Sukkur Institute of Business Administration

Department of Electrical Engineering.

Digital Signal Processing

course lecture notes

Instructor

Dr. Muhammad Asim Ali

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Contents

Chapter 1

Introduction

Outline

The objective of this chapter is to instil the basic definition of signals, systems and digital signal processing.

- (A) Introduction
- (B) Historical background
- (C) Analog signals
- (D) Systems
- (E) Digital signal processing.

1.1 Signals,Systems and Signal Processing

What does "Digital signal processing" mean?
Signal:

- Physical quantity that varies with time, space or any other independent variable.

- Mathematically: Function of one or more independent variables $s_1(t) = 5$, $s_2(t) = 20t^2$
- Examples: Temperature over time t, brightness of image over (x, y) pressure of sound over (x, y, z) or (x, y, z, t).

Signal Processing:

- Passing the signal through a system.
- Examples:

- Modification of the signal (filtering, noise reduction, equalization,...)
- Prediction,transformation to another domain(e.g. Fourier transform)
- Determination of signal statistics mean value, variance and p.d.f.,...

- Properties of the system (e.g. linear/nonlinear) determine the properties of whole processing operation.
- System: Definition also includes:

- software realizations of operations on a signal, which are carried out on a digital computer ($\Rightarrow$ software implementation of system)
- digital hardware realizations (logical circuits) configured such that they are able to perform the processing operations,
or
- most general definition: a combination of both

Digital signal processing: Processing of signals by digital means (software and/or hardware)

Includes

- Conversion from analog to digital domain and back (physical signals are analog)

- Mathematical specification of the processing operations $\Rightarrow$ Algorithm: method or set of rules for implementing the system by a program that performs the corresponding mathematical operations.

- Emphasis on computationally efficient algorithms, which are fast and easily implemented.

Why digital?

Property	Digital	Analog
Precision	generally unlimited	generally limited
	cost, complexity	increase in performance
	precision	drastic increase in cost
Aging	Without problems	Problematic
production costs	Low	Higher
Linear-phase	exactly realizable	approximate realization
frequency response		
Complex Algorithms	realizable	strong limitations

1.2 DSP applications

- Military applications: Targeting, sonar, radar, secure communications.

- Telecommunications: Echo cancellation, speech coding, modems, channel estimation.

- PC and Multimedia applications: audio, video on demand, voice recognition and synthesis.

- Entertainment: Audio, video compression e.g.(mp3, mpeg, dvd, cd player)

- Automotives: active background cancellation, navigation.

Historical perspective

- Sampling theorem(nyquist) 1920's

- PCM was established in 1940's

- Digital filtering, FFT, speech analysis 1960's (MIT, IBM, BELL Labs)

- Adaptive filters and linear prediction 1960's (stanford)

- Digital spectral estimation and speech coding 1970's

- First generation DSP chips from Intel,TI,AT&T, motorola and analog devices

- Low cost DSP late 1980's - high speed and complexity, media applications, low power and portable today.

Examples

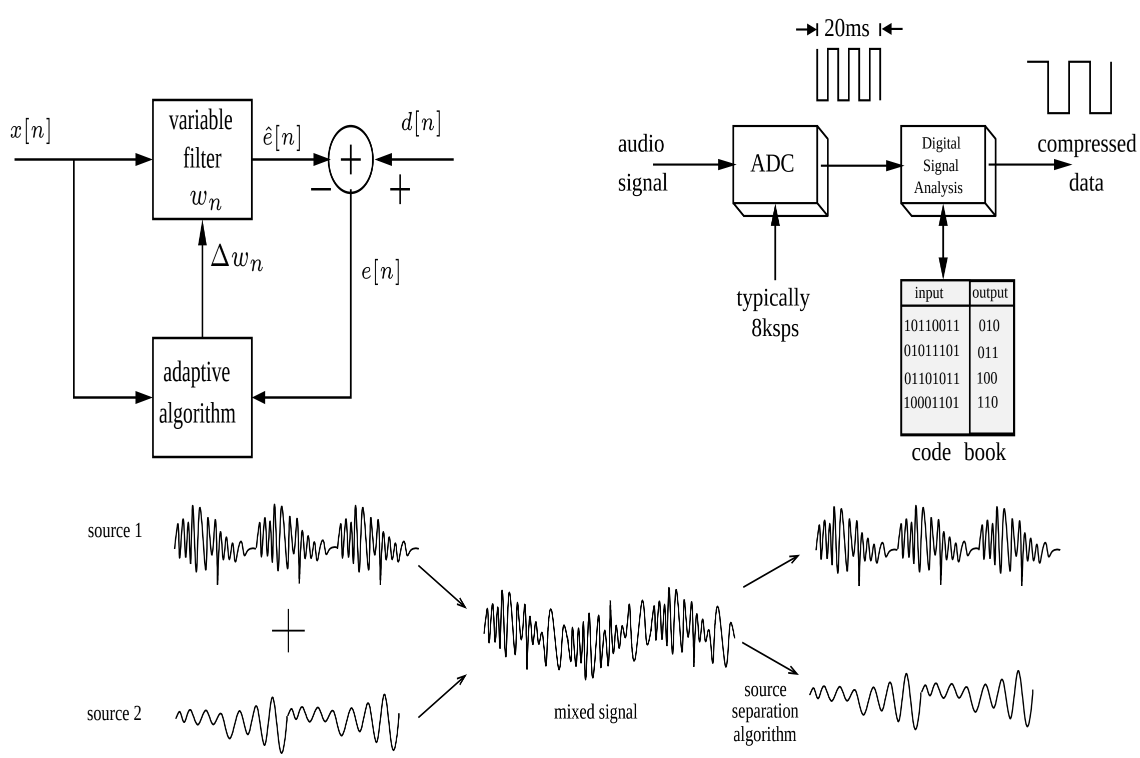


Figure 1.1: Applications of signal processing in real-time applications.

Chapter 2

Sampling and Quantization

Outline

The objective of this chapter is to take a quick review of the elementary concepts essential for this module. This chapter highlights the all important concepts of sampling and quantization. The fundamental concepts covered in this chapter are

- A Types of signals
- B Types of systems
- C Sampling Process
- C Quantization Process
- D Binary number formats
- E Types of A/D
- F D/A Conversion

2.1 Classification of Signals

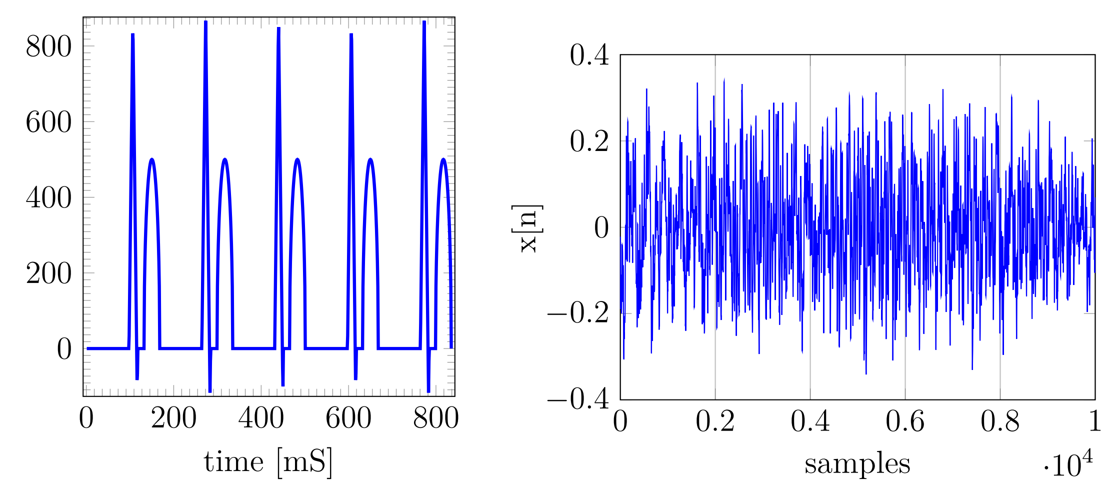
Signal can be classified into periodic and aperiodic functions in time. Another classification refers to continuity or discreteness of time variable. A continuous-time signal has a value $f(t)$ associated with every instant in time. In contrast Discrete-time signal only exists at discrete points $t{=}nT$ of real time, where T is interval between adjacent sampling points.

2.1.1 Continuous-time signals

Continuous time signals are defined for every instance in time. A continuous time signal with continuous amplitude is called analog signal.

Example

Electrocardiogram(ECG), speech signal, musical signal.



(a) Time representation electro cardiogram(ECG) signals.

(b) Time representation of audio signal.

Figure 2.1: Time representation of real time signals.

2.1.2 Discrete-time signals

Discrete time signal is called "sample". Discrete-time signals are generated from parent continuous signals via "sampling". If discrete instants of time at which a discrete-time signal is defined are uniformly spaced, discrete samples can be quantized.

If $v(t)$ can be "determined " by t (possibly uniquely) and if relation can be evaluated despite of a non-random character then $v(t)$ is deterministic.

If $v(t)$ is random signal or if relation of $v(.)$ and t can not be evaluated despite a non-random character then $v(t)$ is / treated as a stochastic signal.

2.1.3 Common test signals

- **Unit impulse**

Unit sample is denoted by $\delta_0[n]$

$$\delta_0[n] = \begin{cases} 1, & n = 0 \\ 0, & \text{elsewhere} \end{cases} \tag{2.1}$$

It is the elementary digital signal, scaled/delayed impulses can represent any arbitrary digital sequence.

$$x[n] = \sum_{k=-\infty}^{+\infty} x[k]\delta_0[n] \tag{2.2}$$

Delay/shift and Sifting properties must be considered.

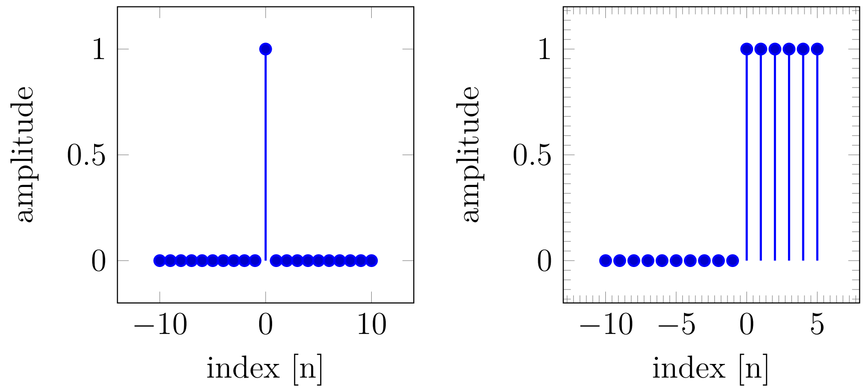
- **Step**

Unit step sequence is denoted by $u[k]$

$$u[n] = \begin{cases} 1, n & \geq 0 \\ 0, & n < 0 \end{cases} \tag{2.3}$$

$\Rightarrow$ extension of unit impulse i.e.

$$u[n] = \sum_{i=0}^{+\infty} \delta_0[n-i] \tag{2.4}$$

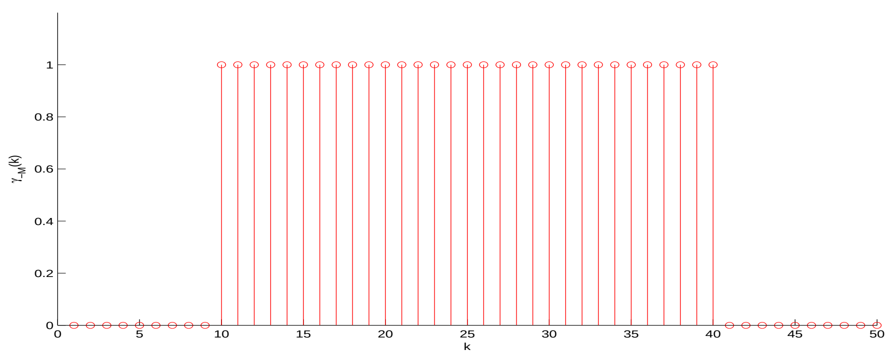


(a) Discrete impulse function. (b) Discrete step functions.

Figure 2.2: Depiction of Discrete-time functions.

- **Rectangular pulses**

$$u_M[n] = \begin{cases} 1, & 0 \leq n \leq M \\ 0, & \text{elsewhere} \end{cases} \tag{2.5}$$



2.2 Classification of systems

System is an operator ”excited“ by input sequences,

creating ”internal”(state)sequences

and output sequences.

Memory

A system is defined to be memoryless if the output $y[n]$ at every value of k depends only on the input $x[n]$.

Example:

$y[n] = [x[n]]^2$

AWGN channel, multi-tap channel, capacitor

Linearity

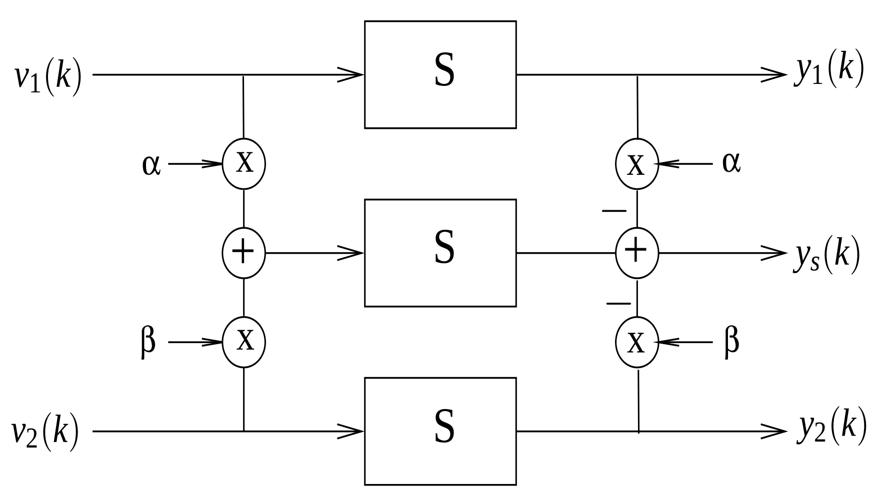
A system is said to be linear if the output $y[n]$ for superposition of two input signals $v_1[n]$ and $v_2[n]$ is also a linear superposition of corresponding individual outputs $y_1[n]$ and $y_2[n]$. i.e.

$$T\{v_1[n] + v_2[n]\} = T\{v_1[n]\} + T\{v_2[n]\} = y_1[n] + y_2[n] \tag{2.6}$$

$$T\{av[n]\} = aT\{v[n]\} = aT\{y[n]\} \tag{2.7}$$

Where a is arbitrary constant, equation (2.6) shows additive superposition and (2.7) shows scaling property. Equations can be generalized

$$T\{av_1[n] + bv_2[n]\} = aT\{v_1[n]\} + bT\{v_2[n]\} \tag{2.8}$$



Obviously $y_3[n] \equiv 0 \quad \forall n \; k \; if \; \alpha_1 s\{v_1[n]\} + \alpha_2 s\{v_2[n]\} = s\{\alpha_1 v_1[n] + \alpha_2 v_2[n]\}$

Example:

$$y(t) = a \cdot v^2(t) + b \cdot v(t) + c$$

Remarks:

- Linear systems are of specially important because of simplicity in mathematical modelling.
- Most real world signal are not linear, on contrary they are “non-linear”.
- For linear systems, however, a widely general theory is applicable- therefore it is often tried to deal with actually non-linear system after lineariztion.

Causality

A system is causal if only for any instance in time n_0 the output $y[n_0]$ depends only on the input sample n_0 and prior.

Example: a device which adds up incoming sample values “accumulator”

$$y[n_0] = \sum_{i=-\infty}^{n_0} v[i] \tag{2.9}$$

Stability

A system is said to be stable in bounded-input bounded-output(BIBO) sense if and only if for every bounded input sequence the system produces a bounded output sequence. Input $v[n]$ is bounded if there exist a finite positive value $M_{1,2}$, such that

$$|v[n]| \leq M_1 < \infty \quad \forall \quad n \tag{2.10}$$

then the system is stable if for every bounded-input there exists a positive finite value.

$$|y[n]| \leq M_2 < \infty \quad \forall \quad n \tag{2.11}$$

Further Reading!
Some text goes here just to illustrate the command

Invertibility

A system is said to be invertible if distinct inputs leads to distinct outputs. If a system is invertible, then an inverse system exists that when cascaded with the original system yields the original input signal.

Example $y_1(t){=}2x(t)$, $y_2(t){=}x^2(t)$.

Further Reading!
Some text goes here just to illustrate the command

2.3 Digital Processing of Continuous Signals

Generation of discrete-time signals from the continuous-time signals

$$p(t) = \sum_{n=-\infty}^{\infty} \delta_0(t - nT) \tag{2.12}$$

Where $\delta_0(t)$ is dirac(delta)function and T is "Sampling Interval"

$$\begin{aligned} v_i(t) &= v_a(t) \sum_{n=-\infty}^{\infty} \delta_0(t - nT) \\ v_i(t) &= \sum_{n=-\infty}^{\infty} v_a(nT)\delta_0(t - nT) \end{aligned} \tag{2.13}$$

How does the **Fourier transform** $F\{v_i(t)\} \circ - - - \bullet V_i(j\omega)$ look like?

Fourier transform of impulse train:

$$p(t) \circ - - - - - - - \bullet P(j\omega) = \frac{2\pi}{T} \sum_{k=-\infty}^{\infty} \delta_0(\omega - k\omega_s) \tag{2.14}$$

where $\omega_s = 2\pi/T$ sampling frequency in radians/sec.

$$V_i(j\omega) = \frac{1}{2\pi} V_a(j\omega) * P(j\omega) \tag{2.15}$$

we finally have the Fourier transform of $v_i(t)$

$$V_i(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} V_a((\omega - k\omega_s)) \tag{2.16}$$

$\Rightarrow$ Periodically repeated copies of $v_a(j\omega)$, shifted by integer multiples of the sampling frequency, scaled with factor of $\frac{1}{T}$.

