Telecommunicatie A (EE2T11)

Lecture 5 overview:

Baseband pulse modulation
- *** Pulse Amplitude Modulation (PAM)**
  - **- natural sampling (gating)**
  - **- flat-top sampling**
- *** Pulse Code Modulation (PCM)**
  - **- noise in PCM systems**

EE2T11 Telecommunicatie A

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Colleges en Werkcolleges Telecommunicatie A

Colleges:

Maandag	14-3	5e en 6e uur, EWI-Pi
Dinsdag	15-3, 29-3	7e en 8e uur, EWI-Pi

Werkcolleges:

Maandag	21-3, 4-4	5e en 6e uur, EWI-Pi

Baseband Pulse Modulation

An analog source signal can be transmitted by means of pulses. The conversion into discrete baseband signals is done in two steps:

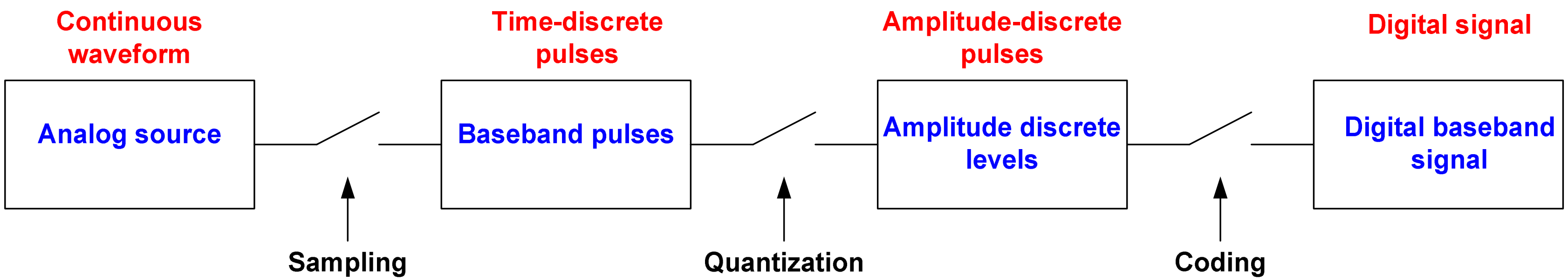
1) conversion into time-discrete amplitude modulated pulses (PAM)
2) conversion of continuous amplitude pulses into amplitude-discrete pulses that can be represented by digital code words (PCM)

The conversion into digital signals is not exact, but an approximation.

Baseband = original frequency band of the source signal (without applying frequency conversion).

Baseband signal = signal in its original frequency band

Conversion of an analog signal into a digital representation



Three processes can be distinguished:

1) Sampling
2) Quantization
3) Coding

Natural sampling (1)

The signal *w(t)* is periodically passed for a time duration τ by an analog switch every $T_s = 1/f_s$ [s] and duty-cycle $d = \tau / T_s$.

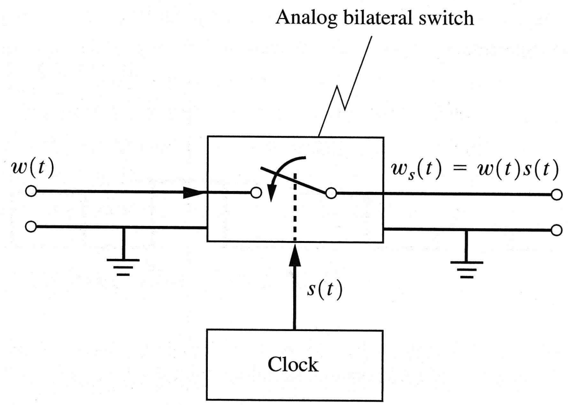


Figure 3–2 Generation of PAM with natural sampling (gating).

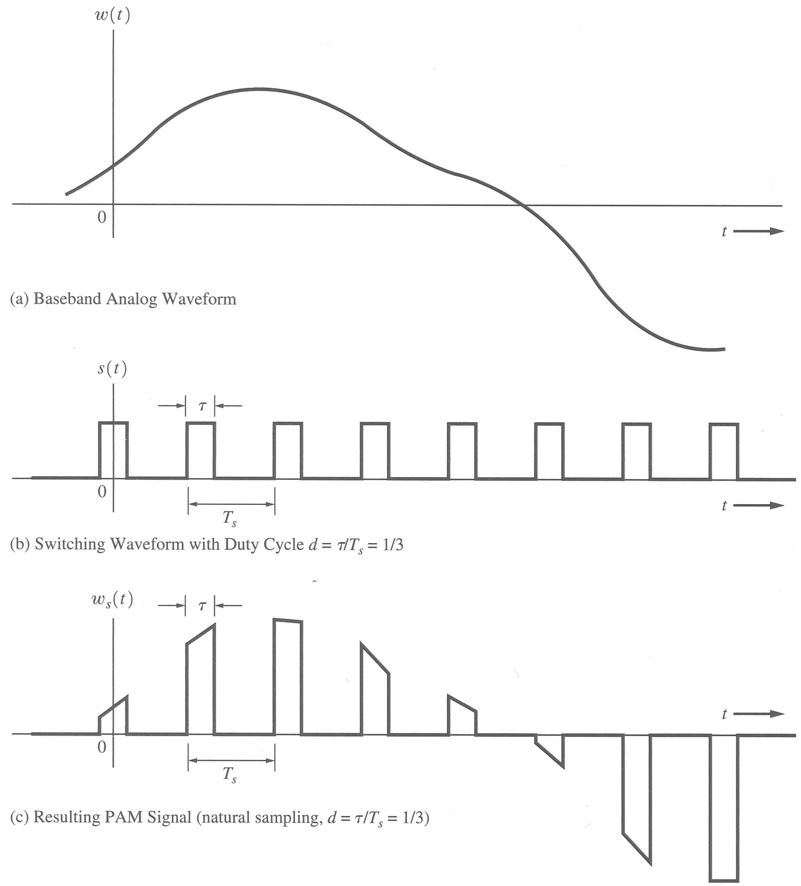


Figure 3–1 PAM signal with natural sampling.

Natural sampling (2)

The minimum sampling frequency is chosen at least equal to the Nyquist frequency:

$$f_{s\_\min} = 2B \quad \text{and} \quad f_s \geq 2B$$

Mathematical description of *natural sampling:*

$$w_s(t) = w(t) \cdot s(t) \qquad s(t) = \sum_{k=-\infty}^{\infty} \Pi\left(\frac{t - kT_s}{\tau}\right)$$

$$W_s(f) = \sum_{n=-\infty}^{\infty} c_n W(f - nf_s)$$

with $c_n = d\dfrac{\sin n\pi d}{n\pi d} = f_s \tau \,\mathrm{sinc}\, n f_s \tau$ and where $d = \dfrac{\tau}{T_s} = f_s \tau$

is the pulse duty-cycle.

