HUST

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ONE LOVE. ONE FUTURE.

FUNDAMENTALS OF OPTIMIZATION

Linear Programming

CONTENT

- **Linear programs**
- Geometric approach
- Simplex method
- Two-phase simplex method
- OR-TOOLS for linear programming
- Programming exercises

LINEAR PROGRAMS

- Standard form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \leq b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \leq b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \leq b_m$$

$$x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n \geq 0$$

LINEAR PROGRAMS

- Standardize general linear programs
  - $f(x) \rightarrow \min \Leftrightarrow -f(x) \rightarrow \max$
  - $g(x) \geq b \Leftrightarrow -g(x) \leq -b$
  - $A = B \Leftrightarrow (A \leq B)$ and $(-A \leq -B)$
  - A variable $x_j \in R$ can be represented by $x_j = x_j^+ - x_j^-$ where $x_j^+, x_j^- \geq 0$

LINEAR PROGRAMS

- Example: Convert a general linear program forms into standard form

  $f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \min$

  $$2x_1 + x_2 \leq 7$$
  $$x_1 + 2x_2 = 8$$
  $$x_1 - x_2 \geq 2$$
  $$x_1, x_2 \in R, x_2 \geq 0$$

- Example: Convert a general linear program forms into standard form
  - Substitution: $x_1 = x_1^+ - x_1^-$

  $f(x_1^+, x_1^-, x_2) = -3x_1^+ + 3x_1^- - 2x_2 \rightarrow \max$

  $2x_1^+ - 2x_1^- + x_2 \leq 7$

  $x_1^+ - x_1^- + 2x_2 \leq 8$

  $-x_1^+ + x_1^- - 2x_2 \leq -8$

  $-x_1^+ + x_1^- + x_2 \leq -2$

  $x_1^+, x_1^-, x_2 \in R, x_1^+, x_1^-, x_2 \geq 0$

CONTENT

- Linear programs
- **Geometric approach**
- Simplex method
- Two-phase simplex method
- OR-TOOLS for linear programming
- Programming exercises

GEOMETRIC APPROACH

- Constraints (inequalities) form a feasible region
- Optimal points will be one of the corners of the feasible region

GEOMETRIC APPROACH

- Example

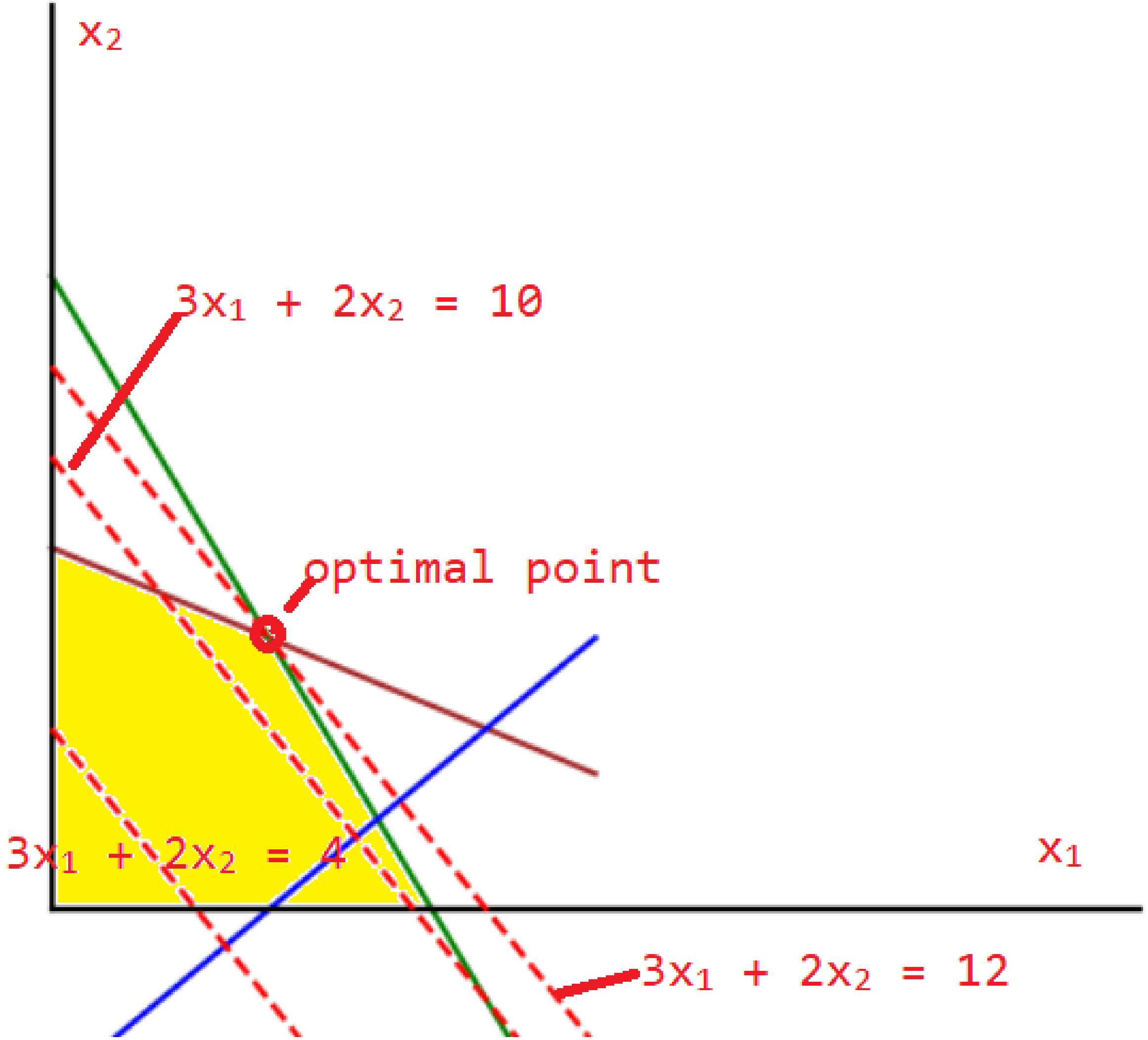
$$
\begin{aligned}
f(x_1, x_2) \quad = \quad & 3x_1 + 2x_2 \rightarrow \max \\
& 2x_1 + x_2 \leq 7 \\
& x_1 + 2x_2 \leq 8 \\
& x_1 - x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

