Viscous regularization of Magnetohydrodynamics (ideal MHD equations)

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Viscous regularization of the Euler equations

Conserved variables $\quad \mathbf{U} := (\rho, \boldsymbol{m}, E)$

Density, momentum, total energy

$\mathbf{m} = \rho\mathbf{u}$

(Compressible) Euler equations

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}_{\text{Euler}}(\mathbf{U}) = 0,$$

where

$$\mathbf{F}_{\text{Euler}} := \begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{m} \otimes \boldsymbol{u} + p\mathbb{I} \\ \boldsymbol{u}(E+p) \end{pmatrix}.$$

Why viscous regularization?

- Positivity of density, internal energy
- Minimum entropy principle

In arbitrary spatial dimensions, PDE level,
=> a starting point for numerics

Godunov scheme can achieve these but it needs exact Riemann solver, which is not fully known for MHD

Viscous regularization of the Euler equations

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}_{\mathrm{Euler}}(\mathbf{U}) - \nabla \cdot \mathbf{F}^{\mathrm{visc}}(\mathbf{U}) = 0$$

what to choose?

Navier-Stokes flux

$$\mathbf{F}^{\mathrm{visc}}_{\mathrm{NS}}(\mathbf{U}) = \begin{pmatrix} 0 \\ 2\mu\nabla^s \boldsymbol{u} + \lambda\nabla\cdot\boldsymbol{u}\mathbb{I} \\ \kappa\nabla T + (2\mu\nabla^s\boldsymbol{u} + \lambda\nabla\cdot\boldsymbol{u}\mathbb{I})\cdot\boldsymbol{u} \end{pmatrix}$$

Temperature $T := p/\rho$ $\qquad \nabla^s\mathbf{u} = \frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^\top)$

(+) Accepted by the physicists
(correct physics, e.g., no mass diffusion, Galilean invariant, …)
(+) A common technique to solve Euler

(-) No mass regularization → positivity of density violated
(-) Minimum entropy principle holds only when $\kappa = 0$
(-) When $\kappa = 0$, no admissible generalized entropy inequalities, à la [Harten,1998]
(-) When $\kappa = 0$, no internal energy regularization => positivity of internal energy
(-) Incompatible with contact waves

[Guermond and Popov, 2014]

$$\mathbf{F}^{\mathrm{visc}}_{\mathrm{GP}}(\mathbf{U}) = \begin{pmatrix} \kappa\nabla\rho \\ \mu\rho\nabla^s\boldsymbol{u} + (\kappa\nabla\rho)\otimes\boldsymbol{u} \\ \kappa\nabla(\rho e) + \frac{\boldsymbol{u}^2}{2}\kappa\nabla\rho + \mu\rho\nabla^s\boldsymbol{u}\cdot\boldsymbol{u} \end{pmatrix}$$

(+) Positivity of density, internal energy
(+) Minimum entropy principles
(+) Compatible with all generalized entropy inequalities
(+) Galilean invariant

(-) Physicists may oppose (?)

$\kappa \geq 0, \mu \geq 0, \lambda$ are small viscosity coefficients

Viscous regularization of the Euler equations

Phenomenological evidence supporting GP flux

Let $\boldsymbol{u}_m = \boldsymbol{u} - \rho^{-1}\kappa\nabla\rho$
mass velocity

(Euler + GP) ≈ (Euler + NS) with two velocities!

Rewrite Euler + GP flux equations as

mass velocity

volume velocity

$$\begin{aligned}
&\partial_t\rho + \nabla\cdot(\boldsymbol{u}_m\rho) = 0\\
&\partial_t\boldsymbol{m} + \nabla\cdot(\boldsymbol{u}_m\otimes\boldsymbol{m}) + \nabla p - \nabla\cdot(\mu\rho\nabla^s\boldsymbol{u}) = 0\\
&\partial_t E + \nabla\cdot(\boldsymbol{u}_m E) - \nabla\cdot(\kappa\rho c_v\nabla T) + \nabla\cdot((p\mathbb{I} - \mu\rho\nabla^s\boldsymbol{u})\cdot\boldsymbol{u}) = 0
\end{aligned}$$

which resembles the Navier-Stokes flux.

These equations coincide with a phenomenological model proposed by Howard Brenner (Physicist, MIT)

References:
[1] H. Brenner, *Fluid mechanics revisited*, Phys. A, 370 (2006), pp. 190–224.

[2] E. Feireisl and A. Vasseur, *New perspectives in fluid dynamics: Mathematical analysis of a model proposed by Howard Brenner*, in New Directions in Mathematical Fluid Mechanics, Adv. Math. Fluid Mech., Birkhäuser-Verlag, Basel, 2010, pp. 153–179.

Viscous regularization of the MHD equations

Conserved variables $\mathbf{U} := (\rho, \boldsymbol{m}, E, \boldsymbol{B})$

Density, momentum, total energy, magnetic field

Ideal MHD equations

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}_{\text{Euler}}(\mathbf{U}) + \nabla \cdot \mathbf{F}_{\text{MHD}}(\mathbf{U}) - \nabla \cdot \mathbf{F}^{\text{visc}}(\mathbf{U}) = 0\,,$$

where

$$\mathbf{F}_{\mathcal{B}} := \begin{pmatrix} 0 \\ -\boldsymbol{\beta} \\ -\boldsymbol{u} \cdot \boldsymbol{\beta} \\ \boldsymbol{u} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{u} \end{pmatrix}, \qquad \boldsymbol{\beta} = \left(-\frac{1}{2}(\boldsymbol{B} \cdot \boldsymbol{B})\mathbb{I} + \boldsymbol{B} \otimes \boldsymbol{B} \right).$$

Maxwell stress tensor

Viscous regularization of the MHD equations

Resistive MHD flux

$$\mathbf{F}^{\text{visc}}_{\text{RMHD}}(\mathbf{U}) := \begin{pmatrix} 0 \\ \boldsymbol{\tau} \\ \boldsymbol{u}\cdot\boldsymbol{\tau} + \kappa\nabla T + \eta\boldsymbol{B}\cdot\left(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top\right) \\ \eta\left(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top\right) \end{pmatrix}$$

(+) Accepted by the physicists

(-) positivity of density violated
(-) Minimum entropy principle holds only when $\kappa = 0$
(-) When $\kappa = 0$, no admissible generalized entropy inequalities
(-) When $\kappa = 0$, positivity of internal energy violated
(-) Incompatible with contact waves

Monolithic flux

$$\mathbf{F}^{\text{visc}}_{\text{monolithic}}(\mathbf{U}) := \begin{pmatrix} \epsilon\nabla\rho \\ \epsilon\nabla\mathbf{m} \\ \epsilon\nabla E \\ \epsilon\nabla\mathbf{B} \end{pmatrix}$$

ϵ is a small positive constant

we have proved

(+) Positivity of density, internal energy
(+) Minimum entropy principles
(+) Compatible with all generalized entropy inequalities
(+) We need this to construct high-order FE PP schemes

(-) Not backed by physics
(-) Galilean invariance violated
(-) Rotational invariance violated

GP + Resistive flux for B + Powell term

On-going work

(+) Positivity of density, internal energy
(+) Minimum entropy principles
(+) Compatible with all generalized entropy inequalities
(+) Consistent with GP flux
(+) Galilean and rotational invariance
(+) Allow (to some extend) violation of $\nabla\cdot\mathbf{B} = 0$

(-) Conservation is lost if the Powell term is added