{1}{3}X_{n+1} + \frac{1}{3}X_n + \frac{1}{3}X_{n-1})(\frac{1}{3}X_{n+k+1} + \frac{1}{3}X_{n+k} + \frac{1}{3}X_{n+k-1})] && (5.4)\\
&= E[\frac{1}{9}X_{n+1}X_{n+k+1} + \frac{1}{9}X_{n+1}X_{n+k} + \frac{1}{9}X_{n+1}X_{n+k-1}] \\
&\quad +E[\frac{1}{9}X_nX_{n+k+1} + \frac{1}{9}X_nX_{n+k} + \frac{1}{9}X_nX_{n+k-1}] \\
&\quad +E[\frac{1}{9}X_{n-1}X_{n+k+1} + \frac{1}{9}X_{n-1}X_{n+k} + \frac{1}{9}X_{n-1}X_{n+k-1}] && (5.5)\\
&= \frac{1}{9}(E[X_nX_{n+k}] + E[X_nX_{n+k-1}] + E[X_nX_{n+k-2}] \\
&\quad +E[X_nX_{n+k+1}] + E[X_nX_{n+k}] + E[X_nX_{n+k-1}] \\
&\quad +E[X_nX_{n+k+2}] + E[X_nX_{n+k+1}] + E[X_nX_{n+k}]) && (5.6)\\
&= \frac{1}{9}(R_X(k) + R_X(k-1) + R_X(k-2) \\
&\quad +R_X(k+1) + R_X(k) + R_X(k-1) \\
&\quad +R_X(k+2) + R_X(k+1) + R_X(k) && (5.7)\\
&= \frac{1}{9}\left(3R_X(k) + 2R_X(k-1) + 2R_X(k+1) + R_X(k-2) + R_X(k+2)\right) \\
&= \begin{cases} \frac{1}{3} & k = 0 \\ \frac{2}{9} & |k| = 1 \\ \frac{1}{9} & |k| = 2 \\ 0 & \text{otherwise} \end{cases} && (5.8)
\end{aligned}$$

11.2.3 Note that there is an error/inconsistency in the question! It is given that $\mu_Y = 1$, but that contradicts $\lim_{n\to\infty} R_Y[n] = \mu_Y^2$. The function that is defined is *not* the autocorrelation, but the autocovariance function:

$$C_Y[n] = \begin{cases} 3 & n = 0 \\ 2 & |n| = 1 \\ 0.5 & |n| = 2 \\ 0 & \text{otherwise} \end{cases} \tag{5.9}$$

Therefore the autocorrelation function becomes $C_Y[n] = R_Y[n] - \mu_Y^2$:

$$R_Y[n] = \begin{cases} 4 & n = 0 \\ 3 & |n| = 1 \\ 1.5 & |n| = 2 \\ 1 & \text{otherwise} \end{cases} \tag{5.10}$$

(a) Using Theorem 11.5:

$$\mu_W = \mu_Y \sum_n h_n = 2\mu_Y = 2 \tag{5.11}$$

(b) The autocorrelation function of a filtered signal is:

$$\begin{aligned} R_W[n] &= \sum_i \sum_j h_i h_j R_Y[n+i-j] & (5.12) \\ &= \sum_{i=0}^{1} \sum_{j=0}^{1} R_y[n+i-j] & (5.13) \\ &= R_Y[n-1] + R_Y[n] + R_Y[n] + R_Y[n+1] & (5.14) \end{aligned}$$

Now we try for different values of n:

$$R_W[n] = \begin{cases} R_Y[-1] + 2R_Y[0] + R_Y[1], & n = 0 \\ R_Y[0] + 2R_Y[1] + R_Y[2], & n = 1 \\ R_Y[-2] + 2R_Y[-1] + R_Y[0], & n = -1 \\ R_Y[1] + 2R_Y[2] + R_Y[3], & n = 2 \\ R_Y[-3] + 2R_Y[-2] + R_Y[-1], & n = -2 \\ R_Y[2] + 2R_Y[3] + R_Y[4], & n = 3 \\ R_Y[-4] + 2R_Y[-3] + R_Y[-2], & n = -3 \\ R_Y[3] + 2R_Y[4] + R_Y[5], & n = 4 \\ \ldots \end{cases} = \begin{cases} 14, & n = 0 \\ 11.5, & n = 1 \\ 11.5, & n = -1 \\ 7, & n = 2 \\ 7, & n = -2 \\ 4.5, & n = 3 \\ 4.5, & n = -3 \\ 4, & n = 4 \\ \ldots \end{cases} = \begin{cases} 14 & n = 0 \\ 11.5 & |n| = 1 \\ 7 & |n| = 2 \\ 4.5 & |n| = 3 \\ 4 & \text{otherwise} \end{cases} \tag{5.15}$$

(c) When we have the autocorrelation function, the variance is just:

$$Var[W_n] = E[W_n^2] - E[W_n]^2 = 14 - 2^2 = 10 \tag{5.16}$$

11.2.8 Note: The answer below is incorrect.

Note that in the definition of $Y_n = a(X_n + Y_{n-1})$ there appears also a Y_{n-1} on the right side of the equation. We have a recursive definition. Using the definition of Y_n we expand:

$$\begin{aligned} Y_n &= aX_n + aY_{n-1} \\ &= aX_n + a(aX_{n-1} + aY_{n_2}) \\ &= aX_n + a^2X_{n-1} + a^2(aX_{n-2} + aY_{n-3}) \\ &= \sum_{i=0}^{n} a^{i+1} X_{n-i} + a^n Y_0 \\ &= \sum_{i=0}^{n} a^{i+1} X_{n-i} & (5.17) \end{aligned}$$

Because we are looking at standard normal distributed X_n we know that $E[X_n] = 0$.

$$E[Y_n] = E[\sum_{i=0}^{n} a^{i+1} X_{n-i}] = \sum_{i=0}^{n} a^{i+1} E[X_{n-i}] = 0 \tag{5.18}$$

To find the autocorrelation function

$$R_Y[m,k] = E\left[(\sum_{i=0}^{m} a^{i+1} X_{m-i})(\sum_{j=0}^{m+k} a^{j+1} X_{m+k-j}) \right] \tag{5.19}$$

we first note that X_n is iid, with a variance of 1, so

$$E[X_i X_j] = \begin{cases} 1 & i = j \\ 0 & \text{otherwise} \end{cases} \tag{5.20}$$

So only the terms in (5.19) survive for which the indices in X_{m-i} and X_{m+k-j} are equal. In the current notation it is not so easy, so I re-index:

$$i' = m - i, \qquad \text{and therefore} \quad i + 1 = m + 1 - i' \tag{5.21}$$

$$j' = m + k - j, \qquad \text{and therefore} \quad j + 1 = m + k + 1 - j' \tag{5.22}$$

