HUST

ĐẠI HỌC BÁCH KHOA HÀ NỘI

HANOI UNIVERSITY OF SCIENCE AND TECHNOLOGY

ONE LOVE. ONE FUTURE.

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FUNDAMENTALS OF OPTIMIZATION

Modelling

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Outline

- Modelling a Combinatorial Optimization Problem
  - Combinatorial Optimization Problem
  - N-Queen problem
  - Sudoku problem
  - Balanced Class Teacher Assignment Problem
  - Class Allocation Problem
  - Traveling Salesman Problem
  - Exercise
- Backtracking algorithm

Combinatorial Optimization Problem

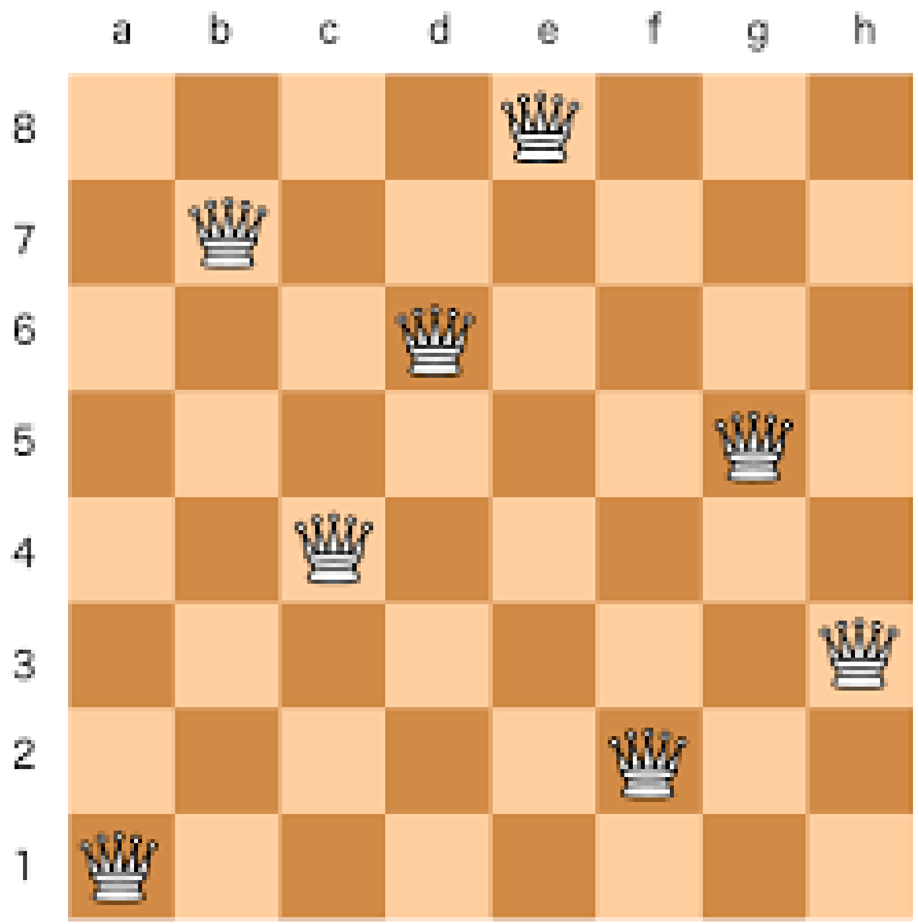
Find a solution (usually a combinatorial configuration) that satisfies a given set of constraints while simultaneously optimizing one or more specified objective functions.

- **Constraint Satisfaction Problem (CSP) = (X, D, C)**
  - $X = \{x_1, \dots, x_n\}$, set of variables
  - $D = \{D_1, \dots, D_n\}$, domains of the variables
  - $C = \{C_1, \dots, C_k\}$, set of constraints
- **Combinatorial Optimization Problem (COP) = (X, D, C, f)**
  - $X = \{x_1, \dots, x_n\}$, set of variables
  - $D = \{D_1, \dots, D_n\}$, domains of the variables
  - $C = \{C_1, \dots, C_k\}$, set of constraints
  - f: objective function

Example: Constraint Satisfaction Problem

Problem N-Queen, $CSP = (X, D, C)$

- Variables: $X = \{x_1, \dots, x_n\}$, in which x_i is the row of the queen in column $i, \forall\, i \in \{1, \dots, n\}$
- Domains: $D_i = D(x_i) = \{1, \dots, n\}, \forall i \in \{1, \dots, n\}$
- Constraints: For all pair (i, j), $1 \le i < j \le n$:
  - $x_i \neq x_j$
  - $x_i + i \neq x_j + j$
  - $x_i - i \neq x_j - j$



