Viscous Regularization And High-order Nonlinear Viscosity Methods For Magnetohydrodynamics

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Why?

Hydrogen bomb since 1950s

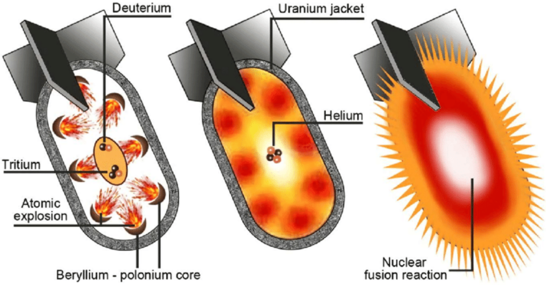

Figure: [Jianan Wang, 2020]

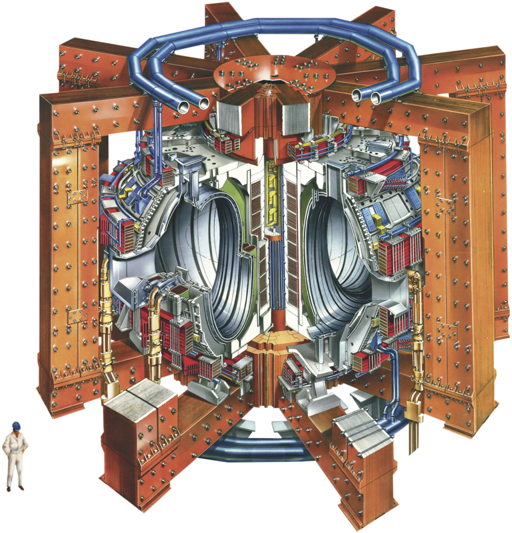

Figure: Tokamak fusion reactor

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China's 'Artificial Sun' Sustains 120 Million Degree Celsius for Over 100 Seconds; Sets New World Record

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The experimental advanced superconducting tokamak (EAST).

(Xinhua/Zhou Mu/IANS)

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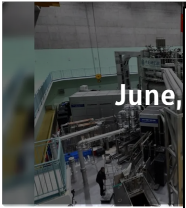

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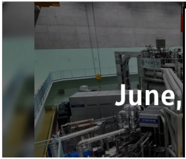

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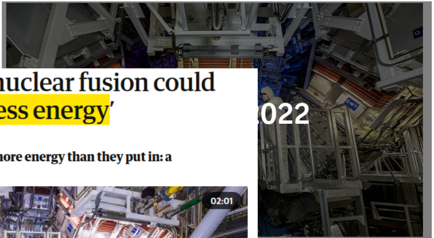

Breakthrough in nuclear fusion could mean 'near-limitless energy'

Researchers managed to release more energy than they put in: a positive gain known as ignition

'Major scientific breakthrough': US recreates fusion – video

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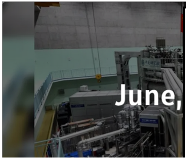

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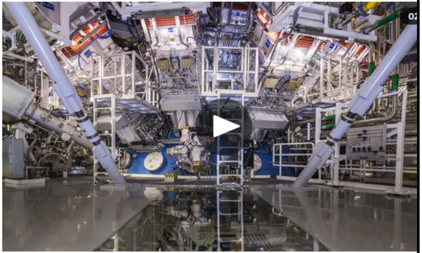

'Major scientific breakthrough': US recreates fusion – video

US scientists achieve net energy gain for second time in nuclear fusion reaction

The Lawrence Livermore National Laboratory's National Ignition Facility achieved the feat using lasers to fuse two atoms

The National Ignition Facility's preamplifier module at the Lawrence Livermore National Laboratory. Photograph: Lawrence Livermore National Laboratory/Reuters

$\mathbf{U} := \big(\rho(\boldsymbol{x},t), \boldsymbol{m}(\boldsymbol{x},t), E(\boldsymbol{x},t), \boldsymbol{B}(\boldsymbol{x},t)\big)^\top,\ (\boldsymbol{x},t) \in \mathbb{R}^d \times \mathbb{R}^+$
Density, momentum, total energy, magnetic field

The ideal MHD equations

$$\partial_t \mathbf{U} + \nabla\cdot \mathbf{F}_{\mathcal{E}}(\mathbf{U}) + \nabla\cdot \mathbf{F}_{\mathcal{B}}(\mathbf{U}) = 0, \tag{1}$$

$$\mathbf{F}_{\mathcal{E}} := \begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{m}\otimes\boldsymbol{u} + p\mathbb{I} \\ \boldsymbol{u}(E+p) \\ 0 \end{pmatrix}, \quad \mathbf{F}_{\mathcal{B}} := \begin{pmatrix} 0 \\ -\boldsymbol{\beta} \\ -\boldsymbol{\beta}\boldsymbol{u} \\ \boldsymbol{u}\otimes\boldsymbol{B} - \boldsymbol{B}\otimes\boldsymbol{u} \end{pmatrix},$$