Using these indices we get (where we assumed $k \geq 0$ for now):

$$\begin{aligned} R_Y[m,k] &= E\left[(\sum_{i'=0}^{m} a^{m+1-i'} X_{i'})(\sum_{j'=0}^{m+k} a^{m+k+1-j'} X_{j'}) \right] \\ &= \sum_{i'=0}^{m} \sum_{j'=0}^{m+k} a^{m+1-i'} a^{m+k+1-j'} E[X_{i'} X_{j'}] \\ &= \sum_{i'=0}^{m} a^{m+1-i'} a^{m+k+1-i'} E[X_{i'}^2] \qquad (5.23) \\ &= \sum_{i=0}^{m} a^{m+1-i} a^{m+k+1-i} \\ &= \sum_{i=0}^{m} a^{2m+2+k-2i} = a^{2m+2+k} \sum_{i=0}^{m} a^{-2i} = a^{2m+2+k} \sum_{i=0}^{m} (a^{-2})^i \qquad (5.24) \end{aligned}$$

Now we have to use Math Fact B.4, from which we can conclude that:

$$\sum_{i=0}^{m} (a^{-2})^i = \frac{1 - (a^{-2})^{m+1}}{1 - a^{-2}} \tag{5.25}$$

So therefore we found:

$$R_Y[m,k] = a^{2m+2+k} \frac{1 - (a^{-2})^{m+1}}{1 - a^{-2}}, \quad \text{for} \quad k \geq 0 \tag{5.26}$$

For $k < 0$ a very similar derivation can be given, only that the sum in (5.23) does not run to m, but just up to $m + k$ (which is smaller than m because $k < 0$. In this situation we obtain:

$$R_Y[m,k] = a^{2m+2+k} \frac{1 - (a^{-2})^{m+k+1}}{1 - a^{-2}}, \quad \text{for} \quad k < 0 \tag{5.27}$$

All in all, from (5.26) and (5.27) we see that R_Y actually depends on k, so Y_n is *not* WSS.

E2 Because X_n is iid, we already know that $E[X_m] = E[X_{m+k}]$.

Now to find the autocovariance function $C_Y[m,k]$, we use the defition again:

$$C_Y[m,k] = E[Y_m Y_{m+k}] - E[Y_m]E[Y_{m+k}] \tag{5.28}$$

First we compute:

$$E[Y_m] = E[X_{n+1}] + E[X_n] + E[X_{n-1}] = 0 \tag{5.29}$$

and second

$$\begin{aligned}
C_Y[m,k] &= E[Y_m Y_{m+k}] - E[Y_m]E[Y_{m+k}] = E[Y_m Y_{m+k}] && (5.30)\\
&= E[(X_{m+1} + X_m + X_{m-1})(X_{m+1+k} + X_{m+k} + X_{m-1+k})] && (5.31)\\
&= E[X_{m+1}X_{m+1+k}] + E[X_{m+1}X_{m+k}] + E[X_{m+1}X_{m+k-1}]\\
&\quad + E[X_m X_{m+1+k}] + E[X_m X_{m+k}] + E[X_m X_{m+k-1}]\\
&\quad + E[X_{m-1}X_{m+1+k}] + E[X_{m-1}X_{m+k}] + E[X_{m-1}X_{m+k-1}]\\
&= R_X(m+1,k) + R_X(m+1,k) + R_X(m+1,k-2)\\
&\quad + R_X(m,k+1) + R_X(m,k) + R_X(m,k-1)\\
&\quad + R_X(m-1,k+2) + R_X(m-1,k+1) + R_X(m-1,k)
\end{aligned}$$

Because X_n is stationary, it is WSS and $C_X[m,k]$ does not depend on m. Furthermore, because $E[X_m] = 0$ we also have that $C_X[k] = R_X[k]$. So we get:

$$\begin{aligned}
C_Y[m,k] &= R_X(,k) + R_X(k-1) + R_X(k-2)\\
&\quad + R_X(k+1) + R_X(k) + R_X(k-1)\\
&\quad + R_X(k+2) + R_X(k+1) + R_X(k)\\
&= 3R_X(k) + 2R_X(k-1) + R_X(k-2) + 2R_X(k+1) + R_X(k+2) && (5.32)
\end{aligned}$$

Now we fill in various values for k:

$$\begin{aligned}
k = -3 &\rightarrow C_Y(-3) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 0 + 0 = 0\\
k = -2 &\rightarrow C_Y(-2) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 0 + 1 = 1\\
k = -1 &\rightarrow C_Y(-1) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 1 + 0 = 2\\
k = 0 &\rightarrow C_Y(0) = 3\cdot 1 + 2\cdot 0 + 0 + 2\cdot 0 + 0 = 3\\
k = 1 &\rightarrow C_Y(1) = 3\cdot 0 + 2\cdot 1 + 0 + 2\cdot 0 + 0 = 2\\
k = 2 &\rightarrow C_Y(2) = 3\cdot 0 + 2\cdot 0 + 1 + 2\cdot 0 + 0 = 1\\
k = 3 &\rightarrow C_Y(2) = 3\cdot 0 + 2\cdot 0 + 0 + 2\cdot 0 + 0 = 0
\end{aligned}$$

So in total:

$$C_Y[m,k] = \begin{cases} 3 & k = 0\\ 2 & |k| = 1\\ 1 & |k| = 2\\ 0 & \text{otherwise} \end{cases} \tag{5.33}$$

E3 Again, because X_n is iid, it is stationary and therefore also WSS. That means that $E[X_n] = E[X_{n+k}] = 0$ (given) and that

$$C_X[m,k] = C_X[k] = R_X[k] - E[X_m]E[X_{m+k}] = R_X[k] = \begin{cases} \sigma^2 & k = 0\\ 0 & \text{otherwise} \end{cases} \tag{5.34}$$

(a) Just fill in:

$$E[Y_n] = \frac{1}{2}E[X_n + Y_{n-1}] = \frac{1}{2}E[X_n] + \frac{1}{2}E[Y_{n-1}] = 0 + \frac{1}{2}E[Y_{n-1}] \tag{5.35}$$

Now we have $E[Y]$ on both sides of the equation. The only solution is

$$E[Y_n] = 0 \tag{5.36}$$

(b) Because X_n are iid, it is most efficient to rewrite Var using Theorem 4.15, pg 173:

$$\begin{aligned}
Var[Y_n] &= E[\frac{1}{2}(X_n + Y_{n-1})] \\
&= Var[\frac{1}{2}(X_n + \frac{1}{2}X_{n-1} + \frac{1}{2}Y_{n-2})] \\
&= Var[\frac{1}{2}(X_n + \frac{1}{2}X_{n-1} + \frac{1}{4}X_{n-2} + \frac{1}{4}Y_{n-3})] \\
&= Var[\frac{1}{2}X_n + \frac{1}{4}X_{n-1} + \frac{1}{8}X_{n-2} + ...] \\
&= \left[\frac{1}{2^2} + \frac{1}{4^2} + \frac{1}{8^2} + ...\right] Var[X_n] && (5.37) \\
&= \sum_{i=1}^{\infty} \left(\frac{1}{4}\right)^i \sigma^2 = \left[\sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i - 1\right] \sigma^2 && (5.38)
\end{aligned}$$

