Exam EE2S31 Signaalbewerking

June 30th, 2015

Answer each question on a **separate sheet**. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Question 1 (10 points)

Consider the following system:

$$\xrightarrow{X[n]} \boxed{h[n]} \xrightarrow{Y[n]}$$

Assume we have a time-discrete input signal $X[n]$ with expected value $E[X] = 1$ and an autocorrelation function given by

$$R_X[k] = \begin{cases} 4 & k = 0 \\ -2 & |k| = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

The impulse response of the linear filter $h[n]$ is defined as

$$h[n] = \begin{cases} 1 & n = 0 \\ \frac{1}{2} & n = 1 \\ 0 & \text{otherwise.} \end{cases} \tag{2}$$

(1 p) (a) Is process X wide sense stationary (WSS)? Motivate your answer.

(1 p) (b) Compute the variance of $X[n]$.

(1 p) (c) Compute the expected value of $Y[n]$.

(3 p) (d) Compute $R_Y[k]$ for $k = -2$, $k = -1$, $k = 0$, $k = 1$, and $k = 2$.

In practice we observe process X as a function of time. Based on these observations, we have to make an estimate of the autocorrelation function given above, i.e., a time average instead of an ensemble average. An often used estimator for the autocorrelation function is given by

$$\bar{R}_X[k] = \frac{1}{N} \sum_{n=1}^{N-k} X[n]X[n+k].$$

In this specific case we estimate $R_X[k]$ using a sequence of $N = 4$ realizations of X that have been observed across time.

(1 p) (e) Under which condition is it allowed to replace ensemble averages with time averages?

(2 p) (f) Compute the expected value of the estimator $\bar{R}_X[k]$ and argue whether or not this estimator is biased.

(1 p) (g) Make a plot of the auto-correlation function $R_X[k]$ given above, and the expected value of the estimated autocorrelation function $\bar{R}_X[k]$.

Question 2 (15 points)

In this assignment we are interested in calculating the autocorrelation function at the output of a filter. Let $X[n]$ be a zero-mean uncorrelated Gaussian process with variance σ_X^2. Let the input-output relation of the filter be given by

$$Y[n] = \frac{1}{2}Y[n-1] + X[n].$$

(1 p) (a) Explain whether this is an IIR or FIR filter and explain how the stochastic process $Y[n]$ is typically called.

(1 p) (b) Give the autocorrelation function $R_X[k]$ of the input and make a plot of $R_X[k]$.

(1 p) (c) Will the output $Y[n]$ be wide sense stationary? Motivate your answer.

The cross-correlation between input $X[n]$ and output $Y[n]$ is given by

$$R_{XY}[k] = h[k] * R_X[k],$$

with $h[k]$ the impulse response of the corresponding filter.

(2 p) (c) Determine the system function $H(z)$ and show that the impulse response is given by $h[n] = \left(\frac{1}{2}\right)^n u[n]$. Notice that the function $u[n]$ denotes the unit-step function.

(3 p) (d) Calculate the cross-correlation $R_{XY}[k]$.

The auto-correlation $R_Y[k]$ of the output $Y[n]$ is given by

$$R_Y[k] = h[k] * h[-k] * R_X[k].$$

The convolution $f[k] = h[k] * h[-k]$ can also be seen as the concatenation of two filters. One with impulse response $h[k]$ and one with impulse response $h[-k]$. The autocorrelation $R_X[k]$ and $R_Y[k]$ are then related to each other by a convolution of $R_X[k]$ with an overall filter that has impulse response $f[k]$.

(3 p) (e) Show by explicitly calculating the convolutions that $f[k] = \frac{4}{3}(\frac{1}{2})^{|n|}$. *Hint: To show this, you might want to use the generalized expression for the geometric series, given by*

$$\sum_{k=a}^{b} r^k = \frac{r^a - r^{b+1}}{1-r}.$$

(2 p) (f) Give the system function $F(z)$ of the overall filter and plot its pole-zero diagram.

(2 p) (g) Calculate the autocorrelation function $R_Y[k]$.

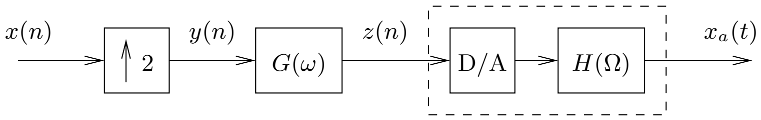

Figure 1: Block diagram of the oversampled D/A convertor.

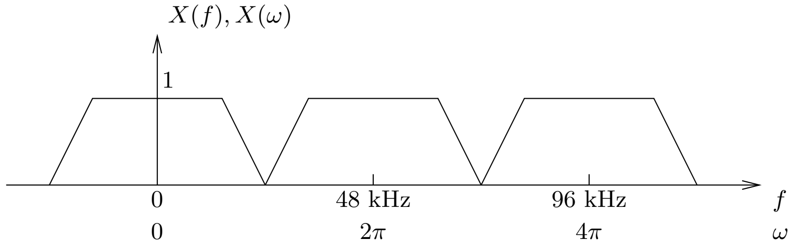

Figure 2: Spectrum of the input signal.

Question 3 (10 points)

Consider the two-times oversampled D/A convertor of which the block diagram is depicted in Figure 1. The spectrum of the input signal x is shown in Figure 2 both as a function of the angular frequency ω (dimensionless) and the frequency f expressed in Hertz (Hz).

(2 p) a) What is the sampling frequency f_s at which the input signal x has been sampled? Motivate your answer.

(2 p) b) Explain in words what the purpose of the different blocks in Figure 1 is and what the advantage is of such an oversampled D/A convertor over a standard (non-oversampled) D/A convertor.

(2 p) c) Assume that the digital filter $G(\omega)$ is a "perfect" brick-wall filter with cut-off frequency $f_c = 24$ kHz. Sketch the spectrum of $y(n)$ and $z(n)$ in Figure 3 both as a function of f and ω.

(2 p) d) Why can't we directly filter out the frequency band $24 - 72$ kHz, so that we don't need the expander which simply insert zero-valued samples in between the samples of $x(n)$, thereby increasing the sample rate of the signal by a factor two.

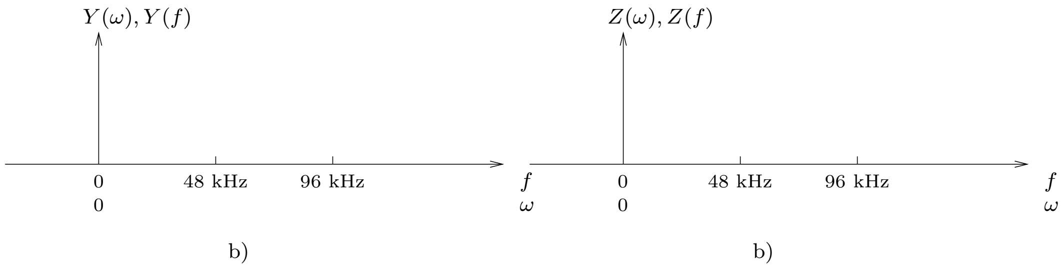

Figure 3: Sketch of frequency domain representation of y and z, both as a function of f and ω.

The actual digital-to-analog conversion takes place in the dashed box in Figure 1.

(2 p) e) What is the maximum transition bandwidth (frequency band between the pass- and stoppand) of the analog interpolation filter $H(\Omega)$ such that we can perfectly reconstruct our audio signal?