Half-time seminar

High-order nonlinear artificial viscosity methods for magnetohydrodynamics

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Outline

1. Background and objective
2. Paper I: A high-order residual-based viscosity finite element method for the ideal MHD equations
3. Paper II: Monolithic parabolic regularization of the MHD equations and entropy principles
4. Paper III: A parameter-free nonlinear viscosity finite element method for conservation laws
5. Summary and outlook

Outline

The system of MHD equations

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  Density, momentum, total energy, magnetic field

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  Density, momentum, total energy, magnetic field
- **MHD equations = Euler equations + Maxwell equations**

$$\partial_t \begin{pmatrix} \rho \\ \boldsymbol{m} \\ E \\ \boldsymbol{B} \end{pmatrix} + \nabla\cdot \begin{pmatrix} \boldsymbol{m} \\ \boldsymbol{m} \otimes \boldsymbol{u} + p\mathbb{I} \\ \boldsymbol{u}(E+p) \\ 0 \end{pmatrix} + \nabla\cdot \begin{pmatrix} 0 \\ -\boldsymbol{\beta} \\ -\boldsymbol{u}\cdot\boldsymbol{\beta} \\ \boldsymbol{u} \otimes \boldsymbol{B} - \boldsymbol{B} \otimes \boldsymbol{u} \end{pmatrix} = 0,$$

where $\boldsymbol{\beta} := \Big(-\frac{1}{2}(\boldsymbol{B}\cdot\boldsymbol{B})\mathbb{I} + \boldsymbol{B} \otimes \boldsymbol{B}\Big)$.

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- **Invariant-domain preserving** schemes: the numerical solution stays in a physically-relevant set as it proceeds in time.
  e.g., $\{\mathbf{U}_h \mid \rho_h > 0, \, p_h > 0, \, \partial_t s_h \geq 0, \, \nabla \cdot \boldsymbol{B}_h = 0\}$.

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- Solve realistic large scale problems in 3D (E.g., tokamak nuclear fusion reactor)

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- **Unstable** for hyperbolic problems

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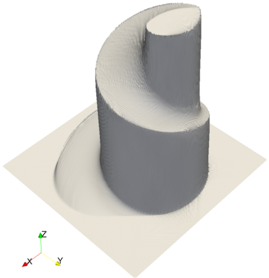
- Example: Kurganov-Petrova-Popov (2007) problem

$$\boldsymbol{f}(u) = (\sin(u), \cos(u))^\top, u_0 = \begin{cases} 3.5\pi & \text{if } \|\boldsymbol{x}\| \le 1, \\ 0.25\pi & \text{otherwise.} \end{cases}$$



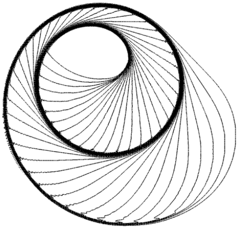
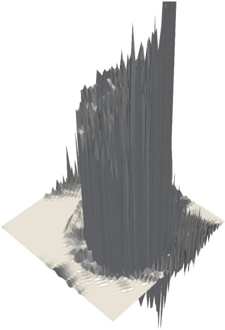


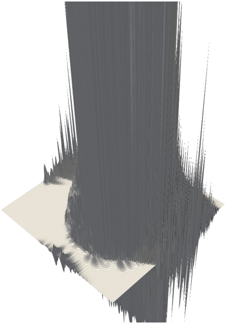
Figure: Reference solution at $t = 1$ [Guermond et. al., 2014]

Residual-based viscosity method

Naive FE discretization



(a) 5845 nodes



(b) 23380 nodes

Central approximations are unstable!

Residual-based viscosity method

- Viscous regularization

$$u_t + \nabla\cdot \boldsymbol{f}(u) = \nabla\cdot (\epsilon(\boldsymbol{x})\nabla u),$$

where $\epsilon(\boldsymbol{x})$ vanishes as the mesh size $h \to 0$.

