Sampling and Reconstruction

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Digital Signal Processing

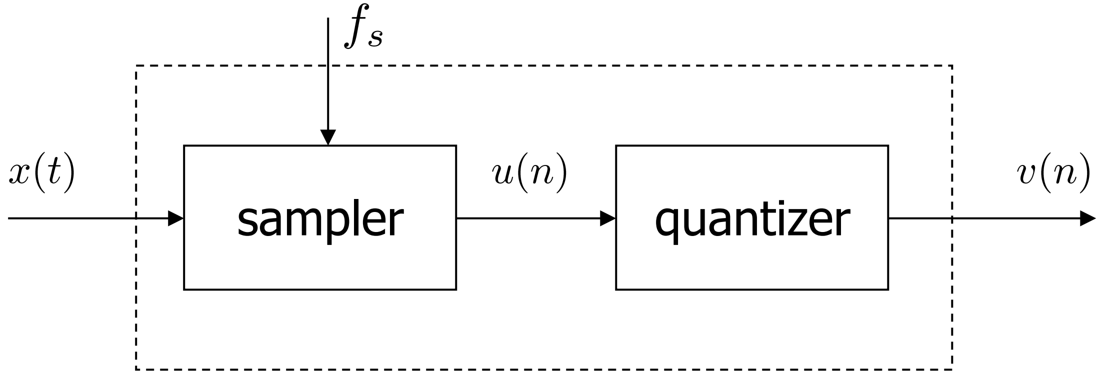
In its most general form, *digital signal processing* (DSP) refers to processing of analog signals by means of discrete-time (discrete-space) operations implemented on digital hardware.

From a system point of view, DSP is concerned with mixed systems:

- the input and output signals are analog
- the processing is done on the equivalent digital signals

Analog-to-Digital Converter

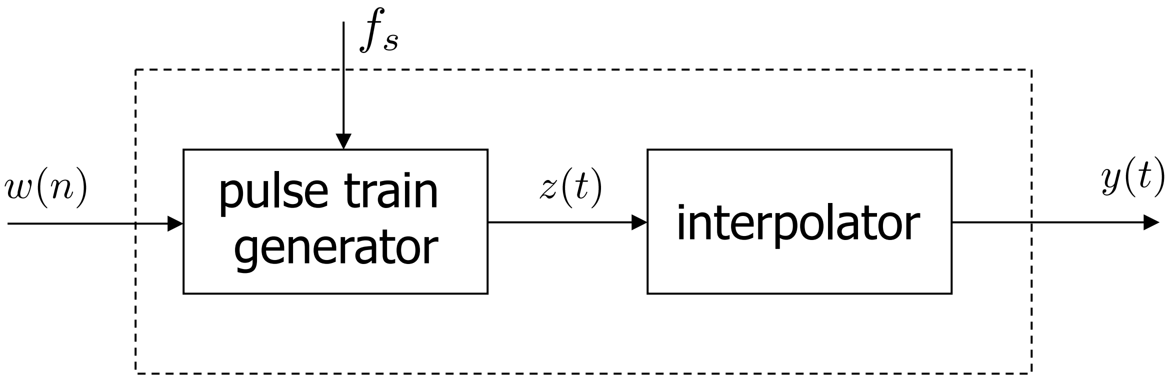
Two-step approach:



- Sampler: $u(n) = x(nT_s)$ where $T_s = 1/f_s$, the sampling period
- Quantizer: $v(n) = (Qu)(n)$, where Q is a (nonlinear) mapping from intervals of the real line (quantization cells) to reproduction levels

Digital-to-Analog Converter

Two-step approach:

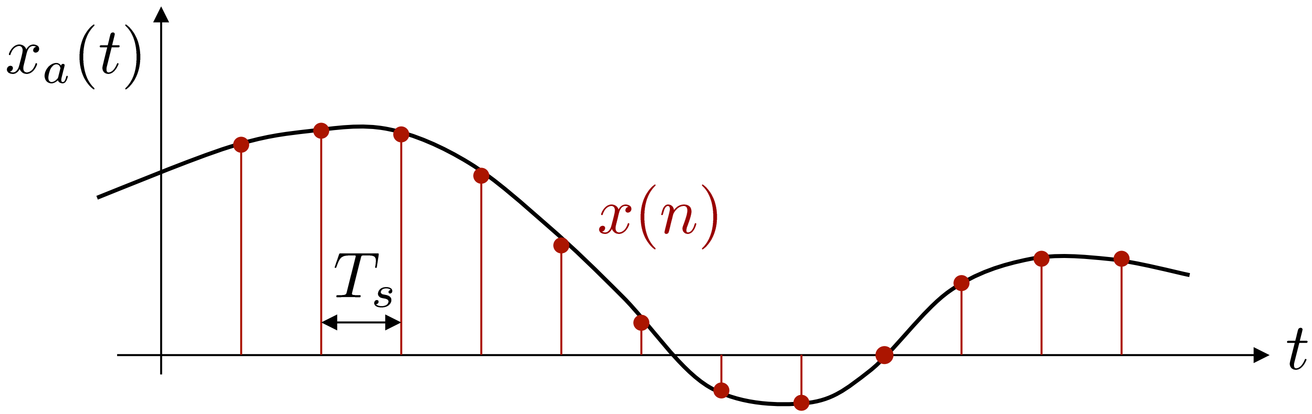


- The pulse-train generator transforms the sequence of numbers $w(n)$ into a sequence of scaled, analog pulses (spaced $T_s = 1/f_s$ seconds apart)
- The interpolator removes high-frequency components in z (via low-pass filtering) to produce a smooth analog output signal

Sampling

To process a continuous-time signal using digital signal processing techniques, it is necessary to convert the signal into a sequence of numbers. This is usually done by *sampling* the analog signal, say $x_a(t)$, periodically every T_s seconds to produce the discrete-time signal $x(n)$ given by

$$x(n) = x_a(nT_s), \quad -\infty < n < \infty$$



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Sampling

Recall that the spectrum of the discrete-time signal x is given by

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n}$$

where the signal x can be recovered from its spectrum by the inverse Fourier transform

$$x(n) = \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega$$

Sampling

If x_a is an aperiodic absolutely integrable signal, its Fourier transform (with $\Omega = 2\pi f$) is given by

$$X_a(\Omega) = \int_{-\infty}^{\infty} x_a(t) e^{-j\Omega t} dt$$

where the signal x_a can be recovered from its spectrum by the inverse Fourier transform

$$x_a(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} X_a(\Omega) e^{j\Omega t} d\Omega$$

The *angular frequency* Ω is expressed in radians per second (rad/s)

Sampling

Key question: What is the relation between $X(\omega)$ and $X_a(\Omega)$, and under what conditions can we recover x_a from $X(\omega)$?

Periodic sampling imposes a relationship between the independent variables t and n in the signals $x_a(t)$ and $x(n)$, respectively

$$t = nT_s = \frac{n}{f_s}$$

and thus between ω and Ω through Fourier transformation.

Sampling

Consider an analog harmonic signal $x_a(t) = e^{j\Omega t}$. What is the relation between the "real" angular frequency Ω and the discrete-time angular frequency ω of the (sampled) discrete-time signal $x(n) = e^{j\omega n}$?

