Fourier Series and Transform (Recap)

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Frequency Decomposition

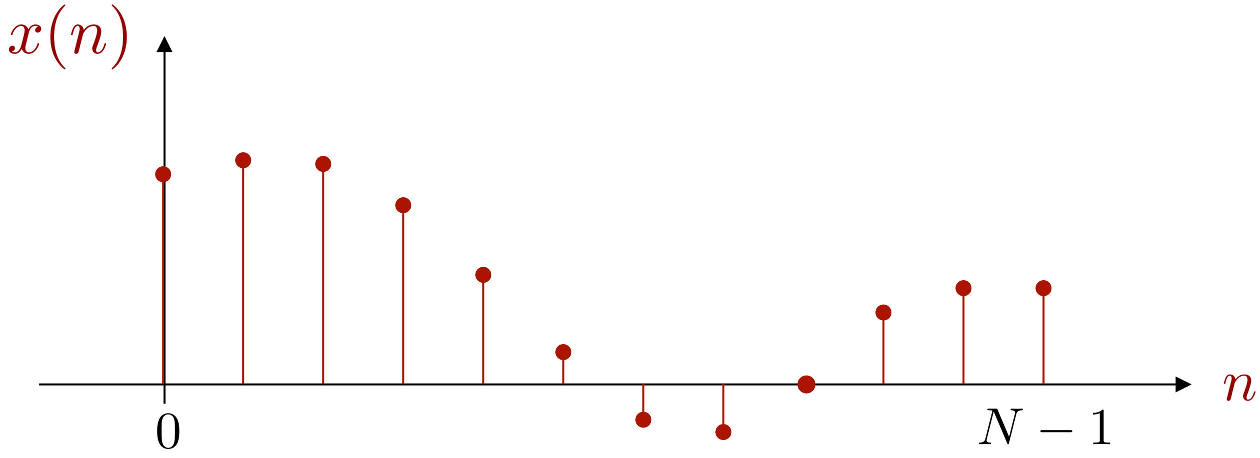
Decomposition of signals in terms of sinusoidal (or complex exponential) components

- For periodic signals, this decomposition is called a *Fourier series*
- For the class of finite energy signals, this decomposition is called the *Fourier transform*

Why important?

- Complex exponentials are eigenfunctions of linear time-invariant systems
- The response of an LTI system to a sinusoidal input signal is a sinusoid of the same frequency but of different amplitude and phase

Signal Decomposition



The discrete-time signal $x \in \mathbb{R}^N$ can be expressed as a linear combination of N basis signals/vectors:

$$\{D^k\delta\}_{k=0}^{N-1} \text{ forms an orthonormal basis in } \mathbb{R}^N \;\Rightarrow\; x = \sum_{k=0}^{N-1} \overset{x(k)}{c_k} D^k \delta$$

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Signal Decomposition

Since complex harmonics are eigenfunctions of LTI systems, a decomposition in terms of harmonics would be very useful as well:

$$x(n) = \sum_{k=0}^{N-1} c_k e^{j\omega_k n} \;\Rightarrow\; y(n) = \sum_{k=0}^{N-1} \lambda_k c_k e^{j\omega_k n}$$

where λ_k is the *eigenvalue* corresponding to the *eigenvector* $e^{j\omega_k(\cdot)}$

Function Spaces

The *inner product* of two (complex-valued) functions x and y defined on an interval $E \subset \mathbb{R}$ is defined by

$$(x, y) = \int_E x(t) y^*(t) dt$$

and the *norm* of a function x is defined by $\|x\| = \sqrt{(x, x)}$.

In order to ensure the existence of the norm and the inner product, we assume that the signals have finite energy, that is, $x \in L^2(E)$ where

$$L^2(E) = \{x : \int_E |x(t)|^2 dt < \infty\}$$

Fourier Series

Theorem: Let $\{e_k\}_{k=1}^{\infty}$ denote a complete orthonormal system in $L^2(E)$ and let $x \in L^2(E)$. Then

$$x = \sum_{k=1}^{\infty} (x, e_k) e_k$$

Moreover, we have (Parseval's identity)

$$\|x\|^2 = \sum_{k=1}^{\infty} |(x, e_k)|^2$$

The series is called the *Fourier series* w.r.t. the system $\{e_k\}$; the coefficients (x, e_k) are called the *Fourier coefficients*.

Trigonometric Systems

Let $E = [-\pi, \pi]$. The following two systems are complete orthonormal systems in $L^2([-\pi, \pi])$:

A:

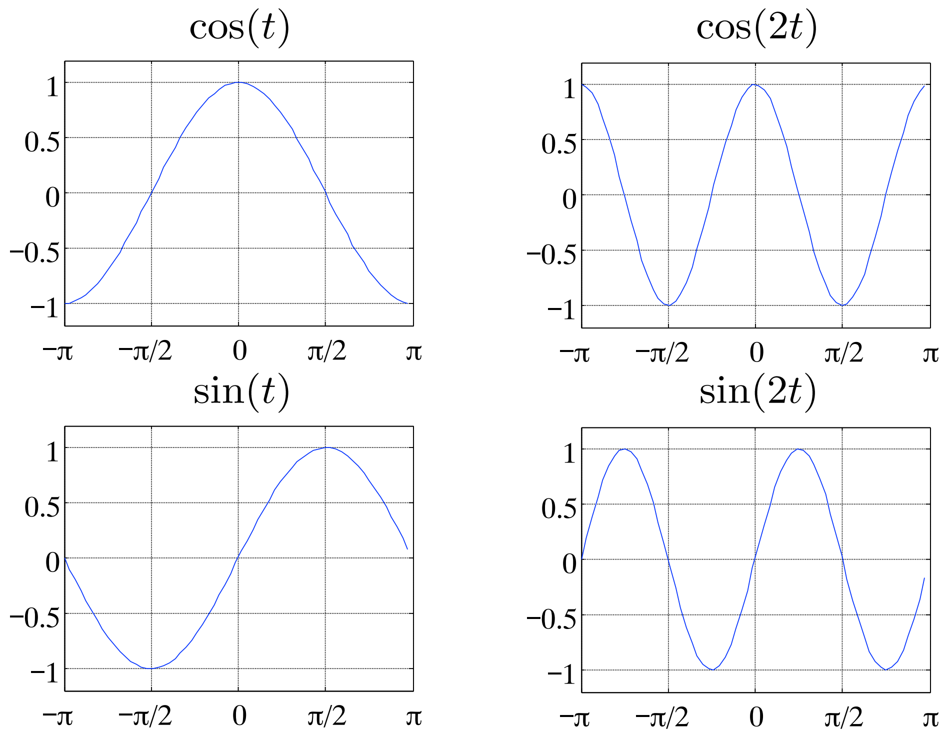
$$\left\{ \frac{e^{jkt}}{\sqrt{2\pi}} \right\}_{k=-\infty}^{\infty}$$

B:

$$\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(t)}{\sqrt{\pi}}, \frac{\sin(t)}{\sqrt{\pi}}, \ldots, \frac{\cos(kt)}{\sqrt{\pi}}, \frac{\sin(kt)}{\sqrt{\pi}}, \ldots \right\}$$

Fourier Series

Example: system B $\left\{ \frac{1}{\sqrt{2\pi}}, \frac{\cos(t)}{\sqrt{\pi}}, \frac{\sin(t)}{\sqrt{\pi}}, \ldots, \frac{\cos(kt)}{\sqrt{\pi}}, \frac{\sin(kt)}{\sqrt{\pi}}, \ldots \right\}$



Fourier Series

Let $x \in L^2([-\pi, \pi])$. The Fourier series of x for system A is given by

$$x(t) = \sum_{k=-\infty}^{\infty} \gamma_k \frac{e^{jkt}}{\sqrt{2\pi}} \quad \text{with} \quad \gamma_k = \left(x, \frac{e^{jkt}}{\sqrt{2\pi}} \right) = \frac{1}{\sqrt{2\pi}} \int_{-\pi}^{\pi} x(t) e^{-jkt} dt$$

To simplify, put $c_k = \frac{\gamma_k}{\sqrt{2\pi}}$, so that the series is given by

$$\boxed{x(t) = \sum_{k=-\infty}^{\infty} c_k e^{jkt} \quad \text{with} \quad c_k = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) e^{-jkt} dt}$$

Fourier Series

Let $x \in L^2([-\pi, \pi])$. The Fourier series of x for system B is given by

$$x(t) = \frac{\xi_0}{\sqrt{2\pi}} + \sum_{k=1}^{\infty} \left(\xi_k \frac{\cos(kt)}{\sqrt{\pi}} + \eta_k \frac{\sin(kt)}{\sqrt{\pi}} \right)$$

To simplify, put $a_0 = \frac{\xi_0}{\sqrt{2\pi}}$, $a_k = \frac{\xi_k}{\sqrt{\pi}}$ and $b_k = \frac{\eta_k}{\sqrt{\pi}}$, so that the series is given by

$$x(t) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(kt) + b_k \sin(kt))$$

Fourier Series

We have

$$a_0 = \frac{1}{2\pi} \int_{-\pi}^{\pi} x(t) dt$$

and

$$a_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x(t) \cos(kt) dt$$

$$b_k = \frac{1}{\pi} \int_{-\pi}^{\pi} x(t) \sin(kt) dt$$

Fourier Series

If we evaluate the Fourier series for system A on the real line $\mathbb{R}$ instead of $[-\pi, \pi]$, the resulting function is 2π-periodic:

$$\sum_{k=-\infty}^{\infty} c_k e^{jk(t+2\pi)} = \sum_{k=-\infty}^{\infty} c_k e^{jkt} e^{jk2\pi} = \sum_{k=-\infty}^{\infty} c_k e^{jkt}$$

Since $\cos(kt)$ and $\sin(kt)$ are 2π-periodic functions, the Fourier series for system B evaluated over the reals is 2π-periodic as well.