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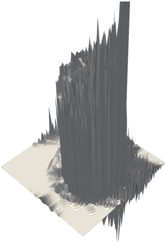
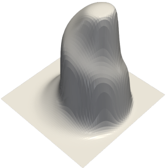
- Residual-based viscosity, [Nazarov, 2013]

$$\epsilon = \min(\epsilon_F, h^2\|R(u)\|_{L^\infty}).$$

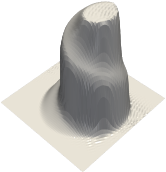
Residual-based viscosity method



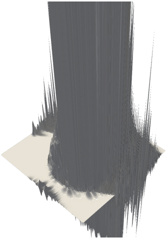
(a) FE, 5845 nodes



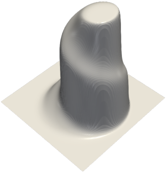
(b) First-order, 5845 nodes



(c) RV, 5845 nodes



(d) FE, 23380 nodes



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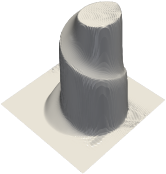

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Residual-based viscosity methods

- Have proven effective in FEM, SEM, FD, RBF settings [Nazarov-Hoffman, 2013, Marras *et. al.*, 2015, Lu *et. al.*, 2019, Stiernström *et. al.*, 2021, Tominec-Nazarov, 2021]

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- The viscosity term does not affect time step limit

Paper I: RV method for MHD

- Viscous regularization

$$u_t + \nabla\cdot \boldsymbol{f}(u) - \boxed{\nabla\cdot(\epsilon\nabla u)} = 0 \qquad \text{(scalar)}$$

$$\downarrow$$

$$\partial_t \mathbf{U} + \nabla\cdot \mathbf{F}_{\mathsf{MHD}}(\mathbf{U}) - \boxed{\nabla\cdot \mathbf{F}_{\mathcal{V}}(\mathbf{U})} = 0 \qquad \text{(system)}$$

Choose $\mathbf{F}_{\mathcal{V}}(\mathbf{U})$ = modified Navier-Stokes flux + resistive MHD flux

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- High-order viscosity

$$\tilde{R} = \max\left(R(\rho_h), R(\boldsymbol{m}_h), R(E_h), R(\boldsymbol{B}_h)\right)$$

$$\epsilon = \min(\epsilon_F, h^2\|\tilde{R}\|_{L^\infty})$$

Convergence study on smooth solutions

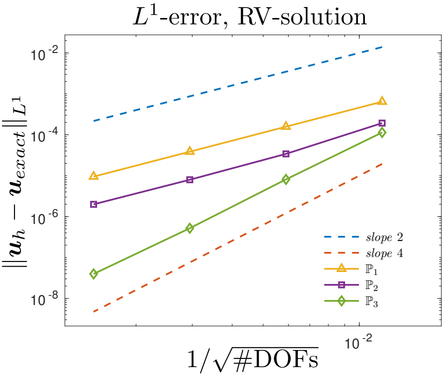
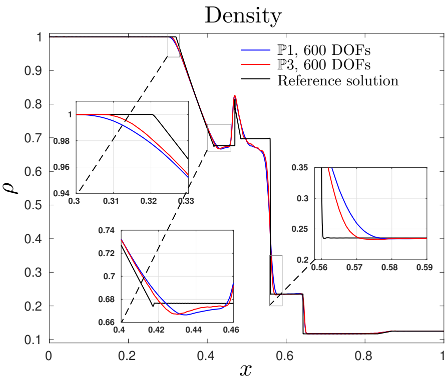
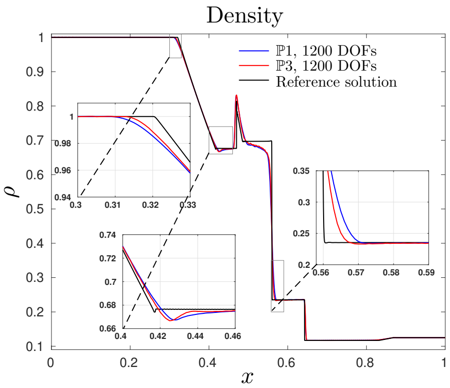


