Telecommunicatie A (EE2T11)

Lecture 6 overview:

Pulse Code Modulation (PCM)
- noise in PCM systems
- μ-law and A-law coding
- differential PCM

Delta modulation (DM)
* principle
* noise in DM systems
* comparison PCM v.s. DM
* variable slope DM

Digital signaling
* mathematical description of digital waveforms
* multilevel signaling
* bandwidth of digital signals

EE2T11 Telecommunicatie A

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Colleges en Werkcolleges Telecommunicatie A

Colleges:

Dinsdag	15-3, 29-3	7e en 8e uur, EWI-Pi

Werkcolleges:

Maandag	21-3, 4-4	5e en 6e uur, EWI-Pi

Pulse Code Modulation

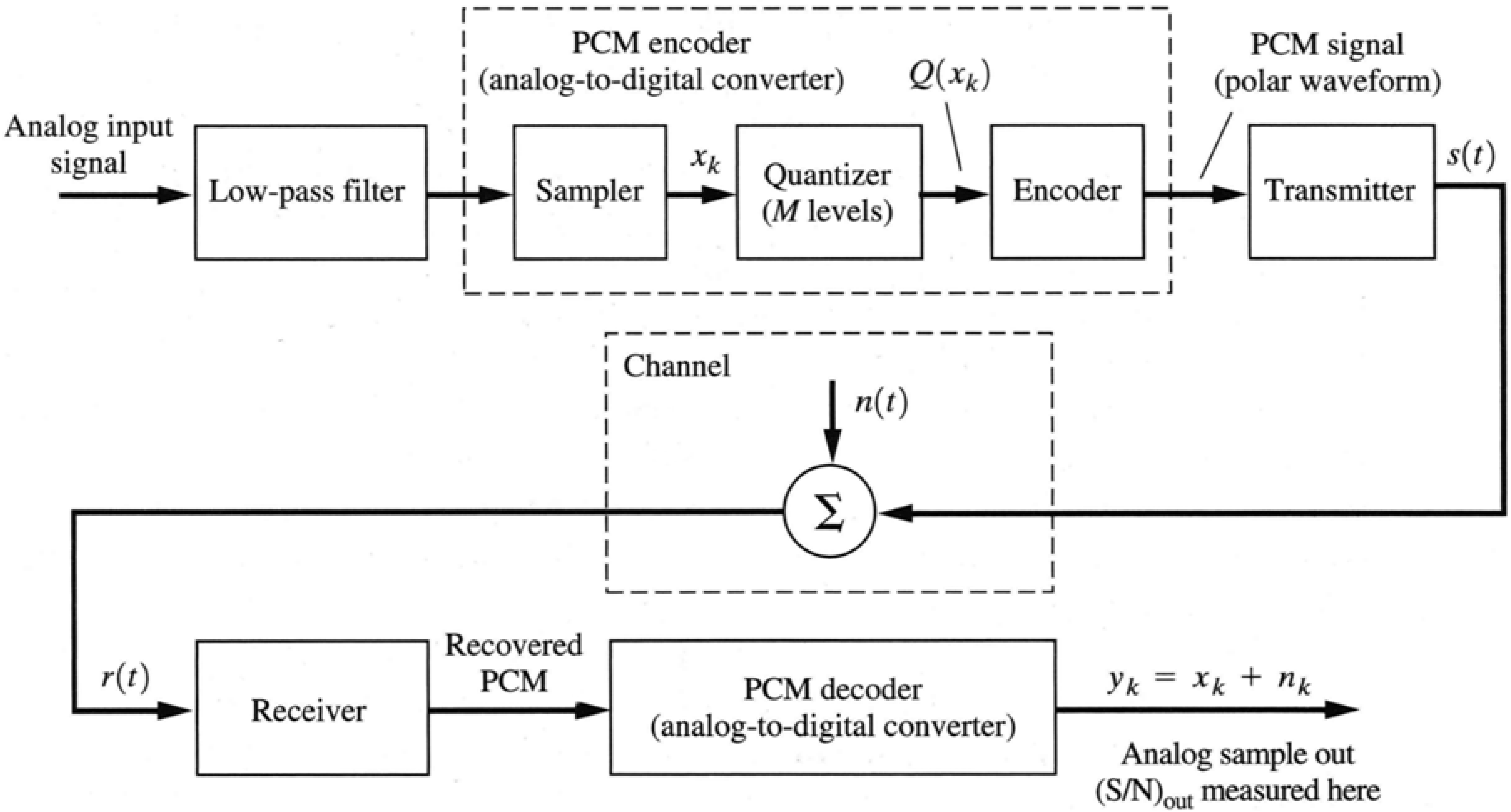

Figure 7–15 PCM communications system.

Noise in a PCM system

In a PCM communication system, the reconstructed signal $y_k = x_k + n_k$ suffers from three sources of noise:

1. Quantization noise: $e_q = Q(x_k) - x_k$

2. Bit Error Noise: $e_b = y_k - Q(x_k)$

   Reconstruction noise due detection errors

3. Overload noise: the input signal is outside the conversion range $-V \leq w(t) \leq V$ of the PCM coder.

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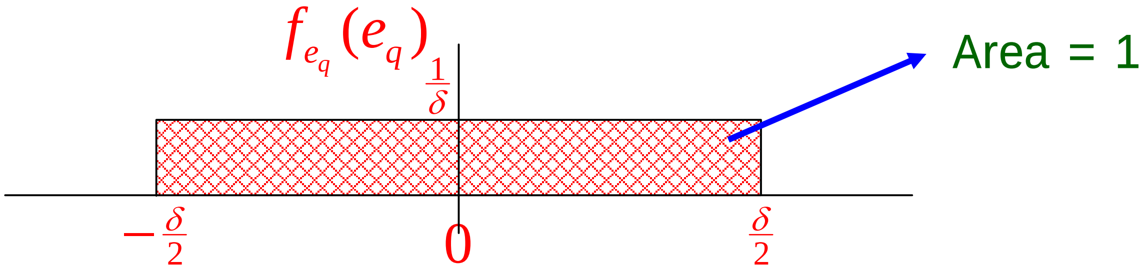
Quantization noise

Quantization noise $e_q = Q(x_k) - x_k$ due to quantization errors is uniformly distributed over $(-\delta/2, \delta/2)$.

The Probability Density Function (PDF) $f_{e_q}(e_q)$ of e_q is given by:



Now we find for the quantization noise power:

$$\overline{e_q^2} = \int_{-\infty}^{\infty} e_q^2 f_{e_q}(e_q) de_q = \int_{-\delta/2}^{\delta/2} e_q^2 \frac{1}{\delta} de_q = \frac{\delta^2}{12} = \frac{V^2}{3M^2}$$

where we used: $\delta = \frac{2 \cdot V}{2^n} = \frac{2 \cdot V}{M}$

Quantization Noise in PCM Systems

Noise in the reconstructed analog signal only due to quantization errors (rounding errors): all bits are received correctly.

Signal-to-Noise Ratio (SNR) for maximum signal level due to quantization errors:

$$\left.\frac{S}{N}\right)_{\max} = \frac{P_{signal-\max}}{P_{noise}} = \frac{V^2}{\overline{e_q^2}} = \frac{V^2}{V^2/3M^2} = 3M^2$$

$M = 2^n \rightarrow$ the number of quantization levels

Signal-to-Noise Ratio (SNR) for average signal level (uniformly distributed) due to quantization errors:

$$\left.\frac{S}{N}\right)_{average} = \frac{P_{signal-av}}{P_{noise}} = \frac{V^2/3}{\overline{e_q^2}} = \frac{V^2/3}{V^2/3M^2} = M^2$$

PCM performance

- With longer PCM words (n bits/sample)
  $\rightarrow$ the number of quantization levels $M = 2^n$ increases
  $\rightarrow$ smaller quantization errors result
  $\rightarrow$ larger SNR due to smaller quantization noise

- Every extra bit:
  $\rightarrow$ the number of quantization levels doubles
  $\rightarrow$ smaller quantization errors result: $\varepsilon_{\max}(n+1) = \varepsilon_{\max}(n)/2$
  $\rightarrow$ 6 dB higher SNR: $SNR(n+1) = SNR(n) + 6\text{ dB}$

$$SNR_{dB} = 6.02 \cdot n + \alpha \quad \text{with } \alpha = \begin{cases} 4.8\text{ dB for maximum signal} \\ 0\text{ dB} \quad \text{for average signal} \end{cases}$$

- The input signal should cover the full input range of the ADC .

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Bit error noise (1)

Noise on the channel may results in bit errors which cause bit error noise in the reconstructed PCM signal: $e_b = y_k - Q(x_k)$

Let the bit error probability be $P_e \; (0 \le P_e \le 1, \text{ with } P_e << 1)$, then the probability of a single error in a PCM-word:

$$P_{ew} = \binom{n}{1} P_e (1 - P_e)^{n-1} \approx n \cdot P_e$$

An error in the j^th bit results in an error voltage of:

$$e_j = 2^{n-j} \cdot \delta = 2^{-(j-1)} V = 2\frac{V}{2^j} \qquad \delta = 2V / 2^n$$

in the received signal.

The value of $\overline{e_j^2}$ averaged over all n bit positions gives the noise power per random bit location.

