

A nodal-based high-order nonlinear viscosity finite element method for MHD

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Vetenskapsrådet



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Outline

- ① Parameter-free first-order method for scalar conservation laws
- ② Extension to systems and higher-order methods
- ③ Application to the system of MHD equations
- ④ Summary and outlook

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1 Parameter-free first-order method for scalar conservation laws

2 Extension to systems and higher-order methods

3 Application to the system of MHD equations

4 Summary and outlook

Motivation

- Vanishing viscous regularization to scalar conservation laws

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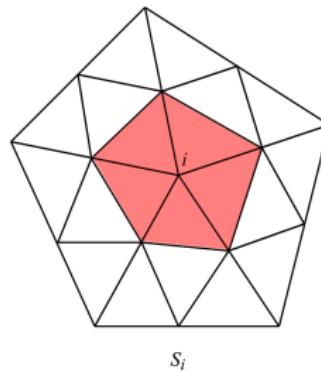
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- Extend for systems

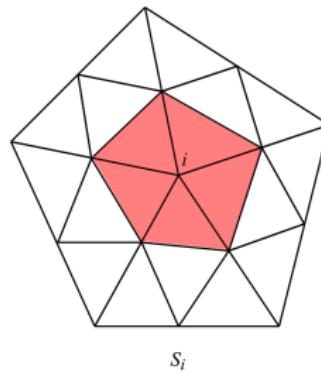
First-order PP method for \mathbb{P}_1



At node i , define

- S_i – support of the Lagrange basis function φ_i

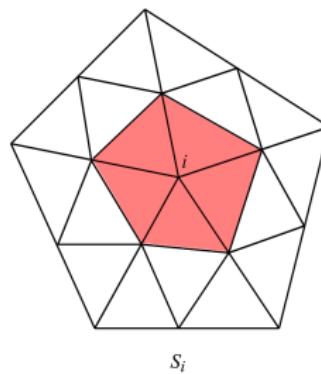
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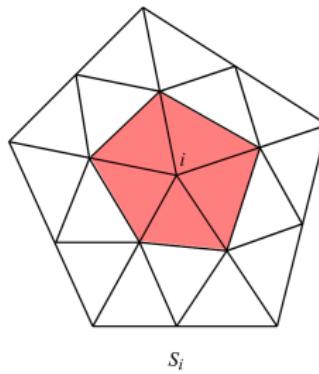
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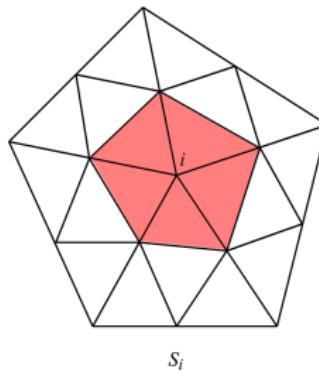
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- m_i – weight of the lumped \mathbb{P}_1 mass matrix at index i

First-order PP method for \mathbb{P}_1

- $C_i := \frac{1}{N_{\text{el}}(S_i)} \max_{K \in S_i, j \neq i, j \in \mathcal{I}(S_i)} \left| \int_K (\mathbb{J}_K^\top \nabla \varphi_j) \cdot (\mathbb{J}_K^\top \nabla \varphi_i) \, d\mathbf{x} \right|^{-1}$

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Theorem

Define the nodal viscosity as

$$\varepsilon_i := \alpha_i C_i \|f'\|_{L^\infty(S_i)} \max_{i \neq j \in \mathcal{I}(S_i)} |\nabla \varphi_j| m_i \quad \sim Ch \|f'\|_{L^\infty(S_i)}.$$

Under a usual CFL condition, the approximation

$$m_i \frac{u_i^{n+1} - u_i^n}{\tau} + \int_{S_i} \nabla \cdot f(u_h^n) \varphi_i \, d\mathbf{x} + \sum_{K \in S_i} \sum_j \varepsilon_j u_j^n b_K(\varphi_j, \varphi_i) = 0$$

is positivity preserving. The form $b_K(\cdot, \cdot)$ is defined next.

