# Random Processes

**Stochastic Processes for EE (EE2511)**

**Lecture 2**

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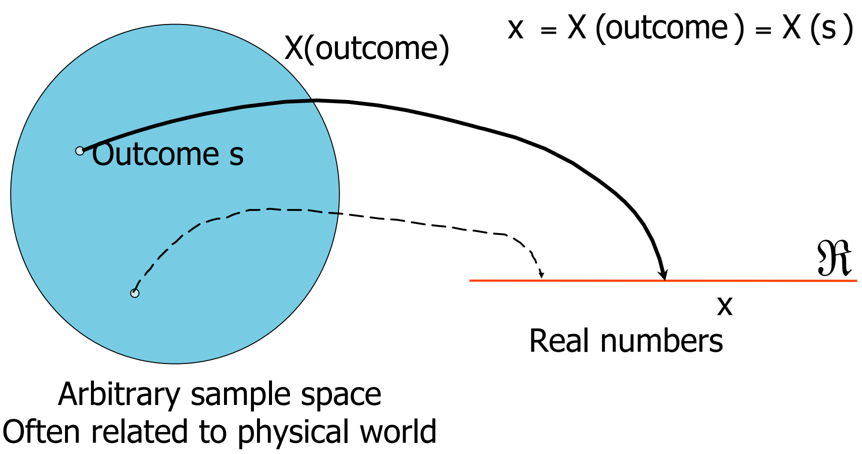
# Definition of Random Variable

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## One experiment, multiple observations

- Experiment S: select student
- Observations: H(s): length of student
  A(s): age of student
  W(s): weight of student

- Obviously these observations are "related"
- Put the observations in a vector
  (H(s), A(s), W(s))

- Outcomes: (180,23,68), (195,21,75), ...

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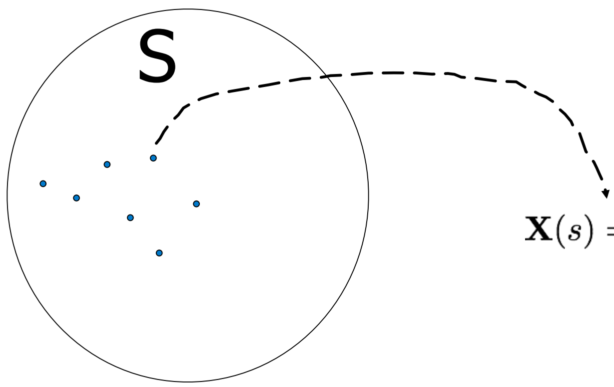
## Multiple Random Variables




$$\mathbf{X}(s) = \begin{bmatrix} X_1(s) \\ X_2(s) \\ \vdots \\ X_N(s) \end{bmatrix}$$

- Chapter 4: N=2 Pairs of RV; Bivariate
- Chapter 5: N>2 Random vectors; Multivariate

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## Pairs of random variables

Discrete RVs:

- (joint) Probability Mass Function (jpdf)
- Marginal PMF can be obtained from the jpdf
- Joint CDF

Continuous RVs:

- joint probability density function, from the joint cumulative distribution function
- Marginal PMF can be obtaine from the jpdf
- Joint CDF





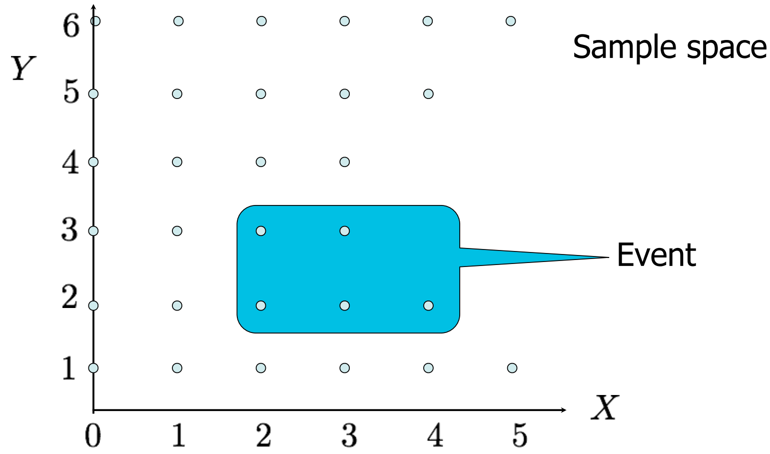
## Pairs of (Discrete) variables

- Example: two dice:

$$\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} |\text{difference between dice}| \\ \text{first dice} \end{pmatrix}$$

- The components of a vector random variable are random variables themselves

# Pairs of variables




- Two dice: $\mathbf{X} = \begin{pmatrix} X \\ Y \end{pmatrix} = \begin{pmatrix} |\text{difference between dice}| \\ \text{first dice} \end{pmatrix}$