This series we can find in the book. Using Math Fact B.5

$$\left[\sum_{i=0}^{\infty} \left(\frac{1}{4}\right)^i - 1\right] \sigma^2 = \left(\frac{1}{1-1/4} - 1\right) \sigma^2 = \left(\frac{4}{3} - 1\right) \sigma^2 = \sigma^2/3 \tag{5.39}$$

(c) For the covariance, we expand Y_n similarly as in (b). In the second step, we use that X_n is iid, and therefore $E[X_n X_{n+k}] = 0$ for $k \neq 0$:

$$\begin{aligned}
Cov[Y_{n+1}, Y_n] &= E[(\frac{1}{2}X_n + \frac{1}{4}X_{n-1} + \frac{1}{8}X_{n-2} + ...)(\frac{1}{2}X_{n-1} + \frac{1}{4}X_{n-2} + \frac{1}{8}X_{n-3} + ...)] \\
&= \sum_{i=1}^{\infty} \frac{1}{2^i}\frac{1}{2^{i-1}} E[X_i^2] = \sum_{i=0}^{\infty} \frac{1}{2^{i+1}}\frac{1}{2^i} E[X_i^2] && (5.40) \\
&= \frac{1}{2}\sum_{i=0}^{\infty} \frac{1}{2^i}\frac{1}{2^i} E[X_i^2] = \frac{1}{2}\sum_{i=0}^{\infty} \frac{1}{4^i} E[X_i^2] && (5.41) \\
&= \frac{1}{2}\frac{1}{1-1/4} E[X_i^2] = \sigma^2/6 && (5.42)
\end{aligned}$$

(d) We now have to combine previous results:

$$\rho_{Y_{n+1},Y_n} = \frac{Cov[Y_{n+1}, Y_n]}{\sqrt{Var[Y_{n+1}]Var[Y_n]}} = \frac{\sigma^2/6}{\sigma^2/3} = \frac{1}{2} \tag{5.43}$$

11.5.1 With the use of Table 11.1, pg 413:

$$\begin{aligned}
S_X(f) &= \int R_X(\tau) e^{-j2\pi f\tau} d\tau \\
&= 10 \int \frac{\sin 2000\pi\tau}{2000\pi\tau} e^{-j2\pi f\tau} d\tau + \frac{10}{2} \int \frac{\sin 1000\pi\tau}{1000\pi\tau} e^{-j2\pi f\tau} d\tau \\
&= 10 \frac{1}{2000} \text{rect}\left(\frac{f}{2000}\right) + 5 \frac{1}{1000} \text{rect}\left(\frac{f}{1000}\right) && (5.44)
\end{aligned}$$

11.8.2 (a) Using Table 11.1, pg 413, we see that the inverse Fourier transform of $S_W(f) = 1$ is $R_W(\tau) = \delta(\tau)$.

(b) Now we can take advantage of the Fourier transform:

$$S_Y(f) = |H(f)|^2 S_W(f) = |H(f)|^2 \tag{5.45}$$

(where $H(f)$ is given in the exercise).

(c) Use the definition:

$$E[Y^2(t)] = \int S_Y(f)df = \int_{-B/2}^{B/2} df = B \tag{5.46}$$

(d)

$$E[Y(t)] = E[W(t)]H(0) = 0 \tag{5.47}$$

11.8.5 (a) The power of a signal can directly be computed using Theorem 11.13:

$$E[X^2(t)] = \int S_X(f)df = \int_{-100}^{100} 1 \cdot 10^{-4} df = 0.02 \tag{5.48}$$

(b) Because we now have everything in the Fourier domain:

$$S_{XY}(f) = H(f)S_X(f) = \begin{cases} \frac{10^{-4}}{100\pi j 2\pi f} & |f| \leq 100 \\ 0 & \text{otherwise} \end{cases} \tag{5.49}$$

(c) Swapping X and Y in $S_{XY}(f)$ means that you swap the X and Y in $R_{XY}(\tau)$. From Theorem 10.14, pg 382, we see that $R_{XY}(\tau) = R_{YX}(-\tau)$. So when we fill this in, in the definition of S_{YX}

$$S_{YX}(f) = \int R_{YX}(\tau)e^{-j2\pi f\tau}d\tau = \int R_{XY}(-\tau)e^{-j2\pi f\tau}d\tau = S^*_{XY}(f) \tag{5.50}$$

where in the last step Table 11.1, pg 413 is used (to find the transform of $g(-\tau)$).

(d)

$$S_Y(f) = H^*(f)S_{XY}(f) = |H(f)|^2 S_X(f) = \begin{cases} \frac{10^4}{10^4\pi^2 + (2\pi f)^2} & |f| \leq 100 \\ 0 & \text{otherwise} \end{cases} \tag{5.51}$$

(e) Just compute the integral:

$$\begin{aligned}
E[Y^2(t)] &= \int S_Y(f)df = \int_{-100}^{100} \frac{10^{-4}}{10^4\pi^2 + 4\pi^2 f^2} df \\
&= \frac{10^{-4}}{\pi^2} \int_{-100}^{100} \frac{1}{10^4 + 4f^2} df \\
&= \frac{10^{-8}}{\pi^2} \int_{-100}^{100} \frac{1}{1 + (0.02f)^2} df \\
&= \frac{10^{-8}}{0.02\pi^2} \int_{-100}^{100} \frac{1}{1 + (0.02f)^2} d(0.02f) \\
&= \frac{10^{-8}}{0.02\pi^2} \left(\tan^{-1}(0.02 \cdot 100) - \tan^{-1}(0.02 \cdot -100)\right) \\
&= \frac{10^{-8}}{0.02\pi^2} 2\tan^{-1}(2) = 1.12 \cdot 10^{-5}
\end{aligned}$$

Week 6

Statistical estimation

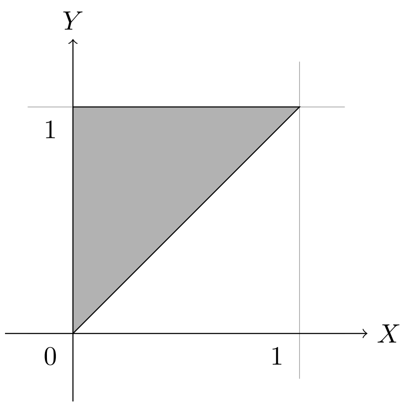
6.1.5 First note that $C_X[m,k] = R_X[m,k] - E[X_m]E[X_{m+k}] = R_X[m,k]$. Then:

$$E[Y_n] = E[X_n/3 + X_{n-1}/3 + X_{n-2}/3] = 0 \tag{6.1}$$

For the variance:

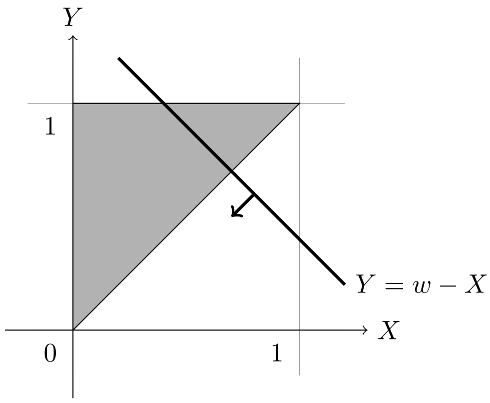
$$\begin{aligned}
Var[Y_n] &= E[Y_n^2] - E[Y_n]^2 = E[Y_n^2] \\
&= \frac{1}{9}E[(X_n + X_{n-1} + X_{n-2})^2] \\
&= \frac{1}{9}E[X_n^2 + X_{n-1}^2 + X_{n-2}^2 + 2X_nX_{n-1} + 2X_nX_{n-2} + 2X_{n-1}X_{n-2}] \\
&= \frac{1}{9}[3R_X(0) + 2R_X(1) + 2R_X(2) + 2R_X(1)] \\
&= \frac{3 + 2/4 + 2/4}{9} = \frac{4}{9}
\end{aligned} \tag{6.2}$$

6.2.1 First a picture of the region where the pdf is defined:

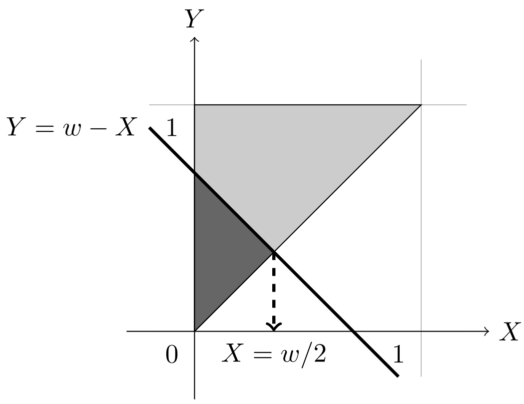


The standard procedure is now to derive the cumulative distribution function $F_W(w)$ for W, and then take the derivative. To find the CDF, we have to find the integration area.

Ok, what values $W = X + Y$ can take? The minimum value for $W = 0$ at $X = 0, Y = 0$. The maximum value is $W = 2$ for $X = Y = 1$. On the diagonal line $Y = 1 - X$ we get $W = 1$. So it seems we have to integrate over the region below the diagonal line $Y = w - X$:



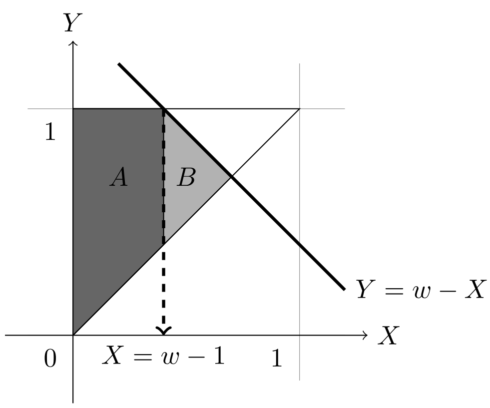
The integration therefore falls apart in two parts, one for $0 \leq w \leq 1$ and another one for $1 < w \leq 2$.



Starting with $0 \leq w \leq 1$, we first integrate over Y, and then over X. The integration of Y runs between x and $w - x$. The integration over X runs until we reach the point that $Y = w - X = X$, or when $x = w/2$:

$$
\begin{aligned}
F_W(w) &= \int_0^{w/2} \int_x^{w-x} 2dydx \\
&= \in_0^{w/2} [2y]_x^{w-x}\, dx = \int_0^{w/2} (2w - 4x)dx \\
&= \left[2wx - 2x^2\right]_0^{w/2} = w^2 - w^2/2 = w^2/2 \qquad (6.3)
\end{aligned}
$$

The second integration is harder, and we have to split the region in two parts, A and B:



Integrating first over Y and then over X:

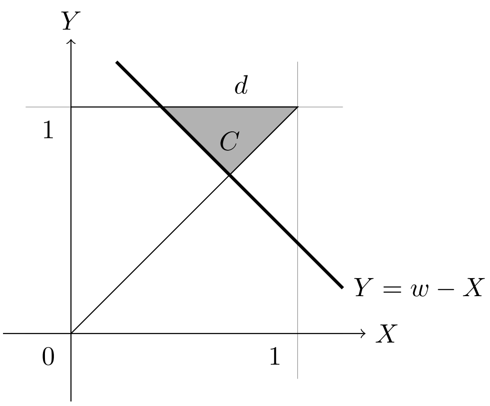
$$
\begin{aligned}
F_W(w) &= A + B = \int_0^{w-1}\int_x^1 2dydx + \int_{w-1}^{w/2}\int_x^{w-x} 2dydx \\
&= \int_0^{w-1} [2y]_x^1\, dx + \int_{w-1}^{w/2} [2y]_x^{w-x}\, dx \\
&= 2\int_0^{w-1}(1-x)dx + 2\int_{w-1}^{w/2}(w-2x)dx \\
&= 2\left[x - \frac{1}{2}x^2\right]_0^{w-1} + 2\left[wx - x^2\right]_{w-1}^{w/2} \\
&= 2(w-1-\frac{1}{2}(w-1)^2) + 2(w\cdot w/2 - (w/2)^2) - 2(w(w-1) - (w-1)^2) \\
&= -\frac{1}{2}w^2 + 2w - 1 \qquad (6.4)
\end{aligned}
$$

So we finally found that:

$$
F_W(w) = \begin{cases} 0 & w < 0 \\ \frac{1}{2}w^2 & 0 \le w < 1 \\ -\frac{1}{2}w^2 + 2w - 1 & 1 \le w < 2 \\ 1 & w \ge 2 \end{cases} \qquad (6.5)
$$

To find the pdf, we have to take the derivative with respect to w:

$$
f_W(w) = \begin{cases} 0 & w < 0 \\ w & 0 \le w < 1 \\ -w + 2 & 1 \le w < 2 \\ 0 & w \ge 2 \end{cases} \qquad (6.6)
$$



Note: from symmetry reasons, you could also have argued that

$$
F_W(w) = 1 - \text{area}\, C \qquad (6.7)
$$

And area C we computed above, that was $\frac{1}{2}w^2$. But beware, that instead of w, we should have used distance d! You can express d in terms of w by $d = 2 - w$, and therefore:

$$
F_W(w) = 1 - \frac{1}{2}d^2 = 1 - (2-w)^2 = -1 + 2w - \frac{1}{2}w^2 \qquad (6.8)
$$

6.6.1 We first have to read the question carefully. We have a random variable $X = W+R = W+3$, and on top of that $A = 12X = 12W + 36$.

(a) $E[X] = E[W + R] = E[W + 3] = 3 + E[W] = 3 + 5 = 8$.

(b) $Var[X] = Var[W + 3] = Var[W] = \frac{(10-0)^2}{12} = 100/12$.

(c) $E[A] = E[\sum_{i-1}^{12} X_i] = 12E[X] = 12 \cdot 8 = 96$.