- Example

$$f(x_1, x_2) = 3x_1 + 2x_2 \rightarrow \max$$
$$2x_1 + x_2 \le 7$$
$$x_1 + 2x_2 \le 8$$
$$x_1 - x_2 \le 2$$
$$x_1, x_2 \ge 0$$

- Example

$$f(x_1, x_2) \quad = \quad 3x_1 + 2x_2 \rightarrow \max$$
$$2x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 8$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \geq 0$$



- Special cases
  - Problem is unbounded
  - Problem does not have feasible solutions

$$\begin{aligned} f(x_1, x_2) = 3x_1 + 2x_2 &\rightarrow \max \\ -2x_1 - x_2 &\leq -7 \\ x_1 - x_2 &\leq 2 \\ x_1, x_2 \in R,\ &x_1, x_2 \geq 0 \end{aligned}$$

$$\begin{aligned} f(x_1, x_2) = 3x_1 + 2x_2 &\rightarrow \max \\ 2x_1 + x_2 &\leq 7 \\ -4x_1 - 2x_2 &\leq -16 \\ x_1, x_2 \in R,\ &x_1, x_2 \geq 0 \end{aligned}$$

GEOMETRIC APPROACH

- Exercise
  - A company must decide to make a plan to produce 2 products P1, P2.
    - The revenue received when selling 1 unit of P1 and P2 are respectively 5$ and 7$
    - The manufacturing cost when producing P1 and P2 are respectively 2$ and 3$
    - The storage cost in warehouses for 1 unit of P1 and P2 are respectively 1$ and 3$
  - Compute the production plan so that
    - Total manufacturing cost is less than or equal to 200$
    - Total storage cost is less than or equal to 150$
    - Total revenue is maximal

CONTENT

- Linear programs
- Geometric approach
- **Simplex method**
- Two-phase simplex method
- OR-TOOLS for linear programming
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STANDARD FORM

- Standard form to standard equational form by adding slack variables $y_1, y_2, \ldots, y_m$

Standard form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \le b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \le b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \le b_m$$

$$x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n \ge 0$$

Standard equality form

$$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$$

$$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 = b_1$$

$$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + y_2 = b_2$$

$$\ldots$$

$$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + y_m = b_m$$

$$x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \ge 0$$

SHORT FORM

- Consider a Linear Program (LP) under a standard equational form

Standard equational form

$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow$ max

$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1$

$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2$

$\ldots$

$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m$

$x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n \geq 0$

$\Rightarrow$

Standard equality form

$f(x) = c^Tx \rightarrow$ max

$Ax = b$

$x \geq 0$

IDEA

$$z = 3x_1 + 2x_2 \rightarrow \max$$
$$2x_1 + x_2 \leq 7$$
$$x_1 + 2x_2 \leq 8$$
$$x_1 - x_2 \leq 2$$
$$x_1, x_2 \in R,\ x_1, x_2 \geq 0$$

$$z = 3x_1 + 2x_2 \rightarrow \max$$
$$2x_1 + x_2 + x_3 = 7$$
$$x_1 + 2x_2 + x_4 = 8$$
$$x_1 - x_2 + x_5 = 2$$
$$x_1, x_2, x_3, x_4, x_5 \in R,\ x_1, x_2, x_3, x_4, x_5 \geq 0$$

$$z = -4/3\, x_3 - 1/3\, x_4 + 12 \rightarrow \max$$
$$x_2 - 1/3x_3 + 2/3x_4 = 3$$
$$-x_3 + x_4 + x_5 = 3$$
$$x_1 + 2/3x_3 - 1/3\, x_4 = 2$$
$$x_1, x_2, x_3, x_4, x_5 \in R,\ x_1, x_2, x_3, x_4, x_5 \geq 0$$

BASIC FEASIBLE SOLUTION

- Consider a Linear Program (LP) under a standard equational form
- Suppose rank(A) = m
- Let B be the matrix of m linearly independent columns (indexed $j_1, j_2, \ldots, j_m$) of A: $B = (A(j_1), A(j_2), \ldots, A(j_m))$
  - Solution x is called a **basic solution** if :
    - $x_j = 0$ for $j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$
    - Remain variables are found by solving this equation:

$$\begin{pmatrix} a_{1,j_1} & a_{1,j_2} & \ldots & a_{1,j_m} \\ a_{2,j_1} & a_{2,j_2} & \ldots & a_{2\ j_m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,j_1} & a_{m,j_2} & \ldots & a_{m,j_m} \end{pmatrix} \begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \ldots \\ x_{j_m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$$

$f(x) = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$

$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1$

$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2$

$\ldots$

$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m$

$x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n \geq 0$

BASIC FEASIBLE SOLUTION

- Let B be the matrix of m linearly independent columns (indexed $j_1, j_2, \ldots, j_m$) of A: $B = (A(j_1), A(j_2), \ldots, A(j_m))$
  - Solution x is called a basic solution if :
    - $x_j = 0$ for $j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$
    - Remain variables are found by solving this equation:

$$\begin{pmatrix} a_{1,j_1} & a_{1,j_2} & \ldots & a_{1,j_m} \\ a_{2,j_1} & a_{2,j_2} & \ldots & a_{2\ j_m} \\ \ldots & \ldots & \ldots & \ldots \\ a_{m,j_1} & a_{m,j_2} & \ldots & a_{m,j_n} \end{pmatrix} \begin{pmatrix} x_{j_1} \\ x_{j_2} \\ \ldots \\ x_{j_m} \end{pmatrix} = \begin{pmatrix} b_1 \\ b_2 \\ \ldots \\ b_m \end{pmatrix}$$

  - B is called a basis
  - $j_1, j_2, \ldots, j_m$: basic indices, $j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$ is called non-basic index
  - $x_{j_1}, x_{j_2}, \ldots, x_{j_m}$: basic variables and x_j ($j \in \{1, 2, \ldots, n\} \setminus \{j_1, j_2, \ldots, j_m\}$) is called non-basic variable
  - A **basic solution** x with $x \geq 0$ is called a **basic feasible solution**