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# Notation

- 1-D case:

$$P(X = x)$$

$$P(X \in B)$$

(B is an event in $\Re$ )

- 2-D case:

$$P(X = x, Y = y) = P_{X,Y}(x, y)$$

$$P(\mathbf{X} \in B)$$

(B is an event in $\Re^2$ )



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## Pairs of variables: joint PMF

| | $X = 1$ | $X = 2$ | $X = 3$ |
|---|---|---|---|
| $Y = 1$ | 0,1 | 0,2 | 0,05 |
| $Y = 2$ | 0,15 | 0,1 | 0,2 |
| $Y = 3$ | 0,05 | 0,1 | 0,05 |

- For discrete RVs, joint-PMF is fully specified with a table

$$P(X = x, Y = y) = P_{X,Y}(x, y)$$





## Pairs of variables: joint PMF

| | $X = 1$ | $X = 2$ | $X = 3$ |
|---|---|---|---|
| $Y = 1$ | 0,1 | 0,2 | 0,05 |
| $Y = 2$ | 0,15 | 0,1 | 0,2 |
| $Y = 3$ | 0,05 | 0,1 | 0,05 |

- Probability of events can easily be computed:

$$P[X > Y] = ?$$

## Pairs of variables: joint PMF

| | $X=1$ | $X=2$ | $X=3$ |
|---|---|---|---|
| $Y=1$ | 0,1 | 0,2 | 0,05 |
| $Y=2$ | 0,15 | 0,1 | 0,2 |
| $Y=3$ | 0,05 | 0,1 | 0,05 |

- Probability of events can easily be computed:

$$\begin{aligned} P[X>Y] &= P[\{(2,1),(3,1),(3,2)\}] \\ &= P_{X,Y}[2,1]+P_{X,Y}[3,1]+P_{X,Y}[3,2] \\ &= 0.45 \end{aligned}$$





## Joint and marginal PMF

- Joint probability mass function

$$P_{X,Y}(X=x,Y=y)=P_{X,Y}(x,y)$$

- Marginal PMF: $P_X(x)$ $\quad$ $P_Y(y)$

$$\begin{aligned} P_X(x) &= P_{X,Y}(x,\text{any}\, y) \\ &= \sum_{\text{all}\, y} P_{X,Y}(x,y) \end{aligned}$$

# Marginal probability

| | $X=1$ | $X=2$ | $X=3$ |
|---|---|---|---|
| $Y=1$ | 0,1 | 0,2 | 0,05 |
| $Y=2$ | 0,15 | 0,1 | 0,2 |
| $Y=3$ | 0,05 | 0,1 | 0,05 |
| | 0,30 | 0,40 | 0,30 |

$$P_X(2) = 0.4$$



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# Illustration

| | | $x_1$ | $x_2$ |
|---|---|---|---|
| | | 0.5 | 0.5 |
| $y_1$ | 0.5 | | |
| $y_2$ | 0.5 | | |

Marginal PMF of X

Marginal PMF of Y

?



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## Illustration

| | | $x_1$ | $x_2$ | Marginal PMF of X |
|---|---|---|---|---|
| | | 0.5 | 0.5 | |
| $y_1$ | 0.5 | 0.5 | 0 | |
| $y_2$ | 0.5 | 0 | 0.5 | |

Marginal PMF of Y





## Illustration

| | | $x_1$ | $x_2$ | Marginal PMF of X |
|---|---|---|---|---|
| | | 0.5 | 0.5 | |
| $y_1$ | 0.5 | 0.25 | 0.25 | |
| $y_2$ | 0.5 | 0.25 | 0.25 | |

Marginal PMF of Y

# Illustration

| | | $x_1$ | $x_2$ |
|---|---|---|---|
| | | 0.5 | 0.5 |
| $y_1$ | 0.5 | 0.1 | 0.4 |
| $y_2$ | 0.5 | 0.4 | 0.1 |

Marginal PMF of X

Marginal PMF of Y



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# Marginal and joint probabilities

- Note, when you know the marginal prob., it does NOT mean that you can derive the joint probabilities!!

Joint probability mass function

Marginal probability mass functions



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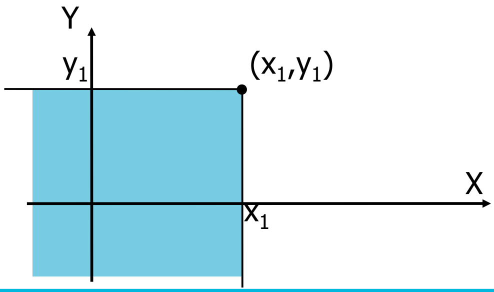
# Continuous random variables

- How to generalize to 2D?
- Again, introduce the joint cumulative distribution function (cdf):