Natural sampling (3)

Proof:

$$w_s(t) = w(t) \cdot s(t)$$

$$s(t) = \sum_{k=-\infty}^{\infty} \Pi\left(\frac{t-kT_s}{\tau}\right) = \sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_s t}$$

See p. 97-100

where $c_n = d\,\dfrac{\sin n\pi d}{n\pi d} = f_s \tau \,\mathrm{sinc}\, n f_s \tau$

$$\Rightarrow S(f) = \mathfrak{F}\{s(t)\} = \sum_{n=-\infty}^{\infty} c_n \delta(f - n f_s)$$

Natural sampling (4)

Proof:

Since: $w_s(t) = w(t) \cdot s(t)$

$$W_s(f) = W(f) * S(f) = W(f) * \sum_{n=-\infty}^{\infty} c_n \delta(f - nf_s)$$

$$= \sum_{n=-\infty}^{\infty} c_n W(f) * \delta(f - nf_s)$$

$$= \sum_{n=-\infty}^{\infty} c_n W(f - nf_s)$$

See also pp. 97-100

with $c_n = d\dfrac{\sin n\pi d}{n\pi d} = d\,\mathrm{sinc}\,nd$ and where $d = \dfrac{\tau}{T_s} = f_s\tau$

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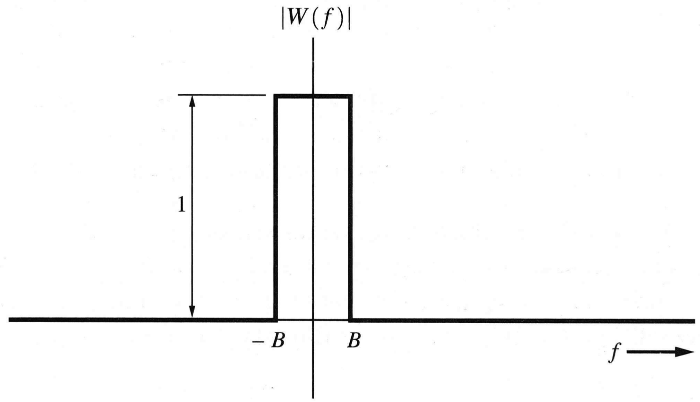
Natural sampling (5)

Here:

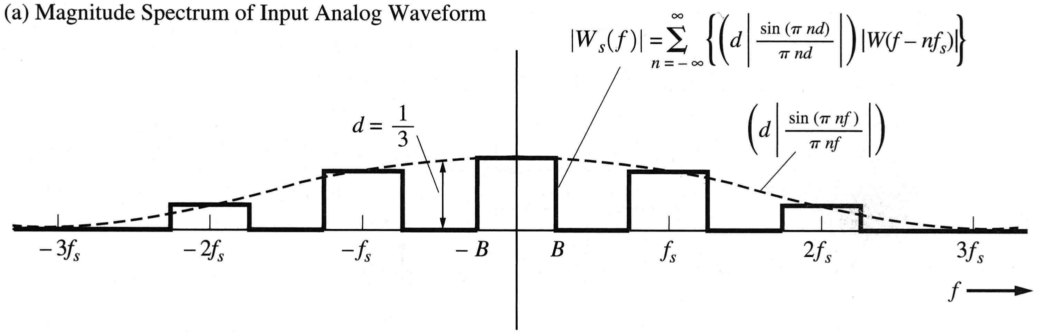
- $f_s = 4B$ (2x over sampling) Nyquist fulfilled

- $d = \frac{1}{3} \Rightarrow c_n = 0 \ \forall \ \frac{n}{3} \neq 0 \in \mathbb{Z}$

The spectrum $W(f)$ is repeated around every multiple of f_s and weighted with a sinc-function depending on n, τ and f_s but not on f.



(a) Magnitude Spectrum of Input Analog Waveform



(b) Magnitude Spectrum of PAM (natural sampling) with $d = 1/3$ and $f_s = 4$ B

Figure 3–3 Spectrum of a PAM waveform with natural sampling.

Signal recovery (1)

Signal retrieval by:

1. lowpass filter with $B < f_{cut-off} < f_s - B$
2. down-converting the spectrum $W(f - nf_s)$ to baseband ($f = 0$) by multiplying with $\cos 2\pi n f_s t$ using a mixer, followed by a LPF.

Why so complicated?

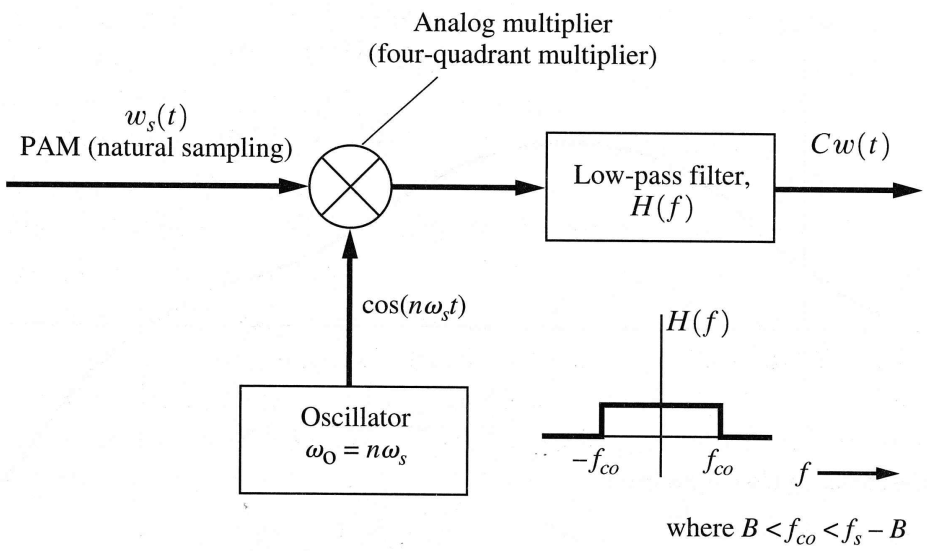


Figure 3–4 Demodulation of a PAM signal (naturally sampled).

Signal recovery (2)

$$w_s(t) = w(t)\cdot\sum_{n=-\infty}^{\infty} c_n e^{jn\omega_s t} \;\; = w(t)\cdot\sum_{n=-\infty}^{\infty} c_n\{\cos n\omega_s t + j\sin n\omega_s t\}$$

$$= w(t)\left\{c_0 + 2\sum_{n=1}^{\infty} c_n \cos n\omega_s t\right\}$$

$$w_s(t)\cos k\omega_s t = w(t)\left\{c_0\cos k\omega_s t + 2\sum_{n=1}^{\infty} c_n \cos n\omega_s t \cos k\omega_s t\right\}$$

$$= w(t)\left\{c_0\cos k\omega_s t + \sum_{n=1}^{\infty} c_n[\cos(n-k)\omega_s t + \cos(n+k)\omega_s t\right\}$$

$\overset{LPF}{=} c_k w(t) +$ other components at high frequencies are removed.