Example

Example

$f(x) = cx \rightarrow \max$

$Ax = b$

$x \geq 0$

$$x = \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{pmatrix} \qquad c = (3, 2, 0, 0, 0)$$

$$A = \begin{pmatrix} 2 & 1 & 1 & 0 & 0 \\ 1 & 2 & 0 & 1 & 0 \\ 1 & -1 & 0 & 0 & 1 \end{pmatrix} \qquad b = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix}$$

$J = (1,2,3,4,5)$ – set of variable indices, $I = (1,2,3)$ - set of constraint indices

$J_B = (3,4,5)$, $J_N = (1,2)$,

$$x_B = \begin{pmatrix} x_3 \\ x_4 \\ x_5 \end{pmatrix} = \begin{pmatrix} 7 \\ 8 \\ 2 \end{pmatrix} \qquad x_N = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad N = \begin{pmatrix} 2 & 1 \\ 1 & 2 \\ 1 & -1 \end{pmatrix}$$

SIMPLEX METHOD: Tabular form

- Consider a linear program under a standard form

$$
\begin{aligned}
&z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max\\
&\quad a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n \le b_1\\
&\quad a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \le b_2\\
&\quad \ldots\\
&\quad a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n \le b_m\\
&\quad b_1, b_2, \ldots b_m \ge 0\\
&\quad x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n \ge 0
\end{aligned}
$$

$$
\begin{aligned}
&z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max\\
&\quad a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 = b_1\\
&\quad a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + y_2 = b_2\\
&\quad \ldots\\
&\quad a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + \quad + y_m = b_m\\
&\quad b_1, b_2, \ldots b_m \ge 0\\
&\quad x_1, x_2, \ldots x_n \in R,\ x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \ge 0
\end{aligned}
$$

We need this additional assumption to ensure we have an initial feasible solution.

	1	2	…	n	n+1	n+2	…	n+m		
0	x_1	x_2	…	x_n	y_1	y_2	…	y_m	z	RHS
1	$a_{1,1}$	$a_{1,2}$	…	$a_{1,n}$	1	0	…	0	0	b_1
2	$a_{2,1}$	$a_{2,2}$	…	$a_{2,n}$	0	1	…	0	0	b_2
…	…	…	…	…	…		…	…	..	…
m	$a_{m,1}$	$a_{m,2}$	…	$a_{m,n}$	0	0	…	1	0	b_m
m+1	$-c_1$	$-c_2$	…	$-c_n$	0	0	..	0	1	0

SIMPLEX METHOD: Tabular form

	1	2	…	n	$n+1$	$n+2$	…	$n+m$	$n+m+1$	
0	x_1	x_2	…	x_n	x_{n+1}	x_{n+2}	…	x_{n+m}	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	…	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	…	$\alpha_{1,n+m}$	$\alpha_{1,n+m+1}$	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	…	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	…	$\alpha_{2,n+m}$	$\alpha_{2,n+m+1}$	β_2
…	…	…	…	…	…		…	…	..	…
m	$\alpha_{m,1}$	$\alpha_{m,2}$	…	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	…	$\alpha_{m,n+m}$	$\alpha_{m,n+m+1}$	β_m
$m+1$	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	…	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	…	$\alpha_{m+1,n+m}$	$\alpha_{m+1,n+m+1}$	β_{m+1}

- $J = \{1, 2, \ldots, n, n+1, \ldots, n+m\}$
- Maintain linear constraints on each row k ($k = 1, 2, \ldots, m+1$):

$$\alpha_{k,1}x_1 + \alpha_{k,2}x_2 + \ldots + \alpha_{k,n}x_n + \alpha_{k,n+1}x_{n+1} + \ldots + \alpha_{k,n+m}x_{n+m} + \alpha_{k,n+m+1}z = \beta_k \qquad (*)$$

- Let R_k be a vector containing elements on row k of the table ($k = 1, 2, \ldots, m+1$)
- Perform linear transformation below, constraint (*) is still satisfied:
  - Replace $R_k = R_k + \delta^* R_i$ ($k, i = 1, 2, \ldots, m+1$), with some constant δ

SIMPLEX METHOD: Tabular form

- **Optimality**

	1	2	...	m	$m+1$	$m+2$	...	$n+m$	$n+m+1$	
0	x_1	x_2	...	x_m	x_{m+1}	x_{m+2}	...	x_{n+m}	z	RHS
1	1	0	...	0	$\alpha_{1,m+1}$	$\alpha_{1,m+2}$	...	$\alpha_{1,n+m}$	0	β_1
2	0	1	...	0	$\alpha_{2,m+1}$	$\alpha_{2,m+2}$	...	$\alpha_{2,n+m}$	0	β_2
...	...	...	...	...	...		...	...	..	...
m	0	0	...	1	$\alpha_{m,m+1}$	$\alpha_{m,m+2}$	...	$\alpha_{m,n+m}$	0	β_m
$m+1$	0	0	...	0	$\alpha_{m+1,m+1}$	$\alpha_{m+1,m+2}$	...	$\alpha_{m+1,n+m}$	1	β_{m+1}

- With $\beta_1, \beta_2, \ldots, \beta_m \geq 0$, $\exists J_B = \{j_1, j_2, \ldots, j_m\}$ such that $\alpha_{m+1,j} = 0, \forall j \in J_B$, $\alpha_{m+1,j} \geq 0\ \forall j \in J \backslash J_B$, columns $j_1, j_2, \ldots, j_m$ forms a unit matrix
- Without loss of generality, suppose that $J_B = \{1, 2, \ldots, n\}$, coefficients $\alpha_{m+1,m+1}, \alpha_{m+1,m+2}, \ldots, \alpha_{m+1,n+m} \geq 0$, columns 1, ..., m forms a unit matrix: $\alpha_{1,1}, \alpha_{2,2}, \ldots, \alpha_{m,m} = 1$
- Constraint (*) is still satisfied. We have $\alpha_{m+1,m+1}x_{m+1} + \alpha_{m+1,m+2}x_{m+2} + \ldots, \alpha_{m+1,n+m}x_{n+m} + z = \beta_{m+1}$
- $z = \beta_{m+1} - (\alpha_{m+1,m+1}x_{m+1} + \alpha_{m+1,m+2}x_{m+2} + \ldots, \alpha_{m+1,n+m}x_{n+m}) \leq \beta_{m+1}$ (because $\alpha_{m+1,m+1}, \alpha_{m+1,m+2}, \ldots, \alpha_{m+1,n+m} \geq 0$ and $x_{m+1}, \ldots, x_{n+m} \geq 0$).

