Resit Exam EE2511 - Stochastic Processes

July 2, 2014

Answer each question on a separate sheet. Make clear in your answer how you reach the final result; the road to the answer is very important. Write your name and student number on each sheet. It is allowed to answer in Dutch or English.

Further is given:

Uniform distribution: for $a < b$:

$$\begin{aligned} f_X(x) &= \begin{cases} \frac{1}{b-a} & a \leq x \leq b \\ 0 & \text{otherwise} \end{cases} \\ E[X] &= \frac{a+b}{2} \\ Var[X] &= \frac{(b-a)^2}{12} \end{aligned}$$

Laplace distribution: for $a > 0$ and $-\infty < b < \infty$:

$$\begin{aligned} f_X(x) &= \frac{a}{2} e^{-a|x-b|} \\ E[X] &= b \\ Var[X] &= \frac{2}{a^2} \end{aligned}$$

Chebyshev inequality: for an arbitrary random variable Y and a constant $c > 0$:

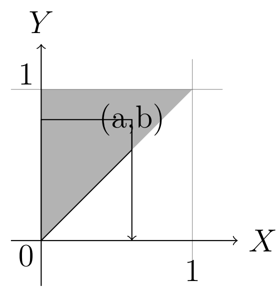
$$P[|Y - \mu_Y| \geq c] \leq \frac{\text{Var}[Y]}{c^2}$$

Erlang's B formula: define the load $\rho = \frac{\lambda}{\mu}$ given the arrival rate λ and service rate μ:

$$P[N = c] = \frac{\rho^c / c!}{\sum_{k=0}^{c} \rho^k / k!}$$

Question 1 - Probabilities (11 p)

A graphical representation is given in the plot below:



(2 p) (a) Integrate over the left-bottom region from (a, b)

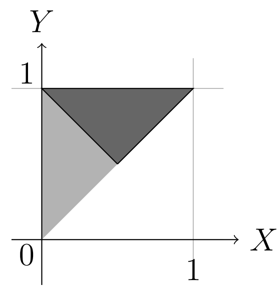
$$
\begin{aligned}
F_{X,Y}(a,b) &= \int_0^a \int_x^b c(x+2y)dy\,dx \\
&= \int_0^a c[xy+y^2]_{y=x}^b dx \\
&= c\int_0^a (xb+b^2-x^2-x^2)dx \\
&= c\int_0^a (b^2+xb-2x^2)dx \\
&= c[b^2x+\frac{1}{2}bx^2-\frac{2}{3}x^3]_0^a \\
&= c(b^2a+\frac{1}{2}ba^2-\frac{2}{3}a^3) \\
&= ca(b^2+\frac{1}{2}ba-\frac{2}{3}a^2)
\end{aligned}
$$

(2 p) (b) c is determined by the fact that it integrates to 1, *or* that $F_{XY}(\infty,\infty) = 1$. Here ∞ is already reached at $a = 1$ and $b = 1$, so

$$ca(b^2+\frac{1}{2}ba-\frac{2}{3}a^2 = c(1^2+\frac{1}{2}-\frac{2}{3}1^2) = c(1\frac{1}{2}-\frac{2}{3}) = c\frac{5}{6} = 1 \qquad (1)$$

So, $c = \frac{6}{5}$.

(2 p) (c) The probability $P[X + Y > 1]$ is the integral over this region:



We have to integrate again:

$$
\begin{aligned}
P &= \int_{1/2}^{1}\int_{1-y}^{y} c(x+2y)dx\,dy \\
&= \int_{1/2}^{1} c[\frac{1}{2}x^2+xy]_{1-y}^{y}dy = \int_{1/2}^{1} c(\frac{1}{2}y^2+y^2)-(\frac{1}{2}(1-y)^2+y(1-y))dy \\
&= c\int_{1/2}^{1}(1.5y^2-\frac{1}{2}(1-2y+y^2)-y+y^2)dy = c\int_{1/2}^{1}(2y^2-\frac{1}{2})dy \\
&= c\left[\frac{2}{3}y^3-\frac{1}{2}y\right]_{1/2}^{1} = c\left[(2/3-1/2)-(2/3\cdot 1/8-1/4)\right] \\
&= c\cdot\frac{4}{12}=\frac{2}{5}=0.4
\end{aligned}
$$