Figure: Convergence history for a smooth problem

Convergence study on discontinuous solutions



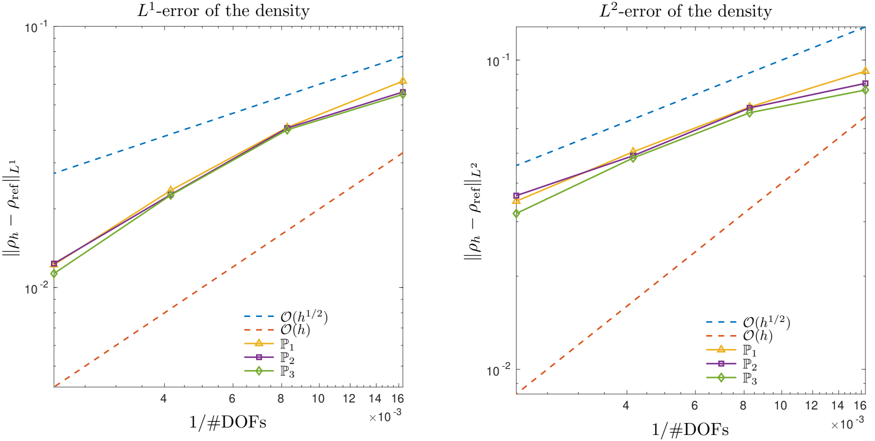
(a) $\mathbb{P}_1, \mathbb{P}_3$ solutions with 600 DOFs



(b) $\mathbb{P}_1, \mathbb{P}_3$ solutions with 1200 DOFs

Figure: RV solutions to the Brio-Wu problem [Brio-Wu, 1988]. Reference solution: Athena solution [Stone *et. al.*, 2008] with 10000 grid points.

Convergence study on the Brio-Wu problem



(a) L^1-error (b) L^2-error

The Orszag-Tang problem

Domain $\Omega = [0,1] \times [0,1]$. Gas constant $\gamma = \frac{5}{3}$.
Initial solution:

$$(\rho_0, \boldsymbol{u}_0, p_0, \boldsymbol{B}_0) =$$
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- at $t = 1.0$ considered transition to turbulence

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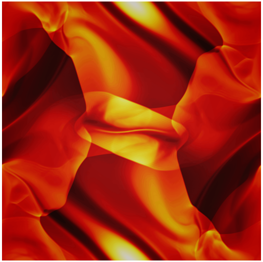
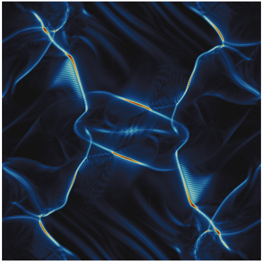
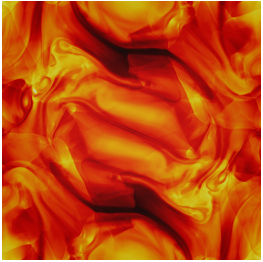


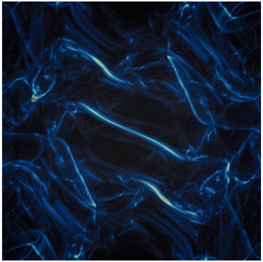
(a) Density at $t = 0.5$



(b) Residual viscosity at $t = 0.5$



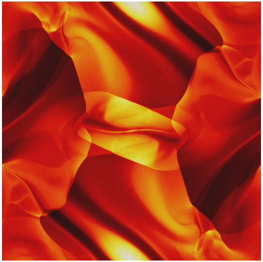
(c) Density at $t = 1.0$



(d) Residual viscosity at $t = 1.0$

Figure: $\mathbb{P}_1$ solution of the Orszag-Tang problem, 450×450 nodes

The Orszag-Tang problem



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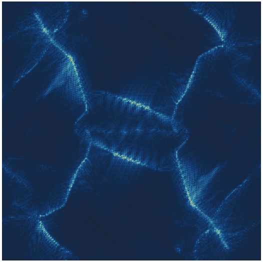