SIMPLEX METHOD: Tabular form

- **Optimality**

	1	2	...	m	$m+1$	$m+2$	...	$n+m$	$n+m+1$	
0	x_1	x_2	...	x_m	x_{m+1}	x_{m+2}	...	x_{n+m}	z	RHS
1	1	0	...	0	$\alpha_{1,m+1}$	$\alpha_{1,m+2}$	...	$\alpha_{1,n+m}$	0	β_1
2	0	1	...	0	$\alpha_{2,m+1}$	$\alpha_{2,m+2}$	...	$\alpha_{2,n+m}$	0	β_2
...	...	...	...	...	...		...	...	..	...
m	0	0	...	1	$\alpha_{m,m+1}$	$\alpha_{m,m+2}$	...	$\alpha_{m,n+m}$	0	β_m
$m+1$	0	0	...	0	$\alpha_{m+1,m+1}$	$\alpha_{m+1,m+2}$	...	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Moreover, there exists a solution (nonnegative values for variables x_1, x_2, ..., x_{n+m}) described below:
  - $x_1 = \beta_1$, $x_2 = \beta_2$, . . ., $x_m = \beta_m$
  - $x_{m+1} = x_{m+2} = . . . = x_{n+m} = 0$

  Satisfying given constraints. Also, the objective value at this solution is equal to the upper bound β_{m+1}. It means that this solution is an optimal solution to the given problem.

SIMPLEX METHOD: Tabular form

- **Simplex step**

	1	2	...	m	m+1		i	...	n+m			
0	x_1	x_2	…	x_m	x_{m+1}	…	x_i	…	x_{n+m}	z	RHS	E
1	1	0	…	0	$\alpha_{1,m+1}$	…	$\alpha_{1,i}$	…	$\alpha_{1,n+m}$	0	β_1	E_1
2	…	…	…	…	…	…	…	…	…	…	…	…
...	0	1	…	0	$\alpha_{k,m+1}$	…	$\alpha_{k,i}$	…	$\alpha_{k,n+m}$	0	β_k	E_k
m	0	0	…	1	$\alpha_{m,m+1}$	…	$\alpha_{m,i}$	…	$\alpha_{m,n+m}$	0	β_m	E_m
m+1	0	0	…	0	$\alpha_{m+1,m+1}$	…	$\alpha_{m+1,i}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}	

- Select column i such that the element on row m+1 (which is $\alpha_{m+1,i}$) is negative minimal
- Compute evaluations (column E): $E_j = +\infty$, if $\alpha_{j,i} \leq 0$, and $E_j = \dfrac{\beta_j}{\alpha_{j,i}}$, if $\alpha_{j,i} > 0$, $j = 1, 2, ..., m$
- Select the row k such that E_k is minimal: if $E_k = +\infty$, then the problem is **unbounded**, otherwise
  - Update:
    - Row $R_k = R_k / \alpha_{k,i}$
    - Row $R_j = R_j - \alpha_{j,i} * R_k$, $j = \{1, 2, ..., m+1\} \setminus \{k\}$

SIMPLEX METHOD: Tabular form

- **Example**

$$
\begin{aligned}
z = \;& 3x_1 + 2x_2 \rightarrow \max \\
& 2x_1 + x_2 \le 7 \\
& x_1 + 2x_2 \le 8 \\
& x_1 - x_2 \le 2 \\
& x_1, x_2 \in R,\ x_1, x_2 \ge 0
\end{aligned}
\quad \Rightarrow \quad
\begin{aligned}
z = \;& 3x_1 + 2x_2 \rightarrow \max \\
& 2x_1 + x_2 + x_3 = 7 \\
& x_1 + 2x_2 + x_4 = 8 \\
& x_1 - x_2 + x_5 = 2 \\
& x_1, x_2, x_3, x_4, x_5 \in R,\ x_1, x_2, x_3, x_4, x_5 \ge 0
\end{aligned}
$$

x_1	x_2	x_3	x_4	x_5	z	RHS	E
2	1	1	0	0	0	7	
1	2	0	1	0	0	8	
1	-1	0	0	1	0	2	
-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

- **Example**

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

- **Example**

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal
- Compute evaluation (column E)

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	7/2
2	1	2	0	1	0	0	8	8/1
3	1	-1	0	0	1	0	2	2/1
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal
- Compute evaluation (column E)
- Select row R3: evaluation is minimal
- Update R3 = R3/1

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	7/2
2	1	2	0	1	0	0	8	8/1
3	1	-1	0	0	1	0	2	2/1
4	-3	-2	0	0	0	1	0	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	2	1	1	0	0	0	7	
2	1	2	0	1	0	0	8	
3	1	-1	0	0	1	0	2	
4	-3	-2	0	0	0	1	0	

- Select column 1: the corresponding element at the last row is minimal
- Compute evaluation (column E)
- Select row R3: evaluation is minimal
- Update R3 = R3/1
- R1 = R1 – 2R3; R2 = R2 – R3; R4 = R4 + 3R3;

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	3/3
2	0	3	0	1	-1	0	6	6/3
3	1	-1	0	0	1	0	2	$+\infty$
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row R1: minimum evaluation

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	3/3
2	0	3	0	1	-1	0	6	6/3
3	1	-1	0	0	1	0	2	$+\infty$
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row R1: minimum evaluation
- Update R1 = R1/3

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	3	1	0	-2	0	3	
2	0	3	0	1	-1	0	6	
3	1	-1	0	0	1	0	2	
4	0	-5	0	0	3	1	6	

- Select column 2: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row R1: minimum evaluation
- Update R1 = R1/3
- R2 = R2 – 3R1; R3 = R3 + R1; R4 = R4 + 5R1

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	$+\infty$
2	0	0	-1	1	1	0	3	3
3	1	0	1/3	0	1/3	0	3	9
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	$+\infty$
2	0	0	-1	1	1	0	3	3
3	1	0	1/3	0	1/3	0	3	9
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation
- Update: R2 = R2/1