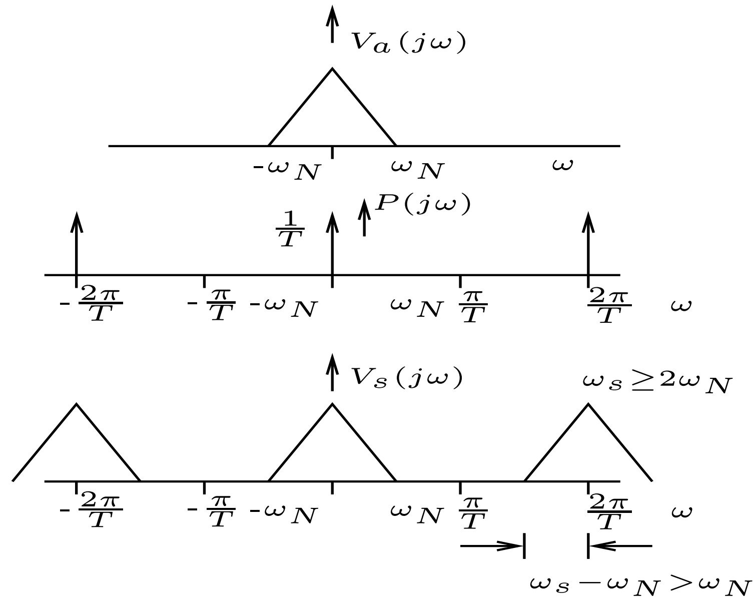


Figure 2.3: Illustration of sampling process in frequency domain.

The concept of sampling in time and frequency domain is illustrated in the following figure. Please note that sampling at very high frequencies yields in high resolution time domain signal while spectral replica of the signal are located wide apart. Reducing the sampling frequency yields in loss of time-domain resolution and separation between spectral replica is reduced.

Sampling Process (Demo).

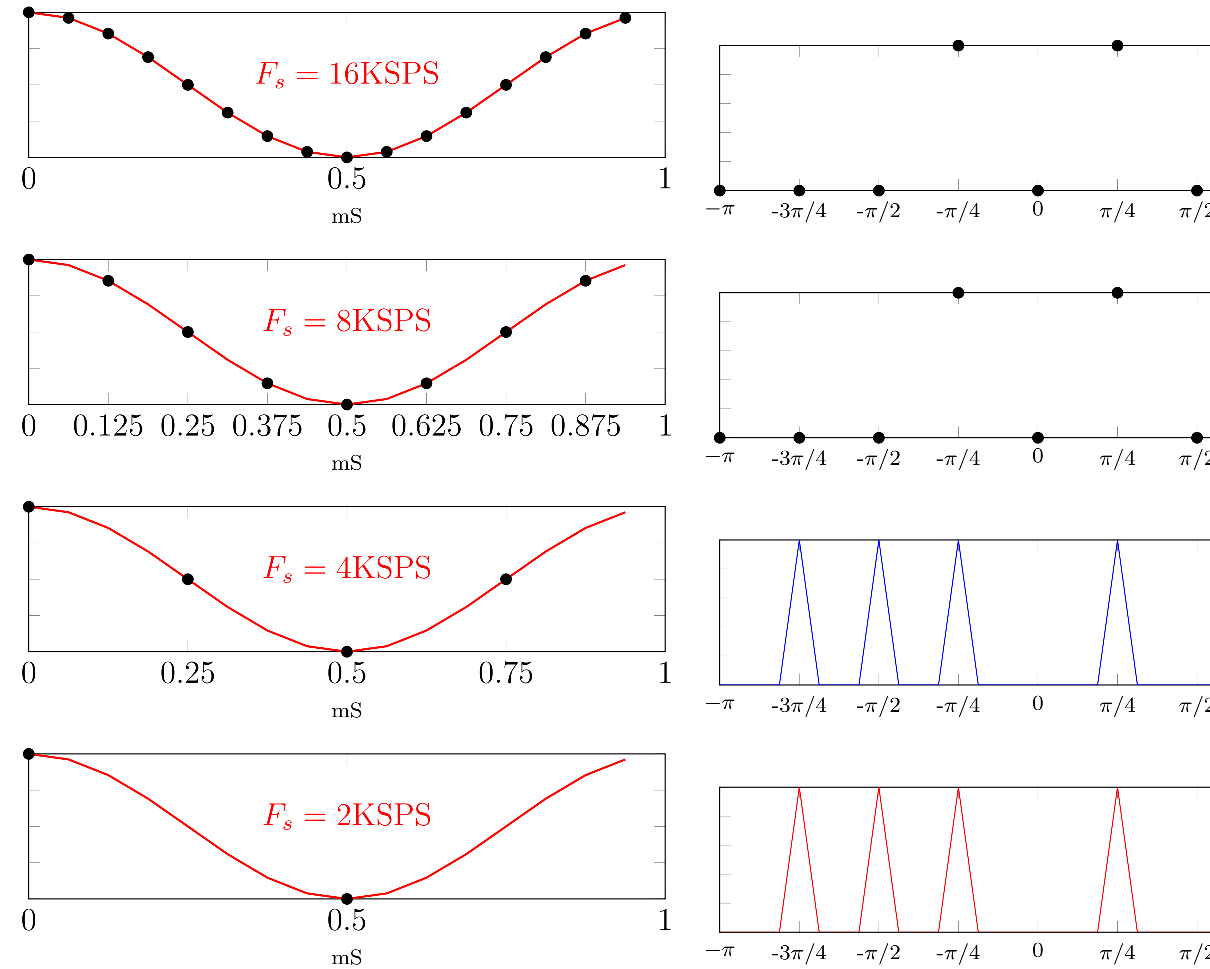


Figure 2.4: Illustration of sampling process for sinusoidal waveform.

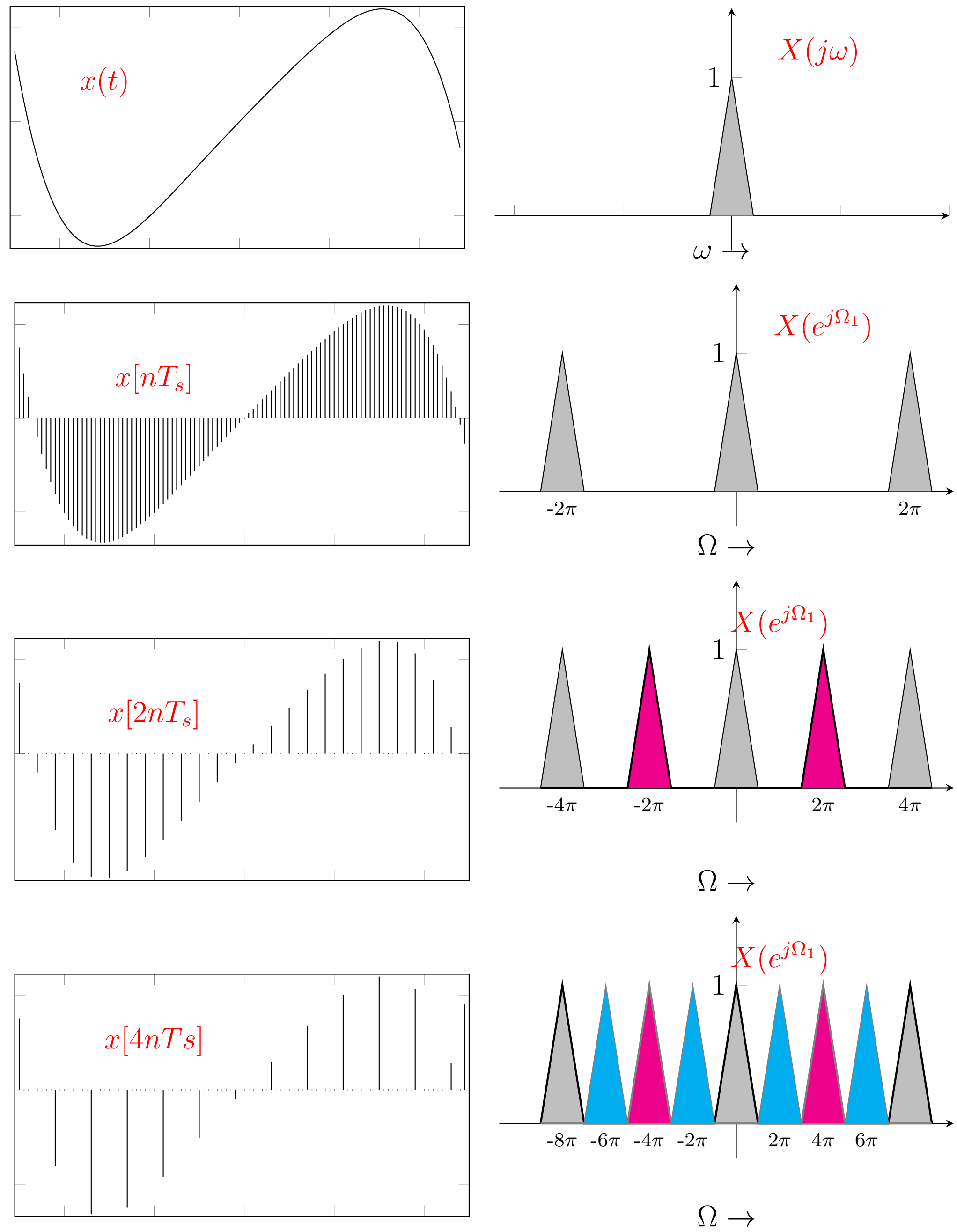


Figure 2.5: Illustration of sampling process in time and its implications in frequency domain.

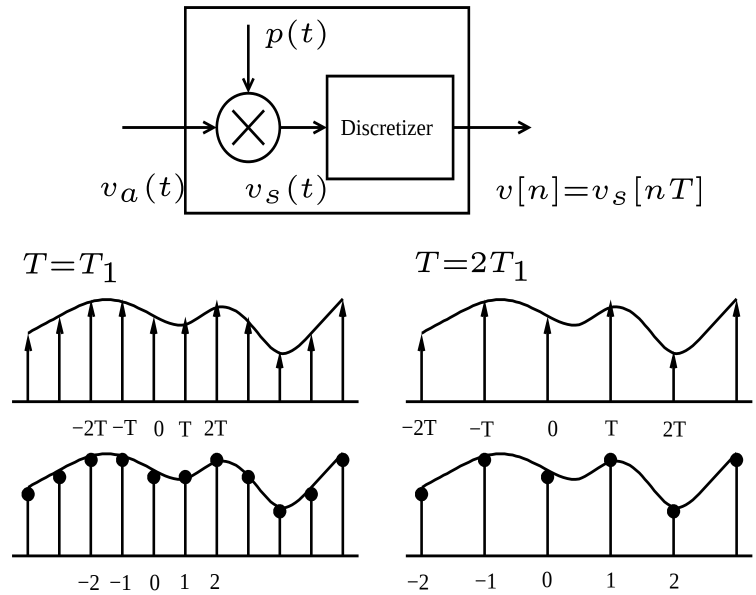


Figure 2.6: Illustration of sampling process in time domain.

Further Reading!
Explain these graphs

1. Fourier transform of a bandlimited analog input signal $V_a(j\omega)$, highest frequency is ω_N.
2. Fourier transform of Dirac impulse train.
3. Result of convolution $P(j\omega) * V_a(j\omega)$. It is evident that when

$$\omega_s - \omega_N \geq \omega_N \quad \text{or} \quad \omega_s > 2\omega_N \tag{2.17}$$

   $\Rightarrow v_a(t)$ can be recovered via ideal low-pass filter.
   if (2.17) does not hold i.e. if $\omega_s < 2\omega_N$ copies of $V_a(j\omega)$ would overlap and signal $v_a(t)$ cannot be recovered by lowpass filtering.⇒"Aliasing". Ideal sampling is practically impossible. $\delta_0(t)$ can be generated only approximately.

discrete samples $x[n]$ do not contain any information about sampling interval.
⇒ Sampling frequency must be known apriori for reconstruction of original signal.

2.4 Reconstruction

Reconstruction of ideally sampled signal by ideal lowpass filtering:

In order to get the input signal $v_a(t)$ after reconstruction filter, i.e. $V_r(j\omega) = V_a(j\omega)$

$$\omega_N < \frac{\omega_s}{2} \quad \text{and} \quad \omega_N < \omega_c < (\omega_s - \omega_N) \tag{2.18}$$

must be satisfied!.Then we have

$$V_r(j\omega) = V_a(j\omega) = V_i(j\omega) \cdot H_r(j\omega) \bullet\!-\!-\!-\!\circ\, v_a(t) = v_r(t) = v_i(t) * h_r(t) \tag{2.19}$$

the frequency response of ideal lowpass filter

$$H_r(j\omega) = \begin{cases} T, & |\omega| < \omega_c \\ 0, & \text{elsewhere} \end{cases}$$

$$\Rightarrow \text{rect}(\frac{\omega}{\omega_c}) \bullet\!-\!-\!-\!\circ\; h_r(t) = \text{sinc}(\omega_s t/2) \tag{2.20}$$

Combining (2.13),(2.20) in (2.19) yields

$$\begin{aligned} v_a(t) &= \int_{\tau:-\infty}^{+\infty} \sum_{n=-\infty}^{+\infty} v_a(nT)\delta_0(\tau - nT)\text{sinc}\Big(\frac{1}{2}\omega_s(t-\tau)\Big)d\tau \\ &= \sum_{n=-\infty}^{+\infty} v_a(nT) \int_{\tau:-\infty}^{+\infty} \delta_0(\tau - nT)\text{sinc}\Big(\frac{1}{2}\omega_s(t-\tau)\Big)d\tau \\ &= \sum_{n=-\infty}^{+\infty} v_a(nT)\text{sinc}\Big(\frac{1}{2}\omega_s(n-kT)\Big) \end{aligned}$$

2.4.1 Sampling Theorem:

Every bandlimited continuous-time signal $v_a(t)$ with $\omega_N < \omega_s/2$ can be uniquely recovered from its samples through:

$$v_a(t) = \sum_{n=-\infty}^{+\infty} v_a(nT)\text{sinc}\Big(\frac{1}{2}\omega_s(t-nT)\Big) \tag{2.21}$$

- equation (2.21) is called the *ideal interpolation formula*, and sinc is termed as *ideal interpolation function*.
- reconstruction of continuous signal using ideal interpolation.

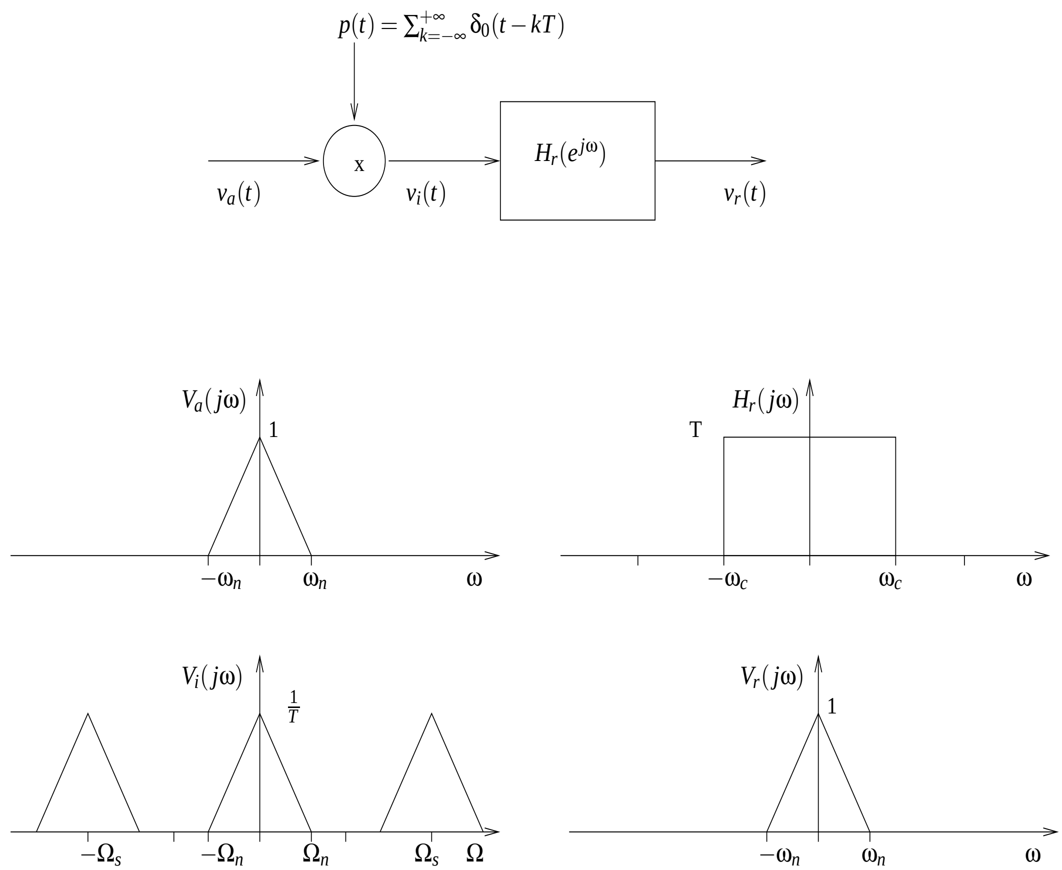


Figure 2.7: Illustration of sampling and reconstruction process.

2.5 Quantization

Conversion carried out by an A/D-converter involves *quantization* of the sampled input signal $v(nT) = v_i(nT)$ and the encoding of result into binary representation.

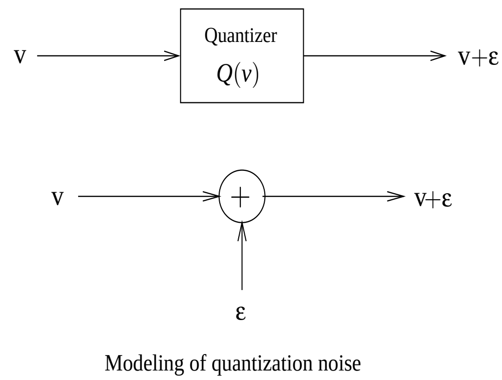


Figure 2.8: Illustration of sampling process.