Example: Constraint Satisfaction Problem

Sudoku Problem, CSP = (X, D, C)

- **Variables**: $X = \{x_{1,1}, \dots, x_{9,9}\}$, where $x_{i,j}$ is the value in cell (i, j), $\forall\ i, j \in \{1,2, \dots, 9\}$
- **Domain**: $D\left(x_{i,j}\right) = \{1, \dots, 9\}, \forall i, j \in \{1,2, \dots, 9\}$
- **Constraints**:
  - The numbers in each column are pairwise distinct:
  $x_{i_1 j} \neq x_{i_2 j}$ for all $1 \leq i_1 < i_2 \leq 9$, $1 \leq j \leq 9$
  - The numbers in each row are pairwise distinct:
  $x_{j i_1} \neq x_{j i_2}$ for all $1 \leq i_1 < i_2 \leq 9$, $1 \leq j \leq 9$
  - The numbers in each 3x3 subgrid are pairwise distinct:
  $x_{3i+i_1,3j+j_1} \neq x_{3i+i_2,3j+j_2}$ for all $0 \leq i, j \leq 2$, $1 \leq i_1, i_2, j_1, j_2 \leq$ satisfying $(i_1, j_1) \neq (i_2, j_2)$

5	3	4	6	7	8	9	1	2
6	7	2	1	9	5	3	4	8
1	9	8	3	4	2	5	6	7
8	5	9	7	6	1	4	2	3
4	2	6	8	5	3	7	9	1
7	1	3	9	2	4	8	5	6
9	6	1	5	3	7	2	8	4
2	8	7	4	1	9	6	3	5
3	4	5	2	8	6	1	7	9

Balanced Class Teacher Assignment

- There are n classes labeled $1, 2, \ldots, n$ that have already been scheduled in a timetable, and they need to be assigned to m teachers labeled $1,2, \ldots, m$
- Each class i has $T(i)$, a list of teachers who can teach it ($i = \{1, \ldots, n\}$), and $c(i)$, the number of credits for the subject of that class.
- Since the timetable has been pre-arranged, there exists a set Q of pairs of classes (i, j) that are scheduled at the same time (these two classes cannot be assigned to the same teacher).
- Find an assignment of classes to teachers such that the maximum total number of credits assigned to any one teacher is minimized.

Balanced Class Teacher Assignment

- Example

Class	0	1	2	3	4	5	6	7	8	9	10	11	12
Credit	3	3	4	3	4	3	3	3	4	3	3	4	4

Teacher	List of classes that the teach can teach
0	0, 2, 3, 4, 8, 10
1	0, 1, 3, 5, 6, 7, 8
2	1, 2, 3, 7, 9, 11, 12

List Q

0	2
0	4
0	8
1	4
1	10
3	7
3	9
5	11
5	12
6	8
6	12

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Balanced Class Teacher Assignment

- Example

Class	0	1	2	3	4	5	6	7	8	9	10	11	12
Credit	3	3	4	3	4	3	3	3	4	3	3	4	4

Teacher	List of classes that the teach can teach
0	0, 2, 3, 4, 8, 10
1	0, 1, 3, 5, 6, 7, 8
2	1, 2, 3, 7, 9, 11, 12

List Q

0	2
0	4
0	8
1	4
1	10
3	7
3	9
5	11
5	12
6	8
6	12

Assignment solution

Teacher	List of classes assigned to the teacher	Total credits
0	2, 4, 8, 10	15
1	0, 1, 3, 5, 6	15
2	7, 9, 11, 12	14

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Balanced Class Teacher Assignment

- Input:
  - Set of classes: $S = \{1, \dots, n\}$
  - Set of teachers: $T = \{1, \dots, m\}$
  - Set of conflict classes: $Q = \{(i_1, j_1), \dots, (i_k, j_k) | i_1, \dots, i_k, j_1, \dots, j_k \in C, i_t \neq j_t \ \forall t \in \{1, \dots, k\}\}$
  - Set of teachers who can give some class $T = \{T_1, \dots, T_n\}$, in which T_i is the set of teachers who can give the class i ($i \in \{1, \dots, n\}$)
  - For each class i ($i \in \{1, \dots, n\}), c(i)$ is its number of credit ($c(i) \in N$)
- Variables:
  - Binary variable x_{ij} ($i \in C, j \in T$) is equal to 1 if the class i is assigned to the teacher j, otherwise the value of this variable is equal to 0;
  - Integral variable $maxcredit$ represents the maximum number of credits for a teacher;
- Domains of variables:
  - $D(x_{ij}) = \{0, 1\}, \forall\, i \in C, j \in T$
  - $D(maxcredit) = \{0, \dots, \sum_{i \in C} c(i)\}$
- Constraints:
  - Each class is assigned to one teacher $\sum_{j \in T_i} x_{ij} = 1, \ \forall\, i \in C$
  - Teacher is not assigned to a class that he cannot teach $x_{ij} = 0, \forall\, i \in C, j \notin T_i$
  - Teacher cannot give two classes in the conflict set $x_{i_1 j} + x_{i_2 j} \leq 1, \forall\, j \in T, (i_1, i_2) \in Q$
  - Relation between variable $maxcredit$ and workload of teacher $\sum_{i \in C} c(i) x_{ij} \leq maxcredit\, , \forall\, j \in T$
- Objective: $Minimize\ maxcredit$