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation
- Update: R2 = R2/1
- R1 = R1 + (2/3)R2; R3 = R3 – (1/3)R2; R4 = R4+(1/3)R2

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	-1/3	2/3	0	0	3	
2	0	0	-1	1	1	0	3	
3	1	0	2/3	-1/3	0	0	2	
4	0	0	4/3	1/3	0	1	12	

SIMPLEX METHOD: Tabular form

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	1/3	0	-2/3	0	1	
2	0	0	-1	1	1	0	3	
3	1	0	1/3	0	1/3	0	3	
4	0	0	5/3	0	-1/3	1	11	

- Select column 5: the corresponding element at the last row is minimal
- Compute evaluations: column E
- Select row 2: minimum evaluation
- Update: R2 = R2/1
- R1 = R1 + (2/3)R2; R3 = R3 – (1/3)R2; R4 = R4+(1/3)R2

	x_1	x_2	x_3	x_4	x_5	z	RHS	E
1	0	1	-1/3	2/3	0	0	3	
2	0	0	-1	1	1	0	3	
3	1	0	2/3	-1/3	0	0	2	
4	0	0	4/3	1/3	0	1	12	

- Optimal solution: $x_1 = 2$, $x_2 = 3$, $x_3 = 0$, $x_4 = 0$, $x_5 = 3$.
- Value of the objective function: 12

Exercises

- Example

$$
\begin{aligned}
f(x_1, x_2) \quad = \quad & 3x_1 + 2x_2 \rightarrow \max \\
& 2x_1 + x_2 \leq 7 \\
& x_1 + 2x_2 \leq 8 \\
& x_1 - x_2 \leq 2 \\
& x_1, x_2 \geq 0
\end{aligned}
$$

Exercise

$$\text{Maximize } Z = 3x_1 + 5x_2$$

Subject to:

$$x_1 + 2x_2 \leq 8$$
$$3x_1 + 2x_2 \leq 12$$
$$x_1, x_2 \geq 0$$

$$\text{Maximize } Z = 4x_1 + 3x_2$$

Subject to:

$$2x_1 + 3x_2 \leq 12$$
$$2x_1 + x_2 \leq 8$$
$$x_1, x_2 \geq 0$$

Exercise

$$\text{Maximize } Z = 3x_1 + 2x_2 + 4x_3$$

Subject to:

$$2x_1 + x_2 + x_3 \leq 8$$
$$x_1 + 2x_2 + 3x_3 \leq 12$$
$$x_1, x_2, x_3 \geq 0$$

$$\text{Maximize } Z = 5x_1 + 4x_2 + 3x_3$$

Subject to:

$$2x_1 + 3x_2 + x_3 \leq 5$$
$$4x_1 + x_2 + 2x_3 \leq 11$$
$$3x_1 + 4x_2 + 2x_3 \leq 8$$
$$x_1, x_2, x_3 \geq 0$$

CONTENT

- Linear programs
- Geometric approach
- Simplex method
- **Two-phase simplex method**
- OR-TOOLS for linear programming
- Programming exercises

TWO-PHASE SIMPLEX METHOD

- **Consider a linear program under a standard equational form**

$$
\begin{aligned}
\text{(LP)}\quad z = \quad & c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max \\
& a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2 \\
& \qquad\qquad \ldots \\
& a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m \\
& \text{Coefficients } b_1, b_2, \ \ldots b_m \geq 0 \\
& x_1, x_2, \ \ldots x_n \in R,\ x_1, x_2, \ \ldots x_n \geq 0
\end{aligned}
$$

TWO-PHASE SIMPLEX METHOD

- The **two-phase Simplex Method** is used when the initial feasible solution is **not obvious** or **difficult to find** — particularly in cases where:
  - The problem has **"≥" (greater than or equal to)** constraints.
  - The problem has **"=" (equality)** constraints.
  - The initial simplex table has **no obvious feasible starting point**.

TWO-PHASE SIMPLEX METHOD

- **Consider a linear program under a standard equational form**

(LP) $z = c_1x_1 + c_2x_2 + \ldots + c_nx_n \rightarrow \max$

$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n = b_1$

$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n = b_2$

$\ldots$

$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n = b_m$

Coefficients $b_1, b_2, \ldots b_m \geq 0$

$x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n \geq 0$

Introduce an auxiliary linear program (ALP) with m artificial variables $y_1, y_2, \ldots, y_m$

(ALP) $g = -y_1 - y_2 - \ldots - y_m \rightarrow \max$

$a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 = b_1$

$a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n + y_2 = b_2$

$\ldots \qquad \ldots$

$a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + \quad +y_m = b_m$

$b_1, b_2, \ldots b_m \geq 0$

$x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \geq 0$

TWO-PHASE SIMPLEX METHOD

$$
\begin{array}{llll}
\text{(ALP)} & g = -y_1 - y_2 - \ldots - y_m \rightarrow \max \\
& a_{1,1}x_1 + a_{1,2}x_2 + \ldots + a_{1,n}x_n + y_1 & & = b_1 \\
& a_{2,1}x_1 + a_{2,2}x_2 + \ldots + a_{2,n}x_n \quad + y_2 & & = b_2 \\
& \qquad \ldots \\
& a_{m,1}x_1 + a_{m,2}x_2 + \ldots + a_{m,n}x_n + & & +y_m = b_m \\
& b_1, b_2, \ldots b_m \geq 0 \\
& x_1, x_2, \ldots x_n \in R, x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_m \geq 0
\end{array}
$$

- Solve the (ALP) by the Simplex Method under the Tabular form: Basis is the column vectors corresponding to artificial variables → obtain an optimal solution (x^*, y^*) and basic indices set is $J_B{}^*$
- Proposition: The original (LP) problem has feasible solutions iff the corresponding (ALP) has an optimal solution (x^*, y^*) in which $y^* = 0$ (**proof ?**)
- If $y^* \neq 0$, then the original (LP) problem does not have feasible solutions
- We consider the case that $y^* = 0$

