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A nodal-based high-order nonlinear viscosity finite element method for MHD

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Outline

1. Parameter-free first-order method for scalar conservation laws
2. Extension to systems and higher-order methods
3. Application to the system of MHD equations
4. Summary and outlook

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Motivation

- Conventional viscous regularization to scalar conservation laws

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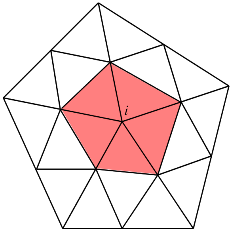
    - Extend for systems

First-order PP method for $\mathbb{P}_1$



S_i

At node i, define

- S_i – support of the Lagrange basis function φ_i

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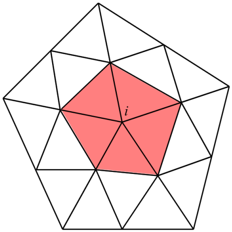

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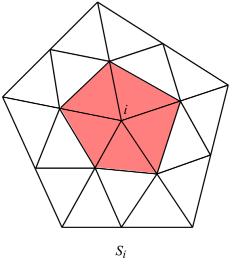

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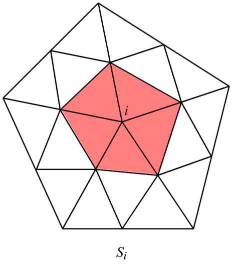

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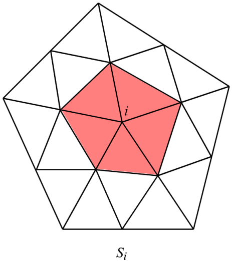

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- m_i – weight of the lumped $\mathbb{P}_1$ mass matrix at index i

First-order PP method for $\mathbb{P}_1$

- $C_i := \frac{1}{N_{\mathrm{el}}(S_i)} \max_{K\in S_i, j\neq i, j\in\mathcal{I}(S_i)} \left| \int_K (\mathbb{J}_K^\top \nabla\varphi_j)\cdot(\mathbb{J}_K^\top \nabla\varphi_i)\,\mathrm{d}\boldsymbol{x} \right|^{-1}$

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Theorem

Define the nodal viscosity as

$$\varepsilon_i := \alpha_i C_i \|\boldsymbol{f}'\|_{L^\infty(S_i)} \max_{i \neq j \in \mathcal{I}(S_i)} |\nabla \varphi_j| m_i.$$

Under a usual CFL condition, the approximation

$$m_i \frac{u_i^{n+1} - u_i^n}{\tau} + \int_{S_i} \nabla \cdot \boldsymbol{f}(u_h^n) \varphi_i \, \mathrm{d}\boldsymbol{x} + \sum_{K \in S_i} \sum_j \varepsilon_j u_j^n b_K(\varphi_j, \varphi_i) = 0$$

is positivity preserving. The form $b_K(\cdot, \cdot)$ is defined next.

Alternative $\mathbb{P}_1$ viscous bilinear form in 2D

For each triangle element K, define $\Phi_K := \widehat{K} \to K$, where $\widehat{K}$ is the equilateral triangle such that $|\widehat{K}| = |K|$. $\mathbb{J}_K$ is the Jacobian matrix of Φ_k.

Tensor-valued viscosity ([Guermond & Nazarov, 2014], edge-wise)

$$b(u,v) := \int_\Omega (\mathbb{J}^\top \nabla u)\cdot(\mathbb{J}^\top \nabla v)\,\mathrm{d}\boldsymbol{x}, \quad \forall u,v \in V_h.$$

(2D) Local bilinear form, for $i,j \in \mathcal{I}(K)$,

$$b_K(\varphi_j,\varphi_i) := \begin{cases} -\frac{1}{n_K-1} & \text{if } j \neq i, \\ 1 & \text{if } j = i, \end{cases}$$

where n_K is the number of nodes in K.