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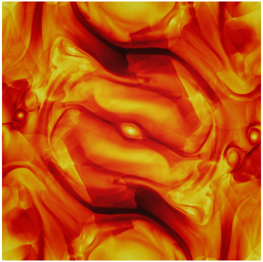

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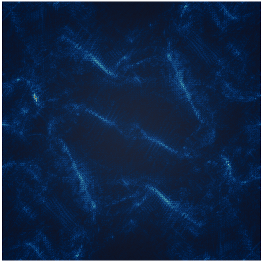

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Monolithic parabolic regularization

$$\partial_t \begin{pmatrix} \rho \\ \boldsymbol{m} \\ E \\ \boldsymbol{B} \end{pmatrix} + \nabla \cdot \mathbf{F}_{\mathsf{MHD}}(\mathbf{U}) - \nabla \cdot \underbrace{\boxed{\begin{pmatrix} \epsilon \nabla \rho \\ \epsilon \nabla \boldsymbol{m} \\ \epsilon \nabla E \\ \epsilon \nabla \boldsymbol{B} \end{pmatrix}}}_{\substack{\text{monolithic} \\ \text{parabolic flux}}} = 0$$

Monolithic parabolic regularization

$$\partial_t \begin{pmatrix} \rho \\ \boldsymbol{m} \\ E \\ \boldsymbol{B} \end{pmatrix} + \nabla\cdot \mathbf{F}_{\mathsf{MHD}}(\mathbf{U}) - \nabla\cdot \underbrace{\boxed{\begin{pmatrix} \epsilon\nabla\rho \\ \epsilon\nabla\boldsymbol{m} \\ \epsilon\nabla E \\ \epsilon\nabla\boldsymbol{B} \end{pmatrix}}}_{\text{monolithic parabolic flux}} = 0$$

Motivation:

- related to well-known existing schemes
- necessary for high-order positivity-preserving FEMs.

Connection to well-known discretizations

Consider

$$\partial_t \rho + \nabla \cdot \boldsymbol{m} = 0.$$

Connection to well-known discretizations

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Lax-Friedrichs method

$$\rho_i^{n+1} = \frac{1}{2}(\rho_{i-1}^n + \rho_{i+1}^n) - \frac{\delta t}{2\delta x}\left(m_{i+1}^n - m_{i-1}^n\right).$$

Let $\epsilon_{LF} = \frac{\delta x^2}{2\delta t}$. Rewrite as

$$\rho_i^{n+1} = \rho_i^n - \frac{\delta t}{2\delta x}\left(m_{i+1}^n - m_{i-1}^n\right) + \epsilon_{LF}\frac{\delta t}{\delta x^2}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n)$$

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which is a discrete analogue of

$$\partial_t \rho + \nabla \cdot \boldsymbol{m} = \epsilon_{LF}\Delta \rho.$$

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Upwind method

$$\rho_i^{n+1} = \rho_i^n - \frac{\delta t}{2\delta x}\left(m_{i+1}^n - m_{i-1}^n\right) + \epsilon_{\mathsf{Up}} \frac{\delta t}{\delta x^2}(\rho_{i+1}^n - 2\rho_i^n + \rho_{i-1}^n),$$

where

$$\epsilon_{\mathsf{Up}} = \frac{1}{2}\delta x \|\boldsymbol{u}\|_{L^\infty(\mathbb{R}^d)},$$

Connection to well-known discretizations

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Paper II: Three theorems at the PDE level

Theorem 1: (Positivity of density)

Assuming sufficient smoothness, the density solution to the monolithic parabolic regularized MHD system satisfies

$$\mathrm{essinf}_{\boldsymbol{x}\in\mathbb{R}^d}\rho(\boldsymbol{x},t) > 0,\ \forall t > 0.$$

Paper II: Three theorems at the PDE level

Theorem 2: (Minimum entropy principle)

Assuming sufficient smoothness and uniform convergence to constant states outside of a compact set, entropy of the solution to the monolithic parabolic regularized MHD system satisfies

$$\inf_{\boldsymbol{x}\in\mathbb{R}^d} s(\boldsymbol{x},t) \geq \inf_{\boldsymbol{x}\in\mathbb{R}^d} s_0(\boldsymbol{x}).$$

