Residual Viscosity Method For Magnetohydrodynamics

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THE IDEAL MHD EQUATIONS

The ideal magnetohydrodynamics (MHD) equations describe conservation of density, momentum, total energy, and magnetic field $\mathsf{U} := \left(\rho(\boldsymbol{x},t), \boldsymbol{m}(\boldsymbol{x},t), E(\boldsymbol{x},t), \boldsymbol{B}(\boldsymbol{x},t)\right)^\top$, $(\boldsymbol{x},t) \in \mathbb{R}^d \times \mathbb{R}^+$,

$$\partial_t \mathsf{U} + \nabla\cdot \mathsf{F}(\mathsf{U}) = 0, \qquad \text{where } \mathsf{F} := \begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{m}\otimes\boldsymbol{u} + p\mathbb{I} \\ \boldsymbol{u}(E+p) \\ 0 \end{pmatrix} + \begin{pmatrix} 0 \\ -\boldsymbol{\beta} \\ -\boldsymbol{\beta}\boldsymbol{u} \\ \boldsymbol{u}\otimes\boldsymbol{B} - \boldsymbol{B}\otimes\boldsymbol{u} \end{pmatrix}, \tag{1}$$

and the Maxwell stress tensor $\boldsymbol{\beta} := \left(-\frac{1}{2}(\boldsymbol{B}\cdot\boldsymbol{B})\mathbb{I} + \boldsymbol{B}\otimes\boldsymbol{B}\right)$.

- Consider $\nabla\cdot\boldsymbol{B} \neq 0$?
  Add Powell term (Godunov form) $\longrightarrow$ recover entropy inequality, Galilean invariance;
- Divergence cleaning: projection, hyperbolic cleaning (GLM-MHD).

VISCOUS REGULARIZATION

Regularize (1) by adding the monolithic parabolic flux

$$\partial_t \mathsf{U} + \nabla\cdot \mathsf{F}(\mathsf{U}) = \nabla\cdot(\epsilon\nabla\mathsf{U}), \tag{2}$$

where ϵ is a vanishing viscosity parameter. The regularized equation (2) is

- a continuous analogue of the upwind, Lax-Friedrichs schemes;
- suitable to apply tensor-valued viscosity.

Theorem (D. & N. [2]). It can be shown that (2)

- preserves positivity of density $\rho > 0$, internal energy $e > 0$;
- fulfills the minimum entropy principle;
- is compatible with all the generalized Harten entropies;
- is Galilean and rotationally invariant.

WHY NOT RESISTIVE MHD MODEL?

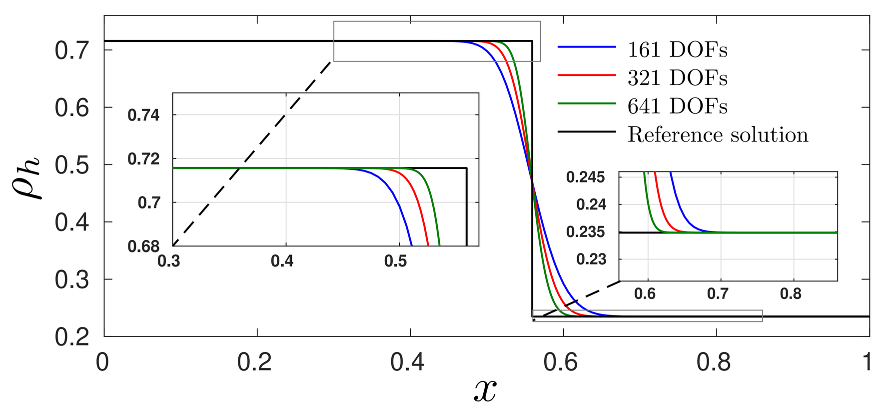
Let κ, μ, η be viscosity parameters. The resistive MHD flux is $\left(0, \boldsymbol{\tau}, \boldsymbol{u}\cdot\boldsymbol{\tau} + \kappa\nabla T + \eta\boldsymbol{B}\cdot(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top), \eta(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top)\right)^\top$, where $\boldsymbol{\tau} = \mu\left(\nabla\boldsymbol{u} + \nabla\boldsymbol{u}^\top\right) - \lambda\nabla\cdot\boldsymbol{u}\mathbb{I}$.