We have

$$e^{j\omega n} = x(n) = x_a(nT_s) = e^{j\Omega T_s n}$$

and we conclude that

$$\boxed{\omega = \Omega T_s = \frac{\Omega}{f_s} = 2\pi \frac{f}{f_s}}$$

Hence, ω is the normalized angular frequency (dimensionless)

Sampling

Relation between $X(\omega)$ and $X_a(\Omega)$:

$$\underbrace{\frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega}_{x(n)} = \underbrace{\frac{1}{2\pi}\int_{-\infty}^{\infty} X_a(\Omega)e^{j\Omega n T_s}d\Omega}_{x_a(nT_s)}$$

$$= \frac{f_s}{2\pi}\int_{-\infty}^{\infty} X_a(\omega)e^{j\omega n}d\omega$$

$$= \frac{f_s}{2\pi}\sum_{k=-\infty}^{\infty}\int_{k2\pi}^{(k+1)2\pi} X_a(\omega)e^{j\omega n}d\omega$$

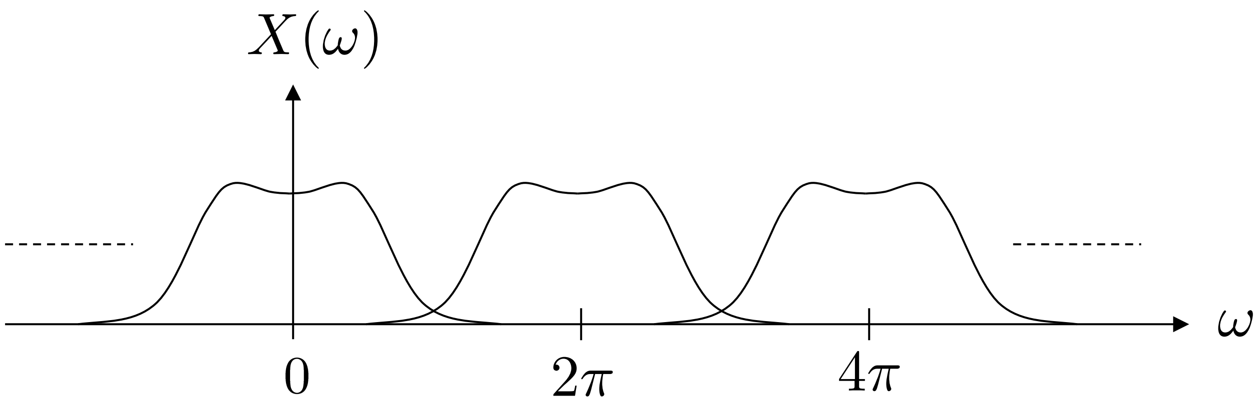
$\omega \rightarrow \omega + k2\pi$

$$= \frac{1}{2\pi}\int_0^{2\pi}\left(f_s\sum_{k=-\infty}^{\infty} X_a(\omega+k2\pi)\right)e^{j\omega n}d\omega$$

Sampling

Hence,

$$X(\omega) = f_s \sum_{k=-\infty}^{\infty} X_a(\omega + k2\pi)$$



The spectrum of the discrete-time signal consists of shifted copies of the (scaled) analog spectrum

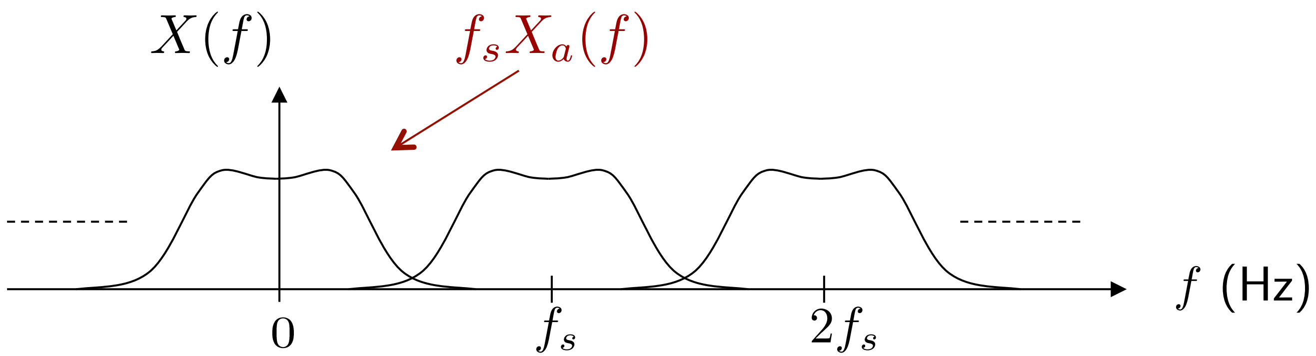
Sampling

Equivalently, we have

$$X(\Omega) = f_s \sum_{k=-\infty}^{\infty} X_a(\Omega + k\Omega_s), \quad \Omega_s = 2\pi f_s$$

