Telecommunications Techniques (EE2T21)

Lecture 13 overview:

Bit error probability for digital modulation techniques

*** Matched filter**

*** Bit Error Probability**

- AWGN: (non-) matched filter

- colored noise

BER in baseband systems

*** unipolar signals**

*** polar signals**

*** bi-polar signals**

EE2T21 Telecommunication Techniques

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Colleges en Instructies Telecommunicatie B

Colleges:

Maandag	30-5, 6-6	5e+6e uur, EWI-CZ Chip

Instructies:

Dinsdag	31-5	7e+8e uur, EWI-CZ Pi
Maandag	13-6	5e+6e uur, EWI-CZ Chip

Signal detection quality (1)

Signal quality at the receiver is determined by:

- received signal power
- noise- and interference power at the receiver input

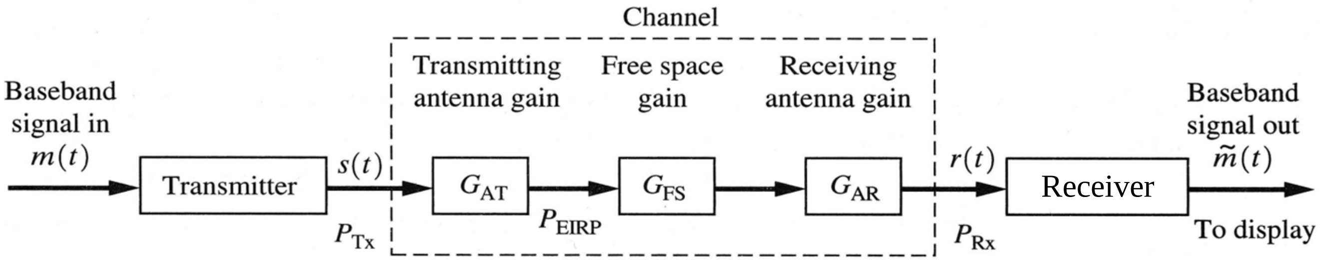


Figure 8–16 Block diagram of a communication system with a free-space transmission channel.

Carrier-to-Noise ratio at the detector input:

$$\frac{C}{N} = \frac{\text{received signal power}}{\text{noise power}}$$

Determines:
- SNR after detection for analogue signals
- bit error probability (BER) for digital signals

Signal detection quality (2)

Digital modulation: relation between C/N and bit error rate (BER): $BER = f(E_b / N_0)$

$$\frac{C}{N} = \frac{E_b / T_b}{N_0 B_N} = \frac{E_b R_b}{N_0 B_N} \quad \Rightarrow \quad R_b = \text{bitrate}$$

$$\frac{E_b}{N_0} = \frac{P_{EIRP} \cdot G_{FS} \cdot G_{AR}}{k \cdot T_{syst} \cdot R_b}$$

$$\frac{E_b}{N_0} = \frac{\text{bit energy}}{\text{noise PSD}}$$

Should be maximized

A larger data rate (with the same quality) can be obtained by:

- increasing EIRP (TX-power or TX-antenna gain)
- increase of the RX-antenna gain
- lower system noise

Matched filter

How can we maximize the SNR for the detection of pulse-like signals at the input of a decision circuit? This will result in the best estimate or reconstruction of the transmitted message! We can calculate C/N using the method of Chapter 8, which assumes an "ideal" rectangular filter. But is this the best filter for pulse-like signals?

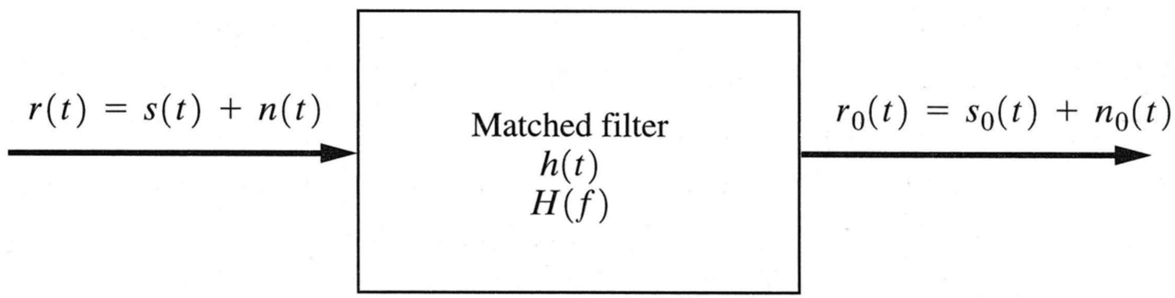
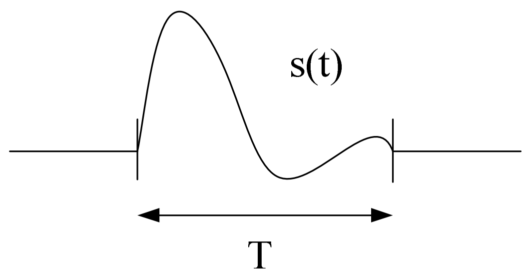


Figure 6–15 Matched filter.



Problem: optimize *h(t)*, *H(f)* so that $\left(\frac{S}{N}\right)_{out} = \frac{\overline{s_o^2(t)}}{\overline{n_o^2(t)}}$ is maximized at $t = t_0$.

The aim is to obtain a sample of a time limited signal (symbol, radar pulse) which is above the noise as much as possible at $t = t_0$.

Note, this filter may distort the signal pulse shape!

Derivation matched filter (1)

Signal at $t = t_0$:

$$s_o(t_0) = h(t) * s(t)\Big|_{t=t_0}$$

Parseval

$$= \int_{-\infty}^{\infty} h(\lambda)s(t_0 - \lambda)d\lambda = \int_{-\infty}^{\infty} H(f)S(f)e^{2\pi j f t_0} df$$

Noise power at $t = t_0$: $\overline{n_o^2(t_0)} = \overline{n_o^2(t)} = \int_{-\infty}^{\infty} |H(f)|^2 P_n(f)df$

Now we find the SNR as:

$$\left(\frac{S}{N}\right)_{out}(t_0) = \frac{\left|\int_{-\infty}^{\infty} H(f)S(f)e^{2\pi j f t_0} df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f)df}$$

Maximize SNR by choosing optimum $H(f)$ for given $S(f)$ and $P_n(f)$.

Matched Filter (MF)

Derivation matched filter (2)

Let us write:

$$\left(\frac{S}{N}\right)_{out}(t_0) = \frac{\left|\int_{-\infty}^{\infty} H(f)S(f)e^{2\pi jft_0}df\right|^2}{\int_{-\infty}^{\infty} |H(f)|^2 P_n(f)df} = \frac{\left|\int_{-\infty}^{\infty} A(f)B(f)df\right|^2}{\int_{-\infty}^{\infty} |A(f)|^2 df}$$

when we choose: $A(f) = H(f)\sqrt{P_n(f)}$ and $B(f) = \dfrac{S(f)e^{2\pi jft_0}}{\sqrt{P_n(f)}}$.

Using Schwartz's inequality:

$$\left|\int_{-\infty}^{\infty} A(f)B(f)df\right|^2 \le \int_{-\infty}^{\infty} |A(f)|^2 df \cdot \int_{-\infty}^{\infty} |B(f)|^2 df$$

where equality holds when: $A(f) = K \cdot B^*(f)$

Derivation matched filter (3)

Then we find: $\left(\frac{S}{N}\right)_{out}(t_0) \leq \int_{-\infty}^{\infty} |B(f)|^2\, df = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{P_n(f)} df$

and the SNR is maximized when $A(f) = K \cdot B^*(f)$, so when:

$$H(f) = K \cdot \frac{S^*(f) e^{-2\pi j f t_0}}{P_n(f)}$$

This is the *Matched filter*.

Discussion: What do we learn from $H(f)$?

$H(f)$ is small for frequencies with a lot of noise and little signal power and $H(f)$ is large where there is a lot of signal power: as could be expected!

The given proof is pure mathematical.

The question is whether such a filter can be implemented in practice?

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Matched filter and white noise

In the case of white noise: $P_n(f) = \frac{N_0}{2}$ and $H(f) = \frac{2K}{N_0} S^*(f) e^{-j\omega t_0}$

Taking the inverse Fourier transform:

$$h(t) = F^{-1}\{H(f)\} = \frac{2K}{N_0} \int_{-\infty}^{\infty} S^*(f) e^{-j\omega t_0} e^{+j\omega t} df$$

$$= \frac{2K}{N_0} \left[\int_{-\infty}^{\infty} S(f) e^{2\pi j f (t_0 - t)} df \right]^* = \frac{2K}{N_0} s^*(t_0 - t)$$

So for a real signal $s(t)$, the impulse response of the matched filter is given by:

$$h(t) = K' s(t_0 - t)$$

The signal shape reversed in time!

SNR after matched-filtering

The SNR achieved after *matched-filtering* is:

$$\left(\frac{S}{N}\right)_{out} = \int_{-\infty}^{\infty} \frac{|S(f)|^2}{N_0/2} df = \frac{2}{N_0} \int_{-\infty}^{\infty} s^2(t) dt = \frac{2E_s}{N_0}$$

related to energy

Parseval

$\Rightarrow$ independent of signal shape

$\Rightarrow$ only depends on the energy in the signal!