- not suitable for numerical purposes due to the unregularized mass equation;
- incompatible with contact waves;
- incompatible with the entropy principles.

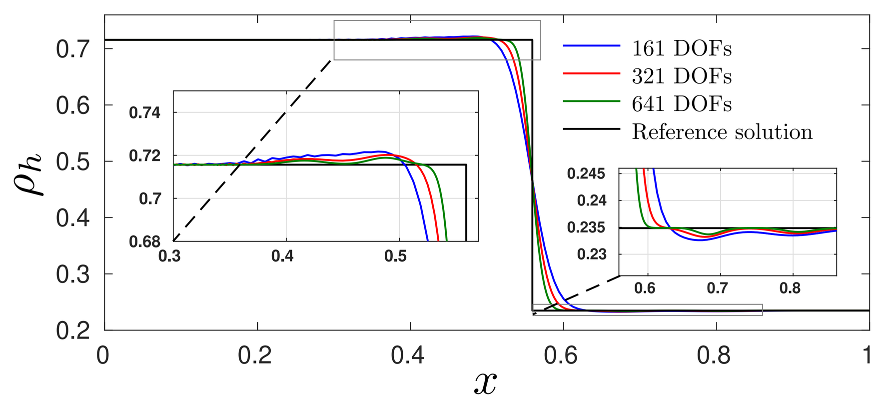
A simple example: Consider a contact wave where there is a jump in the density but other conserved variables are constant. Inserting into (2) gives the mass equation, while the resistive MHD model leads to

$$\nabla\cdot(\kappa p_0 \rho^{-2}\nabla\rho) = 0$$

which only holds if the thermal diffusivity $\kappa = 0$. Numerically, we can see that the resistive MHD flux violates bounds for the contact solutions.