or

$$X(f) = f_s \sum_{k=-\infty}^{\infty} X_a(f + kf_s)$$



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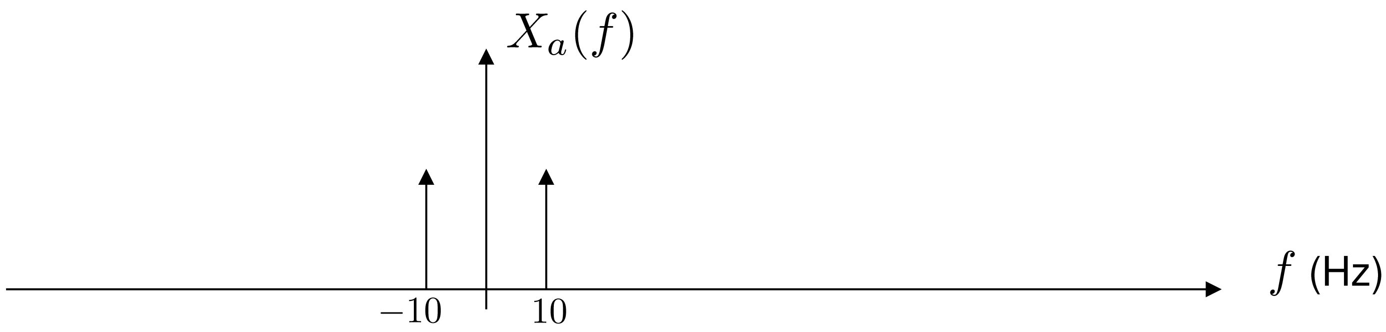
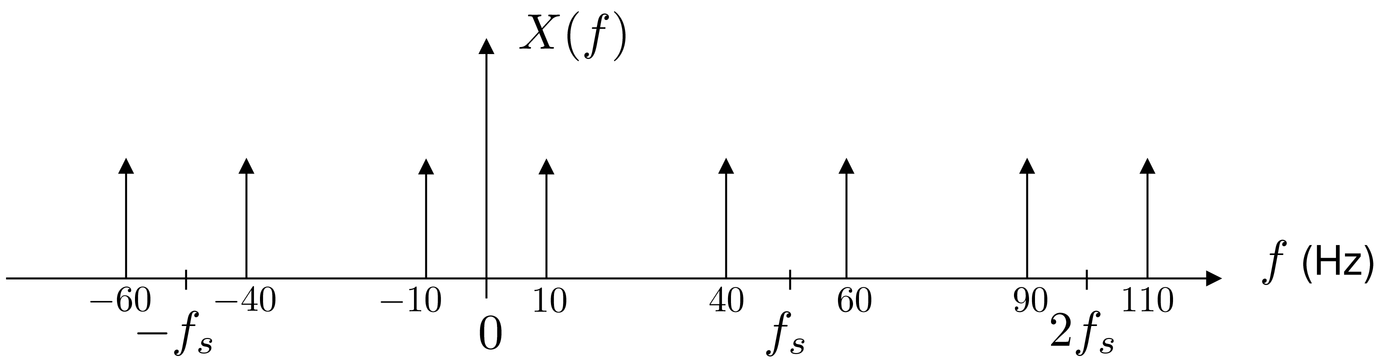
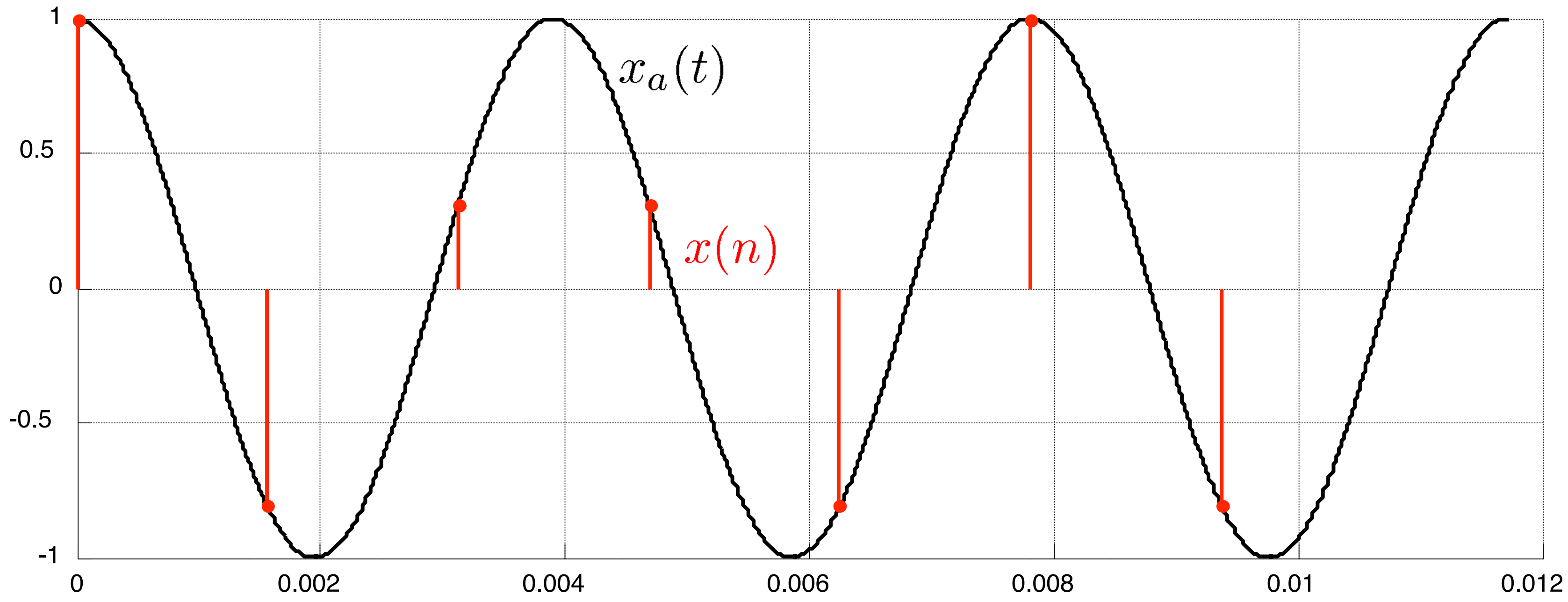
Sampling

Example: $x_a(t) = \cos(2\pi 10 t), \quad f_s = 50$ Hz



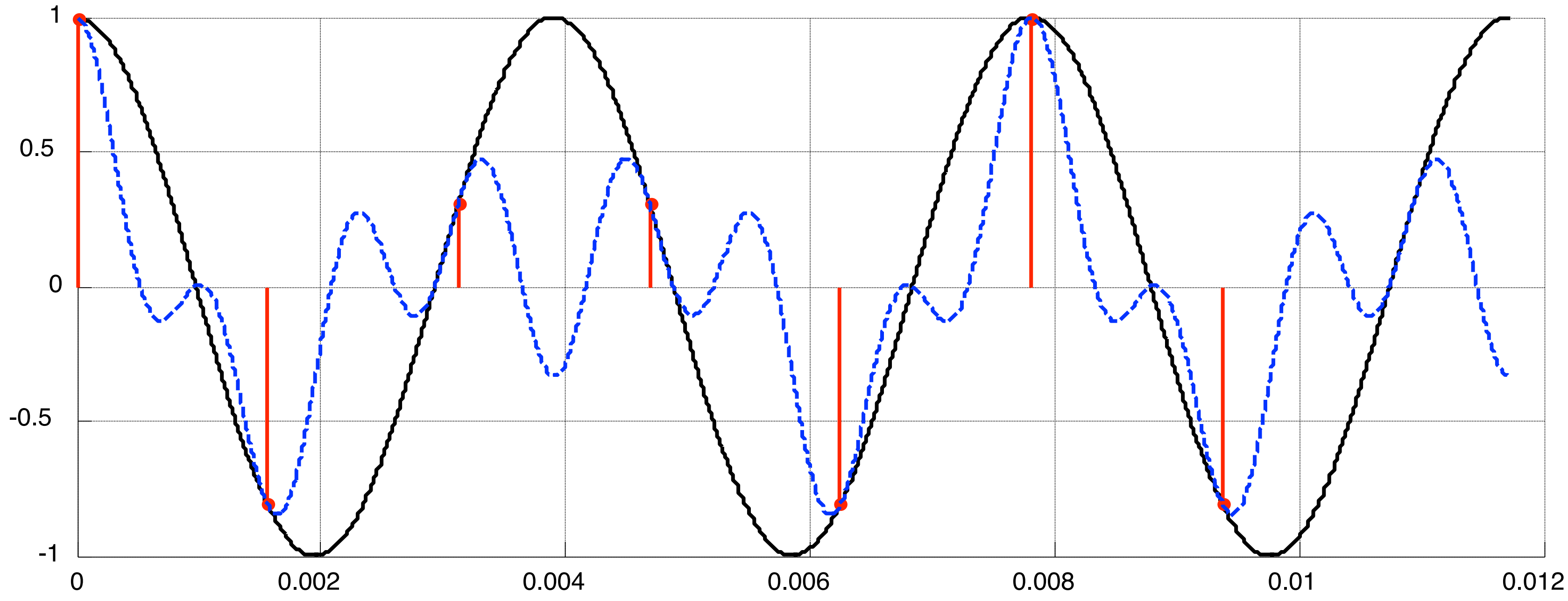


Spectrum of harmonic signals



Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, three cosines with frequencies $(f_0, f_s - f_0, f_s + f_0)$