Hence, Fourier series can be used to decompose periodic functions

Fourier Series

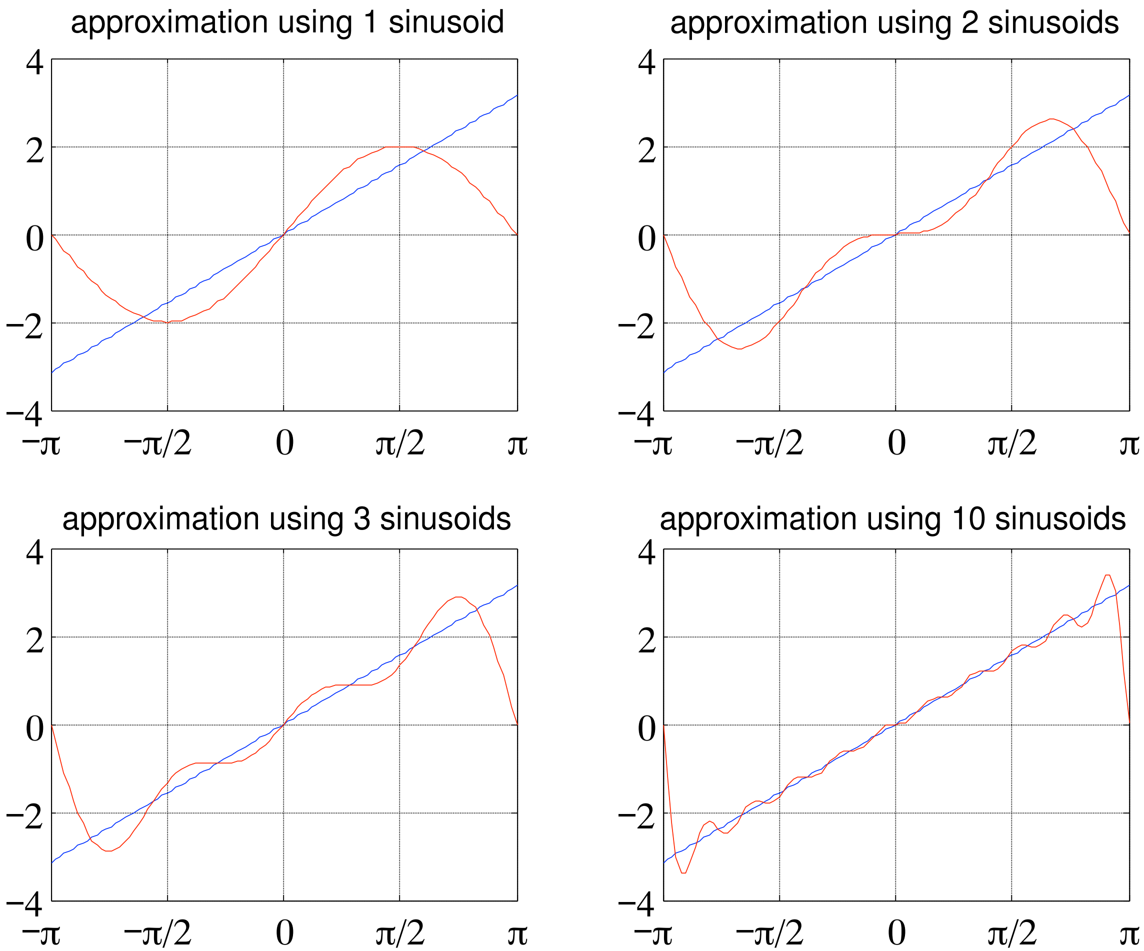
Example: $x(t) = t,\ t \in [-\pi, \pi]$. Since x is odd, $a_k = 0$ for all k

$$\begin{aligned} b_k &= \frac{1}{\pi}\int_{-\pi}^{\pi} t\sin(kt)dt \\ &= \frac{-1}{k\pi} t\cos(kt)\Big|_{-\pi}^{\pi} + \underbrace{\frac{1}{k\pi}\int_{-\pi}^{\pi}\cos(kt)dt}_{=0} = \frac{2}{k}(-1)^{k+1} \end{aligned}$$

