Exam EE2511 - Stochastic Processes

April 16, 2014

Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Further is given:

Uniform distribution: for $a < b$:

$$\begin{aligned} f_X(x) &= \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \\ E[X] &= \frac{a+b}{2} \\ Var[X] &= \frac{(b-a)^2}{12} \end{aligned}$$

Gaussian distribution: for $\sigma > 0$:

$$\begin{aligned} f_X(x) &= \frac{e^{-(x-\mu)^2}/2\sigma^2}{\sigma\sqrt{2\pi}} \\ E[X] &= \mu \\ Var[X] &= \sigma^2 \end{aligned}$$

Chebyshev inequality: for an arbitrary random variable Y and a constant $c > 0$:

$$P[|Y - \mu_Y| \geq c] \leq \frac{\mathrm{Var}[Y]}{c^2}$$

Question 1 - Probabilities

(2 p) (a) Should integrate to 1:

$$\begin{aligned} 1 &= \int_0^1 \int_0^{\sqrt{1-x}} c\, dy\, dx = \int_0^1 [cy]_0^{\sqrt{1-x}} dx \\ &= c\int_0^1 \sqrt{1-x}dx = c[-\frac{2}{3}(1-x)^{3/2}] = c(-\frac{2}{3}\cdot 0 + \frac{2}{3}) = 2c/3 \end{aligned}$$

Therefore $c = 3/2$.

(2 p) (b) Two possibilities:

$$\begin{aligned} P[X+Y>1] &= 1 - P[X+Y<1] = 1 - c \cdot \text{area triangle} \\ &= 1 - 3/2 \cdot 1/2 \cdot 1 \cdot 1 = 1/4 \end{aligned} \tag{1}$$

or a bit more complicated:

$$\begin{aligned} P[X+Y>1] &= \int_0^1 \int_{1-x}^{\sqrt{1-x}} \frac{3}{2} dy\, dx = \frac{3}{2}\int_0^1 (\sqrt{1-x} - 1 + x)dx \\ &= \frac{3}{2}[-\frac{2}{3}(1-x)^{3/2} + \frac{1}{2}x^2 - x]_0^1 \\ &= \frac{3}{2}((0+1/2-1) - (-2/3+0-0)) = \frac{3}{2}(-1/2+2/3) = 1/4 \end{aligned}$$

(2 p) (c) For $E[X]$ we need f_X:

$$f_X(x) = \int_{-\infty}^{\infty} f_{XY}(x,y)dy = \int_0^{\sqrt{1-x}} \frac{3}{2}dy = [\frac{3}{2}]_0^{\sqrt{1-x}} = \frac{3}{2}\sqrt{1-x} \tag{2}$$

Then

$$E[X] = \int_0^1 x \cdot \frac{3}{2}\sqrt{1-x}dx = ...(*) \tag{3}$$

Now we have to do partial integration (like $\int f \cdot g'\, dx = [f \cdot g] - \int f' \cdot g\, dx$):

$$\begin{aligned} \int_0^1 x\sqrt{1-x}dx &= [x \cdot \frac{-2}{3}(1-x)^{3/2}]_0^1 - \int \frac{-2}{3}(1-x)^{3/2}dx \\ &= (0-0) + \frac{2}{3}\int_0^1 (1-x)^{3/2}dx \\ &= \frac{2}{3}[-\frac{2}{5}(1-x)^{5/2}]_0^1 = 0 + \frac{4}{15} \end{aligned}$$

So, we get:

$$E[X] = \frac{3}{2}\frac{4}{15} = \frac{2}{5} \tag{4}$$

In a similar way we do for Y:

$$\begin{aligned} f_Y(y) &= \int_{-\infty}^{\infty} f_{XY}(x,y)dx = \int_0^{1-y^2} \frac{3}{2}dx \\ &= [\frac{3}{2}x]_0^{1-y^2} = \frac{3}{2}(1-y^2) \end{aligned}$$

So:

$$\begin{aligned} E[Y] &= \int_0^1 \frac{3}{2}y(1-y^2)dy = \frac{3}{2}\int_0^1 (y-y^3)dy \\ &= \frac{3}{2}[\frac{1}{2}y^2 - \frac{1}{4}y^4]_0^1 = \frac{3}{2}(1/2-1/4) = \frac{3}{8} \end{aligned} \tag{5}$$

(3 p) (d) Easy:

$$Cov(X,Y) = E[XY] - E[X]E[Y] = E[XY] - \frac{2}{5}\cdot\frac{3}{8} \tag{6}$$

Then:

$$\begin{aligned} E[XY] &= \iint_{-\infty}^{\infty} xy f_{XY}(x,y)dx\,dy = \int_0^1 \int_0^{\sqrt{1-x}} \frac{3}{2}xy\,dy\,dx \\ &= \frac{3}{2}\int_0^1 x[\frac{1}{2}y^2]_0^{\sqrt{1-x}}dx = \frac{3}{2}\int_0^1 x(\frac{1}{2}(1-x))dx \\ &= \frac{3}{4}\int_0^1 (x-x^2)dx = \frac{3}{4}[\frac{1}{2}x^2 - \frac{1}{3}x^3]_0^1 = \frac{3}{4}\cdot\frac{1}{6} = \frac{1}{8} \end{aligned} \tag{7}$$

So, in total we get:

$$Cov(X,Y) = \frac{1}{8} - \frac{2}{5}\cdot\frac{3}{8} = -\frac{1}{40} = -0.025 \tag{8}$$

(2 p) (e) The variables X and Y ARE correlated: covariance is not 0. Therefore the variables X and Y are NOT independent.