Paper II: Three theorems at the PDE level

Theorem 3: (Entropy inequalities)

Let $f(s)$ be a twice differentiable function. We consider the strictly convex generalized entropies $-\rho f(s)$ [Harten *et. al.*, 1998]. Any smooth solution to the monolithic parabolic regularized MHD system satisfies the entropy inequality

$$\partial_t(\rho f(s)) + \nabla \cdot (\boldsymbol{u}\rho f(s) - \epsilon\rho\nabla f(s) - \epsilon f(s)\nabla\rho) \geq 0$$

Resistive MHD flux vs. monolithic parabolic flux

A common MHD model to include resistivity and viscosity

$$\mathsf{F}_{\mathcal{V}}(\mathbf{U}) := \underbrace{\begin{pmatrix} 0 \\ \boldsymbol{\tau} \\ \boldsymbol{u}\cdot\boldsymbol{\tau} + \kappa\nabla T + \eta\boldsymbol{B}\cdot\big(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top\big) \\ \eta\big(\nabla\boldsymbol{B} - \nabla\boldsymbol{B}^\top\big) \end{pmatrix}}_{\text{resistive MHD flux}}.$$

Resistive MHD flux vs. monolithic parabolic flux

An example: a contact wave – velocity, pressure, magnetic field are constant, but density is discontinuous

$$\boldsymbol{u} = \boldsymbol{u}_0, p = p_0, \boldsymbol{B} = \boldsymbol{B}_0, \rho = \rho(x,t).$$

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- The momentum and the magnetic equations are fulfilled trivially.

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- The energy equation gives

$$\nabla \cdot \left(\kappa p_0 \rho^{-2} \nabla \rho\right) = 0.$$

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$\Rightarrow$ incompatible with contact waves if $\kappa \neq 0$.

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  $\Rightarrow$ incompatible with contact waves if $\kappa \neq 0$.
- However, letting $\kappa = 0$ leads to Gibbs phenomenon to numerical solutions

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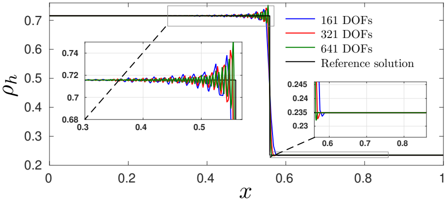
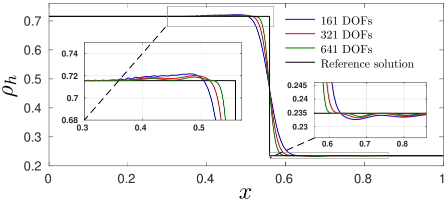
- The momentum and the magnetic equations are fulfilled trivially.
- The energy equation is also fulfilled.

Resistive MHD flux vs. monolithic parabolic flux

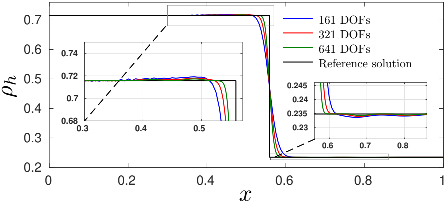
Resistive MHD flux



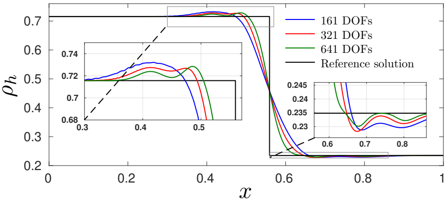
(a) Resistive MHD flux, $\kappa = 0$



(b) Resistive MHD flux, $\kappa = 1$



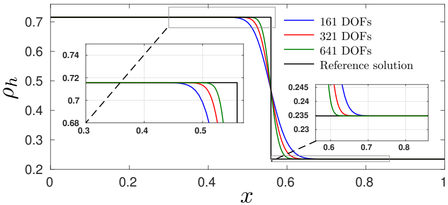
(c) Resistive MHD flux, $\kappa = \frac{1}{2}$