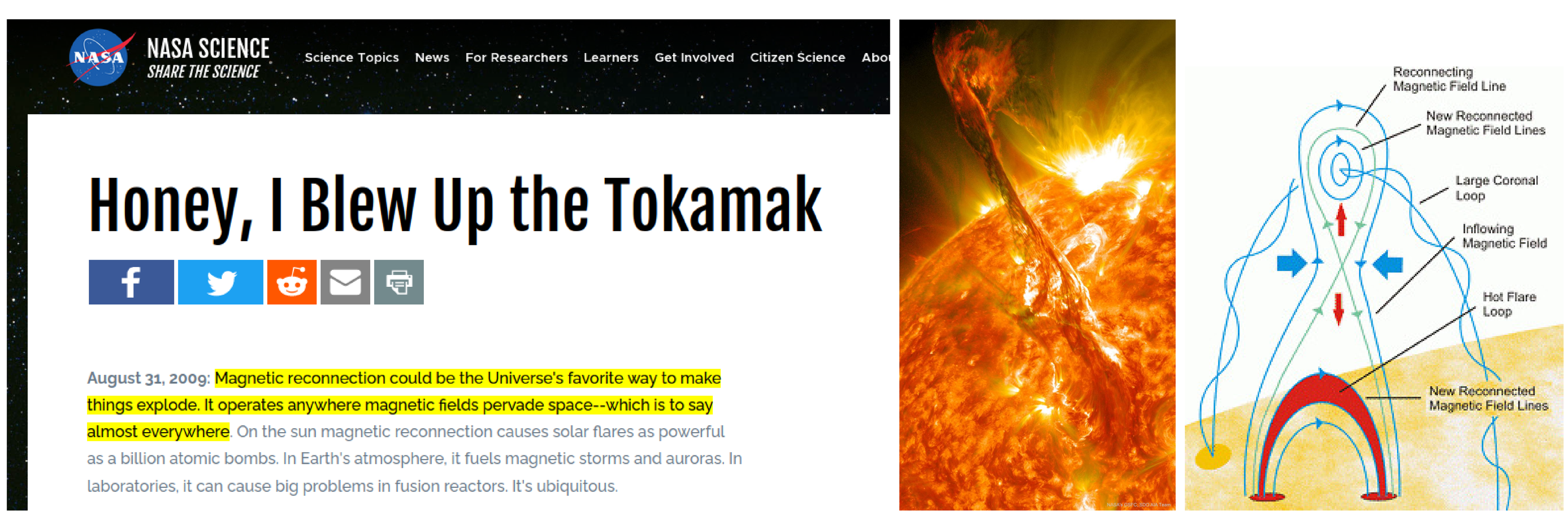


Monolithic flux



Resistive MHD flux

GEOSPACE MAGNETIC RECONNECTION CHALLENGE



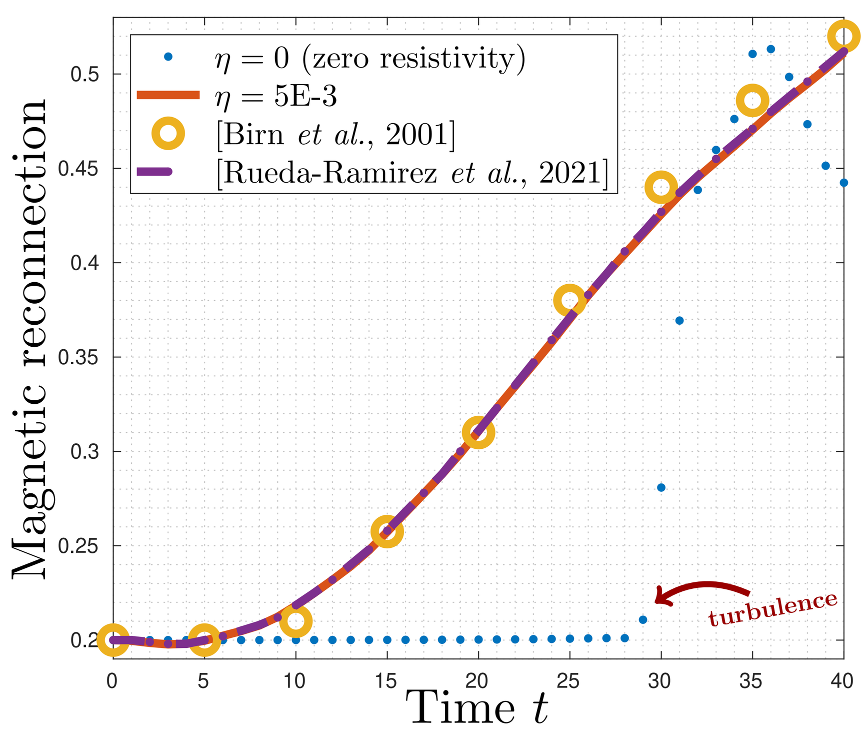
Credit: Science@NASA

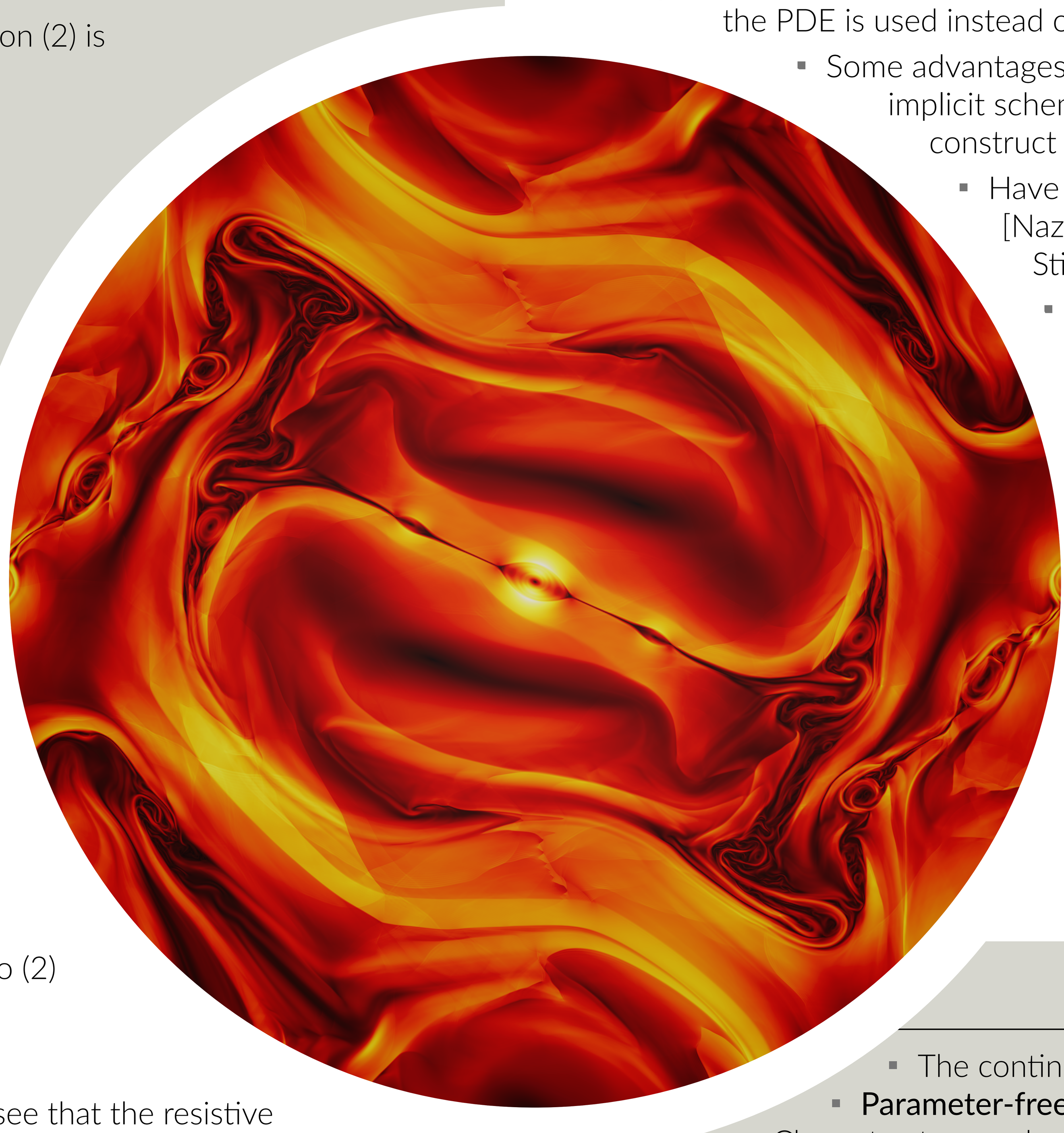
Magnetic reconnection is a phenomenon in plasma physics where magnetic fields pointing opposite to each other break, reconnect, and shoot out plasma. This phenomenon has been observed in many astrophysical events: supernovae, magnetosphere, solar wind, solar eruption, and in fusion devices.