Class Allocation Problem

- n classes labeled by $1, 2, \ldots, n$ need to be allocated in p semesters $1, 2, \ldots p$. Each class i has a credit value of $c(i)$, and its prerequisite conditions are defined by a set Q of pairs (i, j), where subject i must be taken before j. Given the constant $\alpha, \beta, \delta, \gamma$, it is necessary to determine an allocation plan that satisfies the following:
  - The total number of classes assigned to each semester must be greater than or equal to α and less than or equal to β.
  - The total number of credits of the classes assigned to each semester must be greater than or equal to δ and less than or equal to γ
  - For each pair $(i, j) \in Q$, class i must be scheduled in a semester prior to the semester in which class j is scheduled.
- Objective: The maximum number of credits in any one semester must be minimized.

Class Allocation Problem

- Example

Class	1	2	3	4	5	6	7	8	9	10	11	12
Number of credits	2	1	2	1	3	2	1	3	2	3	1	3

3 ≤ Number of classes in each semester ≤ 3
5 ≤ Number of credits in each semester ≤ 7

Set Q

2	1
6	9
5	6
5	8
4	11
6	12
2	7
3	10
5	7
8	11
4	12

Class Allocation Problem

- Example

Class	1	2	3	4	5	6	7	8	9	10	11	12
Number of credits	2	1	2	1	3	2	1	3	2	3	1	3

Set Q

2	1
6	9
5	6
5	8
4	11
6	12
2	7
3	10
5	7
8	11
4	12

$3 \leq$ Number of classes in each semester ≤ 3

$5 \leq$ Number of credits in each semester ≤ 7

Allocation solution

Semester	1	2	3	4
List of classes	2, 5, 3	1, 6,10	4,7,8	9,11,12

Class Allocation Problem

- Input:
  - Set of classes: $C = \{1, \dots, n\}$
  - Set of semesters: $S = \{1, \dots, p\}$
  - Set of precedent classes: $Q = \{(i_1, j_1), \dots, (i_k, j_k) | i_1, \dots, i_k, j_1, \dots, j_k \in C, i_t \neq j_t \; \forall t \in \{1, \dots, k\}\}$
  - Constant $\alpha, \beta, \delta, \gamma$
- Variables:
  - Binary variable x_{ij} ($i \in C, j \in S$) is equal to 1 if the class i is assigned to the semester j, otherwise the value of this variable is equal to 0;
  - Integral variable $maxcredit$ represents the maximum number of credits for a semester
- Constraints:
  - Every class is allocated to some semester $\sum_{j \in S} x_{ij} = 1, \forall i \in C$
  - Number of classes in a semester must be in a range of $[\alpha, \beta]$, it means that $\alpha \leq \sum_{i \in C} x_{ij} \leq \beta, \forall j \in S$
  - Number of credits in a semester must be in a range of $[\delta, \gamma]$, it means that $\delta \leq \sum_{i \in C} c(i) x_{ij} \leq \gamma$
  - For each pair $(i_1, i_2) \in Q$, class i_1 must be scheduled in a semester prior to the semester in which class i_2 is scheduled $\sum_{j \in S} j x_{i_1 j} < \sum_{j \in S} j \, x_{i_2 j}, \forall (i_1, i_2) \in Q$
  - Relation between variable $maxcredit$ and workload in a semester $\sum_{i \in C} c(i) x_{ij} \leq maxcredit, \forall j \in S$
- Objective: $Minimize \; maxcredit$

Traveling Salesman Problem

- A traveler starts from city 1 and needs to visit cities $2, 3, \ldots, n$, passing through each city exactly once before returning to the starting city. The cost of traveling from city i to city j is $c(i, j)$. Calculate the plan for the traveler that results in the minimum total cost.