Relation with $SNR_{in} \Rightarrow$ we measure over a period T and determine the noise power in the signal bandwidth W:

$$\left(\frac{S}{N}\right)_{out} = \frac{2E_s}{N_0} = 2\frac{E_s/T}{N_0 W} \cdot T \cdot W = 2\left(\frac{S}{N}\right)_{in} \cdot T \cdot W$$

time-bandwidth product
of the input signal

Example: Integrate & Dump filter (1)

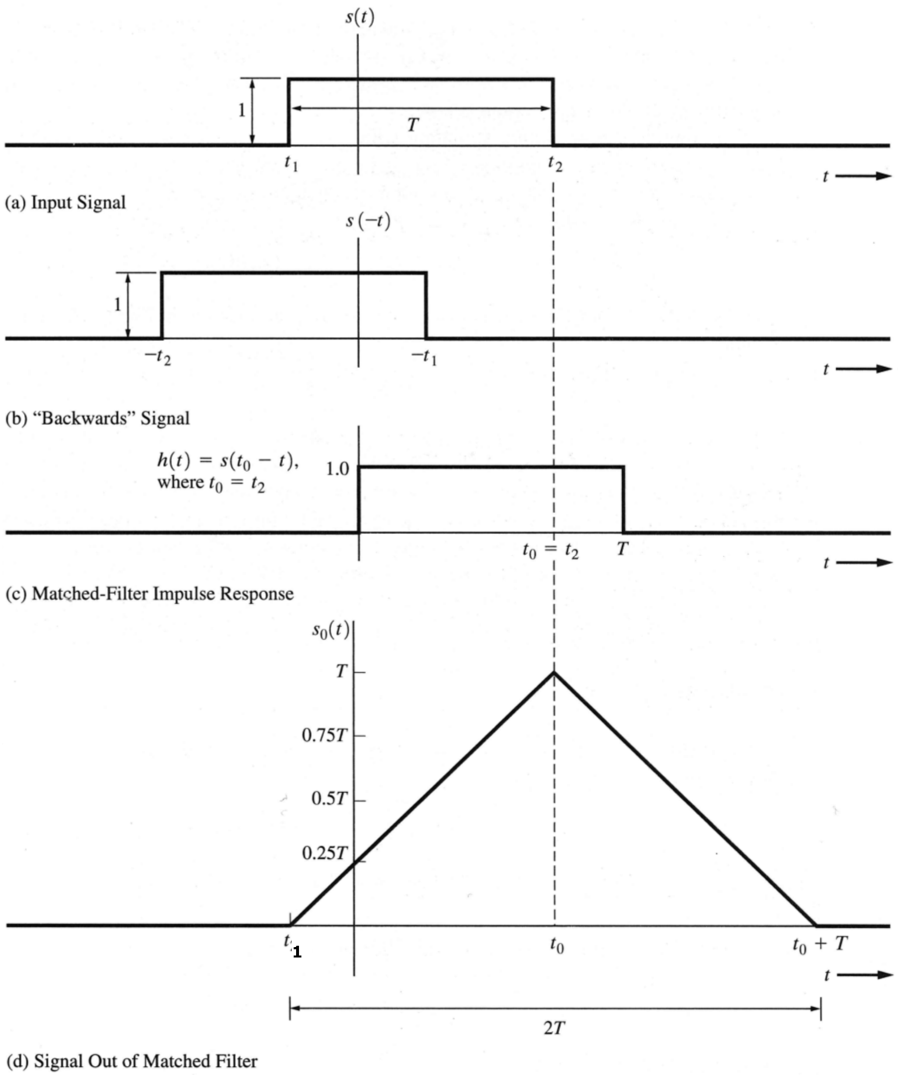

Figure 6–16 Waveforms associated with the matched filter of Example 6–11.

For a rectangular pulse: $h(t) = s(t_0 - t)$

$\Rightarrow$ the output is triangular with a width of $2T$, so a shape different from the input signal pulse, but with maximum SNR on $t = t_0$.

$\Rightarrow$ for causality: $t_0 \geq t_2$ otherwise a response before $t = t_1$ when the signal does not exist!

How can we approximate non-causal filters?

$$\Rightarrow \; r_o(t_0) = r(t) * h(t)\big|_{t_0}$$

$$= \int_{-\infty}^{\infty} r(\lambda) h(t_0 - \lambda) d\lambda$$

$$= \int_{t_0 - T}^{t_0} r(\lambda) d\lambda \;\; \Rightarrow \text{I\&D-filter}$$

Example: Integrate & Dump filter (2)

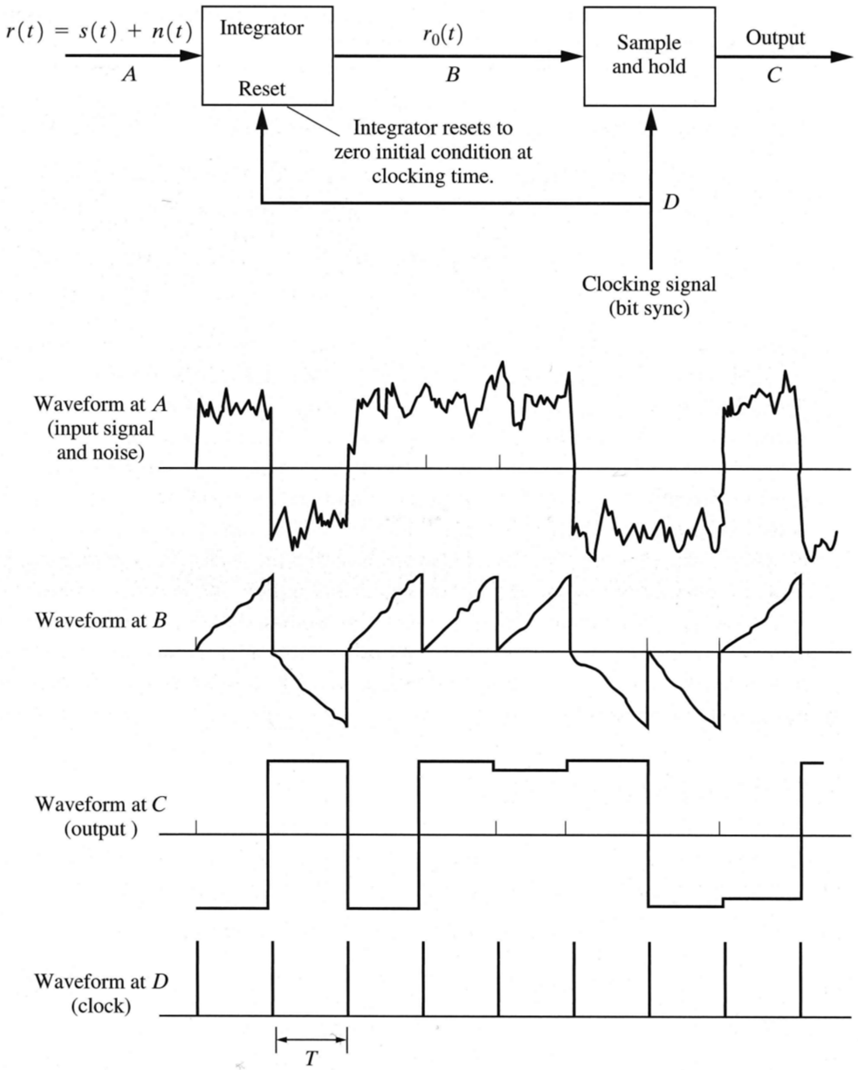

Figure 6–17 Integrate-and-dump realization of a matched filter.

Matched filtering = correlation processing (1)

As we have seen for the I&D-filter:

$$r_o(t_0) = r(t) * h(t)\Big|_{t_0} = \int_{-\infty}^{\infty} r(\lambda)h(t_0 - \lambda)d\lambda$$

Since: $h(t) = \begin{cases} s(t_0 - t) & 0 \le t \le T \\ 0 & \text{otherwise} \end{cases}$

we get:

$$r_o(t_0) = \int_{t_0 - T}^{t_0} r(\lambda)s(\lambda)d\lambda \quad \Rightarrow$$ Correlation processor is matched!

Matched filtering = correlation processing (2)

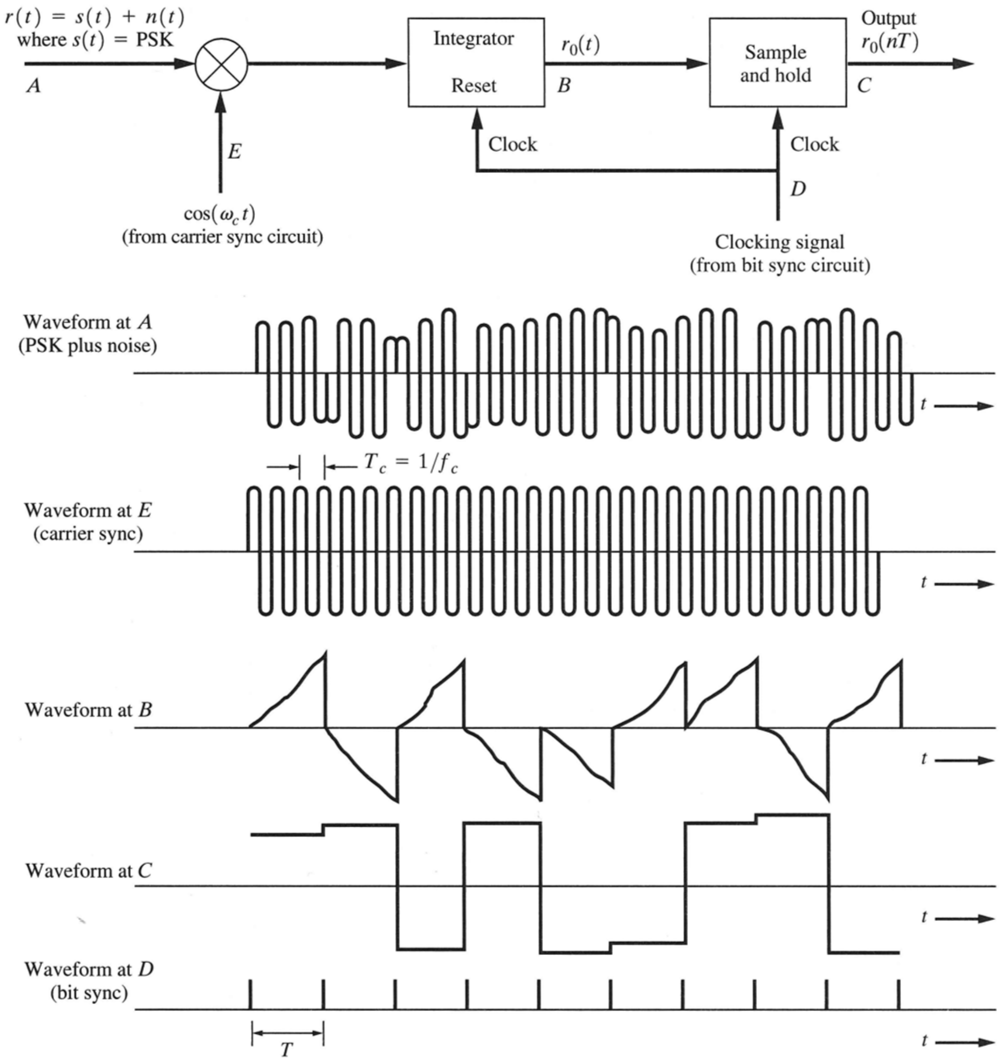

Figure 6–19 Correlation (matched-filter) detection for BPSK.