Although not being the most suitable model for this test, we try the viscous model (2) for the GEM challenge by [Birn et. al., 2001]. For this purpose, the magnetic part of the energy diffusion is separated as

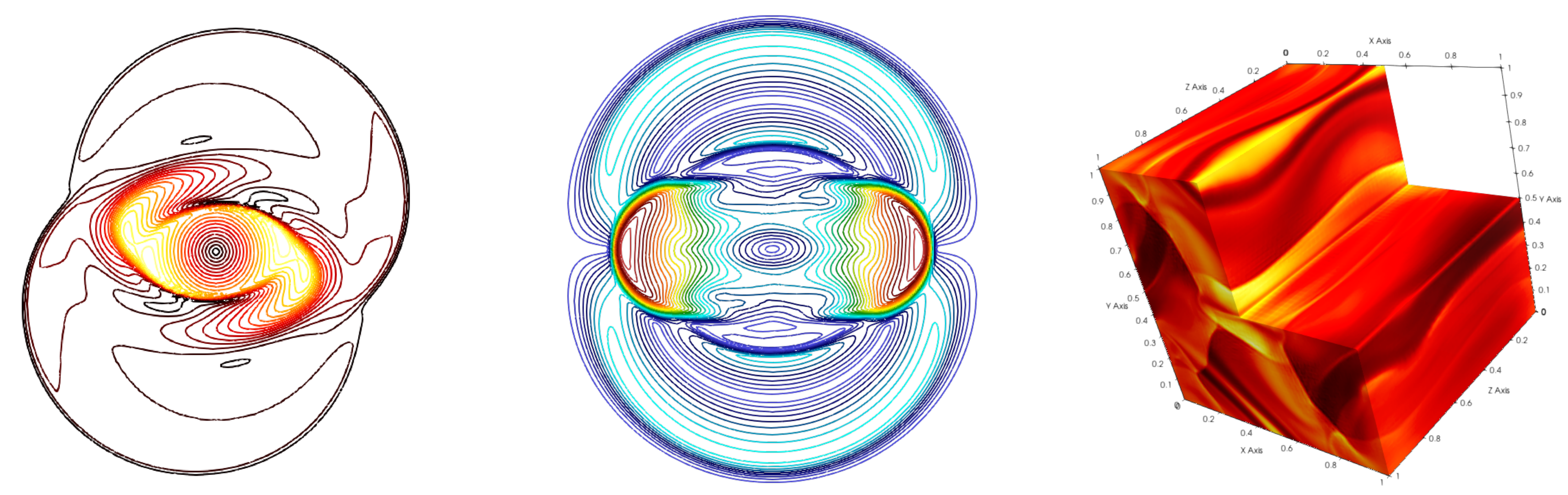
$$(\epsilon\nabla\rho, \epsilon\nabla\boldsymbol{m}, \epsilon\nabla(\rho e + \rho\boldsymbol{u}^2/2) + \eta\nabla\boldsymbol{B}^2/2, \eta\nabla\boldsymbol{B})^\top.$$