- Quantization is a *nonlinear* and *non-invertible* process which realizes the mapping.

$$v(nT) = v[n] \to v_n \ni I \tag{2.22}$$

where v_n is taken from a finite alphabet I.

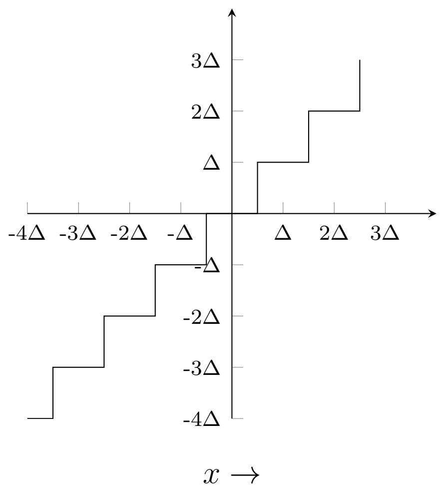
- Signal amplitude range is divided into L intervals I_n, using $L+1$ decision levels $d_1, d_2, \ldots, d_{L+1}$

$$I_n = \{d_n < v[n] \leq d_{n+1}\}, n = 1, 2, \ldots, L$$

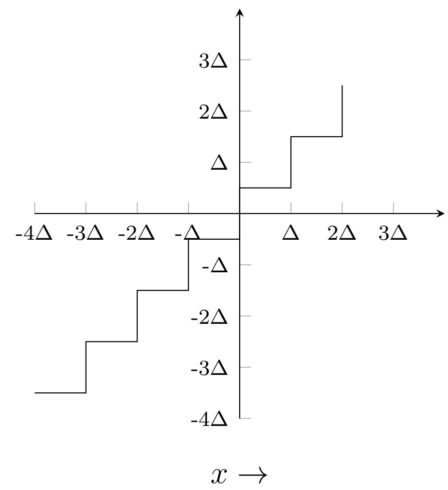
- Mapping in (2.23) is denoted as $v_q[n]{=}Q[v[n]]$.

- Uniform or *linear* quantizers with constant *quantization step size* Δ are very often used in signal processing applications.

$$\Delta = v_{i+1} - v_i = \text{const}, \forall\, i = 1, 2, \ldots, L-1 \tag{2.23}$$



(a) Transfer characteristics of mid-treat quantizer.



(b) Transfer characteristics of mid-rise quantizer.

Figure 2.9: Transfer characteristics of A/D converters.

- Midtreat quantizer: Zero is assigned a quantization level

  Midrise quantizer: Zero is assigned a decision level.

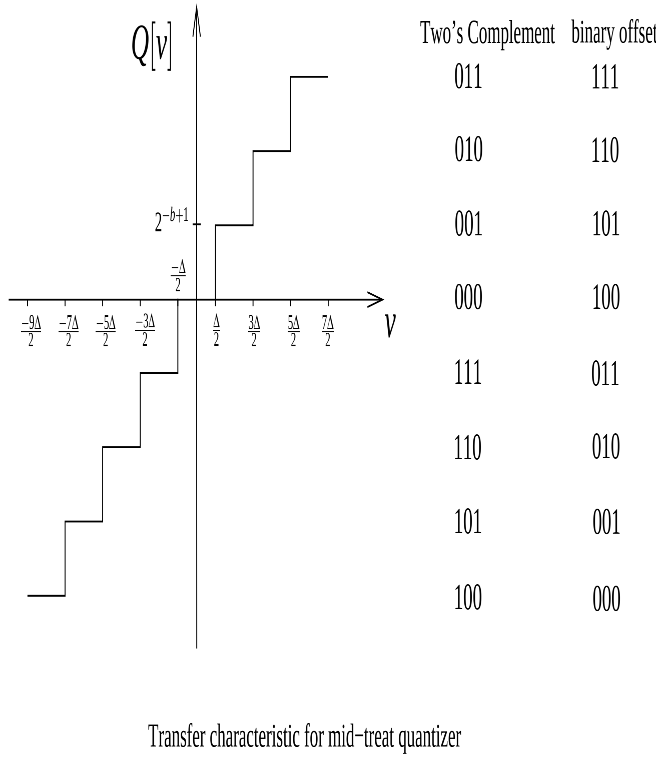


Figure 2.10: Quantization and data encoding process.

The quantization process and its interconnection with numbering format is illustrated in the fig. 2.10.

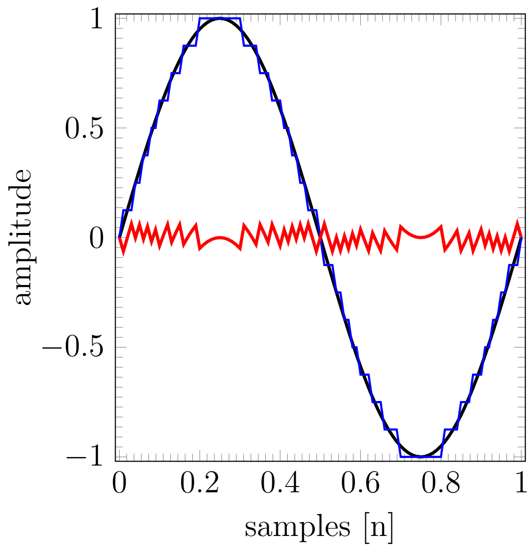
- Quantization error $e_q(k)$ with respect to the unquantized signal

$$-\frac{\Delta}{2} < e_q(n) \leq \frac{\Delta}{2} \tag{2.24}$$

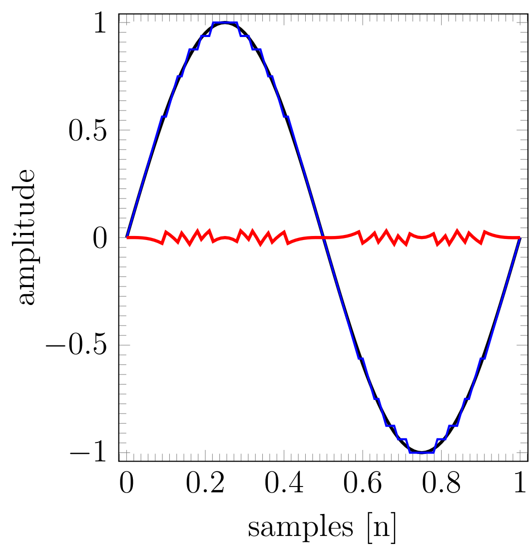
If the dynamic range of the input signal $(v_{\max} - v_{\min})$ is larger than the range of quantizer, the samples exceeding the quantizer range are clipped, which leads to $e_q[n] > \dfrac{\Delta}{2}$.

- Quantization characteristic function for a midtreat quantizer with $L = 8$.

Quantization Process (Demo).



(a) 3 bit quantization of sinusoidal waveform.



(b) 4 bit quantization of sinusoidal waveform.

Figure 2.11: Illustration of quantization noise as a function of ADC resolution.

2.6 Coding and Number Formats

Further Reading!
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The coding process in an A/D converter assigns a binary number to each quantization level.

• with wordlength of b bits we can represent $2^b > L$ binary number to each quantization level.

$$b \geq \log_2(L)$$

• the step size or the resolution of the A/D converter is given as $\Delta = \frac{R}{2^b}$ with range R of the quantizer.

• Two's complement representation is used in most fixed-point DSPs: A b bit binary fraction $[\beta_0\beta_1\beta_2\ldots\beta_{b-1}]$,$\beta_0$ denoting the most significant bit (MSB) and β_{b-1} the least significant bit (LSB), has value

$$v = -\beta_0 + \sum_{l=1}^{b-1}\beta_l 2^{-l} \tag{2.25}$$

• number representation has no influence on the performance of quantization process.

Quantization errors

Quantization error is modelled as noise, which is added to the unquantized signal

Assumptions:

- The quantization error $e_q[n]$ is uniformly distributed over the range $-\frac{\Delta}{2} < e_q[n] \leq \frac{\Delta}{2}$.
- The error sequence $e_q[n]$ is modelled as a stationary white noise sequence.
- The error sequence $e_q[n]$ is uncorrelated with the signal sequence $v[n]$.
- The signal sequence is assumed to have zero mean.

Assumptions do not hold in general, but they are fairly well satisfied for large quantizer word-lengths b.

Table 2.1: Table outlining different binary number formatting techniques.

number	+ve ref	-ve ref	sign magn	2's compl.	offset bin.	1's compl.
$+7$	$+\frac{7}{8}$	$-\frac{7}{8}$	0111	0111	1111	0111
$+6$	$+\frac{6}{8}$	$-\frac{6}{8}$	0110	0110	1110	0110
$+5$	$+\frac{5}{8}$	$-\frac{5}{8}$	0101	0101	1101	0101
$+4$	$+\frac{4}{8}$	$-\frac{4}{8}$	0100	0100	1100	0100
$+3$	$+\frac{3}{8}$	$-\frac{3}{8}$	0011	0011	1011	0011
$+2$	$+\frac{2}{8}$	$-\frac{2}{8}$	0010	0010	1010	0010
$+1$	$+\frac{1}{8}$	$-\frac{1}{8}$	0001	0001	1001	0001
0	$0+$	$0-$	0000	0000	1000	0000
0	$0-$	$0+$	1000	(0000)	(1000)	1111
-1	$-\frac{1}{8}$	$+\frac{1}{8}$	1001	1111	0111	1110
-2	$-\frac{2}{8}$	$+\frac{2}{8}$	1010	1110	0110	1101
-3	$-\frac{3}{8}$	$+\frac{3}{8}$	1011	1101	0101	1100
-4	$-\frac{4}{8}$	$+\frac{4}{8}$	1100	1100	0100	1011
-5	$-\frac{5}{8}$	$+\frac{5}{8}$	1101	1011	0011	1010
-6	$-\frac{6}{8}$	$+\frac{6}{8}$	1110	1010	0010	1001
-7	$-\frac{7}{8}$	$+\frac{7}{8}$	1111	1001	0001	1000
-8	$-\frac{8}{8}$	$+\frac{8}{8}$		(1000)	(0000)	

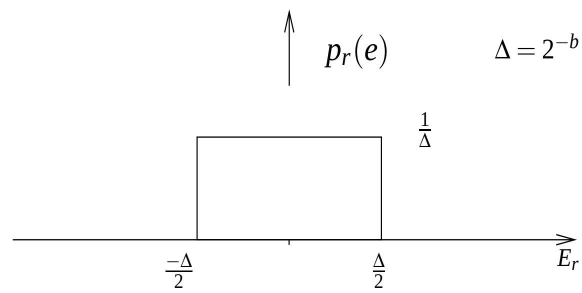


Figure 2.12: Probability density function of noise process.

Effect of quantization error or *quantization noise* on the resulting signal $v_q[n]$ can be evaluated in terms of *signal-to-noise ratio*(SNR) in decibels(dB).

$$SNR = 10log_{10}\Big(\frac{P_v}{P_n}\Big) \tag{2.26}$$

Where P_v denotes the signal power and P_n denotes the power of the quantization noise.

In case of quantization with *rounding*, noise is assumed to be uniformly distributed in the range$(-\Delta/2, \Delta/2)$. $\Rightarrow$Zero mean, and a variance of

$$P_n = \sigma_e^2 = \int_{-\Delta/2}^{\Delta/2} e^2 p(e) de = \frac{1}{\Delta}\int_{-\Delta/2}^{\Delta/2} e^2 de = \frac{\Delta^2}{12} \tag{2.27}$$

Inserting the definition of Δ in (2.27) yields,

$$\sigma_e^2 = \frac{2^{-2b}R^2}{12} \tag{2.28}$$

using (2.26) we obtain

$$\begin{aligned} SNR &= 10\ log_{10}\Big(\frac{\sigma_v^2}{\sigma_e^2}\Big) = 10\ log_{10}\Big(\frac{12 \cdot 2^{2b}\sigma_v^2}{R^2}\Big) \\ &= 6.02b + 10.8 - 20 \cdot log_{10}\Big(\frac{R}{\sigma_v}\Big) \quad \text{dB} \end{aligned} \tag{2.29}$$

Explanation of (2.29)

- σ_v root-mean-square(RMS) amplitude of the signal $v(t)$.
- σ_v $\rightarrow$small$\Rightarrow$ decreasing SNR.
- furthermore not evident from (2.29) σ_v $\rightarrow$large $\Rightarrow$ range R is exceeded.

⇒Signal amplitude has to be carefully matched to the range of the A/D converter. For music and speech a good choice is $\sigma_v = R/4$. Then the SNR of a $b-$bit quantizer can be approximately determined as

$$SNR \;=\; 6.02b - 1.25dB \tag{2.30}$$

Each additional bit in the quantizer increases the signal-to-noise ratio by 6 dB.

Examples:

Narrowband speech: b=8 Bit ⇒SNR=46.9 dB

Music(CD): b=16 Bit ⇒SNR=95.1 dB

Music(Studio): b=24 Bit ⇒SNR=143.2 dB

2.7 A/D converters

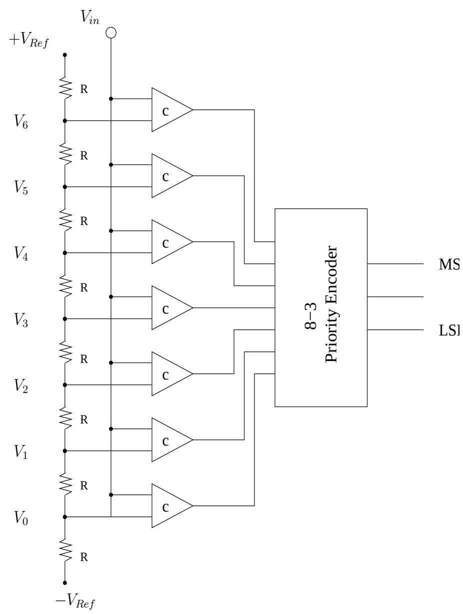
There exist a variety of A/D converter working on different principles. Some of the most commonly used devices are illustrated here in brief.

- Analog input voltage V_A is simultaneously compared with a set of $2^b - 1$ separated references voltages by means of analog comparators⇒ locations of the comparators circuits indicate the range of the input voltage.

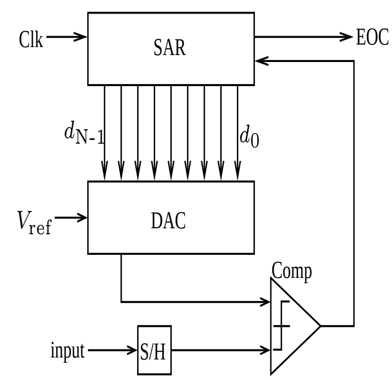
- All output bits are developed simultaneously⇒ very fast conversion. (1-1.5GHz)

- Hardware requirements increase exponentially with linear increase in resolution.

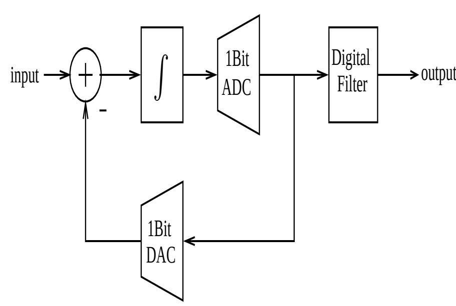
⇒ Flash converters are used for low-resolution(b< 8bits) and high speed conversion applications.



(a) 3-bit Flash ADC block diagram.



(b) Successive Approximation ADC block diagram.



(c) Σ-Δ ADC block diagram

Figure 2.13: Block diagram structure of different AD converters.

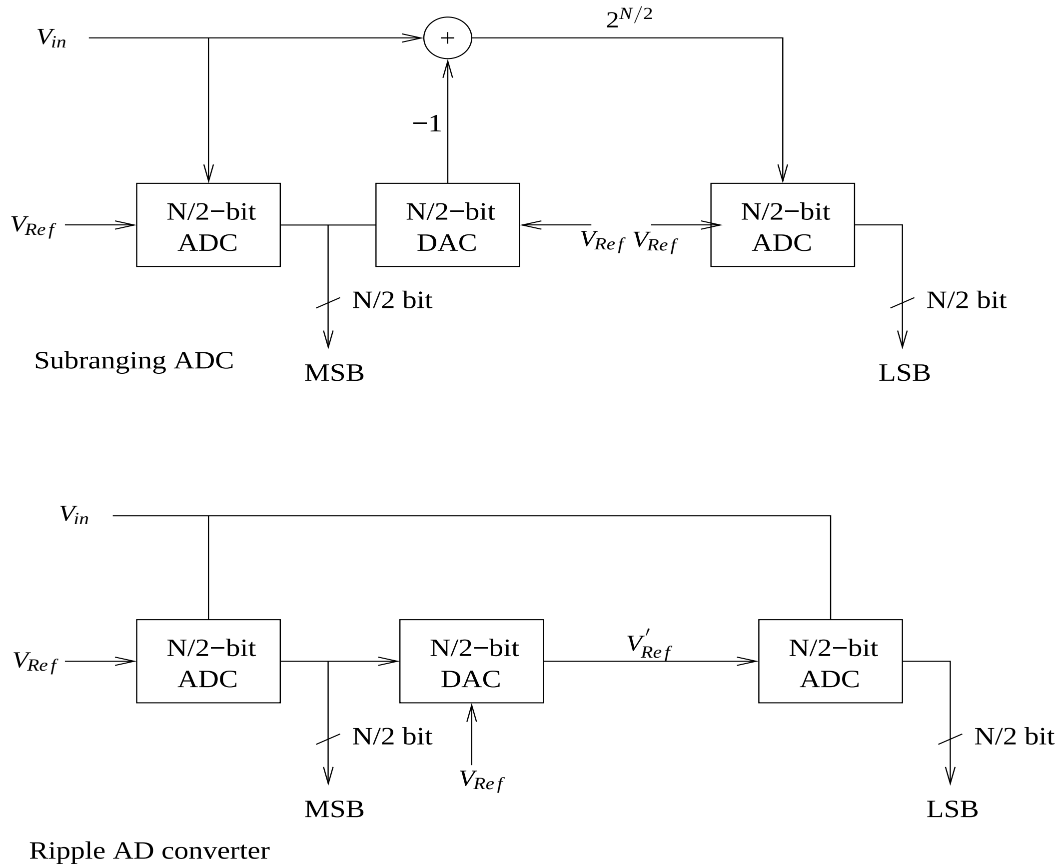


Figure 2.14: Implementations of b-bit ADC through b/2-bits ADCs.

