VV285 RC Part II

Elements of Linear Algebra

"Linear Algebra!"

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Systens of Equations

We briefly return to the theory of solvability of linear systems of equat
$Ax = b$. We define the ***solution set***

$$\text{Sol}(A, b) = \{x \in \mathbb{R}^n \colon Ax = b\}.$$

If $x_0 \in \mathbb{R}^n$ satisfies

$$Ax_0 = b$$

we say that x_0 is a ***particular solution*** of $Ax = b$. The ***associated homogeneous solution set*** is

$$\text{Sol}(A, 0) = \{x \in \mathbb{R}^n \colon Ax = 0\} = \ker A.$$

A very important, fundamental result states:

The solution set of $Ax = b$ is the sum of the homogeneous solution set and a particular solution.

6.1. Lemma. Let $x_0 \in \mathbb{R}^n$ be a particular solution of $Ax = b$. Then

$$\mathrm{Sol}(A, b) = \{x_0\} + \ker A = \{y \in \mathbb{R}^n \colon y = x_0 + x,\ \ x \in \ker A\}.$$

where the sum of sets is understood as in Definition 2.24.

6.8. Theorem. There exists a solution x for $Ax = b$ if and only if $\operatorname{rank} A = \operatorname{rank}(A \mid b)$, where

$$(A \mid b) = \begin{pmatrix} a_{11} & \dots & a_{1n} & b_1 \\ \vdots & & \vdots & \vdots \\ a_{m1} & \dots & a_{mn} & b_m \end{pmatrix} \in \operatorname{Mat}((n+1) \times m).$$

Systens of Equations

$$
\begin{aligned}
& Ax = b \text{ has solution } x \in \mathbb{R}^n \\
\Leftrightarrow\ & b \in \operatorname{ran} A \\
\Leftrightarrow\ & b \in \operatorname{span}\{a_{\cdot 1}, \ldots, a_{\cdot n}\} \\
\Leftrightarrow\ & b \text{ is not independent of } a_{\cdot 1}, \ldots, a_{\cdot n} \\
\Leftrightarrow\ & \dim \operatorname{span}\{a_{\cdot 1}, \ldots, a_{\cdot n}\} = \dim \operatorname{span}\{a_{\cdot 1}, \ldots, a_{\cdot n}, b\} \\
\Leftrightarrow\ & \dim \operatorname{ran} A = \dim \operatorname{ran}(A \mid b) \\
\Leftrightarrow\ & \operatorname{rank} A = \operatorname{rank}(A \mid b)
\end{aligned}
$$

6.4. Fredholm Alternative. Let A be an $n \times n$ matrix. Then

- either $Ax = b$ has a unique solution for any $b \in \mathbb{R}^n$
- or $Ax = 0$ has a non-trivial solution.

6.7. Definition and Theorem. Let $A \in \operatorname{Mat}(m \times n; \mathbb{F})$. Then the column rank is equal to the row rank and we define the ***rank*** of A by

$$\operatorname{rank} A := \text{column rank}\, A = \text{row rank}\, A.$$

Proof.

In the assignments it will be shown that

$$\operatorname{ran} \overline{A}^T = (\ker A)^\perp.$$

Then, using Corollary 3.24 and the dimension formula (4.3),

$$\begin{aligned} \text{row rank}\, A &= \text{column rank}\, A^T = \text{column rank}\, \overline{A}^T = \dim \operatorname{ran} \overline{A}^T \\ &= \dim(\ker A)^\perp = n - \dim \ker A = \dim \operatorname{ran} A = \text{column rank}\, A. \end{aligned}$$

Here we have used that complex conjugation is a linear, bijective map $\mathbb{C} \to \mathbb{C}$ if $\mathbb{C}$ is regarded as a real vector space. $\square$

6.5. Definition. Let $A \in \text{Mat}(m \times n; \mathbb{F})$ be a matrix with columns $a_{\cdot j} \in \mathbb{F}^m$, $1 \leq j \leq n$, and rows $a_{i\cdot} \in \mathbb{F}^n$, $1 \leq i \leq m$. Then we define

- the ***column rank*** of A to be

$$\text{column rank}\, A := \dim \text{span}\{a_{\cdot 1}, \dots, a_{\cdot n}\}$$

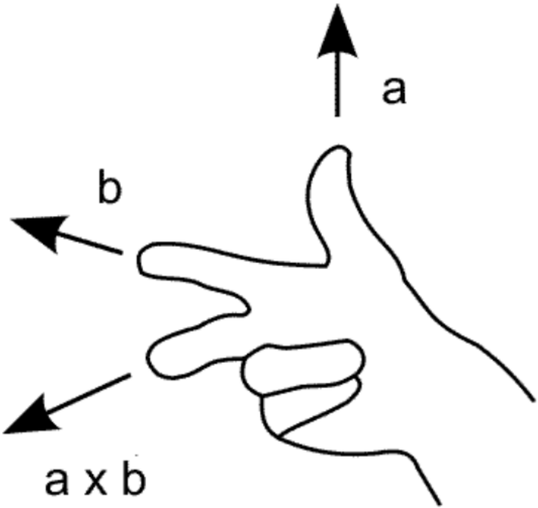
- and the ***row rank*** of A to be

$$\text{row rank}\, A := \dim \text{span}\{a_{1\cdot}, \dots, a_{m\cdot}\}.$$

6.6. Remarks.

- The column rank is the greatest number of independent column vectors $a_{\cdot j}$ that can be selected from all columns. This is analogously true for the row rank.
- $\text{column rank}\, A = \text{row rank}\, A^T$.
- $\text{column rank}\, A = \dim \text{ran}\, A$.

Determinant



Learn Well

And

Have Fun!