Notes for New Constructions of DMPF

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ABSTRACT

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CCS CONCEPTS

• **Theory of computation** → **Cryptographic primitives**.

KEYWORDS

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1 INTRODUCTION

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2 PRELIMINARY

2.1 Basic Notations

Point and multi-point functions. Given a domain size N and Abelian group $\mathbb{G}$, a *point function* $f_{\alpha,\beta} : [N] \to \mathbb{G}$ for $\alpha \in [N]$ and $\beta \in \mathbb{G}$ evaluates to β on input α and to $0 \in \mathbb{G}$ on all other inputs. We denote by $\hat{f}_{\alpha,\beta} = (N, \hat{\mathbb{G}}, \alpha, \beta)$ the representation of such a point function. A t*-point function* $f_{A,B} : [N] \to \mathbb{G}$ for $A = (\alpha_1, \cdots \alpha_t) \in [N]^t$ and $B = (\beta_1, \cdots, \beta_t) \in \mathbb{G}^t$ evaluates to β_i on input α_i for $1 \le i \le t$ and to 0 on all other inputs. Denote $\hat{f}_{A,B}(N, \hat{\mathbb{G}}, t, A, B)$ the representation of such a t-point function. Call the collection of all t-point functions for all t *multi-point functions.*

Enote: MPF. Also representation of groups.

2.2 Distributed Multi-Point Functions

Enote: should directly adapt to multi-point function case

We begin by defining a slightly generalized notion of distributed point functions (DPFs), which accounts for the extra parameter $\mathbb{G}'$. Yaxin: What is $\mathbb{G}'$?

Definition 1 (DPF [3, 7]). *A (2-party)* distributed point function (DPF) *is a triple of algorithms* $\Pi = (\mathsf{Gen}, \mathsf{Eval}_0, \mathsf{Eval}_1)$ *with the following syntax:*

- $\mathsf{Gen}(1^\lambda, \hat{f}_{\alpha,\beta}) \to (k_0, k_1)$*: On input security parameter* $\lambda \in \mathbb{N}$ *and point function description* $\hat{f}_{\alpha,\beta} = (N, \hat{\mathbb{G}}, \alpha, \beta)$*, the (randomized) key generation algorithm* Gen *returns a pair of keys* $k_0, k_1 \in \{0,1\}^*$*. We assume that* N *and* $\mathbb{G}$ *are determined by each key.*
- $\mathsf{Eval}_i(k_i, x) \to y_i$*: On input key* $k_i \in \{0,1\}^*$ *and input* $x \in [N]$ *the (deterministic) evaluation algorithm of server* i*,* Eval_i *returns* $y_i \in \mathbb{G}$*.*

We require Π *to satisfy the following requirements:*

- ***Correctness:*** *For every* λ*,* $\hat{f} = \hat{f}_{\alpha,\beta} = (N, \hat{\mathbb{G}}, \alpha, \beta)$ *such that* $\beta \in \mathbb{G}$*, and* $x \in [N]$*, if* $(k_0, k_1) \leftarrow \mathsf{Gen}(1^\lambda, \hat{f})$*, then*
$$\Pr\left[\sum_{i=0}^{1} \mathsf{Eval}_i(k_i, x) = f_{\alpha,\beta}(x)\right] = 1$$
- ***Security:*** *Consider the following semantic security challenge experiment for corrupted server* $i \in \{0,1\}$*:*
  1. *The adversary produces two point function descriptions* $(\hat{f}^0 = (N, \hat{\mathbb{G}}, \alpha_0, \beta_0), \hat{f}^1 = (N, \hat{\mathbb{G}}, \alpha_1, \beta_1)) \leftarrow \mathcal{A}(1^\lambda)$*, where* $\alpha_i \in [N]$ *and* $\beta_i \in \mathbb{G}$*.*
  2. *The challenger samples* $b \leftarrow \{0,1\}$ *and* $(k_0, k_1) \leftarrow \mathsf{Gen}(1^\lambda, \hat{f}^b)$*.*
  3. *The adversary outputs a guess* $b' \leftarrow \mathcal{A}(k_i)$*.*

  Denote by $\mathsf{Adv}(1^\lambda, \mathcal{A}, i) = \Pr[b = b'] - 1/2$ *the advantage of* $\mathcal{A}$ *in guessing* b *in the above experiment. For every non-uniform polynomial time adversary* $\mathcal{A}$ *there exists a negligible function* ν *such that* $\mathsf{Adv}(1^\lambda, \mathcal{A}, i) \le \nu(\lambda)$ *for all* $\lambda \in \mathbb{N}$*.*

Definition 2 (DMPF). *A (2-party)* distributed multi-point function (DMPF) *is a triple of algorithms* $\Pi = (\mathsf{Gen}, \mathsf{Eval}_0, \mathsf{Eval}_1)$ *with the following syntax:*