2.7.1 Serial to Parallel A/D Converters

Here, two $b/2-$bit flash converters in a serial-parallel configuration are used to reduce the hardware complexity at the expense of slightly higher conversion time.

2.7.2 Characteristics of A/D Converter

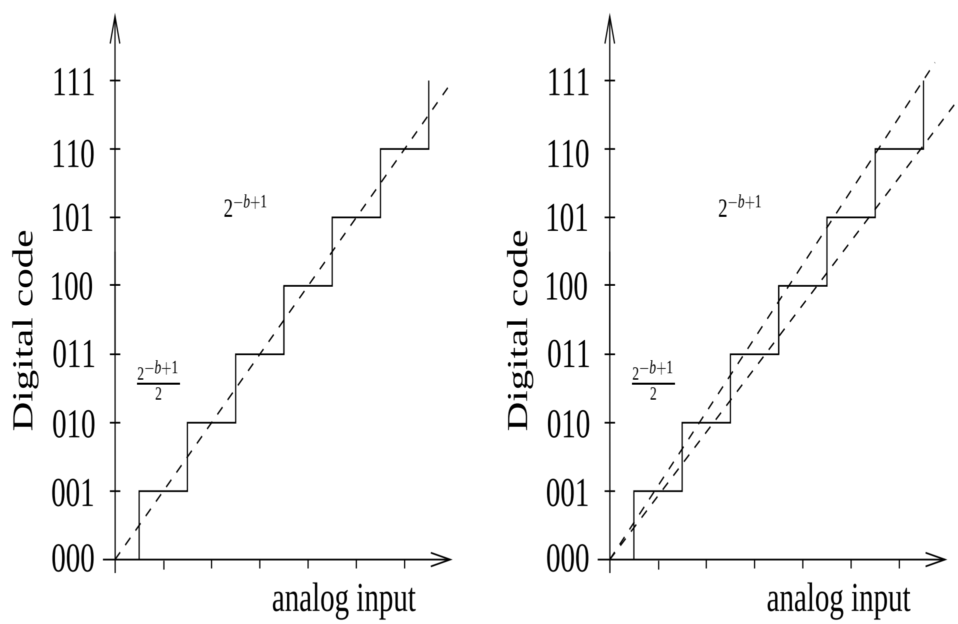


Figure 2.15: Illustration of additive and multiplicative error in A/D converter.

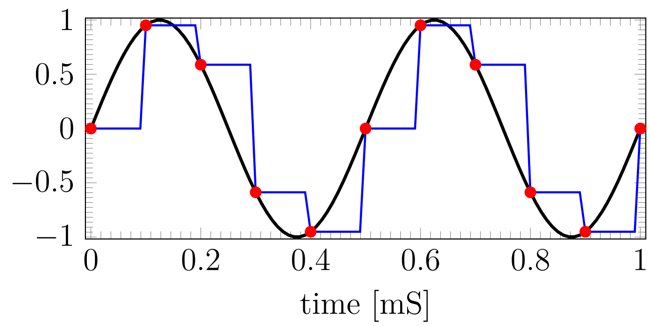
2.8 Digital to Analog Conversion

$$g_{\mathrm{SH}}(t) = \begin{cases} 1, & 0 \leq t \leq T \\ 0, & \text{otherwise} \end{cases} \tag{2.31}$$

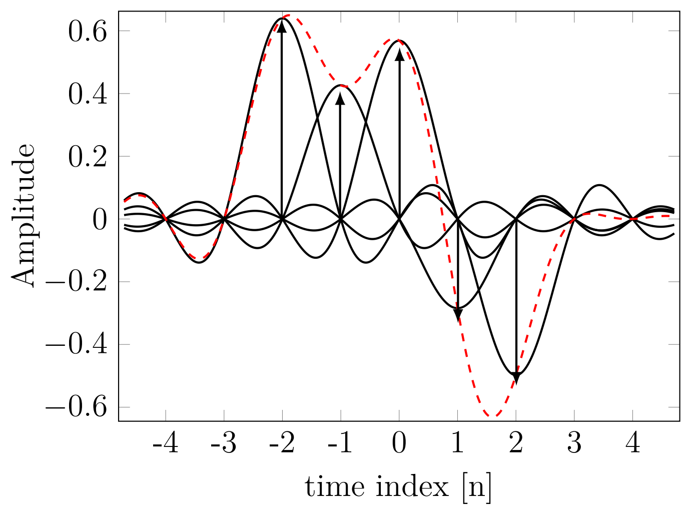
The frequency-domain characteristics are obtained by evaluating the its Fourier transform

$$G_{\mathrm{SH}}(\omega) = \int_{-\infty}^{+\infty} g_{\mathrm{SH}}(t) e^{-j2\pi t} dt = T \frac{\sin \pi F t}{\pi F t} e^{-j2\pi F(T/2)} \tag{2.32}$$

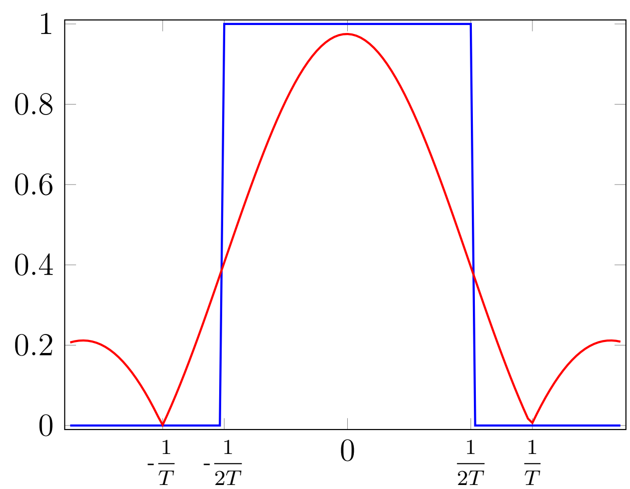
The magnitude of $G_{\mathrm{SH}}(F)$ is illustrated in the fig. 2.16, to compensate the effect of $\sin x/x$ distortion of S/H (aperture effect). The aperture effect attenuation compensation, which reaches the maximum at $2/\pi$ or 4 dB at $F = F_s/2$.



(a) reconstruction through sample and hold circuit.



(b) Reconstruction of signal through ideal lowpass filter.



(c) Spectral representation of ideal and practical reconstruction filter.

Figure 2.16: Frequency response of ideal and simple sample and hold reconstruction filter.

Chapter 3

Fourier Transform

The objective of this chapter is to revisit Fourier series and Integral and to study Discrete-time Fourier Transform, with special emphasis on its properties which provide valuable insights into analysis and manipulation of digital signals.

Introduction

“ Spectral representation ” means generally: representation of a sequence by a “linear superpositions ” of suitable basis sequences, weighted by suitable factors.
“Spectral analysis” means generally: determination of above named weighting factors for a given sequence and basis.
“Spectral synthesis” means generally the above-named linear superposition.
This is very simple yet very useful representation of deterministic and stochastic signals.

3.1 The Fourier transforms

1. Fourier series for continuous-time periodic signals

Table 3.1: Frequency Analysis of Continuous-Time Periodic Signals.

Synthesis equations	$x(t)=\sum_{k=-\infty}^{+\infty} c_k e^{j2\pi k F_0 t}$
Analysis equations	$c_k=\frac{1}{T_p}\int_{T_p} x(t)e^{-2\pi k F_0 t}dt$

2. Fourier transform for continuous-time aperiodic signals

Table 3.2: Frequency Analysis of Continuous-Time Periodic Signals

Synthesis equations	$x(t)=\frac{1}{2\pi}\int_{-\infty}^{+\infty} X(j\omega)e^{j\omega t}d\omega$
Analysis equations	$X(j\omega)=\int_{-\infty}^{+\infty} x(t)e^{-\omega t}dt$

Further Reading!
This a brief review

3.1.1 Definitions

$$v[n], n \in \mathbb{Z}, v \in \mathbb{C} \circ - - - - V(e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} v[n]e^{-j\Omega n} \tag{3.1}$$

Further Reading!
Some text goes here just to illustrate the command

is the“ Spectrum ” of $v[n]$. Equation (3.1) is called the “ Discrete-time Fourier transfo
of $v[n]$.

Inverse transformation

$$V(e^{j\Omega}), \Omega \in \mathbb{R}, V \in \mathbb{C} - - - - \circ\, v[n] = \frac{1}{2\pi}\int_{\Omega=-\pi}^{\pi} V(e^{j\Omega})e^{+j\Omega n}d\Omega \tag{3.2}$$

Further Reading!
Some text goes here just to illustrate the command

Equation (3.2) $\hat{=}$ spectral synthesis i.e. $v[n]$ written as superposition of harmonics of $e^{j\Omega n}$ weighted by $\frac{1}{2\pi} \cdot V(e^{j\Omega})d\Omega$ as “Complex amplitudes”$\Rightarrow V(e^{j\Omega})\hat{=}$ amplitude density

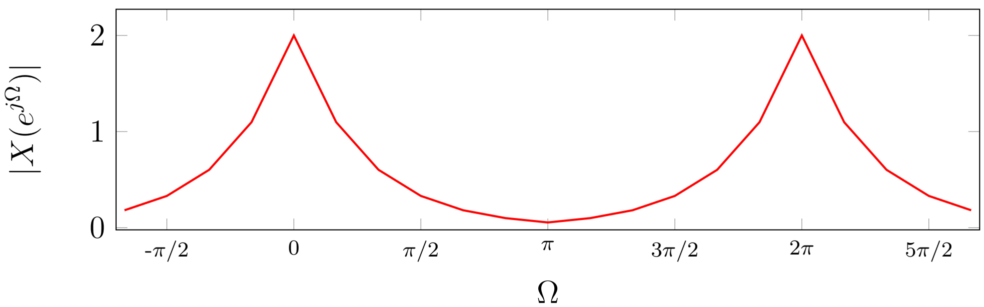


Figure 3.1: Illustration of discrete Fourier transform

3.1.2 Existence

(3.1) $\hat{=}$ finite summation $\Rightarrow$ existence $\Rightarrow |V(e^{j\Omega})| \leq M < \infty$

$$|V(e^{j\Omega})| \overset{3.1}{=} |\sum_{n=-\infty}^{+\infty} v[n]e^{-j\Omega n}| \leq \sum_{n=-\infty}^{+\infty} |v[k]| \cdot |e^{-j\Omega n}| = \sum_{n=-\infty}^{+\infty} |v[n]| \tag{3.3}$$

so: If $\sum_{n=-\infty}^{+\infty} |v[n]| \leq M < \infty$, i.e. $v[n]$ is "absolutely summable" then $|Ve^{j\Omega}| \leq M < \infty$ holds certainly.

Further Reading!
Some text goes here just to illustrate the command

But: This is only a sufficient condition, not a necessary one. Equation (3.3) may be violated while still $V(e^{j\Omega})$ may exist because

$$|V(e^{j\Omega})| \leq \sum_{n=-\infty}^{+\infty} |v[n]|$$

Periodicity

$$\begin{aligned}\text{in 3.1} \quad \Omega = \Omega + \lambda \cdot 2\pi \quad \Rightarrow \quad & \sum_{n=-\infty}^{+\infty} v[n]e^{-jn(\Omega+\lambda\cdot 2\pi)} = V(e^{j(\Omega+\lambda\cdot 2\pi)}) \\ = \; & \underbrace{\sum_{n=-\infty}^{+\infty} v[n]e^{-j\Omega n}}_{V(e^{j\Omega})} \cdot \underbrace{e^{-jn(\lambda\cdot 2\pi)}}_{1}\end{aligned} \tag{3.4}$$

$\rightarrow$ Fourier transform of a sequence is 2π periodic in Ω.

3.2 Characteristics, Rules, Theorems

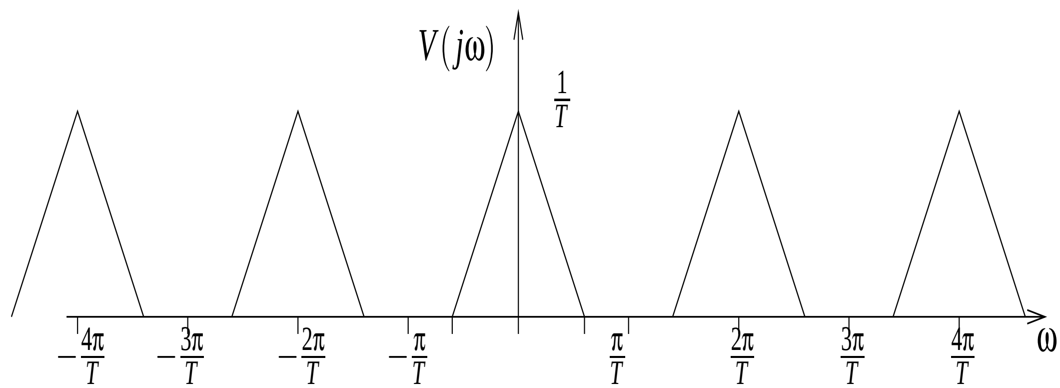
Background

$$\text{Relation of } F\text{−transform} \quad V_0(j\omega) = F\{v_0(t)\} = \int_{t=-\infty}^{+\infty} v_0(t) e^{-j\omega t} dt \tag{3.5}$$

of a continuous signal $v_0(t)$. If $v[n]$ contains equi-spaced samples of $v_0(t)$ then,

$$v_0[nT] = v[n] \circ - - \overset{F_*}{-} - - - V(e^{j\Omega}) = \frac{1}{T} \sum_{\lambda=-\infty}^{+\infty} V_0\Big[j(\omega - \frac{2\pi \cdot \lambda}{T})\Big]_{\Omega=\omega T} \tag{3.6}$$

If we increase T to a large value, it will result in overlapping, known as aliasing.



Further Reading!
Some text goes here just to illustrate the command

a. Linearity: The Fourier transformation is linear in the sense that

$$\text{for} \quad v_{1,2}[n] \circ - - - - - - - V_{1,2}(e^{j\Omega}) \quad \text{and} \quad v[n] = \alpha_1 \cdot v_1[n] + \alpha_2 \cdot v_2[n]$$

$$\text{We have} \quad v[n] \circ - - - - V(e^{j\Omega}) = \alpha_1 \cdot V_1(e^{j\Omega}) + \alpha_2 \cdot V_2(e^{j\Omega}) \tag{3.7}$$

The equivalence holds for the inverse transformation obviously (3.7) says “F_*\{superposit
= superposition of F_*’s”

b. Modulation: Given: $v[n] \circ - - - - V(e^{j\Omega})$

def: $v_1[n] = v[n] \cdot e^{+j\Omega_0 n}$ “ Complex modulation” then from (3.1) we ha

$v_1[n] \circ - - - - V[e^{j(\Omega-\Omega_0)}] \triangleq$ Shift in frequency (3

c. Shift: Given: $v[n] \circ - - - - - - - V(e^{j\Omega})$

$$\begin{aligned} v_1[n] &= v[n-n_0](\underline{\text{ delayed / advanced / shifted signal }}) && (3.9)\\ \text{then from (3.1)} \quad v_1[n] &\circ - - - - - V(e^{j\Omega})e^{-j\Omega n_0} \ \underline{\text{ additional linear phase}} && (3.10) \end{aligned}$$

The " Shift operator " will turn out to be a central element of the discrete systems to be dealt with. It does not alter the signal form - and the spectrum neither, except for adding a linear term $(-k_0, \Omega)$ to the phase angle $\arg\{V(e^{j\Omega})\}$

d. Component symmetries:

It is well established that $v[n] \in \mathbb{C} \rightarrow v[n] = \Re\{v[n]\} + j\Im\{v[n]\} = v_{re}[n] + v_{img}[n]$

Any signal can also be decomposed into an even and odd component:

$$v[n] \rightarrow v_e[n] = \frac{1}{2}[v[n] + v^*[-n]] = v_e[-n] \qquad (3.11)$$

$$\rightarrow v_o[n] = \frac{1}{2}[v[n] - v^*[-n]] = -v_o[-n] \qquad (3.12)$$

- Both decompositions can be combined.

- The same holds for the (continuous) functions $V(e^{j\Omega})$, over Ω.

So: 4 components of $v[n]$ and $V(e^{j\Omega})$ each.

then from (3.1), unique symmetry relations are found:

$$v[n] = v_{Re}[n] + v_{Ro}[n] + jv_{Ie}[n] + jv_{Io}[n]$$

$$V(e^{j\Omega}) = V_{Re}(e^{j\Omega}) + V_{Ro}(e^{j\Omega}) + jV_{Ie}(e^{j\Omega}) + jV_{Io}(e^{j\Omega}) \qquad (3.13)$$

(for proof see exercise)

$v[n] \in \Re \rightarrow v_{Ie}[n] \equiv 0, v_{Io}[n] \equiv 0 \overset{3.11}{\rightarrow} V_{Ie}(e^{j\Omega}) \equiv 0, V_{Ro}(e^{j\Omega}) \equiv 0$

$$V(e^{j\Omega}) = V_{Re}(e^{j\Omega}) + jV_{Io}(e^{j\Omega}) = V^*(e^{-j\Omega}) \qquad (3.14)$$

$\rightarrow$ real-valued signals have Hermitian Fourier transforms. Consequences:

$$|V(e^{j\Omega})| = \pm\sqrt{V_{Re}^2(e^{j\Omega}) + \underbrace{V_{Ie}^2(e^{j\Omega})}_{\text{even due to sqr.}}} = \text{even} \qquad (3.15)$$

$\rightarrow$ The modulus is an even function of Ω if $v[n] \in \Re$

e. Convolution:Linear convolution of continuous function well-known

$$v_1(t) * v_2(t) = \int_{\tau=-\infty}^{\infty} v_1(\tau) \cdot v_2(t-\tau) d\tau = v_2(t) * v_1(t) \tag{3.16}$$

same definitions for any continuous function for any variable e.g. $v_1(j\omega)$ and $v_2(j\omega)$ but for $v_{1,2}(e^{j\Omega})$ their periodicities have to be taken into account, so now the definition for periodic function needed, convolution of two periodic spectra.

$$v_1(e^{j\Omega}) \circledast v_2(e^{j\Omega}) = \int_{\eta=-\pi}^{\pi} v_1(e^{j\eta}) \cdot v_2(e^{j\Omega-\eta}) d\eta = v_2(e^{j\Omega}) \circledast v_1(e^{j\Omega}) \tag{3.17}$$

circular (periodic) convolution results in 2π periodic sequence. Convolution of continuous spectra can be easily extended to discrete case.