Instantaneous sampling (flat-top PAM) (1)

Ideal sampling is applied followed by a hold circuit which holds the sample value for a time duration τ

sample & hold circuit.

$$w_s(t) = \sum_{k=-\infty}^{\infty} w(kT_s)h(t-kT_s)$$

$$h(t) = \Pi\left(\frac{t}{\tau}\right) = \begin{cases} 1, & \text{for } |t| < \frac{\tau}{2} \\ 0, & \text{for } |t| > \frac{\tau}{2} \end{cases} \qquad \tau \le T_s$$

$$W_s(f) = \underbrace{H(f)}_{\text{Filter function}} \cdot \underbrace{f_s \sum_{n=-\infty}^{\infty} W(f-nf_s)}_{\text{Ideal sampling}}$$

Filter function **Ideal sampling**

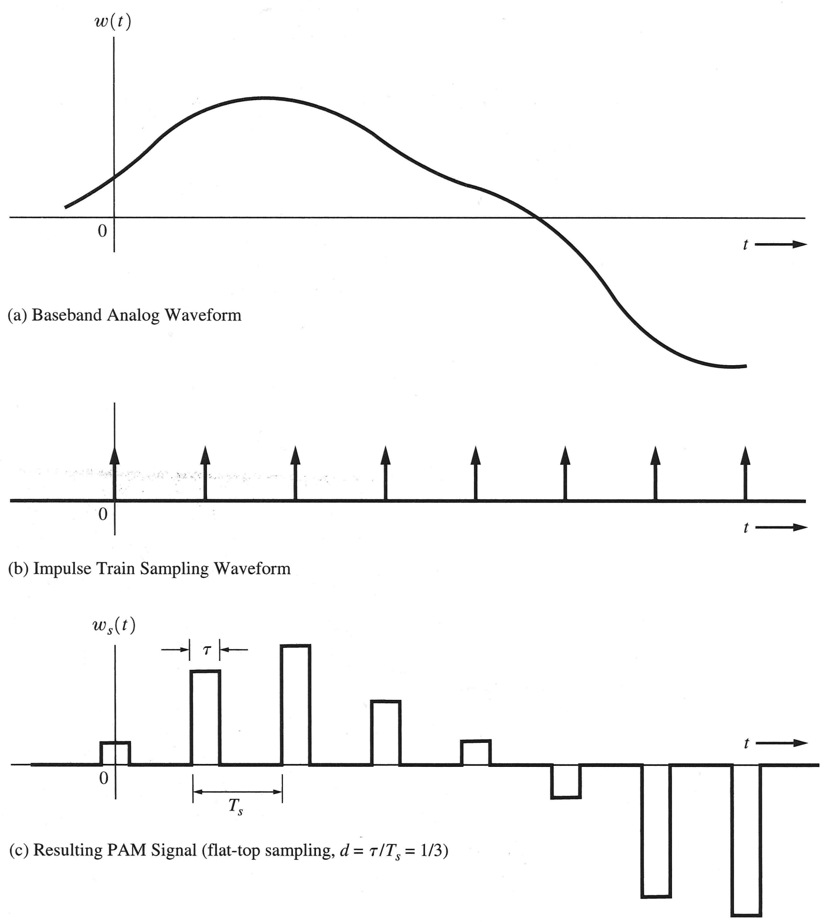


Figure 3–5 PAM signal with flat-top sampling.

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Instantaneous sampling (2)

Proof: $w_s(t) = \sum_{k=-\infty}^{\infty} w(kT_s)h(t-kT_s)$ with $h(t) = \prod\left(\frac{t}{\tau}\right) = \begin{cases} 1, & \text{for } |t| < \frac{\tau}{2} \\ 0, & \text{for } |t| > \frac{\tau}{2} \end{cases}$ $\tau \leq T_s$

$$w_s(t) = \sum_{k=-\infty}^{\infty} w(kT_s)\cdot h(t) * \delta(t-kT_s)$$

$$= h(t) * \sum_{k=-\infty}^{\infty} w(kT_s)\delta(t-kT_s) = h(t) * w(t)\sum_{k=-\infty}^{\infty} \delta(t-kT_s)$$

$$= h(t) * w(t)\sum_{n=-\infty}^{\infty} c_n e^{j2\pi n f_s t} \quad \text{with } c_n = f_s$$

See §2.7

$$W_s(f) = H(f)\cdot\left[W(f) * \sum_{n=-\infty}^{\infty} f_s\delta(f-nf_s)\right]$$

$$= H(f)\cdot \underbrace{f_s\sum_{n=-\infty}^{\infty} W(f-nf_s)}_{\text{Ideal sampling}} \quad \text{where} \quad H(f) = \mathfrak{F}\{h(t)\} = \tau\frac{\sin \pi f \tau}{\pi f \tau}$$

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Instantaneous sampling (3)

Note that:

- for flat-top PAM signal filtering appears outside the sum and filtering is continuous with frequency
- for natural sampling the weight $c_n = d\,\mathrm{sinc}(nd)$ is not a function of f but of nf_s only

- when τ increases $\Rightarrow$ a faster decrease of the sinc-filter function results; this is called the aperture effect
- for small $\tau \Rightarrow$ less aperture effect, but also loss of amplitude (signal power)

Power is divided over more repeated spectra!

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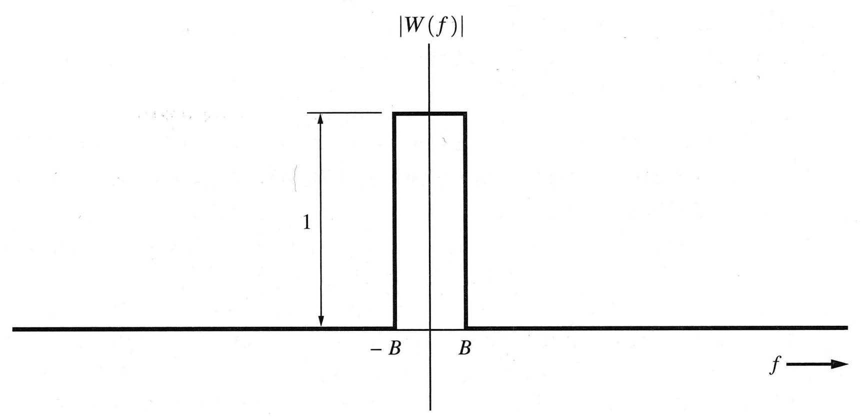
Instantaneous sampling (4)

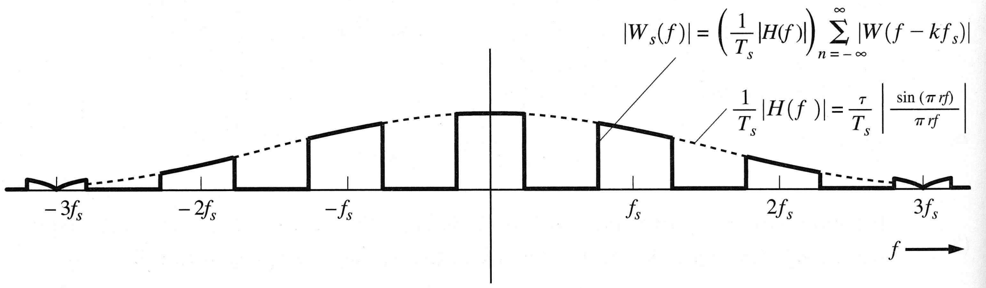
Again:

- $f_s = 4B$ (2x over sampling)
  Nyquist fulfilled
- $d = \frac{\tau}{T_s} = \frac{1}{3}$

The ideal sampled signal spectrum $W_\delta(f)$ is filtered with the sinc-shaped $H(f)$. The individual spectral components are not flat anymore: linear distortion.