(2 p) (d) Easy:

$$
\begin{aligned}
f_X(x) &= \int_{x}^{1} c(x+2y)dy = c[xy+y^2]_x^1 \\
&= c((x+1)-(x\cdot x+x^2)) = c(-x^2+x+1)
\end{aligned}
$$

and

$$
\begin{aligned}
f_Y(y) &= \int_{0}^{y} c(x+2y)dy = c[\frac{1}{2}x^2+2yx]_0^y \\
&= c(\frac{1}{2}y^2+2y^2) = c\cdot\frac{5}{2}y^2 = 3y^2
\end{aligned}
$$

(2 p) (e) Find the expected value and variance of Y,

$$E[Y]=\int_0^1 y\cdot c\frac{5}{2}y^2\,dy = 3[\frac{1}{4}y^4]_0^1=\frac{3}{4}$$

and because $Var(Y) = E[Y^2] - E[Y]^2$ we do:

$$E[Y^2] = \int_0^1 y^2 \cdot 3y^2 \, dy = \frac{3}{5}[y^5]_0^1 = \frac{3}{5}$$

and therefore

$$Var[Y] = E[Y^2] - E[Y]^2 = \frac{3}{5} - (\frac{3}{4})^2 = \frac{3}{80} = 0.0375$$

(1 p) (f) The variables X and Y are not independent because when I know something about X, then I certainly know also something about Y.

Question 2 - Signal Processing (10 p)

$$Y_n = \frac{1}{2}X_{n-1} + X_n + \frac{1}{2}X_{n+1}$$

(1 p) (a) The autocorrelation function $R_X(k) = E[X_m X_{m+k}]$, and we are given that $Var(X_i] = 1 = E[X_i^2] - E[X_i]^2 = E[X_i^2] - a^2$, and therefore $E[X_i^2] = 1 + a^2$.

Because the X_i are independent, it holds that $E[X_i X_{i+k}] = E[X_i]E[X_{i+k}] = a^2$. So:

$$\begin{aligned} R_X(k) &= \begin{cases} E[X_i^2], & k = 0 \\ E[X_i X_{i+k}], & k \neq 0 \end{cases} \\ &= \begin{cases} 1 + a^2, & k = 0 \\ a^2, & k \neq 0 \end{cases} \end{aligned}$$

(1 p) (b) Easy:

$$E[Y_i] = E[\frac{1}{2}X_{n-1} + X_n + \frac{1}{2}X_{n+1}] = \frac{1}{2}E[X_{n-1}] + E[X_n] + \frac{1}{2}E[X_{n+1}] = 2a$$

(3 p) (c) Here we go:

$$C_Y[m,k] = E[Y_m Y_{m+k}] - E[Y_m]E[Y_{m+k}] = E[Y_m Y_{m+k}] - 4a^2$$

$$
\begin{aligned}
E[Y_m Y_{m+k}] &= E[(\frac{1}{2}X_{m-1} + X_m + \frac{1}{2}X_{m+1})(\frac{1}{2}X_{m+k-1} + X_{m+k} + \frac{1}{2}X_{m+k+1}] \\
&= E[\frac{1}{4}X_{m-1}X_{m+k-1} + \frac{1}{2}X_{m-1}X_{m+k} + \frac{1}{4}X_{m-1}X_{m+k+1} \\
&\quad \frac{1}{4}X_m X_{m+k-1} + \frac{1}{2}X_m X_{m+k} + \frac{1}{4}X_m X_{m+k+1} \\
&\quad \frac{1}{4}X_{m+1}X_{m+k-1} + \frac{1}{2}X_{m+1}X_{m+k} + \frac{1}{4}X_{m+1}X_{m+k+1}] \\
&= \frac{1}{4}R_X(k) + \frac{1}{2}R_X(k+1) + \frac{1}{4}R_X(k+2) \\
&\quad \frac{1}{2}R_X(k-1) + R_X(k) + \frac{1}{2}R_X(k+1) \\
&\quad \frac{1}{4}R_X(k-2) + \frac{1}{2}R_X(k-1) + \frac{1}{4}R_X(k) \\
&= 1\frac{1}{2}R_X(k) + R_X(k+1) + R_X(k-1) + \frac{1}{4}R_X(k+2) + \frac{1}{4}R_X(k-2)
\end{aligned}
$$

So we try a few:

$$C_Y[m,0] = 3/2(1+a^2) + a^2 + a^2 + \frac{1}{4}a^2 + \frac{1}{4}a^2 - 4a^2 = 1\frac{1}{2}$$

$$C_Y[m,1] = 3/2a^2 + a^2 + (1+a^2) + \frac{1}{4}a^2 + \frac{1}{4}a^2 - 4a^2 = 1$$

and

$$C_Y[m,2] = 3/2a^2 + a^2 + a^2 + \frac{1}{4}a^2 + \frac{1}{4}(1+a^2) - 4a^2 = \frac{1}{4}$$

and

$$C_Y[m,3] = 3/2a^2 + a^2 + a^2 + \frac{1}{4}a^2 + \frac{1}{4}a^2 - 4a^2 = 0$$

(2 p) (d) Y_n is not iid random sequence, because there is correlation between consecutive timepoints.