Spectral operations on $V_{1,2}(e^{j\Omega})$ are equivalent to the convolutions of $v_{1,2}[n]$ and vice-versa

$$\begin{aligned} F_*\{v[n]\} = F_*\{v_1[n] * v_2[n]\} &= \sum_{n=-\infty}^{+\infty} \sum_{\kappa=-\infty}^{+\infty} v_1[\kappa] \cdot v_2[n-\kappa] e^{-j\Omega k} \\ &= \sum_{\kappa=-\infty}^{+\infty} v_1[\kappa] \sum_{n=-\infty}^{+\infty} v_2[n-\kappa] e^{-j\Omega[n-\kappa]} \cdot e^{-j\Omega\kappa} \\ \text{insert} \quad \nu = n-\kappa & \\ V(e^{j\Omega}) &= \underbrace{\sum_{\kappa=-\infty}^{+\infty} v_1(\kappa) \cdot e^{-j\Omega\kappa}}_{V_1(e^{j\Omega})} \cdot \underbrace{\sum_{\nu=-\infty}^{+\infty} v_2(\nu) \cdot e^{-j\Omega\nu}}_{V_2(e^{j\Omega})} \\ &= V_1(e^{j\Omega}) \cdot V_2(e^{j\Omega}) \end{aligned}$$

so:

$$v_1[n] * v_2[n] \quad \circ \overset{F_*}{- - - -} V_1(e^{j\Omega}) \cdot V_2(e^{j\Omega}) \tag{3.18}$$

$$\Rightarrow \quad F_*^{-1}\{V_1(e^{j\Omega}) \circledast V_2(e^{j\Omega})\} = \frac{1}{2\pi} \int_{\Omega=-\pi}^{\pi} \int_{\eta=-\pi}^{\pi} V_1(e^{j\eta}) \cdot V_2(e^{j(\Omega-\eta)})_{mod 2\pi} \cdot e^{j\Omega k} d\eta \cdot d\Omega$$

$$= v_1[n] \cdot v_2[n] \tag{3.19}$$

f. Difference & Integration

$$\text{given} \quad v[n] \circ - - - V(e^{j\Omega}) \tag{3.20}$$

$$v_1[n] = ?? \; \circ - - - V_1(e^{j\Omega}) = \frac{dV(e^{j\Omega})}{d\Omega} = \frac{d}{d\Omega} \sum_{n=-\infty}^{+\infty} v[n] e^{-j\Omega n} = - \quad jn \cdot v[n]$$

$$\text{so:} \quad -jn \cdot v[n] \circ - - - \frac{d}{d\Omega} V(e^{j\Omega}) \tag{3.21}$$

Inversely as there is no derivative of sequence $v[n]$ we consider difference signal

$$\begin{aligned}
\text{Given:} \qquad v[n] \quad & \circ - - - V(e^{j\Omega}) \\
v_1[n] \quad & = v[n] - v[n-1] \circ - - - V_1(e^{j\Omega}) = ?? \\
V_1(e^{j\Omega}) \quad & = V(e^{j\Omega})\Big[1 - e^{-j\Omega\cdot 1}\Big] - - - \circ\, v[n] - v[n-1] \qquad (3.22)
\end{aligned}$$

g. Parseval's theorem

given

$$v_1[n] \quad \circ - - - - V_1(e^{j\Omega})$$

and

$$v_2[n] \quad \circ - - - - V_2(e^{j\Omega})$$

then

$$\sum_{n=-\infty}^{+\infty} v_1[n] v_2^*[n] = \frac{1}{2\pi}\int_{-\pi}^{+\pi} V_1(e^{j\Omega}) V_2^*(e^{j\Omega}) d\Omega \qquad (3.23)$$

proof:

inserting (3.1) in (3.23).

$$\begin{aligned}
&= \frac{1}{2\pi}\int_{-\pi}^{+\pi} \Big[\sum_{n=-\infty}^{+\infty} v_1[n] e^{j\Omega n}\Big] V_2^*(e^{j\Omega}) d\Omega \qquad (3.24) \\
&= \sum_{n=-\infty}^{+\infty} v_1[n] v_2^*[n]
\end{aligned}$$

$$\sum_{n=-\infty}^{+\infty} |v[n]|^2 = \frac{1}{2\pi}\int_{-\pi}^{+\pi} |V(e^{j\Omega})|^2 d\Omega \qquad (3.25)$$

Sum of signal power = Sum of power of components

Parseval theorem of (3.25) states that the sum of signal power values gives the same as “ the sum ” of component powers.

3.3 Symmetry properties of DTFT

The following properties find extensive application in practice and must be explored further.

$$x[n] \circ\!\!-\!\!\bullet X(e^{j\omega})$$
$$x^*[n] \circ\!\!-\!\!\bullet X^*(e^{-j\omega})$$
$$x[-n] \circ\!\!-\!\!\bullet X^*(e^{j\omega})$$
$$\Re\{x[n]\} = \frac{1}{2}(x[n] + x^*[n]) \circ\!\!-\!\!\bullet \frac{1}{2}(X(e^{j\Omega}) + X^*(e^{-j\Omega}))$$
$$\Im\{x[n]\} = \frac{1}{2}(x[n] - x^*[n]) \circ\!\!-\!\!\bullet \frac{1}{2}(X(e^{j\Omega}) - X^*(e^{-j\Omega}))$$
$$\frac{1}{2}(x[n] + x^*[-n]) \circ\!\!-\!\!\bullet \Re\{X(e^{j\Omega})\}$$
$$\frac{1}{2}(x[n] - x^*[-n]) \circ\!\!-\!\!\bullet j\Im\{X(e^{j\Omega})\}$$

Furthermore for real valued signal $x[n]$ we have

$$X(e^{j\Omega}) = X^*(e^{-j\Omega})$$
$$X_{\mathrm{R}}(e^{j\Omega}) = X_{\mathrm{R}}(e^{-j\Omega})$$
$$X_{\mathrm{I}}(e^{j\Omega}) = -X_{\mathrm{I}}(e^{-j\Omega})$$
$$|X(e^{j\Omega})| = |X(e^{-j\Omega})|$$
$$\angle X(e^{j\Omega}) = -\angle X(e^{-j\Omega})$$

A general table of properties relevant to DTFT are

Sequence	Fourier Transform
$x[n]$	$X(e^{j\Omega})$
$y[n]$	$Y(e^{j\Omega})$
$ax[n]+by[n]$	$aX(e^{j\Omega})+bY(e^{j\Omega})$
$x[n-n_d](n_d \in \text{ integer})$	$e^{-j\Omega n_d}X(e^{j\Omega})$
$e^{j\Omega_0 n}x[n]$	$X(e^{j(\Omega-\Omega_0)})$
$x[-n]$	$X(e^{-j\Omega})$
$nx[n]$	$j\dfrac{dX(e^{j\Omega})}{d\Omega}$
$x[n]*y[n]$	$X(e^{j\Omega})Y(e^{j\Omega})$
$x[n]y[n]$	$\dfrac{1}{2\pi}\int_{\theta:-\infty}^{+\infty} X(e^{j\theta})Y(e^{j(\Omega-\theta)})d\theta$
$\sum_{n:-\infty}^{+\infty} \lvert x[n]\rvert^2$	$\dfrac{1}{2\pi}\int_{\Omega:-\pi}^{\pi} \lvert X(e^{j\Omega})\rvert^2 d\Omega$

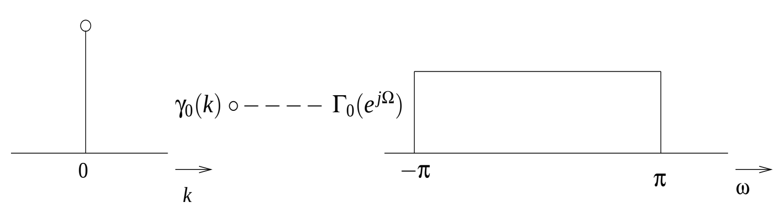
3.3.1 Examples

1. Impulse (dirac function) (see also 2.1.3)

$$\delta_0[n] = \begin{cases} 1, & n=0 \\ 0, & \text{elsewhere} \end{cases}$$

$$\Delta_0(e^{j\Omega}) \overset{3.1}{=} \sum_{n=-\infty}^{+\infty} \delta_0[n]e^{-j\Omega}$$
$$\sum_{n=-\infty}^{+\infty} \delta_0[n]e^{0} = 1\forall\ \Omega$$

$$\delta_0[n] \circ - - - - - - - \Gamma_0(e^{j\Omega}) = 1\forall\ \Omega \tag{3.26}$$



2. Step function:

$$\gamma_{-1}[n] \circ - - - - - - - \sum_{n=0}^{+\infty} \gamma_{-1}[n] e^{-j\Omega n} = \sum_{n=0}^{+\infty} cos(\Omega n) - jsin(\Omega n) =???$$

$$\sum_{n=-\infty}^{+\infty} |\gamma_{-1}[n]| \to \infty$$

$\Rightarrow$ sufficient condition (3.3) is not satisfied $\to$ there is no FT (at least common type FT)

$$\Gamma(e^{j\Omega}) = \frac{1}{1 - e^{-j\Omega}} + \sum_{n=-\infty}^{\infty} \pi \gamma_0(\Omega + 2\pi n)$$

3. Rectangle :

$$\gamma_M[n] \circ\!-\!-\!-\!-\!-\!-\!- R_M(e^{j\Omega}) = \sum_{n=0}^{M-1} 1 \cdot e^{-j\Omega n}$$

Remarks: finite summations → no problems with (absolute) summability. Their F_*- transforms always exist due to **(3.3)**

Some basic formulae

- General summation formula for exponential sequence

$$\sum_{k=k_1}^{k_2} q^k = q^{k_1} \sum_{k=0}^{k_2-k_1} q^k = q^{k_1} \frac{1-q^{k_2-k_1+1}}{1-q} \tag{3.27}$$

- and for $k_2 \to \infty$

$$\sum_{k=k_1}^{\infty} q^k = q^{k_1} \sum_{k=0}^{\infty} q^k = q^{k_1} \frac{1}{1-q} \quad \text{only for } |q| < 1 \tag{3.28}$$

back to rectangular pulse with (3.27)

$$\gamma_M[n] \circ\!-\!-\!-\!-\!-\!-\!- R_M(e^{j\Omega}) = \frac{1-e^{-j\Omega M}}{1-e^{-j\Omega}} = e^{-j\frac{M-1}{2}\Omega} \frac{\sin(\frac{M}{2}\Omega)}{\sin(\frac{\Omega}{2})} \tag{3.29}$$

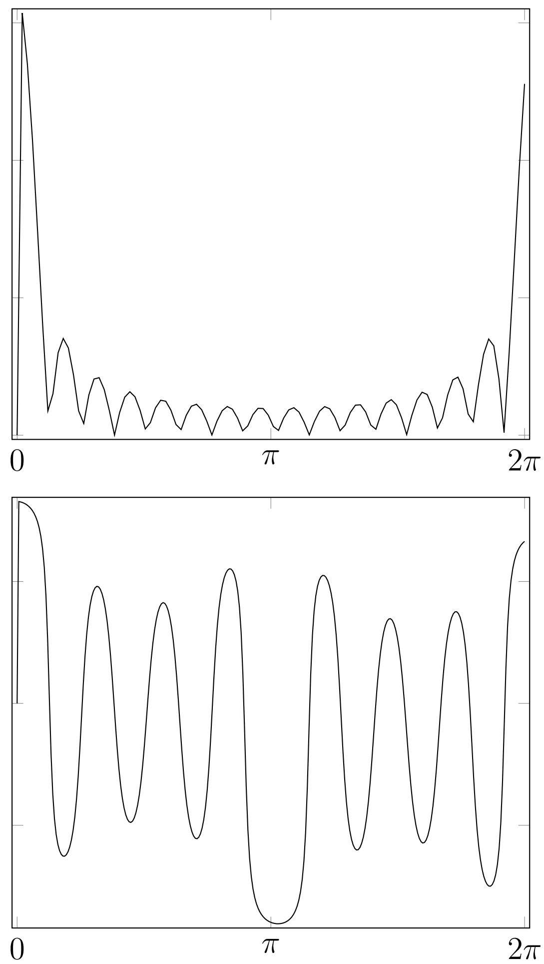


Figure 3.2: The Fourier transform of discrete rect function.

Chapter 4

The Z transforms

The objective of this chapter is to introduce Z and some of the special variants of Fourier transforms, with special emphasis on their properties which provide valuable insights into analysis and manipulation of digital signals.

- Ⓐ Introduction
- Ⓑ Discrete-time Fourier transforms
- Ⓒ Z transforms
- Ⓓ Discrete Fourier transforms
- Ⓔ The fast Fourier transforms.

4.1 Definitions

$$v[n], n \in \mathbb{Z}, v \in \mathbb{C} \circ - - - - V(z) = \sum_{n=-\infty}^{+\infty} v[n] \cdot z^{-n} \triangleq \text{“ Spectrum ”} \quad \text{“ Z-Transform ”} \tag{4.1}$$

inverse transformation

$$V(z), z \in \mathbb{C}, V \in \mathbb{C} \circ - - - - v[n] = \frac{1}{2\pi j} \oint V(z) \cdot z^{+k} \frac{dz}{z} \tag{4.2}$$

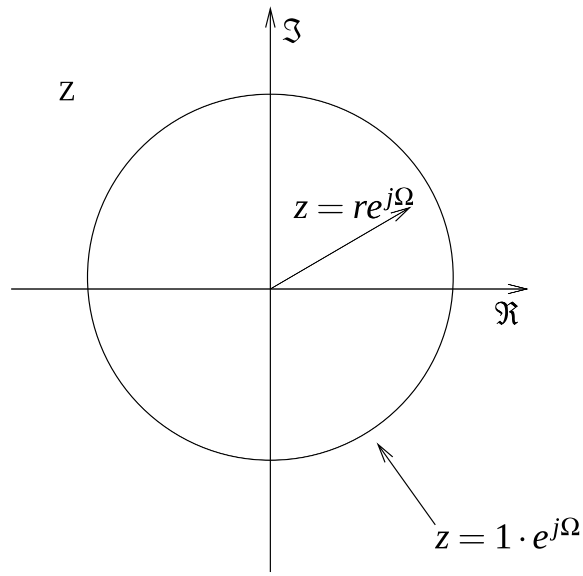


Figure 4.1: Illustration of Z-plane and its connection with Fourier transform.

Closed integration path in z-plane counter-clockwise.

Again(see 4.1,4.2): (4.2)$\triangleq$ $\underline{\text{spectral synthesis}}$, i.e. $v[n]$ as superposition of (now:) general complex exponentials z^n, weighted by $\frac{1}{2\pi j} \cdot V(z) \frac{dz}{z}$ as "$\underline{\text{complex amplitudes}}$" – $V(z) \triangleq$ amplitude densities.

(4.1)$\triangleq$ $\underline{\text{spectral analysis}}$ i.e, determination of suitable weights.

Remarks:

(4.1-4.2) $\triangleq$ generalization of (3.1,3.2) as z^n generalizes $e^{j\Omega n}$, regarding the *unit circle* $z = e^{j\Omega}$ in the z-plane from 4.1

$$V(z = e^{j\Omega}) = \sum_{n=-\infty}^{+\infty} v[n] e^{-j\Omega n} \stackrel{3.1}{=} F_*\{v[n]\}$$

and from (4.2) for a fixed 'r', $z = r \cdot e^{j\Omega} \Rightarrow \frac{dz}{d\Omega} = \frac{d\, re^{j\Omega}}{d\Omega}, \qquad dz = jr \cdot e^{j\Omega} d\Omega$

$$v[n] = \frac{1}{j2\pi} \int_{\Omega-\pi}^{\pi} V(z = e^{j\Omega}) \cdot e^{j\Omega n} \cdot j e^{j\Omega} d\Omega \cdot \frac{1}{e^{j\Omega}} = \frac{1}{2\pi} \int_{\Omega-\pi}^{\pi} v(e^{j\Omega}) e^{j\Omega k} d\Omega$$

Purpose of this generalization?