For PRK, BPSK:

$$s(t) = \begin{cases} A\cos\omega_c t & \text{"1"} \\ -A\cos\omega_c t = A\cos(\omega_c t + \pi) & \text{"0"} \end{cases}$$

We can use: $s'(t) = A\cos\omega_c t$ as template.

and choose:

$$h(t) = A\cos\omega_c(t_0 - t) \qquad t_0 - T \le t \le t_0$$

What does

$$h(t) = s^*(t_0 - t)$$

mean for modulated signals?

Bit error probability in digital modulation systems

Two important design criteria for digital communication systems:

- required bandwidth (bandwidth efficiency)
- performance of the system with noise (power efficiency)

In the previous lecture we defined a number of digital modulation schemes (OOK, BPSK, FSK, QPSK, M-PSK, QAM, etc.)

In the following, we will focus on the detection of these signals and the performance in terms of bit error probability.

First we discuss a general frame work on decision making for baseband signals: What is the probability of an error?

General binary communication system model (1)

$$s(t) = \begin{cases} s_1(t) & 0 \le t \le T \quad \text{"1, mark"} \\ s_2(t) & 0 \le t \le T \quad \text{"0, space"} \end{cases}$$

When $s_1(t) = -s_2(t)$

$\Rightarrow$ the signals are anti-podal

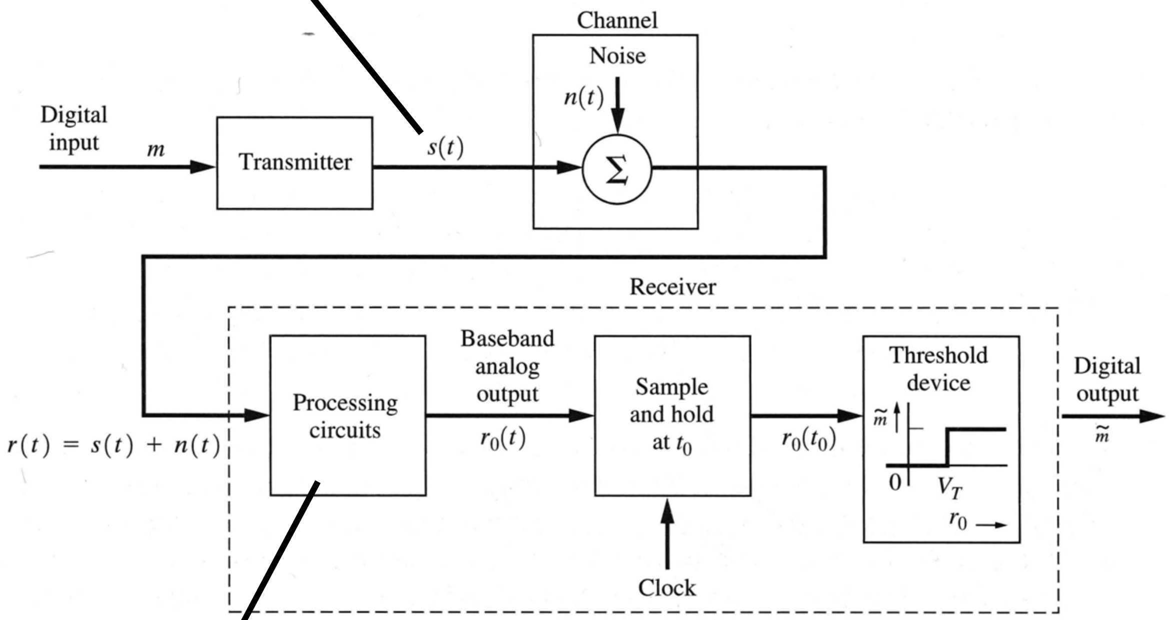

Figure 7–1 General binary communication system.

For bandpass signals the processing circuit contains a superheterodyne receiver (p. 279, fig. 4.29) with a mixer that produces a baseband signal.
This is an analog signal.

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General binary communication system model (2)

A binary signal plus noise at the receiver input results in an analog baseband signal at the input of the sample circuit. The value at sample time t_0 is called the test statistic:

$$r_0 = r_0(t_0) = \begin{cases} r_{01} = \{r_0 \mid s_1\} = s_{01} + n_{01} & \text{"1"} \\ r_{02} = \{r_0 \mid s_2\} = s_{02} + n_{02} & \text{"0"} \end{cases}$$

with corresponding conditional probability density functions $f\{r_0 \mid s_i\}$ for r_{0i} $\{i = 1, 2\}$. Often, these PDF's are Gaussian.

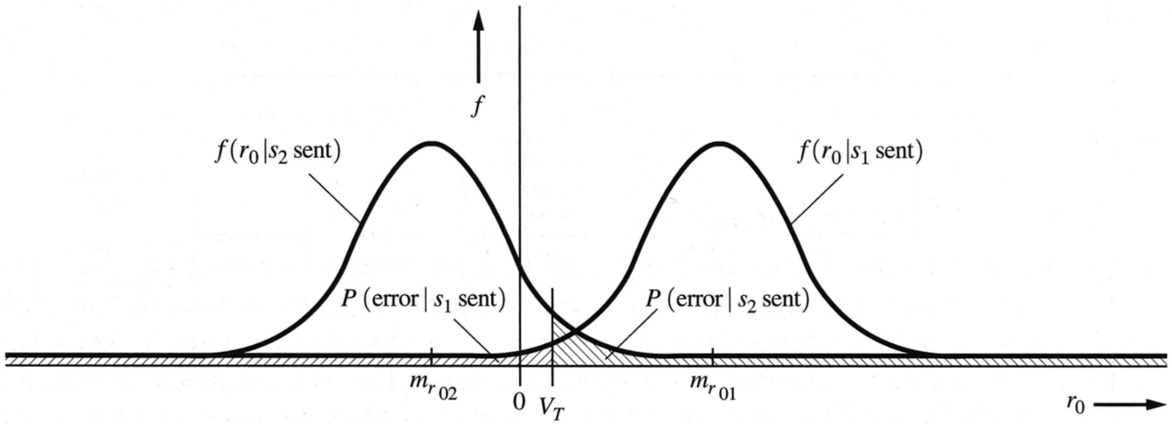

Figure 7–2 Error probability for binary signaling.

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General binary communication system model (3)

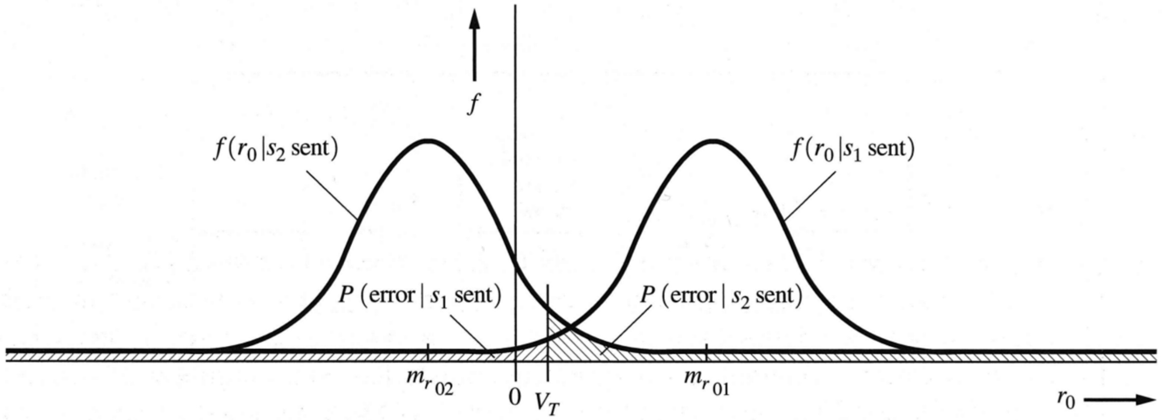

Figure 7–2 Error probability for binary signaling.

Assume: $\left.\begin{matrix} s_{01} > V_T \text{ for } s_1 \\ s_{02} < V_T \text{ for } s_2 \end{matrix}\right\}$, then we find the following

conditional error probabilities:

$$P(error\,|\,s_1) = \int_{-\infty}^{V_T} f(r_0\,|\,s_1)dr_0$$

$$P(error\,|\,s_2) = \int_{V_T}^{\infty} f(r_0\,|\,s_2)dr_0$$

and the total unconditional error probability:

$$P(error) = P(error\,|\,s_1)P(s_1) + P(error\,|\,s_2)P(s_2)$$

Bit error probability is also often indicated as bit error rate (BER).

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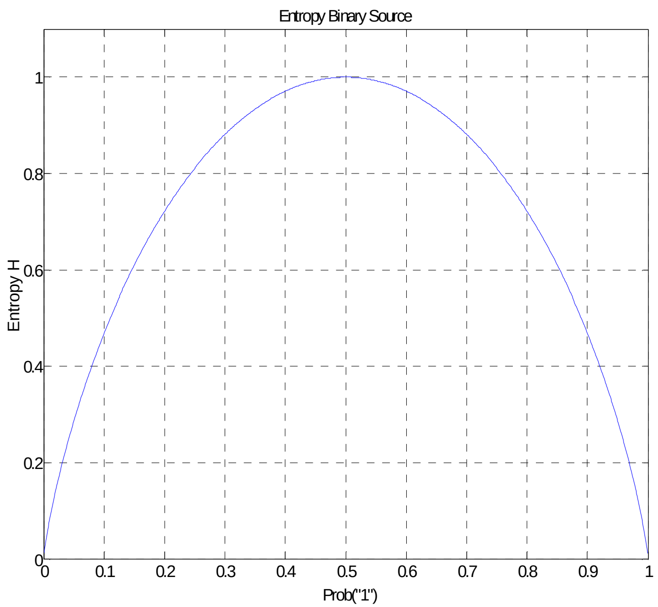
General binary communication system model (4)

Efficient use of the symbols requires a good coding of the source information signal (see eqn. 1-8, p. 17). A measure for the average source information per symbol is the entropy:

$$H = \sum_{j=1}^{2} P_j I_j = -\sum_{j=1}^{2} P_j \log_2 P_j \quad \text{[bits]}$$



The best source statistic (maximum H) has equi-probable symbols: $P_1 = P_2 = 0.5$

With a different source statistic less bits per symbol are transmitted (on average).

