Resit Exam EE2511 - Stochastic Processes

July 2, 2014

Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Further is given:

Uniform distribution: for $a < b$:

$$
\begin{aligned}
f_X(x) &= \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \\
E[X] &= \frac{a+b}{2} \\
Var[X] &= \frac{(b-a)^2}{12}
\end{aligned}
$$

Laplace distribution: for $a > 0$ and $-\infty < b < \infty$:

$$
\begin{aligned}
f_X(x) &= \frac{a}{2} e^{-a|x-b|} \\
E[X] &= b \\
Var[X] &= \frac{2}{a^2}
\end{aligned}
$$

Chebyshev inequality: for an arbitrary random variable Y and a constant $c > 0$:

$$P[|Y - \mu_Y| \geq c] \leq \frac{\text{Var}[Y]}{c^2}$$

Erlang's B formula: define the load $\rho = \frac{\lambda}{\mu}$ given the arrival rate λ and service rate μ:

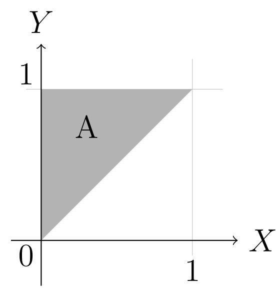
$$P[N = c] = \frac{\rho^c / c!}{\sum_{k=0}^{c} \rho^k / k!}$$

Question 1 - Probabilities (11 p)

The joint probability density function of two variables X and Y is given by:

$$f_{X,Y}(x,y) = \begin{cases} c(x+2y) & 0 \le x \le y \le 1 \\ 0 & \text{otherwise.} \end{cases}$$

A graphical representation is given in the plot below:



(2 p) (a) Show that in area A (see figure) the expression of the cumulative distribution function $F_{X,Y}(x,y)$ is given by:

$$F_{X,Y}(x,y) = cx\left(\frac{1}{2}xy + y^2 - \frac{2}{3}x^2\right)$$

(2 p) (b) Calculate the value of constant c.

(2 p) (c) Calculate the probability $P[Y > \frac{1}{2}]$.

(2 p) (d) Calculate the marginal pdfs of X and Y: $f_X(x)$ and $f_Y(y)$.

(2 p) (e) Find the expected value and variance of Y,

(1 p) (f) Are the variables X and Y independent? Explain your answer.

Question 2 - Signal Processing (10 p)

Consider the sequence $..., X_{-1}, X_0, X_1, ...$ where X_n are independent random variables with $E[X_i] = a$, $Var[X_i] = 1$. The output of a digital filter is $..., Y_{-1}, Y_0, Y_1, ...$ defined by

$$Y_n = \frac{1}{2}X_{n-1} + X_n + \frac{1}{2}X_{n+1}$$

(1 p) (a) Find the autocorrelation function $R_X(k)$.

(1 p) (b) Find the expected value $E[Y_i]$.

(3 p) (c) Compute the autocovariance function $C_Y[m, k]$.

(2 p) (d) Explain if Y_n is iid random sequence or not.

(3 p) (e) Compute the autocorrelation function of the random process $Z_n = Y_n + W_n$, where W_n is a stationary noise process with an expected value $\mu_W = 0$, and an autocorrelation of:

$$R_W(k) = \begin{cases} 4 & k = 0 \\ 2 & |k| = 1 \\ 1 & \text{otherwise.} \end{cases}$$

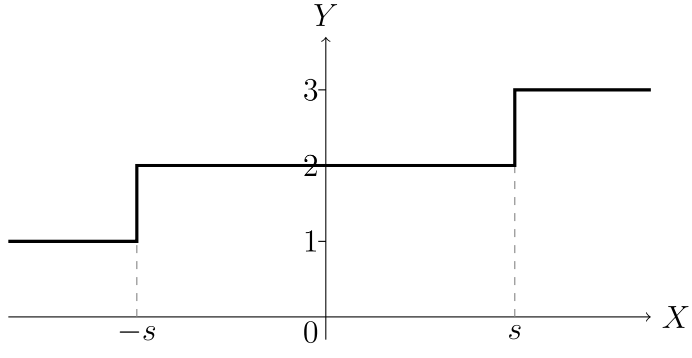
Furthermore, W_n is mutually uncorrelated with Y_n.

Question 3 - Estimation (10 p)

Assume that X has a Laplace distribution, with zero mean ($b = 0$) and scale parameter $a = 2$ (see formula on the first page of this exam). We define a new random variable Y such that:

$$Y = \begin{cases} 1 & \text{if } X \text{ lies in the interval } (-\infty, -s) \\ 2 & \text{if } X \text{ lies in the interval } [-s, +s) \\ 3 & \text{if } X \text{ lies in the interval } [+s, +\infty) \end{cases}$$

A graphical representation is given in the plot below:



(2 p) (a) Determine the cumulative distribution function $F_X(x)$.

(2 p) (b) Calculate s if we give that $P[X \in (-\infty, -s)] = P[X \in [-s, s)]$.

(2 p) (s) Use the Chebyshev inequality to estimate the probability that $X < -s$ or $X > +s$.

(1 p) (d) Is the probability that you estimated in question **(c)** larger, smaller or equal to the probability from question **(b)**? Explain why this is necessarily the case.

(2 p) (e) Calculate the expected value $E[Y]$ and the variance $Var[Y]$ of Y.

(1 p) (f) Calculate the conditional probability $P[X > 2s|Y = 1]$.

Question 4 - Markov Chains (9 p)

A student decides to analyse the way he lives using a Markov Chain with four states: IDLE, STUDY, PARTY and SLEEP. With a probability of 0.8, he starts to study after being idle. With a probability of 0.7, he keeps studying and with a probability of 0.3 he leaves for a party. When in the PARTY state, the student needs to go back to the SLEEP state with a probability of 0.1, or goes back to the IDLE state with a probability of 0.2. Otherwise he keeps partying. In the SLEEP state, he starts to study with a probability of 0.4, or otherwise keeps sleeping.

(2 p) (a) Draw the diagram of this Markov Chain and give the transition matrix.

(2 p) (b) Determine the communicating classes and define whether each class is transient or recurrent. If defined, give the period of each class.

(2 p) (c) Compute the transition probabilities p_1, p_2, p_3 and p_4 after 3 transitions for the following cases: the student starts (i) in the IDLE state, (ii) in the STUDY state, (iii) in the PARTY state, and (iv) in the SLEEP state.

(3 p) (d) Compute the average hour for each state that the students spends for a week consisting of $7 \times 24 = 168$ hours. [**Hint**: You need to compute the limiting state probabilities: $\pi_0, \pi_1, \pi_2, \pi_3$.]