- $\mathsf{Gen}(1^\lambda, \hat{f}_{A,B}) \to (k_0, k_1)$*: On input security parameter* $\lambda \in \mathbb{N}$ *and point function description* $\hat{f}_{A,B} = (N, \hat{\mathbb{G}}, t, A, B)$*, the (randomized) key generation algorithm* Gen *returns a pair of keys* $k_0, k_1 \in \{0,1\}^*$*.*
- $\mathsf{Eval}_i(k_i, x) \to y_i$*: On input key* $k_i \in \{0,1\}^*$ *and input* $x \in [N]$ *the (deterministic) evaluation algorithm of server* i*,* Eval_i *returns* $y_i \in \mathbb{G}$*.*

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$$\Pr\left[\sum_{i=0}^{1} \mathsf{Eval}_i(k_i, x) = f_{\alpha,\beta}(x)\right] = 1$$
- ***Security:*** *Consider the following semantic security challenge experiment for corrupted server* $i \in \{0,1\}$*:*
  1. *The adversary produces two* t*-point function descriptions* $(\hat{f}^0 = (N, \hat{\mathbb{G}}, t, A_0, B_0), \hat{f}^1 = (N, \hat{\mathbb{G}}, t, A_1, B_1)) \leftarrow \mathcal{A}(1^\lambda)$*, where* $\alpha_i \in [N]$ *and* $\beta_i \in \mathbb{G}$*.*
  2. *The challenger samples* $b \leftarrow \{0,1\}$ *and* $(k_0, k_1) \leftarrow \mathsf{Gen}(1^\lambda, \hat{f}^b)$*.*
  3. *The adversary outputs a guess* $b' \leftarrow \mathcal{A}(k_i)$*.*

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We will also be interested in applying the evaluation algorithm on *all* inputs. Given a DMPF $(\mathsf{Gen}, \mathsf{Eval}_0, \mathsf{Eval}_1)$, we denote by $\mathsf{FullEval}_i$ an algorithm which computes Eval_i on every input x. Hence, $\mathsf{FullEval}_i$ receives only a key k_i as input.

2.3 Batch Code

We introduce batch code and probabilistic batch code, which can be used to construct DMPF (see construction 4).

DEFINITION 3 (BATCH CODE[8]). *An (N, M, t, m)-batch code over alphabet Σ is given by a pair of efficient algorithms* $(\mathsf{Encode}, \mathsf{Decode})$ *such that:*

- $\mathsf{Encode}(x \in \Sigma^N) \to (C_1, C_2, \cdots, C_m)$*: Any string $x \in \Sigma^N$ is encoded into an m-tuple of strings $C_1, C_2, \cdots C_m \in \Sigma^*$ (called buckets) of total length M.*
- $\mathsf{Decode}(I, C_1, C_2, \cdots, C_m) \to \{x[i]\}_{i \in I}$*: On input a set I of t distinct indices in $[N]$ and m buckets, recover t coordinates of x indexed by I by reading at most one coordinate from each of the m buckets.*

An (N, M, t, m)-batch code can be represented by an (N, m)-bipartite graph $G = (U, V, E)$ where each edge $(u_i, v_j) \in E$ corresponds to Encode assigning $x[i]$ to the bucket C_j, while it is guaranteed that any subset $S \subseteq U$ such that $|S| = t$ has a perfect matching to V. Yaxin: Add example instantiation (random regular bipartite graph) and explain it is not efficient.

DEFINITION 4 (PROBABILISTIC BATCH CODE (PBC)[1]). *An (N, M, t, m, ϵ)-probabilistic batch code over alphabet Σ is a randomized (N, M, t, m)-batch code that for any string x and any set I of t distinct indices in $[N]$,*

$$\Pr[\mathsf{Decode}_r(I, \mathsf{Encode}_r(x)) \to \{x[i]\}_{i \in I}] = 1 - \epsilon$$

where the probability is taken over the public randomness r and private randomness of Encode *and* Decode *algorithms.*

We mention Cuckoo hashing algorithm[9] as a concrete instantiation of PBC[1].

w-way cuckoo hashing. Given t balls, $m = et$ buckets (e is some expansion parameter that is bigger than 1), and w independent hash functions $h_1, h_2, \cdots, h_w$ mapping each ball to a random bucket, allocates all balls to the buckets such that each bucket contains at most one ball through the following process:

1. Choose an arbitrary unallocated ball b.
2. Choose a random hash function h_i compute the bucket index $h_i(b)$. If this bucket is empty, then allocate b to this bucket and go to step 1. If this bucket is not empty and filled with ball b', then evict b', allocate b to this bucket set b' the current unallocated ball, and repeat step 2.

The algorithm should be given a fixed amount of time to run, or equipped with a loop detection process to guarantee termination. Yaxin: Add asymptotic parameters?

The failure probability of cuckoo hashing. Let's denote the failure probability of w-way cuckoo hashing to be $\epsilon = 2^{-\lambda_{\mathsf{stat}}}$. In practice we usually consider the statistical security parameter λ_{stat} to be 30 or 40. The empirical result in [4] shows for $w = 3$, $m = 16384$, $\lambda_{\mathsf{stat}} = 124.4e - 144.6$ where e is the expansion parameter that $m = et$. For $w = 3$, $m = 8192$, $\lambda_{\mathsf{stat}} = 125e - 145$. However we use cuckoo hashing to construct DMPF for t-point functions, in which case we'd also care about t being small, say 2, 3 or 100, and m should not be too large. In this sense the previous empirical results are not complete. Yaxin: [1] uses $w = 3, e = 1.5, t > 200$ and $\lambda_{\mathsf{stat}} \approx 40$ and claims it follows the analysis from [4], but I don't see how...

