Signal Processing EE2S31

July 2015

Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (13 points)

A joint probability density function of the random variables X and Y is given by

$$f_{X,Y}(x,y) = \begin{cases} c & \text{for } 0 \leq x \leq 2, \text{ and } 0 \leq y \leq 2 - x \\ 0 & \text{elsewhere.} \end{cases}$$

(2 p) (a) Calculate constant c.

(2 p) (b) Compute the marginal pdfs of X and Y and show that they are given by

$$f_X(x) = 2c - cx \text{ for } 0 \leq x \leq 2,$$

and

$$f_Y(y) = 2c - cy \text{ for } 0 \leq y \leq 2.$$

(1 p) (c) Explain whether X and Y are independent.

(1 p) (d) Compute the conditional pdf $f_{X|Y}(X|Y)$.

(2 p) (e) Compute the expected value $E[X]$.

In an experiment we observe realizations of random variable Y, while we want to make an estimate of X. To do so, we can make use of linear or non-linear estimators.

(2 p) (f) Proof that the MMSE estimate for X is given by $E[X|y]$.

(2 p) (g) Calculate $E[X|y]$.

(1 p) (h) As an alternative, one could also calculate the linear MMSE estimator. Given an expression for the linear MMSE estimator in terms of expected values (there is no need to calculate them) and argue which of the two estimators (the linear or the non-linear) leads to a smaller mean-squared error.

Question 2 (14 points)

Let $Y[n]$ be an auto-regressive process that is given by the following input-output relation:

$$Y[n] = aY[n-1] + X[n].$$

The input process $X[n]$ has a variance σ_X^2.

(1 p) (a) Give an expression for the autocorrelation function $R_X[k]$ of the input.

(1 p) (b) Explain for both $Y[n]$ and $X[n]$ whether they form an IID process.

(2 p) (c) Determine the system function $H(z)$ and calculate the impulse response.

(2 p) (d) Calculate the value for a in the above input-output relation, such that the cross-correlation between input and output becomes $R_{XY}[k] = \sigma_X^2(-\frac{1}{4})^n u[k]$.

For the following questions, assume that the impulse response is given by

$$h[n] = \left(\frac{1}{3}\right)^n u[n].$$

The auto-correlation $R_Y[k]$ of the output $Y[n]$ is given by

$$R_Y[k] = h[k] * h[-k] * R_X[k].$$

The convolution $f[k] = h[k] * h[-k]$ can also be seen as the concatenation of two filters. One with impulse response $h[k]$ and one with impulse response $h[-k]$. The autocorrelation $R_X[k]$ and $R_Y[k]$ are then related to each other by a convolution of $R_X[k]$ with an overall filter that has impulse response $f[k]$.

(3 p) (e) Show by explicitly calculating the convolutions that $f[k] = \frac{9}{8}(\frac{1}{3})^{|n|}$.
Hint: To show this, you might want to use the generalized expression for the geometric series, given by

$$\sum_{k=a}^{b} r^k = \frac{r^a - r^{b+1}}{1-r}.$$

(2 p) (f) Give the system function $F(z)$ of the overall filter and plot its pole-zero diagram.

(1 p) (g) Argue whether or not this is a stable filter.

(2 p) (h) Calculate the autocorrelation function $R_Y[k]$.

Question 1

(2 p) (a) $\int_x \int_y f_{X,Y}(x,y)dydx = \int_0^2 \int_0^{2-x} cdydx = \int_0^2 [cy]_0^{2-x} cdx = \left[2cx - c\frac{1}{2}x^2\right]_0^2 = 2c = 1$ Constant c should thus equal $c = 1/2$, as the pdf should integrate to one.
Alternatively, directly compute the area, which is $0.5*2*2*c = 2c = 1$, from which it also follows that $c = 1/2$.

(2 p) (b)

$$\int_y f_{X,Y}(x,y)dy = \int_0^{2-x} cdy = 2c - cx \text{ for } 0 \leq x \leq 2,$$

and

$$\int_x f_{X,Y}(x,y)dx = \int_0^{2-y} cdx = 2c - cy \text{ for } 0 \leq y \leq 2.$$

(1 p) (c) $f_{X,Y}(x,y) \neq f_X(x)f_Y(y)$ and thus are X and Y dependent.

(1 p) (d) $f_{X|Y}(X|Y) = f_{X,Y}(x,y)/f_Y(y) = \frac{1}{2-y}$

(2 p) (e) $E[X] = \int_0^2 x(2c - cx)dx = \int_0^2 (2cx - cx^2)dx = [cx^2 - \frac{1}{3}cx^3]_0^2 = \frac{4c}{3}$.

(2 p) (f)

(2 p) (g) $E[X|y] = \int_0^2 x f_{X|Y}(X|Y)dx = \int_0^2 x\frac{1}{2-y}dx = \frac{2}{2-y}$.

(1 p) (h)

Question 2 (14 points)

Let $Y[n]$ be an auto-regressive process that is given by the following input-output relation:

$$Y[n] = -\frac{1}{4}Y[n-1] + X[n].$$

The input process $X[n]$ has a variance σ_X^2.

(1 p) (a) $R_X[k] = \delta(k)\sigma_X^2$

(1 p) (b) $X[n]$ is an IID process as it consists of uncorrelated Gaussian variables. $Y[n]$ is not independent as each $Y[n]$ depends on the previous $Y[n-1]$.

(2 p) (c) The system function is given by $H(z) = \frac{1}{1+1/4z^{-1}}$. The inverse Z-transform of this is given by $h[n] = \ -\frac{1}{4})^n u[n]$.

(2 p) (d) $P_{XY}(z) = H(z)P_X(z) = \frac{\sigma_X^2}{1-1/3z^{-1}}$ leading to $R_{XY}[k] = \sigma_X^2 \frac{1}{3}^n u[k]$.

(3 p) (e) $f[k]] = \sum_n \ \frac{1}{3})^n u[n] \ \frac{1}{3})^{-(k-n)} u[-(k-n)]$ We can split this into a part for which $k \geq 0$ and a part for which $k < 0$.
For $k < 0$: $f[k]] = \sum_{n=0}^{\infty} \left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right)^{-(k-n)} = \left(\frac{1}{3}\right)^{-k} \sum_{n=0}^{\infty} \ \frac{1}{3})^{2n} = \ \frac{1}{3})^{-k} \frac{1}{1-(\frac{1}{3})^2}$
For $k > 0$: $f[k]] = \sum_{n=k}^{\infty} \left(\frac{1}{3}\right)^n \left(\frac{1}{3}\right)^{-(k-n)} = \left(\frac{1}{2}\right)^{-k} \sum_{n=k}^{\infty} \left(\frac{1}{3}\right)^{2n} = \left(\frac{1}{3}\right)^{-k} \frac{(\frac{1}{3})^{2k}}{1-(\frac{1}{3})^2} =$
$\frac{1}{3})^k \frac{1}{1-(\frac{1}{3})^2}$
The two parts can be taken together resulting in $f[k] = \frac{9}{8}(\frac{1}{3})^{|k|}$.

(2 p) (f) $F(z) = \frac{3z}{z-1/3} \frac{1}{3-z^1}$

(1 p) (g) Argue whether or not this is a stable filter.

(2 p) (h) Using $f[k] = \frac{9}{8}(\frac{1}{3})^{|k|}$ we get $R_Y[k] = \sigma_X^2 \frac{9}{8}(\frac{1}{3})^{|k|}$