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Extensions

- High-order viscosity methods: Residual-based, Entropy-based

$$\varepsilon_{h,i}^{\mathrm{H}} := C_i \alpha_i \min\left(\|\boldsymbol{f}'\|_{L^\infty(S_i)} \max_{i \neq j \in \mathcal{I}(S_i)} |\nabla \varphi_j|,\ |R_i(q_h)| \right) m_i^{(\mathbb{P}_1)}$$

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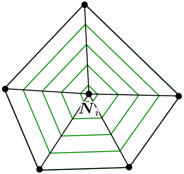
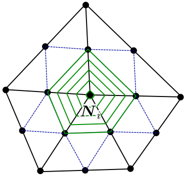
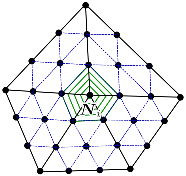
- Other polynomial spaces: Calculate ϵ_i on $\mathbb{P}_1$ sub-mesh



(a) $\mathbb{P}_1$



(b) $\mathbb{P}_2$



(c) $\mathbb{P}_3$

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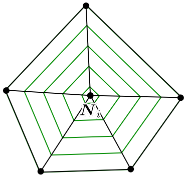

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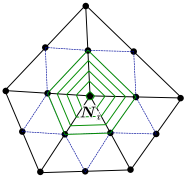
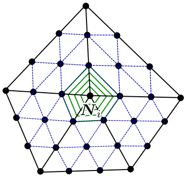


(b) $\mathbb{P}_2$



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- Systems: take maximum of the components

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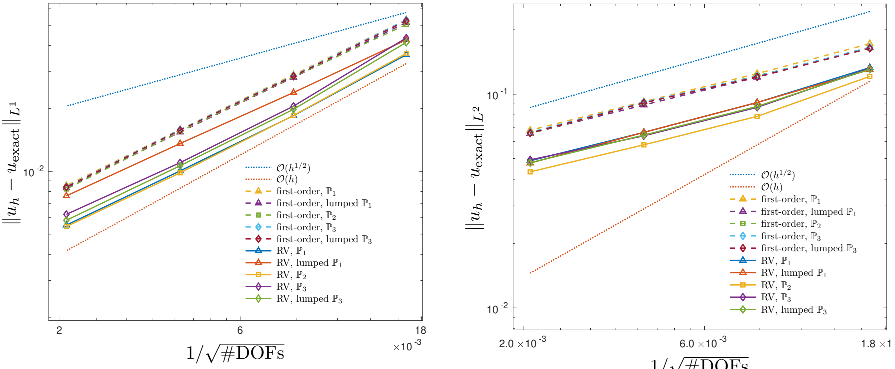
- The viscosity term does not affect time step limit

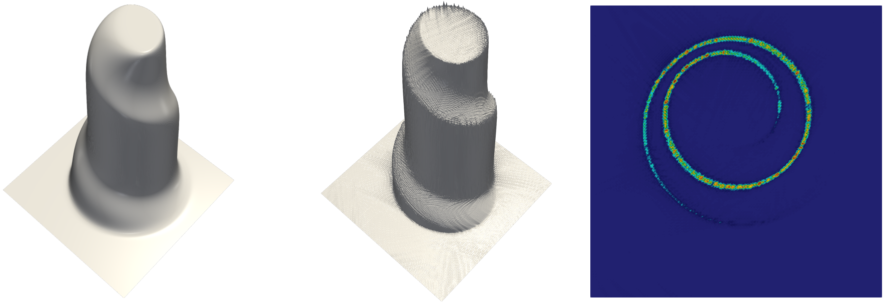
Convergence test on Burger's equation with shocks



(a) L^1-error (b) L^2-error

Figure: First-order $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ shock solutions to the Burger's equation

The KPP problem



(a) First order solution

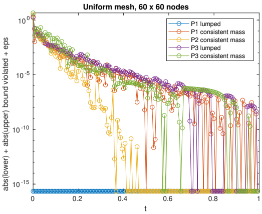
(b) RV solution

(c) RV viscosity

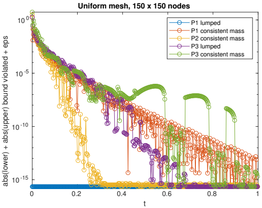
Figure: KPP solution with 300×300 $\mathbb{P}_3$ nodes

The KPP problem

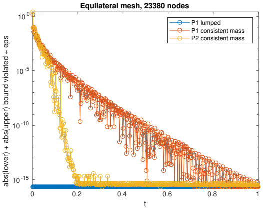
First order solutions



(a) 60×60 mesh



(b) 150×150 mesh



(c) Equilateral mesh

Figure: Discrete maximum principle violation in higher order polynomials.