	1	2	…	n	n+1	n+2	…	n+m	n+m+1	
0	x_1	x_2	…	x_n	y_1	y_2	…	Y_m	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	…	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	…	$\alpha_{1,n+m}$	0	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	…	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	…	$\alpha_{2,n+m}$	0	β_2
…	…	…	…	…	…	. . .	…	…	..	…
m	$\alpha_{m,1}$	$\alpha_{m,2}$	…	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	…	$\alpha_{m,n+m}$	0	β_m
m+1	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	…	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Case 1: J_B^* (set of indices in the basic) does not contain indices of artificial variables
  - Move to the second phase, solve the original (LP) problem
    - Remove columns corresponding to artificial variables: n+1, . . ., n+m
    - Recompute elements on row m+1 (based on the original objective function)

TWO-PHASE SIMPLEX METHOD

	1	2	…	n	$n+1$	$n+2$	…	$n+m$	$n+m+1$	
0	x_1	x_2	…	x_n	Y_1	y_2	…	Y_m	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	…	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	…	$\alpha_{1,n+m}$	0	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	…	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	…	$\alpha_{2,n+m}$	0	β_2
…	…	…	…	…	…	. . .	…	…	..	…
m	$\alpha_{m,1}$	$\alpha_{m,2}$	…	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	…	$\alpha_{m,n+m}$	0	β_m
$m+1$	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	…	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Case 2: J_B^* contains some indices of artificial variables
  - Suppose J_B^* contains index $n+j$ of the artificial variable (y_j), perform the linear transformation to remove index $n+j$ from J_B^* as follows:
    - Consider row k such that the element in row k and column $n+j$ is 1 (column vector corresponding to column $n+j$ is a unit vector)
    - Case 2.1: If all elements on row k, from column 1 to column n are equal to 0 ($\alpha_{k,1} = . . . = \alpha_{k,n} = 0$): it means, the constraint of row k is linear dependent on other constraints → we can remove this row k and column $n+j$ from the table

TWO-PHASE SIMPLEX METHOD

	1	2	…	n	n+1	n+2	…	n+m	n+m+1	
0	x_1	x_2	…	x_n	y_1	y_2	…	Y_m	z	RHS
1	$\alpha_{1,1}$	$\alpha_{1,2}$	…	$\alpha_{1,n}$	$\alpha_{1,n+1}$	$\alpha_{1,n+2}$	…	$\alpha_{1,n+m}$	0	β_1
2	$\alpha_{2,1}$	$\alpha_{2,2}$	…	$\alpha_{2,n}$	$\alpha_{2,n+1}$	$\alpha_{2,n+2}$	…	$\alpha_{2,n+m}$	0	β_2
…	…	…	…	…	…	. . .	…	…	..	…
m	$\alpha_{m,1}$	$\alpha_{m,2}$	…	$\alpha_{m,n}$	$\alpha_{m,n+1}$	$\alpha_{m,n+2}$	…	$\alpha_{m,n+m}$	0	β_m
m+1	$\alpha_{m+1,1}$	$\alpha_{m+1,2}$	…	$\alpha_{m+1,n}$	$\alpha_{m+1,n+1}$	$\alpha_{m+1,n+2}$	…	$\alpha_{m+1,n+m}$	1	β_{m+1}

- Case 2: $J_B{}^*$ contains some indices of artificial variables
    - Case 2.2: There exists a column i such that $\alpha_{k,i} \neq 0$
    - In this optimal table, all artificial variables are equal to 0, so β_k is equal to 0
    - Perform the rotation with the pivot $\alpha_{k,i}$. With this rotation, column RHS is unchanged due to the fact that β_k is equal to 0. Hence β_{m+1} is always 0. It means that the new table corresponds to another optimal solution in which one artificial variable is replaced by an original variable x_i
    - The above procedure is repeated until all artificial variables are removed from the basic
    - We now process the computation as the case 1 (above)

EXAMPLE

(LP) $z = 40x_1 + 10x_2 + 7x_5 + 14x_6 \rightarrow \max$

$$\begin{aligned}
x_1 - x_2 \quad\quad + 2x_5 \quad\quad &= 0 \\
-2x_1 + x_2 \quad\quad - 2x_5 \quad\quad &= 0 \\
x_1 \quad + x_3 \quad + \quad x_5 - x_6 &= 3 \\
x_2 + x_3 + x_4 + 2x_5 + x_6 &= 4
\end{aligned}$$

$x_1, x_2, x_3, x_4, x_5, x_6 \in R,\ x_1, x_2, x_3, x_4, x_5, x_6 \geq 0$

(ALP) $z = -y_1 - y_2 - y_3 - y_4 \rightarrow \max$

$$\begin{aligned}
x_1 - x_2 \quad\quad + 2x_5 \quad\quad + y_1 \quad\quad\quad &= 0 \\
-2x_1 + x_2 \quad\quad - 2x_5 \quad\quad + y_2 \quad\quad &= 0 \\
x_1 \quad + x_3 \quad + \quad x_5 - x_6 \quad\quad + y_3 \quad &= 3 \\
x_2 + x_3 + x_4 + 2x_5 + x_6 \quad\quad\quad + y_4 &= 4
\end{aligned}$$

$x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4 \in R,\ x_1, x_2, x_3, x_4, x_5, x_6, y_1, y_2, y_3, y_4 \geq 0$

	1	2	3				4	5	6		7		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	
1	1	-1	0	0	2	0	1	0	0	0	0	0	
2	-2	1	0	0	-2	0	0	1	0	0	0	0	
3	1	0	1	0	1	-1	0	0	1	0	0	3	
	0	1	1	1	2	1	0	0	0	1	0	4	
4	0	-1	-2	-1	-3	0	0	0	0	0	1	-7	

EXAMPLE

	1	2	3				4	5	6		7		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	E
1	1	-1	0	0	2	0	1	0	0	0	0	0	0
2	-2	1	0	0	-2	0	0	1	0	0	0	0	$+\infty$
3	1	0	1	0	1	-1	0	0	1	0	0	3	3/1
4	0	1	1	1	2	1	0	0	0	1	0	4	4/2
5	0	-1	-2	-1	-3	0	0	0	0	0	1	-7	