$\boldsymbol{\beta}$ is the Maxwell stress tensor:

$$\boldsymbol{\beta} = \Big(-\frac{1}{2}(\boldsymbol{B}\cdot\boldsymbol{B})\mathbb{I} + \boldsymbol{B}\otimes\boldsymbol{B}\Big).$$

In addition:

$$\nabla\cdot \boldsymbol{B} = 0, \text{ and } p = \rho T,$$

where T is the temperature.

Comparison between resistive and monolithic viscous fluxes:
(κ, η, ϵ are all positive constants, $\boldsymbol{\tau}$ is the stress tensor)

$$\mathsf{F}_{\mathcal{V}}(\mathsf{U}) := \begin{pmatrix} 0 \\ \boldsymbol{\tau} \\ \boldsymbol{u}\cdot\boldsymbol{\tau} + \kappa\nabla T + \eta \boldsymbol{B}\cdot(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top) \\ \eta(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top) \end{pmatrix}$$

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$$\mathbf{F}_{\mathcal{V}}(\mathbf{U}) := \begin{pmatrix} \epsilon\nabla\rho \\ \epsilon\nabla\boldsymbol{m} \\ \epsilon\nabla E \\ \epsilon\nabla\boldsymbol{B} \end{pmatrix}$$

Goals and objective

- Investigate viscous regularization of the MHD system

$$\partial_t \mathbf{U} + \nabla\cdot \mathbf{F}_{\mathcal{E}}(\mathbf{U}) + \nabla\cdot \mathbf{F}_{\mathcal{B}}(\mathbf{U}) = \nabla\cdot\left(\epsilon\nabla\mathbf{U}\right),$$

  where $\epsilon > 0$.
- **Investigate mathematically** if the above equation is compatible with thermodynamic properties, such as
  - **positivity of density**
  - **positivity of internal energy**
  - **minimum entropy principle**
- **Investigate numerically** if the above viscous regularization can be used to construct a stable and high-order method.

Positivity of density and internal energy

Theorem (D. & Nazarov 2022)

Assume the solution of the regularized MHD system is sufficiently smooth, and $\boldsymbol{u}$*,* $\nabla\cdot\boldsymbol{u}$ *and* $\partial_t\rho + \nabla\cdot(\rho\boldsymbol{u})$ *are bounded. Then, the density solution satisfies the following positivity property*

$$\operatorname{essinf}_{\boldsymbol{x}\in\mathbb{R}^d} \rho(\boldsymbol{x},t) > 0, \quad \forall t > 0.$$

Theorem (D. & Nazarov 2022)

The specific internal energy e *and the internal energy* ρe *of the regularized MHD sytem always remain positive given that the initial data fulfills* $e > 0$*.*

Minimum entropy principle

Theorem (D. & Nazarov 2022)

Assume sufficient smoothness and that the density and the internal energy uniformly converge to stationary constant states ρ^*, e^* outside of a compact set $\Omega \in \mathbb{R}^d$. The regularized MHD system exhibits a specific entropy function s which satisfies

$$\inf_{\boldsymbol{x} \in \mathbb{R}^d} s(\boldsymbol{x}, t) \geq \inf_{\boldsymbol{x} \in \mathbb{R}^d} s_0(\boldsymbol{x}).$$

Let $f(s)$ is a twice differentiable function. Consider a class of strictly convex generalized entropies in the form $\rho f(s)$.

Theorem (Dao & N. 2022)

Any smooth solution to the regularized MHD system satisfies the entropy inequality

$$\partial_t(\rho f(s)) + \nabla \cdot (\boldsymbol{u}\rho f(s) - \epsilon\rho\nabla f(s) - \epsilon f(s)\nabla\rho) \geq 0.$$

Finite element approximation

- Let $\Omega \subset \mathbb{R}^d$ be an open and bounded domain
- Define a triangulation $\mathcal{T}_h := \{\overline{K_i}\}_{i=1}^{N_e} = \overline{\Omega}$, where $\overline{K_i}$ is the closure of K_i.
- Construct continuous finite element spaces:

$$\mathcal{Q}_h := \{v \in C^0(\overline{\Omega}) \mid v_{|_{K_i}} \in \mathbb{P}_k(K_i),\, \forall i = 1, \ldots, N_e\}, \quad \boldsymbol{\mathcal{V}}_h := [\mathcal{Q}_h]^d.$$