Answer: Spectral representation of signals possible which are not "F_*-Transformable" (see condition 3.3)

4.1.1 Existence

(see sec. 3.1.2): finite $|v(z)|$ $\rightarrow$ $v[n]$ must be absolutely summable.

$$
\begin{aligned}
|V(z)| &= |\sum_{n=-\infty}^{+\infty} v[n]z^{-n}| \leq \sum_{k=-\infty}^{+\infty} |v[n]||z^{-n}| \\
&= \sum_{n=-\infty}^{+\infty} |v[n]|r^{-n} \cdot \underbrace{|e^{-j\Omega n}|}_{\equiv 1 \forall \quad \Omega, n} \\
|V(z)| &\leq \sum_{n=-\infty}^{+\infty} |v[n]r^{-n}| \leq M < \infty
\end{aligned}
$$

additional degree of freedom, choice of $r = |z|$

Still not every signal is allowed.

$$
\begin{aligned}
|V(z)| &\leq \sum_{n=-\infty}^{-1} |v[n]r^{-n}| + \sum_{n=0}^{\infty} |\frac{v[n]}{r^n}| \\
&\leq \sum_{n=1}^{\infty} |v(-n)r^n| + \sum_{n=0}^{\infty} |\frac{v[n]}{r^n}|
\end{aligned} \tag{4.3}
$$

There must exist values of ‘ r ’ such that the summation in equation(4.3) is exactly summable, the values of ‘r’ for which function is summable is called *region of convergence* ROC. The region of convergence for the first sum consists of all the points on a circle of some radius r_1, where $r_1 < \infty$. For the second sum in the equation there must exist values of r large enough such that the product sequence $v[n]/r^n$, $0 \leq n < \infty$ is absolutely summable. $\rightarrow$ ROC of the second term consists of all points outside the circle of radius $r > r_2$. For general, two sided signals $v[n]$, convergence is needed for both $n \leq 0$ and $n \geq 0$.

$$
r_2 < |z| < r_1 \tag{4.4}
$$

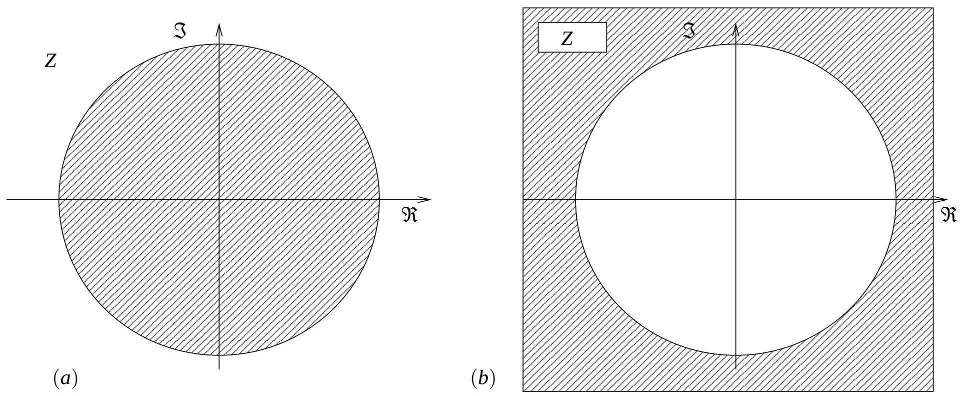
Consequences:

① r_1 and r_2 must exist.

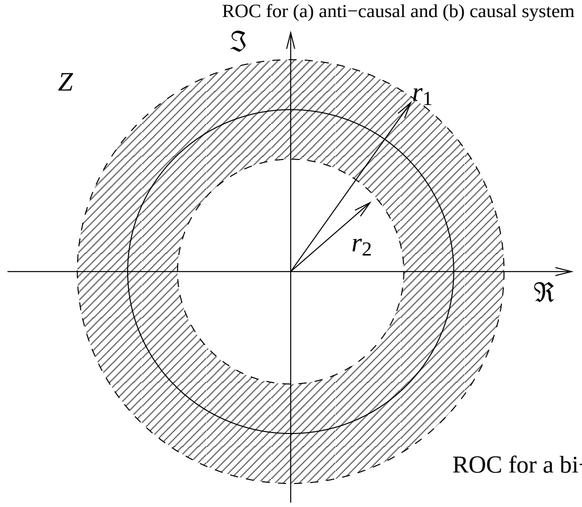
② $r_2 < r_1$ is necessary.

Then: $V(z)$ exists according to (4.1) as the Z-transform of $v[n]$ in a "ring of convergence"

i.e: for z-values in this ring acc. to (4.4) $V(z)$ corresponds to the sum (4.1) $V(z)$ may be evaluated at other points of the z-plane.



ROC for (a) anti−causal and (b) causal system



ROC for a bi−directional signal

Example

1. Bi-directional sequence:

$$x[n] \quad = \quad \sum_{n=0}^{\infty} a^n u[n] + b^n u(-n-1) \tag{4.5}$$

Solution: from (4.1) we have:

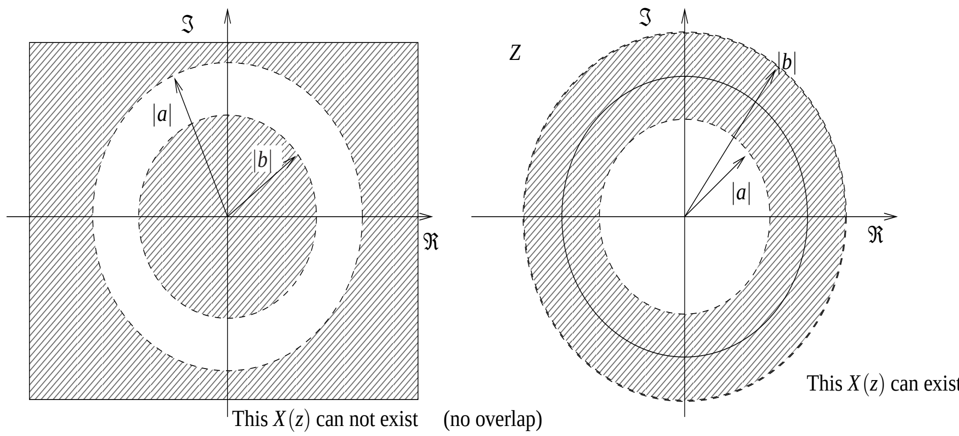
$$X(z) \quad = \quad \sum_{n=0}^{\infty} a^n z^{-n} + \sum_{n=-\infty}^{-1} b^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n + \sum_{l=1}^{\infty} (b^{-1}z)^l$$

The first term in the expression converges if $|az^{-1}| < 1$ or $|z| > |a|$. The second term converges if $|b^{-1}z| < 1$ *or* $|z| < |b|$

So in determining the region of convergence for $X(z)$ we have to consider two different cases.

i. **Case 1**: In the case the two ROC above do not overlap, we can not find values of z for which $X(z)$ converges.

ii. **Case 2**: In this case there exists a ring in the z-plane, where both power series converge simultaneously.



$$\begin{aligned} X(z) &= \frac{1}{1-az^{-1}} - \frac{1}{1-bz^{-1}} \\ &= \frac{b-a}{a+b-z-abz^{-1}} \end{aligned}$$

the ROC of $X(z)$ is $|a| < |z| < |b|$.

2. 1-sided sequence

$$x[n] = a^n u[n] \overset{z}{\Leftrightarrow} X(z) = \frac{1}{1-az^{-1}} \qquad \text{ROC:}|Z| > |a|$$

the ROC in this case is would be one-sided.

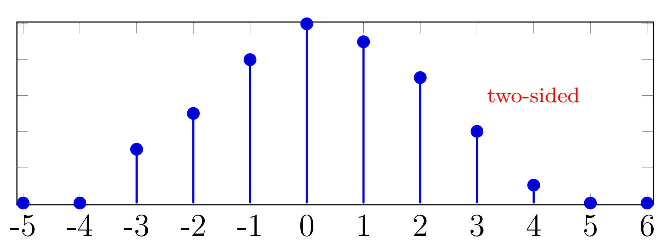


Table 4.1: Common Z-transform Pairs.

Signal $x[n]$	z-Transform $X(z)$	ROC
$\delta[n]$	1	All x
$u[n]$	$\frac{1}{1-z^{-1}}$	$\|z\| > 1$
$au[n]$	$\frac{a}{1-z^{-1}}$	$\|z\| > 1$
$nu[n]$	$\frac{z}{(z-1)^2}$	$\|z\| > 1$
$a^n u[n]$	$\frac{1}{1-az^{-1}}$	$\|z\| > \|a\|$
$na^n u[n]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| > \|a\|$
$-a^n u[-n-1]$	$\frac{az^{-1}}{(1-az^{-1})^2}$	$\|z\| < \|a\|$
$\cos\Omega_0 n u[n]$	$\frac{1-z^{-1}\cos\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}$	$\|z\| > 1$
$\sin\Omega_0 n u[n]$	$\frac{z^{-1}\sin\Omega_0}{1-2z^{-1}\cos\Omega_0+z^{-2}}$	$\|z\| > 1$

4.1.2 Periodicities

From (4.1), $v(z)$ is a-periodic in terms of general z-transform, however from $z = re^{j\Omega}$.

$$\begin{aligned} V(z) &= V(re^{j\Omega}) = \sum_{n=-\infty}^{+\infty} v[n]r^{-n} = \sum_{n=-\infty}^{+\infty} v[n]r^{-n} \underbrace{e^{-j(\Omega+\lambda\cdot 2\pi)n}}_{e^{-j\Omega n}.\underbrace{e^{-j2\pi\lambda}}_{\equiv 1}} \\ &= V(r\,e^{(\quad j\Omega+2\pi\lambda)}) \end{aligned} \tag{4.6}$$

for any fixed value of r.

$V(z)$ is periodic with regard to the angle Ω on a circle with radius r (which includes $r = 1$, unit circle!)

4.2 Inverse Z-transform

Equation (4.2) defines the inversion of z-transform.

$$v[n] = \frac{1}{j2\pi} \oint V(z) z^{n-1} dz$$

The integral is a contour integral over a closed path C that encloses the origin and lies within region of convergence of $V(z)$. C can be taken as a circle in ROC of $V(z)$ in the z-plane.

Possible methods of evaluating z-transform are,

① Direct evaluation of equation (4.2).

② Expansion into a series of terms, in variables z and z^{-1}.

③ Partial-fraction expansion and table lookup.

Here only 3 is treated.

A rational function for which the order of numerator polynomial M is greater then denominator N is called improper function i.e $M > N$. Any improper function can be made proper using long division. Partial-fraction expansion is particularly useful if $V(z)$ is a proper rational function and without loss of generality we consider that $a_0 = 1$

$$V(z) = \frac{\sum_{n=0}^{M} b_n z^{-n}}{\sum_{n=0}^{N} a_n z^{-n}} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2} + \cdots + b_M z^{-M}}{1 + a_1 z^{-1} + a_2 z^{-2} + \cdots + b_N z^{-N}} \tag{4.7}$$

Since numerator and denominator are both polynomials in z, $V(z)$ can be expressed in factored form. The task is to perform partial-fraction expansion to express (4.7) as a sum or product of simple fractions.

$$V(z) = \frac{N(z)}{D(z)} = \frac{b_0}{a_0} z^{-M+N} \frac{(z - z_{0_1})(z - z_{0_2}) \ldots (z - z_{0_M})}{(z - z_{\infty_1})(z - z_{\infty_2}) \ldots (z - z_{\infty_N})} \tag{4.8}$$

This representation can be directly used to show $V(z)$ on a pole-zero plot. Pole zero plot and ROC information can completely describe the system features such as causality, stability, amplitude and phase characteristics. For examples (see exercises).

4.3 Characteristics Rules,Theorems

a. Linearity: as in the Fourier transforms see sec(3.2)

b. Modulation:

$$\begin{aligned} \text{given:} \quad v[n] \quad & \circ ------ V(z) \\ v_1[n] \quad & = v[n] \cdot z_0^n \quad \text{“\underline{Generalized complex modulation}”} \end{aligned}$$

then from (4.1)

$$v_1[n] \quad \circ ------ V(\frac{z}{z_0}) \tag{4.9}$$

c. Shift: Given:

$$\begin{aligned} v[n] \quad & \circ ------ V(z) \\ v_1[n] \quad & = v[n-n_0] \quad \text{“\underline{Delayed / Advanced complex signal}”} \end{aligned}$$

then from (4.1)

$$v_1[n] \circ ------ V(z) \cdot z^{-n_0} \tag{4.10}$$

$\triangleq$ factor $e^{-j\Omega n_0}$, i.e. Linear phase, if $z = e^{j\Omega}$ as in (3.9).

$\Rightarrow$a factor of z^{-n_0} describes a shift (delay) by n_0 (“Shift operator”)

d. Scaling: If

$$x[n] \overset{z}{\longleftrightarrow} X(z), \qquad \text{ROC:} r_1 < |z| < r_2 \tag{4.11}$$

$$a^n x[n] \overset{z}{\longleftrightarrow} X(a^{-1}z), \qquad \text{ROC:} |a| r_1 < |z| < |a| r_2 \tag{4.12}$$

for any constant, real or complex a. Proof: From definition (4.1)

$$\begin{aligned} Z\{a^n x[n]\} &= \sum_{n=-\infty}^{\infty} a^n x[n] z^{-n} \sum_{n=-\infty}^{\infty} x[n](a^{-1}z)^{-n} \\ &= X(a^{-1}z) \end{aligned}$$

e. Convolution & Product:

with definition of convolution in (3.16) we find from (4.1)

$$\begin{aligned} v_{1,2}[n] \quad & \circ --- V_{1,2}(z) \\ Y(z) \quad & = \sum_{n=-\infty}^{+\infty} \Big[\sum_{n=-\infty}^{+\infty} v_1[n] v_2[n-k] \Big] z^{-n} \\ v_1[n] * v_2[n] \quad & = y[n] \circ -- Y(z) = V_1(z) \cdot V_2(z) \triangleq (3.18) \text{ for } z = e^{j\Omega} \end{aligned}$$

ROC is atleast the intersection of $V_1(z)$ and $V_2(z)$

f. Time reversal: If

$$x[n] \longleftrightarrow X(z), \qquad \text{ROC:} r_1 < |z| < r_2$$

then

$$x[-n] \longleftrightarrow X(z^{-1}), \qquad \text{ROC:} \frac{1}{r_2} < |z| < \frac{1}{r_1} \tag{4.13}$$

Proof: From the definition (4.1), we have

$$Z\{x[-n]\} = \sum_{n=-\infty}^{+\infty} x[-n]z^{-n} = \sum_{l=-\infty}^{+\infty} x[l](z^{-1})^{-l} = X(z^{-1}) \tag{4.14}$$

where the change of variable $l = -n$ is made. The ROC of $X(z^{-1})$ is

$$r_1 < |z^{-1}| < r_2 \quad \text{or equivalently} \quad \frac{1}{r_2} < |z| < \frac{1}{r_1} \tag{4.15}$$

$\Rightarrow$ the ROC of $x[n]$ is the inverse of the $x[-n]$

g. Difference & Differentiation:

Given:

$$v[n] \circ - - - - V(z)$$

using the shift theorem(4.10)

Difference:

$$y[n] = v[n] - v[n-1] \circ - - - - Y(z) = V(z)[1 - z^{-1}] \tag{4.16}$$

which compounds to (3.9) for $z = e^{j\Omega}$

Differentiation:

$$-z[\frac{dV(z)}{dz}] - - - - - - \circ\, n \cdot v[n] \tag{4.17}$$

The properties of the Z transform are tabulated in the table below

One-sided z-Transforms

Primarily: Generally, two-sided definition(4.1) is applicable to one-sided signals (with simpler convergence conditions). Possible: Definition of a one-sided (right hand sided) transformation independently of signal values on the other side

$$Z_R\{v[n]\} = \sum_{n=0}^{\infty} v[n]z^{-n} \tag{4.18}$$

Table 4.2: Properties of the Z-Transform.

Property	Time Domain	z-domain	Roc
Notation	$x[n]$	$X(z)$	ROC: $r_2 < \|z\| < r_1$
	$x_1[n]$	$X_1(z)$	ROC_1
	$x_2[n]$	$X_2(z)$	ROC_2
Linearity	$a_1x_1[n] + a_2x_2[n]$	$a_1X_1(z) + a_2X_2(z)$	At least the intersection of ROC_1 and ROC_2.
Time shifting	$x[n-n_0]$	$z^{-n_0}X(z)$	That of $X(z)$ except z=0 if k>0 and z=∞ if k<0.
Scaling	$a^n x[n]$	$X(a^{-1}z)$	$\|a\|r_2<\|z\|<\|a\|r_1$.
Time reversal	$x[-n]$	$X(z^{-1})$	$\frac{1}{r_1}<\|z\|<\frac{1}{r_2}$.
Conjugation	$x^*[n]$	$X^*(z^*)$	ROC
Real part	$\Re\{x[n]\}$	$\frac{1}{2}\Big[X(z)+X^*(z^*)\Big]$	Includes ROC
Real part	$\Re\{x[n]\}$	$\frac{1}{2}j\Big[X(z)-X^*(z^*)\Big]$	Includes ROC
Differentiation in time domain	$nx[n]$	$-z\frac{dX(z)}{dz}$	$r_2<\|z\|<r_1$.
Convolution	$x_1[n]*x_2[n]$	$X_1(z)X_2(z)$	At least the intersection of ROC_1 and ROC_2.
Correlation	$r_{x_1x_2}(l) = x_1(l)*x_1(-l)$	$R_{x_1x_2}(z)$=$X_1(z)X_2(z^{-1})$	At least the intersection of ROC of $X_1(z)$ and $X_2(z^{-1})$.
Multiplication	$x_1[n]\cdot x_2[n]$	$\frac{1}{2\pi j}\oint_c X_1(v)X_2(\frac{z}{v})v^{-1}dv$	At least $r_{1l}r_{2l}<\|z\|<r_{1u}r_{2u}$
Parseval's Theorem	$\sum_{n=-\infty}^{+\infty} x_1[n]x_2^*[n]$	$\frac{1}{2\pi j}\oint_c X_1(v)X_2(\frac{1}{v})v^{-1}dv$	

with same inversion as in (4.2). Characteristics, rules and theorems remain, unchanged too - with simpler convergence consideration($\rightarrow$ circle rather then a ring). Additional considerations with the shift theorem, in the case of shifts "to the left".

$$\begin{aligned} \text{assumed :} \quad v[n] &\equiv 0 \quad \text{for} \quad n < 0 \quad \text{(Right handed sequence)} \\ Z_R\{v[n]\} &= V_R(z) \overset{4.1}{\equiv} Z\{v[n]\} = V(z) \\ \text{Shift:} \quad y[n] &= v[n+n_0] \quad n_0 > 0 \end{aligned}$$

$$\begin{aligned} Y_R(Z) = \sum_{n=0}^{\infty} v[n+n_0]z^{-n} = &\begin{cases} v[n_0]z^0 + v[n_0+1]z^{-1} + v[n_0+2]z^{-2} + \dots \\ -\underbrace{v[0]z^{n_0} + v[1]z^{n_0-1} + \dots + v[n_0-1]z}_{-\sum_{n=0}^{n_0-1} v[n]z^{n_0-n}} \end{cases} \\ = &\quad z^{+n_0}\underbrace{\sum_{n=0}^{\infty} v[n]z^{-n}}_{V_R(z)} - \sum_{n=0}^{n_0-1} v[n]z^{n_0-n} \\ = &\quad z^{n_0} \cdot V_R(z) - z^{n_0}\sum_{n=0}^{n_0-1} v[n]z^{-n} \end{aligned}$$

Important Definitions

For a transform specified by a ratio of polynomials defined as

$$X(z) = \frac{B(z)}{A(z)} = \frac{b_0 + b_1 z^{-1} + \cdots + b_M z^{-M}}{a_0 + a_1 z^{-1} + \cdots + a_N z^{-N}} \tag{4.19}$$

1. The rational function of (4.19) is called *proper* if $a_N \neq 0$ and $M < N$. Which implies that the number of finite zeros is less than finite poles.