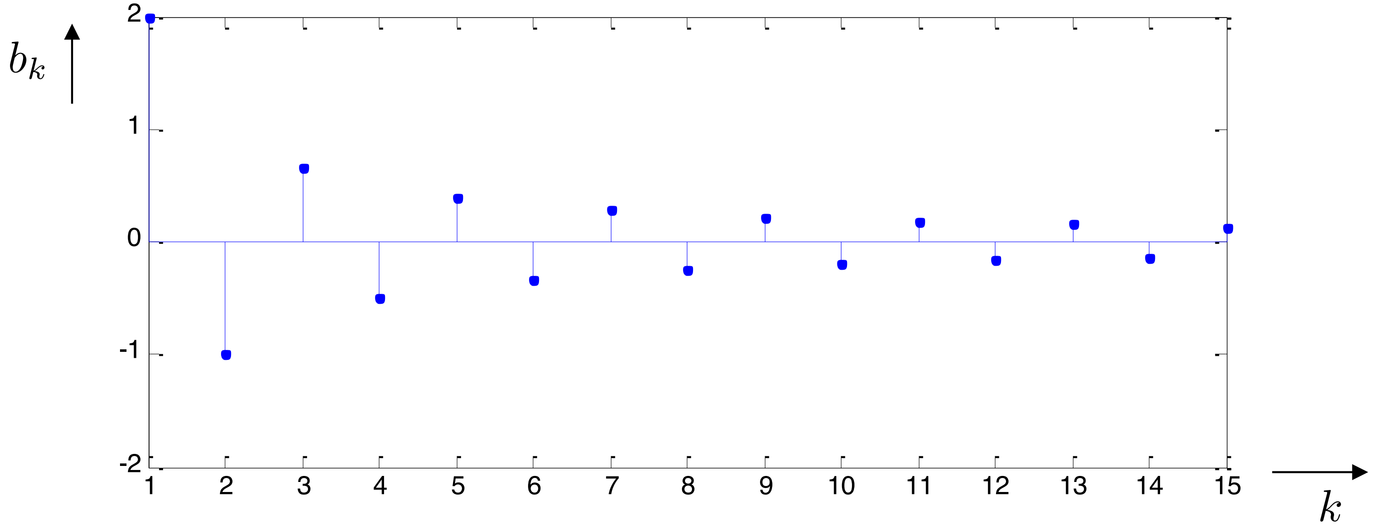
Hence, the Fourier series becomes

$$\frac{2}{1}\sin(t) - \frac{2}{2}\sin(2t) + \frac{2}{3}\sin(3t) - \frac{2}{4}\sin(4t) + \cdots$$

Fourier Series



Fourier Series



- Notice that $\lim_{k \to \infty} b_k = 0$ and that the decay of b_k is of order k^{-1}
- $x(\pi) = \pi$ but substituting $t = \pi$ in the Fourier series yields 0!

Convergence of the Fourier Series

Nth partial sum:

$$S_N(t) = \sum_{k=-N}^{N} c_k e^{jkt} = a_0 + \sum_{k=1}^{N} (a_k \cos(kt) + b_k \sin(kt))$$

We have

$$\lim_{N\to\infty} S_N(t_0) = \frac{1}{2}\left(x(t_0^-) + x(t_0^+)\right)$$

Notice that if x is continuous at $t = t_0$, then $\lim_{N\to\infty} S_N(t_0) = x(t_0)$

In conclusion, the Fourier series converges in *norm* (we have equality in $L^2([\pi,\pi])$), but we only have *pointwise convergence* (or *uniform convergence*) at points where x is continuous!

Fourier Series

Example: $x(t) = |t|,\ t \in [-\pi, \pi]$. Since x is even, $b_k = 0$ for all k

$$a_0 = \frac{1}{\pi}\int_0^{\pi} t dt = \frac{\pi}{2}$$

and

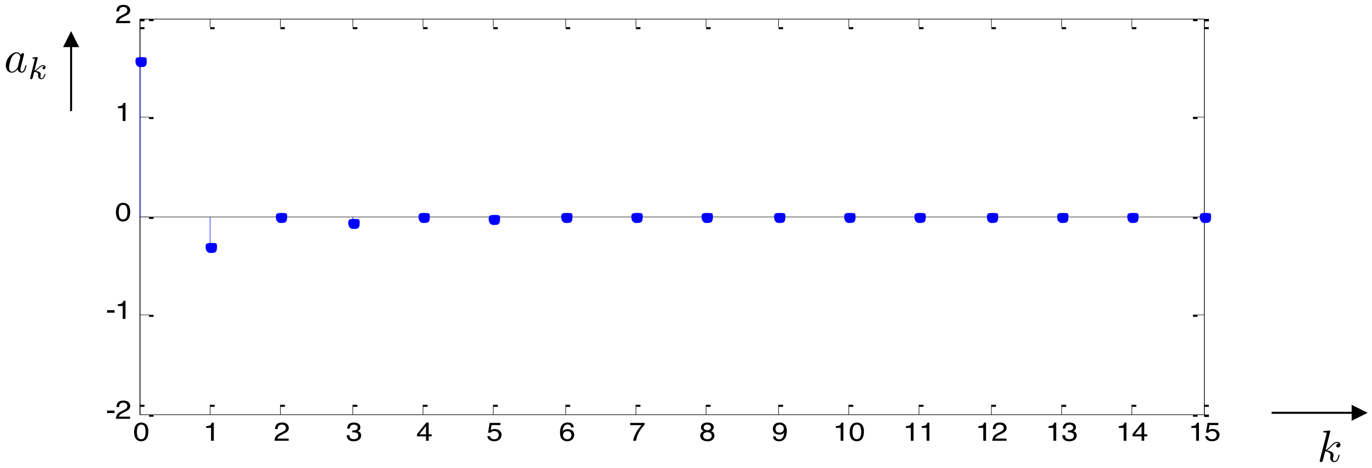
$$a_k = \frac{2}{\pi}\int_0^{\pi} t\cos(kt)dt$$

$$= \underbrace{\frac{2}{k\pi} t\sin(kt)\Big|_0^{\pi}}_{=0} - \frac{2}{k\pi}\int_0^{\pi} \sin(kt)dt = \begin{cases} 0, & k = 2, 4, \ldots \\ \frac{-4}{\pi k^2}, & k \text{ odd} \end{cases}$$

