VV285 RC Part I

Elements of Linear Algebra

"Linear Algebra!"

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Something you need to pay attention to...

Think More and Be Interactive!

- Do think more about the question in "()".
  e.g. "(How to prove?)"
- You are welcome to ask questions in a adequate manner.
- DO MORE PRACTICE

Overview of Linear Algebra

1. Systems of Linear Equations
2. Finite-Dimensional Vector Spaces
3. Inner Product Spaces
4. Linear Maps
5. Matrices
6. Theory of Systems of Linear Equations
7. Determinants

Warm up!

Prove or give a counterexample: If v_1, v_2, v_3, v_4 is a basis of V and U is a subspace of V such that $v_1, v_2 \in U$ and $v_3, v_4 \notin U$, then v_1, v_2 is a basis of U.

Warm up!

Suppose $p_0, p_1, \ldots, p_m$ are polynomials in $\mathcal{P}_m(\mathbb{F})$ such that $p_k(2) = 0$ for each k. Prove that $p_0, p_1, \ldots, p_m$ is not linearly independent in $\mathcal{P}_m(\mathbb{F})$

Overview - Inner Product Spaces

1. Inner Product Spaces
2. Induced Norm
3. Orthogonality & Orthonormal System
4. The Projection Theorem
5. Gram-Schmidt Orthonormalization

Inner Product Space I

Let V be a real or complex vector space. Then a map $\langle \cdot , \cdot \rangle : V \times V \to \mathbb{F}$ is called a scalar product or inner product if for all $u, v, w \in V$ and all $\lambda \in \mathbb{F}$

1. *Positive-definite*
   $\langle v , v \rangle \geq 0$ and $\langle v , v \rangle = 0$ if and only if $v = 0$,
2. *Linearity in the 2nd argument*
   $\langle u , v + w \rangle = \langle u , v \rangle + \langle u , w \rangle$
3. *Linearity in the 2nd argument*
   $\langle u , \lambda v \rangle = \lambda \langle u , v \rangle$
4. *Conjugate symmetry*
   $\langle u , v \rangle = \overline{\langle v , u \rangle}$

The pair $(V, \langle \cdot , \cdot \rangle)$ is called an *inner product space*.

Inner Product Space II

Prove that

1. $$\langle \lambda u \, , \, v \rangle = \overline{\lambda} \langle u \, , \, v \rangle.$$

2. $$\langle u + v \, , \, w \rangle = \langle u \, , \, w \rangle + \langle v \, , \, w \rangle$$

This is called the *conjugate linearity* in the 1st argument.

What if $\mathbb{F} = \mathbb{R}$?

Inner Product Space III

Why is inner product space important?

- allow the rigorous introduction of intuitive geometrical notions such as the length of a vector or the angle between two vectors
- provide the means of defining orthogonality between vectors (zero inner product)
- generalize Euclidean spaces (in which the inner product is the dot product, also known as the scalar product) to vector spaces of any (possibly infinite) dimension, and are studied in functional analysis.
- naturally induces an associated *norm*, thus an inner product space is also a *normed vector space*.

Induced Norm I

Let $(V, \langle \cdot , \cdot \rangle)$ be an inner product space. The map

$$\| \cdot \| : V \to \mathbb{R}, \qquad\qquad \|v\| = \sqrt{\langle\, v\,,\, v\,\rangle}$$

is called the *induced norm* on V.

(How to prove that an induced norm is actually a norm?)

3.3.8. Definition. Let V be a real (complex) vector space. Then a map $\|\cdot\| : V \to \mathbb{R}$ is called a norm if for all $u, v \in V$ and all $\lambda \in \mathbb{R}$ $(\mathbb{C})$,

1. $\|v\| \geq 0$ for all $v \in V$ and $\|v\| = 0$ if and only if $v = 0$,
2. $\|\lambda \cdot v\| = |\lambda| \cdot \|v\|$,
3. $\|u + v\| \leq \|u\| + \|v\|$.

The pair $(V, \|\cdot\|)$ is called a normed vector space or a normed linear space.

Induced Norm II

- In $\mathbb{C}^n$ we can define the inner product

$$\langle x , y \rangle := \sum_{i=1}^{n} \overline{x_i} y_i \qquad\qquad x, y \in \mathbb{C}^n.$$

- In $C([a, b])$, the space of complex-valued, continuous functions on the interval $[a, b]$, we can define an inner product by

$$\langle f , g \rangle := \int_a^b \overline{f(x)} g(x) dx, \qquad\qquad f, g \in C([a, b]).$$

Remark: Pay attention to the conjugate in two definitions. We will study further on the last one in VV286 to establish *Fourier Series*!