(d) Resistive MHD flux, $\kappa = 5$

Resistive MHD flux vs. monolithic parabolic flux

Monolithic flux



Extension to high-order stabilization

Entropy viscosity

- Use the **entropy residual** as a shock detector
- Is calculated on the nodal points

$$\epsilon = \min(\epsilon_F, h^2 \|R_E(u)\|_{L^\infty})$$

Convergence study on smooth solutions

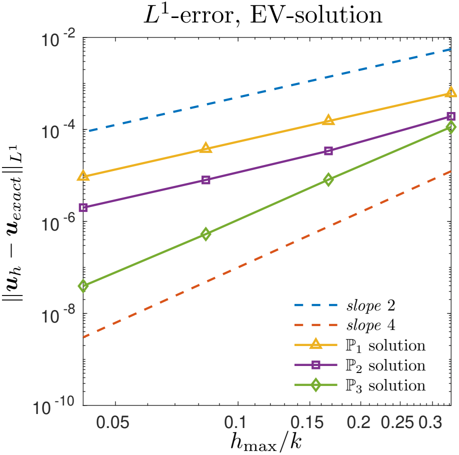
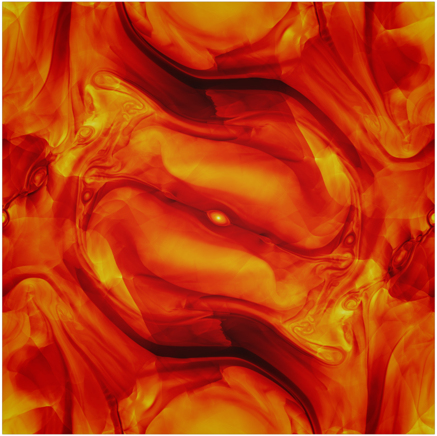


Figure: Convergence history: monolithic flux, entropy viscosity

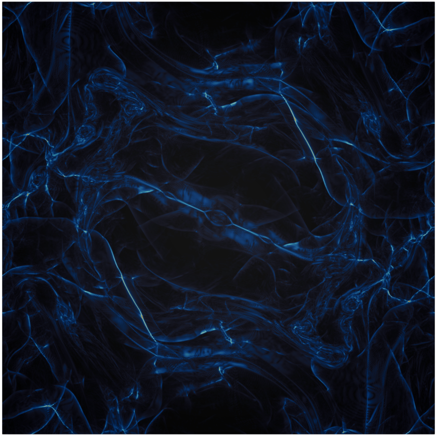
Orszag-Tang problem

EV solution with 4×10^6 $\mathbb{P}_1$ nodes

Density



Artificial viscosity



Orszag-Tang problem

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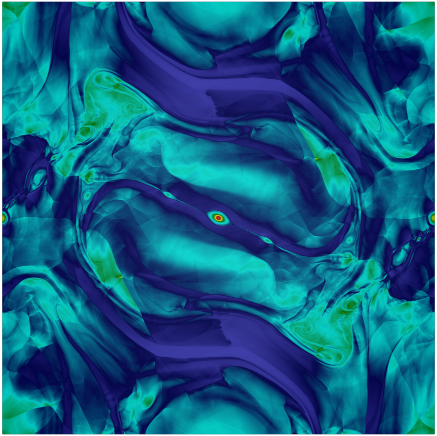
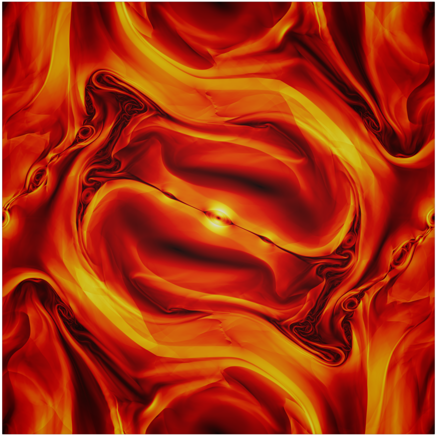
Pressure