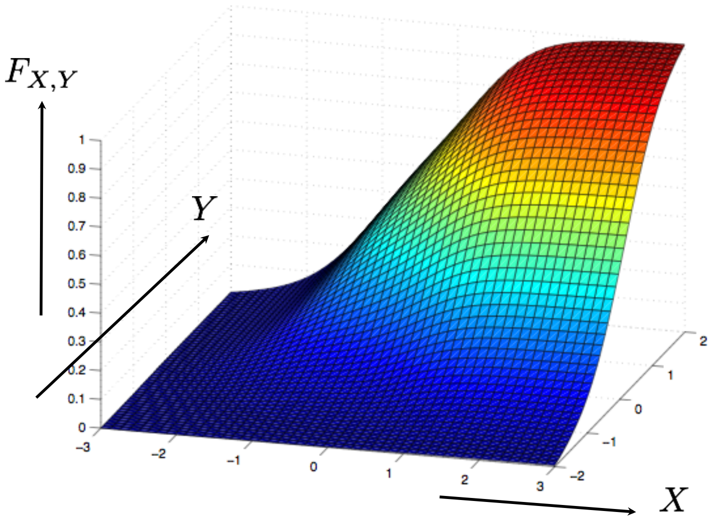
$$F_{X,Y}(x_1, y_1) = P(X \leq x_1, Y \leq y_1)$$







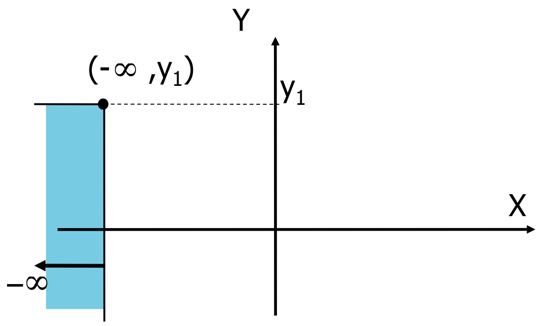

# CDF for continuous variables



# Properties of a joint-CDF

$$F_{X,Y}(-\infty, y_1) = ?$$

$$F_{X,Y}(-\infty, y_1) = P[X \leq -\infty, Y \leq y_1] = 0$$



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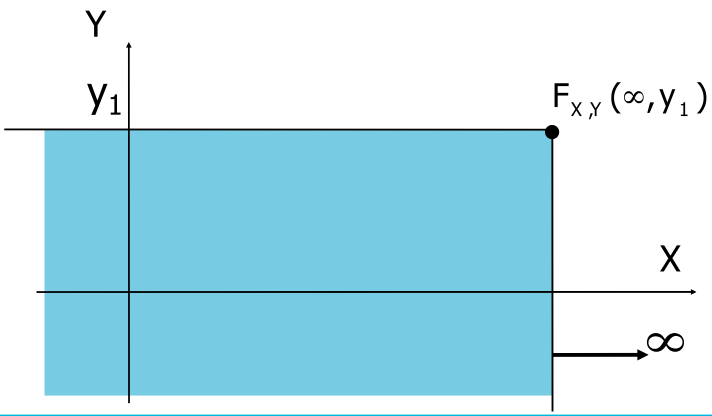
# From j-CDF to ??

$$F_{X,Y}(\infty, y_1) = ?$$



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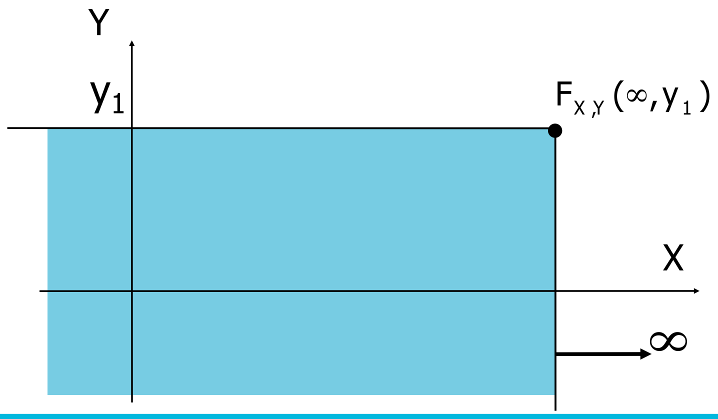
## From j-CDF to ??

$$F_{X,Y}(\infty, y_1) = P[X \leq \infty, Y \leq y_1] = P[Y \leq y_1] = F_Y(y_1)$$



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## Warning

- The determination of the 2-D j-CDF has to be done for all $(x,y)\in\Re\cdot\Re\cdot$
- This is much harder than in 1-D

- For that reason, always start from the definition

$$F_{X,Y}(x_1, y_1) = P(X \leq x_1, Y \leq y_1)$$

and make sure that the CDF is defined for all (x,y) values.