The balls, buckets and hash functions can be represented by a w-regular (t, m)-bipartite graph $G = (U, V, E)$ where each left node has w neighbors, and each edge $(u_i, v_j) \in E$ corresponds to $h_l(i) = j$ for some $1 \le l \le w$. In this graph representation the w-way cuckoo hashing essentially computes a perfect matching from U to V. Therefore one can construct a PBC from cuckoo hashing.

CONSTRUCTION 1 (PBC FROM CUCKOO HASHING). *Given w-way cuckoo hashing as a sub-procedure allocating t balls to m buckets with failure probability ϵ, an (N, wN, t, m, ϵ)-PBC is as follows:*

- $\mathsf{Encode}_r(x \in \Sigma^N) \to (C_1, \cdots, C_m)$*: Use r to determine w independent random hash functions $h_1, h_2, \cdots h_w$ that maps from $[N]$ to $[m]$. Initialize $C_1, \cdots, C_m$ to be empty. For each $i \in [N]$, append $x[i]$ to $C_{h_j(i)}$ for $1 \le j \le w$.*
- $\mathsf{Decode}_r(I, C_1, \cdots, C_m) \to \{x[i]\}_{i \in I}$*: Determine $h_1, \cdots, h_w$ as in* Encode*. For I of size t, allocate I to $[m]$ using w-way cuckoo hashing. For each $i \in I$, fetch $x[i]$ from C_j where i is allocated to j.*

2.4 Oblivious Key-Value Stores

DEFINITION 5 (OBLIVIOUS KEY-VALUE STORES (OKVS)[6, 10]). *An Oblivious Key-Value Stores scheme is a pair of randomized algorithms* $(\mathsf{Encode}_r, \mathsf{Decode}_r)$ *with respect to a statistical security parameter λ_{stat} and a computational security parameter λ, a randomness space $\{0,1\}^\kappa$, a key space $\mathcal{K}$, a value space $\mathcal{V}$, input length t and output length m. The algorithms are of the following syntax:*

- $\mathsf{Encode}_r(\{(k_1, v_1), (k_2, v_2), \cdots, (k_t, v_t)\}) \to P$*: On input t key-value pairs with distinct keys, the encode algorithm with randomness r in the randomness space outputs an encoding $P \in \mathcal{V}^m \cup \bot$.*
- $\mathsf{Decode}_r(P, k) \to v$*: On input an encoding from $\mathcal{V}^m$ and a key $k \in \mathcal{K}$, output a value v.*

We require the scheme to satisfy

- *For all $S \in (\mathcal{K} \times \mathcal{V})^t$, $\Pr_{r \leftarrow \{0,1\}^\kappa}[\mathsf{Encode}_r(S) = \bot] \le 2^{-\lambda_{\mathsf{stat}}}$.*
- *For all $S \in (\mathcal{K} \times \mathcal{V})^t$ and $r \in \{0,1\}^\kappa$ such that $\mathsf{Encode}_r(S) \to P \ne \bot$, it is the case that $\mathsf{Decode}_r(P, k) \to v$ whenever $(k, v) \in S$.*
- ***Obliviousness:** Given any distinct key sets $\{k_1^0, k_2^0, \cdots, k_t^0\}$ and $\{k_1^1, k_2^1, \cdots, k_t^1\}$ that are different, if they are paired with random values then their encodings are computationally indistinguishable, i.e.,*

$$\{r, \mathsf{Encode}_r(\{(k_1^0, v_1), \cdots, (k_t^0, v_t)\})\}_{v_1, \cdots, v_t \leftarrow \mathcal{V}, r \leftarrow \{0,1\}^\kappa}$$
$$\approx_c \{r, \mathsf{Encode}_r(\{(k_1^1, v_1), \cdots, (k_t^1, v_t)\})\}_{v_1, \cdots, v_t \leftarrow \mathcal{V}, r \leftarrow \{0,1\}^\kappa}$$

One can obtain a linear OKVS *if in addition require:*

- ***Linearity:** There exists a function family $\{\mathsf{row}_r : \mathcal{K} \to \mathcal{V}^m\}_{r \in \{0,1\}^\kappa}$ such that $\mathsf{Decode}_r(P, k) = \langle \mathsf{row}_r(k), P \rangle$.*

The Encode process for a linear OKVS is the process of sampling a random P from the set of solutions of the linear system $\{\langle \mathsf{row}_r(k_i), P\rangle = v_i\}_{1\le i\le t}$.

We evaluate an OKVS scheme by its encoding size (output length m), encoding time and decoding time. We stress the following two (linear) OKVS constructions:

CONSTRUCTION 2 (POLYNOMIAL). *Suppose $\mathcal{K} = \mathcal{V} = \mathbb{F}$ is a field. Set*

- $\mathsf{Encode}(\{(k_i, v_i)\}_{1\le i\le t}) \to P$ *where P is the coeffients of a $(t-1)$-degree $\mathbb{F}$-polynomial g_P that $g_P(k_i) = v_i$ for $1 \le i \le t$.*
- $\mathsf{Decode}(P, k) \to g_P(k)$.