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Bit error noise (2)

For a single bit error in a PCM-word, the average bit error noise is given by:

$$\overline{e_b^2} = \overline{[y_k - Q(x_k)]^2} = nP_e\overline{e_j^2} \qquad (7.77)$$

with $M = 2^n = 2V/\delta$ and the finite series $\sum_{n=0}^{N} a^n = \frac{a^{N+1}-1}{a-1}$

we find:

$$\overline{e_j^2} = \frac{1}{n}\sum_{j=1}^{n}(\delta \cdot 2^{n-j})^2 \overset{k=n-j}{=} \frac{\delta^2}{n}\sum_{k=0}^{n-1}4^k$$

$$= \frac{\delta^2}{n}\frac{4^n-1}{4-1} = \frac{4}{3}\frac{V^2}{n}\frac{M^2-1}{M^2}$$

With (7.77) we find: $\overline{e_b^2} = nP_e\overline{e_j^2} = \frac{4}{3}V^2P_e\frac{M^2-1}{M^2} \qquad (7.81)$

Total noise power and SNR (1)

Additional noise appears in the reconstructed analog signal due to:

1. Quantization errors (rounding errors)
2. Bit errors during the detection process

Combining the results for $\overline{e_q^2}$ and $\overline{e_b^2}$, we find for $SNR_{pk\,out}$:

$$SNR_{pk\ out} = \frac{V^2}{\overline{n_k^2}} = \frac{V^2}{\overline{e_q^2} + \overline{e_b^2}} = \frac{V^2}{\frac{V^2}{3M^2} + \frac{4}{3}V^2 P_e \frac{M^2-1}{M^2}}$$

$$= \frac{1}{\frac{1}{3M^2} + \frac{4}{3}P_e \frac{M^2-1}{M^2}} = \frac{3M^2}{1+4P_e(M^2-1)}$$

$P_e =$ bit error probability

$M = 2^n \rightarrow$ the number of quantization levels

For $M^2 >> 1$: $\quad SNR_{pk\ out} = \dfrac{3M^2}{1+4P_e M^2}$

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Total noise power and SNR (2)

If $P_e = 0$, (only quantization noise):

$$SNR_{pk\ out} = 3M^2 = 10 \cdot {}^{10}\log 3M^2 = 6.02n + 4.8\ [\text{dB}]$$

Signal-to-Noise Ratio (SNR) for average signal level:

$$\left.\frac{S}{N}\right)_{average} = \frac{M^2}{1 + 4(M^2 - 1)P_e} = M^2\Big|_{P_e=0}$$

6dB-Rule: adding one bit doubles $M \rightarrow \Delta SNR \approx 6\ \text{dB}$

Total noise power and SNR (2)

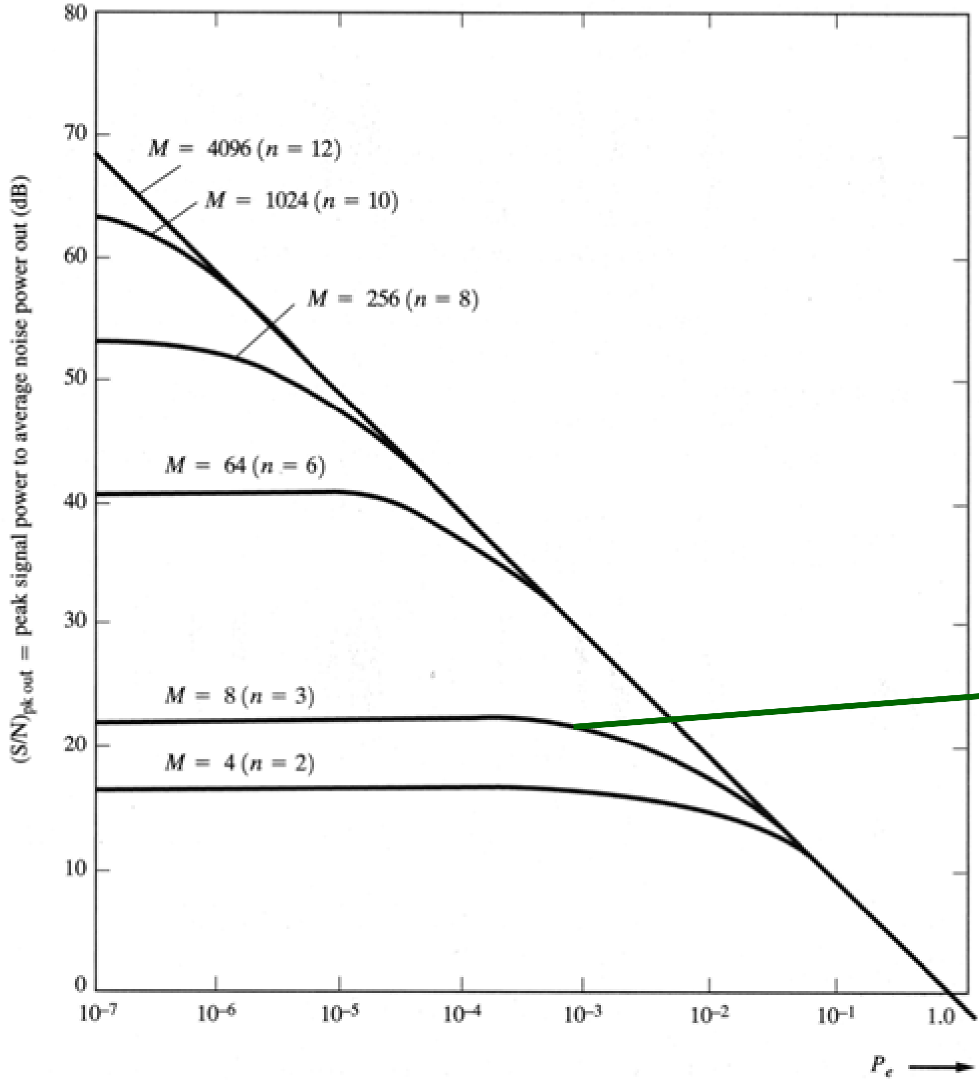

Figure 7–17 $(S/N)_{out}$ of a PCM system as a function of P_e and the number of quantizer steps M.

$$\left.\frac{S}{N}\right)_{\max} = \frac{3M^2}{1+4(M^2-1)P_e}$$

$$4P_e M^2 >> 1$$

$$\Downarrow$$

$$\text{SNR} \approx \frac{3}{4P_e}$$

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Bit error noise (5)

In example 3.4, p. 173: A telephone voice signal (300 – 3400 Hz) is sampled with f_s = 8 ksamp/s and PCM coded with n = 8 bit/samp.

Then: R_b = 64 kbit/s, B_{min} = R_b/2 = 32 kHz (sinc pulses) and $SNR_{pk\ out} = 3(2^8)^2 = 52.9\ \mathrm{dB}$.

Does this design make sense when the BER $P_e = 10^{-4}$ due to noise on the transmission channel?

Answer:

For $P_e = 10^{-4}$, fig. 7-17 clearly shows that taking n > 6 is useless.

Using the **6dB-Rule**: $SNR_{out} = 6.02n + \alpha\ [\mathrm{dB}]$ with $\alpha = 4.77$ for peak SNR and $\alpha = 0$ for average SNR. For n = 6 we find:

$$SNR_{pk\ out} = 6.02n + 4.77 = 40.9\ \mathrm{dB}$$

PCM: advantages/disadvantages

Advantages

1. cheap electronics
2. transparant for different types of information:
   multiplexing of different data; analog as well as digital
3. relative insensitive for errors due to noise
4. error detection/correction possible
5. simple data encryption

Disadvantages

1. Requires much more bandwidth than needed for analog signals

PCM System

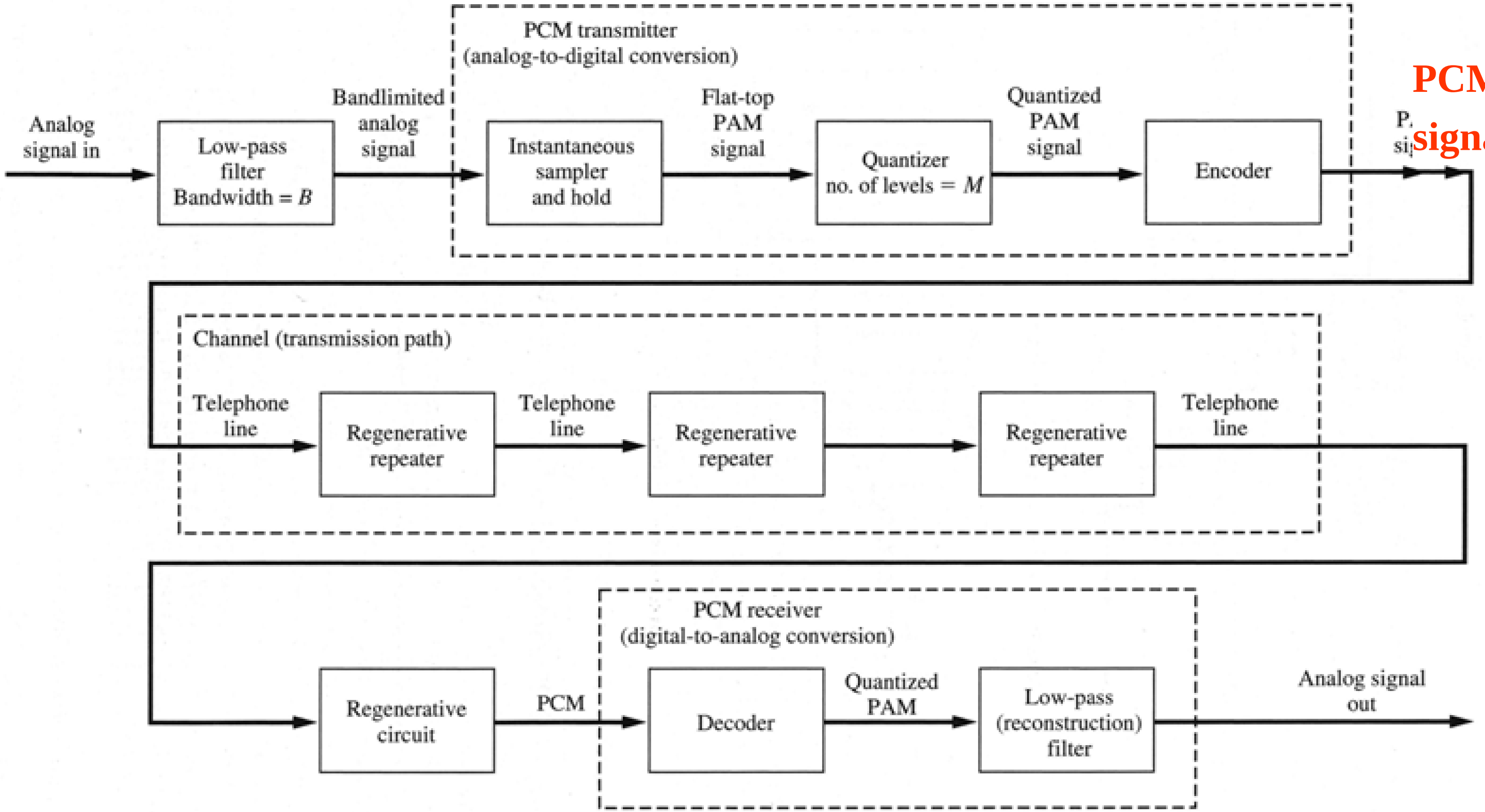


Figure 3–7 PCM trasmission system.