Alternative \mathbb{P}_1 viscous bilinear form in 2D

For each triangle element K , define $\Phi_K := \hat{K} \rightarrow K$, where \hat{K} is the equilateral triangle such that $|\hat{K}| = |K|$. \mathbb{J}_K is the Jacobian matrix of Φ_K .

Tensor-valued viscosity ([Guermond & Nazarov, 2014], edge-wise)

$$b(u, v) := \int_{\Omega} (\mathbb{J}^{\top} \nabla u) \cdot (\mathbb{J}^{\top} \nabla v) \, dx, \quad \forall u, v \in V_h.$$

(2D) Local bilinear form, for $i, j \in \mathcal{I}(K)$,

$$b_K(\varphi_j, \varphi_i) := \begin{cases} -\frac{1}{n_K-1} & \text{if } j \neq i, \\ 1 & \text{if } j = i, \end{cases}$$

where n_K is the number of nodes in K .

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Extensions

- High-order viscosity methods: Residual-based, Entropy-based

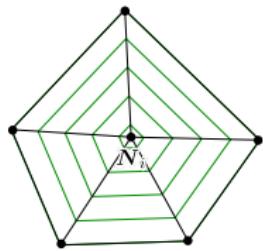
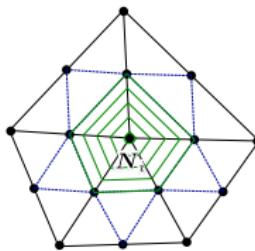
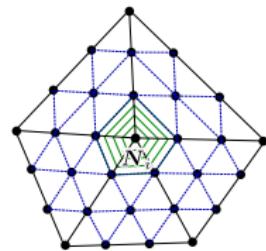
$$\varepsilon_{h,i}^H := C_i \alpha_i \min \left(\|f'\|_{L^\infty(S_i)}, \max_{i \neq j \in \mathcal{I}(S_i)} |\nabla \varphi_j|, |R_i(q_h)| \right) m_i^{(\mathbb{P}_1)}$$

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- Other polynomial spaces: Calculate ϵ_i on \mathbb{P}_1 sub-mesh

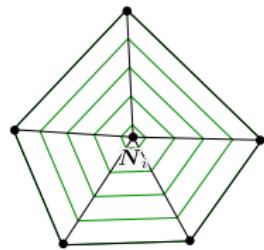
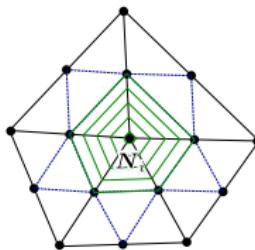
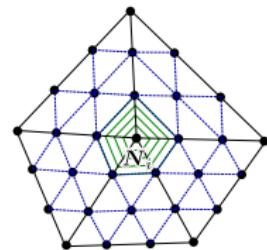
(a) \mathbb{P}_1 (b) \mathbb{P}_2 (c) \mathbb{P}_3

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- Systems: take maximum of the components

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- **The viscosity term does not affect time step limit**