(d) $\sigma_A = \sqrt{Var[A]} = \sqrt{12Var[X]} = \sqrt{12 \cdot 100/12} = 10$ because

$$Var[A] = Var[\sum_{i=1}^{12} X_i] = \sum_{i=1}^{12} Var[X_i] = 12Var[X] \tag{6.9}$$

(e) With a bit of goodwill, we may say that 12 repetitions are sufficient to use the central limit theorem:

$$P[A > 116] = 1 - \Phi(\frac{116 - 96}{10}) = 0.0228 \tag{6.10}$$

(where we have used the table on page 123 in the book to find the values for Φ).

(f)

$$P[A < 86] = \Phi(\frac{86 - 96}{10}) = 0.1587 \tag{6.11}$$

7.1.2 (a) It is given that X_i are uniform random variables. For the uniform random variable, you can find in the tables in the Appendix A in the book that:

$$\mu_X = E[X] = \frac{a + b}{2} = 7 \tag{6.12}$$

and

$$Var[X] = \frac{(b - a)^2}{12} = 3 \tag{6.13}$$

Now we have two equations, with two unknowns. Solving gives:

$$a = 4 \qquad b = 10 \tag{6.14}$$

Note that the pdf should integrate to one, and the constant should therefore be $1/(10 - 4)$. The pdf becomes:

$$f_X(x) = \begin{cases} \frac{1}{6} & 4 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases} \tag{6.15}$$

(b) Realizing that we are looking at a sum of 16 random variables, we can use Theorem 7.1, and:

$$Var[M_{16}(X)] = \frac{Var[X]}{16} = \frac{3}{16} \tag{6.16}$$

(c)

$$P[X_1 \geq 9] = \int_9^\infty f_{X_1}(x)dx = \int_9^{10} \frac{1}{6}dx = \frac{1}{6} \tag{6.17}$$

(d) From question (b) we see that the variance of M_{16} is smaller than the variance of X_1. When the variance is smaller, it is less likely that realizations appear that are larger than 9. So I would expect $P[M_{16}(X) > 9]$ to be smaller than $P[X_1 > 9]$.

So, 16 looks like a large number, so we apply the central limit theorem. We have to rephrase the problem in terms of sums of random variables, and use Definition 6.2, pg 259.

$$P[M_{16} > 9] = P[(X_1+...+X_{16})/16 > 9] = P[X_1+...+X_{16} > 144] = 1-P[X_1+...+X_{16} \leq 144] \tag{6.18}$$

Then we apply

$$P[M_{16} > 9] = 1 - \Phi\left(\frac{144 - 16\mu_X}{\sqrt{16}\sigma_X}\right) = 1 - \Phi(4.61) = 2.01 \cdot 10^{-6} \tag{6.19}$$

So indeed, this is much smaller than we obtained in (c).

7.2.4 Let us first consider the number of dice rolls that we need to obtain the *first* snake eyes. At each roll of the two dice, we have a probability of $p = 1/6 \cdot 1/6$ of obtaining snake eyes. The number of rolls we need for the first snake eyes has therefore a geometric distribution, with a change of success of 1/36:

$$P_X(x) = \begin{cases} p(1-p)^{x-1} & x = 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{6.20}$$

From the appendix in the book we find that $E[X] = 1/p$ and $Var[X] = (1-p)/p^2$. Now we want to wait for the *third* occurrance of snake eyes. Because the dice are thrown independently, we can sum the number of rolls of three independent geometrical random variables:

$$R = X + Y + Z \tag{6.21}$$

where X, Y and Z are each random variables that indicate the number of rolls of finding snake eyes for the first time. Therefore:

$$E[R] = E[X] + E[Y] + E[Z] = 3 \cdot 1/p = 108 \tag{6.22}$$
$$Var[R] = Var[X] + Var[Y] + Var[Z] = 3 \cdot (1-p)/p^2 = 3680 \tag{6.23}$$

(a) The Markov inequality states:

$$P\left[X \geq c^2\right] \leq \frac{E[X]}{c^2} \tag{6.24}$$

For our situation, it means:

$$P\left[R \geq 250\right] \leq \frac{E[X]}{250} = \frac{108}{250} = 0.43 \tag{6.25}$$

(b) The Chebychev inequality states:

$$P[|Y - \mu_Y| \geq c] \leq \frac{Var[Y]}{c^2} \tag{6.26}$$

which means for us:

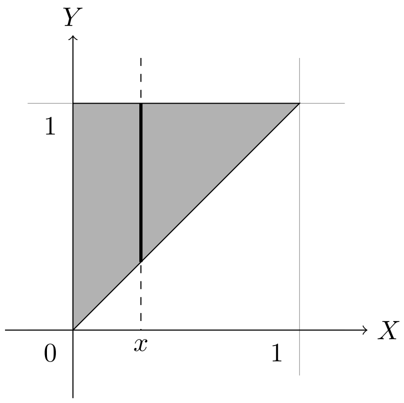
$$P[P \geq 250] =]P[|R - 108| \geq 250 - 108] \leq \frac{3780}{142^2} = 0.19 \tag{6.27}$$

(c) The true distribution requires the convolution of the geometric distributions:

$$P_R(r) = P_X(x) * P_Y(y) * P_Z(z) \tag{6.28}$$

This may be a bit too much for a homework exercise, so we skip this...

9.1.2 (a) First we make a picture:



Then we use the definition, and we integrate over y:

$$\begin{aligned} f_X(x) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_x^1 6(y-x)dy \\ &= \left[3y^2 - 6xy\right]_x^1 = 3x^2 - 6x + 3 \end{aligned} \tag{6.29}$$

(b)

$$\hat{x}_B = E[X] = \int_0^1 x(3x^2 - 6x + 3)dx = \left[\frac{3}{4}x^4 - 2x^3 + \frac{3}{2}x^2\right]_0^1 = \frac{1}{4} \tag{6.30}$$

(c) Define event $A : X < 0.5$. The probability of this event is:

$$P[X < 0.5] = \int_0^{0.5} (3x^2 - 6x + 3)dx = [x^3 - 3x^2 + 3x]_0^{0.5} = \frac{7}{8} \tag{6.31}$$

Then the minimum MSE estimate becomes:

$$\hat{X}_A = E[X|A] = \frac{8}{5}\int_0^{0.5} x(3x^2 - 6x + 3)dx = \frac{8}{5}\left[\frac{3}{4}x^4 - 2x^3 + \frac{3}{2}x^2\right]_0^{0.5} = \frac{55}{8} \tag{6.32}$$

(d) Again the definition, but now integrate over x:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} = \begin{cases} \int_0^y 6(y-x)dx, & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} = \begin{cases} \left[6xy - 3x^2\right]_0^y & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \\ &= \begin{cases} 3y^2 & 0 < y < 1 \\ 0 & \text{otherwise} \end{cases} \end{aligned} \tag{6.33}$$

