Spectral Analysis of Finite-Length Signals

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Spectral Analysis

Recall that the spectrum of the discrete-time signal x is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

where the signal x can be recovered from its spectrum by the inverse Fourier transform

$$x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega$$

In order to compute $X(\omega)$, we need infinite many data samples:

- implementation by a numeric computer or DSP not possible

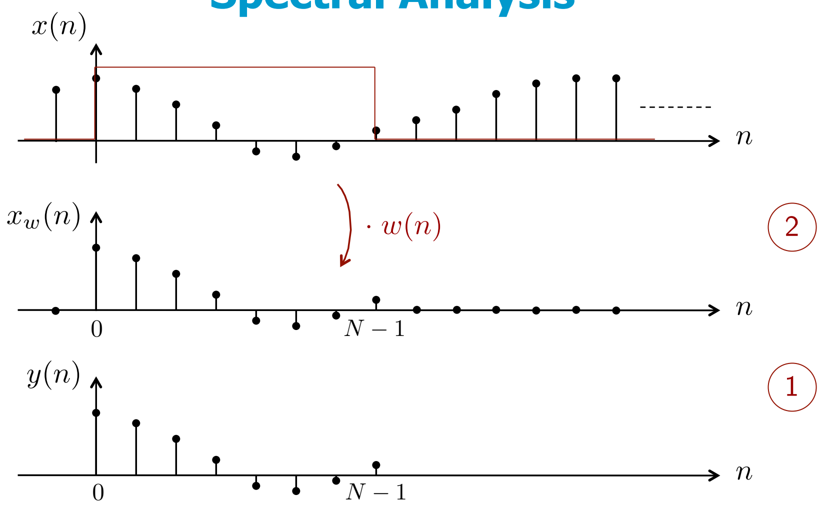
Spectral Analysis

We will approximate the Fourier transform (FT) by using the *discrete Fourier transform (DFT)*, which works with finite-length data sequences

1) we start by investigating the relation between the FT and the DFT when the signal to be transformed has finite support, that is, has a finite number of non-zero elements

2) we then study the relation between the FT of a signal with infinite support and one in which the signal is first time-windowed (thus having finite support)

Spectral Analysis



Spectral Analysis

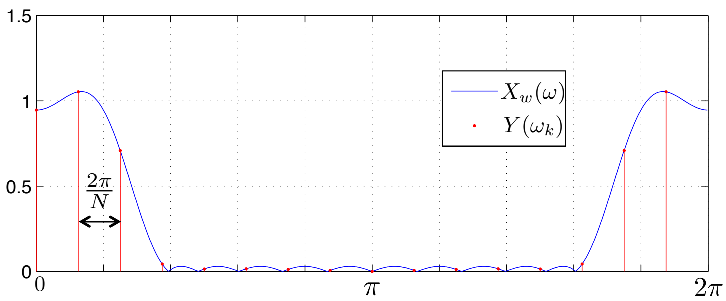
Step 1: DFT of y:

$$\begin{aligned} Y(\omega_k) &= \sum_{n=0}^{N-1} y(n) e^{-j\omega_k n} \\ &= \sum_{n=-\infty}^{\infty} x_w(n) e^{-j\omega_k n} \\ &= X_w(\omega_k), \quad \omega_k = \frac{2\pi}{N}k, \; k = 0, \ldots, N-1 \end{aligned}$$

Hence, $Y(\omega_k)$ is composed of samples of the frequency transform $X(\omega)$, where the samples are taken at frequencies $\omega_k = \frac{2\pi}{N}k, \; k = 0, \ldots, N-1$.

Spectral Analysis

Example:



What is the effect of padding zeros to y, thereby making it of length $L \geq N$?

Spectral Analysis

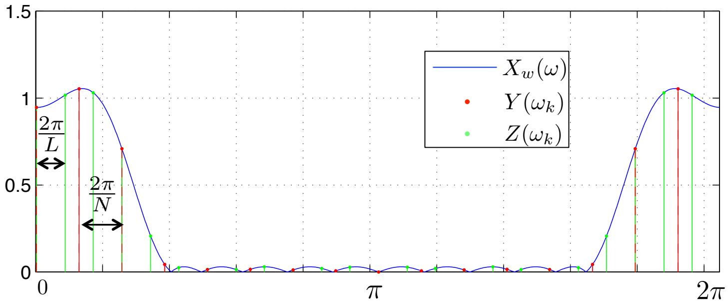
Let $z(n) = x_w(n)$ for $n = 0, \ldots, L-1$. We then have

$$Z(\omega_k) = \sum_{n=0}^{L-1} z(n) e^{-j\omega_k n}$$
$$= \sum_{n=-\infty}^{\infty} x_w(n) e^{-j\omega_k n}$$
$$= X_w(\omega_k), \quad \omega_k = \frac{2\pi}{L} k,\ k = 0, \ldots, L-1$$

Zero-padding gives us more samples of the underlying frequency-continuous spectrum!

Spectral Analysis

Example:



Spectral Analysis

Step 2: DFT of x_w:

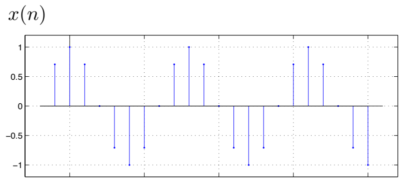
$$\begin{aligned} X_w(\omega) &= \sum_{n=-\infty}^{\infty} x_w(n) e^{-j\omega n} \\ &= \sum_{n=-\infty}^{\infty} (x \cdot w)(n) e^{-j\omega n} \\ &= (X * W)(\omega), \quad \omega \in [0, 2\pi) \end{aligned}$$

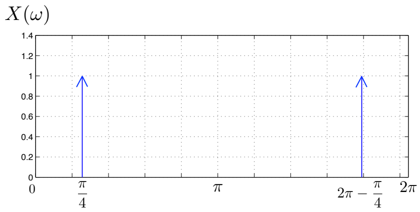
Hence, $X_w(\omega)$ is given by the convolution of the frequency transforms X and W evaluated at the frequency ω

Spectral Analysis

Example:

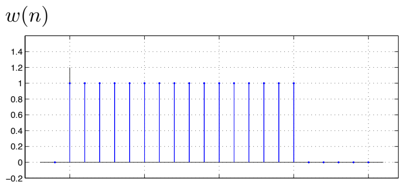
$$x(n) = \cos\left(\frac{\pi}{4} n\right), \; n \in \mathbb{R} \quad \Longleftrightarrow \quad X(\omega) = \pi\delta\left(\omega - \frac{\pi}{4}\right) + \pi\delta\left(\omega + \frac{\pi}{4}\right)$$

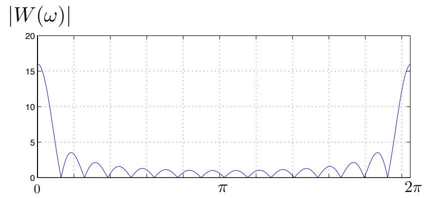




Spectral Analysis

$$w(n) = \begin{cases} 1, & n = 0, \dots, N-1 \\ 0, & \text{otherwise} \end{cases} \iff |W(\omega)| = \frac{\sin(N\omega/2)}{\sin(\omega/2)}$$



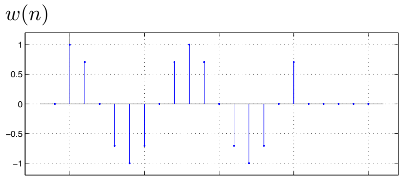


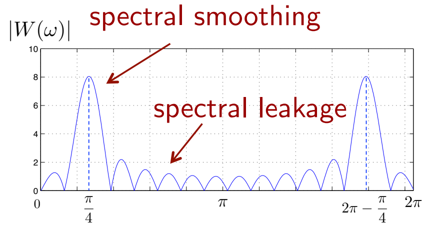
Spectral Analysis

$$x_w(n) = \begin{cases} \cos\left(\frac{\pi}{4}n\right), & n = 0, \dots, N-1 \\ 0, & \text{otherwise} \end{cases} \iff |X_w(\omega)| = |(X*W)(\omega)|$$

where

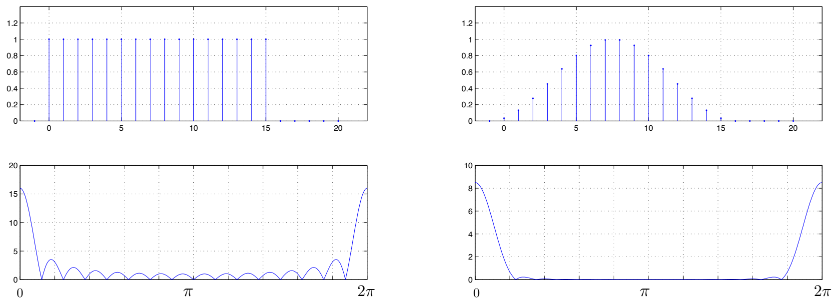
$$(W * \delta_{\omega_0})(\omega) = \int_0^{2\pi} W(\nu)\delta(\omega - \nu - \omega_0)d\nu = W(\omega - \omega_0)$$





Window Functions

We can use other than rectangular windows that have a better frequency decay (less spectral leakage)

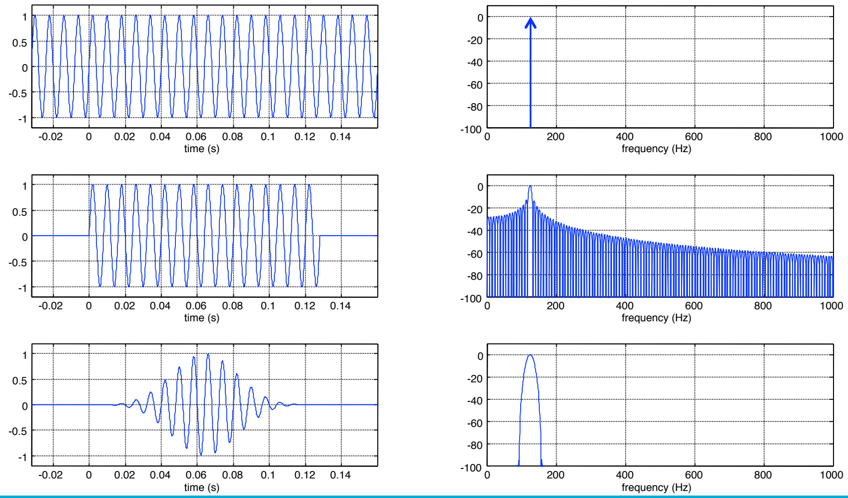


Fourier Transform

We would like to design windows having rapid descent of $W(f)$. Let $w \in L^1(\mathbb{R})$:

- If w is p times differentiable and all derivatives are in $L^1(\mathbb{R})$, then
$$w^{(p)}(t) \overset{\mathcal{F}}{\longleftrightarrow} (j2\pi f)^p W(f)$$
- Applying the Riemann-Lebesgue lemma on $w^{(p)}$, we conclude
$$\lim_{f\to\pm\infty} f^p W(f) = 0$$
so that regularity of w translates in rapid descent of W.

Window Functions



Window Functions

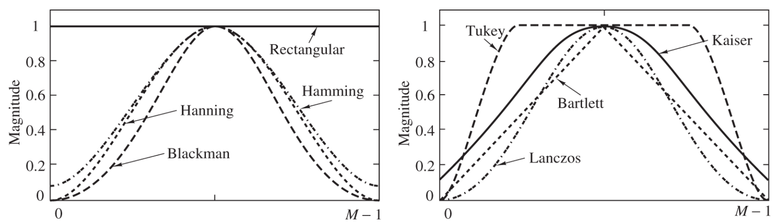


Figure 10.2.3 Shapes of several window functions.

Window Functions

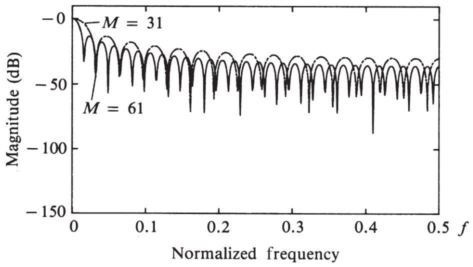


Figure 10.2.2 Frequency response for rectangular window of lengths (a) $M = 31$, (b) $M = 61$.

Window Functions

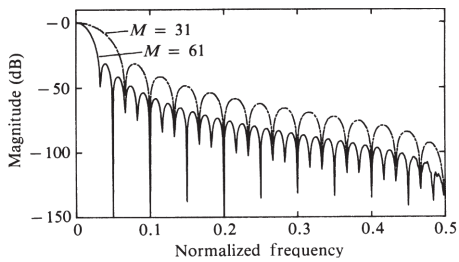


Figure 10.2.4 Frequency responses of Hanning window for (a) $M = 31$ and (b) $M = 61$.

Window Functions

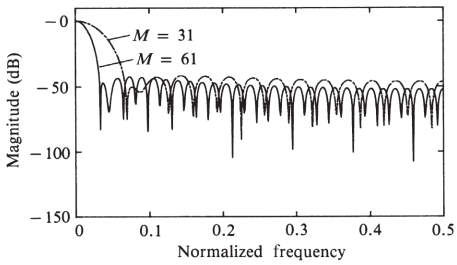


Figure 10.2.5 Frequency responses for Hamming window for (a) $M = 31$ and (b) $M = 61$.

Window Functions

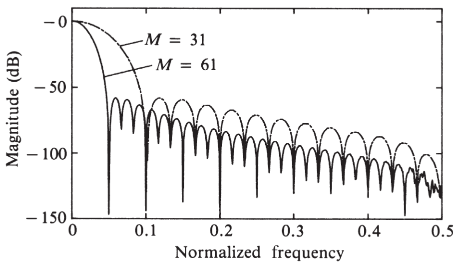


Figure 10.2.6 Frequency responses for Blackman window for (a) $M = 31$ and (b) $M = 61$.

Window Functions

How can we reduce the spectral smoothing?

Example: rectangular window

$$w(n) = \begin{cases} 1, & n = 0, \ldots, N-1 \\ 0, & \text{otherwise} \end{cases} \quad \Longleftrightarrow \quad |W(\omega)| = \frac{\sin(N\omega/2)}{\sin(\omega/2)}$$

Hence, we have zero-crossings at $\omega_k = \frac{2\pi}{N}k, \; k = 1, \ldots, N-1$

$\Rightarrow$ transition width of the main lobe is $\frac{4\pi}{N}$ ← reduced by increasing N!

Window Functions

window type	**width main lobe**	**peak side lobe (dB)**
Rectangular	$\frac{4\pi}{N}$	-13
Bartlett	$\frac{8\pi}{N}$	-25
Hanning	$\frac{8\pi}{N}$	-31
Hamming	$\frac{8\pi}{N}$	-41
Blackman	$\frac{12\pi}{N}$	-57

Window Functions

Example: $x(n) = w(n)\,(\cos(0.25\pi n) + \cos(0.35\pi n))$