Convergence test on Burger's equation with shocks

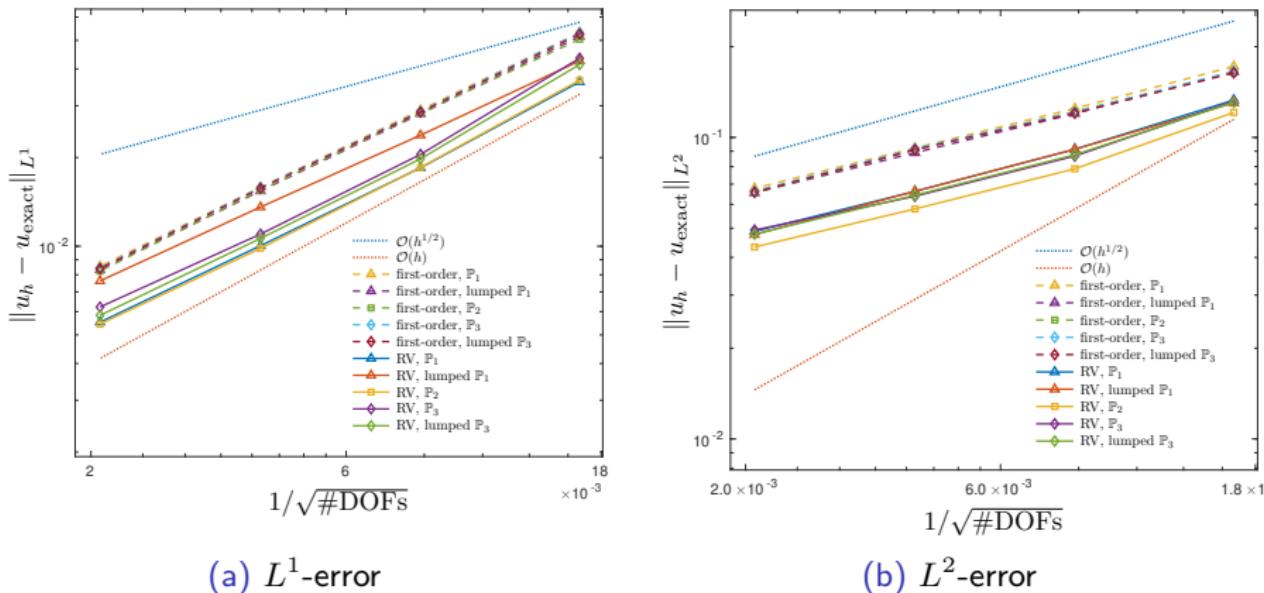
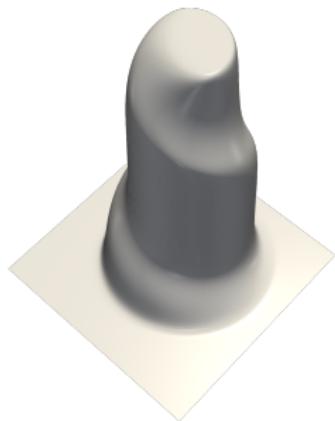
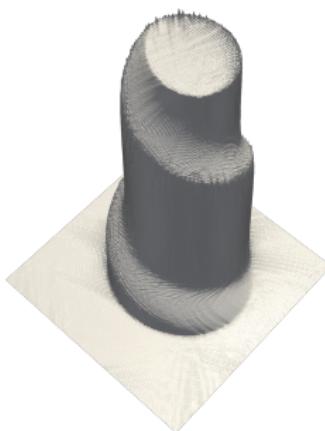


Figure: First-order \mathbb{P}_1 , \mathbb{P}_2 , \mathbb{P}_3 shock solutions to the Burger's equation