Then, the numerical resistivity coefficient η is assembled as the maximum of the physical and the artificial viscosity at every nodal point. The obtained result is comparable with resistive MHD models.





NUMERICAL RESULTS TO POPULAR BENCHMARKS



MHD Rotor problem, Mach number

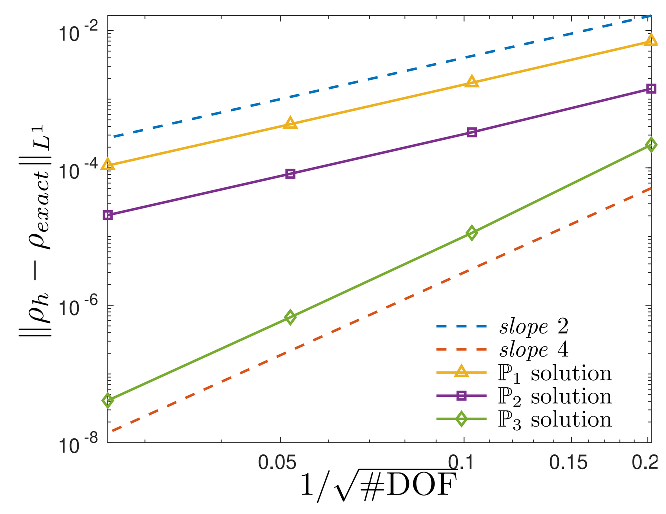
MHD Blast problem, Velocity

3D Orszag-Tang problem, Density

Solutions using $\mathbb{P}_3$ Lagrange basis

HIGH-ORDER NONLINEAR VISCOSITY

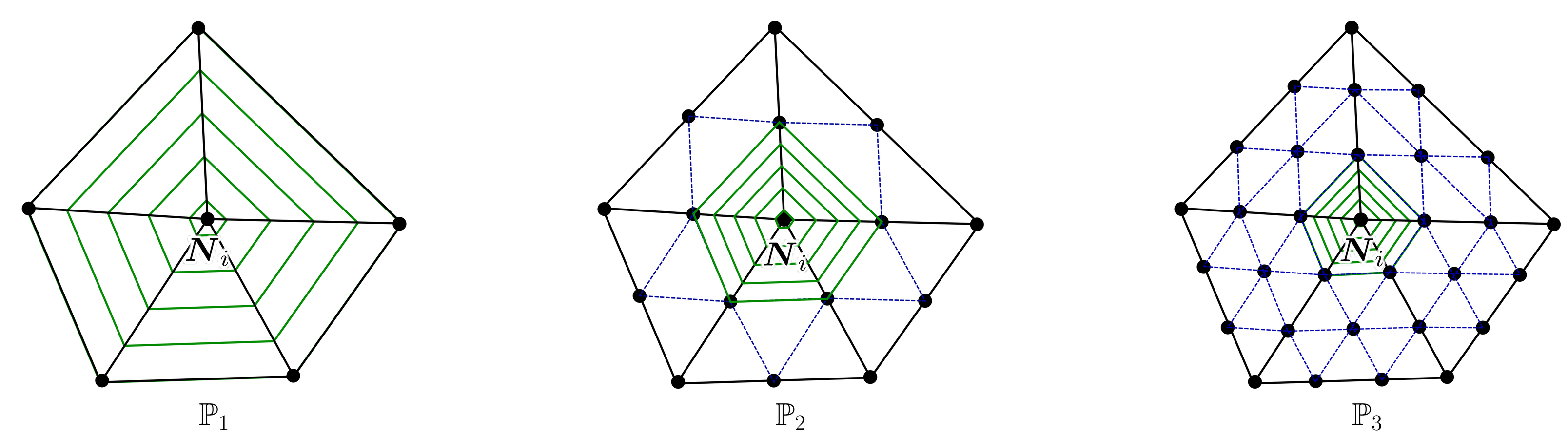
- Residual-based viscosity (or residual viscosity) is used to make the method high-order.
- Originated from the **Entropy Viscosity** method and the **Streamline Diffusion** method, but the residual of the PDE is used instead of the entropy residual.
- Some advantages: (+) Provable convergence to the unique entropy solution for some implicit schemes [Nazarov, 2013]; (+) For many systems/applications, it is hard to construct entropy residuals but the PDE residual is always available.
- Have proven effective in FEM, SEM, FD, RBF settings [Nazarov & Hoffman, 2013, Marras *et al.*, 2015, Lu *et al.*, 2019, Stiernström *et al.*, 2021, Tominec & Nazarov, 2021].
- Fully discrete entropy inequalities for all entropy functionals.
- The viscosity term does not affect time step limit.