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# Probability density function

- Analogous to the 1D case, for the 2D case:

$$f_{X,Y}(x,y) = \frac{d^2 F_{X,Y}(x,y)}{dx\,dy} = \frac{d^2 P(X \le x, Y \le y)}{dx\,dy}$$

- This is the **joint**-pdf, or j-pdf

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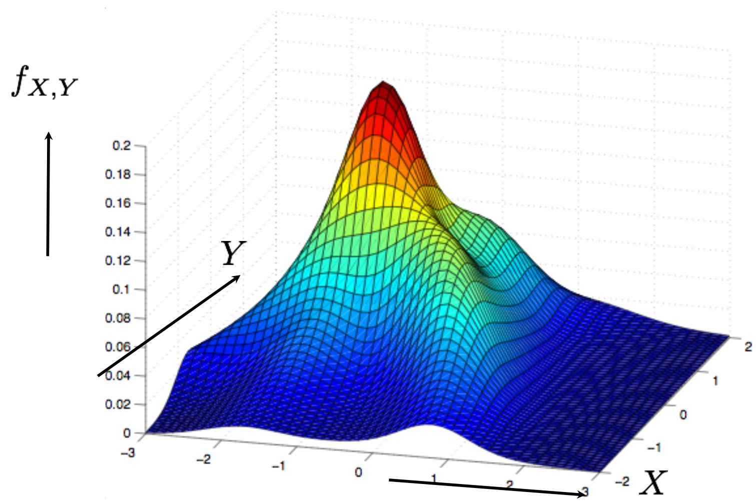
# j-PDF for continuous variables

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## j-PDF ⇔ j-CDF

- 1-D Case

$$F_X(a) = \int_{-\infty}^{a} f_X(x)dx$$

- 2-D Case

$$\underline{F_{X,Y}(a,b) = \int_{-\infty}^{a}\int_{-\infty}^{b} f_{X,Y}(x,y)dydx}$$

$P(X \leq a, Y \leq b)$

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## Marginal pdf's

- Analogous to the marginal probability mass functions, we define marginal probability density functions:

$$f_X(x) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dy$$

$$f_Y(y) = \int_{-\infty}^{\infty} f_{X,Y}(x,y)dx$$

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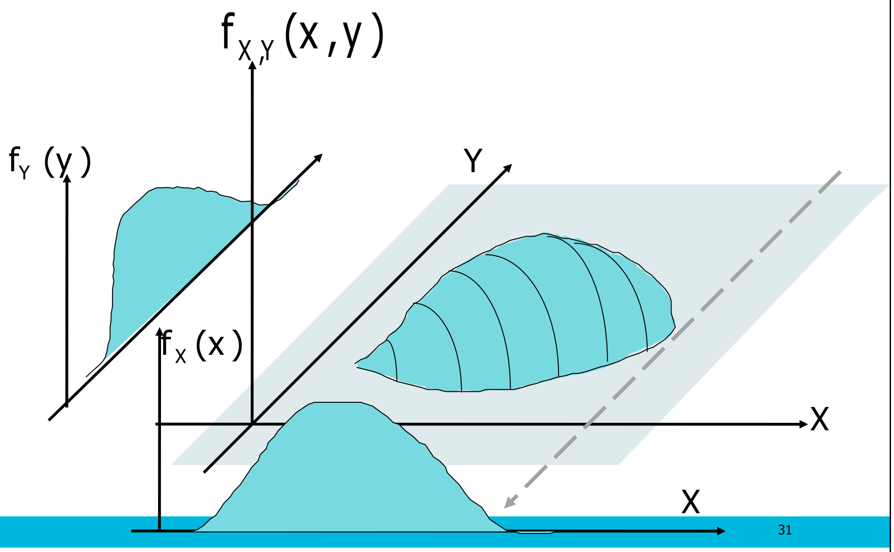
# Illustration



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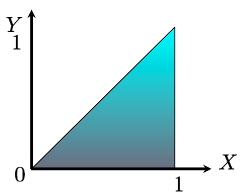
# Example marginal pdf

$$f_{X,Y}(x,y) = \begin{cases} 6y, & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$




- What are the marginal pdfs?



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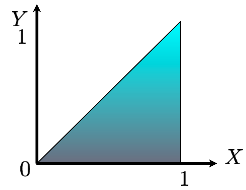
## Example marginal pdf

$$f_{X,Y}(x,y) = \begin{cases} 6y, & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$




- What are the marginal pdfs?

$$f_X(x) = \int_0^x 6y dy = 3x^2, \quad 0 \le x \le 1$$
$$f_Y(y) = \int_y^1 6y dx = 6y(1-y), \quad 0 \le y \le 1$$