Fourier Series

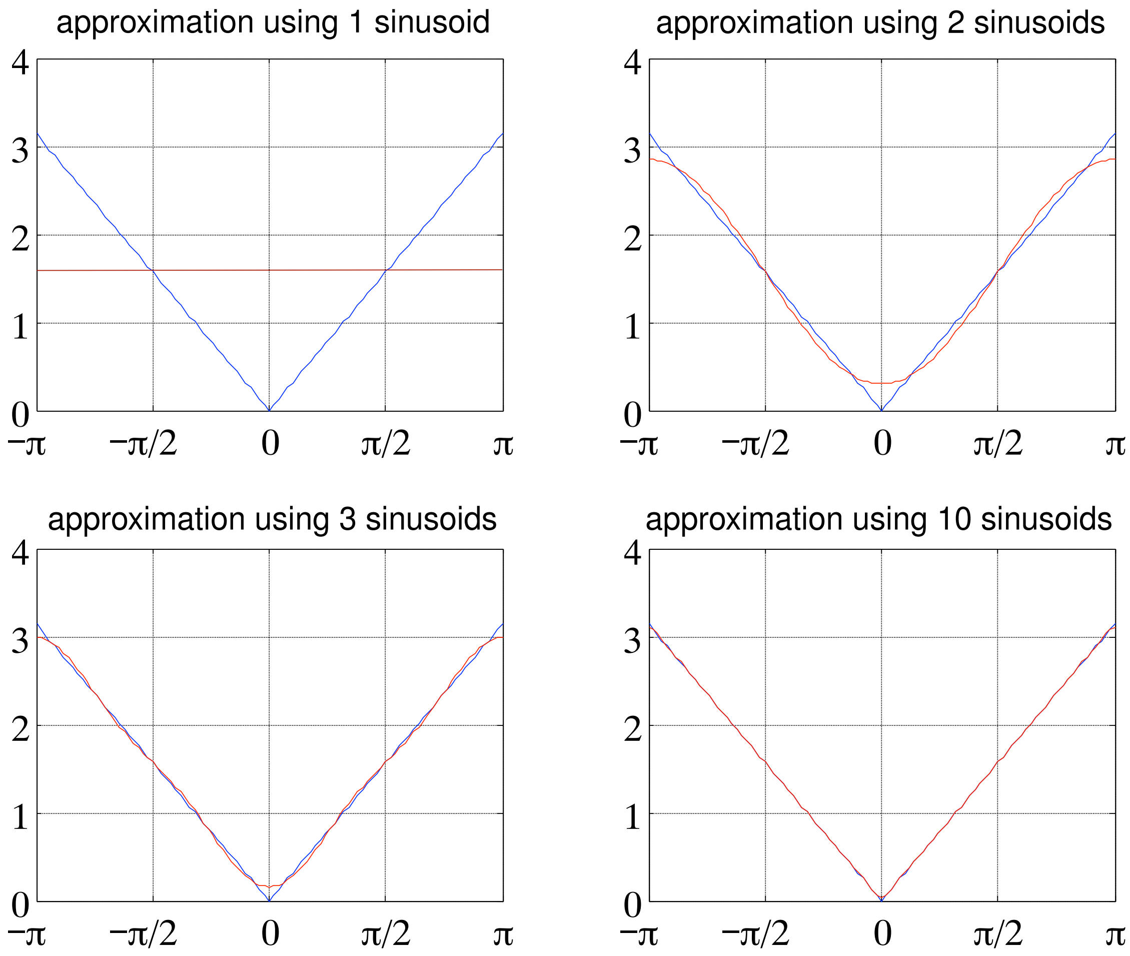
The Fourier series becomes

$$\frac{\pi}{2} - \frac{4}{\pi}\left(\frac{\cos(t)}{1^2} + \frac{\cos(3t)}{3^2} + \frac{\cos(5t)}{5^2} + \cdots\right)$$



Notice that $\lim_{k\to\infty} a_k = 0$ and that the decay of a_k is of order k^{-2}

Fourier Series



Fourier Transform

Some observations:

- Fourier series to represent periodic signals
- Periodic signals have a line spectrum ($f_k = \frac{k}{2\pi}$)
- The line spacing is equal to the fundamental frequency
- What happens if the fundamental period becomes infinite (fundamental frequency zero)? $\longrightarrow$ continuous spectrum?

From Fourier Series to Fourier Intergral

Fourier Transform

Let $x \in L^1([-\pi,\pi])$, the space of absolutely integrable functions, and assume x is so "nice" that the Fourier series converges pointwise to $x(t)$ for all $t \in [-\pi,\pi]$.

$$\sum_{k=-\infty}^{\infty} c_k e^{jkt'} \quad \text{with} \quad c_k = \frac{1}{2\pi}\int_{-\pi}^{\pi} x(t')e^{-jkt'}dt'$$

Step 1: transformation from $[-\pi,\pi]$ to $[-\frac{T}{2},\frac{T}{2}]$: $\quad t' = \frac{2\pi}{T}t$

$$f_k = \frac{k}{T}$$

$$\sum_{k=-\infty}^{\infty} c_k e^{j\frac{2\pi k}{T}t} \quad \text{with} \quad c_k = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} x(t)e^{-j\frac{2\pi k}{T}t}dt$$