Non-uniform quantizing

Voice signals have a highly non-uniform amplitude distribution.
Low values have a much higher probability than the higher values.
However, we want to understand the "soft speaking lady" with the same quality as "the barking General".

Solution: use a variable step size $\delta \Rightarrow$ non-uniform quantization.
Increase the number of quantization levels for lower amplitudes at the cost of those at higher amplitudes.

At lower amplitudes the step size is decreased (smaller quantization error), whereas it is increased for higher amplitudes $\Rightarrow$ the resulting SNR becomes almost independent of the input level or all input levels are about equally understandable.

Non-uniform quantizing: μ-law and A-law companding

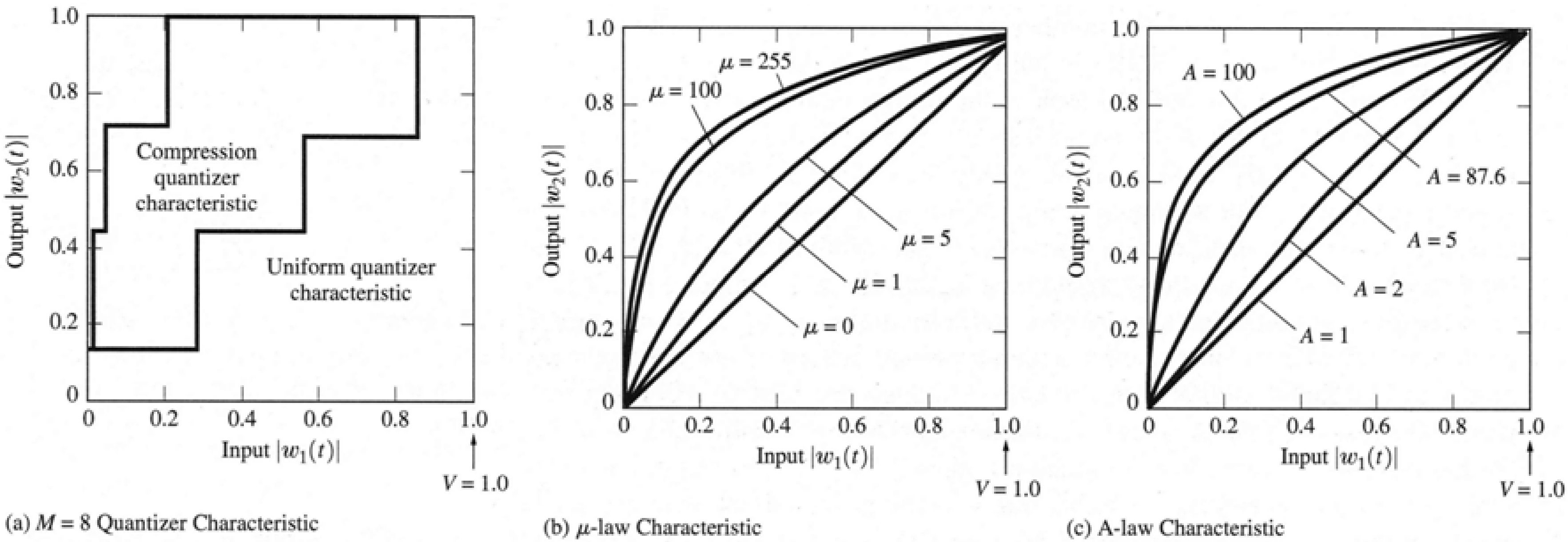


Figure 3–9 Compression characteristics (first quadrant shown).

US: μ -law quantizer

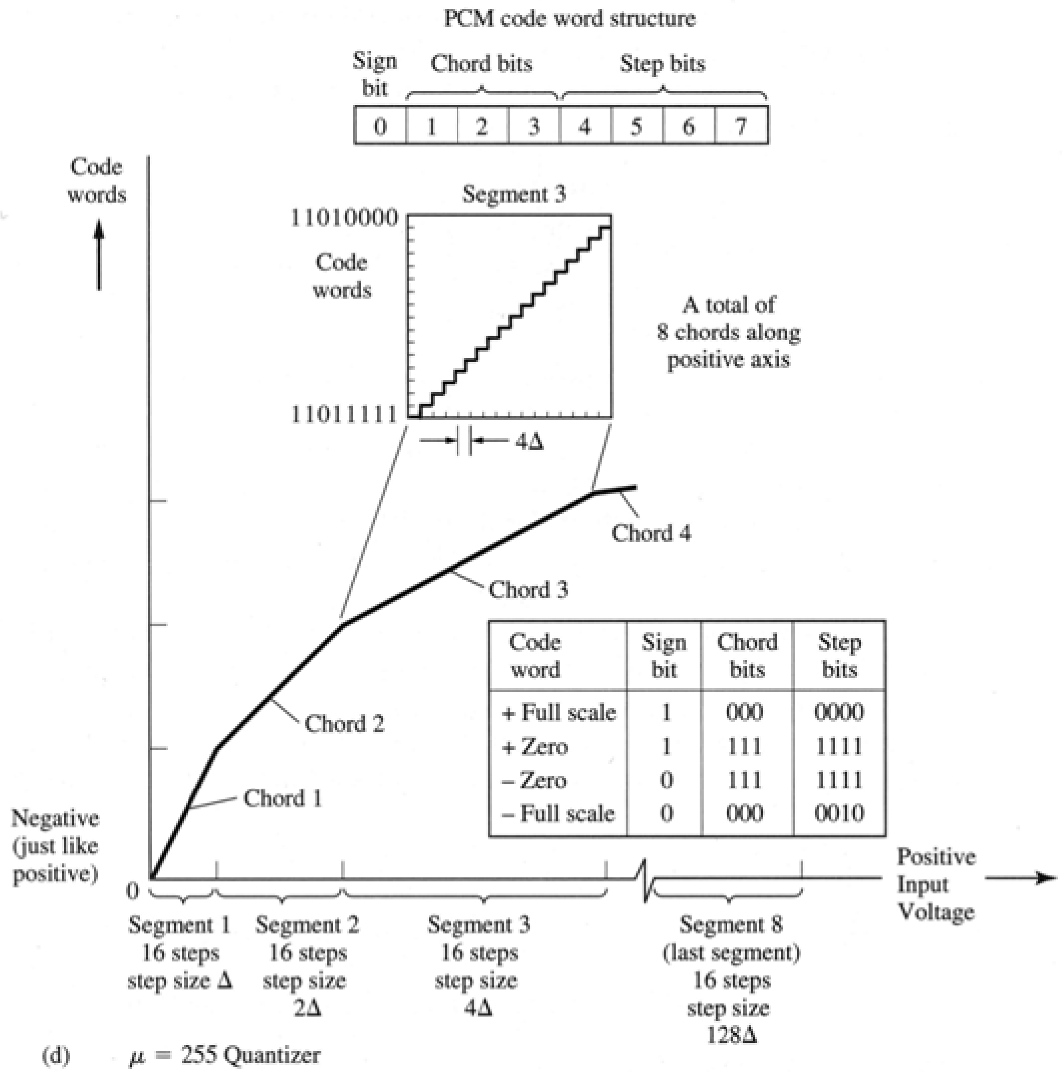
$$|w_{out}(t)| = \frac{\ln(1+\mu|w_{in}(t)|)}{\ln(1+\mu)}$$

$$|w_{in}| \le 1$$

EU: A-law quantizer

$$|w_{out}(t)| = \begin{cases} \dfrac{A|w_{in}(t)|}{1+\ln A} & 0 \le |w_{in}(t)| \le \dfrac{1}{A} \\[2ex] \dfrac{1+\ln(A|w_{in}(t)|)}{1+\ln A} & \dfrac{1}{A} \le |w_{in}(t)| \le 1 \end{cases}$$

Non-uniform quantizing: μ-law and A-law companding



Code word	Sign bit	Chord bits	Step bits
+ Full scale	1	000	0000
+ Zero	1	111	1111
− Zero	0	111	1111
− Full scale	0	000	0010

Figure 3–9 Continued

The non-uniform quantizer is usually approximated by piecewise linear parts. For each part a different step size is chosen.