Maximum surprise for the detector means that most uncertainty is removed by a successful detection.

Additive White Gaussian Noise

Additive White Gaussian Noise:

- is additive to the signal, i.e. independent of that signal
- has a flat (white) power spectral density $P_n(f) = N_0/2$ (double sided)
- its amplitude has a Gaussian (Normal) probability density function with a variance $\sigma_0^2 = \overline{n_0^2}$ equal to the noise power in the observed bandwidth.

When the noise process at the receiver input is a zero mean WSS Gaussian process, and the receiver is linear, then the noise at the output will also be Gaussian. Only the Gaussian statistic has this characteristic.

Bit error probability (1)

Test statistic: $r_0 = s_0 + n_0$ with $s_0 = \begin{cases} s_{01} = \overline{r_0(s = s_1)} \\ s_{02} = \overline{r_0(s = s_2)} \end{cases}$

$\Rightarrow$ *conditional pdf*: $$f(r_0 \mid s_i) = \frac{1}{\sigma_0\sqrt{2\pi}} e^{\left\{\frac{-(r_0 - s_{0i})^2}{2\sigma_0^2}\right\}}$$

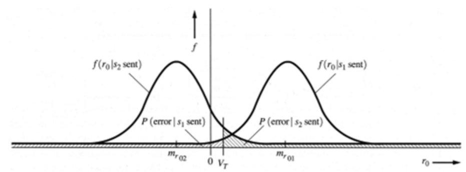


Figure 7–2 Error probability for binary signaling.

where $\sigma_0^2 = \overline{n_0^2}$ is the noise power within the detection bandwidth. Now we find:

$$P_e = P_1 \int_{-\infty}^{V_T} \frac{1}{\sigma_0\sqrt{2\pi}} e^{\left\{\frac{-(r_0 - s_{01})^2}{2\sigma_0^2}\right\}} dr_0 + P_2 \int_{V_T}^{\infty} \frac{1}{\sigma_0\sqrt{2\pi}} e^{\left\{\frac{-(r_0 - s_{02})^2}{2\sigma_0^2}\right\}} dr_0$$

Bit error probability (2)

Using $\lambda = \dfrac{r_0 - s_{0i}}{\sigma_0}$ and $P_1 = P_2 = \dfrac{1}{2}$

we find:

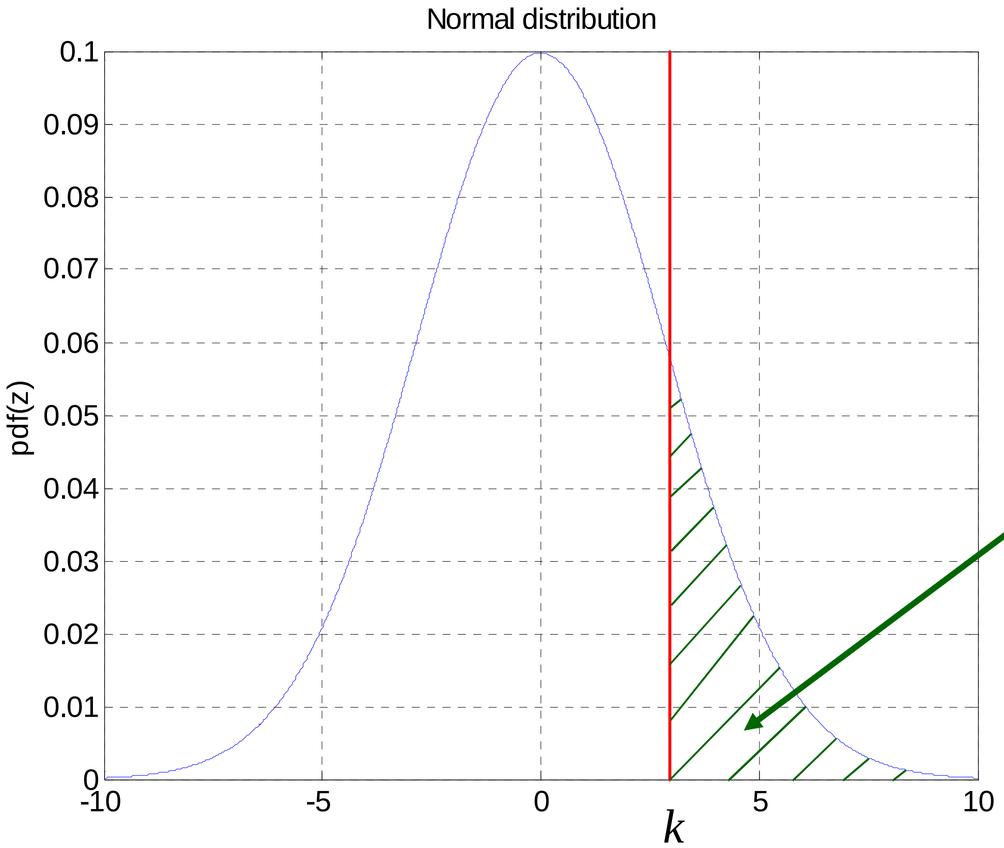
$$P_e = \frac{1}{2}\int_{-\infty}^{\frac{-(V_T - s_{01})}{\sigma_0}} \frac{1}{\sqrt{2\pi}} e^{\left\{\frac{-\lambda^2}{2}\right\}} d\lambda + \frac{1}{2}\int_{\frac{(V_T - s_{02})}{\sigma_0}}^{\infty} \frac{1}{\sqrt{2\pi}} e^{\left\{\frac{-\lambda^2}{2}\right\}} d\lambda$$

$$= \frac{1}{2} Q\left(\frac{s_{01} - V_T}{\sigma_0}\right) + \frac{1}{2} Q\left(\frac{V_T - s_{02}}{\sigma_0}\right)$$

$$Q(k) = \frac{1}{\sqrt{2\pi}}\int_{k}^{\infty} \exp\left(-\frac{\lambda^2}{2}\right) d\lambda$$

Decision threshold

Q-function (1)



$$Q(k) = \frac{1}{\sqrt{2\pi}} \int_k^{\infty} \exp\left(-\frac{\lambda^2}{2}\right) d\lambda$$

$$Q(-k) = 1 - Q(k)$$

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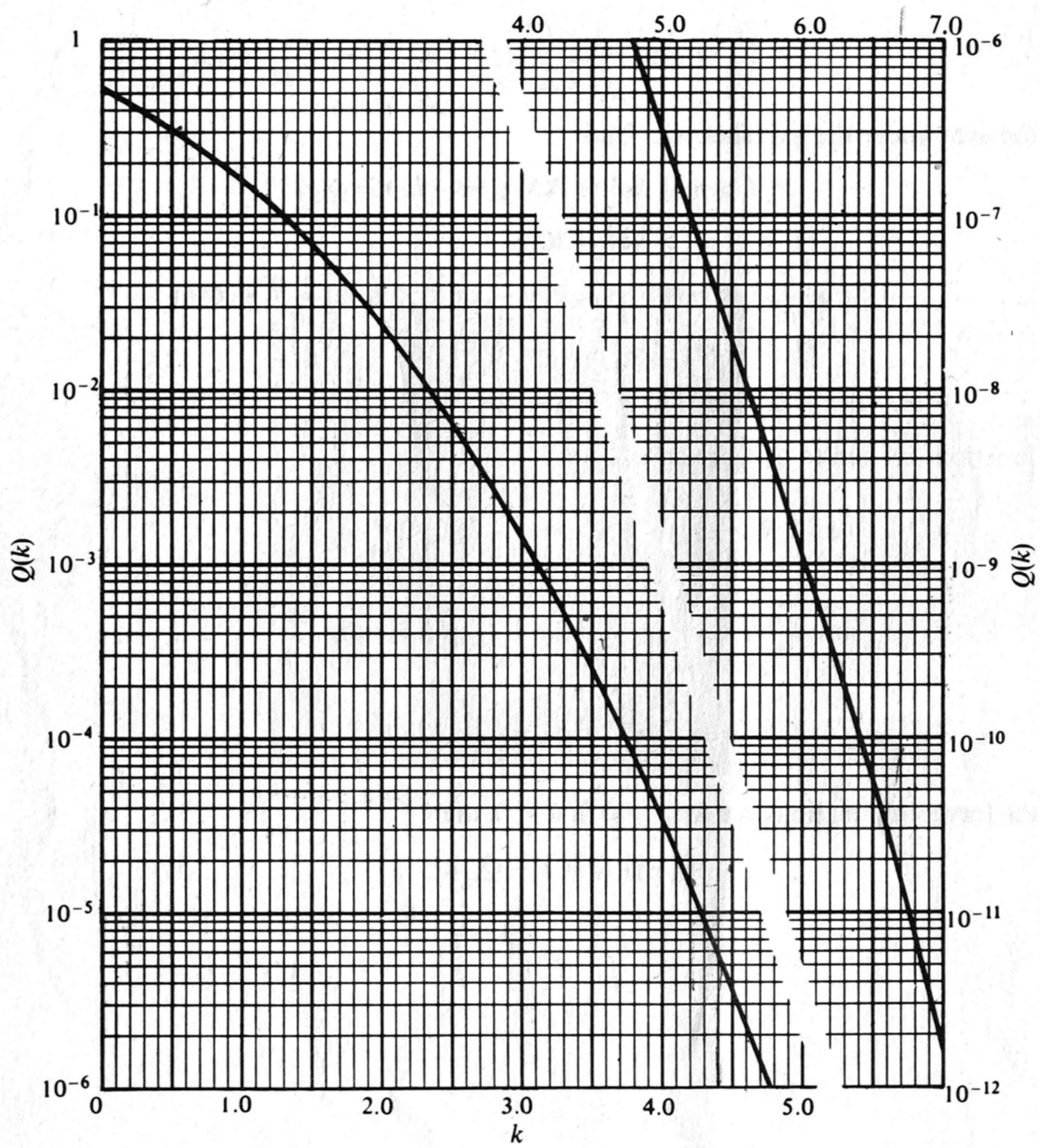
Q-function (2)



For larger values of k the Q-function can be approximated by:

$$Q(k) = \frac{1}{\sqrt{2\pi}} \int_k^{\infty} e^{-\frac{\lambda^2}{2}} d\lambda$$

$$\approx \frac{1}{k\sqrt{2\pi}} e^{-k^2/2}$$

which is quite accurate for $k > 3$.