Convergence of smooth solutions using the residual viscosity method

SUMMARY

- The continuous model satisfies positivity and entropy principles;
- **Parameter-free** viscosity construction;
- Cheap to store and compute, easy to implement, high-order accurate;
- Robustness tested against challenging MHD benchmarks.



Compute ε_i on $\mathbb{P}_1$ submeshes

PARAMETER-FREE VISCOSITY CONSTRUCTION

Tensor-valued viscosity ([Guermond & Nazarov, 2014]). For $i, j \in \mathcal{I}(K)$– set of all nodes in element K,

$$b_K(\varphi_j, \varphi_i) := \begin{cases} -\frac{1}{n_K - 1} & \text{if } j \neq i, \\ 1 & \text{if } j = i, \end{cases}$$

where φ_j is the corresponding basis function and n_K is the number of nodes in K.

Define the nodal viscosity as

$$\varepsilon_i := \alpha_i C_i \|\boldsymbol{f}'\|_{L^\infty(S_i)} \max_{i\neq j\in\mathcal{I}(S_i)} |\nabla\varphi_j| m_i^{(\mathbb{P}_1)}, \qquad \sim Ch\|\boldsymbol{f}'\|_{L^\infty(S_i)}$$

where α_i, C_i are pre-computed from the mesh, and $m_i^{(\mathbb{P}_1)}$ is the lumped mass of the $\mathbb{P}_1$ submesh.

Theorem (D. & N.). Under a usual CFL condition, the approximation

$$m_i \frac{u_i^{n+1} - u_i^n}{\tau} + \int_{S_i} \nabla\cdot\boldsymbol{f}(u_h^n)\varphi_i \,\mathrm{d}\boldsymbol{x} + \sum_{K\in S_i}\sum_{j\in\mathcal{I}(S_i)} \varepsilon_j u_j^n b_K(\varphi_j, \varphi_i) = 0$$

is positivity preserving.

The high-order viscosity is calculated from the nodal residual R_i,

$$\varepsilon_i^{\mathrm{H}} := C_i\alpha_i \min\left(\|\boldsymbol{f}'\|_{L^\infty(S_i)} \max_{i\neq j\in\mathcal{I}(S_i)} |\nabla\varphi_j|,\ |R_i|\right) m_i^{(\mathbb{P}_1)}.$$

References

[1] Tuan Anh Dao and Murtazo Nazarov. A high-order residual-based viscosity finite element method for the ideal MHD equations. *J. Sci. Comput.*, 92(3):Paper No. 77, 24, 2022.

[2] Tuan Anh Dao and Murtazo Nazarov. Monolithic parabolic regularization of the MHD equations and entropy principles. *Comput. Methods Appl. Mech. Engrg.*, 398:Paper No. 115269, 25, 2022.

Full paper

Note

*The center figure is the magnetic pressure solution to the Orszag-Tang problem at time $t = 1.0$.