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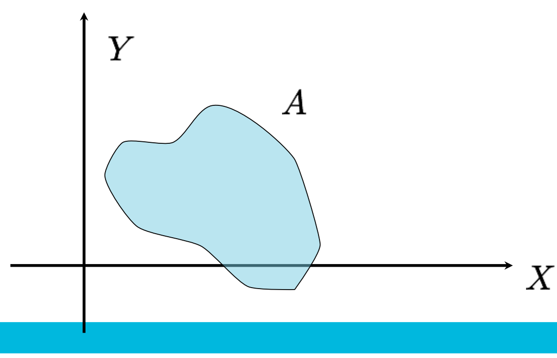
## Probability of events

- Event A: $P(A) = \iint_A f_{X,Y}(x,y) dx\, dy$
- Can be very hard due to boundaries of the event A

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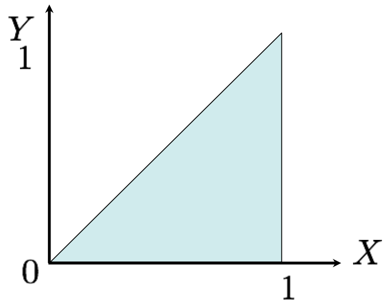
## Example




$f_{XY}(x,y) = c$

(in the blue area)

- Value of c? $\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f_{XY}(x,y)dy\,dx = \int_0^1\int_0^x c dy\,dx = 1$
  $\rightarrow c = 2$

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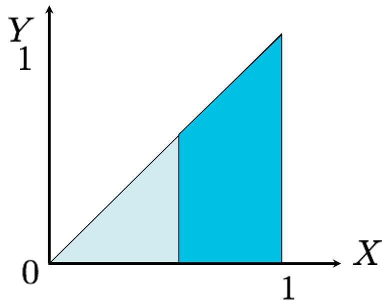
## Example




$f_{XY}(x,y) = c$

(in the blue area)

- Value of c? $\rightarrow c = 2$
- $P[X > 1/2]$ ? $\int_{0.5}^1\int_0^x c dy\,dx = \ldots$

$\rightarrow \ldots = 3/4$

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## Pairs of random variables

- Independence
- Conditional PMF
- Expectation, variance and COvariance
- Correlation



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## Independent Random variables

- X and Y are (stochastically) independent if for any events A and B:

$$P(X \in A, Y \in B) = P(X \in A) \cdot P(Y \in B)$$

- Consequence:

$$X, Y \text{ independent} \Leftrightarrow P_{X,Y}(x,y) = P_X(x)P_Y(y)$$
$$X, Y \text{ independent} \Leftrightarrow f_{X,Y}(x,y) = f_X(x)f_Y(y)$$



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# Test procedure (PMFs)

- Assume we are given $P_{X,Y}(x,y)$
- Compute the marginal PMFs

$$P_X(x) = \sum_{\text{all } y} P_{X,Y}(x,y) \qquad P_Y(y) = \sum_{\text{all } x} P_{X,Y}(x,y)$$

- Test if

$$P_{X,Y}(x,y) = P_X(x) \cdot P_Y(y)$$

for all x,y

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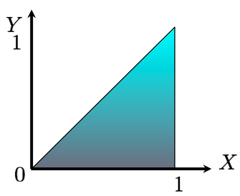
# Example independence

$$f_{X,Y}(x,y) = \begin{cases} 6y, & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$




$$f_X(x) = \int_0^x 6y dy = 3x^2, \quad 0 \le x \le 1$$
$$f_Y(y) = \int_y^1 6y dx = 6y(1-y), \quad 0 \le y \le 1$$

- Are X and Y independent?

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## Conditional PMF/PDF

- $P_{X,Y}(x,y)$ contains all information about the behavior of
  - X
  - Y
  - X and Y jointly
  - dependency
- What about the influence of X on the behavior of Y, and vice versa?

$$P_{X|Y}(x|y) \quad \text{and} \quad P_{Y|X}(y|x)$$
$$f_{X|Y}(x|y) \quad \text{and} \quad f_{Y|X}(y|x)$$

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## Conditional PMF/PDF

- Behavior of a random variable, given (the outcome of) another variable
- Discrete

$$P_{X|Y}(x|y) = \frac{P_{X,Y}(x,y)}{P_Y(y)}$$

- Continuous (is a little tricky, because denominator cannot be a probability of an outcome)

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

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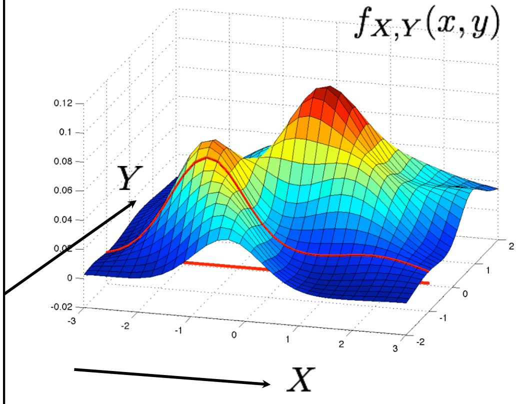
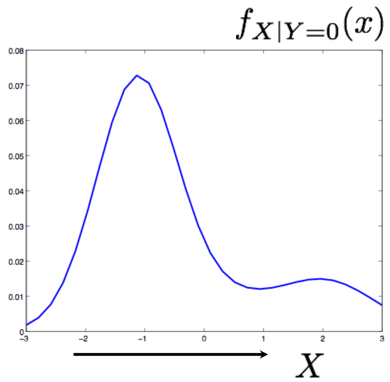
# Conditional pdf