2. The rational function of (4.19) is called *improper* if $a_N \neq 0$ and $M < N$. Improper rational function can always be written as the sum of a polynomial and a proper rational function.

Polynomial division and partial fraction expansion.

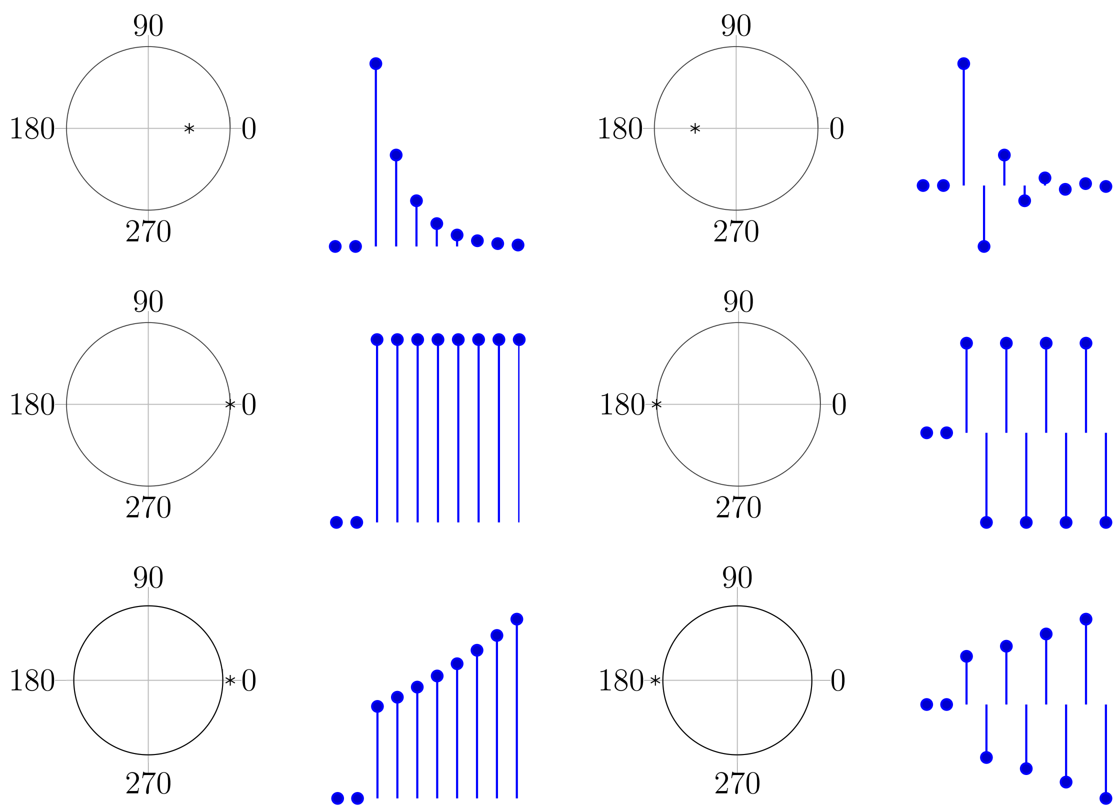


Figure 4.2: Pole zero plots and corresponding impulse response for a first order system.

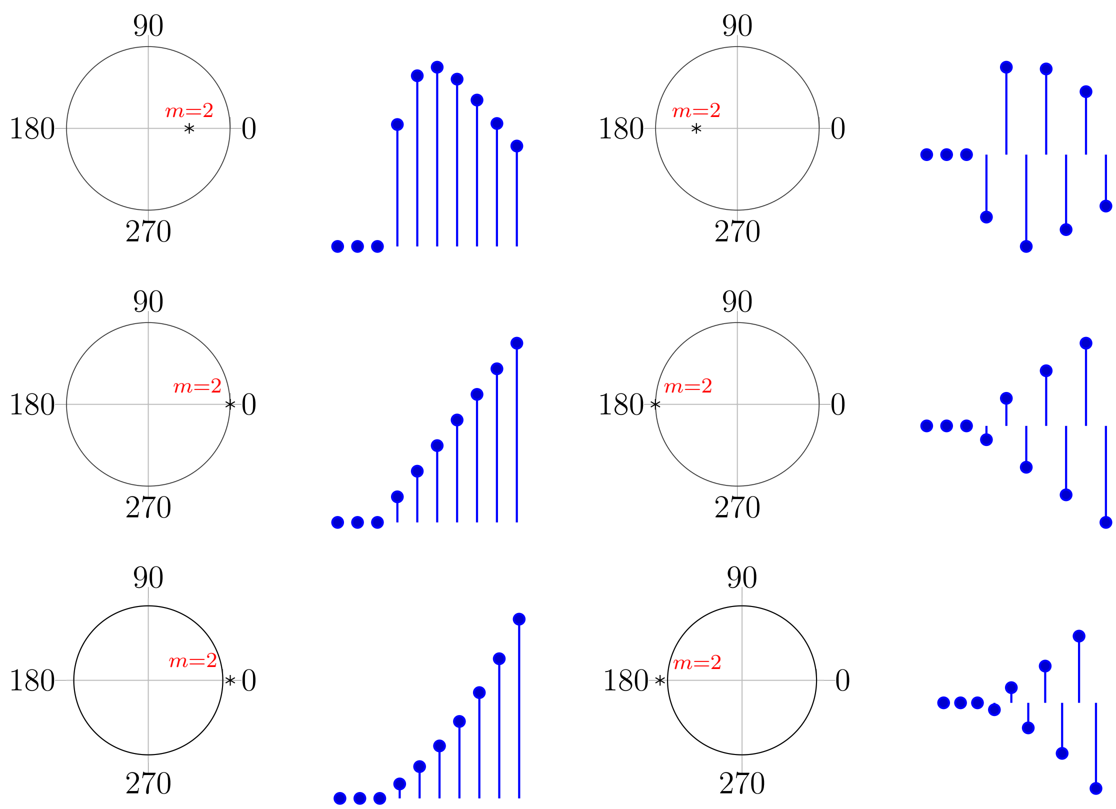


Figure 4.3: Pole zero plots and corresponding impulse response for a second order system.

Chapter 5

Efficient Implementation

The objective of this chapter is to introduce Discrete Fourier transform and to study the computationally efficient techniques.

- Ⓐ Introduction
- Ⓑ Discrete Fourier transforms
- Ⓒ The Leakage effect
- Ⓓ Circular Convolution
- Ⓔ Decimation in Time/Frequency schemes
- Ⓕ Special Topics.

Introduction

5.1 Discrete Fourier Transforms

5.1.1 Definitions

Definitions of discrete Fourier transforms

$$v[n] \circ - - - - - - - \bullet V_M[k] = \sum_{n=0}^{M-1} v[n] W_M^{nk} \tag{5.1}$$

Further Reading!

Some text goes here just to illustrate the command

$$V_M[k] \bullet\!-\!-\!-\!-\!-\!-\!-\circ v[n] = \frac{1}{M}\sum_{k=0}^{M-1} V[k]W_M^{-nk} \tag{5.2}$$

where $W_M = e^{-j2\pi/M}$, M is the no. of DFT points

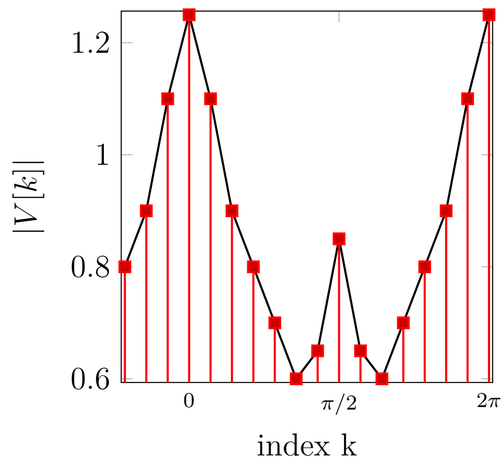


Figure 5.1: Illustration of discrete Fourier transform

5.1.2 Background:

If we evaluate the DTFT from (3.1) at $\Omega{=}2\pi k/N$, we obtain

$$X\left(\frac{2\pi}{N}k\right) = \sum_{n=-\infty}^{+\infty} x[n]e^{-j2\pi kn/N} \qquad k{=}0,1\dots,N{-}1 \tag{5.3}$$

The summation in (5.3) can be subdivided into an infinite number of summations, where each sum contains N terms, i.e.

$$\begin{aligned} X\Big(\frac{2\pi}{N}k\Big) &= \dots + \sum_{n=-N}^{-1} x[n]e^{-j2\pi kn/N} + \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \\ &+ \sum_{n=N}^{2N+1} x[n]e^{-j2\pi kn/N} + \dots \\ &= \sum_{l=-\infty}^{+\infty}\sum_{n=lN}^{lN+N-1} x[n]e^{-j2\pi kn/N} \end{aligned} \tag{5.4}$$

Further Reading!

Some text goes here just to illustrate the command

If we change the index in the inner summation from n to $n - lN$ and interchange the order of summation, we obtain the results

$$X\Big(\frac{2\pi}{N}k\Big) = \sum_{n=0}^{N-1}\left[\sum_{l=-\infty}^{+\infty} x(n-lN)\right]e^{-j2\pi kn/N} \tag{5.5}$$

for the values k=$0, 1, 2, \dots, N-1$, the signal

$$x_p[n] = \sum_{l=-\infty}^{+\infty} x[n-lN] \tag{5.6}$$

which is just the periodic repetition of $x[n]$ every N samples, is clearly periodic with fundamental period N. Consequently, it can be expanded in a Fourier series as

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} c_k e^{j2\pi kn/N}, \qquad n{=}0, 1, \dots, N-1 \tag{5.7}$$

with Fourier coefficients

$$c_k = \frac{1}{N}\sum_{n=0}^{N-1} x_p[n] e^{-j2\pi kn/N} \qquad k{=}0, 1, \dots, N-1 \tag{5.8}$$

Therefore

$$x_p[n] = \frac{1}{N}\sum_{k=0}^{N-1} X\Big(\frac{2\pi}{N}k\Big) e^{j2\pi kn/N} \qquad n{=}0, 1, \cdots, N-1 \tag{5.9}$$

The equation (5.9) provides the reconstruction of the periodic signal from $x_p[n]$ from the samples of $X[\Omega]$. Now lets try to obtain the Fourier series coefficients $X[k]$

$$\sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)rn} = \sum_{n=0}^{N-1} \frac{1}{N} \sum_{k=0}^{N-1} X[n] e^{j(2\pi/N)(k-r)n} \tag{5.10}$$

After interchanging the order of the summation on the right hand side in (5.10) we have

$$\sum_{n=0}^{N-1} x[n] e^{-j(2\pi/N)rn} = \sum_{k=0}^{N-1} X[k] \Bigg[\frac{1}{N} \sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n}\Bigg] \tag{5.11}$$

The following identity expresses the orthogonality of the complex exponentials

$$\frac{1}{N}\sum_{n=0}^{N-1} e^{j(2\pi/N)(k-r)n} = \begin{cases} 1, & k-r = mN \text{ } m \text{ is an integer,} \\ 0, & \text{otherwise.} \end{cases} \tag{5.12}$$

Further Reading!

Some text goes here just to illustrate the command

This identity can be easily assessed from fig. 5.2.

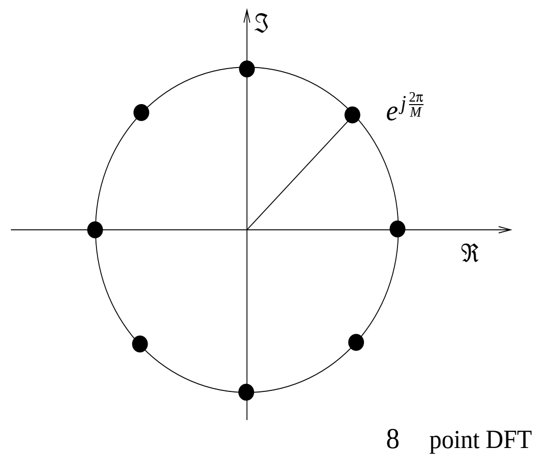


Figure 5.2: Illustration of 8 point DFT circle.

5.2 Properties

a. Periodicity:

$$\text{If } v_M[n] \circ\!-\!-\!-\!\bullet\, V_M[k] \quad \text{then}$$
$$v[n+M] = v[n] \quad \forall\ n$$
$$V[k+M] = V[k] \quad \forall\ \mu$$

This property is evident from (5.1) and (5.2). $\Rightarrow$ M point DFT of a finite length sequence $v[n]$ is equivalent to M point DFT of a periodic sequence $v_p[n]$ with period M and,

$$v_p[n] = \sum_{l=-\infty}^{\infty} v[n-lM] \tag{5.13}$$

$\Rightarrow$ like DFS, DFT assumes the signal to infinite and periodic, the coefficients are calculated over one interval. *Relation between DFS and DFT should be explored further.*

b. Linearity: Linearity property holds for DFT:

$$\begin{aligned} \text{if} \quad v_1[n] &\circ \; - - - \bullet \, V_1[k] \quad \text{and} \\ v_2[n] &\circ \; - - - \bullet \, V_2[k] \\ \text{then} \quad \alpha v_1[n] + \beta v_2[n] &\circ \; - - - \bullet \, \alpha V_1[k] + \beta V_2[k] \end{aligned} \tag{5.14}$$

If $v_1[n]$ has length M_1 and $v_2[n]$ has length M_2, then maximum length of $v[n]$ is $\max[M_1, M_2]$, which means shorter sequence must be appended with additional zeros ($\Rightarrow$ Zero-padding).

c. Symmetry: Any given sequence can be decomposed into even and odd components, a sequence is said to be
even: if $v([n] = v[-n] = v[M-n]$
odd: if $v[n] = -v[-n] = -v[M-n]$

$$\begin{aligned} \text{If} \quad v[n] \in \mathbb{C} \text{ then} & \\ v^*[n] &\circ \; - - - \bullet \, V^*[M-k] \\ \text{and} & \\ v^*[M-n] &\circ \; - - - \bullet \, V^*[k] \end{aligned}$$

A complex sequence $v[n]$ can be decomposed in to even and odd components

$$v_{\text{even}}[n] = \frac{1}{2}[v[n] + v^*[n]] \tag{5.15}$$

$$v_{\text{odd}}[n] = \frac{1}{2}[v[n] - v^*[n]] \tag{5.16}$$

For a real valued sequence $v[M-n] = v^*[n] = v[-n] \Rightarrow |V[M-k]| = |V[k]|$ and $\angle V[M-k] = -\angle V[k]$

d. Shift & Modulation:

$$\begin{aligned} \text{if} \qquad v[n] &\circ \; - - - \bullet \, V[k] \quad \text{then} \\ v[n-l] &\circ \; - - - \bullet \, V[k] e^{\frac{-j2\pi lk}{M}} \end{aligned} \tag{5.17}$$

and

$$v[n] e^{\frac{j2\pi n k_0}{M}} \circ \; - - - \bullet \, V[k-k_0] \tag{5.18}$$

Table 5.1: Symmetry properties of DFT

N-Point Sequence $x[n]$ $0 \leq n \leq N-1$	DFT
$x[n]$	$X[k]$
$x^*[n]$	$X^*[N-k]$
$x^*[N-n]$	$X^*[k]$
$x_{\mathrm{R}}[n]$	$X_{ce}[k] = \frac{1}{2}[X[k] + X^*[n-k]]$
$jx_{\mathrm{I}}[n]$	$X_{co}[k] = \frac{1}{2}[X[k] - X^*[n-k]]$
$x_{ce}[n] = \frac{1}{2}[x[n] + x^*[N-n]]$	$X_{\mathrm{R}}[K]$
Any Real signal	$X[k] = X^*[k]$
$x[n]$	$X_{\mathrm{R}}[K] = X_{\mathrm{R}}[N-k]$
	$X_{\mathrm{I}}[K] = -X_{\mathrm{I}}[N-k]$
	$\lvert X[k]\rvert = \lvert X[N-k]\rvert$
	$\angle X[k] = -\angle X[N-k]$

e. Time reversal:

$$\text{if} \qquad v(k) \circ - - - \bullet\, V[k] \quad \text{then}$$

$$v[-n] = v[M-n] \circ - - - \bullet\, V[M-k]$$

$$DFT\{v(M-k)\} = \sum_{k=0}^{M-1} v[M-n] e^{-j\frac{2\pi nk}{M}} \tag{5.19}$$

change index from n to $m = M - n$,then

$$= \sum_{m=0}^{M-1} v[m] e^{-j\frac{2\pi(m-M)k}{M}} = \sum_{m=0}^{M-1} v(m) e^{j\frac{2\pi mk}{M}}$$

$$V(M-k) = \sum_{m=0}^{M-1} v(m) e^{-j\frac{2\pi m(M-k)}{M}} \tag{5.20}$$

Further Reading!