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- Find $\mathbf{U}_h(t) := (\rho_h(t), \boldsymbol{m}_h(t), E_h(t), \boldsymbol{B}_h(t))^\top \in \mathcal{C}^1(\mathbb{R}^+, \mathcal{Q}_h \times \boldsymbol{\mathcal{V}}_h \times \mathcal{Q}_h \times \boldsymbol{\mathcal{V}}_h)$ such that

$$\begin{aligned}(\partial_t \mathbf{U}_h, \mathbf{V}_h) + (\nabla\cdot \mathbf{F}_{\mathcal{E}}(\mathbf{U}_h), \mathbf{V}_h) + (\nabla\cdot \mathbf{F}_{\mathcal{B}}(\mathbf{U}_h), \mathbf{V}_h)& \\ +(\mathbf{F}_{\mathcal{V}}^m(\mathbf{U}), \nabla \mathbf{V}_h) - (\boldsymbol{n}\cdot\mathbf{F}_{\mathcal{V}}^m(\mathbf{U}), \mathbf{V}_h)_{\partial\Omega} = 0&,\end{aligned} \tag{2}$$

for all test functions $\mathbf{V}_h \in \mathcal{Q}_h \times \boldsymbol{\mathcal{V}}_h \times \mathcal{Q}_h \times \boldsymbol{\mathcal{V}}_h$,

- Use 4th order Strong-Stability-Preserving Runge-Kutta scheme
- For every time-level t_n and every node $\boldsymbol{N}_i$ compute the largest eigenvalue of the MHD system:

$$\lambda^n_{\max,i} := \max_{j=1,8} |\lambda^n_{j,i}|.$$

- Define the time-step using the following CFL condition:

$$\Delta t_n := \mathsf{CFL} \frac{\min_{\boldsymbol{N}_i \in \mathcal{V}} h_i}{\max_{\boldsymbol{N}_i \in \mathcal{V}} \lambda^n_{\max,i}}.$$

- Compute first order viscosity:

$$\boxed{\epsilon^n_i := \frac{1}{2} h_i \lambda^n_{\max,i}.}$$

- Consider $\nabla\cdot \boldsymbol{B} \neq 0$? $\longrightarrow$ add Powell term (Godunov form)
- Divergence cleaning: projection, hyperbolic cleaning (GLM-MHD)
- H-curl conforming elements for the magnetic field

Comparison between resistive and monolithic viscous fluxes:

$$\mathsf{F}^r_{\mathcal{V}}(\mathsf{U}) := \begin{pmatrix} 0 \\ \boldsymbol{\tau} \\ \boldsymbol{u}\cdot\boldsymbol{\tau} + \kappa\nabla T + \eta\boldsymbol{B}\cdot\left(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top\right) \\ \eta\left(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top\right) \end{pmatrix}$$

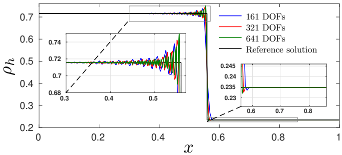
$$\mathsf{F}_{\mathcal{V}}(\mathsf{U}) := \begin{pmatrix} \epsilon\nabla\rho \\ \epsilon\nabla\boldsymbol{m} \\ \epsilon\nabla E \\ \epsilon\nabla\boldsymbol{B} \end{pmatrix}$$

Consider the following initial condition:

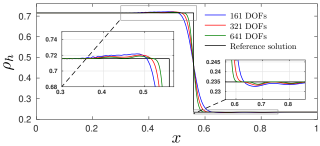
- Let the domain be $\Omega = [0, 1]$.
- The gas constant is $\gamma = 2$.
- The initial data:
  - velocity $\boldsymbol{u} = (u_x, u_y)$, $u_x = 0.5915470932$, $u_y = -1.5792628803$
  - pressure $p = 0.5122334291$
  - magnetic field $\boldsymbol{B} = (B_x, B_y)$, $B_x = 0.75$, $B_y = -0.5349102426$
  - density

$$\rho = \begin{cases} 0.7156521382, & \boldsymbol{x} \in [0, 0.5], \\ 0.2348529760, & \boldsymbol{x} \in (0.5, 1]. \end{cases}$$