Induced Norm III

By *the Cauchy-Schwarz inequality*, we define the *angle* $\alpha(u, v) \in [0, \pi]$ *between u and v* by

$$\cos \alpha(u, v) = \frac{\langle u, v \rangle}{\|u\| \|v\|}. \tag{1}$$

We are particularly interested in the case that $\alpha = \pi/2$. i.e. $\langle u, v \rangle = 0$. Therefore, we introduce *orthogonality*.

Exercise

Try to prove yourself!

- For real inner product space:

$$< u, v >= \frac{||u+v||^2 - ||u-v||^2}{4}$$

- For inner product spaces V_1, V_2,... V_m,

$$< (u_1, ...u_m), (v_1, ...v_m) >=< u_1, v_1 > + ... + < u_m, v_m >$$

Orthogonality I

Let $(V, \langle \cdot , \cdot \rangle)$ be an inner product vector space.

1. Two vectors $u, v \in V$ are called *orthogonal* or *perpendicular* if $\langle u , v \rangle = 0$. We then write $u \perp v$.
2. We call
$$M^{\perp} := \left\{ v \in V : \underset{m \in M}{\forall} \langle m , v \rangle = 0 \right\}$$
the *orthogonal complement* of a set $M \subset V$.

For short, we sometimes write $v \perp M$ instead of $v \in M^{\perp}$ or $v \perp m$ for all $m \in M$.

Remark: The orthogonal complement $M^{\perp}$ is a subspace of V. (How to prove?)

Orthonormal Systems & Bases I

Let $(V, \langle \cdot\,, \cdot \rangle)$ be an inner product vector space. A tuple of vectors $(v_1, v_2, \dots, v_r) \in V$ is called a *(finite) orthonormal system* if

$$\langle\, v_j\,,\, v_k \,\rangle = \delta_{jk} := \begin{cases} 1 & \text{for } j = k, \\ 0 & \text{for } j \neq k, \end{cases}, \qquad\qquad j, k = 1, \dots, r,$$

i.e., if $\|v_k\| = 1$ and $v_j \perp v_k$ for $j \neq k$.

Let $(V, \langle \cdot\,, \cdot \rangle)$ be a finite-dimensional inner product vector space and $\mathcal{B} = (e_1, \dots, e_n)$ a basis of V. If $\mathcal{B}$ is also an orthonormal system, we say that $\mathcal{B}$ is an *orthonormal basis* (ONB).

Orthonormal Systems & Bases II

Let $(V, \langle \cdot, \cdot \rangle)$ be a finite-dimensional inner product vector space and $\mathcal{B} = \{e_1, \ldots, e_n\}$ an orthonormal basis of V. Then

$$\|v\|^2 = \sum_{i=1}^{n} |\langle v, e_i \rangle|^2$$

for any $v \in V$.

Remark: Parseval's Theorem gives a alternative way to calculate a vector's induced norm.

Linear Functionals on Inner Product Space

A linear functional on V is a linear map from V to F. In other words, a linear functional is an element of $\mathbf{L}(V, F)$

Example

$$\phi(p) = \int_{-1}^{1} p(t) cos(t) dt$$

is a linear functional on $P_2(R)$

Riesz Representation Theorem Suppose V is finite-dimensional and ϕ is a linear functional on V. Then there is a unique vector $u \in V$ such that:

$$\phi(v) = \langle v, u \rangle$$

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Proof:

First we show there exists a vector $u \in V$ such that $phi(v) = \langle v, u \rangle$ for every $v \in V$. Let $e_1, \cdots, e_n$ be an orthornormal basis of V. Then

$$\begin{aligned}
\phi(v) &= \phi(\langle v, e_1 \rangle e_1 + ... + \langle v, e_n \rangle e_n) \\
&= \langle v, e_1 \rangle \phi(e_1) + ... + \langle v, e_n \rangle \phi(e_n) \\
&= \langle v, \overline{\phi(e_1)} e_1 + ... \overline{\phi(e_n)} e_n \rangle
\end{aligned}$$

for every $v \in V$, where the first equality comes from 6.30. Thus setting

$$u = \overline{\phi(e_1)} e_1 + \cdots \overline{\phi(e_n)} e_n \tag{2}$$

we have $\phi(v) = \langle v, u \rangle$ for every $v \in V$, as desired. Now we prove that only one vector $u \in V$ has the desired behavior.Suppose u_1, $u_2 \in V$ are such that

$$\phi(v) = \langle v, u_1 \rangle = \langle v, u_2 \rangle$$

for every $v \in V$. Then

$$0 = \langle v, u_1 \rangle - \langle v, u_2 \rangle = \langle v, u_1 - u_2 \rangle$$

for every $v \in V$. Taking $v = u_1 - u_2$ shows that $u_1 - u_2 = 0$. In other words, $u_1 = u_2$, completing the proof of the uniqueness part of the result.