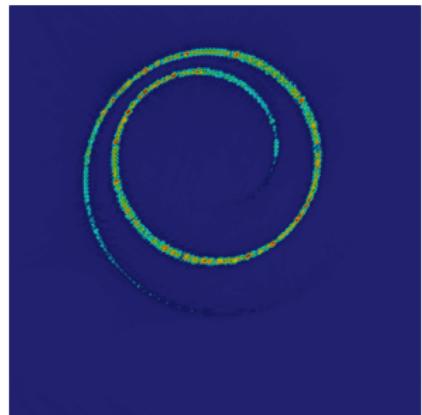
The KPP problem



(a) First order solution



(b) RV solution

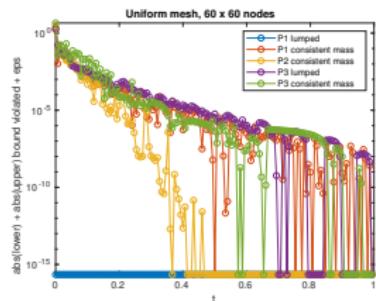
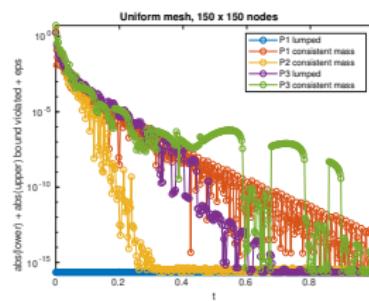
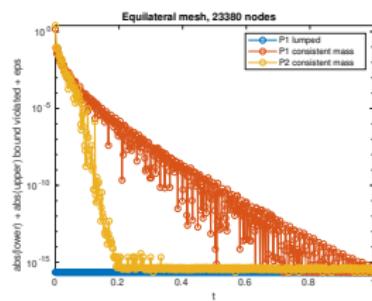


(c) RV viscosity

Figure: KPP solution with $300 \times 300 \mathbb{P}_3$ nodes

The KPP problem

First order solutions

(a) 60×60 mesh(b) 150×150 mesh

(c) Equilateral mesh

Figure: Discrete maximum principle violation in higher order polynomials.

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The system of MHD equations

$\mathbf{U} := (\rho(\mathbf{x}, t), \mathbf{m}(\mathbf{x}, t), E(\mathbf{x}, t), \mathbf{B}(\mathbf{x}, t))^{\top}$, $(\mathbf{x}, t) \in \mathbb{R}^d \times \mathbb{R}^+$
 Density, momentum, total energy, magnetic field

The ideal MHD equations

$$\partial_t \mathbf{U} + \nabla \cdot (\mathbf{F}_{\mathcal{E}}(\mathbf{U}) + \mathbf{F}_{\mathcal{B}}(\mathbf{U})) = 0, \quad (1)$$

$$\mathbf{F}_{\mathcal{E}} := \begin{pmatrix} \mathbf{m} \\ \mathbf{m} \otimes \mathbf{u} + p\mathbb{I} \\ \mathbf{u}(E + p) \\ 0 \end{pmatrix}, \quad \mathbf{F}_{\mathcal{B}} := \begin{pmatrix} 0 \\ -\boldsymbol{\beta} \\ -\boldsymbol{\beta}\mathbf{u} \\ \mathbf{u} \otimes \mathbf{B} - \mathbf{B} \otimes \mathbf{u} \end{pmatrix},$$

$\boldsymbol{\beta}$ is the Maxwell stress tensor:

$$\boldsymbol{\beta} = \left(-\frac{1}{2}(\mathbf{B} \cdot \mathbf{B})\mathbb{I} + \mathbf{B} \otimes \mathbf{B} \right).$$

Monolithic parabolic regularization to the MHD equations

Regularize (1)

$$\partial_t \mathbf{U} + \nabla \cdot \mathbf{F}_{\mathcal{E}}(\mathbf{U}) + \nabla \cdot \mathbf{F}_{\mathcal{B}}(\mathbf{U}) = \nabla \cdot (\epsilon \nabla \mathbf{U}), \quad (2)$$

The regularized equation (2) is

- a continuous analogue of Upwind, Lax-Friedrichs schemes
- suitable to apply the new viscosity method