Traveling Salesman Problem

- Input:
  - n number of cities
  - $c(i,j)$ traveling cost from the city i to the city j
- Variables:
  - Binary variable x_{ij} with $i,j \in \{1,\dots,n\}, i \neq j$ is equal to 1 if the traveler go from the city i to the city j in the optimal plan, otherwise the value of this variable is equal to 0;
- Constraints:
  - For each city, the traveler goes in once and goes out once

$$\sum_{j \in \{1,\dots,n\}} x_{ij} = \sum_{j \in \{1,\dots,n\}} x_{ji} = 1, \forall i \in \{1,\dots,n\}$$

  - No subtour

$$\sum_{i,j \in S, i \neq j} x_{ij} \leq |S| - 1, \forall S \subset \{1,\dots,n\}$$

- Objective: $Minimize \ \sum_{i,j \in \{1,\dots,n\}, i \neq j} c(i,j) x_{ij}$

Exercises

- Modelling the problem in your mini-project

Outline

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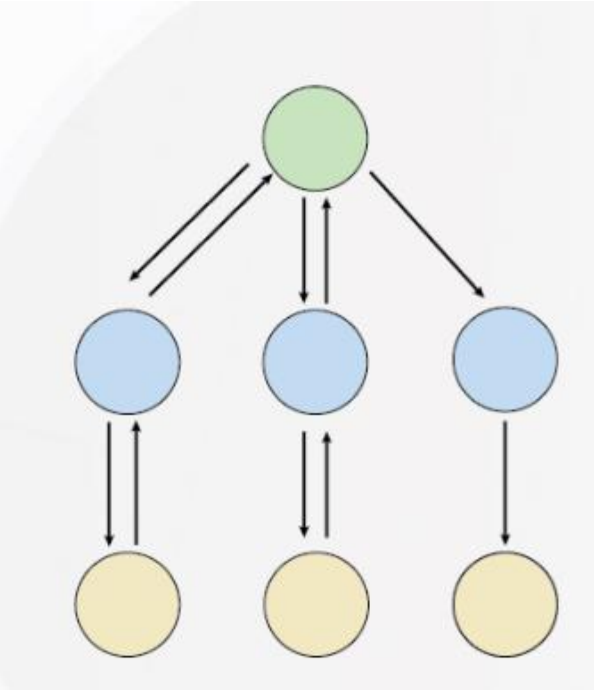
  - N-Queen problem
  - Sudoku problem
  - Balanced Class Teacher Assignment Problem
  - Class Allocation Problem
  - Traveling Salesman Problem
  - Exercise
- **Backtracking algorithm**

Backtracking Algorithm

- Backtracking is a problem-solving algorithmic technique that involves finding a solution incrementally by trying **different options** and **undoing** them if they lead to a **dead end**. It is commonly used in situations where you need to explore multiple possibilities to solve a problem, like searching for a path in a maze or solving puzzles like Sudoku. When a dead end is reached, the algorithm backtracks to the previous decision point and explores a different path until a solution is found or all possibilities have been exhausted.