(3 p) (e) The autocorrelation of $Z_n = Y_n + W_n$:

$$
\begin{aligned}
R_Z(m,k) &= E[(Y_m + W_m)(Y_{m+k} + W_{m+k})] \\
&= E[Y_m Y_{m+k}] + E[Y_m W_{m+k}] + E[W_m Y_{m+k}] + E[W_m W_{m+k}] \\
&= R_Y(k) + 0 + 0 + R_W(k)
\end{aligned}
$$

because W is mutually uncorrelated with Y. Now fill in some values for k:

$$R_Z(k) = \begin{cases} 1.5 + 4a^2 + 4 & k = 0 \\ 1 + 4a^2 + 2 & |k| = 1 \\ \frac{1}{4} + 4a^2 + 1 & |k| = 2 \\ 4a^2 + 1 & |k| > 2 \end{cases}$$

Question 3 - Estimation (10 p)

(2 p) (a) With $a = 2$ and $b = 0$:

$$f_X(x) = e^{-2|x|} \tag{2}$$

so

$$F_X(x) = \int_{-\infty}^{x} e^{-2|u|} du \tag{3}$$

Make a distinction for $x < 0$ and $x > 0$. For $x < 0$:

$$F_X(x) = \int_{-\infty}^{x} e^{2u} du = [\frac{1}{2}e^{2u}]_{-\infty}^{x} = \frac{1}{2}e^{2x}$$

For $x > 0$:

$$F_X(x) = \frac{1}{2} + \int_{0}^{x} e^{-2u} du = \frac{1}{2} + [-\frac{1}{2}e^{-2u}]_0^x = 1 - \frac{1}{2}e^{-2x}$$

(2 p) (b) First:

$$P[X \in (-\infty, -s)] = F_X(-s) = \frac{1}{2}e^{-2s} \tag{4}$$

and

$$P[X \in [-s, s)] = F_X(s) - F_X(-s) = 1 - \frac{1}{2}e^{-2s} - \frac{1}{2}e^{-2s} \tag{5}$$

If we equate them:

$$\begin{aligned}
\frac{1}{2}e^{-2s} &= 1 - e^{-2s} \\
\frac{3}{2}e^{-2s} &= 1 \\
e^{-2s} &= \frac{2}{3} \\
-2s &= \log\frac{2}{3} \\
s &= -\frac{1}{2}\log\frac{2}{3} \approx 0.2027
\end{aligned}$$

(2 p) (c) Use Chebyshev:

$$P[|X - \mu_X| \geq s] \leq \frac{\mathrm{Var}[X]}{s^2} = \frac{2/a^2}{s^2} = \frac{1}{2s^2} \approx 12.16(!)$$

(1 p) (d) Because we chose s such that the probability in each of the three parts is equal, we know that $P(X < -s) + P(X > +s) = 2/3$. That is already pretty likely, but Chebyshevs inequality can only say that with a probability MORE than 1 it will happen. That is because Chebyshev holds for all possible distributions.

(2 p) (e) We have a discrete distribution, with three different values:

$$f_Y(y) = \begin{cases} \frac{1}{3} & y = 1, 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

So:

$$E[Y] = 1 \cdot \frac{1}{3} + 2 \cdot \frac{1}{3} + 3 \cdot \frac{1}{3} = 2$$

and:

$$E[Y^2] = 1^2 \cdot \frac{1}{3} + 2^2 \cdot \frac{1}{3} + 3^2 \cdot \frac{1}{3} = \frac{14}{3} = 4\frac{2}{3}$$

and therefore $Var[Y] = E[Y^2] - E[Y]^2 = 4\frac{2}{3} - 4 = \frac{2}{3}$.

(1 p) (f) Ha! Notice that this can never happen, so $P[X > 2c|Y = 1] = 0$.

Question 4 - Markov Chains (9 p)

Bla.

(1 p) (a) Bla

(1 p) (b) Bla

Given the Markov chain shown below: