Telecommunicatie A (EE2T11)

Lecture 2 overview:

Linear time-invariant systems (review)
Distortion
Bandlimited signals and noise
Sampling theorem
Bandwidths definitions

EE2T11 Telecommunicatie A
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February 1, 2016

Colleges en Werkcolleges Telecommunicatie A

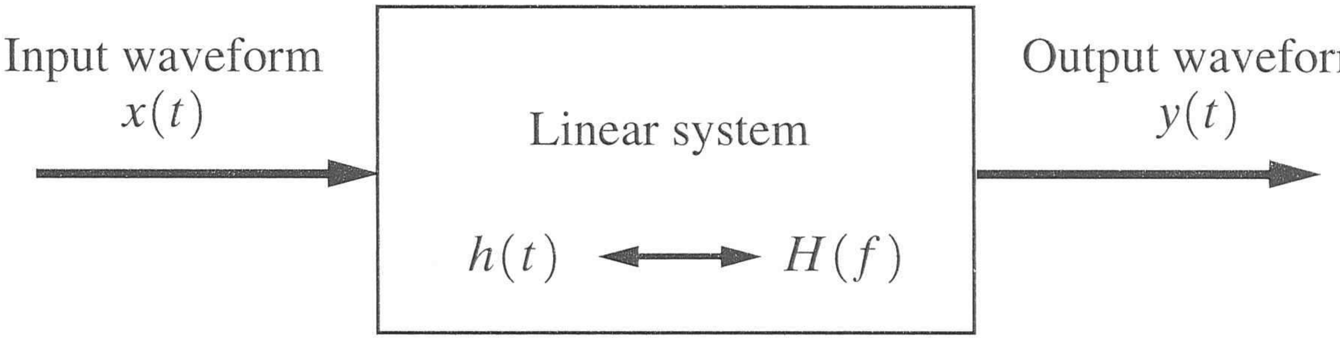
Colleges:

Maandag	15-2, 22-2, 7-3, 21-3	5e en 6e uur, EWI-Pi
Dinsdag	1-3, 15-3	7e en 8e uur, EWI-Pi

Werkcolleges:

Maandag	29-2, 14-3, 4-4	5e en 6e uur, EWI-Pi

Linear systems (1)



Some descriptions for the input		Some descriptions for the output
$X(f)$	"Voltage" spectrum	$Y(f)$
$R_x(\tau)$	Autocorrelation function	$R_y(\tau)$
$\mathcal{P}_x(f)$	Power spectral density	$\mathcal{P}_y(f)$

Figure 2–14 Linear system.

Linear systems (2)

1. Definition of a linear system:

$$y = \mathfrak{L}\{x\}$$

$$\mathfrak{L}\{a_1 x_1 + a_2 x_2\} = a_1 \mathfrak{L}\{x_1\} + a_2 \mathfrak{L}\{x_2\} = a_1 y_1 + a_2 y_2$$

2. Impulse response:

$$\delta(t) \quad \leftrightarrow \quad h(t)$$

$$y(t) = x(t) * h(t) = \int_{-\infty}^{\infty} x(\tau) h(t-\tau) d\tau$$

Linear systems (3)

3. Time-invariance:
   the system behavior does not change with time

$$y(t-t_0)=\mathfrak{L}\{x(t-t_0)\}$$

4. Transfer function

$$H(f)=\mathfrak{F}\{h(t)\}=\frac{Y(f)}{X(f)},\quad Y(f)=\mathfrak{F}\{y(t)\}=X(f)H(f)$$

$H(f)$ is a complex function:

$$H(f)=\underbrace{|H(f)|}_{\text{amplitude characteristic}}\exp(j\cdot \underbrace{ang\{H(f)\}}_{\text{phase characteristic}})=\mathrm{Re}\{H(f)\}+j\,\mathrm{Im}\{H(f)\}$$

$$ang\{H(f)\}=\arctan\left(\frac{\mathrm{Im}\{H(f)\}}{\mathrm{Re}\{H(f)\}}\right)$$

Linear systems (4)

Since $h(t)$ is a real function: - $|H(f)| \rightarrow$ even
- $ang\{H(f)\} \rightarrow$ odd

For practical systems:
$h(t)$ is causal.

How can we determine $|H(f)|$ and $ang\{H(f)\}$?
What is the output signal for $x(t) = A\cos\omega_0 t$?

5. Power spectral density (or power spectrum)

$$P_x(f) = \lim_{T\to\infty} \frac{|X_T(f)|^2}{T} = \lim_{T\to\infty} \frac{X_T(f)X^*_T(f)}{T} \text{ [W/Hz]} \Rightarrow \text{ direct way}$$

$$P_x(f) = \mathfrak{F}\{R_x(\tau)\} \text{ [W/Hz]} \Rightarrow \text{ indirect way (Wiener-Khintchine theorem)}$$

6. Power transfer function

$$G(f) = \frac{P_y(f)}{P_x(f)} = \frac{|Y(f)|^2}{|X(f)|^2} = |H(f)|^2$$

Power spectrum sine-wave (1)

Let $w(t) = A\sin\omega_0 t = \dfrac{A}{2j}\left(e^{j\omega_0 t} - e^{-j\omega_0 t}\right)$. What is $P_w(f)$?

Since $W(f) = \frac{A}{2j}[\delta(f - f_0) - \delta(f + f_0)]$, we cannot determine $P_w(f)$ in a direct way using $P_w(f) = |W(f)|^2$!

Why not?

We need to use the indirect (correct) way via the autocorrelation function:

$$P_w(f) = \mathfrak{F}\{R_w(\tau)\}$$

Power spectrum sine-wave (2)

For $w(t) = A\sin\omega_0 t$:

See also Couch pp. 63-64

$$R_w(\tau) =< w(t)w(t+\tau) >$$

$$= \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} A^2 \sin\omega_0 t \sin\omega_0(t+\tau)dt$$

$$= \frac{A^2}{2}\cos\omega_0\tau \lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2} dt - \frac{A^2}{2}\lim_{T\to\infty}\frac{1}{T}\int_{-T/2}^{T/2}\cos(2\omega_0 t + \omega_0\tau)dt$$

$$= \frac{A^2}{2}\cos\omega_0\tau = \frac{A^2}{4}\left(e^{j\omega_0\tau} + e^{-j\omega_0\tau}\right)$$

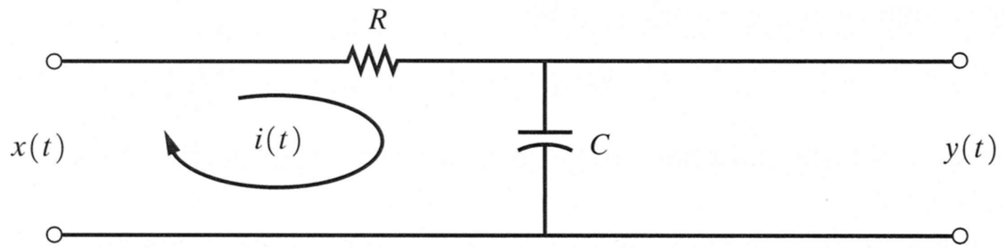
Now we find: $P_w(f) = \mathfrak{F}\{R_w(\tau)\} = \frac{A^2}{4}[\delta(f - f_0) + \delta(f + f_0)]$

and the total power is: $P_w = \int_{-\infty}^{\infty} P_w(f)df = \frac{A^2}{2}$ as expected.

Example: First order RC-lowpass filter (1)

$$x(t) = R \cdot i(t) + y(t)$$

$$i(t) = C \frac{dy(t)}{dt}$$



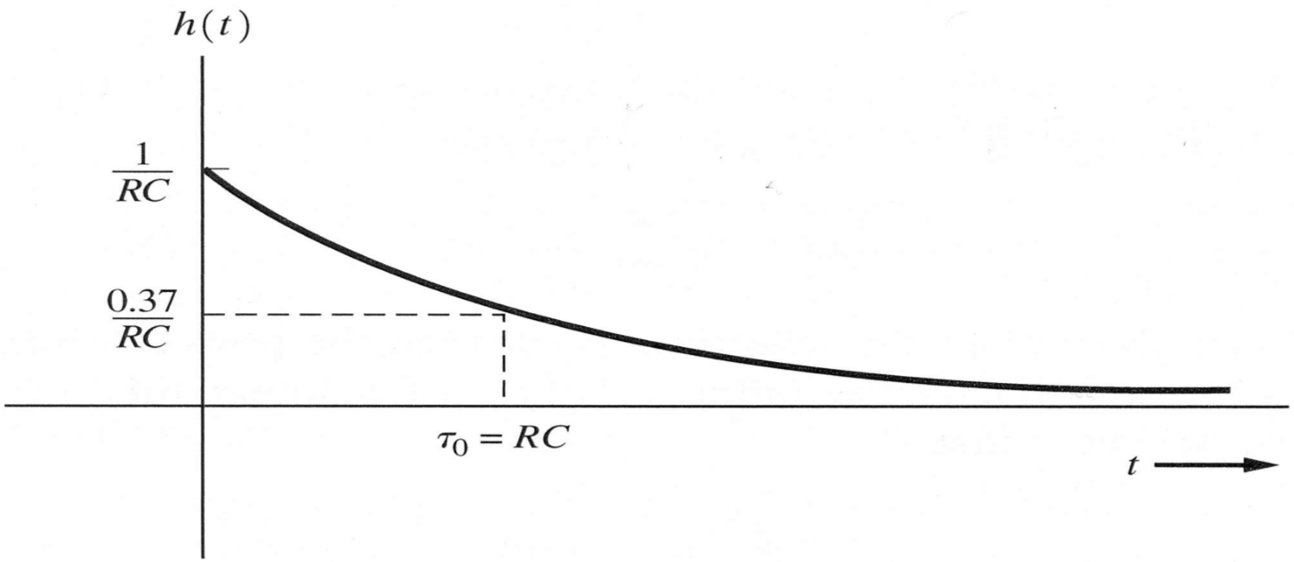
(a) RC Low-Pass Filter

$$\Rightarrow \quad RC \tfrac{dy}{dt} + y = x \quad \xrightarrow{\mathfrak{F}} RC \cdot j2\pi f \cdot Y(f) + Y(f) = X(f)$$

$$\Rightarrow \quad H(f) = \frac{Y(f)}{X(f)} = \frac{1}{1 + j2\pi fRC} = \frac{1}{1 + j \cdot f / f_0}$$

with $f_0 = \dfrac{1}{2\pi RC} = \dfrac{1}{2\pi\tau_0}$ and $\tau_0 = RC$ is the time constant.

First order RC-lowpass filter (2)



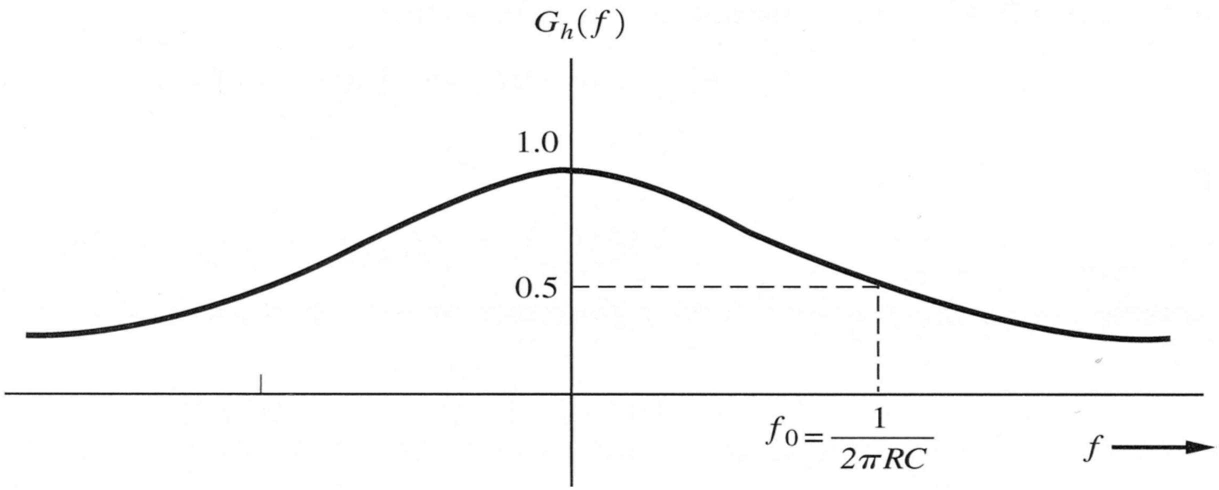
Impulse response:

See Table 2.2

$$h(t) = \mathfrak{F}^{-1}\{H(f)\} = \frac{1}{\tau_0}\exp\left(\frac{-t}{\tau_0}\right) \qquad t > 0$$

with $\tau_0 = RC$ is the time constant of the filter.

First order RC-lowpass filter (3)



Power transfer function:

$$G(f) = |H(f)|^2 = \frac{1}{1+(2\pi f)^2 R^2 C^2} = \frac{1}{1+(f/f_0)^2}$$

For $f = f_0 \Rightarrow G(f_0) = \frac{1}{2}$, f_0 is the -3 dB frequency.

Distortion-free transmission

Requirements for distortion-free transmission:

$$y(t) = A \cdot x(t - T_d) \quad |A| > 0,\ T_d \geq 0$$

or

$$Y(f) = A \cdot X(f)\exp\{-2\pi jfT_d\}$$

So $H(f) = \frac{Y(f)}{X(f)} = A\exp\{-2\pi jfT_d\} \Rightarrow \phi(f) = -2\pi fT_d$

Over the frequency band in which the signal is contained:

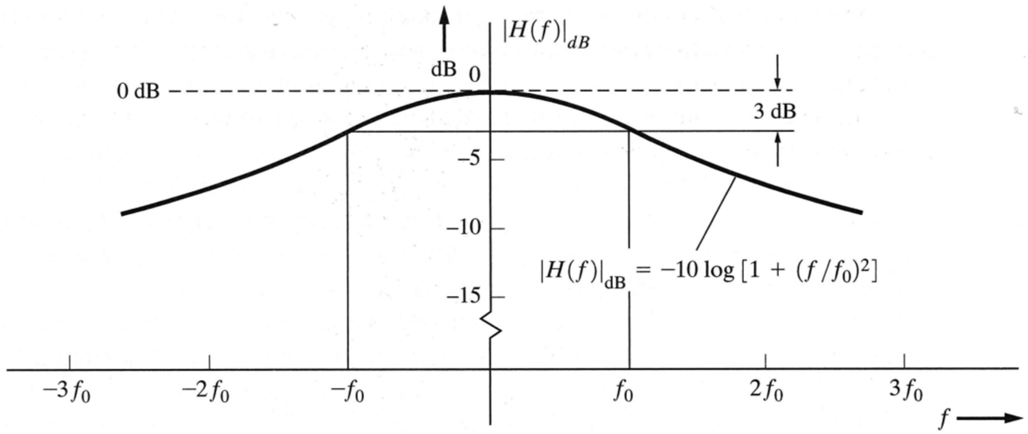
- the response should be flat: $|H(f)| = |A| > 0$
- the phase decreases linear with frequency $\Rightarrow$ constant delay for all frequency components:

$$T_d = \frac{-1}{2\pi f} ang\{H(f)\} = -\frac{1}{2\pi f}\phi(f)$$

Distortion of the RC-filter (1)

$$H(f) = \frac{1}{1 + j2\pi fRC}$$

$$= \frac{1}{1 + j \cdot f / f_0}$$



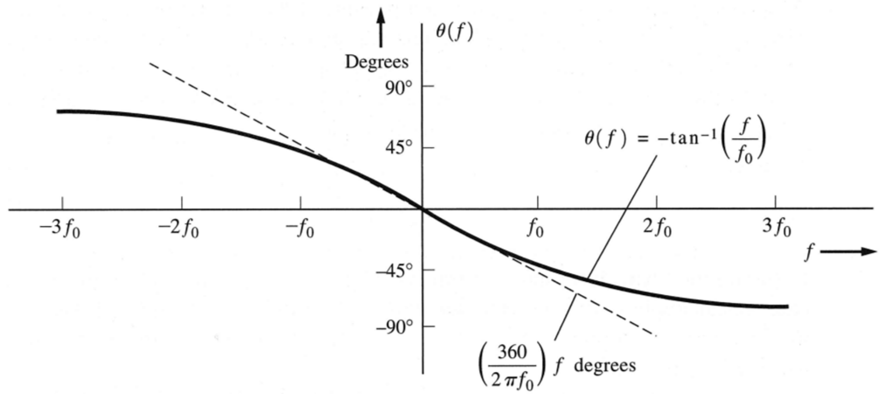
(a) Magnitude Response

Amplitude response:

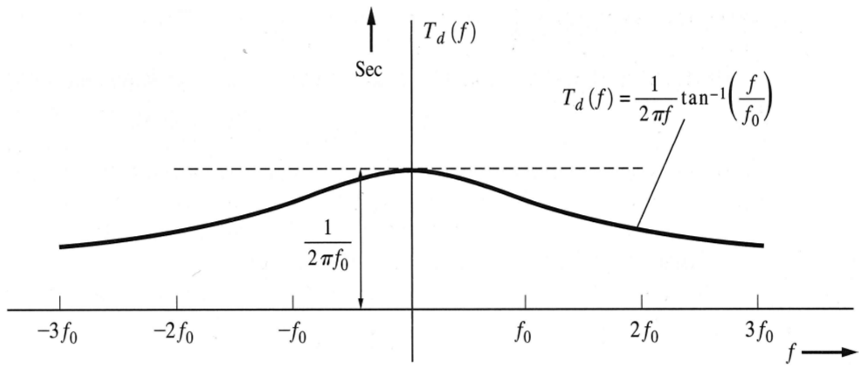
$$|H(f)| = \sqrt{H(f) \cdot H^*(f)} = \frac{1}{\sqrt{1 + (f/f_0)^2}} \quad \rightarrow \text{ even function}$$

At 10% deviation $\Rightarrow |H(f)| = 0.9 \rightarrow f \simeq 0.48 f_0$

Distortion of the RC-filter (2)



(b) Phase Response



(c) Time Delay

Phase and delay response:

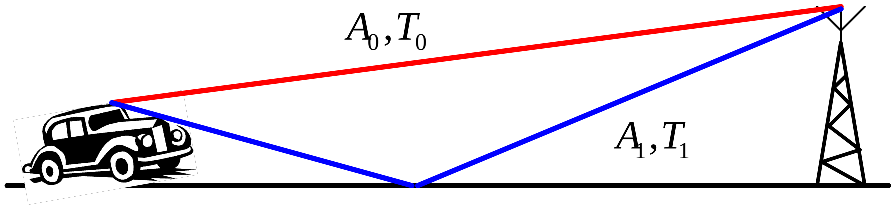
$$\phi(f) = -\arctan\frac{f}{f_0} = -\frac{f}{f_0} + O\left(\frac{f}{f_0}\right)^3 \quad \rightarrow \text{ odd function}$$

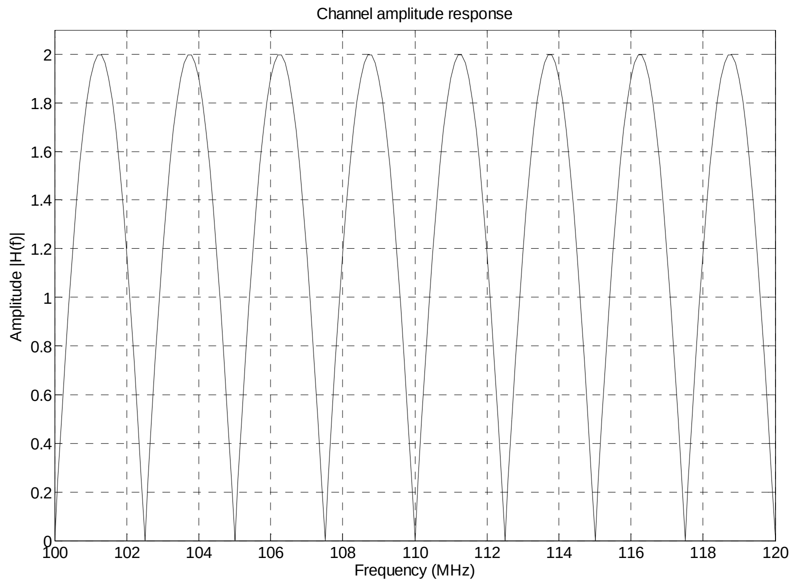
$$T_d(f) = \frac{-1}{2\pi f}\phi(f) = \frac{1}{2\pi f_0} + O\left(\frac{f}{f_0}\right)^2 \quad \rightarrow \text{ even function}$$

A practical design is always a compromise and never ideal.

For $f = 0.5 f_0 \Rightarrow \Delta\phi \simeq 2^o \approx\ 8\%$

Time variant system: multipath channel





$$h(t) = A_0\delta(t - T_0) - A_1\delta(t - T_1)$$

$$H(f) = A_0 \exp\{-2\pi jfT_0\} - A_1 \exp\{-2\pi jfT_1\}$$

$A_0 = A_1 = 1$
$T_0 = 0.6\ \mu s$
$T_1 = 1\ \mu s$

When moving, T_0 and T_1 change and so does $H(f)$.

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Band-limited signals and noise (1)

A wave form *w(t)* is absolutely band-limited, if:

$$W(f) = \mathfrak{F}\{w(t)\} = 0 \qquad \text{for} \qquad |f| \geq B_0$$

and time limited, if:

$$w(t) = 0 \qquad \text{for} \qquad |t| > T_0$$

Theorem: absolutely time-limited signals cannot be also absolutely band-limited and vise versa.

Uncertainty relation:

$$B_0 \cdot T_0 \geq \frac{1}{2} \quad \text{or} \quad B_0 = \frac{\alpha}{T_0} \quad \text{with} \quad \alpha \geq \frac{1}{2}$$

What determines the value of α?

Band-limited signals and noise (2)

Why are band-limited signals important in the Telecommunications?

1. Band-limited signals allow multiplexing in the frequency domain:
   efficient use of the available bandwidth.

2. Band-limited signals can be completely represented
   by a set of discrete-time sample values:

signal sampling

This allows for *digitizing* and *digital signal processing* and *multiplexing* of signals in time (TDM = time division multiplexing).

Sampling theorem (1)

Every physical signal can be expressed as:

$$w(t) = \sum_{n=-\infty}^{\infty} a_n \varphi_n(t)$$

$$\operatorname{sinc} x = \frac{\sin \pi x}{\pi x}$$

With sample function

$$\varphi_n(t) = \frac{\sin \pi f_s (t - n/f_s)}{\pi f_s (t - n/f_s)} = \frac{\sin \pi f_s (t - nT_s)}{\pi f_s (t - nT_s)} = \operatorname{sinc}(f_s (t - nT_s))$$

with coefficients

$$a_n = f_s \int_{-\infty}^{\infty} w(t) \varphi_n(t) dt$$ and $$f_s \int_{-\infty}^{\infty} \varphi_m(t) \varphi_n(t) dt = \begin{cases} 1 & \text{for } m = n \\ 0 & \text{for } m \neq n \end{cases}$$

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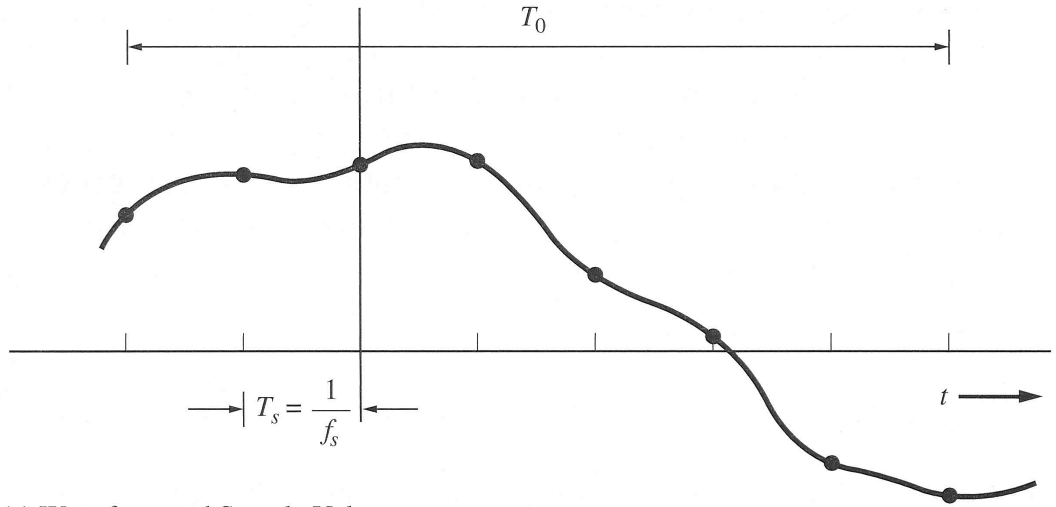
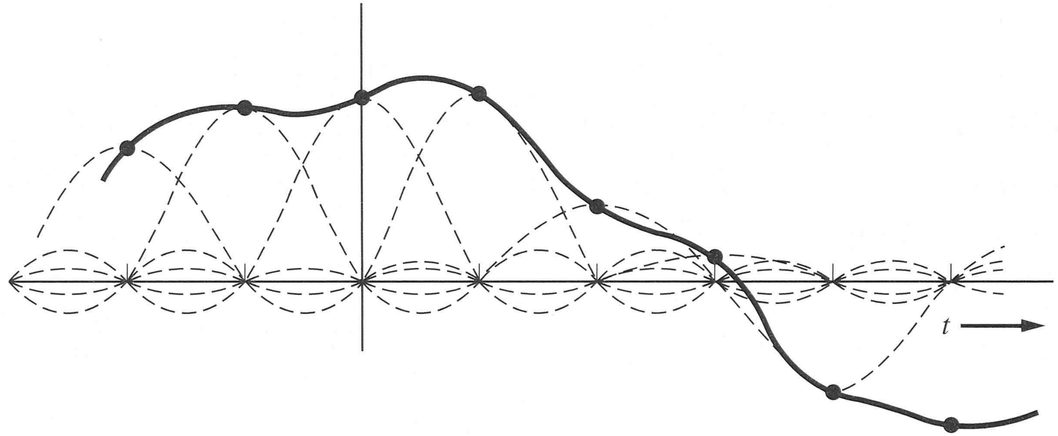


(a) Waveform and Sample Values



(b) Waveform Reconstructed from Sample Values

Figure 2–17 Sampling theorem.

Sampling theorem (2)

If the signal $w(t)$ is band-limited in B [Hz] and the sample frequency

$$f_s \geq 2B$$

then

$$a_n = w(n / f_s) = w(nT_s)$$

with f_s the sample frequency and T_s the sample time.
The set $\{a_n\}$ is a complete representation of the signal $w(t)$.

The lowest possible sample frequency: $f_s = 2B$

is called the *Nyquist frequency*.

Dimensionality of a signal

The minimum number of samples required to represent a time-continuous signal $w(t)$ with a bandwidth of B [Hz] over a period T_0 is equal to:

$$N = \frac{T_0}{T_s} = f_s \cdot T_0 \geq 2B \cdot T_0$$

N is the number of dimensions needed to describe the waveform $w(t)$ over the period T_0 .

Using the uncertainty relation $BT_0 \geq \frac{1}{2}$ it follows: $N \geq 1$

What does it mean when $N = 1$?

Ideal sampling (1)

In ideal sampling, we use the δ-function as sample function instead of the sinc-function!

$$w_s(t) = w(t)\sum_{k=-\infty}^{\infty}\delta(t-kT_s) \quad \text{with} \quad T_s = \frac{1}{f_s}, \;\; f_s \ge 2B$$

$$= \sum_{k=-\infty}^{\infty} w(kT_s)\delta(t-kT_s)$$

Using (2-115): $\displaystyle\sum_{k=-\infty}^{\infty}\delta(t-kT_s) = \sum_{n=-\infty}^{\infty} c_n e^{2\pi jnf_s t} = f_s \sum_{n=-\infty}^{\infty} e^{2\pi jnf_s t}$

we find:

$$w_s(t) = w(t)\sum_{n=-\infty}^{\infty}\tfrac{1}{T_s}e^{2\pi jnf_s t}$$

What is $W_s(f)$?

Ideal sampling (2)

Take the Fourier transform of $w_s(t) = w(t)\sum_{n=-\infty}^{\infty}\frac{e^{2\pi jnf_s t}}{T_s}$

$$W_s(f) = \mathfrak{F}\{w_s(t)\} = \frac{1}{T_s}W(f) * \mathfrak{F}\{\sum_{n=-\infty}^{\infty} e^{2\pi jnf_s t}\}$$

$$= \frac{1}{T_s}W(f) * \sum_{n=-\infty}^{\infty}\delta(f - nf_s)$$

$$= f_s\sum_{n=-\infty}^{\infty}W(f - nf_s)$$

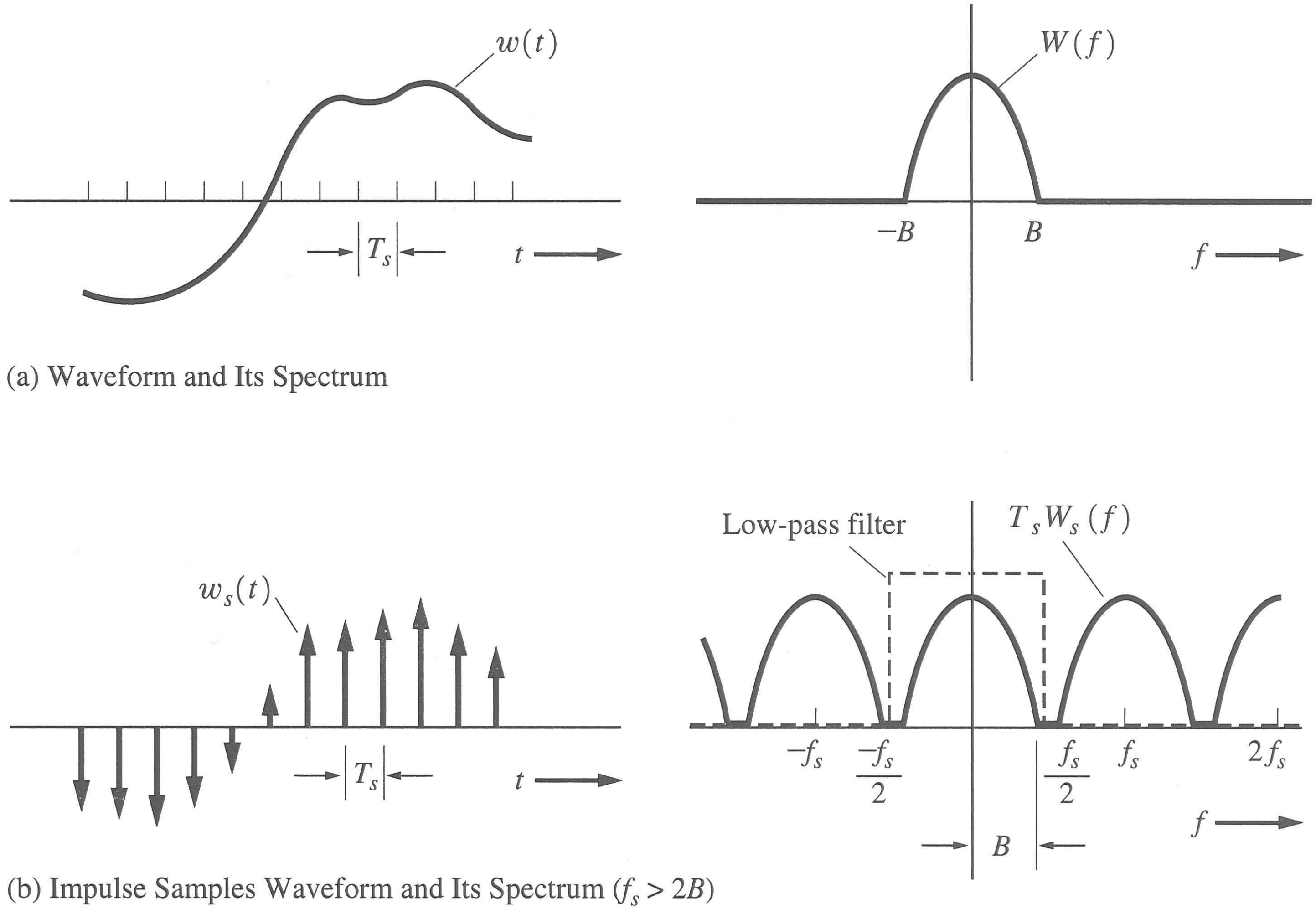
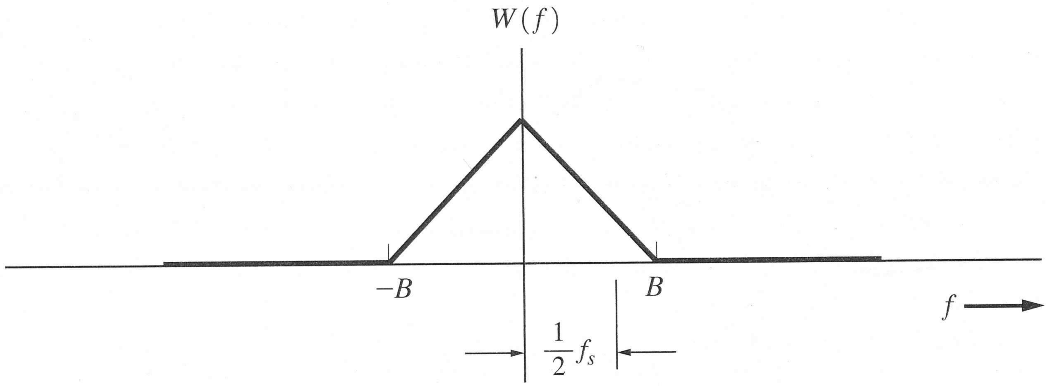
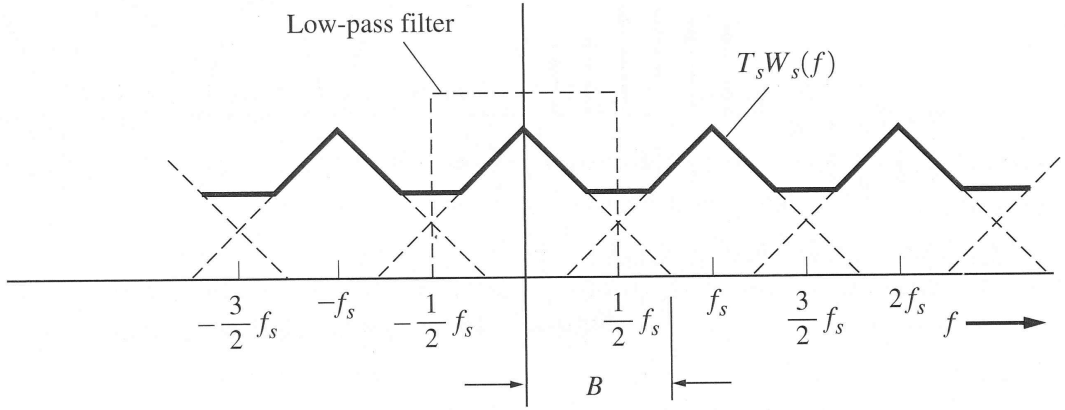


Figure 2–18 Impulse sampling.



(a) Spectrum of Unsampled Waveform



(b) Spectrum of Impulse Sampled Waveform ($f_s < 2B$)

Figure 2–19 Undersampling and aliasing.

General dimensionality theorem

Any real band-limited signal with a bandwidth B [Hz] and duration T_0 [s], can be completely represented by N independent samples.
The dimensionality of the signal is N, with: $N \geq 2B \cdot T_0$.

Two consequences:

1. At least N independent (not necessarily periodic) numbers are required to represent a signal with bandwidth B over a period T_0.

$$N \geq 2B \cdot T_0 \quad \Rightarrow \quad f_s \geq 2B.$$

2. If we have N independent numbers, e.g. symbols or code words, the required bandwidth to transmit this information in a time T_0 is:

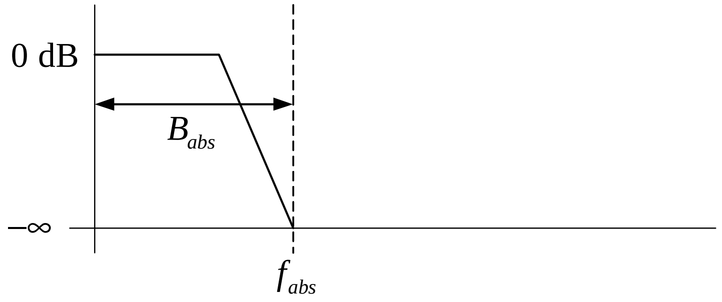
$$B \geq \frac{N}{2T_0} \quad [\mathrm{Hz}]$$

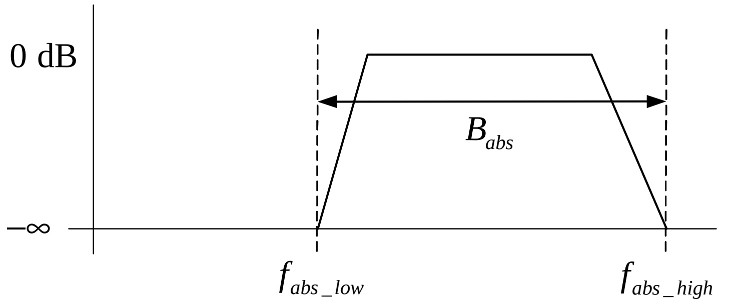
So if we are able to reduce N, we can reduce B, e.g. by using a richer alphabet!

Bandwidth definitions (1)

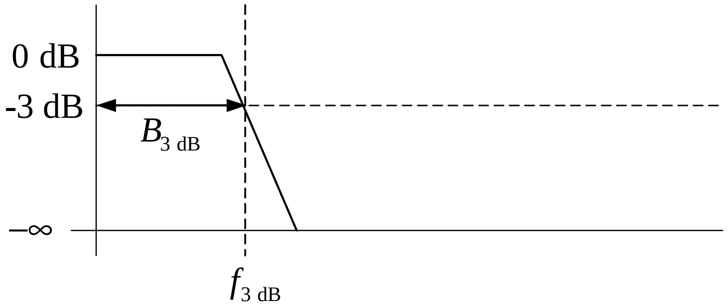
All bandwidths are defined for the single sided spectrum: i.e. the positive frequencies.

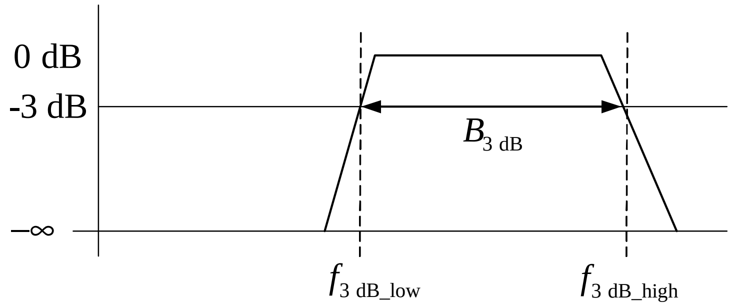
Absolute bandwidth





3 dB bandwidth





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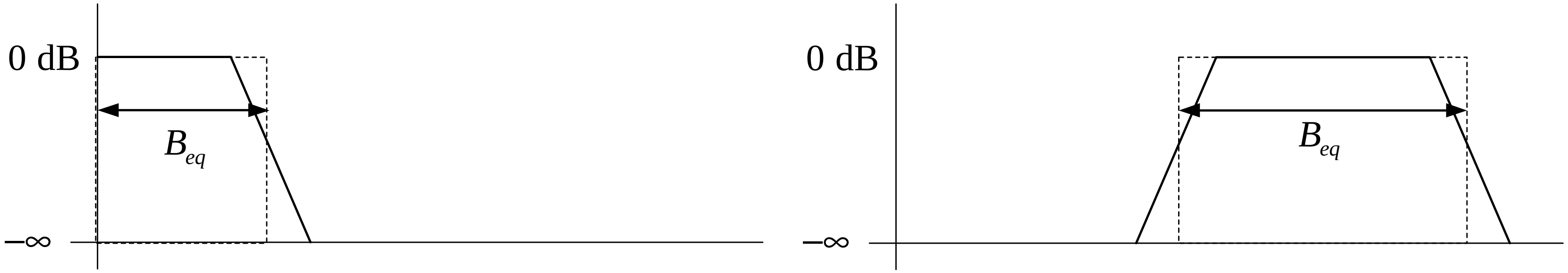
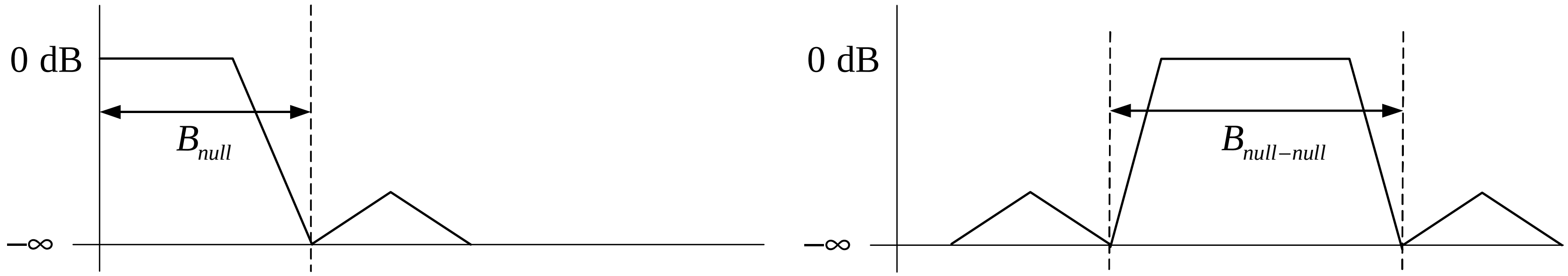
Bandwidth definitions (2)

Equivalent (noise)bandwidth: $B_{eq} = \int_0^{\infty} \frac{|H(f)|^2}{|H(0)|^2} df$

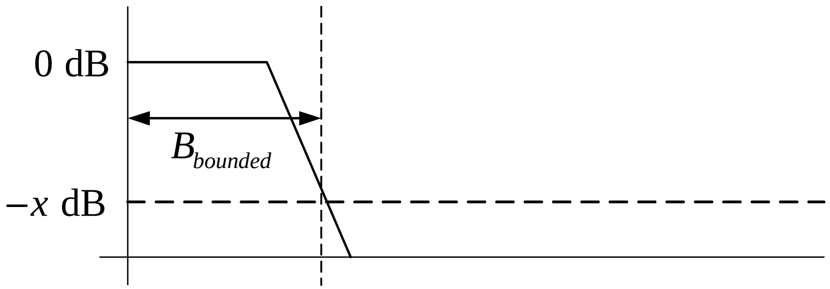


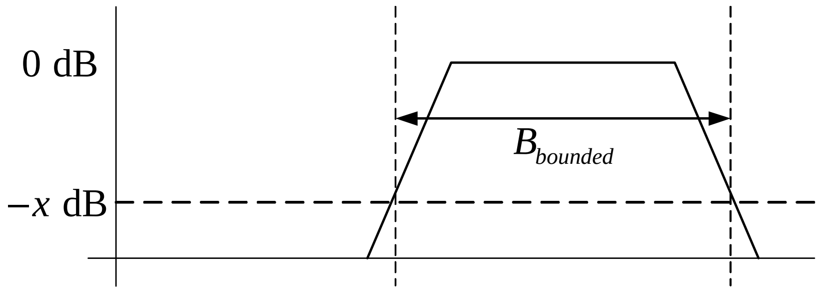
Null-null bandwidth



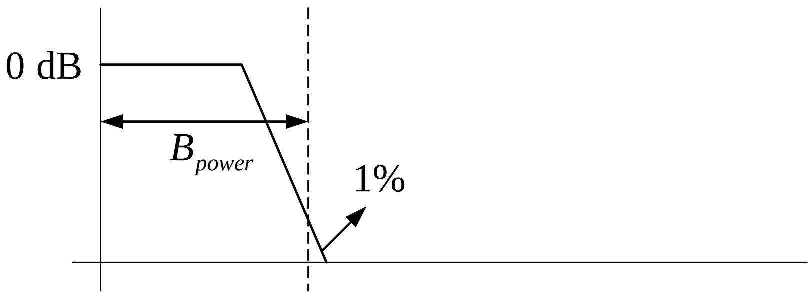
Bandwidth definitions (3)

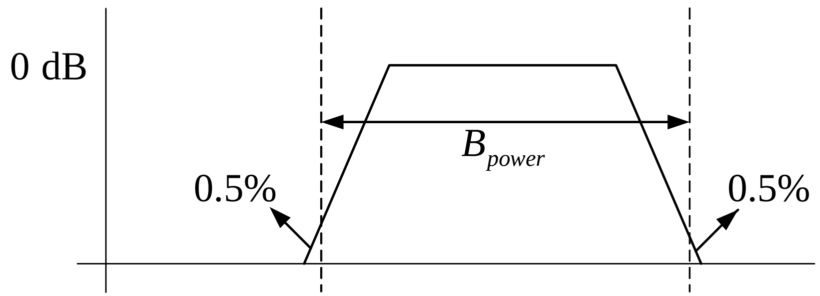
Bounded spectrum bandwidth





Power bandwidth





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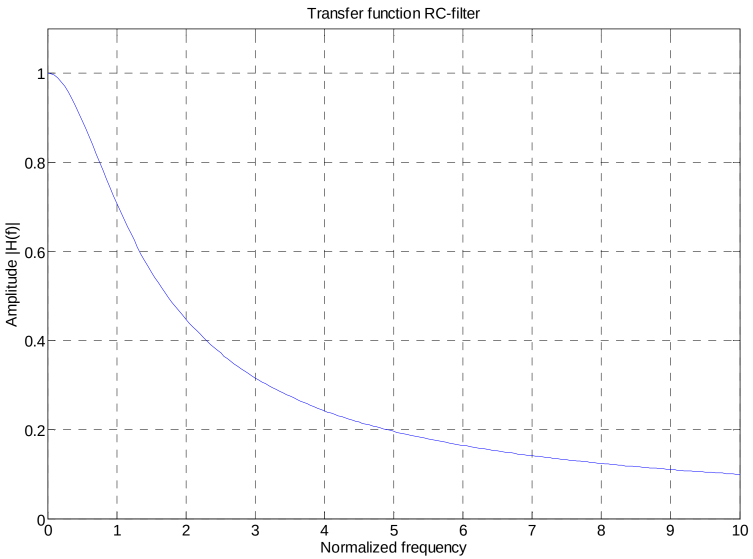
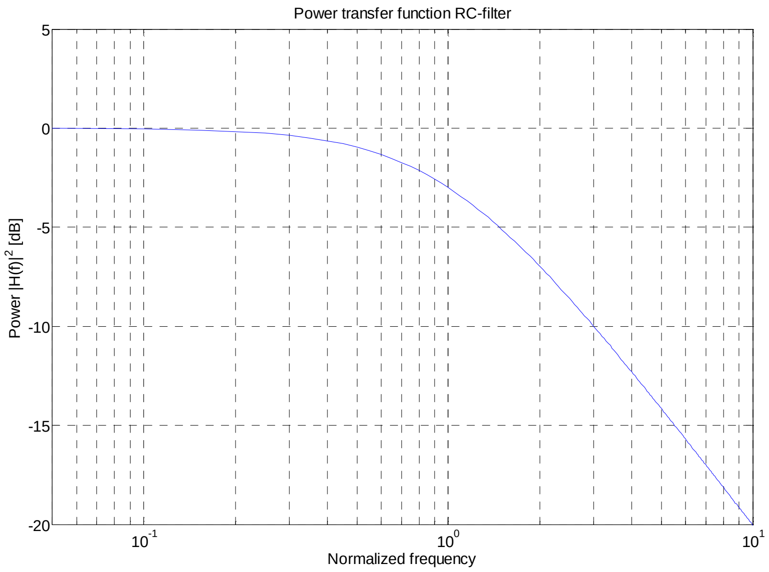
RC-filter bandwidths (1)

Transfer function: $H(f) = \dfrac{1}{1 + j \cdot f / f_0}$ $\qquad f_0 = \dfrac{1}{2\pi RC}$





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RC-filter bandwidths (2)

Transfer function: $H(f) = \dfrac{1}{1 + j \cdot f / f_0}$ $\quad f_0 = \dfrac{1}{2\pi RC}$

- absolute bandwidth: $B_{abs} = \infty$

- 3 dB bandwidth: $|H(f_{3\text{ dB}})|^2 = \dfrac{1}{2} \rightarrow B_{3\text{ dB}} = f_0 = \dfrac{1}{2\pi RC}$

- equivalent noise bandwidth:

$$B_{eq} = \int_0^\infty \frac{|H(f)|^2}{|H(0)|^2} df = \int_0^\infty \frac{1}{1 + (f / f_0)^2} df$$

$$= f_0 \arctan \frac{f}{f_0} \Bigg|_0^\infty = \frac{\pi f_0}{2} = \frac{1}{4RC}$$

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RC-filter bandwidths (3)

- null-null bandwidth: $B_{null} = \infty$

- N_dB bounded spectrum bandwidth (e.g. N = -50):

$$|H(f)|^2 = \frac{1}{1+(f/f_0)^2} = 10^{-5} \Leftrightarrow \left(\frac{f}{f_0}\right)^2 = 10^5 - 1 \simeq 10^5 \quad \Rightarrow \quad B_{-50\text{ dB}} \simeq 316\, f_0$$

- Power bandwidth / occupied bandwidth:

$$\int_0^{B_{99\%}} \frac{1}{1+(f/f_0)^2} df = 0.99 B_{eq} \quad \Leftrightarrow \quad f_0 \arctan \frac{B_{99\%}}{f_0} = 0.99 \frac{\pi f_0}{2}$$

$$\Rightarrow B_{99\%} = f_0 \tan\left(0.99 \frac{\pi}{2}\right) \simeq 63.7\, f_0$$

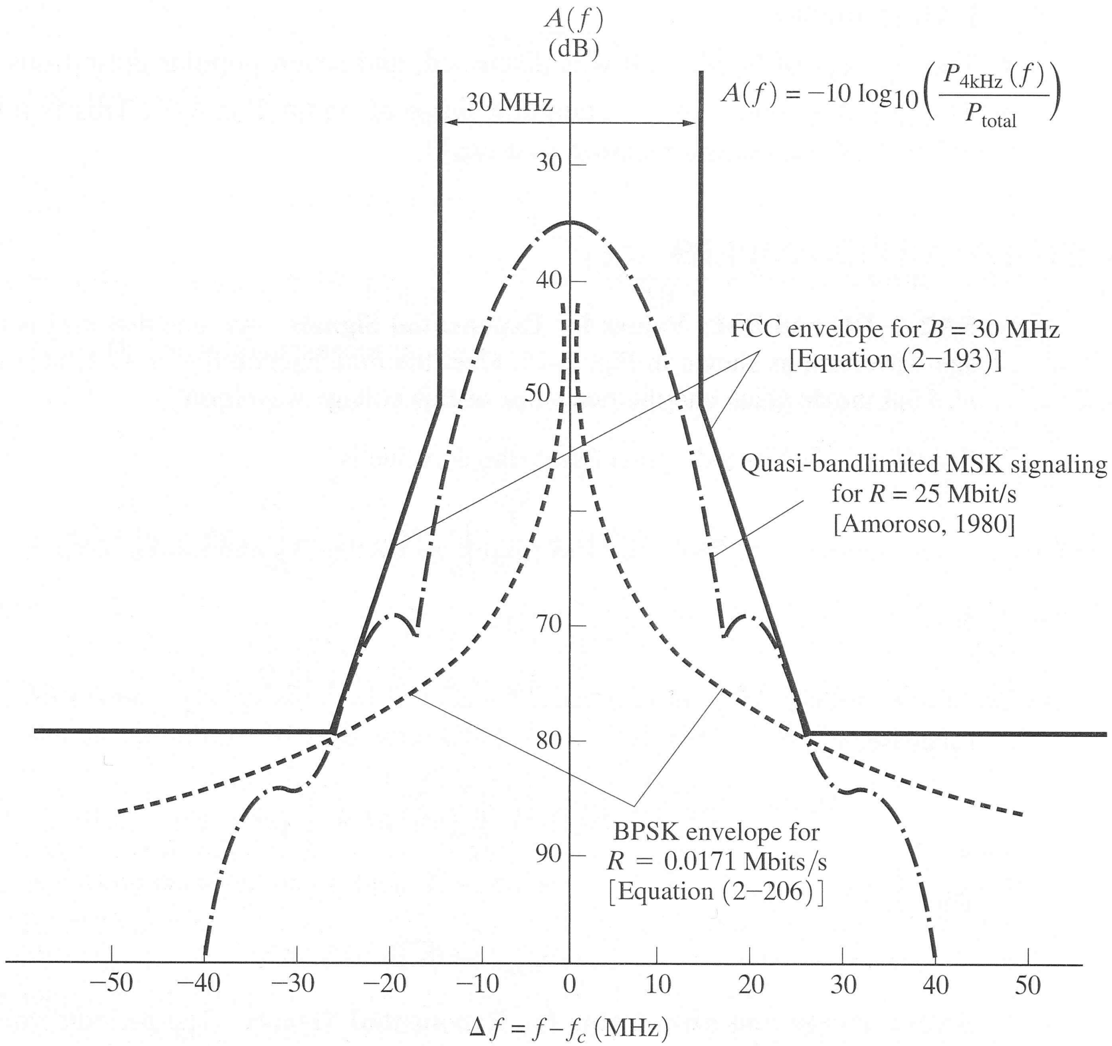

Figure 2–24 FCC-allowed envelope for $B = 30$ MHz.

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