(a) Magnitude Spectrum of Input Analog Waveform



(b) Magnitude Spectrum of PAM (flat-top sampling), $\tau/T_s = 1/3$ and $f_s = 4B$

Figure 3–6 Spectrum of a PAM waveform with flat-top sampling.

Remarks Pulse Amplitude Modulation

1. For *natural sampling*, the individual spectral components do have a flat frequency response $\Rightarrow$ rather simple signal recovery
2. For *flat-top PAM*, the individual spectral components do not have a flat frequency response $\Rightarrow$ complicated signal recovery.
   The filtering by $H(f)$ results in linear signal distortion, especially for large aperture τ. A correction filter (equalizer) can be used after signal recovery. The ideal one is $H^{-1}(f)$.
3. The bandwidth required for PAM is much larger than needed for the baseband signal with bandwidth B because of the narrow pulses for $\tau/T_s << 1$. Thus also a large receiver bandwidth (with good amplitude and phase response) is needed which will pass more noise.

Time Division Multiplexing

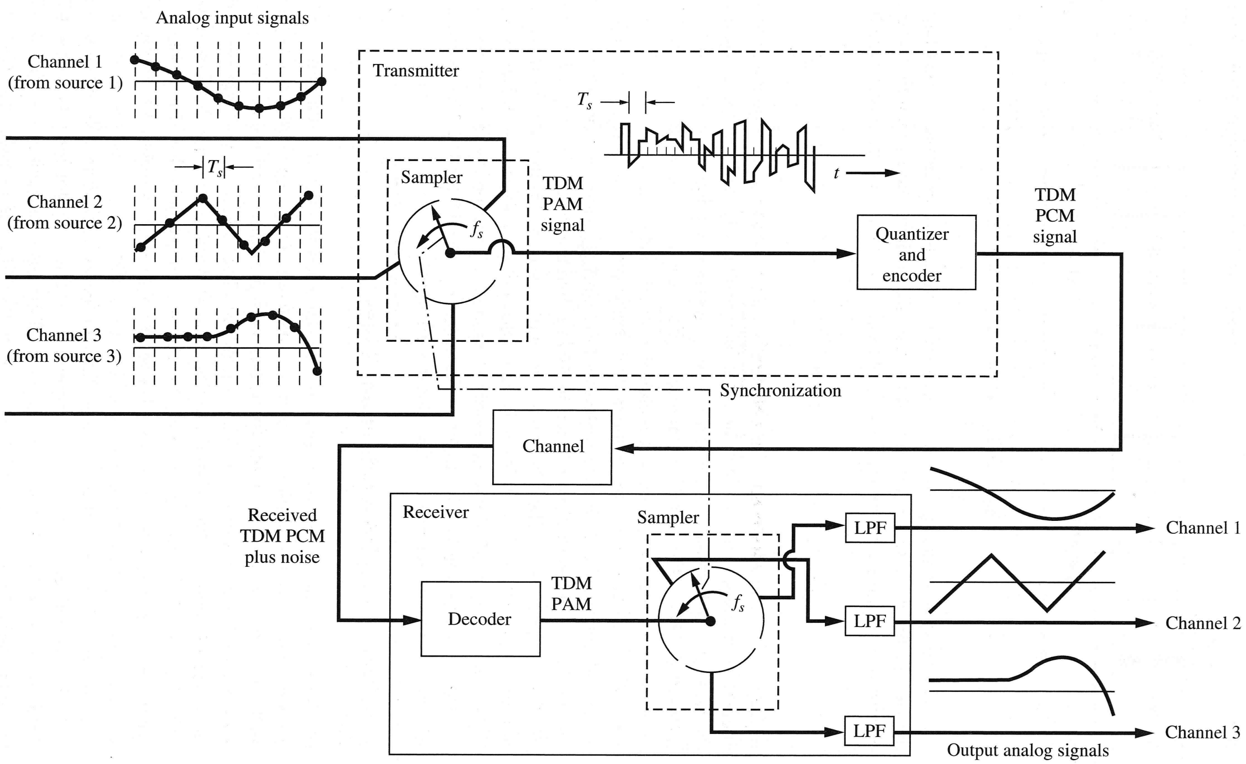
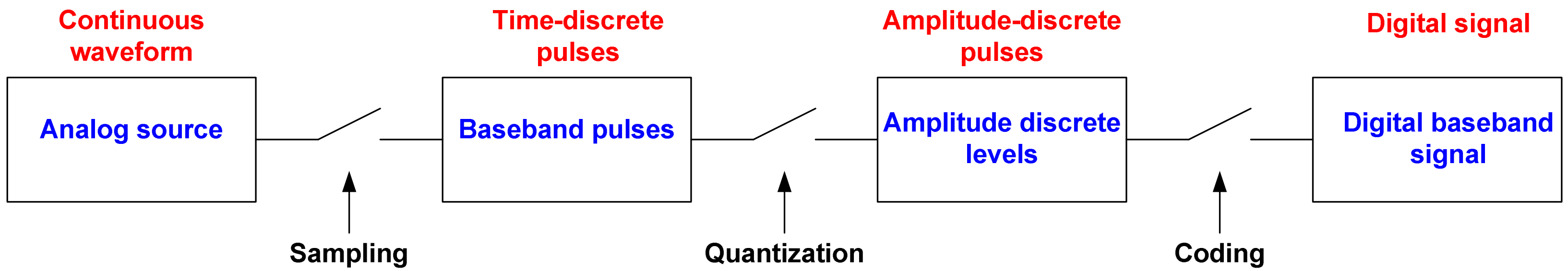


Figure 3–35 Three-channel TDM PCM system.

PAM pulses from different sources can be multiplexed in time (TDM = Time Division Multiplexing). Note that this results in overlapping frequency spectra. At the receive site, pulses belonging to the desired source have to be selected: accurate synchronization required.