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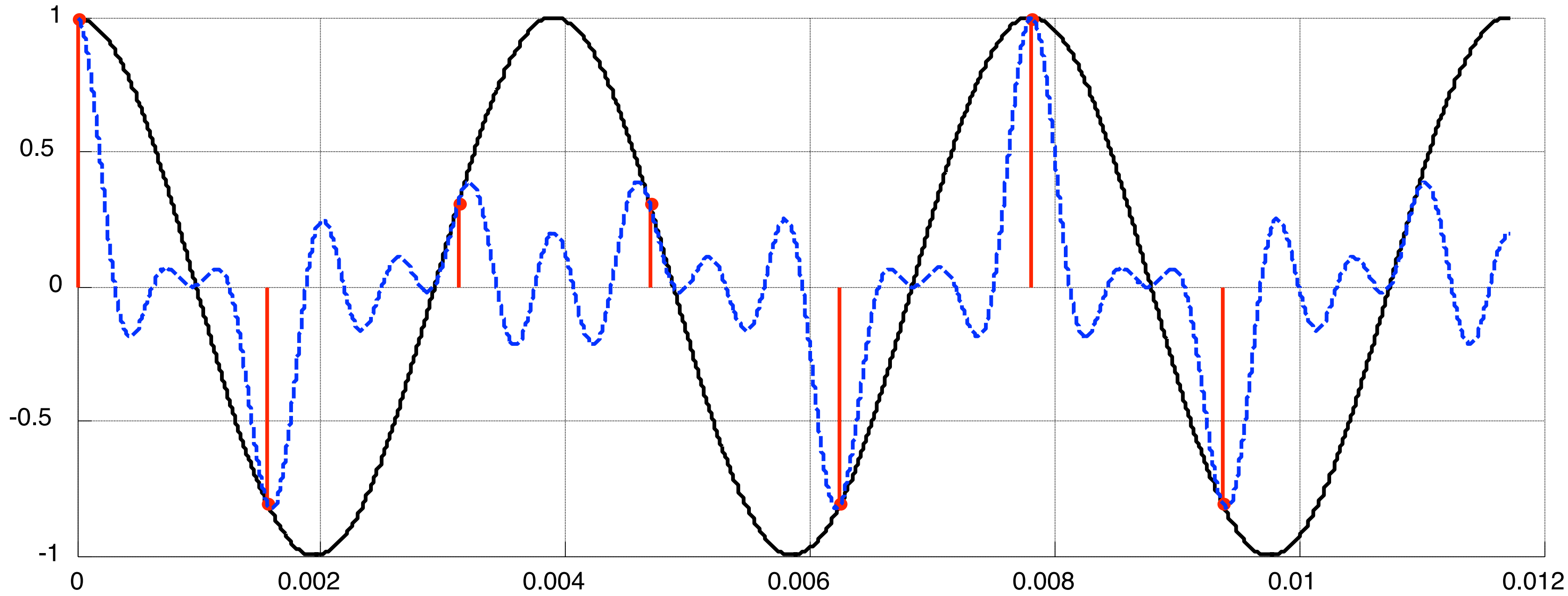
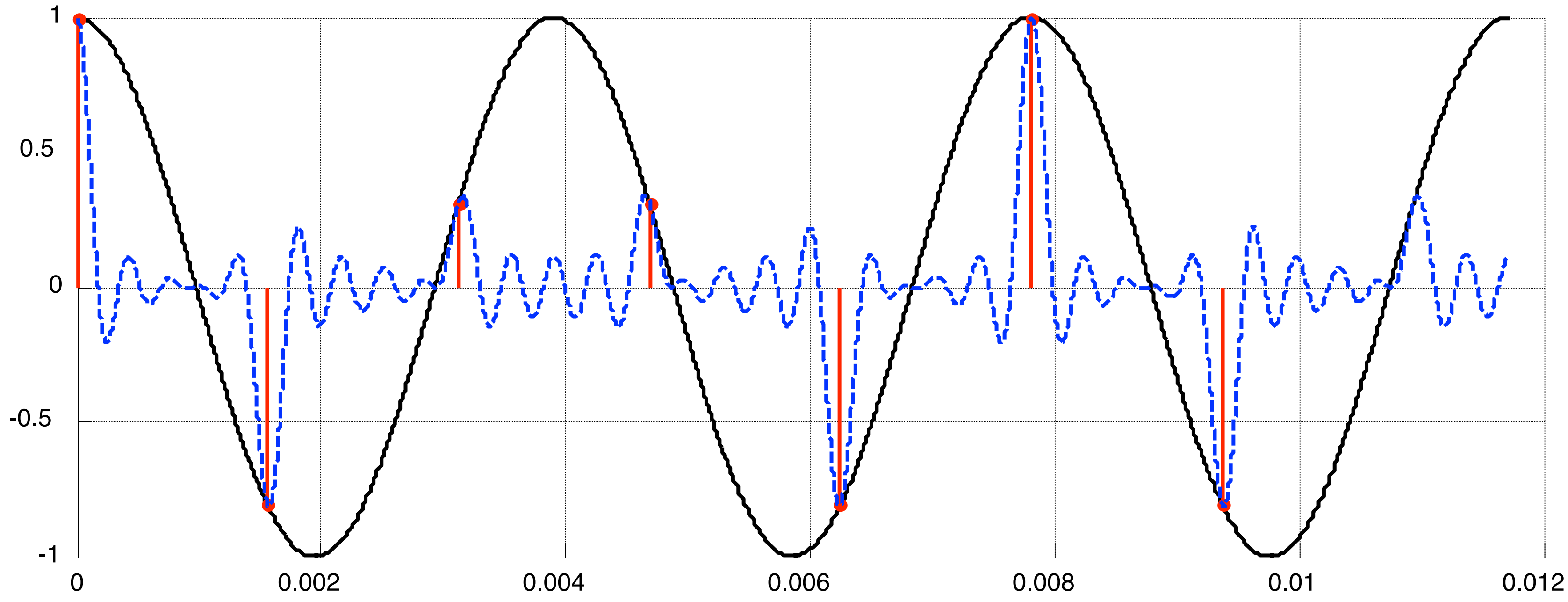
Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, five cosines with frequencies $(f_0, f_s - f_0, f_s + f_0, 2f_s - f_0, 2f_s + f_0)$



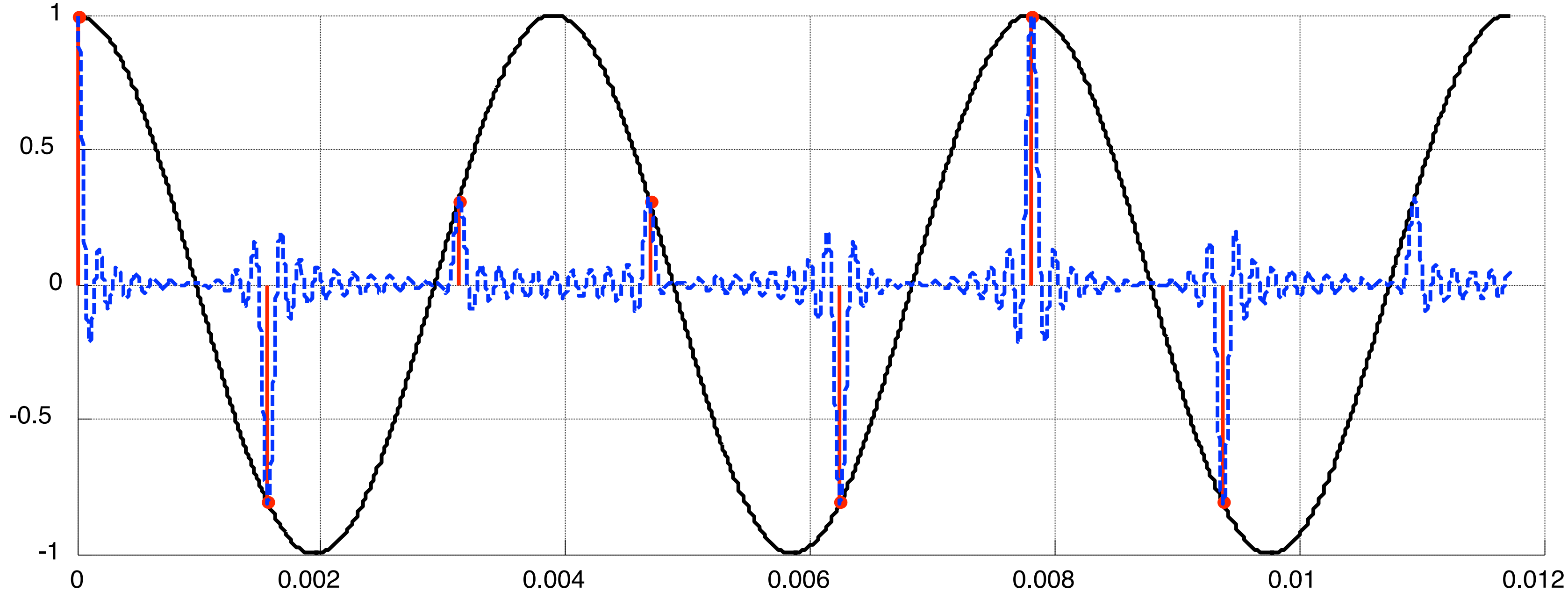
Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, nine cosine terms