The polynomial OKVS possesses an optimal encoding size $m = n$, but the Encode process is a polynomial interpolation which is only known to be achieved in time $O(t \log^2 t)$. The time for a single decoding is $O(t)$ and that for batched decodings is (amortized) $O(\log^2 t)$.

An alternative construction that has near optimal encoding size but much better running time is as follows.

CONSTRUCTION 3 (3-HASH GARBLED CUCKOO TABLE (3H-GCT)[6, 10]). *Suppose $\mathcal{V} = \mathbb{F}$ is a field. Set $\mathsf{row}_r(k) := \mathsf{row}_r^{\mathsf{sparse}}(k)||\mathsf{row}_r^{\mathsf{dense}}(k)$ where $\mathsf{row}_r^{\mathsf{sparse}}$ outputs a uniformly random weight-w vector in $\{0,1\}^{m_1}$, and $\mathsf{row}_r^{\mathsf{dense}}(k)$ outputs a short dense vector in $\mathbb{F}^{m_2}$.*

- $\mathsf{Encode}(\{(k_i, v_i)\}_{1\le i\le t}) \to P$ *where P is solved from the system $\{\langle \mathsf{row}_r(k_i), P\rangle = v_i\}_{1\le i\le t}$ using the triangulation algorithm in [10].*
- $\mathsf{Decode}(P, k) \to \langle \mathsf{row}_r(k), P\rangle$.

We denote $m_1 = et$, where e is an expandion parameter indicating the rough blowup to store t pairs. In practice the number of dense columns m_2 is usually set to a small constant.

This OKVS construction features a linear encoding time, constant decoding time (the constant relates to w and m_2) while having a linear encoding size.

Yaxin: TBD: Carefully(!) recompute the comparison table for OKVS.

We'll mostly use the expansion parameter e and the number of dense columns $m_2 := \hat{g}$ (where $\hat{g}$ is a parameter relating to the equation system solving process) according to the analysis in [10]: Given w, t and λ_{stat}, the choices of the e and $\hat{g}$ are fixed through the following steps:

- Set $e^* = \begin{cases} 1.223 & w = 3 \\ 1.293 & w = 4 \\ 0.1485w + 0.6845 & w \ge 5 \end{cases}$.
- Compute $\alpha := 0.55 \log_2 t + 0.093w^3 - 1.01w^2 + 2.92w - 0.13$.
- $e := e^* + 2^{-\alpha}(\lambda_{\mathsf{stat}} + 9.2)$.
- $\hat{g} := \frac{\lambda_{\mathsf{stat}}}{(w-2)\log_2(et)}$.

Yaxin: Fix t and λ_{stat}, we want to find the best choice of w. The adavantageous choices of w in [10] are $w = 3$ and $w = 5$. From the first sight when w is smaller e can be smaller but $\hat{g}$ will be larger. Since e stands for number of $\mathbb{F}$-ADD's and $\hat{g}$ stands for number of $\mathbb{F}$-MULT's in decoding, previously I thought $\hat{g}$ is the dominating factor of Decode running time. However table 1 in [10] suggests that $w = 3$ outruns nearly all of other choices of w while $w = 5$ is almost 3 times slower in decoding time. This may suggest there are some other heavy computations other than $\mathbb{F}$-MULT that need to be considered when evaluating running time.

The range of t previous literature [6, 10] have considered in their empirical results are also limited, which will be one of our problems. We want to cover small t, say $t < 100$, while previous literature aiming for constructing PSI protocols usually consider very large t.

One may also let row_r^{dense} output a short dense vector in $\{0,1\}^{m_2}$, which avoids multiplication of large field elements in the encoding and decoding processes. To achieve same level of security one could simply set $m_2 = \hat{g} + \lambda_{\mathsf{stat}}$, as proposed in [6, 10]. Yaxin: TBD: mention some connections to cuckoo hashing?

3 NEW DMPF CONSTRUCTIONS

In this section, we display two new constructions of DMPF that follow the same paradigm shown in fig. 1.

We begin by introducing the DMPF paradigm in fig. 1, which is based on the idea of the DPF construction in [3]. Each key $k_b (b = 0, 1)$ generated by $\mathsf{Gen}(\hat{f}_{A,B})$ can span a height-n (n is the input length of $\hat{f}_{A,B}$) complete binary tree T_b (call it the evaluation tree), with which party b can evaluate the input $x = x_1 \cdots x_n$ by starting from the root of this tree, on the ith layer going left if $x_i = 0$ and going right if $x_i = 1$, until reaching a leaf node then computing the result according to this leaf node.

Each node of this tree is associated with a λ-bit seed and a l-bit sign. For a parent node on the ith layer with seed and sign, its children's seeds and signs are generated by $\mathsf{PRG}(\mathsf{seed}) \oplus \mathsf{Correction}$, where the Correction is determined by the parent node's position, its sign and a correction word $CW^{(i)}$ associated with that layer (computed by the method $\mathsf{Correct}()$). On a leaf node on the last layer, its seed will generate a random element in the output group, which will be corrected by adding a Correction determined by the leaf node's position, its sign and the last correction word $CW^{(n+1)}$ (computed by the method $\mathsf{ConvCorrect}()$).