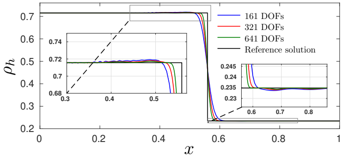
Contact line problem



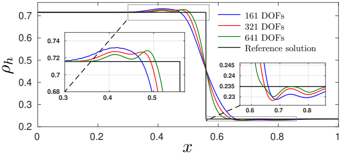
(a) $\kappa = 0$



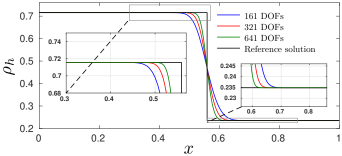
(b) $\kappa = 1$



(c) $\kappa = 0.5$



(d) $\kappa = 5$



(e) monolithic

Entropy viscosity

- For every time level t_n and every node $\boldsymbol{N}_i$ compute an entropy residual:

$$R_i^n(t) := \sum_{K \in \mathcal{T}_h} \frac{1}{|K|} \int_K |D_t S_h^n + \nabla \cdot (\boldsymbol{u}_h^n S_h^n)| \varphi_i \, \mathrm{d}\boldsymbol{x},$$

  where S_h^n is an entropy functional for the MHD system.
- Compute the low-order viscosity:

$$\boxed{\epsilon_i^{\mathsf{L},n} := \frac{1}{2} h_i \lambda_{\max,i}^n.}$$

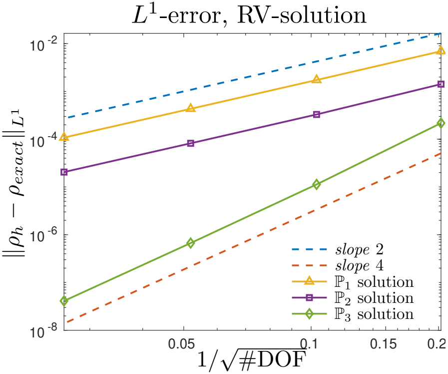
- Compute the high-order viscosity:

$$\boxed{\epsilon_i^{\mathsf{H},n} = c_E h_i^2 \frac{|R_i^n|}{\|\overline{R_h^n} - R_h^n\|_{\infty,\Omega}},}$$

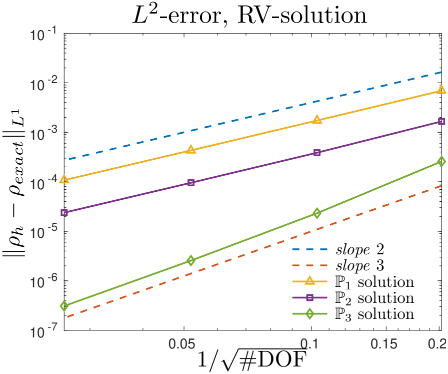
- Compute the entropy viscosity:

$$\boxed{\epsilon_i^n = \min(\epsilon_i^{\mathsf{L},n}, \epsilon_i^{\mathsf{H},n}).}$$

Smooth wave propagation, [Wu and Shu, 2018].



(a) L^1-error



(b) L^2-error

Figure: $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ convergence for smooth solutions.

Brio-Wu shock-tube using entropy viscosity

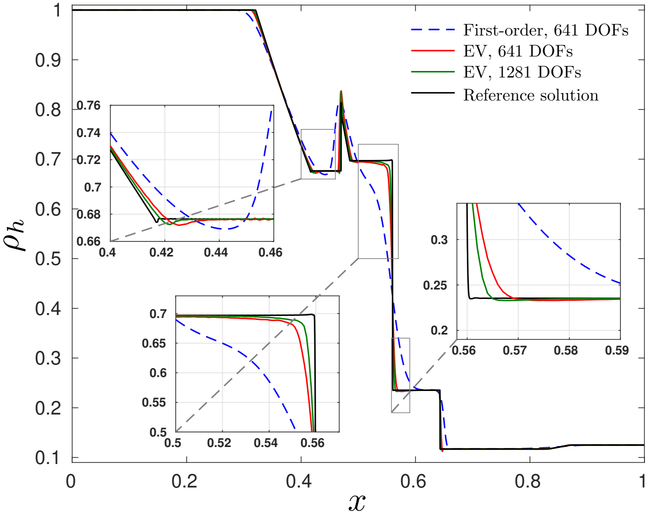


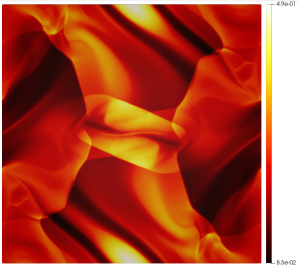
Figure: $\widehat{t} = 0.1$.