$$f_{X|Y=0}(x) = \frac{f_{X,Y}(x, y=0)}{f_Y(y=0)}$$





# Expectation

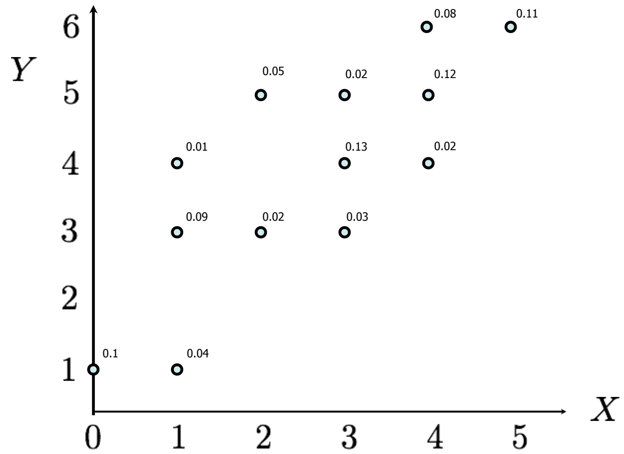
- Expectation of a random vector is a vector
- Compute element-wise the expectation:

$$E[\mathbf{X}] = E\left[\begin{pmatrix} X \\ Y \end{pmatrix}\right] = \begin{bmatrix} E[X] \\ E[Y] \end{bmatrix}$$

$$= \begin{bmatrix} \int_{-\infty}^{\infty} x f_X(x) dx \\ \int_{-\infty}^{\infty} y f_Y(y) dy \end{bmatrix}$$

- With variance it is slightly more involved...

# COvariance




- Characterize the joint behavior of two random variables



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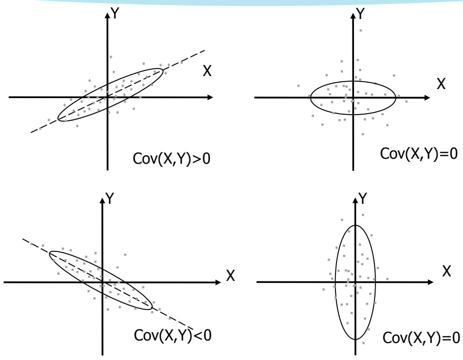
# Covariance of X and Y

- Describes the joint dispersion (global dependency) of X and Y
- Product of the fluctuations of X and Y

$$Cov(X,Y) = E[(X-E[X])(Y-E[Y])] = E[XY] - E[X]E[Y]$$



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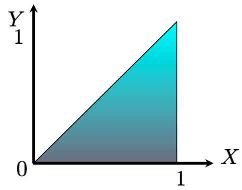
## Example covariance

$$f_{X,Y}(x,y) = \begin{cases} 6y, & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$




- What is the covariance between X and Y?

$$Cov(X,Y) = E[XY] - E[X]E[Y]$$

$$E[XY] = ?$$

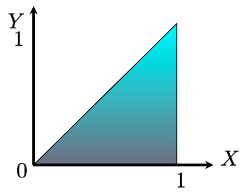
$$E[X] = ?$$

$$E[Y] = ?$$



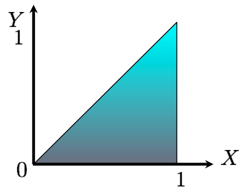
## Example covariance

$$f_{X,Y}(x,y) = \begin{cases} 6y, & 0 \le y \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$




$$E[XY] = \int_0^1 \int_y^1 xy\, 6y\, dx\, dy = 0.4$$

## Example covariance




$$f_{X,Y}(x,y) = \begin{cases} 6y, & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[XY] = \int_0^1 \int_y^1 xy\,6y\,dx\,dy = 0.4$$

$$f_X(x) = \int_0^x 6ydy = 3x^2, \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_y^1 6ydx = 6y(1-y), \quad 0 \leq y \leq 1$$

$$E[X] = \int_0^1 x\,3x^2\,dx = 0.75 \qquad E[Y] = \int_0^1 y\,6y(1-y)dy = 0.5$$

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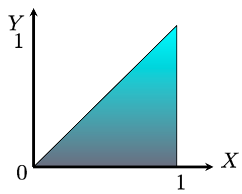
## Example covariance




$$f_{X,Y}(x,y) = \begin{cases} 6y, & 0 \leq y \leq x \leq 1 \\ 0 & \text{otherwise} \end{cases}$$

$$E[XY] = \int_0^1 \int_y^1 xy\,6y\,dx\,dy = 0.4$$

$$f_X(x) = \int_0^x 6ydy = 3x^2, \quad 0 \leq x \leq 1$$

$$f_Y(y) = \int_y^1 6ydx = 6y(1-y), \quad 0 \leq y \leq 1$$

$$E[X] = \int_0^1 x\,3x^2\,dx = 0.75 \qquad E[Y] = \int_0^1 y\,6y(1-y)dy = 0.5$$

$$Cov(X,Y) = 0.4 - 0.75 \cdot 0.5 = 0.025$$

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## Covariance and correlation