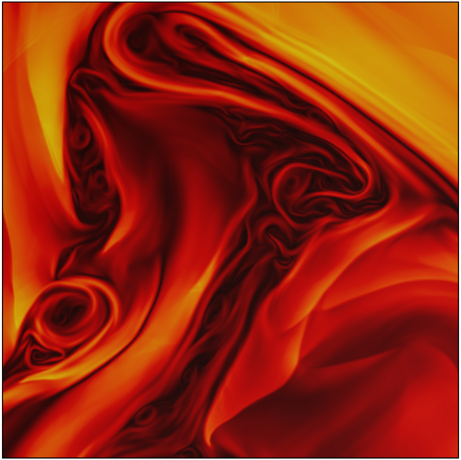


Magnetic pressure $B_h^2/2$



Orszag-Tang problem

Zoomed-in magnetic pressure at $t = 1.0$



Outline

Viscous regularization methods revisited

- Conventional viscous regularization to scalar conservation laws

$$u_t + \nabla\cdot \boldsymbol{f}(u) = \nabla\cdot (\epsilon \nabla u)$$

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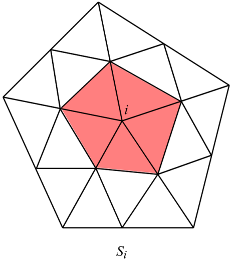
  - Extend to high-order parameter-free methods.

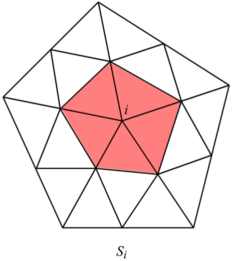
First-order method for $\mathbb{P}_1$



At node i, define

- S_i – support of the Lagrange basis function φ_i

First-order method for $\mathbb{P}_1$



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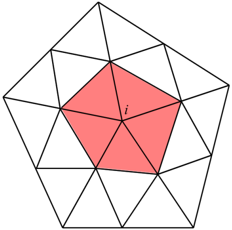

S_i

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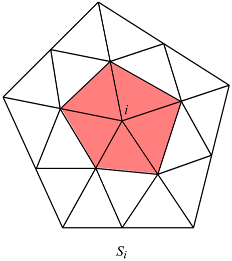

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- m_i – weight of the lumped $\mathbb{P}_1$ mass matrix at index (i, i)

First-order methods for $\mathbb{P}_1$

Theorem 1: (Nodal viscosity ϵ_i)

Define the nodal viscosity as

$$\varepsilon_i := \alpha_i C_i \|\boldsymbol{f}'\|_{L^\infty(S_i)} \max_{i \neq j \in \mathcal{I}(S_i)} |\nabla \varphi_j| \int_{S_i} \varphi_i \,\mathrm{d}\boldsymbol{x}, \qquad (\star)$$

where C_i is a mesh function.

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where C_i is a mesh function. Under a CFL condition, the approximation

$$m_i \frac{u_i^{n+1} - u_i^n}{\tau} + \int_{S_i} \nabla \cdot \boldsymbol{f}(u_h^n) \varphi_i \, \mathrm{d}\boldsymbol{x} + \int_{S_i} \varepsilon_i (\mathbb{J}_K^\top \nabla u_h^n) \cdot (\mathbb{J}_K^\top \nabla \varphi_i) \, \mathrm{d}\boldsymbol{x} = 0$$

fulfills the discrete maximum principle.

First-order methods for $\mathbb{P}_1$

Theorem 2: (Nodal viscosity $\sum \epsilon_j$)

Under a CFL condition, the approximation

$$m_i \frac{u_i^{n+1} - u_i^n}{\tau} + \int_{S_i} \nabla \cdot \boldsymbol{f}(u_h^n)\varphi_i \, \mathrm{d}\boldsymbol{x} + \int_{S_i} (\mathbb{J}_K^\top \nabla u_h^{\varepsilon,n}) \cdot (\mathbb{J}_K^\top \nabla \varphi_i) \, \mathrm{d}\boldsymbol{x} = 0,$$

where $u_h^{\varepsilon,n} := \sum_j \varepsilon_j u_j^n \varphi_j$, ε_j is calculated by $(\star)$, also fulfills the discrete maximum principle.

First-order methods for $\mathbb{P}_1$

Theorem 3: (ε_h)

Under a CFL condition, the approximation

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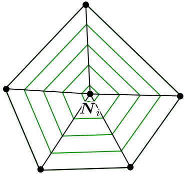
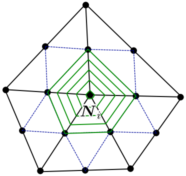
This is equivalent to **the weak formulation of a viscous regularization**!

Extensions

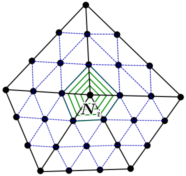
- Other polynomial spaces



(a) $\mathbb{P}_1$



(b) $\mathbb{P}_2$



(c) $\mathbb{P}_3$

Extensions

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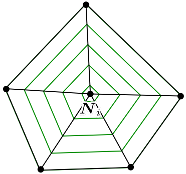

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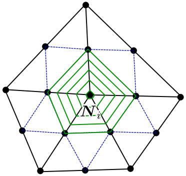
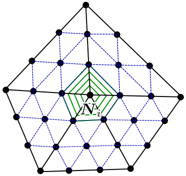


(b) $\mathbb{P}_2$

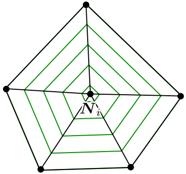


(c) $\mathbb{P}_3$

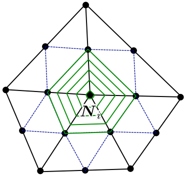
- System

Extensions

- Other polynomial spaces



(a) $\mathbb{P}_1$



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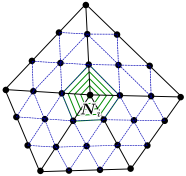
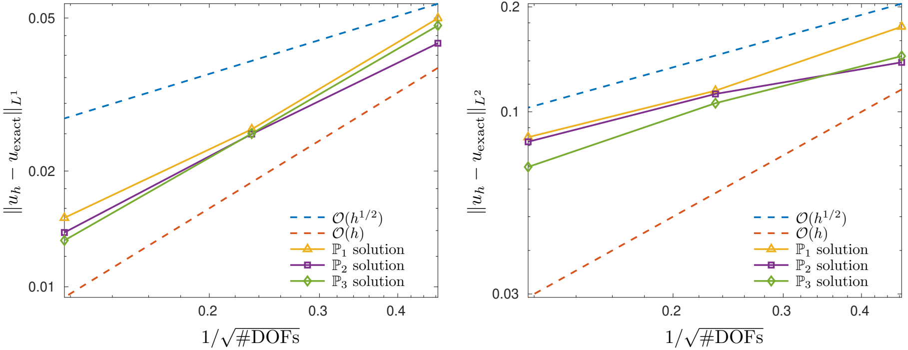


(c) $\mathbb{P}_3$

- System
- High-order viscosity methods: RV, EV

$$\varepsilon^{\mathrm{H}}_{h,i} := C_i \alpha_i \int_{S_i} \min\left(\max_{j\in\mathcal{I}(i)} |\boldsymbol{f}'(q_h^n)\cdot\nabla\varphi_j|,\ |R_i(q_h)|\right) \varphi_i \, \mathrm{d}\boldsymbol{x}$$

Convergence test on Burger's equation with shocks



(a) L^1-error (b) L^2-error

Figure: First-order $\mathbb{P}_1$, $\mathbb{P}_2$, $\mathbb{P}_3$ shock solutions to the Burger's equation

Outline

Summary of the papers

- Paper I

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  - Propose a robust shock-capturing FE method for MHD.

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  - The method is numerically tested up to $\mathbb{P}_3 \Rightarrow$ Fourth order accurate.

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  - The method works in 1D, 2D, 3D.

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Thank you for your listening!