Call any path from the root a leaf corresponding to an input string in A an accepting path. To force the correctness, we maintain the following invariance on the evaluation trees T_0, T_1 of the two parties:

- If a node is not on any accepting path, then T_0 and T_1 assign to it with the same seed and sign.
- If a node is on an accepting path, then T_0 and T_1 assign to it with different signs that controls the corrections on its children (or on the output if the node is on the last layer).

The paradigm contains four methods (GenCW, GenConvCW, Correct, ConvCorrect) and the sign length l to be determined by different constructions. We make the following restrictions on the methods in order to guarantee the invariance on the evaluation trees:

$\mathsf{M}(\bar{x}, \mathsf{sign}, CW) = \sum_{i=1}^{l} \mathsf{sign}[i] \cdot \mathsf{M}(\bar{x}, 0^{i-1}0^{l-i}, CW)$ for all $\mathsf{M} \in \{\mathsf{Correct}, \mathsf{ConvCorrect}\}$, input $\bar{x}$ and CW.

3.1 Big-State DMPF

Displayed in fig. 2. TBD: explain

Figure 1: The paradigm of our DMPF schemes. We leave the PRG expand length l, methods Initialize, GenCW, GenConvCW, Correct, ConvCorrect **to be determined by specific constructions.**

Public parameters:
The t-point function family $\{f_{A,B}\}$ with t an upperbound of the number of nonzero points, input domain $[N] = \{0,1\}^n$ and the output group $\mathbb{G}$.
Suppose there is a public PRG $G : \{0,1\}^\lambda \to \{0,1\}^{2\lambda+2l}$. Parse $G = G_0 \| G_1$ to the left half and right half.
Suppose there is a public PRG $G_{\text{convert}} : \{0,1\}^\lambda \to \mathbb{G}$.

procedure $\textsc{Gen}(1^\lambda, \hat{f}_{A,B})$
Denote $A = (\alpha_1, \cdots, \alpha_t)$ in lexicographical order, $B = (\beta_1, \cdots, \beta_t)$. If $|A| < t$, extend A to size-t with arbitrary $\{0,1\}^n$ strings and B with 0's.
For $0 \le i \le n-1$, let $A^{(i)}$ denote the sorted and deduplicated list of i-bit prefixes of strings in A. Specifically, $A^{(0)} = [\epsilon]$.
For $0 \le i \le n-1$ and $b = 0, 1$, initialize empty lists $\text{seed}_b^{(i)}$ and $\text{sign}_b^{(i)}$.
$\text{Initialize}(\{\text{seed}_b^{(0)}, \text{sign}_b^{(0)}\}_{b=0,1})$.
for $i = 1$ to n **do**
$CW^{(i)} \leftarrow \text{GenCW}(i, A, \{\text{seed}_b^{(i-1)}, \text{sign}_b^{(i-1)}\}_{b=0,1})$.
for $k = 1$ to $|A^{(i-1)}|$ and $z = 0, 1$ **do**
Compute $C_{\text{seed},b} \| C_{\text{sign}^0,b} \| C_{\text{sign}^1,b} \leftarrow \text{Correct}(A^{(i-1)}[k], \text{sign}_b^{(i-1)}[k], CW^{(i)})$ for $b = 0, 1$.
if $A^{(i-1)}[k] \| z \in A^{(i)}$ **then**
Append the first λ bit of $G_z(\text{seed}_b^{(i-1)}[k]) \oplus (C_{\text{seed},b} \| C_{\text{sign}^z,b})$ to $\text{seed}_b^{(i)}$ and the rest to $\text{sign}_b^{(i)}$.
end if
end for
end for
$CW^{(n+1)} \leftarrow \text{GenConvCW}(A, B, \{\text{seed}_b^{(n)}, \text{sign}_b^{(n)}\}_{b=0,1})$.
Set $k_b \leftarrow (\text{seed}_b^{(0)}, \text{sign}_b^{(0)}, CW^{(1)}, CW^{(2)}, \cdots, CW^{(n+1)})$.
return (k_0, k_1).
end procedure

procedure $\textsc{Eval}_b(1^\lambda, k_b, x)$
Parse $k_b = ([\text{seed}], [\text{sign}], CW^{(1)}, CW^{(2)}, \cdots, CW^{(n+1)})$.
Denote $x = x_1 x_2 \cdots x_n$.
for $i = 1$ to n **do**
$C_{\text{seed}} \| C_{\text{sign}^0} \| C_{\text{sign}^1} \leftarrow \text{Correct}(x_1 \cdots x_{i-1}, \text{sign}, CW^{(i)})$.
$\text{seed} \| \text{sign} \leftarrow (\text{seed} \oplus C_{\text{seed}}) \| (\text{sign} \oplus C_{\text{sign}^{x_i}})$.
end for
return $(-1)^b \cdot \left(G_{\text{convert}}(\text{seed}) + \text{ConvCorrect}(x, \text{sign}, CW^{(n+1)})\right)$.
end procedure