Consider the following initial condition:

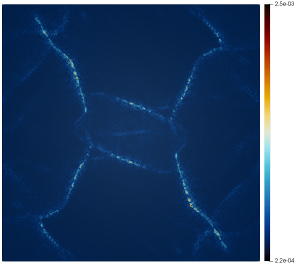
- Let the domain be $\Omega = [0,1] \times [0,1]$.
- The gas constant is $\gamma = \frac{5}{3}$.
- The initial data:

$$(\rho_0, \boldsymbol{u}_0, p_0, \boldsymbol{B}_0) = \left(\frac{25}{36\pi}, (-\sin(2\pi y), \sin(2\pi x)), \frac{5}{12\pi}, \left(-\frac{\sin(2\pi y)}{\sqrt{4\pi}}, \frac{\sin(4\pi x)}{\sqrt{4\pi}} \right) \right).$$

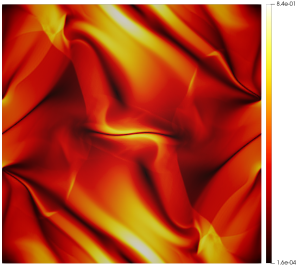
- Final time: $\widehat{t} = 1$.



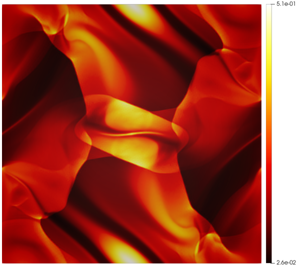
(a) Density ρ_h



(b) Entropy viscosity μ_h



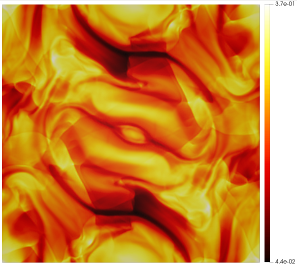
(c) Magnetic pressure $\frac{1}{2}\boldsymbol{B}_h^2$



(d) Thermodynamic pressure p_h

Entropy viscosity solution of the Orszag-Tang problem at time $\widehat{t} = 0.5$, 90000 $\mathbb{P}_3$ nodes.

Orszag-Tang problem



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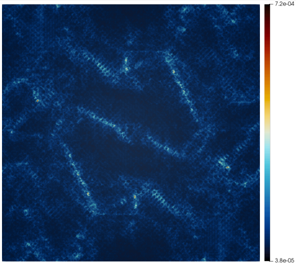
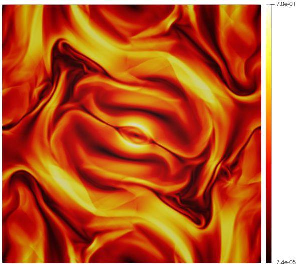


(b) Entropy viscosity μ_h



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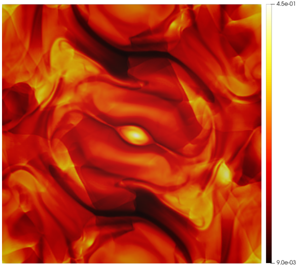
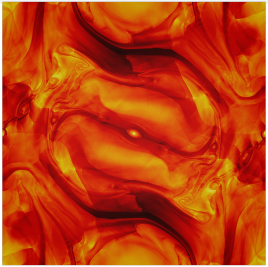


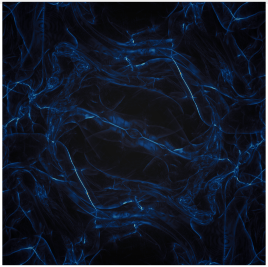
(d) Thermodynamic pressure p_h

Entropy viscosity solution of the Orszag-Tang problem at time $\widehat{t} = 1$, 90000 $\mathbb{P}_3$ nodes.

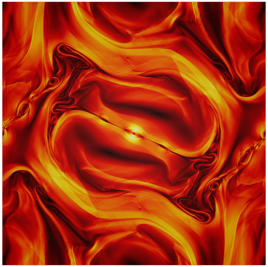
Orszag-Tang problem



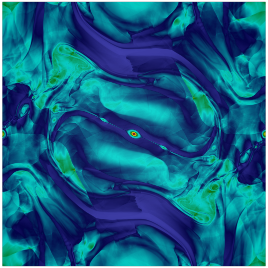
(a) Density ρ_h



(b) Entropy viscosity μ_h



(c) Magnetic pressure $\frac{1}{2}\boldsymbol{B}_h^2$



(d) Thermodynamic pressure p_h

Entropy viscosity solution of the Orszag-Tang problem at time $\widehat{t} = 1$, 20mln $\mathbb{P}_3$ nodes.

Thank you!

[D. & Nazarov, CMAME, 2022]

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