Spectrum of harmonic signals

$x(t)$, $x(nT_s)$, 21 cosine terms



Sampling

If the spectrum of the analog signal is band limited to, say B Hz, and the sampling frequency satisfies $f_s > 2B$, we have

$$X(f) = f_s X_a(f) \quad \text{for } |f| \leq \frac{f_s}{2}$$

since the periodic repetition of the spectrum of X_a does not introduce spectral overlap

In this case, we can perfectly reconstruct the original analog signal by scaling the input spectrum by f_s^{-1} and removing all spectral components $|f| > \frac{f_s}{2}$

$$X_a(f) = \begin{cases} f_s^{-1} X(f), & |f| \leq \frac{f_s}{2} \\ 0, & |f| > \frac{f_s}{2} \end{cases}$$

Reconstruction

$$x_a(t) = \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X_a(f) e^{j2\pi f t} df$$

no spectral overlap

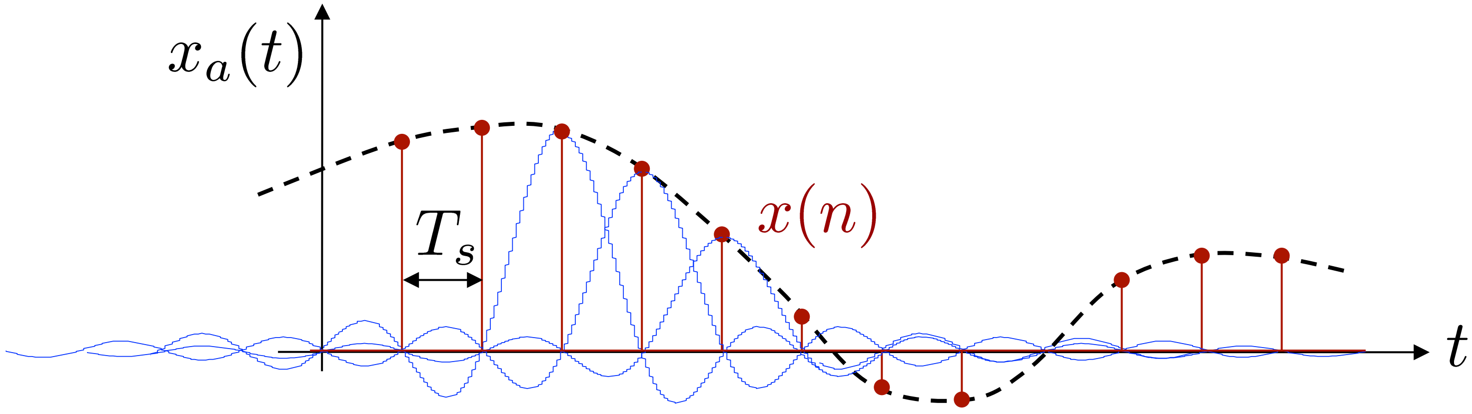
$$= T_s \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} X(f) e^{j2\pi f t} df$$

$$= T_s \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} \left(\sum_{n=-\infty}^{\infty} x(n) e^{-j2\pi \frac{f}{f_s} n} \right) e^{j2\pi f t} df$$

$$= T_s \sum_{n=-\infty}^{\infty} x(n) \int_{-\frac{f_s}{2}}^{\frac{f_s}{2}} e^{j2\pi f (t - nT_s)} df$$

$$= \sum_{n=-\infty}^{\infty} x(n) \frac{\sin\left(\frac{\pi}{T_s}(t - nT_s)\right)}{\frac{\pi}{T_s}(t - nT_s)} = \sum_{n=-\infty}^{\infty} x(n) g(t - nT_s)$$

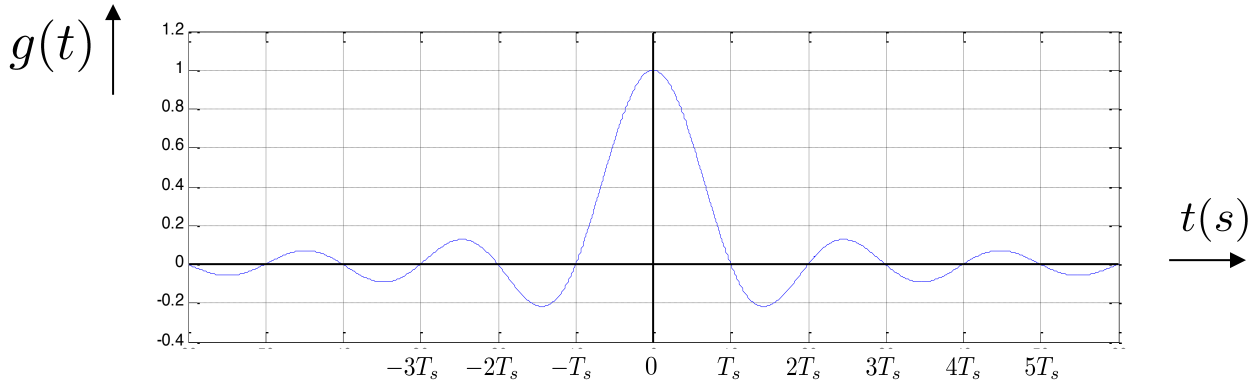
Reconstruction



Hence, $x_a(t)$ is a linear combination of time-shifted signals $g(t-nT_s)$ weighted by the sample values $x(n)$

The family $\{g(t-nT_s)\}_{n=-\infty}^{\infty}$ forms an orthonormal basis for band-limited signals, assuming that $f_s > 2B$

Reconstruction



The function g is called an *interpolation* function:

- at $t = kT_s$, the interpolation function $g(t - nT_s)$ is zero, except for $k = n$, where it is 1, and consequently, $x_a(t)$ evaluated at $t = kT_s$ is simply the sample $x_a(kT_s)$
- all other points $t \neq kT_s$ are weighted sums of time-shifted versions of the interpolation function

Reconstruction

Since the Fourier transform of g is given by

$$g(t) = \frac{\sin(\frac{\pi}{T_s}t)}{\frac{\pi}{T_s}t} \quad \stackrel{\mathcal{F}}{\longleftrightarrow} \quad G(f) = \begin{cases} f_s^{-1}, & |f| \leq \frac{f_s}{2} \\ 0, & |f| > \frac{f_s}{2} \end{cases}$$

we conclude that the ideal interpolator scales the input spectrum by a factor f_s^{-1} and removes frequencies components for $|f| > \frac{f_s}{2}$. Basically, the ideal interpolator acts as a frequency window that removes the discrete-time spectral periodicity to generate an aperiodic continuous-time signal spectrum