Non-uniform quantizing: μ-law and A-law companding

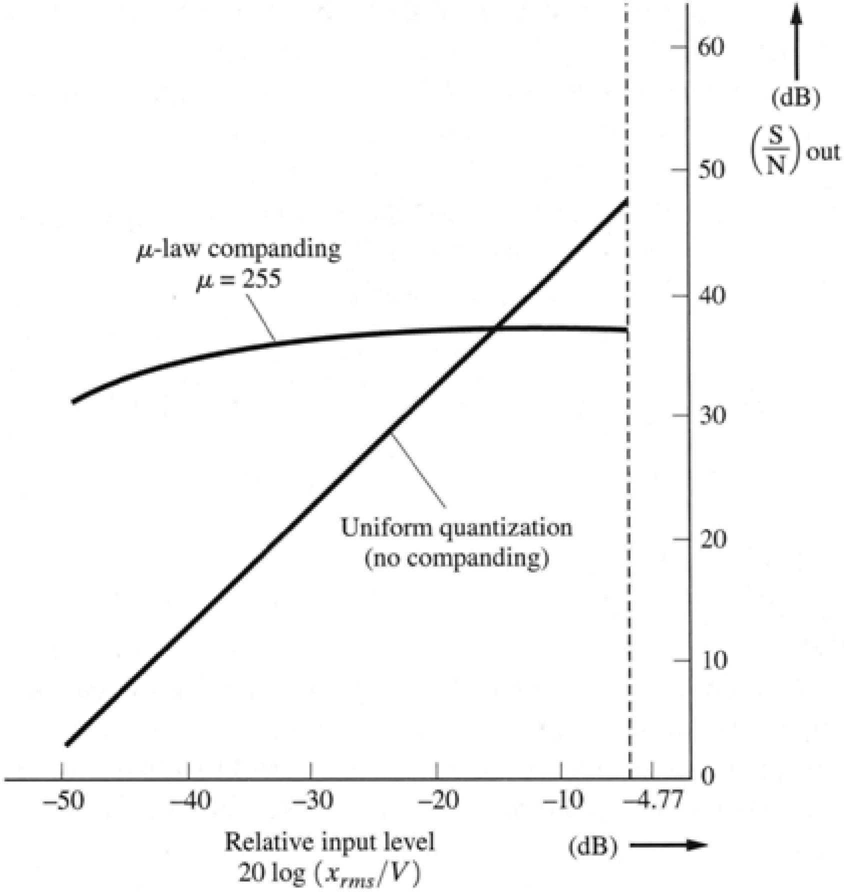


Figure 3–10 Output SNR of 8-bit PCM systems with and without companding.

The non-uniform quantizer can be implemented by a compressor devise with μ -law or A-law characteristics followed by a uniform quantizer.

In the reconstruction an expander has to be used to prevent distortion.

Differential PCM

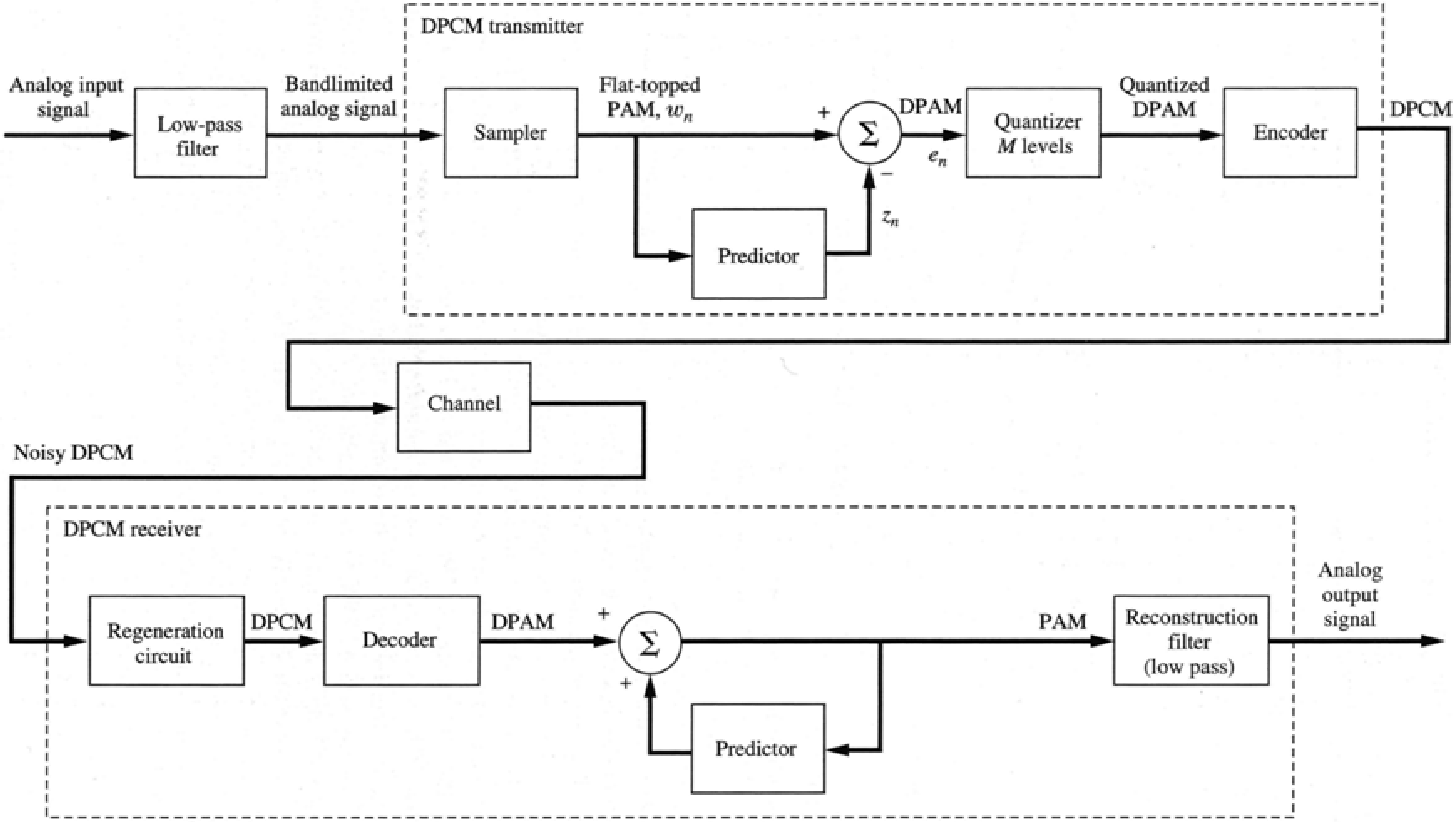


Figure 3–29 DPCM, using prediction from samples of input signal.

Delta modulation (1)

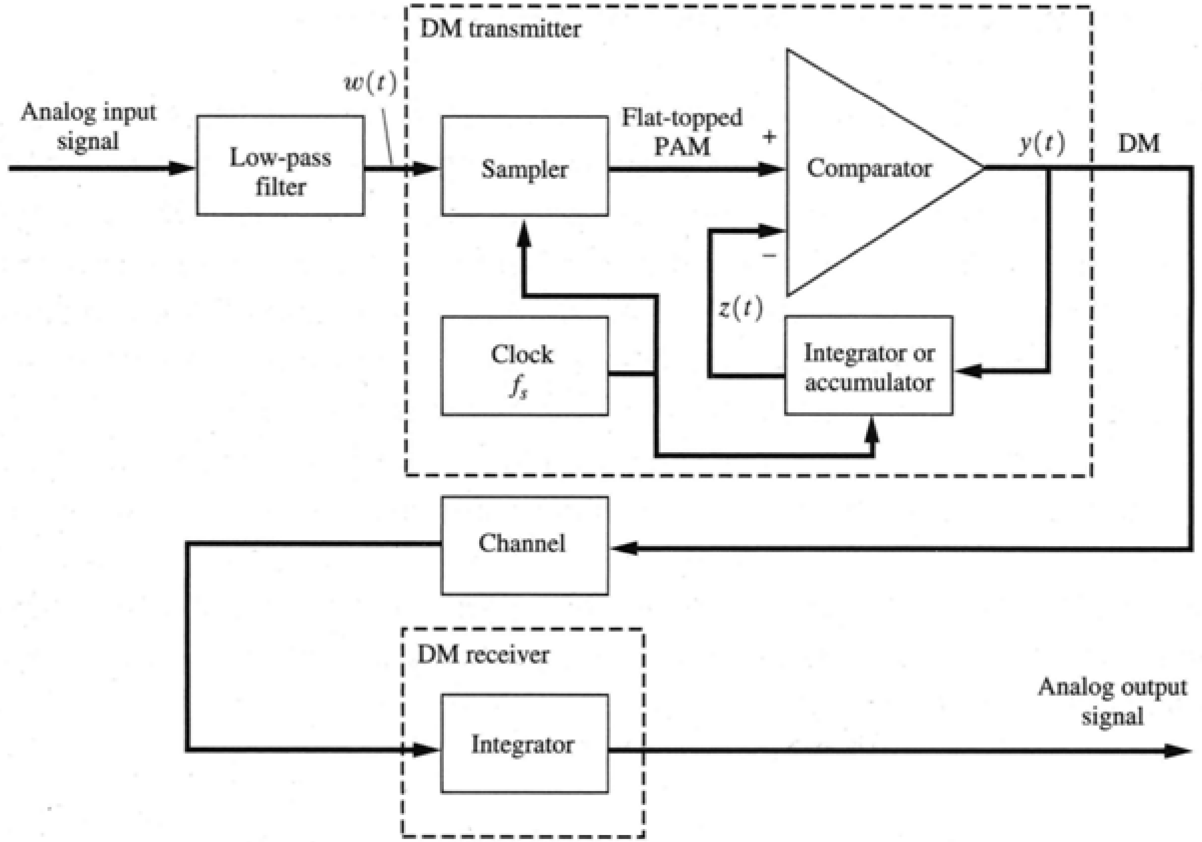


Figure 3–31 DM system.

In simple Delta Modulation one bit per sample is transmitted indicating whether the current level is higher/lower then the reconstructed signal which is used as reference level.