- Two related concepts:

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$
$$= E[XY] - E[X]E[Y]$$

correlation

- $Cov(X,Y) = 0 \quad \Leftrightarrow \quad$ X and Y are uncorrelated
- $E[XY] = 0 \quad \Leftrightarrow \quad$ X and Y are orthogonal

- Correlation **coefficient**: $\rho_{X,Y} = \dfrac{Cov(X,Y)}{\sigma_X \sigma_Y}$

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## Independence ⇨ uncorrelated

- Property if X and Y are <u>independent</u>

$$E[h(X)g(Y)] = \iint_{-\infty}^{\infty} h(X)g(Y)f_{XY}(x,y)dxdy$$
$$= \iint_{-\infty}^{\infty} h(X)g(Y)f_X(x)f_Y(y)dxdy$$
$$= \int_{-\infty}^{\infty} h(X)f_X(x)dx \int_{-\infty}^{\infty} g(Y)f_Y(y)dy$$
$$= E[h(X)]E[g(Y)]$$

- Correlation $E[XY] = E[X]E[Y]$
- Covariance $Cov(X,Y) = E[XY] - E[X]E[Y] = 0$

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## Independent vs. uncorrelated

X and Y are independent random variables

$$f_{X,Y}(x,y) = f_X(x) f_Y(y)$$

$$\Downarrow \qquad \not\Uparrow$$

X and Y are uncorrelated random variables

$$Cov(X,Y) = 0$$



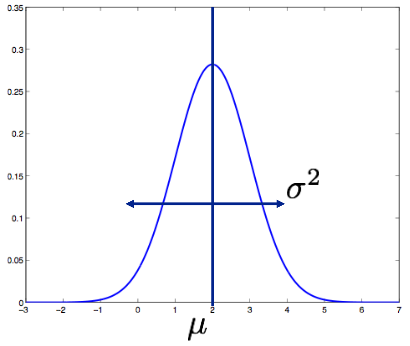
## Independent vs. uncorrelated

- Make sure you understand:
  - the difference between
    - covariance, correlation, correlation coefficient
    - uncorrelated and orthogonal
  - independence implies uncorrelated
  - uncorrelated does NOT automatically imply independence
  - correlation implies dependence
  - dependence does not automatically imply correlated

# Gaussian random variable

- When we have a single Gaussian random variable:

$$f_X(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-(x-\mu)^2/2\sigma^2}$$

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# Multivariate Gaussian

- When we have a **pair** of Gaussian random variables, and we write the pair $\mathbf{X} = \begin{bmatrix} X_1 \\ X_2 \end{bmatrix}$

$$f_{\mathbf{X}}(\mathbf{x}) = \frac{1}{2\pi \det(C_{\mathbf{X}})^{1/2}} \exp\left( -\frac{1}{2}(\mathbf{x} - \mu_{\mathbf{X}})' C_{\mathbf{X}}^{-1} (\mathbf{x} - \mu_{\mathbf{X}}) \right)$$

- Here, the mean vector $\mu_{\mathbf{X}} = \begin{bmatrix} E[X_1] \\ E[X_2] \end{bmatrix}$
- and the **covariance matrix**:

$$C_{\mathbf{X}} = \begin{bmatrix} Var[X_1] & Cov(X_1, X_2) \\ Cov(X_2, X_1) & Var[X_2] \end{bmatrix}$$