procedure $\textsc{FullEval}_b(1^\lambda, k_b)$
Parse $k_b = (\text{seed}^{(0)}, \text{sign}^{(0)}, CW^{(1)}, CW^{(2)}, \cdots, CW^{(n+1)})$.
For $1 \le i \le n$, $\text{Path}^{(i)} \leftarrow$ the lexicographical ordered list of $\{0,1\}^i$. $\text{Path}^{(0)} \leftarrow [\epsilon]$.
for $i = 1$ to n **do**
for $k = 1$ to 2^{i-1} **do**
$C_{\text{seed}} \| C_{\text{sign}^0} \| C_{\text{sign}^1} \leftarrow \text{Correct}(\text{Path}(i-1)[k], \text{sign}^{(i-1)}[k], CW^{(i)})$.
$\text{seed}^{(i)}[2k] \| \text{sign}^{(i)}[2k] \leftarrow G_0(\text{seed}^{(i-1)}[k]) \oplus (C_{\text{seed}} \| C_{\text{sign}^0})$.
$\text{seed}^{(i)}[2k+1] \| \text{sign}^{(i)}[2k+1] \leftarrow G_1(\text{seed}^{(i-1)}[k]) \oplus (C_{\text{seed}} \| C_{\text{sign}^1})$.
end for
end for
for $k = 1$ to 2^n **do**
$\text{Output}[k] \leftarrow (-1)^b \cdot \left(G_{\text{convert}}(\text{seed}^{(n)}[k]) + \text{ConvCorrect}(\text{Path}[k], \text{sign}^{(n)}[k], CW^{(n+1)})\right)$.
end for
return Output.
end procedure

Figure 2: The parameter l and methods' setting that turns the paradigm of DMPF in fig. 1 into the big-state DMPF.

Set $l \leftarrow t$, the upperbound of $|A|$.
procedure INITIALIZE($\{\mathsf{seed}_b^{(0)}, \mathsf{sign}_b^{(0)}\}_{b=0,1}$)
For $b = 0, 1$, let $\mathsf{seed}_b^{(0)} = [r_b]$ where $r_b \xleftarrow{\$} \{0,1\}^\lambda$.
For $b = 0, 1$, set $\mathsf{sign}_b^{(0)} = [b \| 0^{t-1}]$.
end procedure

procedure GENCW($i, A, \{\mathsf{seed}_b^{(i-1)}, \mathsf{sign}_b^{(i-1)}\}_{b=0,1}$)
Let $\{A^{(i)}\}_{0 \le i \le n}$ be defined as in fig. 1.
Sample a list CW of t random strings from $\{0,1\}^{\lambda+2t}$.
for $k = 1$ to $|A^{(i-1)}|$ **do**
Parse $G(\mathsf{seed}_b^{(i-1)}[k]) = \mathsf{seed}_b^0 \| \mathsf{sign}_b^0 \| \mathsf{seed}_b^1 \| \mathsf{sign}_b^1$, for $b = 0, 1$, $\mathsf{seed}_b^0, \mathsf{seed}_b^1 \in \{0,1\}^\lambda$ and $\mathsf{sign}_b^0, \mathsf{sign}_b^1 \in \{0,1\}^t$.
Compute $\Delta\mathsf{seed}^c = \mathsf{seed}_0^c \oplus \mathsf{seed}_1^c$ and $\Delta\mathsf{sign}^c = \mathsf{sign}_0^c \oplus \mathsf{sign}_1^c$ for $c = 0, 1$.
Denote path $\leftarrow A^{(i-1)}[k]$.
if both path$\|z$ for $z = 0, 1$ are in $A^{(i)}$ **then**
$d \leftarrow$ the index of path$\|0$ in $A^{(i)}$.
$CW[d] \leftarrow r \| \Delta\mathsf{sign}^0 \oplus e_d \| \Delta\mathsf{sign}^1 \oplus e_{d+1}$ where $r \xleftarrow{\$} \{0,1\}^\lambda$, $e_d = 0^{d-1}10^{t-d}$.
else
Let z be such that path$\|z \in A^{(i)}$.
$d \leftarrow$ the index of path$\|z$ in $A^{(i)}$.
$$CW[d] \leftarrow \begin{cases} \Delta\mathsf{seed}^1 \| \Delta\mathsf{sign}^0 \oplus e_d \| \Delta\mathsf{sign}^1 & z = 0 \\ \Delta\mathsf{seed}^0 \| \Delta\mathsf{sign}^0 \| \Delta\mathsf{sign}^1 \oplus e_d & z = 1 \end{cases}.$$
end if
end for
return CW.
end procedure

procedure GENCONVCW($A, B, \{\mathsf{seed}_b^{(n)}, \mathsf{sign}_b^{(n)}\}$)
Sample a list CW of t random $\mathbb{G}$-elements.
for $k = 1$ to $|A|$ **do**
$\Delta g \leftarrow G_{\mathsf{convert}}(\mathsf{seed}_0^{(n)}[k]) - G_{\mathsf{convert}}(\mathsf{seed}_1^{(n)}[k])$.
$CW[k] \leftarrow (-1)^{\mathsf{sign}_0^{(n)}[k][k]}(\Delta g - B[k])$.
end for
return CW.
end procedure

procedure CORRECT($\bar{x}, \mathsf{sign}, CW$)
return $C_{\mathsf{seed}} \| C_{\mathsf{sign}^0} \| C_{\mathsf{sign}^1} \leftarrow \sum_{i=1}^t \mathsf{sign}[i] \cdot CW[i]$, where C_{sign^0} and C_{sign^1} are t-bit.
end procedure

procedure CONVCORRECT(x, sign, CW)
return $\sum_{i=1}^t \mathsf{sign}[i] \cdot CW[i]$.
end procedure

3.2 Batch-Code DMPF

We display the construction of DMPF from black-box usage of DPF basing on PBC with appropriate parameters, which has been discussed in previous literature[2, 5].