Outline

The system of MHD equations

$\mathbf{U} := \big(\rho(\boldsymbol{x},t), \boldsymbol{m}(\boldsymbol{x},t), E(\boldsymbol{x},t), \boldsymbol{B}(\boldsymbol{x},t)\big)^\top$, $(\boldsymbol{x},t) \in \mathbb{R}^d \times \mathbb{R}^+$
Density, momentum, total energy, magnetic field

The ideal MHD equations

$$\partial_t \mathbf{U} + \nabla\cdot \big(\mathbf{F}_{\mathcal{E}}(\mathbf{U}) + \mathbf{F}_{\mathcal{B}}(\mathbf{U})\big) = 0, \tag{1}$$

$$\mathbf{F}_{\mathcal{E}} := \begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{m}\otimes\boldsymbol{u} + p\mathbb{I} \\ \boldsymbol{u}(E+p) \\ 0 \end{pmatrix}, \quad \mathbf{F}_{\mathcal{B}} := \begin{pmatrix} 0 \\ -\boldsymbol{\beta} \\ -\boldsymbol{\beta}\boldsymbol{u} \\ \boldsymbol{u}\otimes\boldsymbol{B} - \boldsymbol{B}\otimes\boldsymbol{u} \end{pmatrix},$$

$\boldsymbol{\beta}$ is the Maxwell stress tensor:

$$\boldsymbol{\beta} = \Big(-\frac{1}{2}(\boldsymbol{B}\cdot\boldsymbol{B})\mathbb{I} + \boldsymbol{B}\otimes\boldsymbol{B}\Big).$$

Monolithic parabolic regularization to the MHD equations

Regularize (1) by adding $\nabla\cdot(\epsilon\nabla\mathbf{U})$,

$$\partial_t\mathbf{U} + \nabla\cdot\mathbf{F}_{\mathcal{E}}(\mathbf{U}) + \nabla\cdot\mathbf{F}_{\mathcal{B}}(\mathbf{U}) = \nabla\cdot(\epsilon\nabla\mathbf{U}), \tag{2}$$

where ϵ is a small artificial viscosity parameter.

The regularized equation (2) is

- a continuous analogue of the well-known upwind, Lax-Friedrichs schemes;
- suitable to apply the new viscosity method.

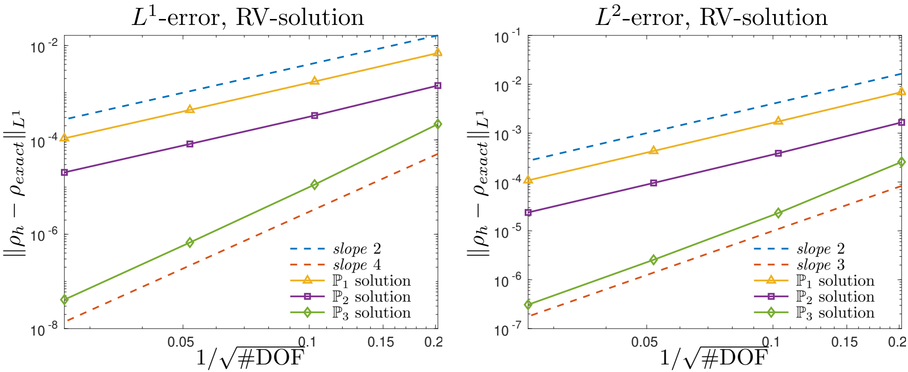
At the PDE level, we have shown that (2)

- is compatible with all the generalized entropies [Harten, 1998];
- fulfills the minimum entropy principles;
- preserves $\rho > 0, p > 0$.

(See [D. & Nazarov, CMAME, 2022])

Convergence test on a smooth problem

Smooth wave propagation, [Wu and Shu, 2018].



(a) L^1-error (b) L^2-error

Figure: $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ convergence for smooth solutions.