Question 2 - Signal Processing

(4 p) (a) We use information about the uniform distribution from the first page: $a = 1 - \sqrt{3}$, and $b = 1 + \sqrt{3}$.

1. The expected value $E[X_n] = (a + b)/2 = 1$,
2. The variance $Var[X_n] = (b - a)^2/12 = (2\sqrt{3})^2/12 = 1$,
3. The covariance for $k = 0$ is actually the variance: $C_X[m, 0] = 1$. For $k \neq 0$, we have the covariance between two independent variables: $C_X(m, k) = E[X_m X_{m+k}] - E[X_m]E[X_{m+k}] = E[X_m]E[X_{m+k}] - E[X_m]E[X_{m+k}] = 0$.
4. Both $E[X_n]$ and $C_X(m, k)$ do not depend on m, therefore WSS.

(1 p) (b) Use definition:

$$R_X(m, k) = C_X(m, k) + E[X_m]E[X_{m+k}] = C_X(m, k) + 1$$

Let Y_n denote a random sequence defined as:

$$Y_n = aX_{n-1} + bX_{n-2}$$

where a and b are constants.

(2 p) (c)

$$E[Y_n] = aE[X_{n-1}] + bE[X_{n-2}] = a + b$$

(2 p) (d)

$$\begin{aligned}
R_Y[m, k] &= E[Y_m Y_{m+k}] = E[(aX_{n-1} + bX_{n-2})(aX_{n+k-1} + bX_{n+k-2})] \\
&= a^2 E[X_{m-1}X_{m+k-1}] + b^2 E[X_{m-2}X_{m+k-2}] \\
&\quad + abE[X_{m-1}X_{m+k-2}] + abE[X_{m-2}X_{m+k-1}] \\
&= (a^2 + b^2)R_X[m, k] + abR_X[m, k-1] + abR_X[m, k-1] \\
&= \begin{cases} 2(a^2 + b^2) + 2ab & k = 0 \\ (a^2 + b^2) + 3ab & |k| = 1 \\ (a^2 + b^2) + 2ab & |k| > 1 \end{cases}
\end{aligned}$$

(1 p) (e) Again both $E[Y_n]$ and $C_Y(m, k)$ do not depend on m, therefore WSS.

(2 p) (f) Independent time samples: noise. Correlated samples: temperatures of consecutive days.

Question 3 - Estimation

(1 p) (a) Use Table 3.1:

$$P(t > 60) = P(\frac{t-35}{10} > \frac{60-35}{10}) = P(z > 2.5) = 1 - P(z < 2.5) = 1 - 0.99379 = 0.006 \tag{9}$$

(2 p) (b) Chebyshev:

$$P[|Y - \mu_Y| \geq c] \leq \frac{Var[Y]}{c^2} \tag{10}$$

Fill in:

$$P[|Y - 35| \geq (60 - 35)] \leq \frac{Var[Y]}{25^2} = 0.16 \tag{11}$$

(1 p) (c) Chebyshev works on any distribution, and therefore is much more loose. (a) uses the *true* distribution, and therefore is the most tight.

(2 p) (d) Find μ such that:

$$\begin{aligned} P[t > 60] &\leq 0.01 \\ P[\frac{t-\mu}{10} > \frac{60-\mu}{10}] &\leq 0.01 \\ P[z > \frac{60-\mu}{10}] &\leq 0.01 \\ P[z < \frac{60-\mu}{10}] &\geq 0.99 \end{aligned}$$

From the table we find that $\frac{60-\mu}{10} = 2.33$, or $\mu = 36.7$.

(1 p) (e) Total travel time:

$$E[T] = E[\sum_{i=1}^{21} t_i] = 21 \cdot 35 = 735 \tag{12}$$

(2 p) (f) If we assume that each travel is independent, and we have 21 travels, then we may want to use the central limit theorem! With n variables with mean μ_t and variance σ_t^2, the mean

$$M_n \sim N(\mu_t, \frac{\sqrt{n\sigma_t^2}}{n}) = N(\mu_t, \sqrt{\sigma_t^2/n}) \tag{13}$$

For us:

$$P[M_n > 60] = P[\frac{M_n - \mu_t}{\sigma_M} > \frac{60 - 35}{\sqrt{10^2/21}}] = 1 - P[z < \frac{25}{2.18}] \approx 1 - 1 = 0 \tag{14}$$

Question 4 - Markov Chains

A worker in a factory can be idle (State 0) or busy (State 1) each hour. This situation is modelled using a Markov chain with a transition matrix **P** as given below.

$$P = \begin{bmatrix} 0.2 & 0.8 \\ 0 & 1 \end{bmatrix} \tag{15}$$

(2 p) (a) Compute the state probabilities after 4 hours. Initially (at hour 0), the worker starts in the idle state.

(2 p) (b) Compute the state probabilities after n transitions.

(1 p) (c) Compute the stationary state probabilities, that is π_0 and π_1.

For another day, the worker's states are modelled with a new Markov chain and the transition matrix is given by Q:

$$Q = \begin{bmatrix} 0.2 & 0.8 \\ 0.1 & 0.9 \end{bmatrix} \tag{16}$$

(1 p) (d) Compute the stationary state probabilities, that is π_0 and π_1, for Q.

(2 p) (e) The worker is paid hourly based on his/her state: 25 Euro's if he/she is busy and 2 Euro's, otherwise. Assuming that the worker starts in the idle state, compute the amount of wage after 4 hours using Q.

(2 p) (f) Draw and classify both of the Markov chains, and find the periods for each class.