R1 = R1/2; R2 = R2 + 2R1; R3 = R3 – R1; R4 = R4 – 2R1; R5 = R5 + 3R1

	1	2	3				4	5	6		7		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	
1	1/2	-1/2	0	0	1	0	1/2	0	0	0	0	0	
2	-1	0	0	0	0	0	1	1	0	0	0	0	
3	1/2	1/2	1	0	0	-1	-1/2	0	1	0	0	3	
4	-1	2	1	1	0	1	-1	0	0	1	0	4	
5	3/2	-5/2	-2	-1	0	0	3/2	0	0	0	1	-7	

EXAMPLE

	1	2	3				4	5	6		7		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	E
1	1/2	-1/2	0	0	1	0	1/2	0	0	0	0	0	+∞
2	-1	0	0	0	0	0	1	1	0	0	0	0	+∞
3	1/2	1/2	1	0	0	-1	-1/2	0	1	0	0	3	6
4	-1	2	1	1	0	1	-1	0	0	1	0	4	4/2
5	3/2	-5/2	-2	-1	0	0	3/2	0	0	0	1	-7	

R4 = R4/2; R1 = R1+(1/2)R4; R2 = R2; R3 = R3 – (1/2)R4; R5 = R5 + (5/2)R4

	1	2	3				4	5	6		7		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	
1	1/4	0	1/4	1/4	1	1/4	1/4	0	0	1/4	0	1	
2	-1	0	0	0	0	0	1	1	0	0	0	0	
3	3/4	0	3/4	-1/4	0	-5/4	-1/4	0	1	-1/4	0	2	
4	-1/2	1	1/2	½	0	1/2	-1/2	0	0	1/2	0	2	
5	1/4	0	-3/4	¼	0	5/4	1/4	0	0	5/4	1	-2	

EXAMPLE

	1	2	3				4	5	6		7		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	E
1	1/4	0	1/4	1/4	1	1/4	1/4	0	0	1/4	0	1	4
2	-1	0	0	0	0	0	1	1	0	0	0	0	$+\infty$
3	3/4	0	3/4	-1/4	0	-5/4	-1/4	0	1	-1/4	0	2	8/3
4	-1/2	1	1/2	½	0	1/2	-1/2	0	0	1/2	0	2	4
5	1/4	0	-3/4	¼	0	5/4	1/4	0	0	5/4	1	-2	

R3 = R3/(3/4); R1 = R1-(1/4)R3; R2 = R2; R4 = R4 – (1/2)R3; R5 = R5 + (3/4)R3

	1	2	3				4	5	6		7		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	
1	0	0	0	1/3	1	2/3	1/3	0	-1/3	1/3	0	1/3	
2	-1	0	0	0	0	0	1	1	0	0	0	0	
3	1	0	1	-1/3	0	-5/3	-1/3	0	4/3	-1/3	0	8/3	
4	-1	1	0	2/3	0	4/3	-1/3	0	-2/3	2/3	0	2/3	
5	1	0	0	0	0	0	0	0	1	1	1	0	

EXAMPLE

	1	2	3	4	5	6	7	8	9	10	11		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	E
1	0	0	0	1/3	1	2/3	1/3	0	-1/3	1/3	0	1/3	
2	-1	0	0	0	0	0	1	1	0	0	0	0	
3	1	0	1	-1/3	0	-5/3	-1/3	0	4/3	-1/3	0	8/3	
4	-1	1	0	2/3	0	4/3	-1/3	0	-2/3	2/3	0	2/3	
5	1	0	0	0	0	0	0	0	1	1	1	0	

The basic index set JB* = {2, 3, 5, 8}, where column 8 corresponds to the artificial variable y2. In this column, the element corresponding to row R2 is 1. This row has an RHS of 0 (because in the optimal solution, this RHS value equals y2, which is 0). Additionally, there is an element in column 1 (the column corresponding to the original variable) equal to -1 (which is nonzero), so we perform a pivot operation on this element (row 2, column 1), specifically:

- R2 = R2/ (-1)
- R1 = R1; R3 = R3 – R2; R4 = R4 + R2; R = R5 – R2

EXAMPLE

	1	2	3	4	5	6	7	8	9	10	11		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	E
1	0	0	0	1/3	1	2/3	1/3	0	-1/3	1/3	0	1/3	
2	-1	0	0	0	0	0	1	1	0	0	0	0	
3	1	0	1	-1/3	0	-5/3	-1/3	0	4/3	-1/3	0	8/3	
4	-1	1	0	2/3	0	4/3	-1/3	0	-2/3	2/3	0	2/3	
5	1	0	0	0	0	0	0	0	1	1	1	0	

- Pivot row 2, column 1: R2 = R2/ (-1)
- R1 = R1; R3 = R3 – R2; R4 = R4 + R2; R5 = R5 – R2

	1	2	3	4	5	6	7	8	9	10	11		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	
1	0	0	0	1/3	1	2/3	1/3	0	-1/3	1/3	0	1/3	
2	1	0	0	0	0	0	-1	-1	0	0	0	0	
3	0	0	1	-1/3	0	-5/3	2/3	1	4/3	-1/3	0	8/3	
4	0	1	0	2/3	0	4/3	-4/3	-1	-2/3	2/3	0	2/3	
5	0	0	0	0	0	0	1	1	1	1	1	0	

EXAMPLE

	1	2	3	4	5	6	7	8	9	10	11		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	E
1	0	0	0	1/3	1	2/3	1/3	0	-1/3	1/3	0	1/3	
2	1	0	0	0	0	0	-1	-1	0	0	0	0	
3	0	0	1	-1/3	0	-5/3	2/3	1	4/3	-1/3	0	8/3	
4	0	1	0	2/3	0	4/3	-4/3	-1	-2/3	2/3	0	2/3	
5	0	0	0	0	0	0	1	1	1	1	1	0	

- We obtain the optimal solution for the first phase: no artificial variables are basic variables. Therefore, we eliminate the columns corresponding to artificial variables and proceed to the second phase. We retain the coefficients in the table (rows 1–4 and columns 1–6) and use the objective function of the original problem to recalculate row 5.