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Fourier Transform

Step 2: Take the limit for $T \to \infty$. Define

$$X_T(f_k) = \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) e^{-j2\pi \frac{k}{T} t} dt$$

so that $c_k = \frac{1}{T} X_T(\frac{k}{T})$. Now let $T \to \infty$ and we have

$$X(f) = \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt$$

and

$f_k = \frac{k}{T}$ $\qquad$ $\Delta f = f_{k+1} - f_k = \frac{1}{T}$

$$x(t) = \sum_{k=-\infty}^{\infty} X_T\left(\frac{k}{T}\right) e^{j2\pi \frac{k}{T} t} \frac{1}{T} \xrightarrow{T \to \infty} \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df$$

Fourier Transform

Fourier transform pair:

$$\boxed{\begin{aligned} x(t) &= \int_{-\infty}^{\infty} X(f) e^{j2\pi f t} df \\ X(f) &= \int_{-\infty}^{\infty} x(t) e^{-j2\pi f t} dt \end{aligned}}$$

For sequences $x \in L^2(\mathbb{R})$ which are not absolutely integrable, we define X as $X = \lim_{n \to \infty} X_n$, where $x = \lim_{n \to \infty} x_n$, $x_n \in L^1(\mathbb{R})$.

Fourier Transform

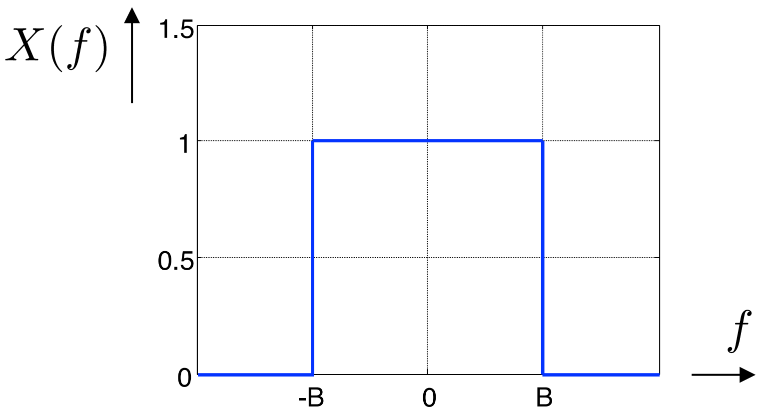
Example:

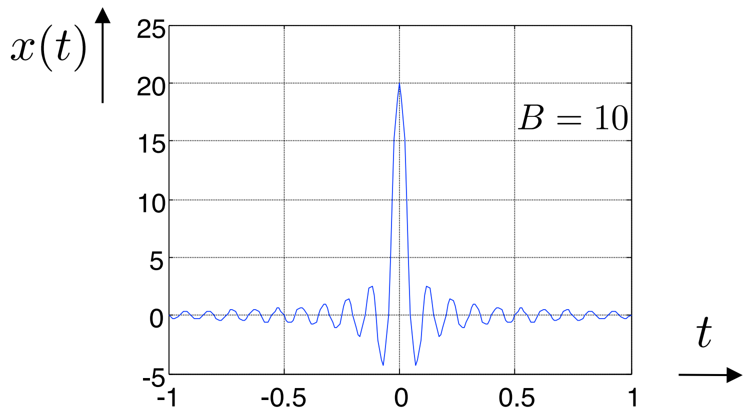
$$X(f) = \begin{cases} 1, & |f| < B \\ 0, & \text{otherwise} \end{cases}$$

so that

not absolutely integrable

$$x(t) = \int_{-B}^{B} e^{j2\pi f t} df = \frac{1}{j2\pi t} e^{j2\pi f t} \Big|_{-B}^{B} = \frac{\sin(2\pi B t)}{\pi t}$$





Fourier Series for Discrete-Time Signals

The inner product of two (complex-valued) sequences x and y in $\mathbb{C}^N$ is defined as

$$(x, y) = \sum_{n=0}^{N-1} x(n) y^*(n)$$

and the *norm* of a sequence x is defined by $\|x\| = \sqrt{(x, x)}$.

The family $\left\{ \dfrac{e^{j\frac{2\pi k}{N}n}}{\sqrt{N}} \right\}_{k=0}^{N-1}$ forms an orthonormal basis for $\mathbb{C}^N$.

Fourier Series for Discrete-Time Signals

Fourier series:

$$x(n) = \sum_{k=0}^{N-1} (x, e_k) e_k = \sum_{k=0}^{N-1} c_k e^{j\frac{2\pi k}{N}n}$$

where

N-periodic

$$c_k = \frac{1}{N} \sum_{n=0}^{N-1} x(n) e^{-j\frac{2\pi k}{N}n} = c_{k+N}$$

Usually called the discrete Fourier transform

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Discrete-Time Fourier Transform

Some observations:

- The line spacing of the spectrum of a finite-length discrete-time signal is equal to $\frac{1}{N}$
- The signal is completely determined by its N frequency samples $f_k = \frac{k}{N},\ k = 0, \ldots, N-1$, so that $0 \leq f_k < 1 - \frac{1}{N}$

Similar to what we did with continuous-time signals, we can make the fundamental period N infinite:

- The line spectrum becomes a continuous spectrum
- The signal is completely characterized by its Fourier transform defined on $f \in [0, 1]$

Discrete-Time Fourier Transform

Similar to what we did for continuous-time signals, we assume that $x \in l^1(\mathbb{R})$, the space of absolutely summable sequences.