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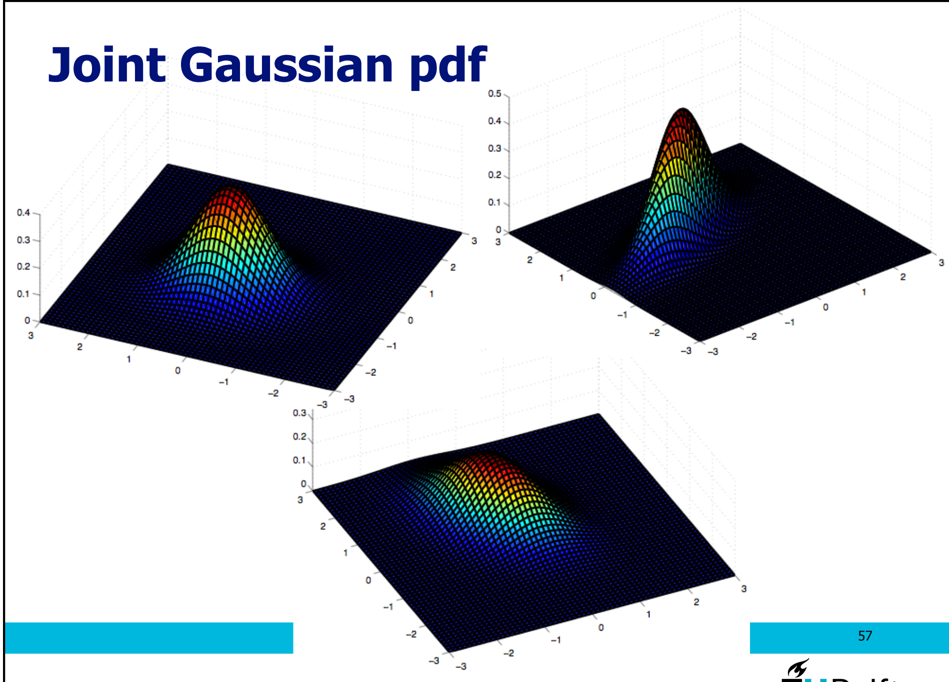
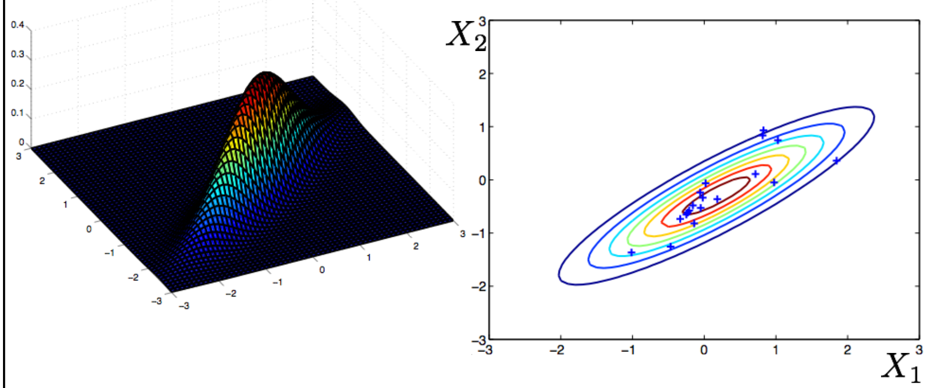
# Joint Gaussian pdf





# Samples from a (2D) Gaussian pdf




- 20 samples drawn from a Gaussian pdf with

$$\mu_{\mathbf{x}} = \begin{pmatrix} 0.2 \\ -0.3 \end{pmatrix} \qquad C_{\mathbf{x}} = \begin{pmatrix} 1.2 & 0.8 \\ 0.8 & 0.7 \end{pmatrix}$$

# Covered Today

- Chapter 4
- Key terms
  - Random vectors
  - Joint and marginal PMF and PDF
  - Probability of events
  - Expectation, variance and covariance
  - Independence and uncorrelated
  - Multivariate Gaussian
- Next week:
  - Stochastic processes

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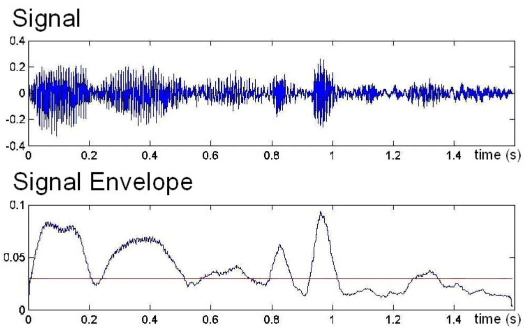
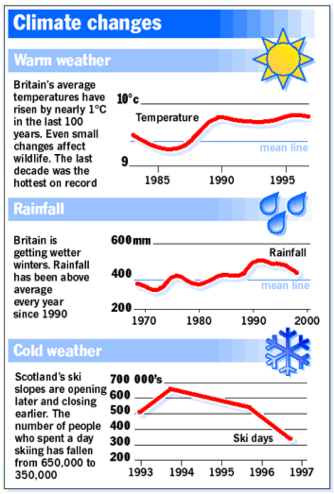
# Almost there...




“I am am bi dex trous.”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# Before Next Time

Exercises corresponding to lecture 2:

- 2nd ed:  4.1.1, 4.2.1, 4.2.2, 4.3.2, 4.4.1, 4.5.2, 4.5.6, 4.7.8
- 3rd ed: 5.1.1, 5.2.1,  5.2.2, 5.3.2, (5.4.1, 5.5.3,  5.5.9, 5.5.8

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