Q-function: Couch pp. 700-701, 722-725, and cover page in the back.

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Bit error probability (3)

What is the optimum decision threshold level?

For which V_T is P_e minimized?

$$P_e = P_1 \int_{-\infty}^{V_T} \frac{1}{\sigma_0 \sqrt{2\pi}} e^{\left\{ \frac{-(r_0 - s_{01})^2}{2\sigma_0^2} \right\}} dr_0$$

$$+ P_2 \int_{V_T}^{\infty} \frac{1}{\sigma_0 \sqrt{2\pi}} e^{\left\{ \frac{-(r_0 - s_{02})^2}{2\sigma_0^2} \right\}} dr_0$$

Determine V_T for which $dP_e / dV_T = 0$.

Using *Leibnitz's* rule we find:

$$\frac{dP_e}{dV_T} = \frac{P_1}{\sigma_0 \sqrt{2\pi}} e^{\left\{ \frac{-(V_T - s_{01})^2}{2\sigma_0^2} \right\}} - \frac{P_2}{\sigma_0 \sqrt{2\pi}} e^{\left\{ \frac{-(V_T - s_{01})^2}{2\sigma_0^2} \right\}} = 0$$

Bit error probability (4)

Manipulation of:

$$\frac{P_1}{\sigma_0\sqrt{2\pi}}e^{\left\{\frac{-(V_T-s_{01})^2}{2\sigma_0^2}\right\}}-\frac{P_2}{\sigma_0\sqrt{2\pi}}e^{\left\{\frac{-(V_T-s_{02})^2}{2\sigma_0^2}\right\}}=0$$

results in:

$$2\sigma_0^2\ln\left(\frac{P_1}{P_2}\right)=-2V_T(s_{01}-s_{02})+s_{01}^2-s_{02}^2$$

$$\Rightarrow \quad V_T=\frac{-\sigma_0^2}{s_{01}-s_{02}}\ln\left(\frac{P_1}{P_2}\right)+\frac{s_{01}+s_{02}}{2}$$

and we find for $P_1=P_2=0.5$: $V_T=\dfrac{s_{01}+s_{02}}{2}$

the optimum threshold!

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Bit error probability (5)

Now we find for the bit error probability:

$$P_e = \frac{1}{2} Q\left(\frac{-V_T + s_{01}}{\sigma_0} \right) + \frac{1}{2} Q\left(\frac{V_T - s_{02}}{\sigma_0} \right)$$

with $V_T = \dfrac{s_{01} + s_{02}}{2}$

$$P_e = Q\left(\sqrt{\frac{(s_{01} - s_{02})^2}{4\sigma_0^2}} \right) = Q\left(\frac{| s_{01} - s_{02} |}{2\sigma_0} \right)$$

Also for non-matched filters.

This holds in general: for any type of filter!!

The BER is further minimized by maximizing the argument of the Q-function:

$$\frac{[s_{01}(t_0) - s_{02}(t_0)]^2}{\sigma_0^2} = \frac{s_d^2(t_0)}{\sigma_0^2}$$

with $s_d(t_0) \triangleq s_{01}(t_0) - s_{02}(t_0)$, is the difference signal.

Bit error probability (6)

This requires a "*matched filter*" (see section 6.8, p. 486 - 494).

$$h(t) = \frac{2K}{N_0}[s_1(t_0 - t) - s_2(t_0 - t)]$$

This gives a maximum SNR at time instant t_0.

The optimal SNR is now (6.159):

$$\left.\frac{\overline{s_d^2(t_0)}}{\sigma_0^2}\right|_{opt} = \int_{-\infty}^{\infty} \frac{|S_d(f)|^2}{P_n(f)} df = \frac{2}{N_0} \int_{-\infty}^{\infty} |S_d(f)|^2 \, df$$

With *Parseval's* theorem we find:

$$\left.\frac{\overline{s_d^2(t_0)}}{\sigma_0^2}\right|_{opt} = \frac{2}{N_0} \int_0^T s_d^2(t) dt = \frac{2E_d}{N_0} \quad \text{where} \quad E_d \triangleq \int_0^T [s_{01}(t) - s_{02}(t)]^2 dt$$

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Bit error probability (7)

Now we find for the BER with a *matched filter*:

$$P_e = Q\left(\sqrt{\frac{(s_{01}-s_{02})^2}{4\sigma_0^2}}\right) = Q\left(\sqrt{\frac{E_d}{2N_0}}\right)$$

This is the best we can do with white Gaussian noise!!!

The *matched filter* for rectangular pulses with duration T is given by:

$$h(t) = \begin{cases} \prod\left(\frac{T-t}{T}\right) & 0 \le t \le T \\ 0 & \text{other } t \end{cases} \qquad H(f) = T\frac{\sin \pi fT}{\pi fT} = T \,\text{sinc}\, fT$$

with equivalent noise bandwidth $B_{eq} = \dfrac{1}{2T}$.

and: $$\frac{(s_{01}-s_{02})^2}{4\sigma_0^2} = \frac{s_d^2}{4\sigma_0^2} = \frac{s_d^2}{4B_{eq}N_0} = \frac{2Ts_d^2}{4N_0} = \frac{E_d}{2N_0}$$

Colored noise (1)

When the noise is non-white, maybe due to filtering, we cannot apply the previous theory unless we apply a pre-whitening filter:

$$H_{pw}(f) = \frac{1}{\sqrt{P_n(f)}} \quad \Rightarrow \quad h_{pw}(t) = F^{-1}\{H_{pw}(f)\}$$

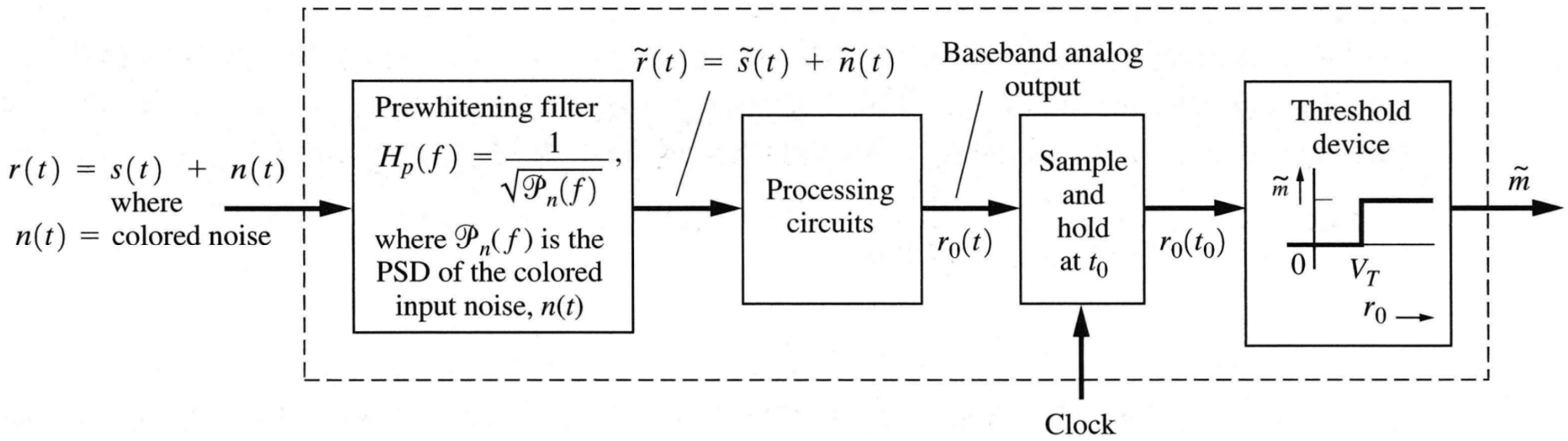

Figure 7–3 Matched-filter receiver for colored noise.

and the filtered signal becomes: $\tilde{s}(t) = s(t) * h_{pw}(t)$

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Colored noise (2)

Due to the pre-whitening filter, the filtered signal $\tilde{s}(t) = s(t) * h_p(t)$ will be dispersed in time:

- loss of signal power since part of it will be outside the interval: $0 \leq t \leq T$

- Inter-symbol interference (ISI, see also Ch. 3, pp. 176-185)

This problem can be solved by using shorter pulses and thus concentrating the power in time: requires more bandwidth!

Resume: optimal detection of binary signals

In general for additive white Gaussian noise (AWGN) and arbitrary type of filter:

$$P_e = Q\left(\sqrt{\frac{(s_{01}-s_{02})^2}{4\sigma_0^2}}\right) = Q\left(\frac{|s_{01}-s_{02}|}{2\sigma_0}\right)$$

s_{01} and s_{02} are the sampled values without noise.

and the optimum decision threshold is: $V_T = \dfrac{s_{01}+s_{02}}{2}$
if $P_1 = 1 - P_2 = 0.5$.

For the matched filter: $P_e = Q\left(\sqrt{\dfrac{E_d}{2N_0}}\right)$

with $E_d \triangleq \int_0^T [s_{01}(t_0) - s_{02}(t_0)]^2 dt$ is the "difference symbol" energy.