Fourier transform pair:

$$x(n) = \int_0^1 X(f)e^{j2\pi fn}df$$

$$X(f) = \sum_{n=-\infty}^{\infty} x(n)e^{-j2\pi fn}$$

For sequences $x \in l^2(\mathbb{R})$ which are not absolutely summable, we define X as $X = \lim_{n\to\infty} X_n$, where $x = \lim_{n\to\infty} x_n$, $x_n \in l^1(\mathbb{R})$.

Discrete-Time Fourier Transform

Equivalently, by putting $\omega = 2\pi f$, we have:

$$\boxed{\begin{aligned} x(n) &= \frac{1}{2\pi}\int_0^{2\pi} X(\omega)e^{j\omega n}d\omega \\ X(\omega) &= \sum_{n=-\infty}^{\infty} x(n)e^{-j\omega n} \end{aligned}}$$

Notice that $X(\omega+k2\pi) = X(\omega)$, that is, $X(\omega)$ is periodic with period 2π. As a consequence, the integration interval can be taken over *any* interval of lentgh 2π, e.g. $[0, 2\pi]$ or $[-\pi, \pi]$.

Relation Different Transforms

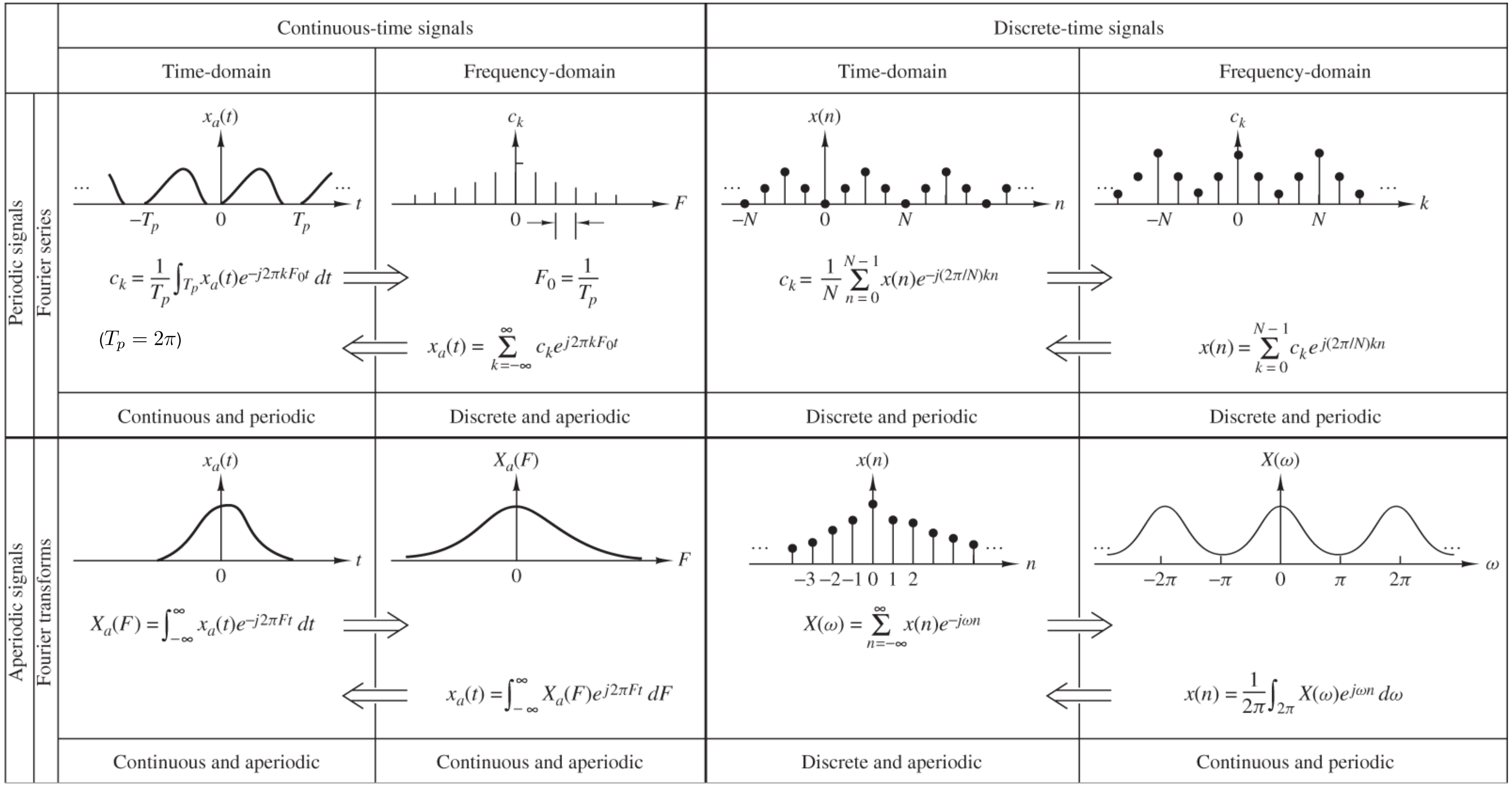

Figure 4.3.1 Summary of analysis and synthesis formulas.