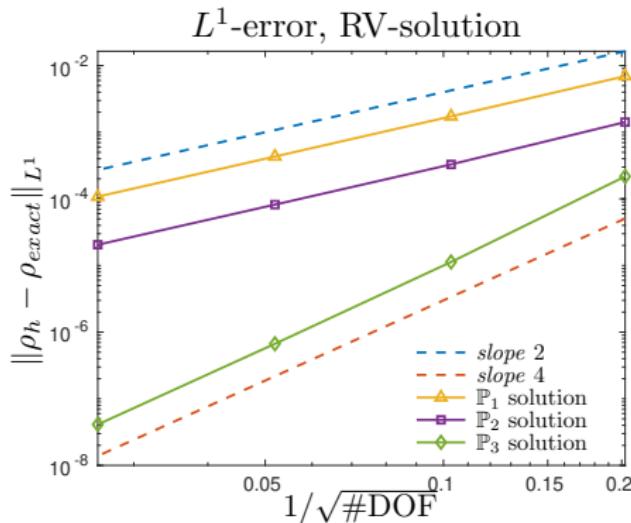
Theorem [D. & Nazarov, CMAME, 2022]

It can be shown that (2)

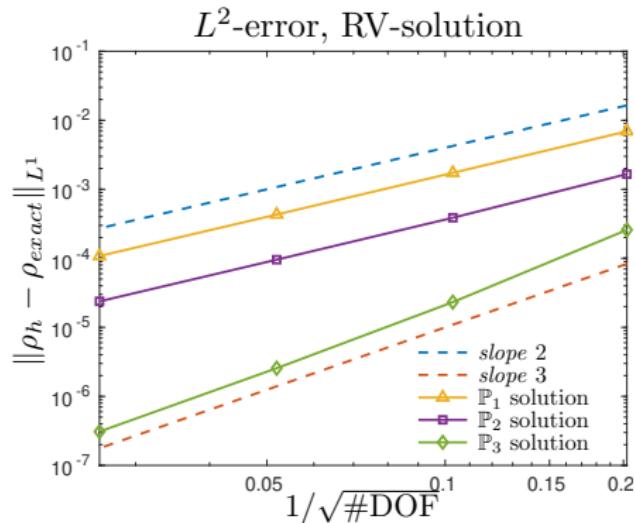
- preserves $\rho > 0, p > 0$
- fulfills the minimum entropy principle
- is compatible with all the generalized Harten entropies
- Consider $\nabla \cdot \mathbf{B} \neq 0 ? \rightarrow$ add Powell term (Godunov form)
- Divergence cleaning: projection, hyperbolic cleaning (GLM-MHD)

Convergence test on a smooth problem

Smooth wave propagation, [Wu and Shu, 2018].



(a) L^1 -error



(b) L^2 -error

Figure: \mathbb{P}_1 , \mathbb{P}_2 , \mathbb{P}_3 convergence for smooth solutions. Sub-optimal for \mathbb{P}_2 , see [Ainsworth, JCP, 2014].

The Orszag-Tang problem [Orszag & Tang, 1998]

Domain $\Omega = [0, 1] \times [0, 1]$. Gas constant $\gamma = \frac{5}{3}$.

Initial solution:

$$(\rho_0, \mathbf{u}_0, p_0, \mathbf{B}_0) = \left(\frac{25}{36\pi}, (-\sin(2\pi y), \sin(2\pi x)), \frac{5}{12\pi}, \left(-\frac{\sin(2\pi y)}{\sqrt{4\pi}}, \frac{\sin(4\pi x)}{\sqrt{4\pi}} \right) \right)$$

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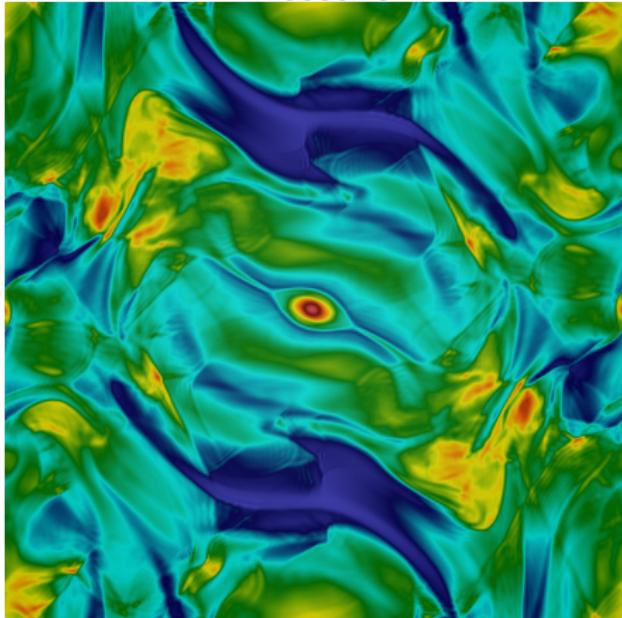
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- at $t = 1.0$ considered transition to turbulence

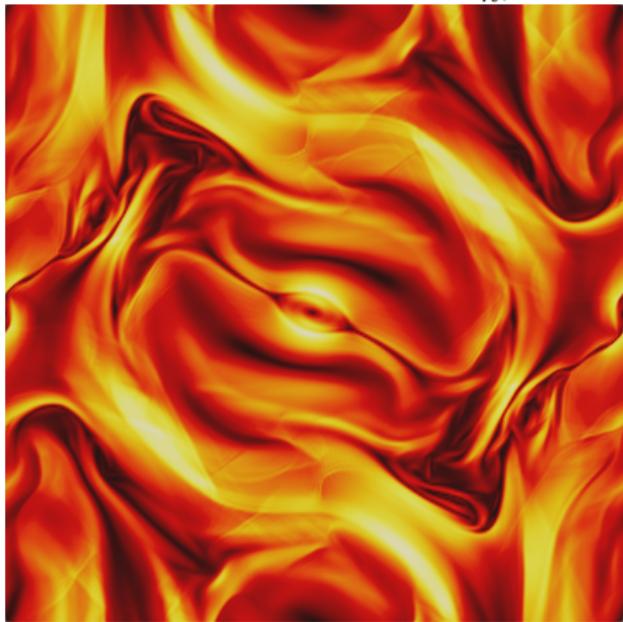
Orszag-Tang problem [Orszag & Tang, 1998]

\mathbb{P}_3 RV solution, 100×100 quasi-uniform mesh

Pressure



Magnetic pressure $B_h^2/2$



The MHD Rotor problem [Balsara & Spicer, 1998]

Domain $\Omega = [0, 1] \times [0, 1]$. Gas constant $\gamma = 1.4$.

Initial solution:

$$p_0 = 1, \mathbf{B}_0 = \left(\frac{5}{\sqrt{4\pi}}, 0 \right)$$

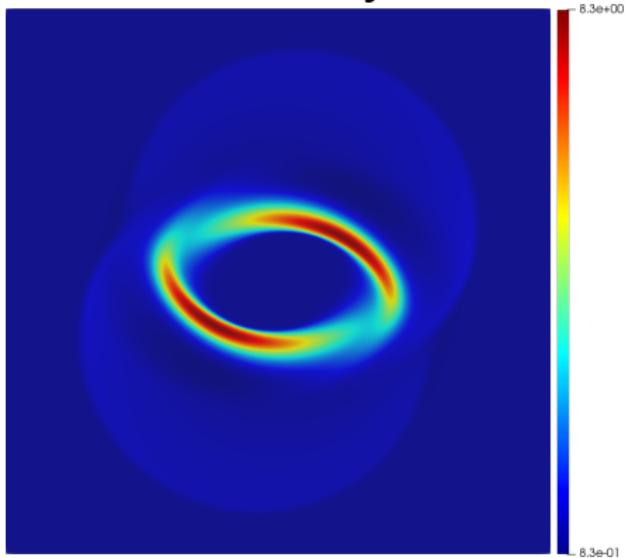
$$(\rho_0, \mathbf{u}_0) = \begin{cases} \left(10, \left(\frac{2}{r_0} \left(\frac{1}{2} - y \right), \frac{2}{r_0} \left(x - \frac{1}{2} \right) \right)^\top \right) & \text{if } r < r_0, \\ \left(1 + 9f, \left(f \frac{2}{r} \left(\frac{1}{2} - y \right), f \frac{2}{r} \left(x - \frac{1}{2} \right) \right)^\top \right) & \text{if } r_0 \leq r < r_1, \\ (1, (0, 0)^\top) & \text{otherwise,} \end{cases}$$

where $r := \sqrt{(x - 0.5)^2 + (y - 0.5)^2}$.

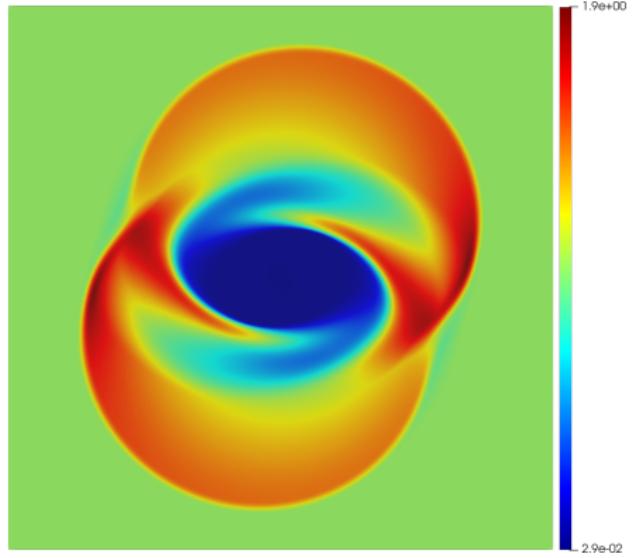
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Density



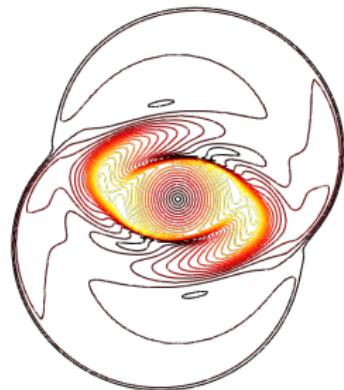
Pressure



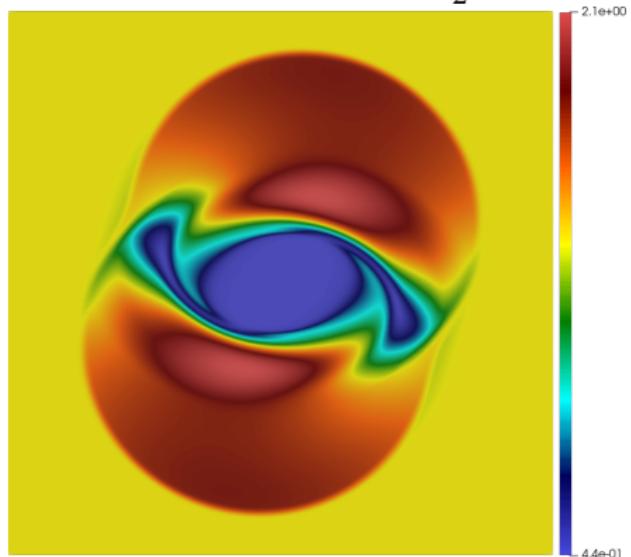
The MHD Rotor problem [Balsara & Spicer, 1998]

\mathbb{P}_3 RV solution, 100×100 quasi-uniform mesh

Mach number



Magnetic pressure $\frac{1}{2} B^2$



Astrophysical jets [Wu & Shu, 2018]

Domain $\Omega = [-0.75, 0.75] \times [0, 1.5]$. Gas constant $\gamma = 1.4$.

Initial solution: ambient

$$p_0 = 1, \mathbf{B}_0 = \left(0, \sqrt{200}\right)^\top, \rho_0 = 0.1\gamma, \mathbf{u}_0 = (0, 0)^\top$$

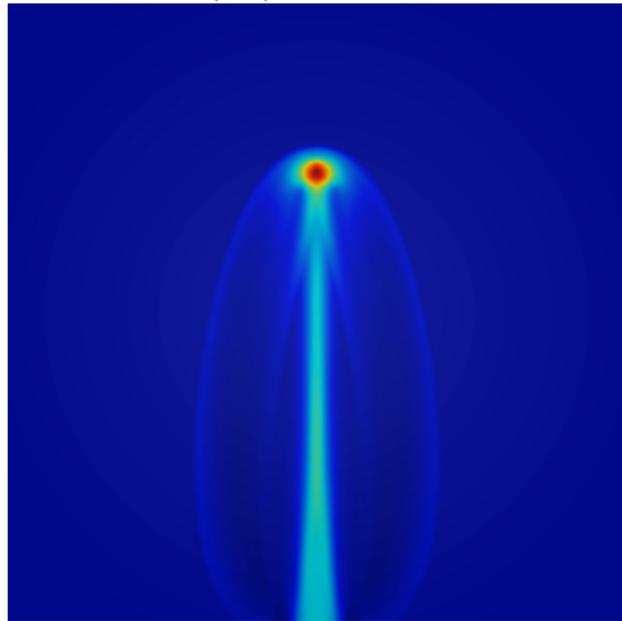
On the inlet $(x, y) \in [-0.05, 0.05] \times 0$: Mach 800

$$\mathbf{u}_0 = (0, 800)^\top, \rho = \gamma.$$

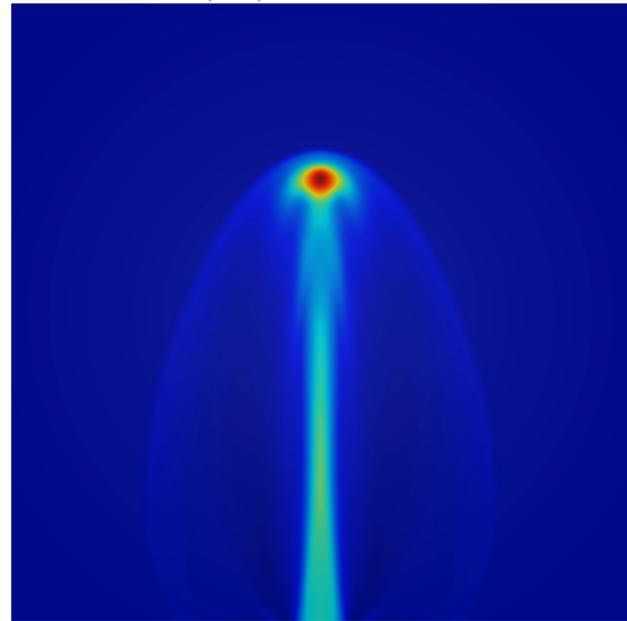
Astrophysical jets [Wu & Shu, 2018]

RV solution with 371447 \mathbb{P}_1 nodes, lumped mass

$$|B| = \sqrt{200}$$



$$|B| = \sqrt{2000}$$



Outline

1 Parameter-free first-order method for scalar conservation laws

2 Extension to systems and higher-order methods

3 Application to the system of MHD equations

4 Summary and outlook

Summary

- **Parameter-free** first-order PP FE method for scalar equations
- Extensions for systems, higher order methods
- Cheap to store and compute, easy to implement
- Tested against several MHD benchmarks

Outlook

Future works:

- Invariant-domain properties at the discrete level

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Future works:

- Invariant-domain properties at the discrete level
- Maximum wave speed for MHD, exact or reasonable upper bound
- Conflicting objectives: conservativeness, divergence, positivity, entropy stability
- High-order invariant-domain preserving FE methods for MHD

Thank you for your listening!