Pulse Code Modulation (1)

Conversion of an analog signal into a digital representation



Three processes can be distinguished:

1) Sampling
2) Quantization
3) Coding

Pulse Code Modulation (2)

PCM consists of three basic operations:

1. Signal sampling $\rightarrow$ discrete-time analog pulses

2. Quantization of the amplitude
   $\rightarrow$ discrete-time and discrete-amplitude pulses

3. Coding
   $\rightarrow$ digital words are assigned to the discrete-time pulses that represent discrete-amplitude levels

Pulse Code Modulation (3)

Pulse Code Modulation ≈ Analog-to-Digital Conversion (ADC)

* An analog value is represented by an n-bits digital word.
  An n-bits word can represent $M = 2^n$ discrete amplitude levels.
* During quantization a constant value is required: flat-top PAM
* The analog (continuous) amplitude values are rounded to the nearest discrete value that can be represented.
* Rounding results in a quantization error of $\varepsilon \leq \delta / 2$ where $\delta = V_{pp} / M$ is the distance between two successive quantization levels.
* Quantization errors result in quantization noise.

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Pulse Code Modulation (4)

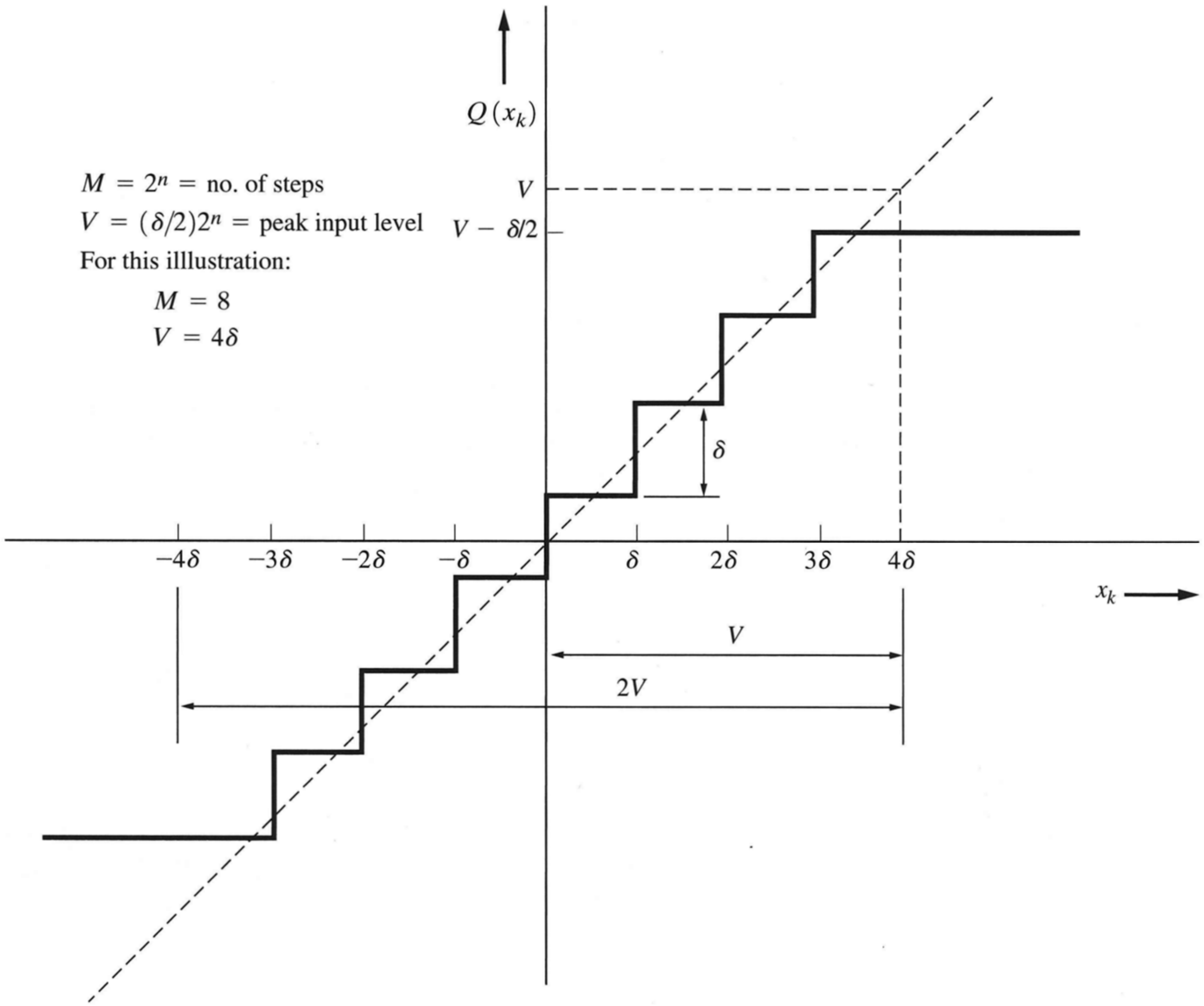
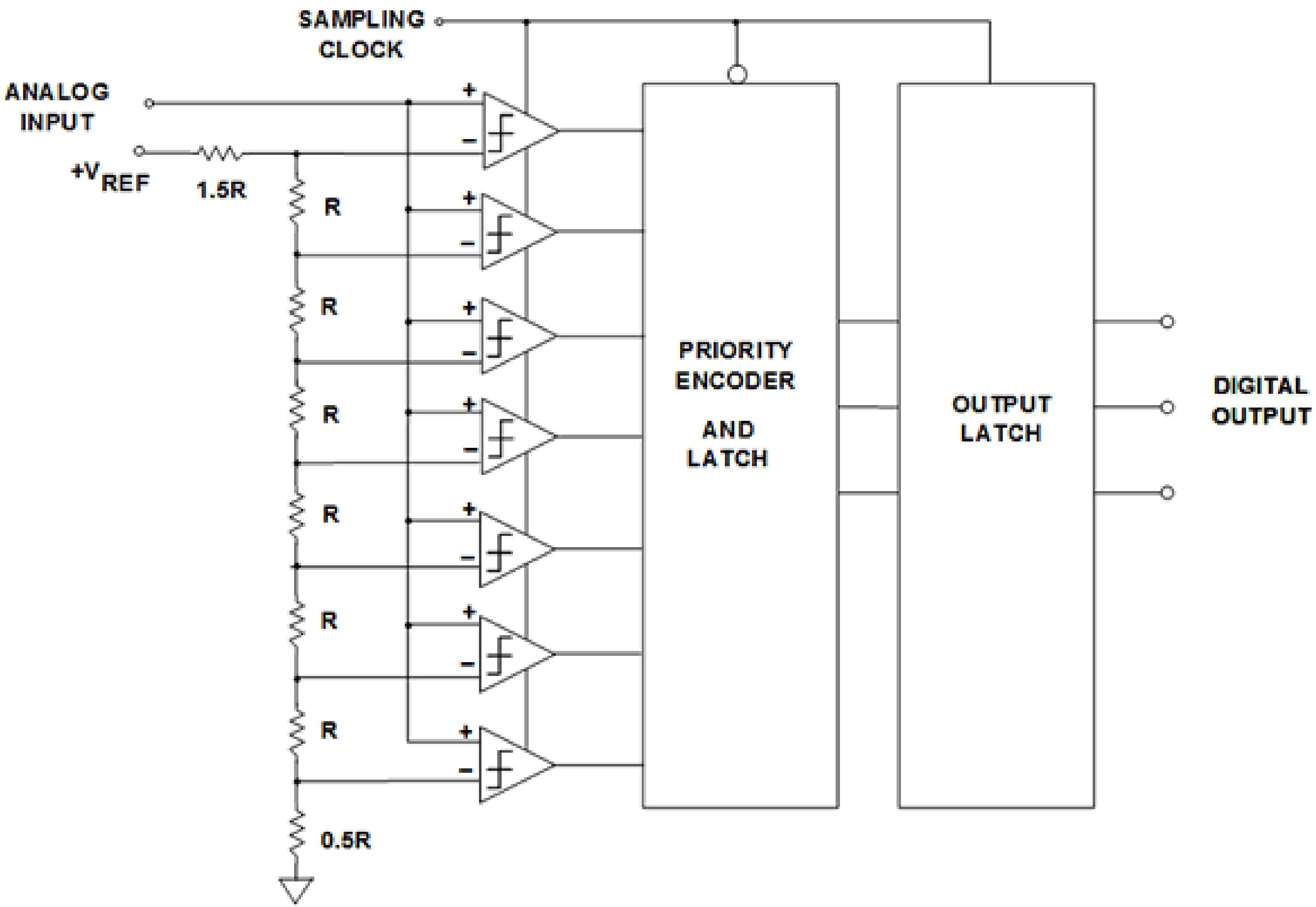


