Exam EE2511 - Stochastic Processes

April 16, 2014

Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Further is given:

Uniform distribution: for $a < b$:

$$
\begin{aligned}
f_X(x) &= \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \\
E[X] &= \frac{a+b}{2} \\
Var[X] &= \frac{(b-a)^2}{12}
\end{aligned}
$$

Gaussian distribution: for $\sigma > 0$:

$$
\begin{aligned}
f_X(x) &= \frac{e^{-(x-\mu)^2}/2\sigma^2}{\sigma\sqrt{2\pi}} \\
E[X] &= \mu \\
Var[X] &= \sigma^2
\end{aligned}
$$

Chebyshev inequality: for an arbitrary random variable Y and a constant $c > 0$:

$$P[|Y - \mu_Y| \geq c] \leq \frac{\text{Var}[Y]}{c^2}$$

Erlang's B formula: define the load $\rho = \frac{\lambda}{\mu}$ given the arrival rate λ and service rate μ:

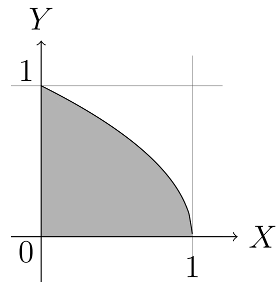
$$P[N = c] = \frac{\rho^c / c!}{\sum_{k=0}^{c} \rho^k / k!} \tag{1}$$

Question 1 - Probabilities (11 p)

The joint probability density function of two variables X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} c & 0 \le x \le 1, \quad 0 \le y \le \sqrt{1-x} \\ 0 & \text{otherwise.} \end{cases}$$

A graphical representation is given in the plot below:



(2 p) (a) Calculate the value of constant c.

(2 p) (b) Calculate the probability $P[X + Y > 1]$.

(2 p) (c) Calculate the expected value of X, $E[X]$ and the expected value of Y, $E[Y]$.

(3 p) (d) Calculate the covariance between X and Y, $Cov[X,Y]$.

(2 p) (e) Are the variables X and Y correlated? Are the variables X and Y independent? Explain your answers.

Question 2 - Signal Processing (12 p)

Consider the sequence $..., X_{-1}, X_0 X_1, ...$ where X_n are independent random variables with the probability density function

$$f_{X_n}(x) = \begin{cases} \frac{1}{2\sqrt{3}} & 1-\sqrt{3} \le x \le 1+\sqrt{3} \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(4 p) (a) Show or argue that

1. The expected value of X_n is $E[X_n] = 1$,
2. The variance of X_n is $Var[X_n] = 1$,
3. The covariance of X_n is $C_X[m,k] = \begin{cases} 1, & k = 0, \forall m \\ 0 & \text{otherwise.} \end{cases}$
4. The random sequence X_n is wide sense stationary.

(1 p) (b) Compute the autocorrelation function $R_X[m,k]$ of the random sequence.

Let Y_n denote a random sequence defined as:

$$Y_n = aX_{n-1} + bX_{n-2} \tag{3}$$

where a and b are constants.

(2 p) (c) Compute the expected value of Y_n.

(2 p) (d) Compute the auto-correlation function $R_Y[m,k]$ of Y_n.

(1 p) (e) Explain if Y_n is wide sense stationary.

(2 p) (f) Give a real-world example of a time signal for which the time samples are independent. Also give an example of a time signal for which two successive samples have positive correlation.

Question 3 - Estimation (9 p)

Assume that the travel time of a student, from home to the university, can be modeled by a Gaussian distribution. The average travel time is 35 minutes, with a standard deviation of 10 minutes.

(1 p) (a) What is the probability that the travel time will be larger than one hour? (Use Table 3.1 on page 6.)

(2 p) (b) Use the Chebyshev inequality to estimate the probability that the travel time will be larger than one hour.

(1 p) (c) Is the probability that you estimated in question **(b)** larger, smaller or equal to the probability from question **(a)**? Explain why this is necessarily the case.

(2 p) (d) Assume that there is an exam at 9:00. At what time does the student have to leave such that with 99% probability he will be in time for the exam? (Assume that the standard deviation of the travel time distribution does not change.)

(1 p) (e) What is expected *total* travel time for a whole month (with 21 working days)?

(2 p) (f) What is the probability that the travel time, *averaged over a month (21 days)*, is larger than one hour?

Question 4 - Markov Chains (8 p)

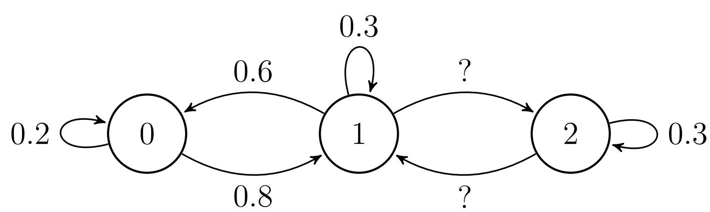
A fast food company has 3 registers to serve the customers. If the registers are all busy, the new customers do not wait but just leave the place. Each customer arrives following a Poisson process with an incoming rate of 4 per minute. Each register serves the customer on average for 8 minutes.

(1 p) (a) With what type of queue can this fast food company be modelled? Explain your answer briefly.

(1 p) (b) What is the probability for an incoming customers to leave the place?

(1 p) (c) How many registers should there be to have the blocking probability of 0.01?

Given the Markov chain shown below:



(1 p) (c) Find the transition matrix P.

(2 p) (d) Calculate the state probabilities after n transitions. Assume that initial state is 0.

(1 p) (e) Calculate the stationary probabilities, π_0, π_1, π_2.

(1 p) (f) Find the communicating classes and determine for each class if it is recurrent or transient. Determine whether the classes are periodic or aperiodic. If periodic, find the period.

z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$	z	$\Phi(z)$
0.00	0.5000	0.50	0.6915	1.00	0.8413	1.50	0.9332	2.00	0.97725	2.50	0.99379
0.01	0.5040	0.51	0.6950	1.01	0.8438	1.51	0.9345	2.01	0.97778	2.51	0.99396
0.02	0.5080	0.52	0.6985	1.02	0.8461	1.52	0.9357	2.02	0.97831	2.52	0.99413
0.03	0.5120	0.53	0.7019	1.03	0.8485	1.53	0.9370	2.03	0.97882	2.53	0.99430
0.04	0.5160	0.54	0.7054	1.04	0.8508	1.54	0.9382	2.04	0.97932	2.54	0.99446
0.05	0.5199	0.55	0.7088	1.05	0.8531	1.55	0.9394	2.05	0.97982	2.55	0.99461
0.06	0.5239	0.56	0.7123	1.06	0.8554	1.56	0.9406	2.06	0.98030	2.56	0.99477
0.07	0.5279	0.57	0.7157	1.07	0.8577	1.57	0.9418	2.07	0.98077	2.57	0.99492
0.08	0.5319	0.58	0.7190	1.08	0.8599	1.58	0.9429	2.08	0.98124	2.58	0.99506
0.09	0.5359	0.59	0.7224	1.09	0.8621	1.59	0.9441	2.09	0.98169	2.59	0.99520
0.10	0.5398	0.60	0.7257	1.10	0.8643	1.60	0.9452	2.10	0.98214	2.60	0.99534
0.11	0.5438	0.61	0.7291	1.11	0.8665	1.61	0.9463	2.11	0.98257	2.61	0.99547
0.12	0.5478	0.62	0.7324	1.12	0.8686	1.62	0.9474	2.12	0.98300	2.62	0.99560
0.13	0.5517	0.63	0.7357	1.13	0.8708	1.63	0.9484	2.13	0.98341	2.63	0.99573
0.14	0.5557	0.64	0.7389	1.14	0.8729	1.64	0.9495	2.14	0.98382	2.64	0.99585
0.15	0.5596	0.65	0.7422	1.15	0.8749	1.65	0.9505	2.15	0.98422	2.65	0.99598
0.16	0.5636	0.66	0.7454	1.16	0.8770	1.66	0.9515	2.16	0.98461	2.66	0.99609
0.17	0.5675	0.67	0.7486	1.17	0.8790	1.67	0.9525	2.17	0.98500	2.67	0.99621
0.18	0.5714	0.68	0.7517	1.18	0.8810	1.68	0.9535	2.18	0.98537	2.68	0.99632
0.19	0.5753	0.69	0.7549	1.19	0.8830	1.69	0.9545	2.19	0.98574	2.69	0.99643
0.20	0.5793	0.70	0.7580	1.20	0.8849	1.70	0.9554	2.20	0.98610	2.70	0.99653
0.21	0.5832	0.71	0.7611	1.21	0.8869	1.71	0.9564	2.21	0.98645	2.71	0.99664
0.22	0.5871	0.72	0.7642	1.22	0.8888	1.72	0.9573	2.22	0.98679	2.72	0.99674
0.23	0.5910	0.73	0.7673	1.23	0.8907	1.73	0.9582	2.23	0.98713	2.73	0.99683
0.24	0.5948	0.74	0.7704	1.24	0.8925	1.74	0.9591	2.24	0.98745	2.74	0.99693
0.25	0.5987	0.75	0.7734	1.25	0.8944	1.75	0.9599	2.25	0.98778	2.75	0.99702
0.26	0.6026	0.76	0.7764	1.26	0.8962	1.76	0.9608	2.26	0.98809	2.76	0.99711
0.27	0.6064	0.77	0.7794	1.27	0.8980	1.77	0.9616	2.27	0.98840	2.77	0.99720
0.28	0.6103	0.78	0.7823	1.28	0.8997	1.78	0.9625	2.28	0.98870	2.78	0.99728
0.29	0.6141	0.79	0.7852	1.29	0.9015	1.79	0.9633	2.29	0.98899	2.79	0.99736
0.30	0.6179	0.80	0.7881	1.30	0.9032	1.80	0.9641	2.30	0.98928	2.80	0.99744
0.31	0.6217	0.81	0.7910	1.31	0.9049	1.81	0.9649	2.31	0.98956	2.81	0.99752
0.32	0.6255	0.82	0.7939	1.32	0.9066	1.82	0.9656	2.32	0.98983	2.82	0.99760
0.33	0.6293	0.83	0.7967	1.33	0.9082	1.83	0.9664	2.33	0.99010	2.83	0.99767
0.34	0.6331	0.84	0.7995	1.34	0.9099	1.84	0.9671	2.34	0.99036	2.84	0.99774
0.35	0.6368	0.85	0.8023	1.35	0.9115	1.85	0.9678	2.35	0.99061	2.85	0.99781
0.36	0.6406	0.86	0.8051	1.36	0.9131	1.86	0.9686	2.36	0.99086	2.86	0.99788
0.37	0.6443	0.87	0.8078	1.37	0.9147	1.87	0.9693	2.37	0.99111	2.87	0.99795
0.38	0.6480	0.88	0.8106	1.38	0.9162	1.88	0.9699	2.38	0.99134	2.88	0.99801
0.39	0.6517	0.89	0.8133	1.39	0.9177	1.89	0.9706	2.39	0.99158	2.89	0.99807
0.40	0.6554	0.90	0.8159	1.40	0.9192	1.90	0.9713	2.40	0.99180	2.90	0.99813
0.41	0.6591	0.91	0.8186	1.41	0.9207	1.91	0.9719	2.41	0.99202	2.91	0.99819
0.42	0.6628	0.92	0.8212	1.42	0.9222	1.92	0.9726	2.42	0.99224	2.92	0.99825
0.43	0.6664	0.93	0.8238	1.43	0.9236	1.93	0.9732	2.43	0.99245	2.93	0.99831
0.44	0.6700	0.94	0.8264	1.44	0.9251	1.94	0.9738	2.44	0.99266	2.94	0.99836
0.45	0.6736	0.95	0.8289	1.45	0.9265	1.95	0.9744	2.45	0.99286	2.95	0.99841
0.46	0.6772	0.96	0.8315	1.46	0.9279	1.96	0.9750	2.46	0.99305	2.96	0.99846
0.47	0.6808	0.97	0.8340	1.47	0.9292	1.97	0.9756	2.47	0.99324	2.97	0.99851
0.48	0.6844	0.98	0.8365	1.48	0.9306	1.98	0.9761	2.48	0.99343	2.98	0.99856
0.49	0.6879	0.99	0.8389	1.49	0.9319	1.99	0.9767	2.49	0.99361	2.99	0.99861

Table 3.1 The standard normal CDF $\Phi(y)$.