Baseband signaling: review

Some line codes:

- Unipolar NRZ
- Polar NRZ
- Bipolar NRZ

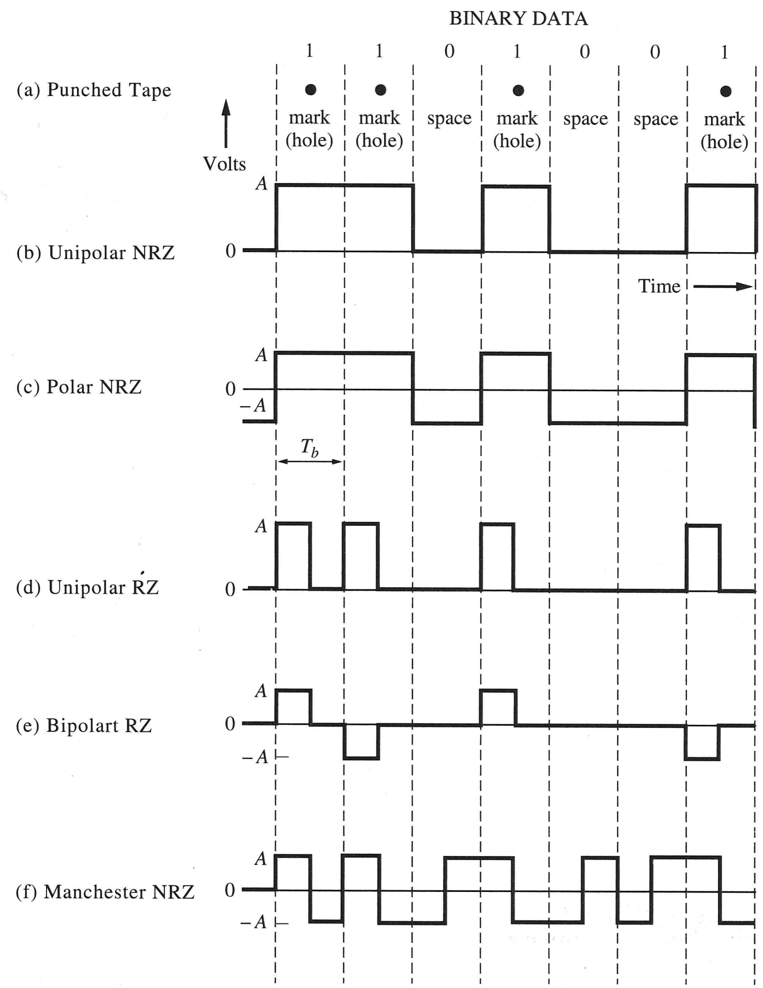

Figure 3–15 Binary signaling formats.

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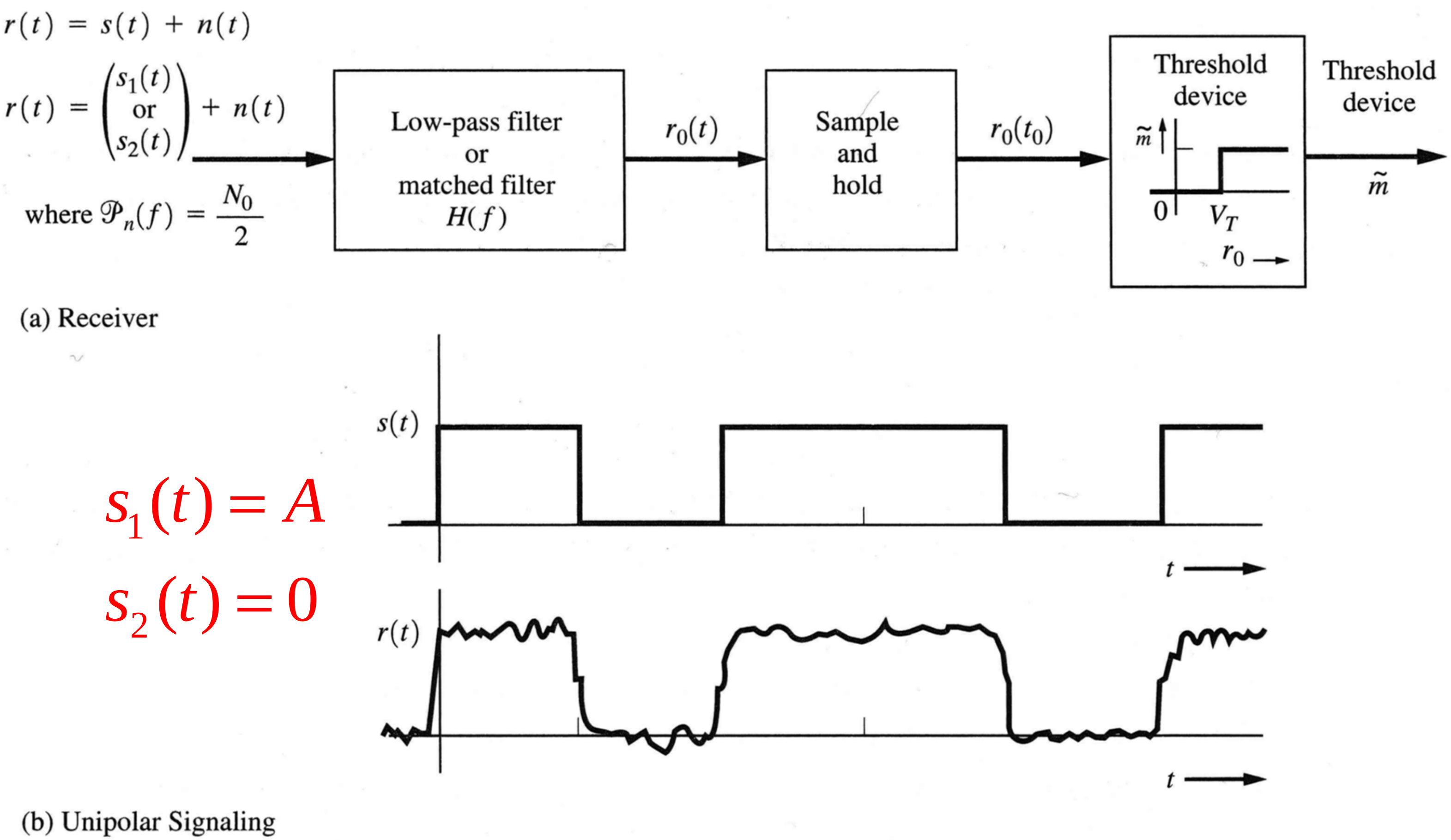
Unipolar baseband signal (1)



$$s_1(t) = A$$

$$s_2(t) = 0$$

Unipolar baseband signal (2)

1. For a non-matched filter: $B_{eq} > \frac{2}{T}$ (no ISI at sample moment $t = t_0$)

- sample values: $\left.\begin{array}{l} s_{01}(t_0) \simeq A \\ s_{02}(t_0) \simeq 0 \end{array}\right\} \Rightarrow V_{T,opt} = \frac{A}{2}$

- noise variance: $\sigma_0^2 = \frac{N_0}{2} \cdot 2B_{eq} = N_0 B_{eq}$

$$\Rightarrow \quad P_e = Q\left(\left|\frac{s_{01} - s_{02}}{2\sigma_0}\right|\right) = Q\left(\sqrt{\frac{(s_{01} - s_{02})^2}{4\sigma_0^2}}\right) = Q\left(\sqrt{\frac{A^2}{4N_0 B_{eq}}}\right)$$

What happens when:
- B_{eq} is increased?
- B_{eq} is decreased?

Exercise: BER unipolar signal (1)

For a received unipolar signal we find:

$$A = 2\ \mathrm{V}$$

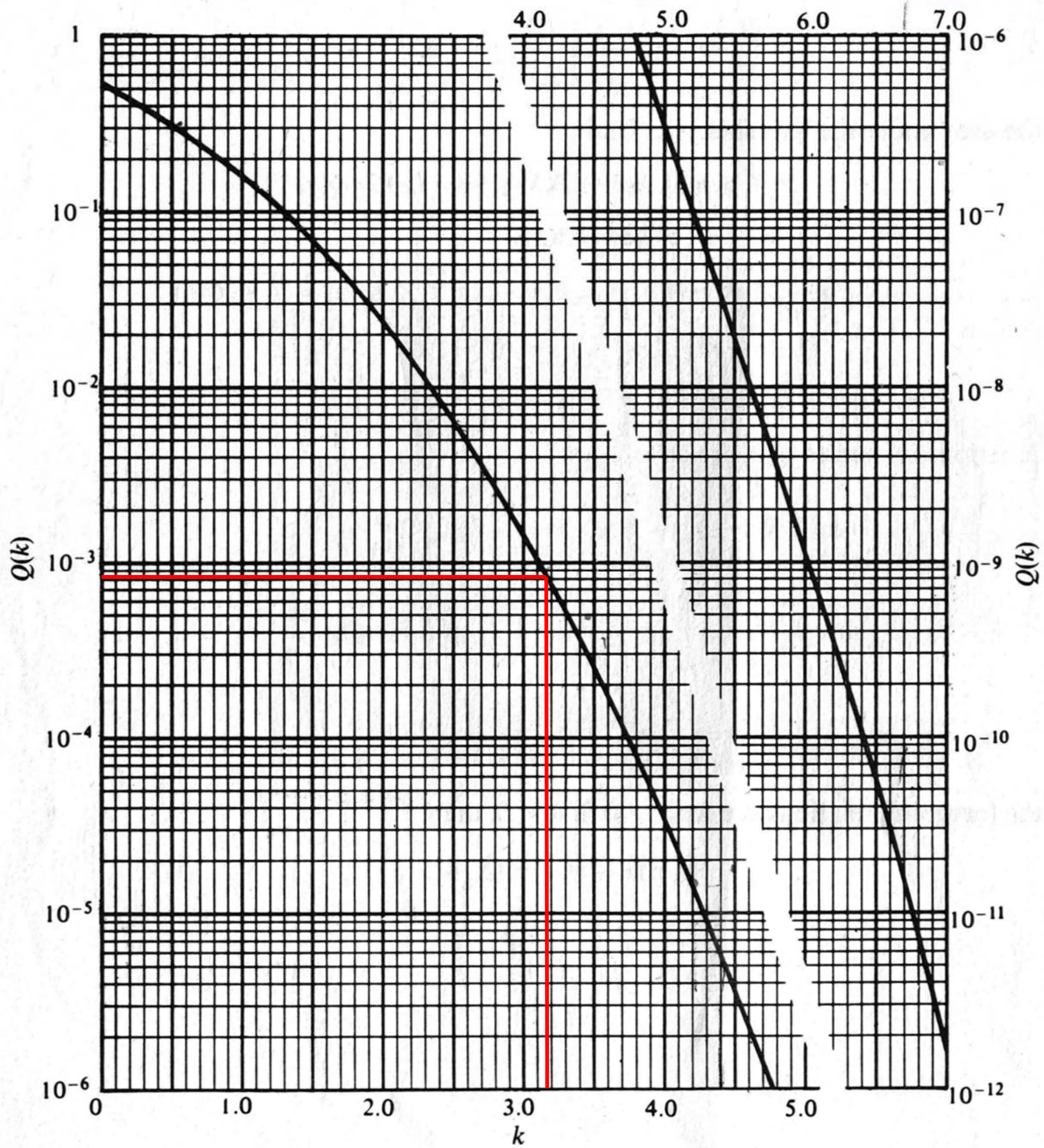
$$\sigma_0^2 = 0.1\ \mathrm{V}^2$$

What is the BER?