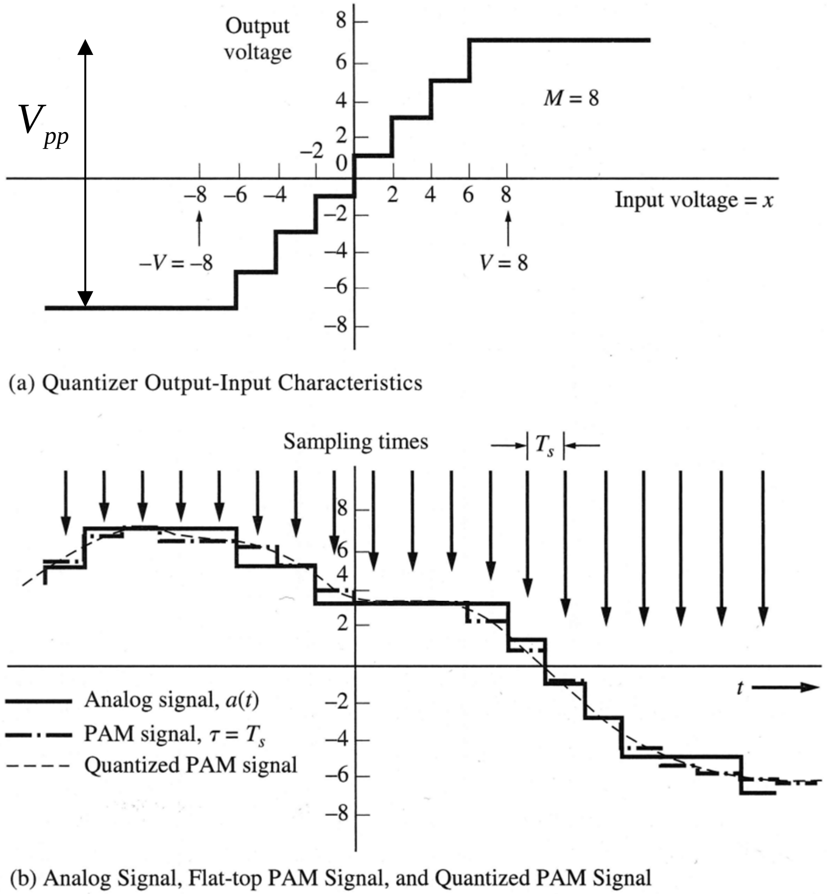
Figure 7–16 Uniform quantizer characteristic for $M = 8$ (with $n = 3$ bits in each PCM word).

Example AD-converter



3-bit All Parallel FLASH AD-converter

Pulse Code Modulation (5)



Quantization errors result in non-linear distortion.

What is the effect of an extra bit?

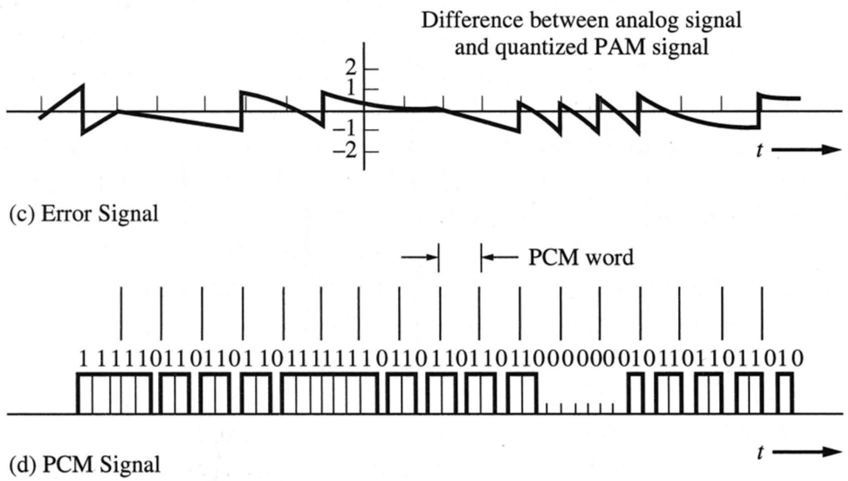


Figure 3–8 Illustration of waveforms in a PCM system.

Analysis of the PCM signal (1)

Let a signal sample x_k of $-V \leq x(t) \leq +V$ taken at $t = kT_s$ be quantized and coded in the n-bits word $\mathbf{a_k} = (a_{k1}, a_{k2},\ \dots\ , a_{kn})$.

With polar signaling: $a_{kj} \in \{-1,+1\}$ and the number of quantization levels is $M = 2^n$.