Exercise*

For $u \in V$, Let ϕu denote the linear functional on V defined by:

$$(\phi u)(v) = \langle v, u \rangle$$

- Show that if $\mathbf{F} = \mathbf{R}$, then ϕ is a linear map from V to V'.
- Show that if $\mathbf{F} = \mathbf{C}$ and V $\neq \{0\}$. then ϕ is not a linear map.
- Show that ϕ is injective.
- Suppose $\mathbf{F}=\mathbf{R}$ and V is finite-dimensional. Use the first and third parts and a dimension-counting argument to show that ϕ is isomorphism from V to V'

Projection Theorem I

Let $(V, \langle \cdot , \cdot \rangle)$ be a (possibly infinite-dimensional) inner product vector space and $(e_1, e_2, \ldots, e_r)$, $r \in \mathbb{N}$, be an orthonormal system in V. Denote $U := \operatorname{span}\{e_1, \ldots, e_r\}$.
Then for every $v \in V$ there exists a unique representation

$$v = u + w \qquad\qquad \text{where } u \in U \text{ and } w \in U^{\perp}$$

and $u = \sum_{i=1}^{r} \langle e_i , v \rangle e_i$, $w := v - u$. The vector

$$\pi_U v := \sum_{i=1}^{r} \langle e_i , v \rangle e_i$$

is called the *orthogonal projection* of v onto U.

Projection Theorem II

The projection theorem essentially states that $\pi_U v$ **always exists** and is independent of the choice of the orthonormal system (it depends only on the span U of the system).

Moreover, it generalize the idea of projection:

$$\pi_{e_i} v \longrightarrow \pi_U v$$

A vector in an inner product space can be decomposed not only on its *orthonormal basis* but also on its *subspaces*.

Exercise

- Prove that $cos(nx)$ are mutually independent
- Prove that $sin(nx)$ are mutually independent
- Find a set of orthogonal basis
- Express function $y = rect(x + 2n)$ with the basis you find

Gram-Schmidt Orthonormalization

Just remember how to do it.

$$w_1 := \frac{v_1}{\|v_1\|}$$

$$w_k := \frac{v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j}{\left\| v_k - \sum_{j=1}^{k-1} \langle w_j, v_k \rangle w_j \right\|}, \quad k = 2, \ldots, n$$

How to use Gram-Schmidt Orthonormalization to obtain *Legendre polynomials*?

Linear Map

A map $L : U \leftarrow V$ is said to be *linear* if it is both

homogeneous, i.e.

$$L(\lambda u) = \lambda L(u)$$

additive,i,e.

$$L(u+v) = L(u) + L(v)$$

structural-perserving

$$\begin{tikzcd}
U \arrow[r, "L"] \arrow[d, "\lambda\odot"'] & V \arrow[d, "\lambda\boxdot"] \\
U & V \arrow[l, "L^{-1}"']
\end{tikzcd}$$

Linear Map

Examples

- We let 0 denote the function that takes each element of some vector space to the additive identity of another vector space.
$$0v = 0$$
- The *identity map*, denoted I, is the function on some vector space that takes each element to itself.
$$Iv = v$$
- Define $D \in L(P(R), P(R))$ by
$$Dp = p^{'}$$
- *backward shift*
Define $T \in L(F^{\infty}, F^{\infty})$ by
$$T(x_1, x_2, x_3, \cdots) = (x_2, x_3, x_4, ...)$$

4.4. Theorem. Let U, V be real or complex vector spaces and $(b_1, \ldots, b_n)$ a basis of U (in particular, it is assumed that $\dim U = n < \infty$). Then for every n-tuple $(v_1, \ldots, v_n) \in V^n$ there exists a unique linear map $L\colon U \to V$ such that $Lb_k = v_k$, $k = 1, \ldots, n$.

4.6. Examples.

(i) If V is a real or complex vector space and $(b_1, \ldots, b_n)$ a basis of V, then the ***coordinate map***

$$\varphi\colon V \to \mathbb{F}^n, \qquad v = \sum_{k=1}^{n} \lambda_k b_k \mapsto \begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_n \end{pmatrix}$$

is linear (and bijective).