The Orszag-Tang problem [Orszag & Tang, 1998]

Domain $\Omega = [0, 1] \times [0, 1]$. Gas constant $\gamma = \frac{5}{3}$.
Initial solution:

$$(\rho_0, \boldsymbol{u}_0, p_0, \boldsymbol{B}_0) =$$
$$\left(\frac{25}{36\pi}, (-\sin(2\pi y), \sin(2\pi x)), \frac{5}{12\pi}, \left(-\frac{\sin(2\pi y)}{\sqrt{4\pi}}, \frac{\sin(4\pi x)}{\sqrt{4\pi}}\right)\right)$$

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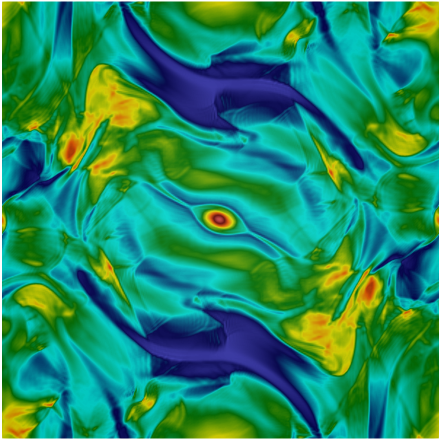
- at $t = 1.0$ considered transition to turbulence

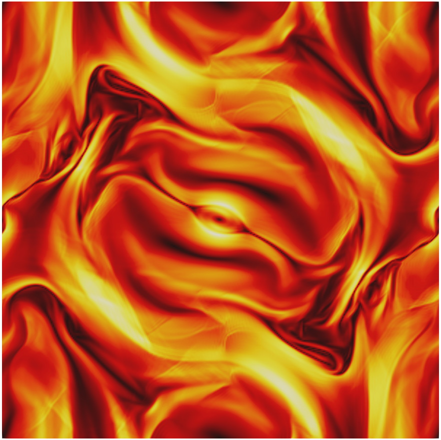
Orszag-Tang problem [Orszag & Tang, 1998]

$\mathbb{P}_3$ RV solution, 100×100 quasi-uniform mesh

Pressure



Magnetic pressure $B_h^2/2$



The MHD Rotor problem [Balsara & Spicer, 1998]

Domain $\Omega = [0,1] \times [0,1]$. Gas constant $\gamma = 1.4$.
Initial solution:

$$p_0 = 1, \boldsymbol{B}_0 = \left(\frac{5}{\sqrt{4\pi}}, 0\right)$$

$$(\rho_0, \boldsymbol{u}_0) = \begin{cases} \left(10, \left(\frac{2}{r_0}\left(\frac{1}{2} - y\right), \frac{2}{r_0}\left(x - \frac{1}{2}\right)\right)^\top\right) & \text{if } r < r_0, \\ \left(1 + 9f, \left(f\frac{2}{r}\left(\frac{1}{2} - y\right), f\frac{2}{r}\left(x - \frac{1}{2}\right)\right)^\top\right) & \text{if } r_0 \le r < r_1, \\ \left(1, (0,0)^\top\right) & \text{otherwise,} \end{cases}$$

where $r := \sqrt{(x-0.5)^2 + (y-0.5)^2}$.

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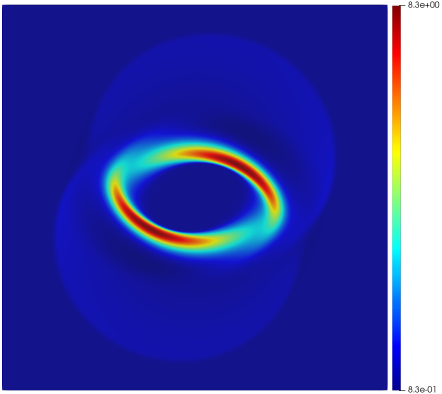
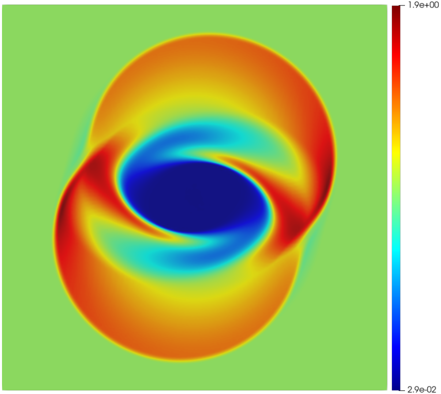
Density