(e) Again:

$$\hat{y}_B = E[Y] = \int_0^1 y \cdot 3y^2 dy = \left[\frac{3}{4}y^4\right]_0^1 = \frac{3}{4} \tag{6.34}$$

(f) Finally, the event $Y > 0.5$ has probability:

$$P[Y > 0.5] = \int_{0.5}^1 3y^2 dy = [y^3]_{0.5}^1 = 1 - 0.5^3 = \frac{7}{8} \tag{6.35}$$

This gives for our minimum MSE:

$$E[Y|Y > 0.5] = \frac{8}{7}\int_{0.5}^1 y \cdot f_{Y|Y>0.5}(y)dy \tag{6.36}$$

$$= \int_{0.5}^1 y \cdot \frac{8}{7}3y^2\, dy \tag{6.37}$$

$$= \frac{8}{7}\int_{0.5}^1 \frac{3}{4}y^3 dy = \frac{8}{7}[y^4]_{0.5}^1 \tag{6.38}$$

Week 7

Markov chains

12.1.1 This is not so hard, all data is available:

$$P = \begin{bmatrix} 1/2 & 1/2 & 0 \\ 1/2 & 1/2 & 0 \\ 1/4 & 1/4 & 1/2 \end{bmatrix} \tag{7.1}$$

12.2.2 To find the n-step probabilities, we have to compute P^n. The easiest is to do an eigenvalue decomposition of P: $P = SDS^{-1}$, so that we can easily compute $P = SD^nS^{-1}$. For the eigenvalue decomposition, we need the eigenvalues and eigenvectors. First we find the eigenvalues by solving:

$$\det(P - \lambda\mathcal{I}) = 0 \tag{7.2}$$

So:

$$\begin{aligned} \det\left(\begin{bmatrix} 1/2-\lambda & 1/2 & 0 \\ 1/2 & 1/2-\lambda & 0 \\ 1/4 & 1/4 & 1/2-\lambda \end{bmatrix}\right) &= 0 = (0.5-\lambda)^3 - 0.5 \cdot 0.5(0.5-\lambda)) \\ &= (0.5-\lambda)\lambda(1-\lambda) \end{aligned} \tag{7.3}$$

The eigenvalues are therefore $\lambda_1 = 1$, $\lambda_2 = 0.5$ and $\lambda_3 = 0$. Now we have to find the corresponding eigenvectors:

$$(P - \lambda_1\mathcal{I})s_1 = \begin{bmatrix} 1/2-\lambda & 1/2 & 0 \\ 1/2 & 1/2-\lambda & 0 \\ 1/4 & 1/4 & 1/2-\lambda \end{bmatrix} s_1 \tag{7.4}$$

$$= \begin{bmatrix} 1/2-1 & 1/2 & 0 \\ 1/2 & 1/2-1 & 0 \\ 1/4 & 1/4 & 1/2-1 \end{bmatrix} s_1 = \begin{bmatrix} -1/2 & 1/2 & 0 \\ 1/2 & -1/2 & 0 \\ 1/4 & 1/4 & -1/2 \end{bmatrix} s_1 = 0$$

Call the individual components of s: $s_1 = (a, b, c)$, then we have to solve the following equations:

$$-0.5a + 0.5b = 0 \tag{7.5}$$

$$0.5a - 0.5b = 0 \tag{7.6}$$

$$0.25a + 0.25b - 0.5c = 0 \tag{7.7}$$

This gives the solution $a = b = c = 1$, so

$$s_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \tag{7.8}$$

In a similar way we obtain:

$$s_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \qquad s_3 = \begin{bmatrix} -1 \\ 1 \\ 0 \end{bmatrix} \tag{7.9}$$

Combining everything gives:

$$P = SDS^{-1} = \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0.5 \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 1 \end{bmatrix} \tag{7.10}$$

and thus:

$$\begin{aligned} P^n = SD^nS^{-1} &= \begin{bmatrix} 1 & -1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & (0.5)^n \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 & 0 \\ 1 & 0 & 0 \\ 1 & 0 & (0.5)^n \end{bmatrix} \begin{bmatrix} 0.5 & 0.5 & 0 \\ -0.5 & 0.5 & 0 \\ -0.5 & -0.5 & 1 \end{bmatrix} \\ &= \begin{bmatrix} 0.5 & 0.5 & 0 \\ 0.5 & 0.5 & 0 \\ 0.5-(0.5)^{n+1} & 0.5-(0.5)^{n+1} & (0.5)^n \end{bmatrix} \end{aligned} \tag{7.11}$$

12.5.2 Now we know the transition matrix P, and we have to solve

$$\pi = \pi P \tag{7.12}$$

Because P is a transition matrix (each row adds up to one), this matrix equation defines 2 independent equations. To solve three variables, we need the last independent equation:

$$\pi_0 + \pi_1 + \pi_2 = 1 \tag{7.13}$$

The set of equations becomes:

$$\begin{aligned} \pi_0 &= 0.5\pi_0 + 0.5\pi_1 + 0.25\pi_2 \\ \pi_2 &= 0.5\pi_2 \\ 1 &= \pi_0 + \pi_1 + \pi_2 \end{aligned}$$

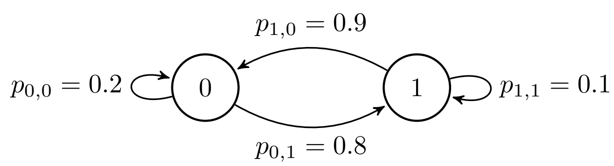
From the second equation, we see that $\pi_2 = 0$, so we are left with:

$$\begin{aligned} \pi_0 &= 0.5\pi_0 + 0.5\pi_1 \\ 1 &= \pi_0 + \pi_1 \end{aligned}$$

Solving this gives $\pi_0 = \pi_1 = 0.5$, and therefore:

$$\pi = \begin{bmatrix} 0.5 & 0.5 & 0 \end{bmatrix} \tag{7.14}$$

12.1.2 The Markov chain becomes:



and the state transition matrix becomes:

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0.9 & 0.1 \end{bmatrix} \tag{7.15}$$

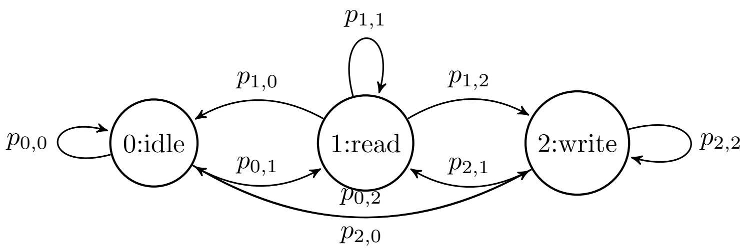
12.2.1 Again, for the n-step probabilities we need an eigenvalue decomposition. From example 12.6, pg.449, we can directly see that the eigenvalues become $\lambda_1 = 1$ and $\lambda_2 = -0.7$. Also the eigenvectors can directly be obtained:

$$P = SDS^{-1} = \begin{bmatrix} 1 & \frac{-0.8}{1.7} \\ 1 & \frac{0.9}{1.7} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & -0.7 \end{bmatrix} \begin{bmatrix} \frac{0.9}{1.7} & \frac{0.8}{1.7} \\ -1 & 1 \end{bmatrix}$$

and thus

$$\begin{aligned} P^n &= \begin{bmatrix} 1 & \frac{-0.8}{1.7} \\ 1 & \frac{0.9}{1.7} \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & (-0.7)^n \end{bmatrix} \begin{bmatrix} \frac{0.9}{1.7} & \frac{0.8}{1.7} \\ -1 & 1 \end{bmatrix} \\ &= \frac{1}{1.7}\left(\begin{bmatrix} 0.9 & 0.8 \\ 0.9 & 0.8 \end{bmatrix} + (0.7)^n \begin{bmatrix} 0.8 & -0.8 \\ 0.9 & -0.8 \end{bmatrix} \right) \end{aligned} \tag{7.16}$$

12.1.5 We start with example 12.3, page 446. We have a Markov chain like this:



where the states are 0: idle, 1: read, 2: write.