The reconstructed value $Q(x_k)$ for sample x_k is given by:

$$Q(x_k) = V\sum_{j=1}^{n} a_{kj}\left(\tfrac{1}{2}\right)^j = \frac{\delta}{2}\sum_{j=1}^{n} a_{kj}\, 2^{n-j} \quad \text{since } V = 2^n\frac{\delta}{2} = 2^{n-1}\delta$$

Analysis of the PCM signal (2)

Example: reconstruction of the all-ones word $\mathbf{a_k} = (+1, +1, \ \dots \ , +1)$ results in

$$Q(x_k) = V \sum_{j=1}^{n} (\tfrac{1}{2})^j = V\left(\frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + \dots + \frac{1}{2^n}\right)$$

$$= \frac{V}{2}\left(1 + \frac{1}{2} + \frac{1}{2^2} + \dots + \frac{1}{2^{n-1}}\right) = \frac{V}{2}\left(\frac{(\tfrac{1}{2})^n - 1}{\tfrac{1}{2} - 1}\right) = V - \frac{V}{2^n} = V - \frac{\delta}{2}$$

were $\delta = \frac{2V}{2^n} = \frac{V}{2^{n-1}}$ is the step size and we use:

$$\sum_{n=0}^{N} a^n = \frac{a^{N+1} - 1}{a - 1}$$

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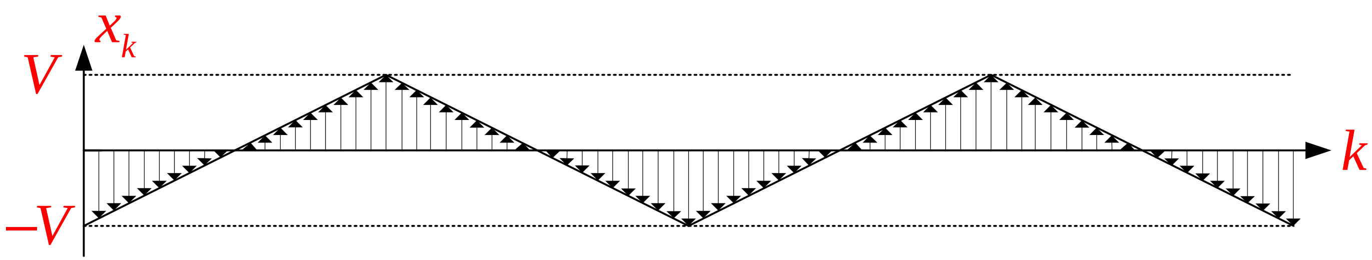
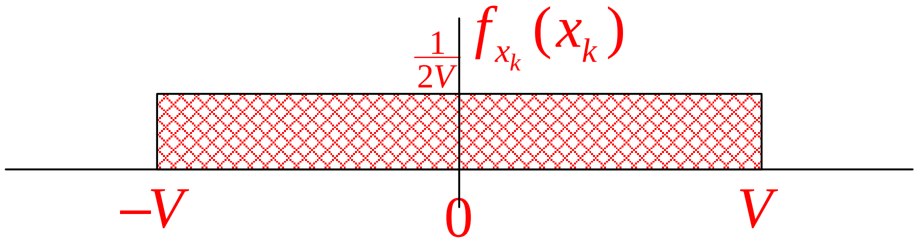
Analysis of the PCM signal (3)

The average power of a signal with uniformly distributed amplitudes:



with PDF:



$$\overline{x_k^2} = \int_{-V}^{V} x_k^2 f_{x_k}(x_k) dx_k = 2\int_0^V \frac{x_k^2}{2V} dx_k = \frac{1}{V} \left.\frac{x_k^3}{3}\right|_0^V = \frac{V^3}{3 \cdot V} = \frac{V^2}{3}$$

For: 1. a square wave $\rightarrow$ peak power = effective power = V^2

2. a sine wave $\rightarrow$ peak power = V^2, effective power = $\dfrac{V^2}{2}$

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Pulse Code Modulation (6)

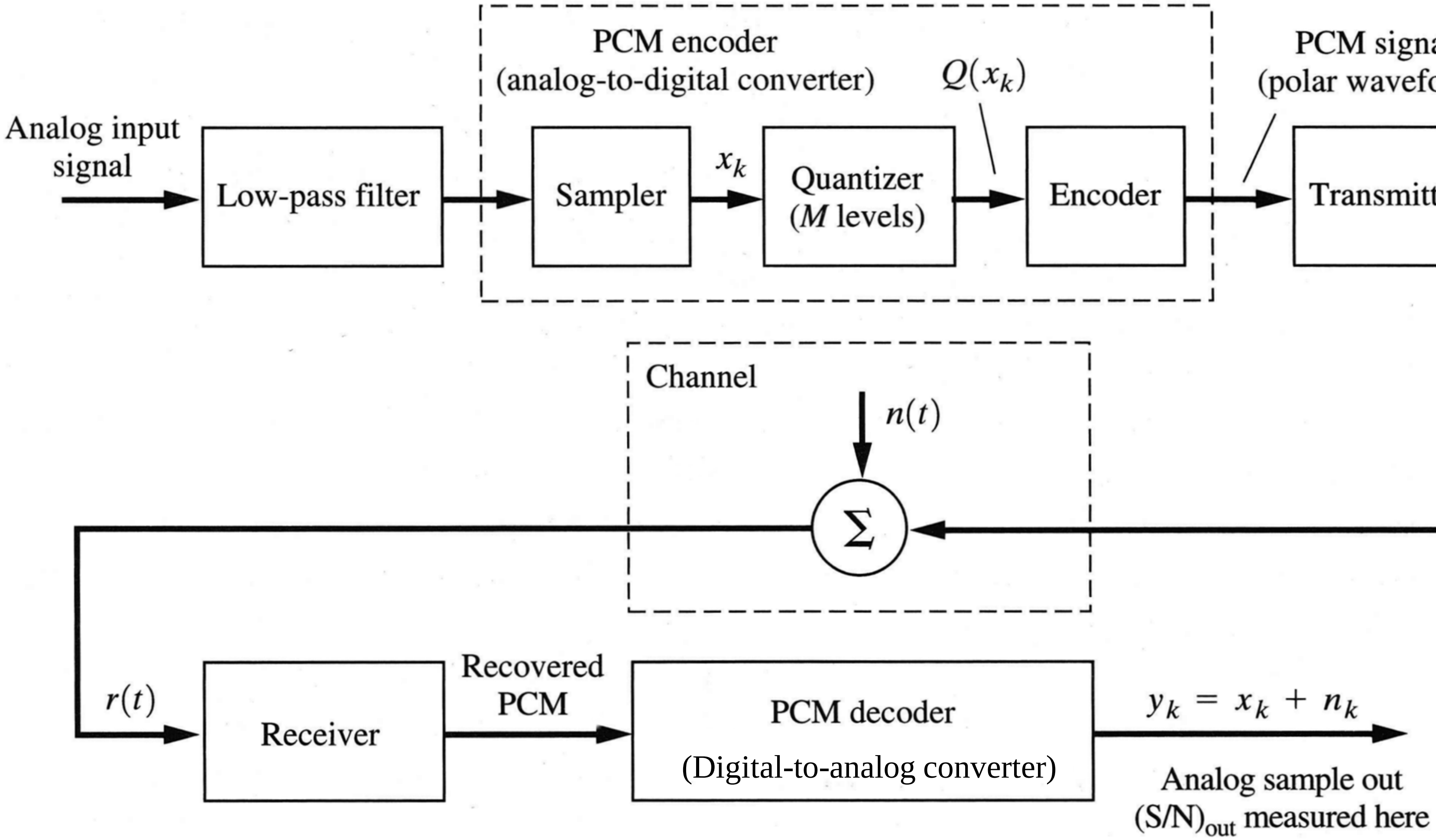

Figure 7–15 PCM communications system.