Pressure



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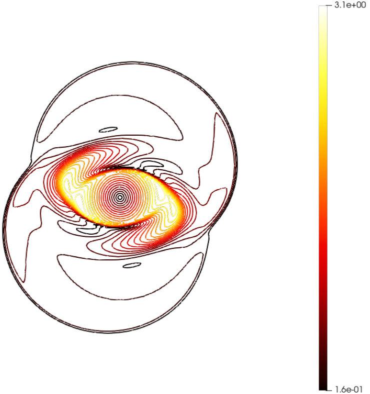
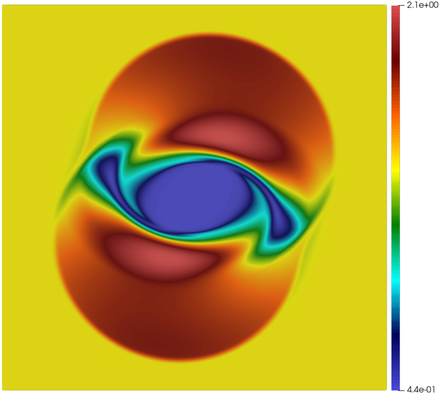
$\mathbb{P}_3$ RV solution, 100×100 quasi-uniform mesh

Mach number



Magnetic presure $\frac{1}{2}\boldsymbol{B}^2$



Astrophysical jets [Wu & Shu, 2018]

Domain $\Omega = [-0.75, 0.75] \times [0, 1.5]$. Gas constant $\gamma = 1.4$.
Initial solution: ambient

$$p_0 = 1, \boldsymbol{B}_0 = \left(0, \sqrt{200}\right)^\top, \rho_0 = 0.1\gamma, \boldsymbol{u}_0 = (0, 0)^\top$$

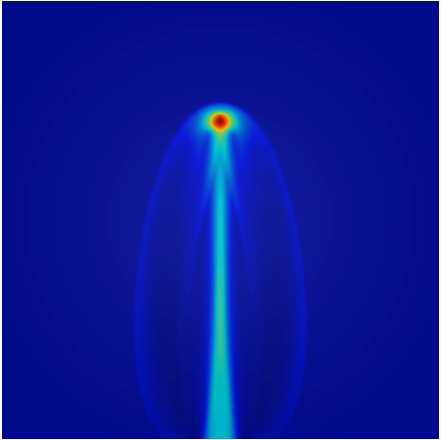
On the inlet $(x, y) \in [-0.05, 0.05] \times 0$: Mach 800

$$u_0 = (0, 800)^\top, \rho = \gamma.$$

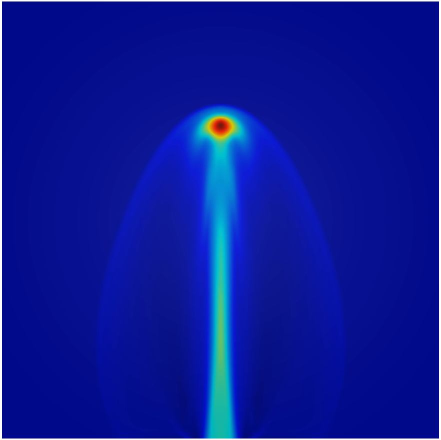
Astrophysical jets [Wu & Shu, 2018]

RV solution with 371447 $\mathbb{P}_1$ nodes, lumped mass

$|\boldsymbol{B}| = \sqrt{200}$



$|\boldsymbol{B}| = \sqrt{2000}$



Outline

Summary

- PP parameter-free first-order FE methods for scalar equations.
- Extensions for systems, higher order methods.
- Tested against several MHD benchmarks.

Outlook

Future works:

- Invariant-domain properties at the discrete level

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Future works:

- Invariant-domain properties at the discrete level
- Maximum wave speed for MHD, exact or reasonable upper bound
- Conflicting objectives: conservativeness, divergence, positivity, entropy stability
- High-order invariant-domain preserving FE methods for MHD

Thank you for your listening!