Noise sources in Delta modulation (1)

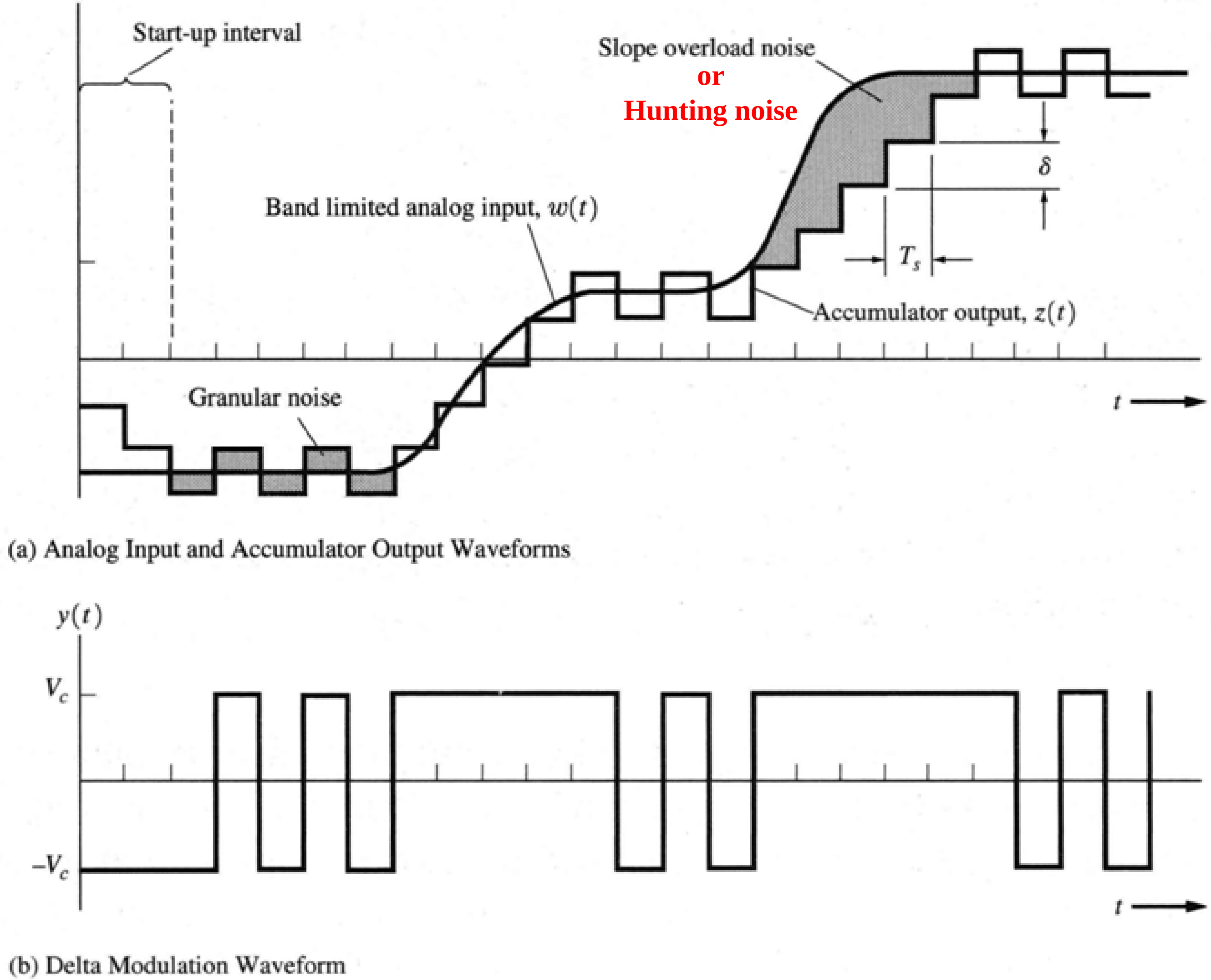


Figure 3–32 DM system waveforms.

Noise sources in Delta modulation (2)

Noise sources in DM are:

1. Slope overload noise or hunting noise:
   $\Rightarrow$ the reference signal cannot follow quick changes of the input signal because the step size δ is too small
   $\Rightarrow$ can be decreased by choosing a larger δ

2 Granular noise:
   $\Rightarrow$ happens when the reference signal is slightly different from the (nearly constant) input signal,
   $\Rightarrow$ decreases with choosing a smaller δ

Noise sources in Delta modulation (3)

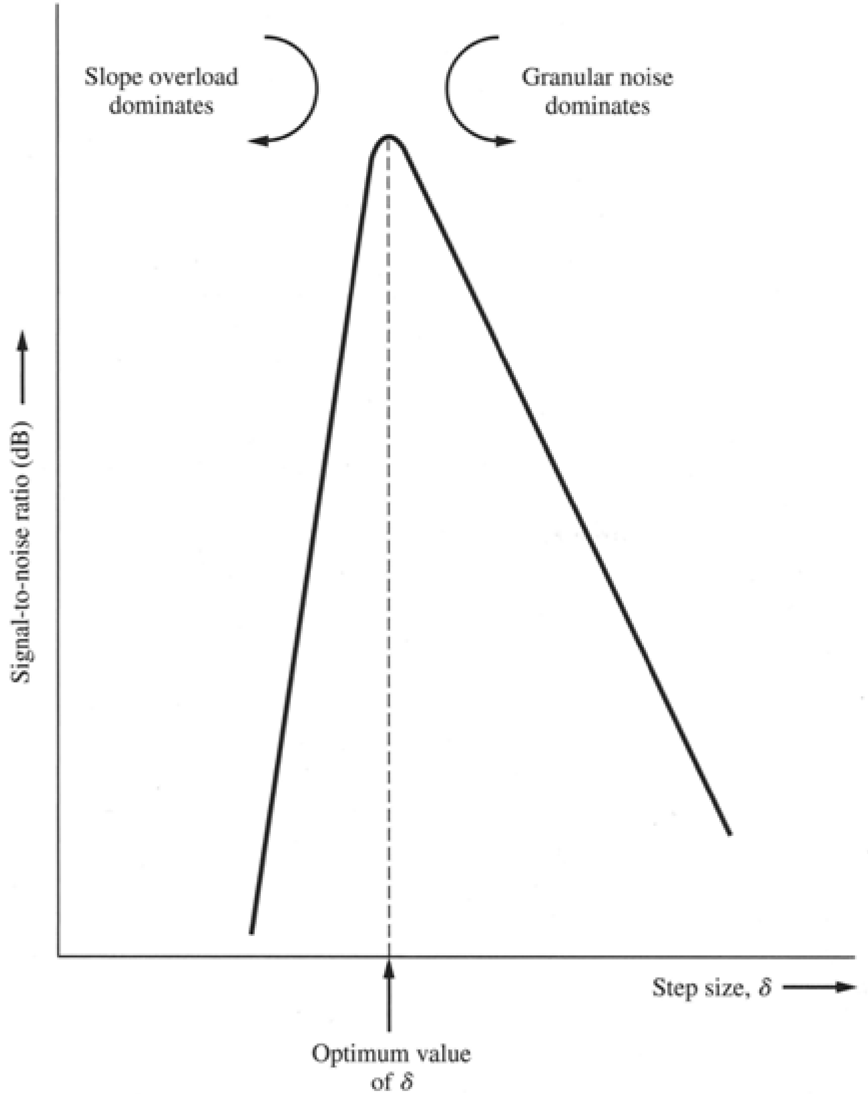


Figure 3–33 Signal-to-noise ratio out of a DM system as a function of step size.

For a given sample frequency, there is an optimum step size δ which maximizes the SNR.

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Design of a DM system (1)

The maximum slope of signal *w(t)* that can be followed by the DM is:

$$\left.\frac{dw(t)}{dt}\right|_{\max} = \frac{\delta}{T_s} = \delta f_s$$

For a sine wave $w(t) = A \sin \omega_a t$

we find the maximum slope from the derivative:

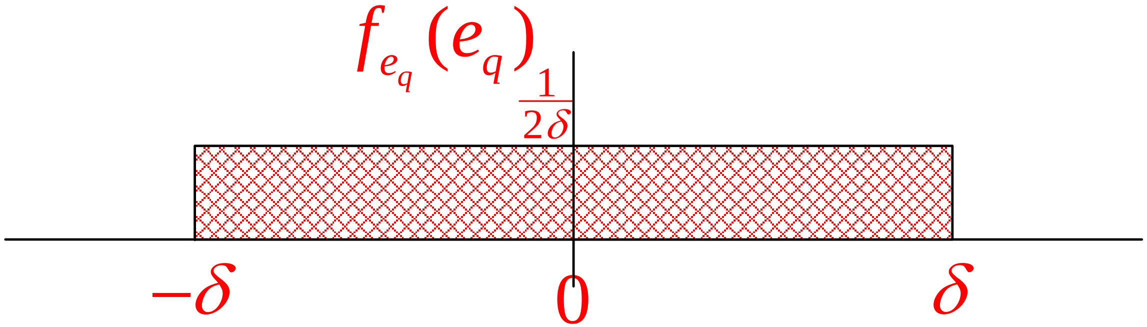
$$\left.\frac{dw}{dt}\right|_{\max} = A\omega_a \cos \omega_a t\Big|_{\max} = A\omega_a$$

No slope overloading if $\delta > \dfrac{\omega_a A}{f_s} = \dfrac{2\pi f_a A}{f_s}$

However, if δ is chosen too large, granular noise will be dominant!

Noise in Delta Modulation (1)

The maximum quantization error due to granular noise in DM is δ (only $\delta/2$ in PCM), which is uniformly distributed over $(-\delta, +\delta)$.

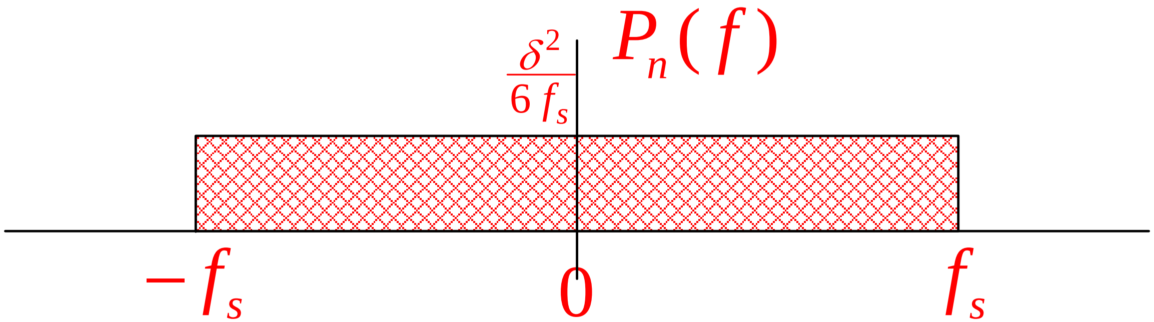


and we find the average noise power as:

$$\overline{e_q^2} = \int_{-\infty}^{\infty} e_q^2 f_{e_q}(e_q) de_q = \int_{-\delta}^{\delta} e_q^2 \frac{1}{2\delta} de_q = \frac{\delta^2}{3} \qquad \left(\overline{e_q^2} = \frac{\delta^2}{12} \quad \text{for PCM} \right)$$

Noise in Delta Modulation (2)

This noise power spectral density is flat over $[-f_s, f_s]$:



$$P_n(f) = \frac{\delta^2}{6f_s} \qquad -f_s \le f \le +f_s$$

If there is no slope overloading!!!

The noise power in a bandwidth $B \le f_s$ is given by:

$$N = \langle n^2 \rangle = \int_{-B}^{B} P_n(f)\,df = \frac{\delta^2 B}{3f_s}$$

SNR for a sine wave with DM

The power of a sine wave $w(t) = A\sin\omega_a t$ is $S = \overline{w^2(t)} = \dfrac{A^2}{2}$

For the noise power we found:

$$N = \frac{\delta^2 B}{3 f_s} = \frac{4\pi^2 A^2 f_a^{\,2} B}{3 f_s^{\,3}} \quad \Rightarrow \quad \delta = \frac{2\pi f_a A}{f_s} \text{ :just no slope overloading}$$

and the SNR is

$$B = f_a$$

$$SNR_{out} = \frac{S}{N} = \frac{3}{8\pi^2}\frac{f_s^{\,3}}{f_a^{\,2} B} \quad \Rightarrow \quad SNR_{out\_\max} = \frac{3}{8\pi^2}\frac{f_s^{\,3}}{f_a^{\,3}}$$

$$SNR_{out} = \text{Const.* (Oversampling factor)}^3$$

A 2x increase of f_s results in an 8x increase of SNR (9 dB !) because δ becomes smaller and the noise power is spread out over a larger bandwidth.

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SNR for voice signals with DM

For voice signals with a maximum frequency of 4 kHz, most of the power is found near 800 Hz. Dimensioning a DM system based on a sine wave at 4 kHz with maximum amplitude will be over-kill.

Hardly any slope overloading will occur when we choose the step size as:

$$\delta = \frac{2\pi 800 W_p}{f_s}$$

with $W_p = \max\{w(t)\}$

and the SNR is $SNR_{out} = \frac{<w^2(t)>}{N} = \frac{3 f_s^3}{(1600\pi)^2 B} \frac{<w^2(t)>}{W_p^2}$

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Comparison of PCM v.s. DM (1)

For PCM:

- bit rate $R_b = n \cdot f_s \geq 2nB$
- with an extra bit:
  - $\rightarrow$ the number of quantization levels $M = 2^n$ doubles
  - $\rightarrow$ quantization errors reduce with a factor 2

$$SNR_{dB} = 6.02 \cdot n + \alpha \quad \text{with } \alpha = \begin{cases} 4.8 \text{ dB for maximum signal} \\ 0 \text{ dB} \quad \text{for average signal} \end{cases}$$

$\rightarrow$ SNR increases with 6 dB

For DM:

- bit rate: $R_b = f_s$
- SNR increases with 9 dB when the bit rate ($= f_s$) doubles

Doubling the bit rate results in:

- $6n$ dB gain for PCM
- 9 dB gain for DM

Comparison of PCM v.s. DM (2)

Let for the transmission of a HiFi audio signal with a bandwidth of 20 kHz, an average SNR of 50 dB (not too much) be required.

With PCM:

- sample frequency $f_s \geq 2B = 40 \text{ kHz}$
- from $SNR_{av} = M^2 = 10^5 \Rightarrow M \geq \sqrt{10^5}$

the required word length is: $n = \left\lceil {}^2\log M \right\rceil = \left\lceil {}^2\log\sqrt{10^5} \right\rceil = \lceil 8.30 \rceil = 9 \ \dfrac{\text{bits}}{\text{sample}}$

and the resulting bit rate is $R_b = nf_s = 9 \cdot 40 \cdot 10^3 = 360 \text{ kbit/s}$

With DM:

- from the SNR expression it follows:

$$f_s = \sqrt[3]{\frac{SNR_{av} \cdot f_a^3 \cdot 8\pi^2}{3}} = \sqrt[3]{\frac{10^5 \cdot (20 \cdot 10^3)^3 \cdot 8\pi^2}{3}} = 2.72 \text{ MHz}$$

- the required bit rate is $R_b = f_s = 2.72 \text{ Mbit/s}$

Almost 8x as much as for PCM!

Continuously Variable Slope DM (1)

In order to minimize the effect of slope overload noise, the step size δ can be made adaptive: Adaptive Delta Modulation (ADM).

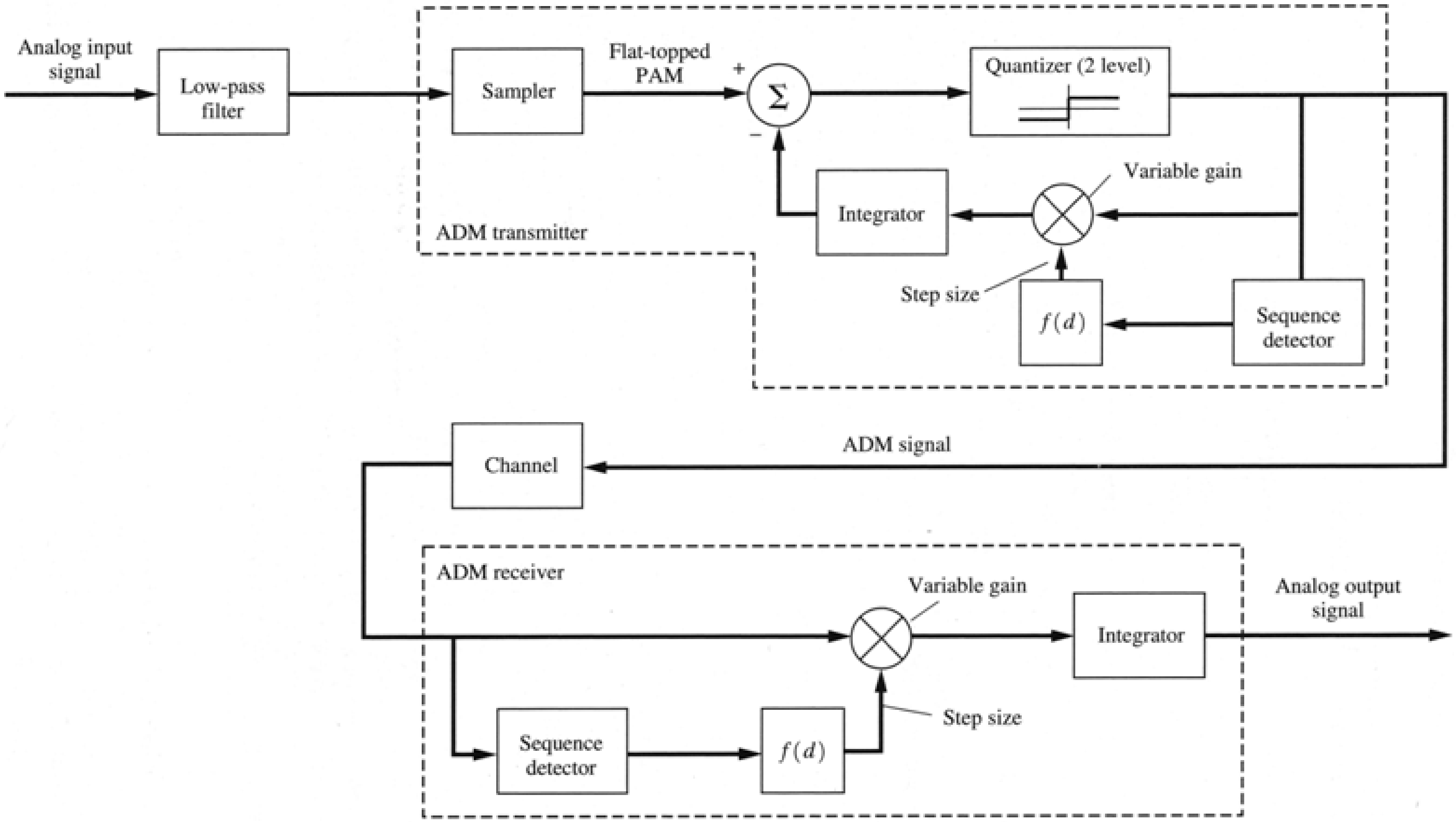

Figure 3–34 ADM system.

Continuously Variable Slope DM (2)

TABLE 3–7 STEP-SIZE ALGORITHM

Data Sequence[a]				Number of Successive Binary 1's or 0's	Step-Size Algorithm, $f(d)$
×	×	0	1	1	δ
×	0	1	1	2	δ
0	1	1	1	3	2δ
1	1	1	1	4	4δ

[a] ×, do not care.

- when strings of 3 consecutive 1's or 0's occur: step size $\rightarrow 2\delta$

- when strings of 4 consecutive 1's or 0's occurs: step size $\rightarrow 4\delta$

- when the bit value changes: step size $\rightarrow \delta$

Digital signaling (1)

- How do we describe the waveform of a signal carrying digital information?
- What is the bandwidth of such a waveform?

The waveform of N symbols (N dimensions) of a digital signal can be written as a sequence of N orthogonal terms:

$$w(t) = \sum_{k=1}^{N} w_k \varphi_k(t) \qquad 0 \le t \le T_0$$

w_k represents a symbol value which contains the digital information.
$\varphi_k(t)$ are orthogonal functions (in general analog waveforms), like a time slot, frequency, phase or a code, with:

$$\int \varphi_k(t)\varphi_l^*(t)dt = \begin{cases} 1 & \text{for } k = l \\ 0 & \text{for } k \neq l \end{cases} \qquad k = 1, 2, \ldots\ldots N$$

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Digital signaling (2)

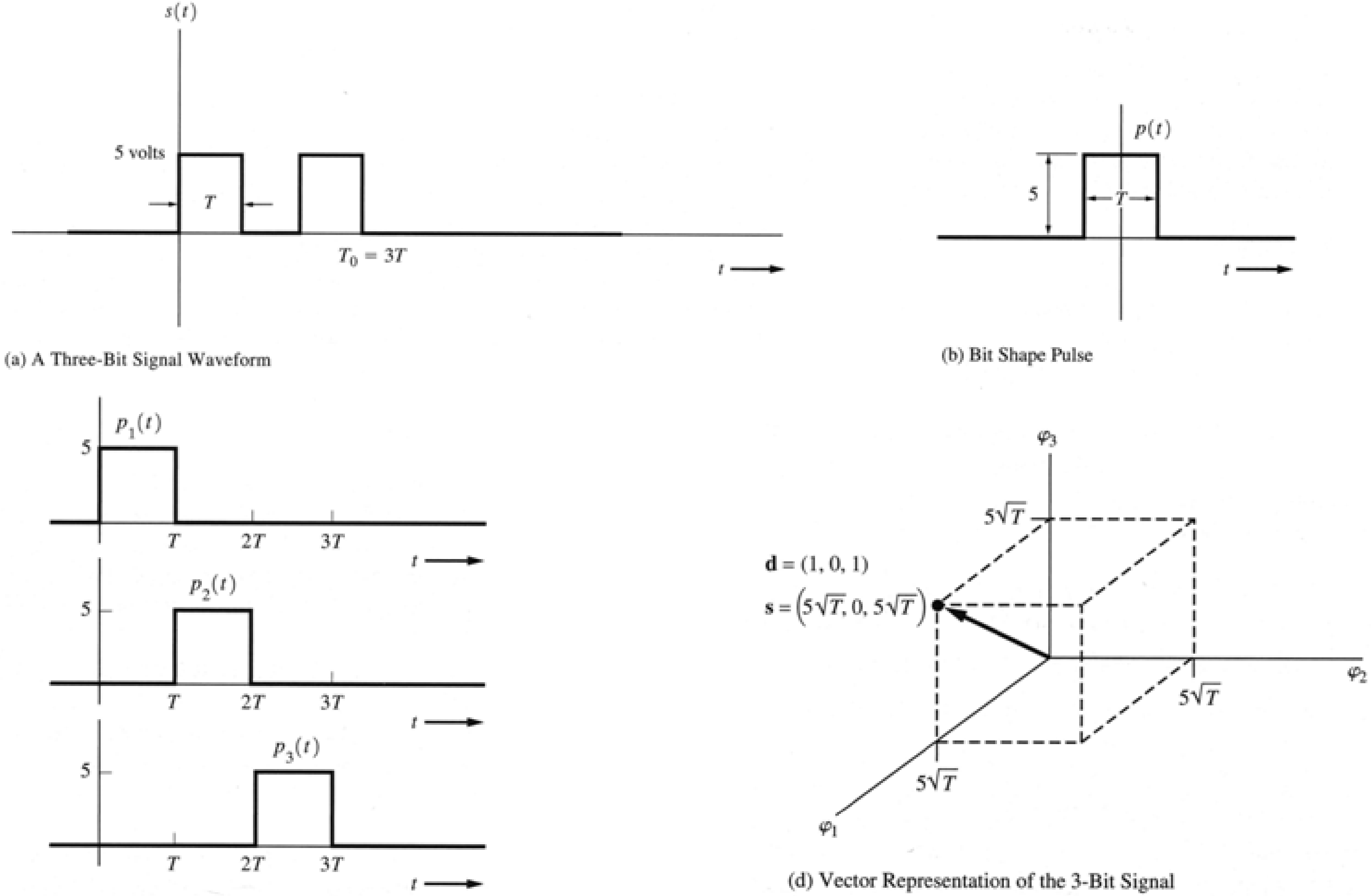


Figure 3–11 Representation for a 3-bit binary digital signal.

Digital signaling (3)

The functions $\varphi_k(t)$ (k = 1, .. N) span an N-dimensional space.
A message of N symbols can be represented by a vector in this N-dimensional space:

$$\underline{w} = \sum_{k=1}^{N} w_k \varphi_k$$

How can we detect the data after transmission over a channel?
By correlation with the complex conjugate of the orthogonal functions:

$$w_k = \int_0^{T_0} \underline{w} \cdot \varphi_k^*(t) dt \quad \text{for } k = 1, 2, .., N$$

This is optimal detection for a channel with Additive White Gaussian Noise (AWGN): "matched-filter" detector.

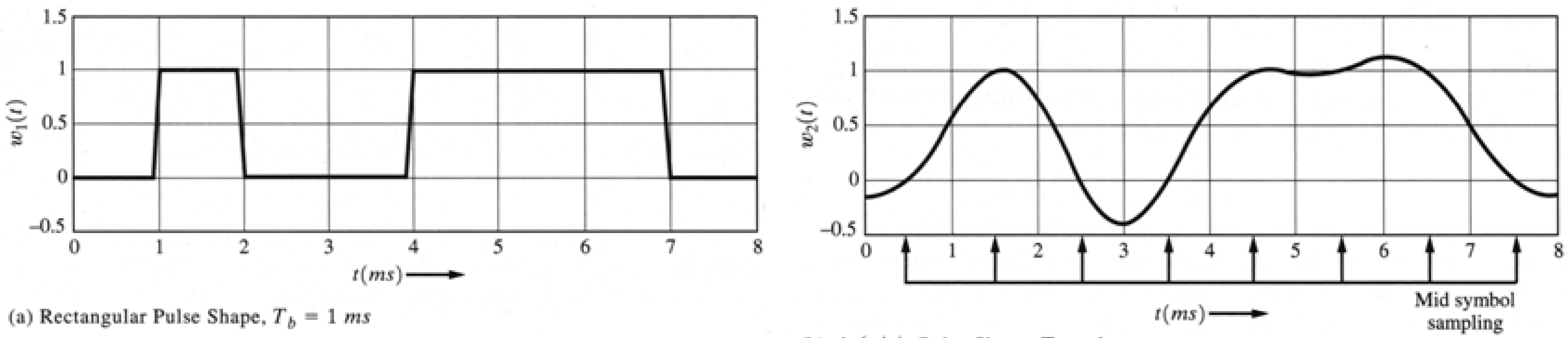
Digital signaling (5)

The bandwidth required to transmit a digital signal does not only depend on N and T_0, but also on the selected waveform $\varphi_k(t)$.

From the dimensionality theorem we find:

$$B_T \geq \frac{N}{2T_0} = \frac{D}{2} \quad [\text{Hz}]$$

Minimum transmission bandwidth is obtained for $\varphi_k \equiv$ sinc-pulse.



(a) Rectangular Pulse Shape, $T_b = 1\ ms$

(b) $\sin(x)/x$ Pulse Shape, $T_b = 1\ ms$

Figure 3–12 Binary signaling (computed).

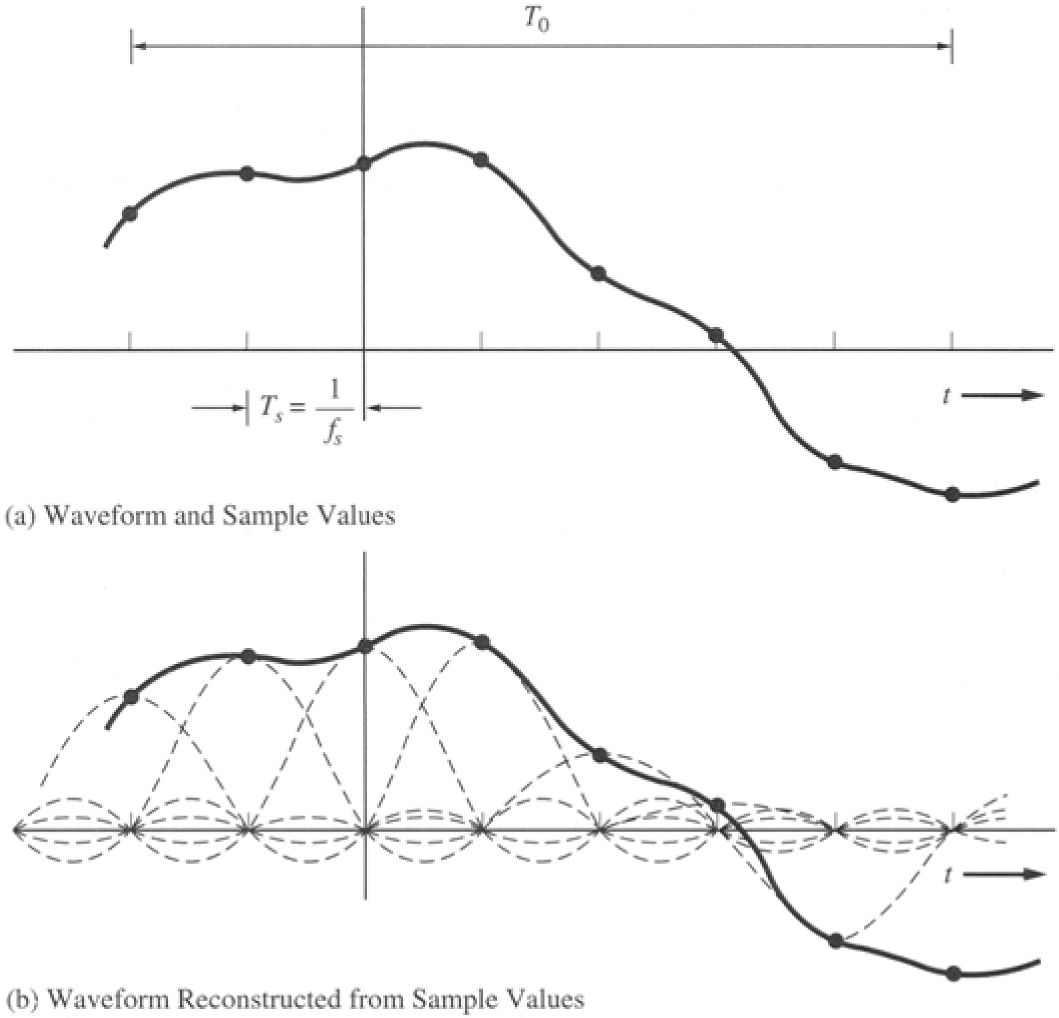


Figure 2–17 Sampling theorem.

Figure 2-17.b shows a periodic sequence of sinc pulses.

The repetition frequency is

$$f_s = \frac{1}{T_s}$$

Digital signaling (4)

What is the bitrate of a digital source transmitting N symbols in a time span T_0 from an alphabet with size M?

For a digital source transmitting N symbols in a time span T_0 from an alphabet with size M, we find:

- the symbol rate: $D = \dfrac{N}{T_0}$ baud (symb/s)
- the number of bits transmitted per symbol is: $n = {}^2\log M$
- the bit rate: $R = \dfrac{n \cdot N}{T_0} = n \cdot D$ bit/s

Digital signaling (6)

In multi-level signaling, the amplitude of the orthogonal functions can take on one of multiple discrete values. In this way multiple bits per symbol can be transmitted.

Multi-level signaling can be exploited to reduce the dimensionality of the signal and still transmit the same information:
- in a shorter time than T_0 but using the same BW.
- in the same T_0 but using less BW.

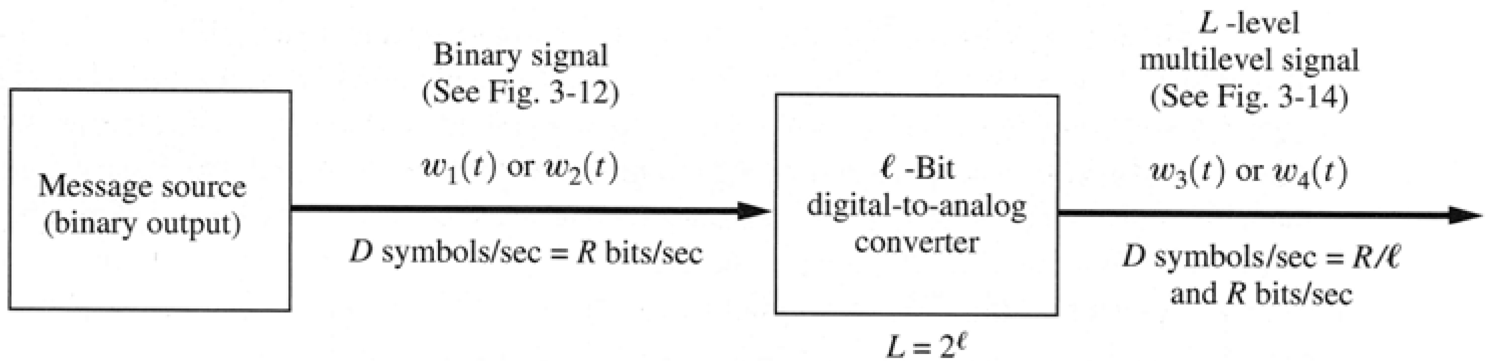
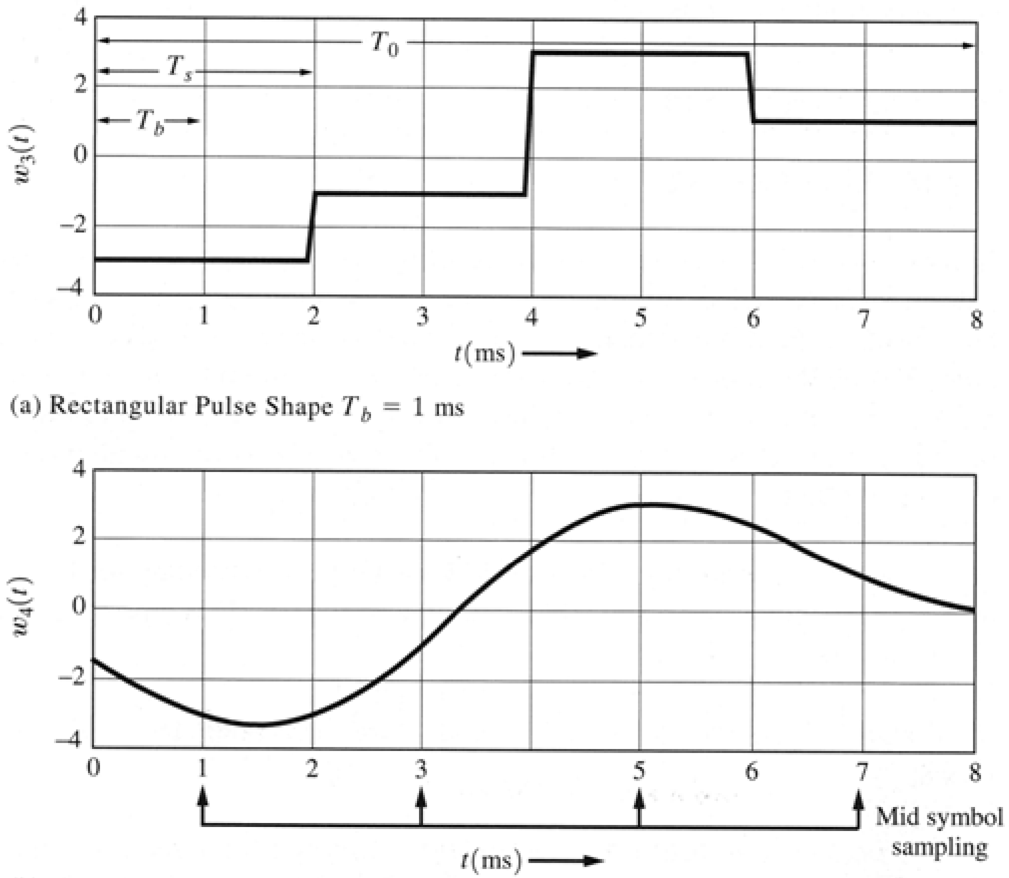


Figure 3–13 Binary-to-multilevel signal conversion.

Digital signaling (7)



(a) Rectangular Pulse Shape $T_b = 1$ ms

(b) sin(x)/x Pulse Shape, $T_b = 1$ ms

Figure 3–14 $L = 4$-level signaling (computed).

Digital signaling: example (1)

Digital source: every T_0 = 8 ms, N = 1 symbol is transmitted out of M = 256 possible symbols.

Symbol rate: $D = \frac{N}{T_0} = \frac{1}{8 \cdot 10^{-3}} = 125$ baud (symb/s)

Bit rate $R = \frac{nN}{T_0} = nD = \frac{8}{8 \cdot 10^{-3}} = 1 \text{ kbit/s}$

A symbol represents $n = {}^{2}\log M = 8$ bits.

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Digital signaling: example (2)

Binary signaling: number of levels L = 2
- per binary symbol $l = {}^2\log L = 1$ bit is transmitted
- one information symbol contains n = 8 bits

$\Rightarrow$ channel symbol rate: $D' = \dfrac{nN}{lT_0} = \dfrac{nN}{T_0} = R = 1\ \text{kbaud} \equiv 1\ \text{kbit/s} \Rightarrow T_s = T_b$

• Rectangular pulses (steep skirts $\rightarrow$ high frequencies)
- $BW_{abs} = \infty$, first null-bandwidth = 1 kHz
- detection moment is less critical

• Sinc-pulses (rounded pulses)
- $BW_{abs} = D'/2 = 500\ \text{Hz} = R/2$
- detection moment is critical

$$B_{T\_\min} \geq \frac{D'}{2} = \frac{1}{2T_s} = 500\ \text{Hz}$$

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Digital signaling: example (3)

Multi-level signaling: number of levels L
- per binary symbol $l = {}^2\log L$ bits are transmitted
  For example: $L = 4 \Rightarrow l = 2$ bits/symbol:
  two successive bits determine one channel symbol: $T_s = 2T_b$

11 $\rightarrow$ +3 V
10 $\rightarrow$ +1 V
00 $\rightarrow$ -1 V
01 $\rightarrow$ -3 V

Symbol rate: $D'' = \dfrac{nN}{lT_0} = \dfrac{8}{2 \cdot 8 \cdot 10^{-3}} = 500 \text{ Baud}$

Bit rate: $R'' = lD'' = R = 1 \text{ kbit/s}$ (does not change of course)

Bandwidth: $B_T \geq \dfrac{D''}{2} = \dfrac{nN}{2lT_0} = 250 \text{ Hz}$

By using multi-level signaling, the number of dimensions, and thus the required Bandwidth, is reduced:

$$D_{\min} = \frac{N_{\min}}{T_0} = \frac{1}{T_0} = 125 \text{ Baud for } L = 256$$

$$B_{\min} = \frac{D_{\min}}{2} = 62.5 \text{ Hz (for sinc-pulsen)}$$