$$P_e = Q\left(\sqrt{\frac{(s_{01}-s_{02})^2}{4\sigma_0^2}}\right) = Q\left(\sqrt{\frac{A^2}{4N_0B_{eq}}}\right)$$

$$= Q\left(\sqrt{\frac{4}{4\cdot 0.1}}\right) = Q\left(\sqrt{10}\right) = Q(3.16) \simeq 8\cdot 10^{-4}$$

Exercise: BER unipolar signal (2)



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Unipolar baseband signal (2)

2. With a matched filter we obtain maximum achievable SNR at the sample moment $t = t_0$.

with $P_e = Q\left(\sqrt{\dfrac{E_d}{2N_0}}\right) = Q\left(\sqrt{\dfrac{E_b}{N_0}}\right) = Q\left(\sqrt{\dfrac{A^2T}{2N_0}}\right)$

where $E_d \triangleq \int_0^T [s_1(t) - s_2(t)]^2 \, dt = A^2T$

The average amount of energy transmitted per bit is:

$$E_b = \frac{(A^2T + 0)}{2} = \frac{A^2T}{2} = \frac{E_d}{2} \quad \Rightarrow \quad E_d = 2E_b$$

and the optimum decision threshold is: $V_{T,opt} = \dfrac{AT}{2}$

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BER for binary modulations with matched filter detection

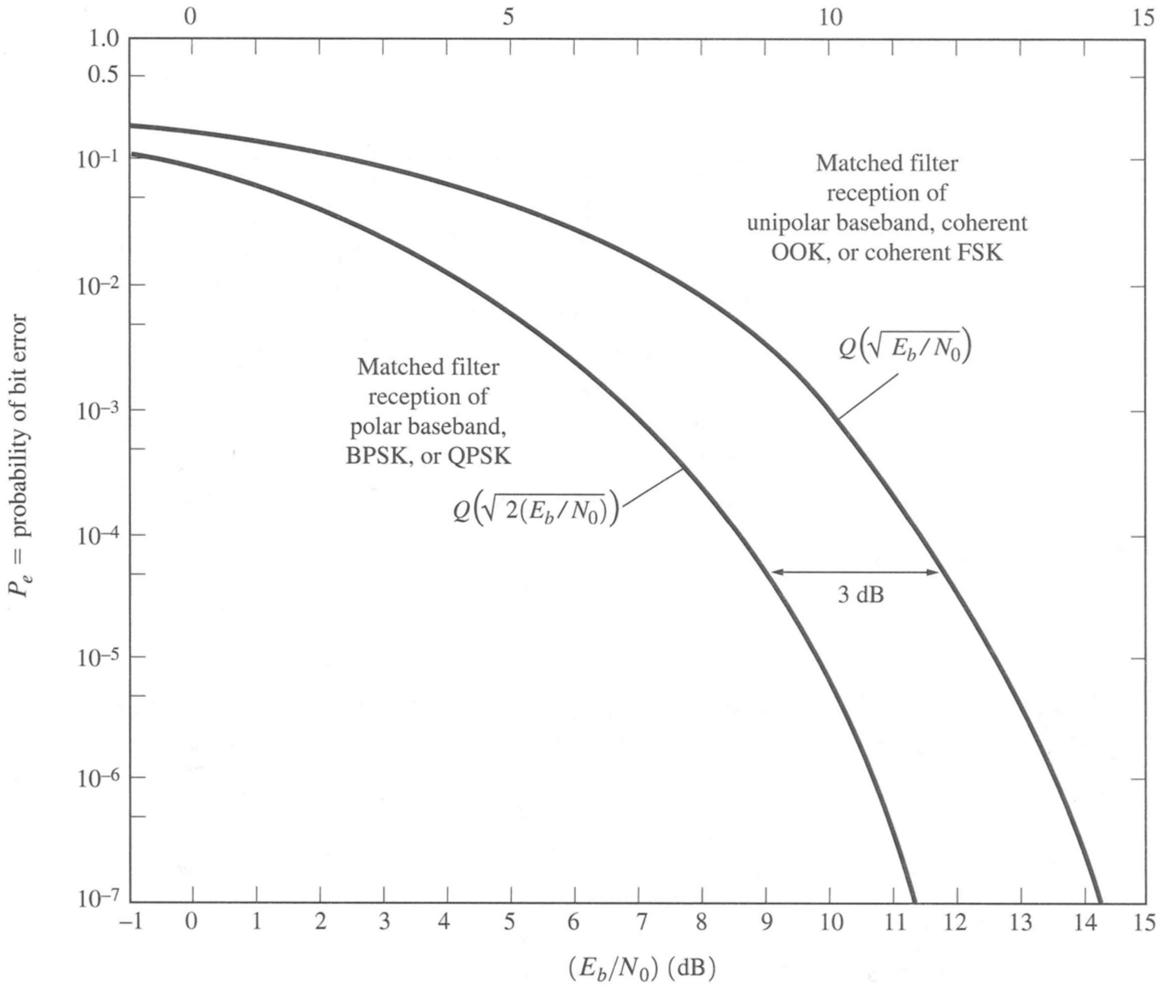
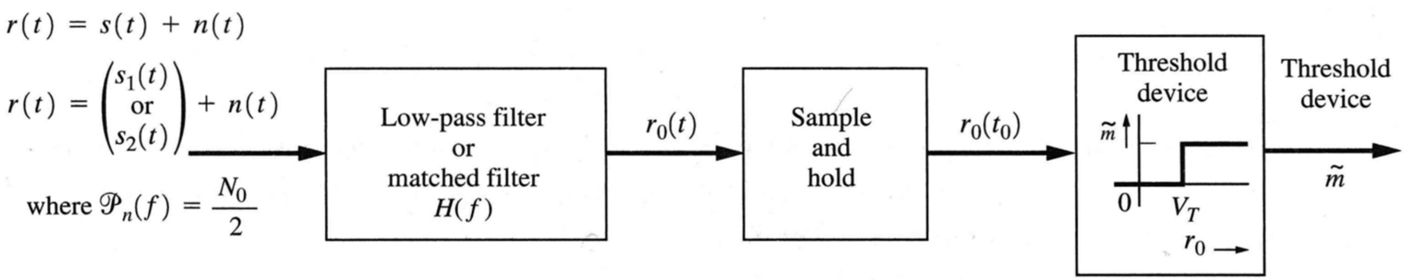


Figure 7–5 P_e for matched-filter reception of several binary signaling schemes.

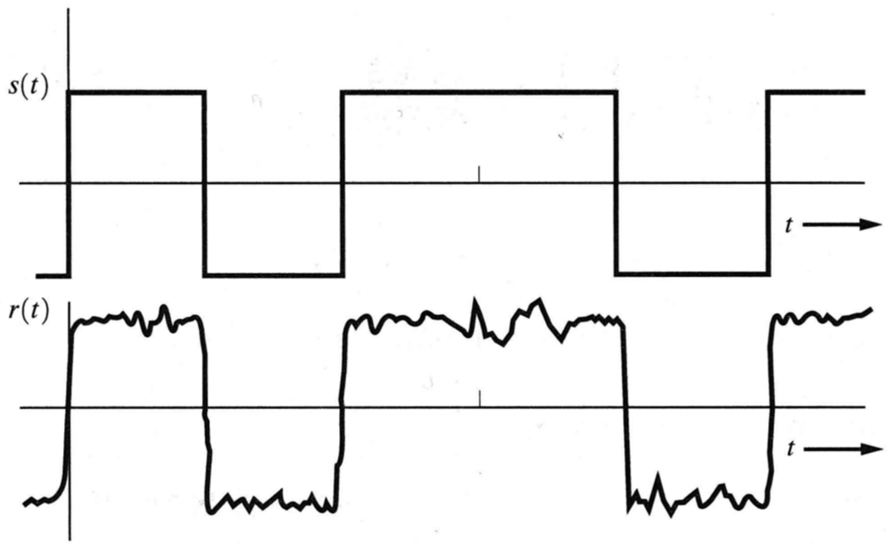
Polar baseband signal (1)



(a) Receiver

$$s_1(t) = A$$

$$s_2(t) = -A$$



(c) Polar Signaling

Figure 7–4 Receiver for baseband binary signaling.

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Polar baseband signal (2)

1. For a non-matched filter: $B_{eq} > \frac{2}{T}$

- sample values: $\left.\begin{array}{l} s_{01}(t_0) \simeq A \\ s_{02}(t_0) \simeq -A \end{array}\right\} \Rightarrow V_{T,opt} = \frac{s_{01}(t_0) + s_{02}(t_0)}{2} = 0$

Important for fading channels!

- noise variance: $\sigma_0^2 = \frac{N_0}{2} \cdot 2B_{eq} = N_0 B_{eq}$

$$\Rightarrow \quad P_e = Q\left(\sqrt{\frac{(s_{01} - s_{02})^2}{4\sigma_0^2}}\right) = Q\left(\sqrt{\frac{A^2}{N_0 B_{eq}}}\right)$$

Note:

- 2x more power effiecient

and

- 4x more PEP efficient!