CONSTRUCTION 4 (DMPF FROM DPF). *Given DPF for any domain of size no larger than N and output group $\mathbb{G}$, and an (N, M, t, m, ϵ)-PBC with alphabet $\Sigma = \mathbb{G}$, we can construct a DMPF scheme for t-point functions with domain size N and output group $\mathbb{G}$ as follows:*

- $\mathsf{Gen}(1^\lambda, \hat{f}_{A,B}) \to (k_0, k_1)$*: Suppose $A = \{\alpha_1, \cdots, \alpha_t\}$ and $B = \{\beta_1, \cdots, \beta_t\}$. Let $TT \in \mathbb{G}^N$ be the truth table of $\hat{f}_{A,B}$. Compute* $\mathsf{Encode}(TT) \to (C_1, \cdots, C_m)$ *according to the PBC. Then run* $\mathsf{Decode}(A, C_1, \cdots, C_m)$ *to determine a perfect matching from A to $\{C_1, \cdots, C_m\}$. For $1 \le i \le m$, let $f_i : [|C_i|] \to \mathbb{G}$ be the following:*
  - *If C_i is assigned none of A by the perfect matching, then set f_i to be the all-zero function.*
  - *If exactly one α_j of A is assigned to the lth position of C_i, then set f_i to be the point function that outputs β_j on l and 0 elsewhere.*

  For $1 \le i \le m$, invoke DPF.$\mathsf{Gen}(1^\lambda, f_i) \to (k_0^i, k_1^i)$. *Set $(k_0, k_1) = (\{k_0^i\}_{i\in[m]}, \{k_1^i\}_{i\in[m]})$. If* Decode *fails then run* Encode *and* Decode *again with fresh randomness.*
- $\mathsf{Eval}_b(k_b, x) \to y_b$*: Follow* $\mathsf{Encode}(TT)$ *to determine the positions $l_{j_1}, l_{j_2}, \cdots, l_{j_s}$ such that the xth entry of TT is sent to the l_{j_i}-th position of C_{j_i}. Compute $y_b = \sum_{i=1}^s$ DPF.*$\mathsf{Eval}_b(k_b^{j_i}, l_i)$.

The scheme is correct with overwhelming probability and has distinguish advantage $< 2\epsilon$.

Note that if one use batch code instead of PBC then the DMPF scheme perfectly correct and secure. When instantiating PBC from w-way cuckoo hashing, the *key generation time* is roughly the time needed for computing cuckoo hashing algorithm plus the total time of all DPF.$\mathsf{Gen}(1^\lambda, f_i)$. The *evaluation time* is roughly the total time of all DPF.$\mathsf{Eval}_b(k_b^{j_i}, l_i)$. Similarly, the *full-domain evaluation time* is roughly the total time of all DPF.$\mathsf{FullEval}_b(k_b^j)$ for $j = 1, \ldots, m$.

3.3 OKVS-based DMPF

Displayed in fig. 3. TBD: explain

3.4 Comparison

Comparison table dependent to PRG & $\mathbb{F}$-MUL(list the formulas?)
analyze tradeoff
distributed gen advantage

3.5 Distributed Key Generation

4 APPLICATIONS

4.1 PCG for OLE from Ring-LPN

Characterize parameters
show nonregular optimization
plug in new DMPF and show overall optimization

4.2 PSI-WCA

plug in new DMPF and analyze advantage interval
plug in distributed gen

Table 1: Keysize and running time comparison for different DMPF constructions for domain size N, t accepting points and computational security parameter λ. We leave this table with the abstraction of (probabilistic) batch code in the second column and the abstraction of OKVS in the last column, and plug in concrete instantiations later. m in the second column stands for the number of buckets used in batch code, and d stands for the number of buckets that an input is mapped to (we only consider regular degree because this is the case in most instantiations).