EXAMPLE

	1	2	3	4	5	6	7	8	9	10	11		
0	x_1	x_2	x_3	x_4	x_5	x_6	y_1	y_2	y_3	y_4	z	RHS	E
1	0	0	0	1/3	1	2/3	1/3	0	-1/3	1/3	0	1/3	
2	1	0	0	0	0	0	-1	-1	0	0	0	0	
3	0	0	1	-1/3	0	-5/3	2/3	1	4/3	-1/3	0	8/3	
4	0	1	0	2/3	0	4/3	-4/3	-1	-2/3	2/3	0	2/3	
5	0	0	0	0	0	0	1	1	1	1	1	0	

	1	2	3	4	5	6	7		
0	x_1	x_2	x_3	x_4	x_5	x_6	z	RHS	
1	0	0	0	1/3	1	2/3	0	1/3	
2	1	0	0	0	0	0	0	0	
3	0	0	1	-1/3	0	-5/3	0	8/3	
4	0	1	0	2/3	0	4/3	0	2/3	
5	0	0	0	9	0	4	1	9	

- **Example 1 (exercise in class)**

$$
\text{(LP)}\quad z = x_1 + 2x_2 - x_3 + x_4 \rightarrow \max
$$

$$
\begin{aligned}
x_1 + x_2 - x_3 - x_4 &= 4\\
x_1 + x_3 + x_4 &= 7\\
2x_1 + x_2 &= 2\\
x_1, x_2, x_3, x_4 \in R,\ x_1, x_2, x_3, x_4 &\geq 0
\end{aligned}
$$

TWO-PHASE SIMPLEX METHOD

- **Example 2 (exercise in class)**

(LP) $z = x_1 + 2x_2 - x_3 + x_4 \rightarrow \max$

$$
\begin{aligned}
x_1 + x_2 - x_3 - x_4 &= 4 \\
x_1 + x_3 + x_4 &= 7 \\
x_1 - x_2 - x_3 &= 2
\end{aligned}
$$

$x_1, x_2, x_3, x_4 \in R,\ x_1, x_2, x_3, x_4 \geq 0$

TWO-PHASE SIMPLEX METHOD

- **Example 3 (exercise in class)**

$$
\text{(LP)}\quad z = 40x_1 + 10x_2 + 7x_5 + 14x_6 \rightarrow \max
$$

$$
\begin{aligned}
x_1 - x_2 + 2x_5 &= 0 \\
-2x_1 + x_2 - 2x_5 &= 0 \\
x_1 + x_3 + x_5 - x_6 &= 3 \\
x_2 + x_3 + x_4 + 2x_5 + x_6 &= 4 \\
x_1, x_2, x_3, x_4, x_5, x_6 \in R,\ x_1, x_2, x_3, x_4, x_5, x_6 &\geq 0
\end{aligned}
$$

CONTENT

- Linear programs
- Geometric approach
- Simplex method
- Two-phase simplex method
- **OR-TOOLS for linear programming**
- Programming exercises

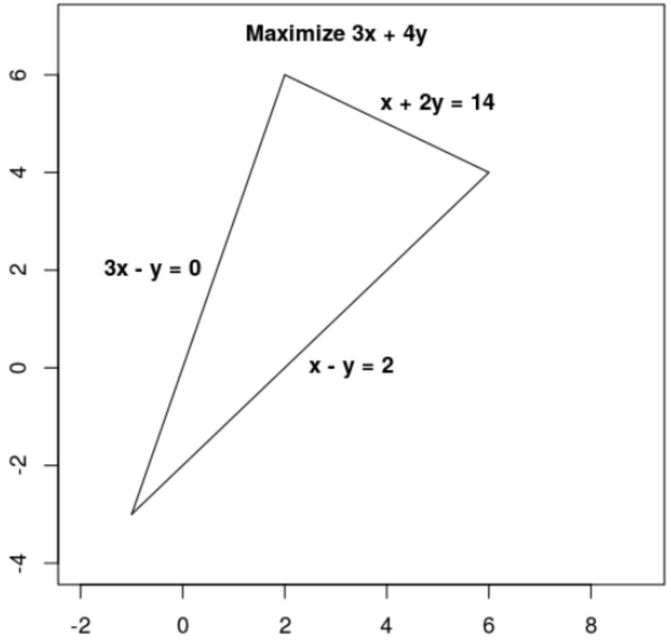
INSTALL

- https://developers.google.com/optimization/install

```
$ python -m pip install ortools
```

Example

Maximize 3x + 4y subject to the following constraints:
$x + 2y \le 14$
$3x - y \ge 0$
$x - y \le 2$



To solve a LP problem, your program should include the following steps:

1. Import the linear solver wrapper,
2. declare the LP solver,
3. define the variables,
4. define the constraints,
5. define the objective,
6. call the LP solver; and
7. display the solution

Example

```python
from ortools.linear_solver import pywraplp


def LinearProgrammingExample():
    """Linear programming sample."""
    # Instantiate a Glop solver, naming it LinearExample.
    solver = pywraplp.Solver.CreateSolver("GLOP")
    if not solver:
        return

    # Create the two variables and let them take on any non-negative value.
    x = solver.NumVar(0, solver.infinity(), "x")
    y = solver.NumVar(0, solver.infinity(), "y")

    print("Number of variables =", solver.NumVariables())

    # Constraint 0: x + 2y <= 14.
    solver.Add(x + 2 * y <= 14.0)

    # Constraint 1: 3x - y >= 0.
    solver.Add(3 * x - y >= 0.0)

    # Constraint 2: x - y <= 2.
    solver.Add(x - y <= 2.0)

    print("Number of constraints =", solver.NumConstraints())

    # Objective function: 3x + 4y.
    solver.Maximize(3 * x + 4 * y)
```

```python
# Solve the system.
print(f"Solving with {solver.SolverVersion()}")
status = solver.Solve()

if status == pywraplp.Solver.OPTIMAL:
    print("Solution:")
    print(f"Objective value = {solver.Objective().Value():0.1f}")
    print(f"x = {x.solution_value():0.1f}")
    print(f"y = {y.solution_value():0.1f}")
else:
    print("The problem does not have an optimal solution.")

print("\nAdvanced usage:")
print(f"Problem solved in {solver.wall_time():d} milliseconds")
print(f"Problem solved in {solver.iterations():d} iterations")
```

CONTENT

- Linear programs
- Geometric approach
- Simplex method
- Two-phase simplex method
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- **Programming exercises**

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THANK YOU !

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