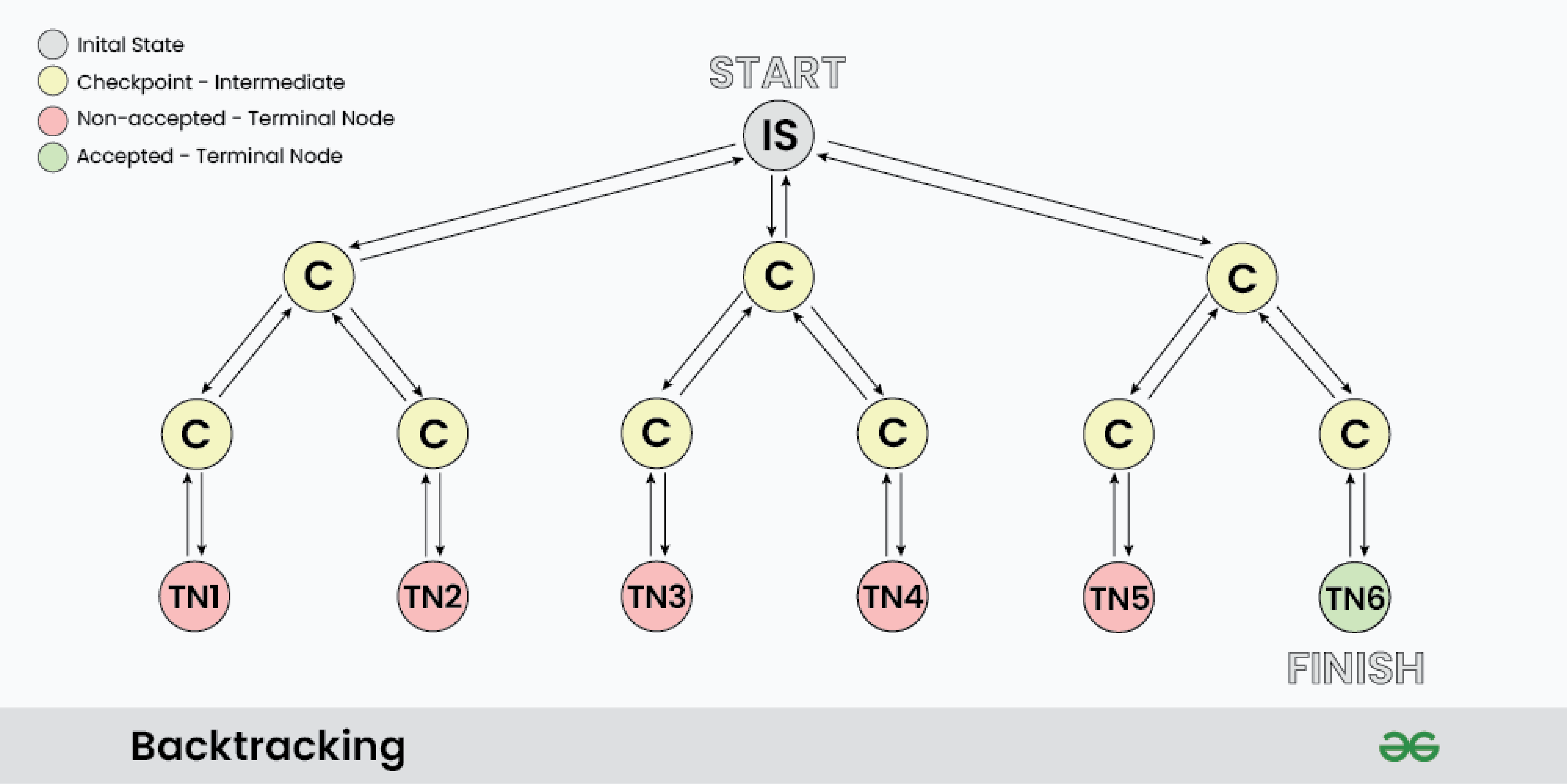


- Backtracking can be defined as a general algorithmic technique that considers searching every possible combination in order to solve a computational problem.

Backtracking Algorithm

- **Candidate**: A candidate is a potential choice or element that can be added to the current solution.
- **Solution**: The solution is a valid and complete configuration that satisfies all problem constraints.
- **Partial Solution**: A partial solution is an intermediate or incomplete configuration being constructed during the backtracking process.
- **Decision Space**: The decision space is the set of all possible candidates or choices at each decision point.
- **Decision Point**: A decision point is a specific step in the algorithm where a candidate is chosen and added to the partial solution.
- **Feasible Solution**: A feasible solution is a partial or complete solution that adheres to all constraints.
- **Dead End**: A dead end occurs when a partial solution cannot be extended without violating constraints.
- **Backtrack**: Backtracking involves undoing previous decisions and returning to a prior decision point.
- **Search Space**: The search space includes all possible combinations of candidates and choices.
- **Optimal Solution**: In optimization problems, the optimal solution is the best possible solution.

Backtracking Algorithm



Recursive Technique for Backtracking Algorithm

```
TRY(k)
  Begin
    Foreach v in A_k
      if check(v,k) /* Check for feasibility of assigning v to x_k  */
        Begin
          x_k = v;
          [Update data structure D]
          if(k = n) save a feasible solution;
          else TRY(k+1);
          [Recovery D]
        End
  End
Main()
Begin
  TRY(1);
End
```

Generate binary string

```python
n = 3
x = [-1] * n
def Try(k):
    if k == n:
        print(x)
    else:
        for i in range(2):
            x[k] = i
            Try(k+1)

Try(0)
```

```
[0, 0, 0]
[0, 0, 1]
[0, 1, 0]
[0, 1, 1]
[1, 0, 0]
[1, 0, 1]
[1, 1, 0]
[1, 1, 1]
```

Generate permutation of a set

```python
n = 3
x = [-1] * n
visited = [False] * (n)

def Try(k):
    if k == n:
        print(x)
        return

    for i in range(0, n):
        if visited[i] == True:
            continue

        x[k] = i
        visited[i] = True
        Try(k+1)
        visited[i] = False

Try(0)
```

```
[0, 1, 2]
[0, 2, 1]
[1, 0, 2]
[1, 2, 0]
[2, 0, 1]
[2, 1, 0]
```

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THANK YOU !