Noise in a PCM system

In a PCM communication system, the reconstructed signal $y_k = x_k + n_k$ suffers from three sources of noise:

1. Quantization noise: $e_q = Q(x_k) - x_k$

2. Bit Error Noise: $e_b = y_k - Q(x_k)$

   Reconstruction errors due to detection errors

3. Overload noise: the input signal is outside the conversion range $-V \le x(t) \le V$ of the PCM coder.

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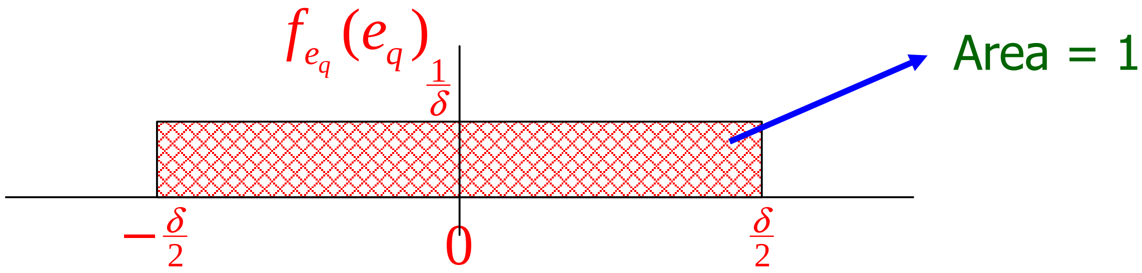
Quantization noise

The quantization noise $e_q = Q(x_k) - x_k$ due to quantization errors is uniformly distributed over $(-\delta/2, \delta/2)$.

The Probability Density Function (PDF) $f_{e_q}(e_q)$ of e_q is given by:



Now we find for the quantization noise power:

$$\overline{e_q^2} = \int_{-\infty}^{\infty} e_q^2 f_{e_q}(e_q) de_q = \int_{-\delta/2}^{\delta/2} e_q^2 \frac{1}{\delta} de_q = \frac{\delta^2}{12} = \frac{V^2}{3M^2}$$

where we used: $\delta = \dfrac{2 \cdot V}{2^n} = \dfrac{2 \cdot V}{M}$

Quantization Noise in PCM Systems

In case all bits are received correctly, noise in the reconstructed analog signal occurs only due to quantization errors (rounding errors).

Signal-to-Noise Ratio (SNR) at maximum signal level due to quantization errors:

$$\left.\frac{S}{N}\right)_{\max} = \frac{P_{signal-\max}}{P_{noise}} = \frac{V^2}{\overline{e_q^2}} = \frac{V^2}{V^2/3M^2} = 3M^2$$

$M = 2^n \rightarrow$ the number of quantization levels

Signal-to-Noise Ratio (SNR) for average signal level (uniformly distributed) due to quantization errors:

$$\left.\frac{S}{N}\right)_{average} = \frac{P_{signal-av}}{P_{noise}} = \frac{V^2/3}{\overline{e_q^2}} = \frac{V^2/3}{V^2/3M^2} = M^2$$

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PCM performance (1)

- With longer PCM words (n bits/sample)
  $\rightarrow$ the number of quantization levels $M = 2^n$ increases
  $\rightarrow$ smaller quantization errors result
  $\rightarrow$ larger SNR due to smaller quantization noise

- Every extra bit:
  $\rightarrow$ the number of quantization levels doubles
  $\rightarrow$ smaller quantization errors result: $\delta_{\max}(n+1) = \delta_{\max}(n)/2$
  $\rightarrow$ 6 dB higher SNR: $SNR(n+1) = SNR(n) + 6 \text{ dB}$

$$SNR_{dB} = 6.02 \cdot n + \alpha \quad \text{with } \alpha = \begin{cases} 4.8 \text{ dB for maximum signal} \\ 0 \text{ dB} \quad \text{for average signal} \end{cases}$$

- The input signal should cover the full input range of the ADC .

PCM performance (2)

Knowing that $f_s \geq 2B$ we find for the data rate of a PCM signal:

$$R_{PCM} = n \cdot f_s \geq 2nB$$

Analog bandwidth

The dimensionality theorem shows for the bandwidth required to transmit a digital (PCM) signal:

$$B_{PCM} \geq \frac{N}{2T_0} = \frac{n\,f_s}{2} = \frac{R_{PCM}}{2} \geq nB$$

The transmission bandwidth depends on the pulse shape used.

PCM is not very bandwidth efficient (Table 3.2). By using "smart" pulses (multi-level signaling) the transmission bandwidth can be reduced.

PCM performance (3)

TABLE 3–2 PERFORMANCE OF A PCM SYSTEM WITH UNIFORM QUANTIZING AND NO CHANNEL NOISE

Number of Quantizer Levels Used, M	Length of the PCM Word, n (bits)	Bandwidth of PCM Signal (First Null Bandwidth)[a]	Recovered Analog Signal-Power-to-Quantizing-Noise Power Ratios (dB)	
			$(S/N)_{\text{pk out}}$	$(S/N)_{\text{out}}$
2	1	$2B$	10.8	6.0
4	2	$4B$	16.8	12.0
8	3	$6B$	22.8	18.1
16	4	$8B$	28.9	24.1
32	5	$10B$	34.9	30.1
64	6	$12B$	40.9	36.1
128	7	$14B$	46.9	42.1
256	8	$16B$	52.9	48.2
512	9	$18B$	59.0	54.2
1,024	10	$20B$	65.0	60.2
2,048	11	$22B$	71.0	66.2
4,096	12	$24B$	77.0	72.2
8,192	13	$26B$	83.0	78.3
16,384	14	$28B$	89.1	84.3
32,768	15	$30B$	95.1	90.3
65,536	16	$32B$	101.1	96.3

[a] B is the absolute bandwidth of the input analog signal.

Example: CD-player (1)

Determine for a compact disk (CD):

1. What is the sample frequency?
2. What is the maximum baseband freqency?
3. What is the bitrate when we use 16 bit words per sample?
4. What is the first-null bandwidth (when we use square pulses)?
5. What is dynamic range of the system (difference between largest and smallest signal level)?

Example: CD-player (2)

$f_s \geq 2B = 44.1\ \text{kHz}$

$n = 16$ bits/sample

$\Rightarrow \quad R_b = 2 f_s n = 2 \cdot 44.1 \cdot 10^3 \cdot 16 \simeq 1.4\ \text{Mbit/s}$

Maximum baseband frequency: $\dfrac{f_s}{2} = 22.05\ \text{kHz}$

With square pulses:

- sinc² spectrum
- 1st null bandwidth $\approx 1.4\ \text{MHz}$

Dynamic range: $M^2 = (2^{16})^2 = 2^{32} \equiv 96\ \text{dB}$

$SNR_{\max} = 3M^2 \cong 101\ \text{dB}$

Conventional analog recording: $BW \simeq 2 \cdot 15\ \text{kHz}$

$\text{Dynamic range} \simeq 70\ \text{dB}$