Polar baseband signal (2)

2. With a matched filter we obtain maximum SNR at the sample moment $t = t_0$.

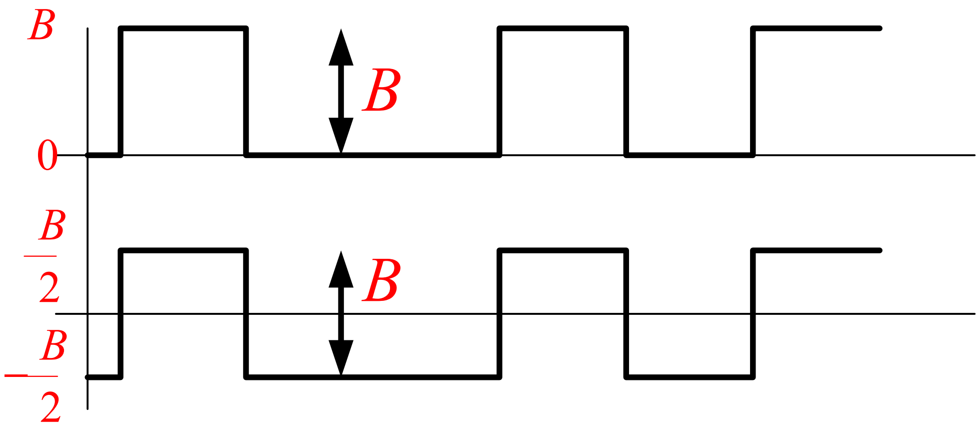
The difference signal $s_d = 2A \Rightarrow E_d \triangleq \int_0^T (2A)^2 dt = 4A^2T = 4E_b$

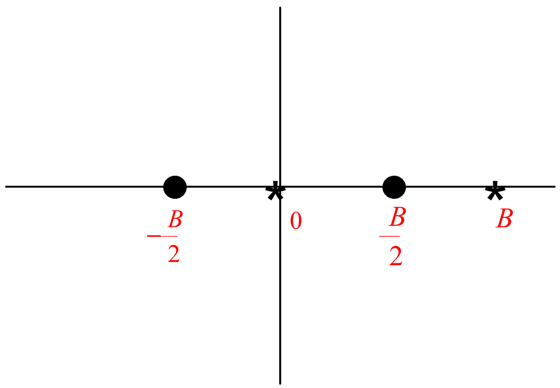
and $P_e = Q\left(\sqrt{\dfrac{E_d}{2N_0}}\right) = Q\left(\sqrt{\dfrac{2E_b}{N_0}}\right) = Q\left(\sqrt{\dfrac{2A^2T}{N_0}}\right)$

where $E_b = \dfrac{E_d}{4} = A^2T$ and $V_{T,opt} = 0$

What is the essential reason for the difference in BER performance between uni-polar and polar signaling?

Unipolar v.s. polar signaling





Unipolar: $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = B^2 T \Rightarrow E_b = \frac{B^2 T}{2} = \frac{E_d}{2},\ E_d = 2E_b$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{E_b}{N_0}}\right) = Q\left(\sqrt{\frac{B^2 T}{2N_0}}\right)$$

Polar: $E_d = \int_0^T [s_1(t) - s_2(t)]^2 dt = B^2 T \Rightarrow E_b = \frac{B^2 T}{4} = \frac{E_d}{4},\ E_d = 4E_b$

$$\Rightarrow P_e = Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = Q\left(\sqrt{\frac{2E_b}{N_0}}\right) = Q\left(\sqrt{\frac{B^2 T}{2N_0}}\right)$$

Anti-podal signals!!!

$(s_{01}-s_{02})^2$ is maximized with minimum power.

Bipolar baseband signal (1)

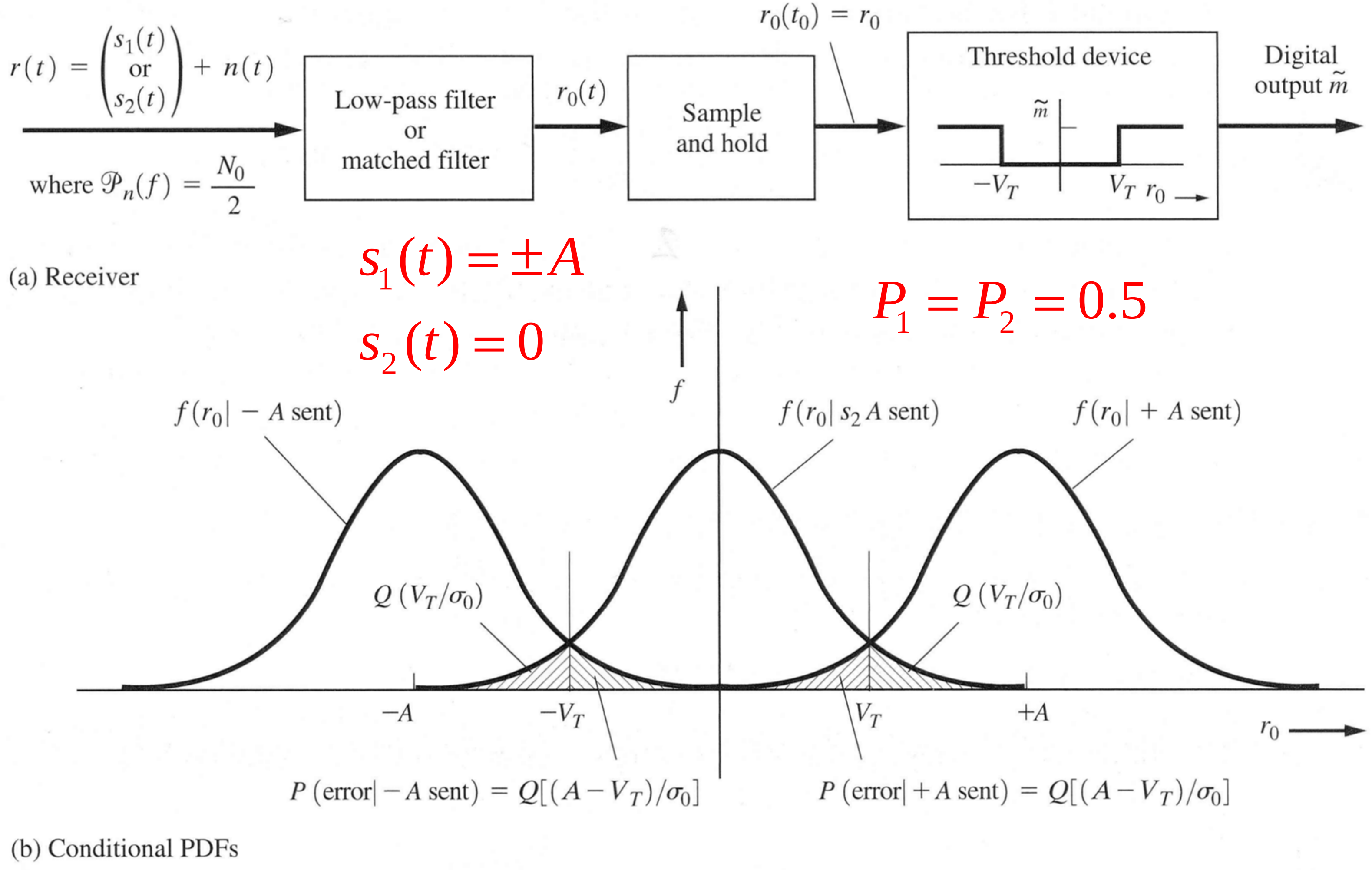


Figure 7–6 Receiver for bipolar signaling.

$$s_1(t) = \pm A$$

$$s_2(t) = 0$$

$$P_1 = P_2 = 0.5$$

Bipolar baseband signal (2)

1. For any non-matched filter: $B_{eq} > \frac{2}{T}$

- sample values: $\left.\begin{array}{ll} s_{01}(t_0) \simeq \pm A & \text{for "1"} \\ s_{02}(t_0) \simeq \ 0 & \text{for "0"} \end{array}\right\} \Rightarrow V_{T,opt} = \begin{cases} \dfrac{s_{01a} + s_{02}}{2} = \dfrac{-A}{2} \\ \dfrac{s_{01b} + s_{02}}{2} = \dfrac{A}{2} \end{cases}$

- noise variance: $\sigma_0^2 = \frac{N_0}{2} \cdot 2B_{eq} = N_0 B_{eq}$

Now two (optimum) decision levels are required!

Now $P_e = P(\varepsilon \mid A)P(A) + P(\varepsilon \mid -A)P(-A) + P(\varepsilon \mid 0)P(0)$

$$\approx \tfrac{1}{4}Q\left(\tfrac{A-V_T}{\sigma_0}\right) + \tfrac{1}{4}Q\left(\tfrac{A-V_T}{\sigma_0}\right) + 2\cdot\tfrac{1}{2}Q\left(\tfrac{V_T}{\sigma_0}\right) \overset{V_{T,opt}}{=} \tfrac{3}{2}Q\left(\tfrac{V_T}{\sigma_0}\right)$$

What has been neglected?

For each of the steps we use: $P_e = Q\left(\sqrt{\dfrac{(s_{01} - s_{02})^2}{4\sigma_0^2}}\right)$

and $(s_{01} - s_{02})^2 = A^2$

Polar baseband signal (2)

1. Using: $V_T = \frac{A}{2}$ we find $P_e = \frac{3}{2} Q\left(\frac{A}{2\sigma_0}\right) = \frac{3}{2} Q\left(\sqrt{\frac{A^2}{4N_0 B_{eq}}}\right)$

The same as unipolar with doubled error probability in "0"-s.

2. With a matched filter we obtain maximum SNR at the sample moment $t = t_0$.

The difference signal $s_d = s_{01} - s_{02} = A \Rightarrow E_d \triangleq \int_0^T (A)^2 dt = A^2 T = 2E_b$

and $P_e = \frac{3}{2} Q\left(\sqrt{\frac{E_d}{2N_0}}\right) = \frac{3}{2} Q\left(\sqrt{\frac{E_b}{N_0}}\right) = \frac{3}{2} Q\left(\sqrt{\frac{A^2 T}{2N_0}}\right)$

where $E_b = \frac{E_d}{2} = \frac{A^2 T}{2}$ and $V_{T,opt} = \pm \frac{AT}{2}$

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