Now the most tricky part comes. What does this mean: "... each read or write operation reads or writes an entire file and that files contain a geometric number of sectors with mean 50." This means that the length N of a file has geometric distribution:

$$P_N(n) = \begin{cases} (1-p)^{n-1}p & n = 1, 2, ... \\ 0 & \text{otherwise} \end{cases} \tag{7.17}$$

Furthermore, it is given that the average length of the file is 50. For a geometric distribution holds that $E[N] = 1/p$, and therefore we can conclude that $p = 1/50$.

This can be easily modeled in a Markov chain. When we happen to be in a Read state, at each time step with probability $(1-p)$ we keep reading, and with probability p we reached the end of the file, and we jump out of the state Read. The same holds for the Write state. With this we can conclude:

$$p_{1,1} = 1 - 1/50 = \frac{980}{1000}, \quad p_{2,2} = 1 - 1/50 = \frac{980}{1000} \tag{7.18}$$

For the Idle state, something very similar holds, but here it is given that the average time is 500, and therefore

$$p_{0,0} = 1 - 1/500 = \frac{998}{1000} \tag{7.19}$$

Next, what does it mean that: "After an idle period, the system is equally likely to read or write a file". This means that after the system has left Idle, the probability to go to Read or to go to Write are the same. So $p_{0,1} = p_{0,2}$. And because the sum of the probabilities should be one $p_{0,0} + p_{0,1} + p_{0,2} = (1 - 1/500) + p_{0,1} + p_{0,2} = 1$, we can solve and find:

$$p_{0,1} = p_{0,2} = \frac{1}{2} \cdot \frac{1}{500} = \frac{1}{1000} \tag{7.20}$$

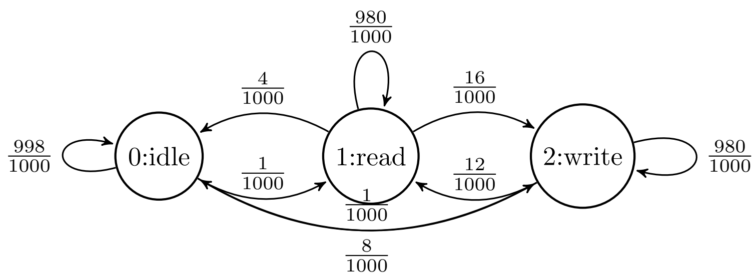
The next sentence says "Following the completion of a read, a write follows with probability 0.8.". That means that after the read state has been left (which was happening with probability 1/50), with probability of 0.8 it will jump to the write state. And therefore it has to jump with probability 0.2 to the idle state:

$$p_{1,2} = \frac{1}{50} \cdot 0.8 = \frac{16}{1000} \quad p_{1,0} = \frac{1}{50} \cdot 0.2 = \frac{4}{1000} \tag{7.21}$$

Finally, the sentence "On completion of a write operation, a read operation follows with probability 0.6" indicates that

$$p_{2,1} = \frac{1}{50} \cdot 0.6 = \frac{12}{1000} \quad p_{2,0} = \frac{1}{50} \cdot 0.4 = \frac{8}{1000} \tag{7.22}$$

So, the resulting Markov chain therefore becomes:



To be complete, this ends up in the following transition matrix:

$$P = \frac{1}{1000} \begin{bmatrix} 998 & 1 & 1 \\ 4 & 980 & 16 \\ 8 & 12 & 980 \end{bmatrix} \tag{7.23}$$

12.5.1 We have to solve:

$$\pi = \pi \begin{bmatrix} 0.998 & 0.001 & 0.001 \\ 0.004 & 0.98 & 0.016 \\ 0.008 & 0.012 & 0.98 \end{bmatrix} = \pi \frac{1}{1000} \begin{bmatrix} 998 & 1 & 1 \\ 4 & 980 & 6 \\ 8 & 12 & 980 \end{bmatrix} \tag{7.24}$$

with the extra equation that $\pi_0 + \pi_1 + \pi_2 = 1$. So then we solve:

$$\begin{cases} 1000\pi_0 = 998\pi_0 + 4\pi_1 + 8\pi_2 \\ 1000\pi_1 = \pi_0 + 980\pi_1 + 12\pi_2 \\ \pi_0 + \pi_1 + \pi_2 = 1 \end{cases} \tag{7.25}$$

Rewrite the first two equations:

$$\begin{cases} \pi_0 = 2\pi_1 + 4\pi_2 \\ 20\pi_1 = \pi_0 + 12\pi_2 \end{cases} \tag{7.26}$$

Substitute the first into the second:

$$20\pi_1 = 2\pi_1 + 4\pi_2 + 12\pi_2 \rightarrow \pi_1 = \frac{8}{9}\pi_2 \tag{7.27}$$

Substitute this into the last equation of (7.25):

$$2 \cdot \frac{8}{9}\pi_2 + 4\pi_2 + \frac{8}{9}\pi_2 + \pi_2 = 1 \tag{7.28}$$

$$(\frac{24}{9} + 5)\pi_2 = \frac{69}{9}\pi_2 = 1 \tag{7.29}$$

$$\pi_2 = \frac{9}{69} \tag{7.30}$$

This gives for $\pi_1 = \frac{8}{69}$ and $\pi_0 = \frac{52}{69}$. So:

$$\pi = \left[\frac{52}{69}, \frac{8}{69}, \frac{9}{69}\right] \tag{7.31}$$

12.5.6 In this story, you have to realize that in order to jump from one state to the other, you have to consider *two* cars at the same time. Let us first define the states:

state 0 : front teller busy, rear teller idle

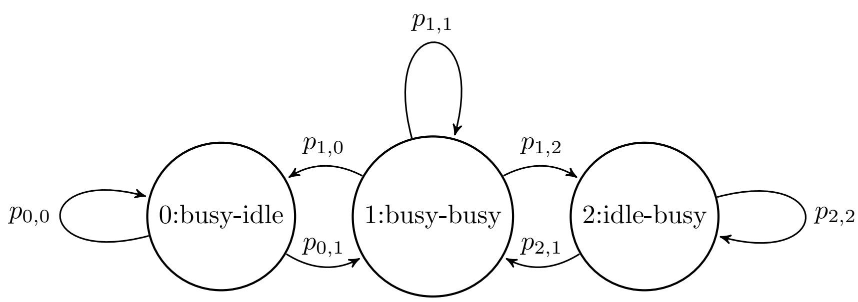
state 1 : front teller busy, rear teller busy

state 2 : front teller idle, rear teller busy

state 3 : front teller idle, rear teller idle

The first remark is, that when both tellers are idle (state 3), immediately two cars drive up to the empty tellers. So actually, this state is not used, and can be removed from the analysis.