Sampling and Reconstruction

Aliasing occurs when the signal is sampled at a rate which is too low. For real signals, the effect can be described by folding of the frequency axis

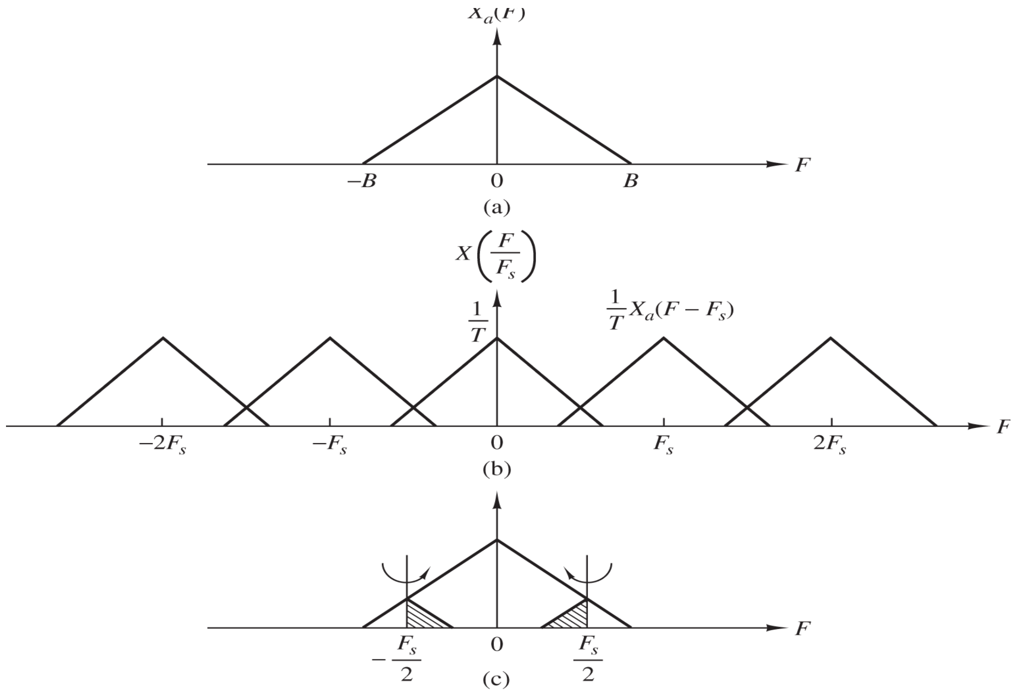
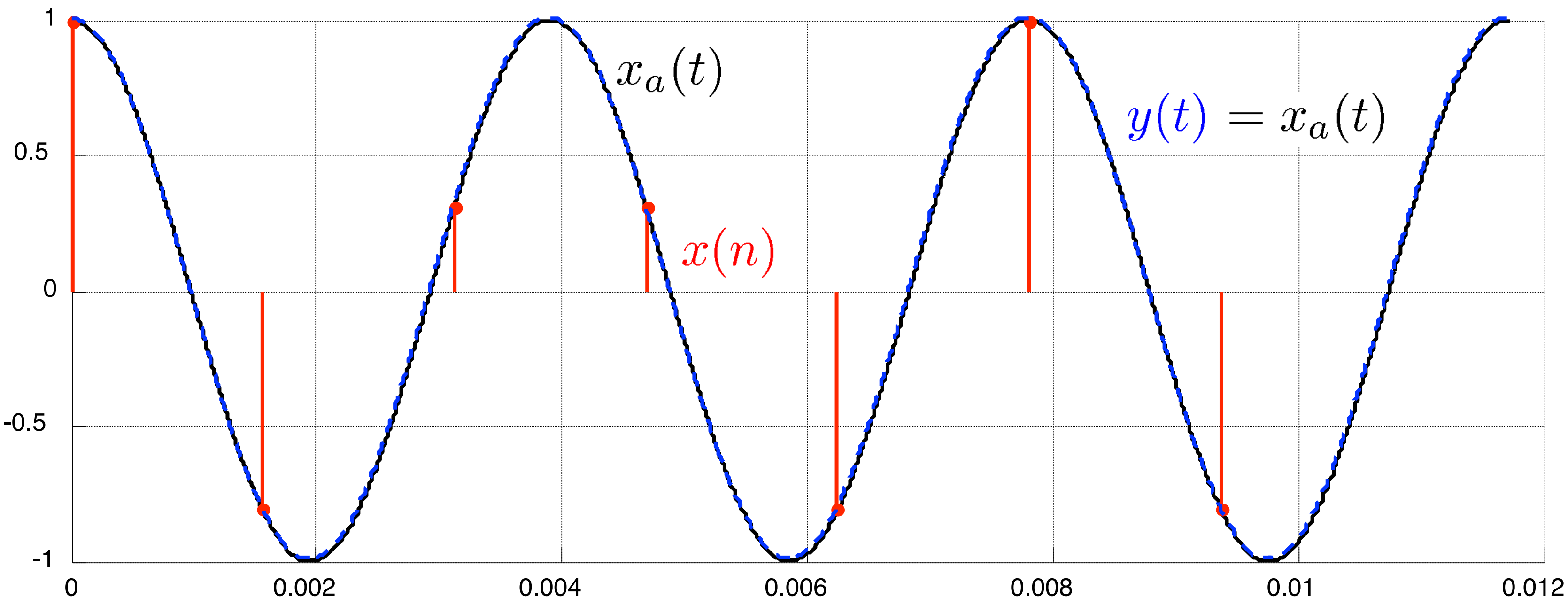


Figure 6.1.3 Illustration of aliasing around the folding frequency.