Relation to the $\mathcal{Z}$-Transform

Recall that the z-transform of a sequence x was given by

$$X(z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

where $r_2 < |z| < r_1$ is the region of convergence. If $X(z)$ converges for $|z| = 1$, we have on the unit circle ($z = e^{j\omega},\ -\pi \leq \omega < \pi$):

$$X(\omega) = \sum_{n=-\infty}^{\infty} x(n) e^{-j\omega n}$$

In conclusion, if the unit circle is contained in the ROC, the Fourier transform of x is obtained by evaluating the z-transform $X(z)$ on the unit circle!

Relation to the $\mathcal{Z}$-Transform

The inverse z-transform, where C is taken to be the unit circle, is given by

$$
\begin{aligned}
x(n) &= \frac{1}{2\pi j}\oint_C X(z) z^{n-1} dz \\
&= \frac{1}{2\pi j}\oint_C X(\omega) e^{j\omega(n-1)} \underbrace{d\left(e^{j\omega}\right)}_{j e^{j\omega} d\omega} \\
&= \frac{1}{2\pi}\int_{-\pi}^{\pi} X(\omega) e^{j\omega n} d\omega
\end{aligned}
$$

which is consistent with the results obtained before.

Relation to the $\mathcal{Z}$-Transform

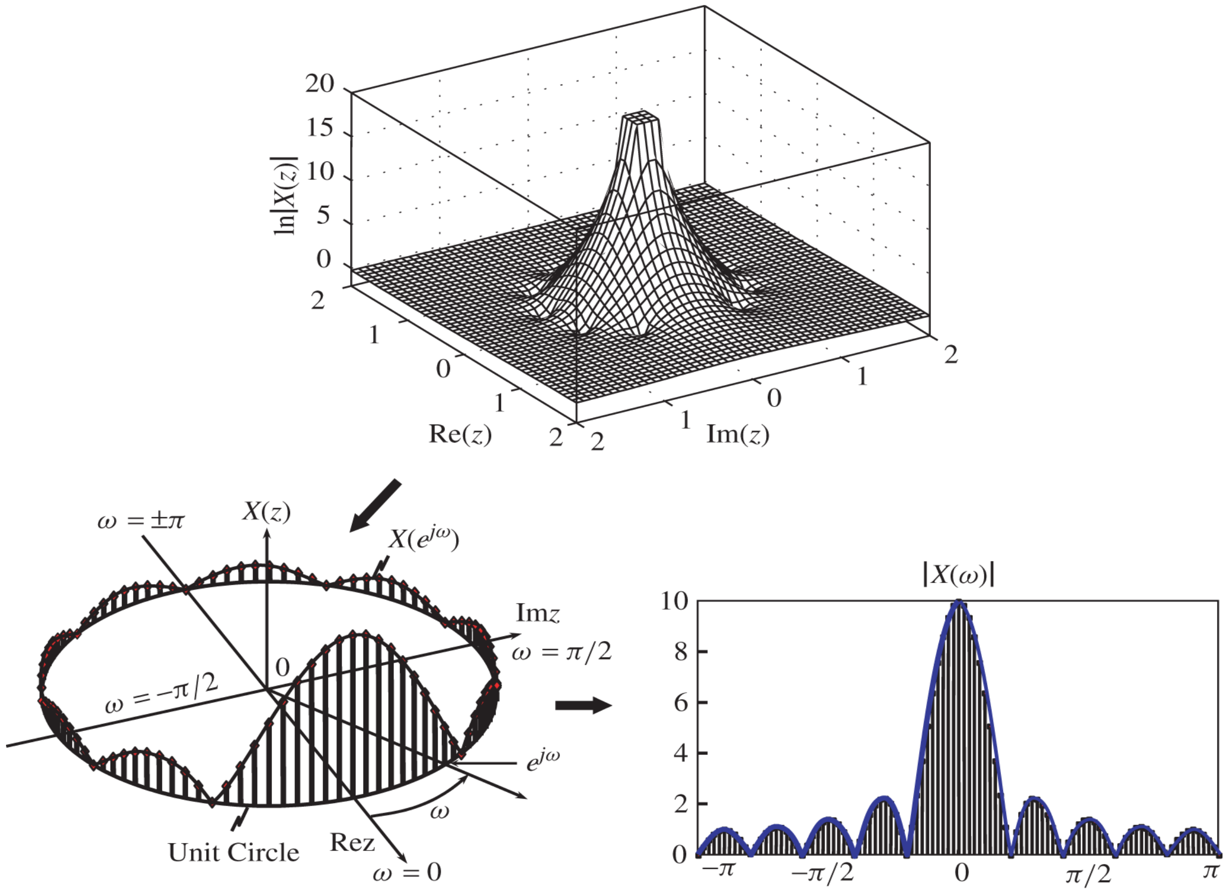

Figure 4.2.9 relationship between $X(z)$ and $X(\omega)$ for the sequence in Example 4.2.4, with $A = 1$ and $L = 10$