The resulting Markov chain will therefore look like this:



Now realize that the probability that we change from state 1 to state 2, is the probability that the first car finishes his business, times the probability that the second car is still not finished (stays busy).

What is the probability that a car finishes its business? It is given that the service time has a geometric distribution with a mean of 120 seconds. Like in question 12.1.5, it means that with probability $p = 1/120$ a car leaves its busy state, and stays busy with probability $1 - p = 119/120$.

So the probability that we change from state 1 to state 2, is

$$p_{1,2} = p(1-p) = \frac{1}{120} \cdot \frac{119}{120} \tag{7.32}$$

The same holds for changing from state 1 to state 0:

$$p_{1,0} = p(1-p) = \frac{1}{120} \cdot \frac{119}{120} \tag{7.33}$$

The probability of staying in state 1, is therefore:

$$p_{1,1} = 1 - p_{1,0} - p_{1,2} \tag{7.34}$$

Next, when we are in state 0, we can stay in this state, or we can go back to state 1 (but not to 2!). What is the probability of staying in state 0? We stay in state 0 only when the (front) car is still being served, which had a probability of $p_{0,0} = 1 - p = 119/120$. The probability of leaving state 0 is therefore $p_{0,1} = p$. The same holds for state 2: $p_{2,2} = 119/120$ and $p_{2,1} = p$.

The overall state transition matrix therefore becomes:

$$P = \begin{bmatrix} 1-p & p & 0 \\ p(1-p) & 1-2p(1-p) & p(1-p) \\ 0 & p & 1-p \end{bmatrix} \quad \text{where} \quad p = \frac{1}{120} \tag{7.35}$$

To find the stationary probabilities, we have to solve:

$$\begin{aligned} \pi P &= \pi & (7.36) \\ \pi_0 + \pi_1 + \pi_2 &= 1 & (7.37) \end{aligned}$$

From the first matrix equality I choose the first and last equation:

$$\begin{aligned} \pi_0(1-p) + \pi_1 p(1-p) + 0 &= \pi_0 & (7.38) \\ 0 + \pi_1 p(1-p) + \pi_2 &= \pi_2 & (7.39) \\ \pi_0 + \pi_1 + \pi_2 &= 1 & (7.40) \end{aligned}$$

The first equation gives us $\pi_0(1-p) + \pi_1 p(1-p) = \pi_0 \;\rightarrow\; \pi_0 \cdot -p + \pi_1 p(1-p) = 0$ and therefore $\pi_0 = \pi_1(1-p)$. Similarly, the second equation gives $\pi_2 = \pi_1(1-p)$. Combining these two results gives that $\pi_0 = \pi_2$.

Finally, we use equation (7.40):

$$\begin{aligned} \pi_1(1-p) + \pi_1 + \pi_1(1-p) &= 1 \\ (2(1-p)+1)\pi_1 &= 1 \\ \pi_1 = \frac{1}{2(1-p)+1} &= \frac{1}{3-2p} & (7.41) \end{aligned}$$

Combining everything gives: $\pi = (\frac{1-p}{3-2p}, \frac{1}{3-2p}, \frac{1-p}{3-2p})$, and the stationary probability that two tellers are busy is $\frac{1}{3-2p}$.

12.10.1 When we define the normalized load $\rho = \lambda/\mu$, and we have $c = 2$ servers, then the blocking probability is:

$$P_N(2) = \frac{\rho^2/2!}{\sum_{k=0}^{2} \rho^k/k!} = \frac{\frac{1}{2}\rho^2}{1 + \rho + \frac{1}{2}\rho^2} \leq 0.1 \tag{7.42}$$

First solve the equality:

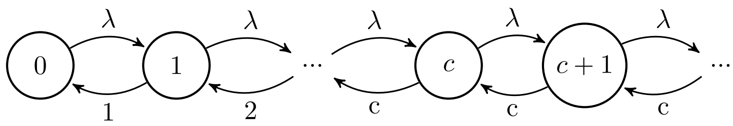
$$\begin{aligned} 5\rho^2 &= 1 + \rho + \frac{1}{2}\rho^2 & (7.43) \\ \frac{9}{2}\rho^2 - \rho - 1 = 0 & & (7.44) \end{aligned}$$

The solutions are:

$$\rho = \frac{+1 \pm \sqrt{1^2 - 4 \cdot 9/2 \cdot -1}}{2 \cdot 9/2} = \frac{1 \pm \sqrt{19}}{9} \tag{7.45}$$

Because ρ should be larger than zero, we find that $\rho = (1+\sqrt{19})/9 = 0.595$. For a normalized load larger than that, we will get a blocking probability larger than 0.1.

12.10.4 The Markov chain looks like this:



When we are in the limiting states, it should hold that:

$$p_{i-1}\lambda_{i-1} = p_i\mu_i \tag{7.46}$$

$$\sum p_i = 1 \tag{7.47}$$

So from the first equation we get trivially that $p_i = \frac{\lambda_{i-1}}{\mu_i}p_{i-1}$, a recursive definition in terms of p_i. Because our Markov chain consists of basically two parts, a part with varying μ_i for $i < c$ and a part with fixed $\mu_i = c$ for $i \geq c$, we have two parts in the definition for p_i:

$$\begin{aligned} p_i &= \begin{cases} \frac{\lambda}{i}p_{i-1} & 1 < i < c-1 \\ \frac{\lambda}{c}p_{i-1} & i \geq c \end{cases} \\ &= \begin{cases} p_0 \prod_{k=1}^{i} \frac{\lambda}{k} & 1 < i < c-1 \\ p_0 \prod_{k=1}^{c-1} \frac{\lambda}{k} \cdot \prod_{k=c}^{i} \frac{\lambda}{c} & i \geq c \end{cases} \end{aligned}$$

To get the limiting state probabilities, this sum $\sum_{i=0}^{\infty} p_i$ should converge:

$$\sum_{i=0}^{\infty} p_i = \sum_{i=0}^{c-1} p_i + \sum_{i=c}^{\infty} p_i = \text{something finite} + \sum_{i=c}^{\infty} p_i \tag{7.48}$$

We have to take special care for p_i with $i \to \infty$ (so obviously $i > c$), and make sure that it converges. This only converges when $|\frac{\lambda}{c}| < 1$, or $\lambda < c$.