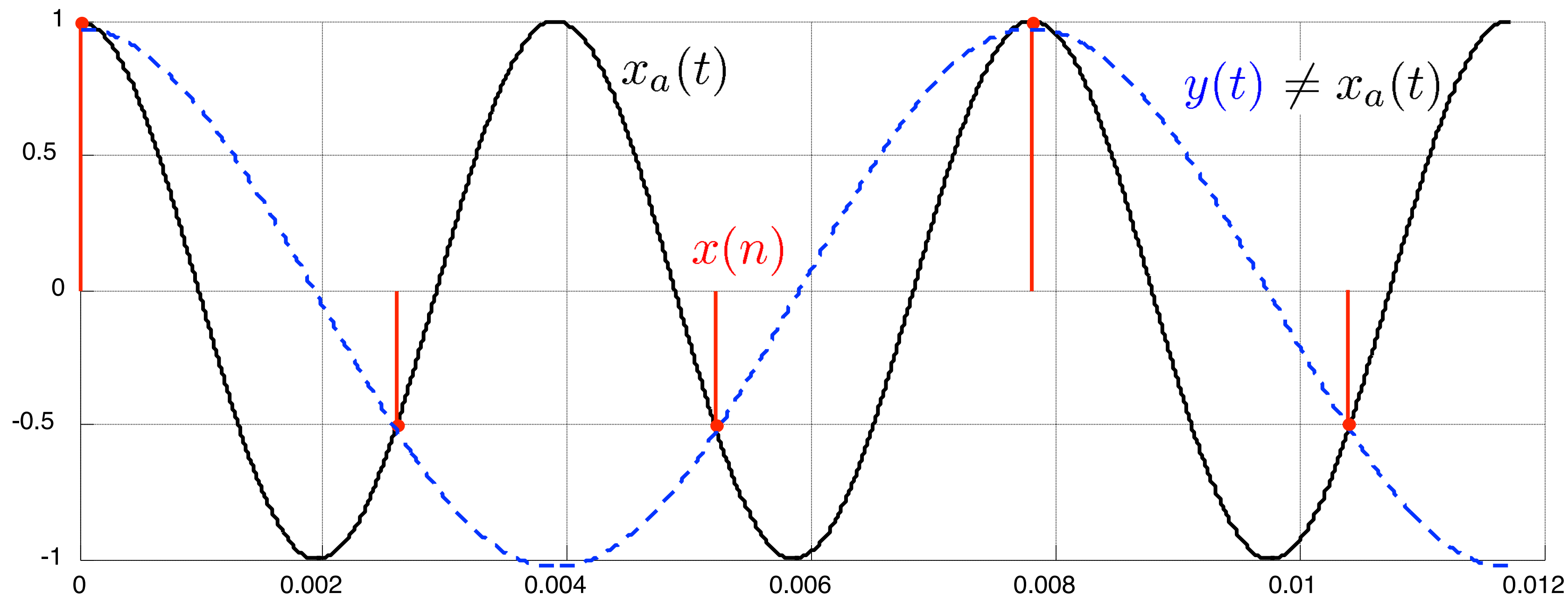
Sampling and Reconstruction

$f_s > 2f_0$:



Sampling and Reconstruction

$f_s < 2f_0$:



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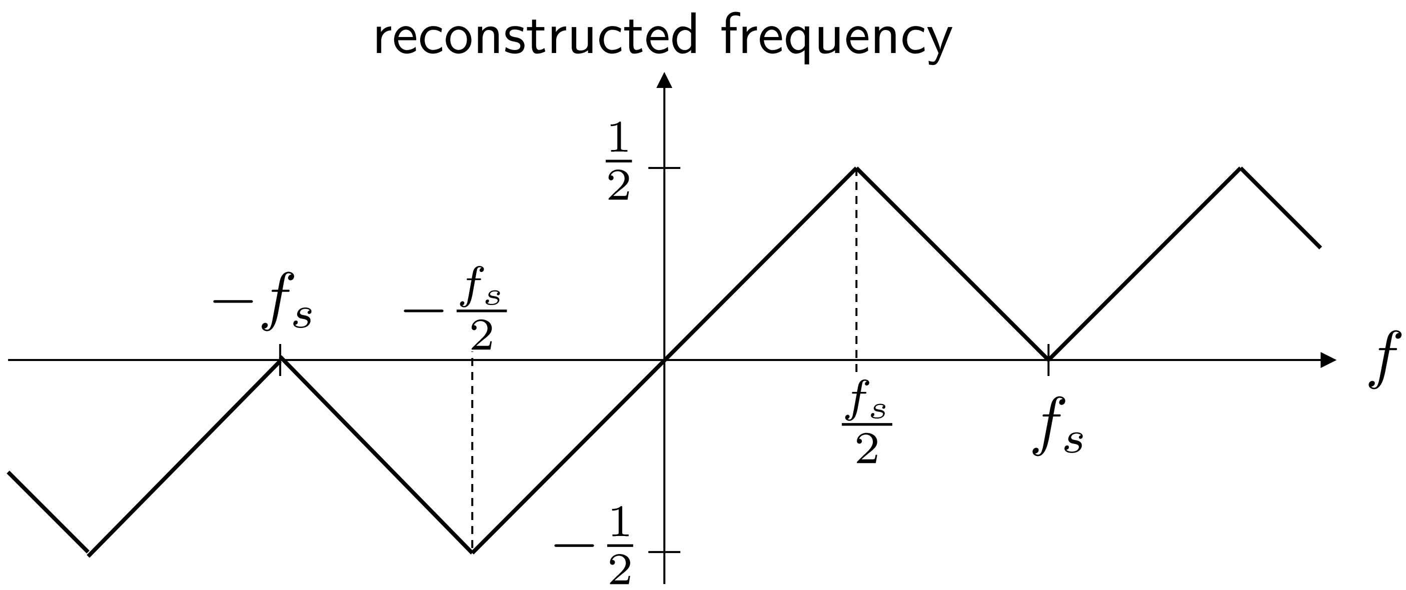
Sampling and Reconstruction



Example: frequency sweep 0 - 4 kHz

$f_s = 8$ kHz

$f_s = 2$ kHz

Anti-Aliasing Filter

In order to avoid aliasing effects, a prefilter (or anti-aliasing filter) is used

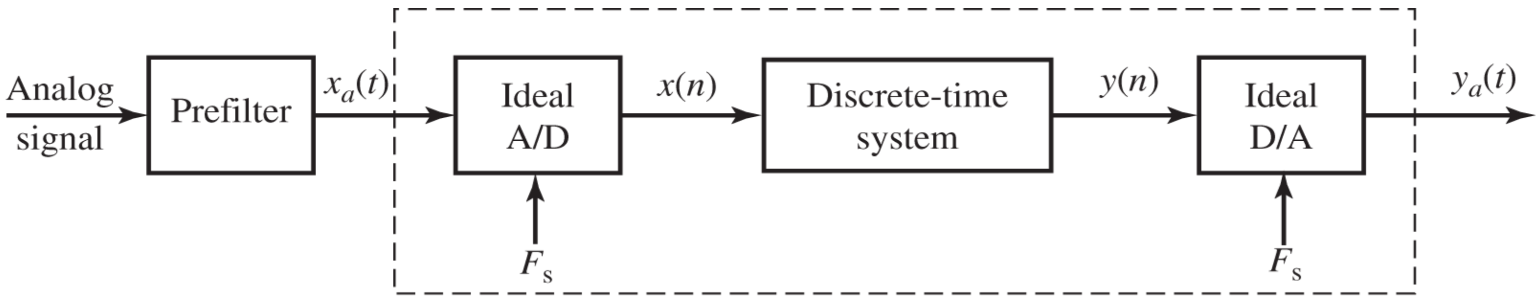


Figure 6.2.1 System for the discrete-time processing of continuous-time signals.

original

with prefilter

without prefilter

original

with prefilter

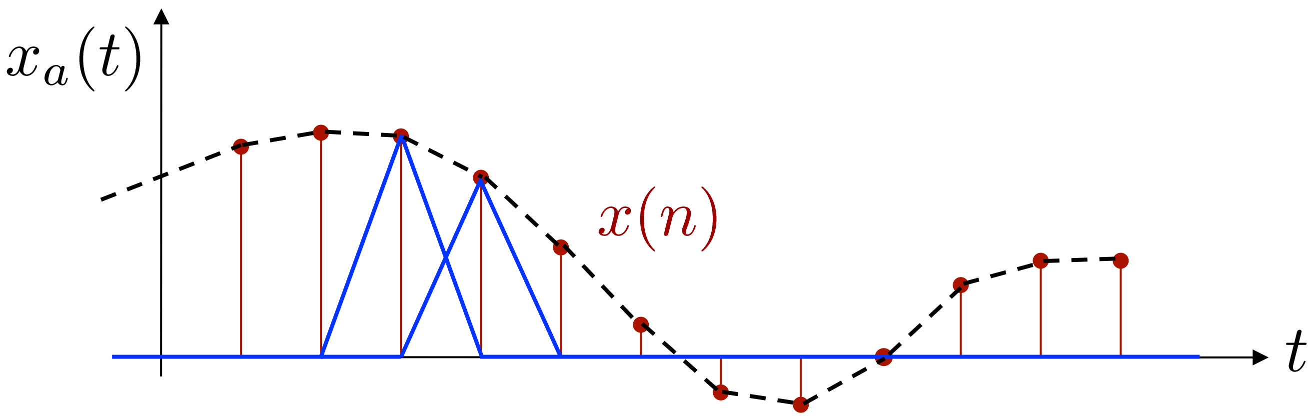
without prefilter

Practical Interpolation

The ideal interpolator is not a practical interpolator. Why?