	$t\times$ DPF	MPFSS from (probabilistic) batch code[2][11][5][1]	Big-state DMPF	OKVS-DMPF	
	keysize	$t(\lambda+2)\log N$	$m\lambda\log(N/m)$	$t(\lambda+2t)\log N$	$\log N\times$OKVS code size
Gen()	Dominating operations	$2t\log N\times$ PRG	$2m\log(dN/m)\times$PRG Finding a matching of t inputs to m buckets	$2t\log N\times$PRG	$2t\log N\times$PRG, $\log N\times$OKVS Encoding
	Cheap operations	$O(t\lambda\log N)$	$O(m\lambda\log(dN/m))$	$O(t(\lambda+t)\log N)$	$O(t\lambda\log N)$
Eval()	Dominating operations	$t\log N\times$PRG	$d\log(dN/m)\times$PRG Finding all buckets an input is mapped to	$\log N\times$PRG	$\log N\times$PRG, $\log N\times$OKVS Decoding
	Cheap operations	$O(t\lambda\log N)$	$O(d\lambda\log(dN/m))$	$O((\lambda+t)\log N)$	$O(\lambda\log N)$
FullEval()	Dominating operations	$tN\times$PRG	$dN\times$PRG Finding the input sequence in every bucket	$N\times$PRG	$N\times$PRG, $N\times$ OKVS Decoding
	Cheap operations	$O(t\lambda N)$	$O(d\lambda N)$	$O((\lambda+t)N)$	$O(\lambda N)$

Figure 3: The parameter l and methods' setting that turns the paradigm of DMPF in fig. 1 into the OKVS-based DMPF.

Set $l \leftarrow 1$.
For $1 \le i \le n$, let OKVS_i be an OKVS scheme (definition 5) with key space $\mathcal{K} = \{0,1\}^{i-1}$, value space $\mathcal{V} = \{0,1\}^{\lambda+2}$ and input length t.
let $\mathsf{OKVS}_{\mathsf{convert}}$ be an OKVS scheme with key space $\mathcal{K} = \{0,1\}^n$, value space $\mathcal{V} = \mathbb{G}$ and input length t.

procedure Initialize($\{\mathsf{seed}_b^{(0)}, \mathsf{sign}_b^{(0)}\}_{b=0,1}$)
  For $b = 0,1$, let $\mathsf{seed}_b^{(0)} = [r_b \xleftarrow{\$} \{0,1\}^\lambda]$ and $\mathsf{sign}_b^{(0)} = [b]$.
end procedure

procedure GenCW($i, A, \{\mathsf{seed}_b^{(i-1)}, \mathsf{sign}_b^{(i-1)}\}_{b=0,1}$)
  Let $\{A^{(i)}\}_{0\le i\le n}$ be defined as in fig. 1.
  Sample a list V of t random strings from $\{0,1\}^{\lambda+2}$.
  for $k = 1$ to $|A^{(i-1)}|$ **do**
    Parse $G(\mathsf{seed}_b^{(i-1)}[k]) = \mathsf{seed}_b^0 \| \mathsf{sign}_b^0 \| \mathsf{seed}_b^1 \| \mathsf{sign}_b^1$, for $b = 0,1$, $\mathsf{seed}_b^0, \mathsf{seed}_b^1 \in \{0,1\}^\lambda$ and $\mathsf{sign}_b^0, \mathsf{sign}_b^1 \in \{0,1\}$.
    Compute $\Delta\mathsf{seed}^c = \mathsf{seed}_0^c \oplus \mathsf{seed}_1^c$ and $\Delta\mathsf{sign}^c = \mathsf{sign}_0^c \oplus \mathsf{sign}_1^c$ for $c = 0,1$.
    Denote path $\leftarrow A^{(i-1)}[k]$.
    if both path$\|z$ for $z = 0,1$ are in $A^{(i)}$ **then**
      $V[k] \leftarrow r \| \Delta\mathsf{sign}^0 \oplus 1 \| \Delta\mathsf{sign}^1 \oplus 1$, where $r \xleftarrow{\$} \{0,1\}^\lambda$.
    else
      Let z be such that path$\|z \in A^{(i)}$.
      $V[k] \leftarrow \Delta\mathsf{seed}^1 \| \Delta\mathsf{sign}^0 \oplus (1-z) \| \Delta\mathsf{sign}^1 \oplus z$.
    end if
  end for
  return $\mathsf{OKVS}_i.\mathsf{Encode}(\{A^{(i-1)}[k], V[k]\}_{1\le k\le |A^{(i-1)}|})$.
end procedure

procedure GenConvCW($A, B, \{\mathsf{seed}_b^{(n)}, \mathsf{sign}_b^{(n)}\}$)
  Sample a list V of t random $\mathbb{G}$-elements.
  for $k = 1$ to $|A|$ **do**
    $\Delta g \leftarrow G_{\mathsf{convert}}(\mathsf{seed}_0^{(n)}[k]) - G_{\mathsf{convert}}(\mathsf{seed}_1^{(n)}[k])$.
    $V[k] \leftarrow (-1)^{\mathsf{sign}_0^{(n)}[k][k]}(\Delta g - B[k])$.
  end for
  return $\mathsf{OKVS}_{\mathsf{convert}}(\{A[k], V[k]\}_{1\le k\le t})$.
end procedure

procedure Correct($\bar{x}, \mathsf{sign}, CW$)
  return $C_{\mathsf{seed}} \| C_{\mathsf{sign}^0} \| C_{\mathsf{sign}^1} \leftarrow \mathsf{sign} \cdot \mathsf{OKVS}_i.\mathsf{Decode}(CW, \bar{x})$, where C_{sign^0} and C_{sign^1} are bits.

4.3 Heavy-hitters

private heavy-hitter
or parallel ORAM?

5 ACKNOWLEDGMENTS

tbd

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A BATCH-CODE DMPF SCHEME

B SECURITY PROOFS