4.7. Examples.

(ii) Let V be a real or complex vector space. Then $\mathscr{L}(V, \mathbb{F})$ is known as the ***dual space*** of V and denoted by V^*. The dual space of V is of course itself a vector space.

Let $\dim V = n < \infty$ and $\mathscr{B} = (b_1, \ldots, b_n)$ be a basis of V. Then for every $k = 1, \ldots, n$ there exists a unique map

$$b_k^* : V \to \mathbb{F}, \qquad b_k^*(b_j) = \delta_{jk} = \begin{cases} 1, & j = k, \\ 0, & j \neq k. \end{cases}$$

It turns out (see exercises) that the tuple of maps $\mathscr{B}^* = (b_1^*, \ldots, b_n^*)$ is a basis of $V^* = \mathscr{L}(V, \mathbb{F})$ (called the ***dual basis*** of $\mathscr{B}$) and thus $\dim V^* = \dim V = n$.

4.8. Definition. Let U, V be real or complex vector spaces and $L \in \mathscr{L}(U, V)$. Then we define the range of L by

$$\operatorname{ran} L := \left\{ v \in V : \underset{u \in U}{\exists}\; v = Lu \right\}$$

and the ***kernel*** of L by

$$\ker L := \{ u \in U : \; Lu = 0 \}.$$

It is easy to see that $\operatorname{ran} L \subset V$ and $\ker L \subset U$ are subspaces.

4.9. Remark. It is not difficult to see that $L \in \mathscr{L}(U, V)$ is injective if and only if $\ker L = \{0\}$.

Linear Map

4.12. Definition. Two vector spaces U and V are called ***isomorphic***, written $U \cong V$, if there exists an isomorphism $\varphi \colon U \to V$.

4.13. Lemma. Two finite-dimensional vector spaces U and V are isomorphic if and only if they have the same dimension:

$$U \cong V \quad \Leftrightarrow \quad \dim U = \dim V$$

4.14. Dimension Formula. Let U, V be real or complex vector spaces, $\dim U < \infty$. Let $L \in \mathscr{L}(U, V)$. Then

$$\dim \operatorname{ran} L + \dim \ker L = \dim U. \tag{4.3}$$

4.15. Corollary. Let U, V be real or complex finite-dimensional vector spaces with $\dim U = \dim V$. Then a linear map $L \in \mathscr{L}(U, V)$ is injective if and only if it is surjective.

Proof.

$$\begin{aligned} L \text{ injective} \quad &\Leftrightarrow \quad \ker L = \{0\} \\ &\Leftrightarrow \quad \dim \ker L = 0 \\ &\Leftrightarrow \quad \dim \operatorname{ran} L = \dim U = \dim V \\ &\Leftrightarrow \quad \operatorname{ran} L = V \\ &\Leftrightarrow \quad L \text{ surjective} \end{aligned}$$

□

Linear Map

4.16. Definition. Let $(U, \|\cdot\|_U)$ and $(V, \|\cdot\|_V)$ be normed vector spaces. Then a linear map $L\colon U \to V$ is said to be ***bounded*** if there exists some constant $c > 0$ (called a ***bound*** for L) such that

$$\|Lu\|_V \le c \cdot \|u\|_U \qquad \text{for all } u \in U. \tag{4.6}$$

Exercise

Give an example of a function $\phi : C \to C$ such that

$$\phi(w + z) = \phi(w) + \phi(z)$$

for all $w, z \in C$ but ϕ is not linear. Here C is a complex vector space.

Exercise

Suppose $v_2, ..., v_m$ is a linearly dependent list of vectors in V. Suppose also that $W \neq \{0\}$. Prove that there exist $w_1, \cdots, w_m \in W$ such that no $T \in L(V, W)$ satisfies $Tv_k = w_k$ for each k=1,...m.

Injectivity and Surjectivity

A function $T : V \rightarrow W$ is called *injective* if Tu=Tv imples u=v
A function $T : V \rightarrow W$ is called *injective* if $\ker T = \{0\}$
A function $T : V \rightarrow W$ is called *surjective* if $\operatorname{ran} T = W$
Suppose V and W are finite-dimensional vector spaces such that dimV $>$ dimW. Then no linear map from V to W is injective.
Suppose V and W are finite-dimensional vector spaces such that dimV $<$ dimW. Then no linear map from W to V is surjective.

Learn Well

And

Have Fun!