Alternatively, we can use a linear interpolator scheme:

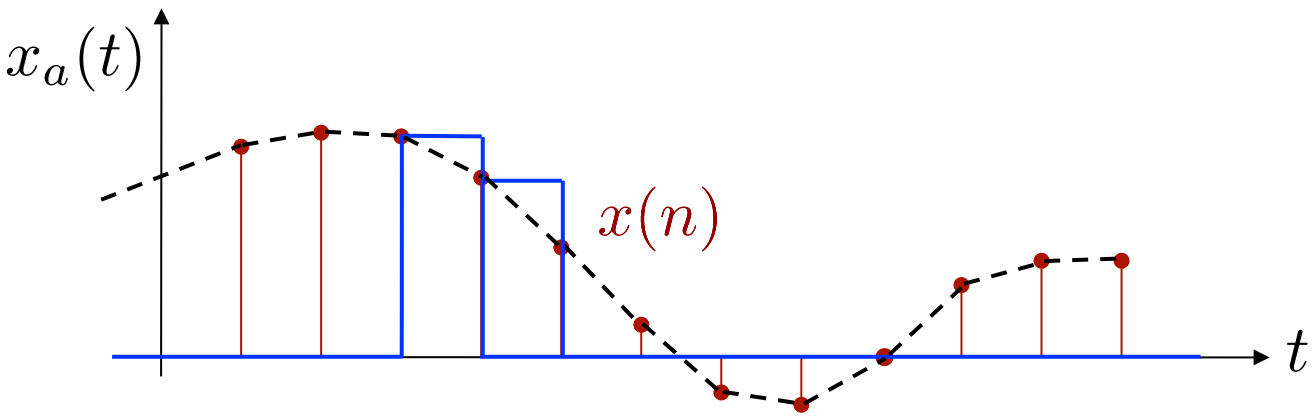
$$\hat{x}_a(t) = \sum_{n=-\infty}^{\infty} x(n) g_{\text{lin}}(t - nT_s), \quad g_{\text{lin}}(t) = \begin{cases} 1 - \frac{|t|}{T_s}, & |t| < T_s \\ 0, & \text{otherwise} \end{cases}$$



Practical Interpolation

Even simpler: sample-and-hold (capacity to buffer the voltage)

$$\hat{x}_a(t) = \sum_{n=-\infty}^{\infty} x(n) g_{\text{sh}}(t - nT_s), \quad g_{\text{sh}}(t) = \begin{cases} 1, & 0 \le t \le T_s \\ 0, & \text{otherwise} \end{cases}$$



Practical Interpolation

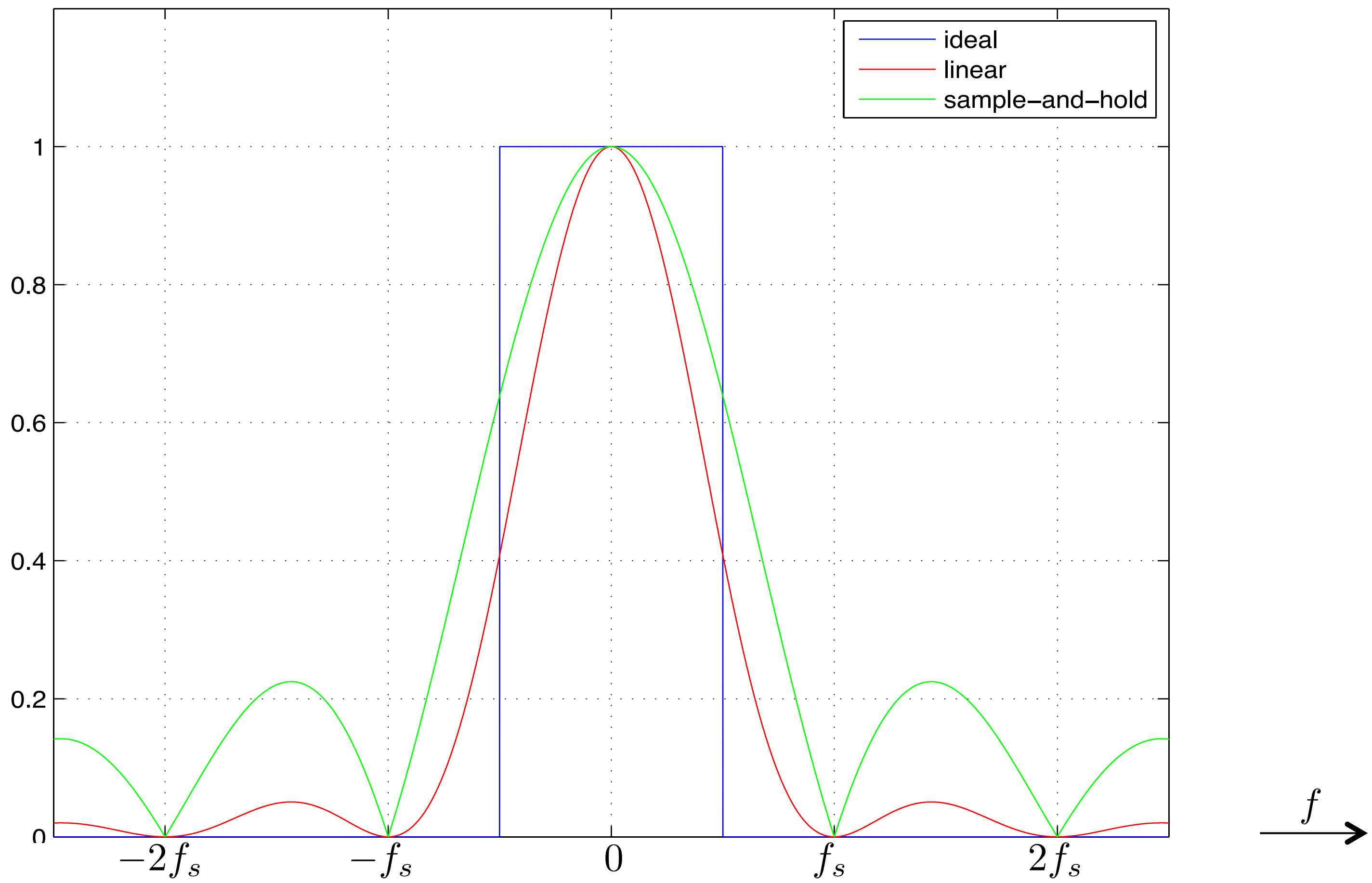
Again, similar to what we derived for the ideal interpolator, we now have that the Fourier transform of the output of the (practical) interpolator is given by $\hat{X}_a(f) = X(f)G_{(\cdot)}(f)$, where

$$G_{\text{ideal}}(f) = \begin{cases} f_s^{-1}, & |f| \leq \frac{f_s}{2} \\ 0, & |f| > \frac{f_s}{2} \end{cases}$$

$$G_{\text{lin}}(f) = T_s \left(\frac{\sin(\pi f T_s)}{\pi f T_s} \right)^2$$

$$G_{\text{sh}}(f) = T_s \frac{\sin(\pi f T_s)}{\pi f T_s} e^{-i\pi f T_s}$$

Practical Interpolation



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