Some text goes here just to illustrate the command

5.2.1 Linear and circular convolution

linear convolution of two sequences $v_1(k)$ and $v_2(k)$,$k \in \mathbb{Z}$

$$\begin{aligned} y_l[n] &= v_1[n] * v_2[n] = v_2[n] * v_1[n] \\ &= \sum_{\kappa=-\infty}^{+\infty} v_1[\kappa] v_2[n-\kappa] = \sum_{\kappa=-\infty}^{+\infty} v_2[\kappa] v_1[n-\kappa] \end{aligned} \tag{5.21}$$

circular convolution of two periodic sequences $v_1[n]$ and $v_2[n]$, $n = \{0, 1, \ldots, M_{1,2} - 1\}$ with the same period $M_1 = M_2 = M$ and $k_0 \in \mathbb{Z}$

$$\begin{aligned} y_c[n] &= v_1[n] \circledast v_2[n] = v_2[n] \circledast v_1[n] \\ &= \sum_{\kappa=-n_0}^{n_0+M-1} v_1(\kappa) v_2(n-\kappa) = \sum_{\kappa=-n_0}^{n_0+M-1} v_2(\kappa) v_1(n-\kappa) \end{aligned} \tag{5.22}$$

5.3 DFT and Circular Convolution

Inverse transform of a finite-length sequence $v[n]$. $k, n = 0, \ldots, M-1$

$$v[n] \circ - - - - - - - \bullet V_M[k] \bullet - - - - - - - \circ v[n] = v[n + \lambda M] \tag{5.23}$$

$\Rightarrow$ DFT of a finite-length sequence and its periodic extension is identical circular convolution property $(n, k = 0, \ldots, M-1)$, for two finite length sequences $v_1[n]$ and $v_2[n]$.

$$y[n] = v_1[n] \circledast v_2[n] \circ - - - - - - - \bullet Y[k] = V_{1_M}[k] \cdot V_{2_M}[k] \tag{5.24}$$

Proof:

$$\begin{aligned} \text{IDFT of} \quad y[n] &= \frac{1}{M} \sum_{k=0}^{M-1} Y[k] W_M^{-nk} \\ &= \frac{1}{M} \sum_{k=0}^{M-1} V_{1_M}[k] \cdot V_{2_M}[k] W_M^{-nk} \end{aligned}$$

Inserting DFT definition of V_{1_M} and V_{2_M}

$$\begin{aligned} y[n] &= \frac{1}{M} \sum_{k=0}^{M-1} \Big[\sum_{n=0}^{M-1} v_1[n] W_M^{-kn} \Big] \Big[\sum_{l=0}^{M-1} v_2(l) W_M^{-kl} \Big] W_M^{-nk} \\ &= \frac{1}{M} \sum_{n=0}^{M-1} v_1(n) \sum_{l=0}^{M-1} v_2(l) \Big[\sum_{k=0}^{M-1} W_M^{-k(k-n-l)} \Big] \end{aligned} \tag{5.25}$$

Terms in bracket summation over the unit circle

$$\sum_{k=0}^{M-1} W_M^{-k(k-n-l)} = \begin{cases} M\ for \quad l = k-n+\lambda M, \lambda \in Z \\ 0, otherwise \end{cases} \tag{5.26}$$

Substituting (5.26) in (5.25) yields

$$\begin{aligned} y[n] &= \frac{1}{M}\sum_{n=0}^{M-1} v_1[n] \underbrace{\sum_{\lambda=-\infty}^{+\infty} v_2[k-n+\lambda M]}_{v_2([k-n])_M \text{(periodic extension)}} \\ &= \sum_{n=0}^{M-1} v_1(n) v_2([k-n])_M && (5.27) \\ &= v_1[n] \circledast v_2[n] && (5.28) \end{aligned}$$

5.4 Frequency analysis of stationary signals

Spectral analysis of analog signal

• Antialiasing lowpass filtering and sampling with $\omega_s \geq \omega_m$.

• For practical purposes (delay, complexity), limitation of the signal duration to the time interval $T_0 = LT$(where L is the number of samples and T is sampling interval).

Limitation of signal duration to T_0 can be modelled as multiplication of input signal $v[n]$ with a rectangular window $w[n]$

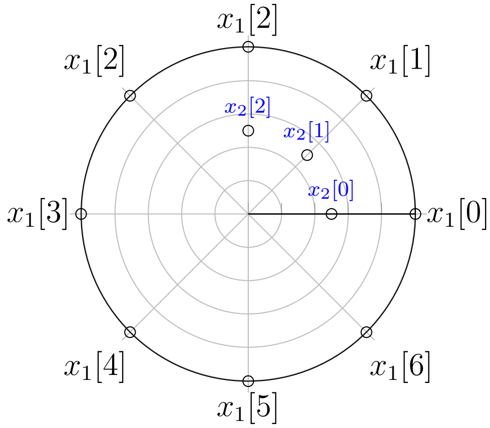
$$\tilde{v}[n] = v[n] \cdot w[n] \quad \text{with} \quad w[n] = \begin{cases} 1, & 0 \leq n \leq L-1 \\ 0, & \text{otherwise} \end{cases} \tag{5.29}$$

Suppose that the input sequence just consists of a single sinusoid i.e. $v[n] = cos(\Omega_0 n)$, the Fourier transform is

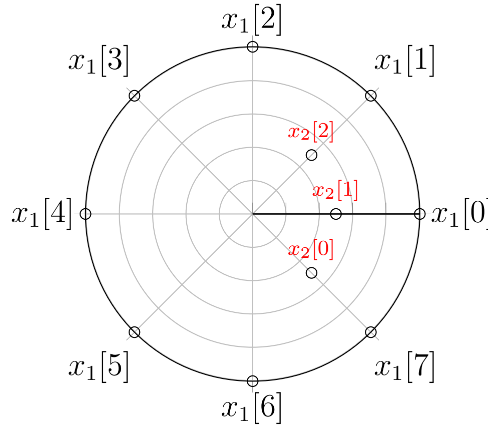
$$V(e^{j\Omega}) = \pi(\delta_0(\Omega - \Omega_0) + \delta_0(\Omega + \Omega_0)) \tag{5.30}$$

For window function $w[n]$ the Fourier transform can be obtained as

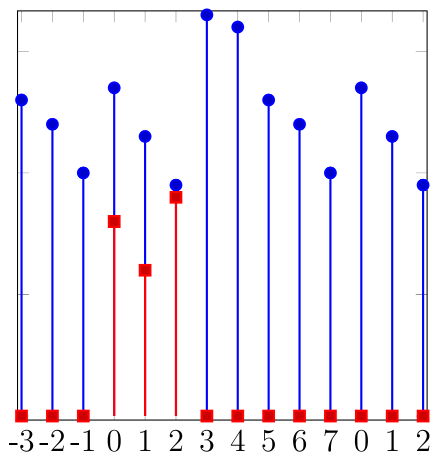
$$W(e^{j\Omega}) = \sum_{n=0}^{L-1} e^{-j\Omega n} = \frac{1 - e^{-j\Omega L}}{e^{-j\Omega}} = e^{-j\frac{L-1}{2}\Omega} \frac{\sin(\frac{L}{2})\Omega}{\sin(\frac{\Omega}{2})} \tag{5.31}$$



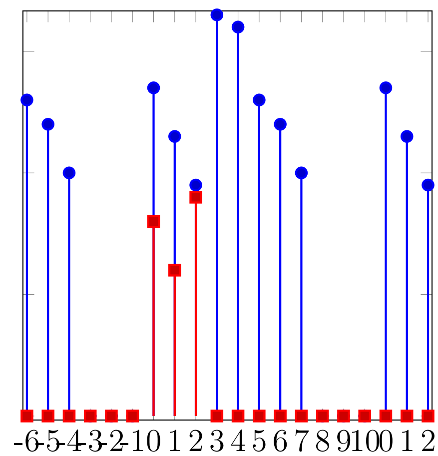
(a) Illustration of cyclic convolution process.



(b) Illustration of cyclic convolution process.



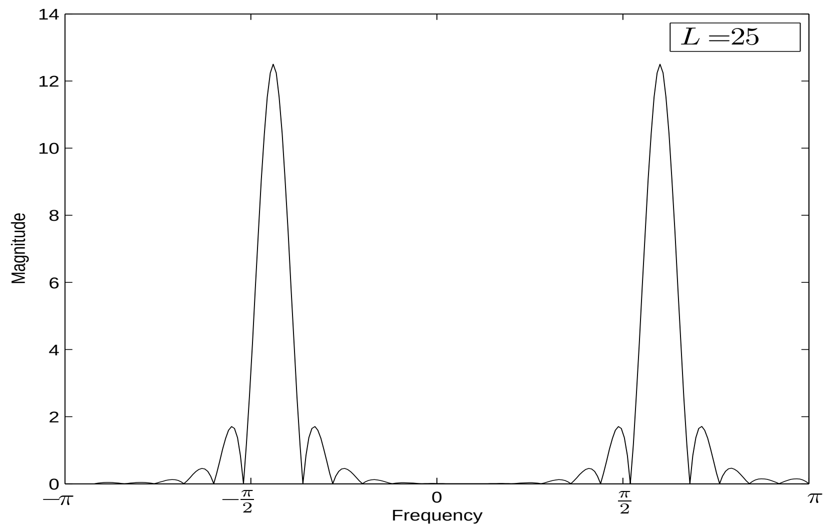
(c) Illustration of cyclic convolution process.



(d) Illustration of linear convolution through cyclic convolution.

We finally have

$$
\begin{aligned}
\hat{V}(e^{j\Omega}) &= \frac{1}{2\pi}\Big[V(e^{j\Omega}) \circledast W(e^{j\Omega})\Big] \\
&= \frac{1}{2}\Big[W(e^{j(\Omega-\Omega_0)}) + W(e^{j(\Omega+\Omega_0)})\Big] \qquad (5.32)
\end{aligned}
$$



Windowed spectrum of $V(e^{j\Omega})$ is not localized to one frequency instead it is spread out to over whole frequency range.

$\Rightarrow$ **spectral leakage**

The first zero-crossing of $W(e^{j\Omega})$ occurs at $\Omega_z = \pm 2\pi/L$.

•The larger the number of sampling points L (thus also the width of rectangular window) the smaller becomes Ω_z(and thus also the main lobe of spectrum).

• $\Rightarrow$Decreasing the frequency resolution leads to an increase of time resolution and vice versa (duality of time and frequency).

In practice DFT is used in order to obtain a sampled representation of spectrum according to $\hat{V}(e^{j\Omega k})$, $k = 0, \ldots, M-1$.

Special case:

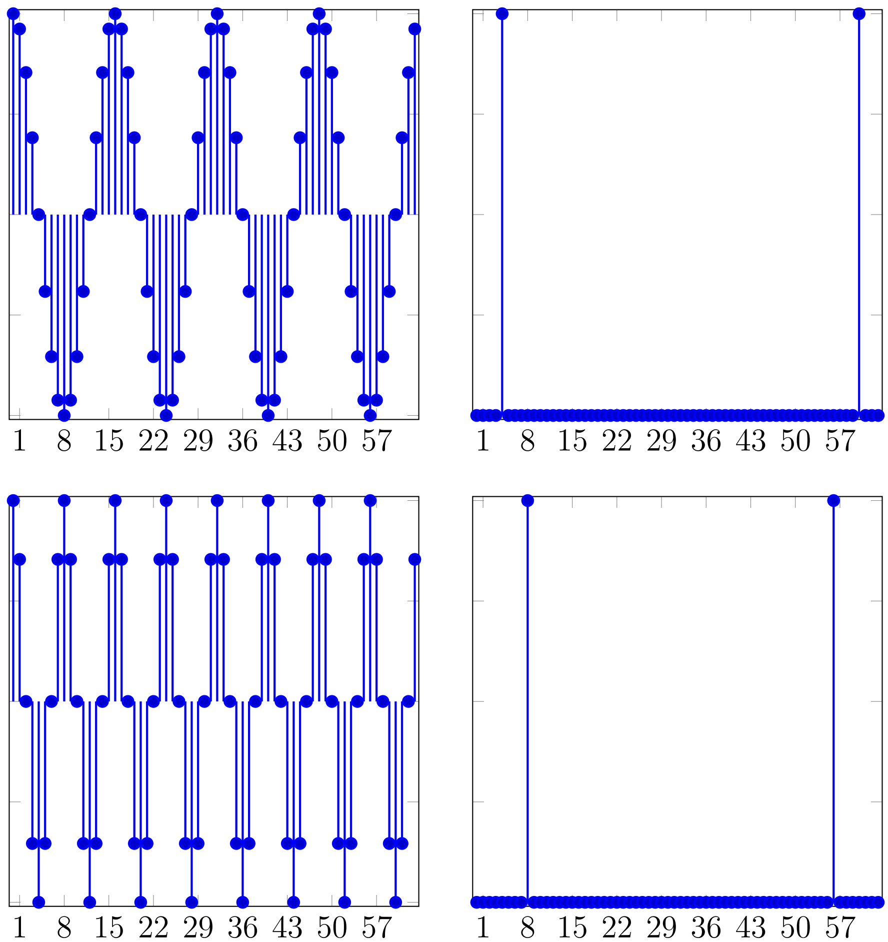
$M = L$ and $\Omega_0 = \dfrac{2\pi}{M}\nu, \nu = 0, \ldots, M-1$

Then the Fourier transform is exactly zero at the sampled frequencies Ω_μ except when $\mu = \nu$

Example:$M = 64$ samples, $k = 0, \ldots, M-1$ and rectangular window $w[n]$.

$$v_0[n] = \cos\left[5\frac{2\pi}{M}n\right], v_1[n] = \cos\left[\left(5\frac{2\pi}{M} + \frac{\pi}{M}\right)n\right] \tag{5.33}$$

• for $v_0[n] \circ - - - \hat{V}_0(e^{j\Omega_k}) = V_0(e^{j\Omega_k}) \circledast W(e^{j\Omega_k}) = 0$ for $\Omega_k = 2\pi k/M$ for $k = 5$, since Ω_0 is exactly an integer multiple of $2\pi/M$.


1 8 15 22 29 36 43 50 57
1 8 15 22 29 36 43 50 57
1 8 15 22 29 36 43 50 57
1 8 15 22 29 36 43 50 57

⇒ Periodic repetition of $v_0[n]$ leads to a $\underline{\text{ pure cosine sequence }}$.

• for $v_1[n]$ slight increase in $\Omega_0 \neq 2\pi\bar{k}/M$ for $\bar{k} \in Z \Rightarrow \hat{V}_0(e^{j\Omega_k}) \neq 0$ for $\Omega_k = 2\pi k/M$, periodic repetition is not a cosine sequence anymore.

⇒ It is also important to note that when sequence $x[n]$ has a finite duration $L \leq N$, then $x_p[n]$ is simply a periodic repetition of $x[n]$, where $x_p[n]$ over a single period is defined as

$$x_p[n] = \begin{cases} x[n], & 0 \leq n \leq L-1 \\ 0, & 0 \leq n \leq N-1 \end{cases} \tag{5.34}$$

Consequently, the frequency samples $X(2\pi k/N)$, k=$0, 1, \ldots, N-1$, uniquely represent the finite-duration sequence $x[n]$. Since $x[n] {\equiv} x_p[n]$ over a single period (padded by $N{-}L$ zeros).

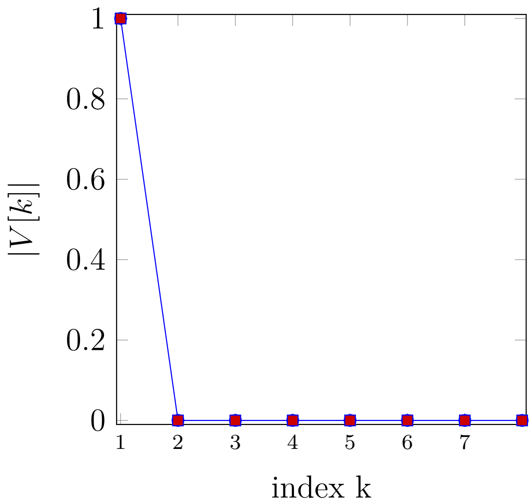
It is important to note that zero padding does not provide any additional information about the spectrum $X(\Omega)$ of the sequence $\{x[n]\}$. The finite-duration sequence $x[n]$of length L [i.e. $x[n] = 0$ for $n < 0$ and $n \geq L$] has a Fourier transform

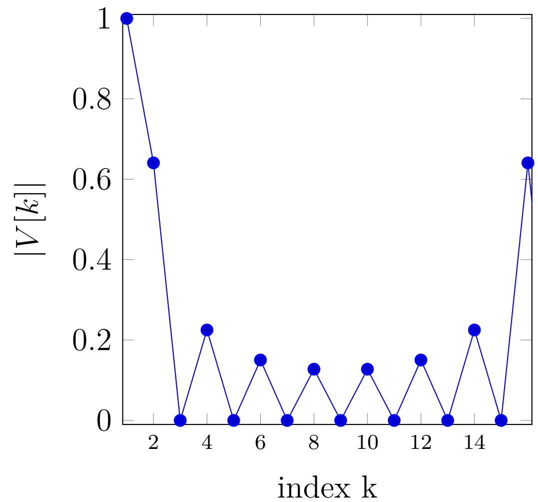
$$X(\Omega) = \sum_{n=0}^{L-1} x[n]e^{-j\Omega n}, \qquad 0 \leq \Omega \leq 2\pi \tag{5.35}$$

where the upper and lower limit in the summation reflects the fact that $x[n]$=0 outside the range $0 \leq n \leq N{-}1$. When we sample $X(\Omega)$ at equally spaced frequencies Ω_k=$2\pi k/N$, k=$0, \ldots, N{-}1$, where $N \geq L$, the resultant samples are

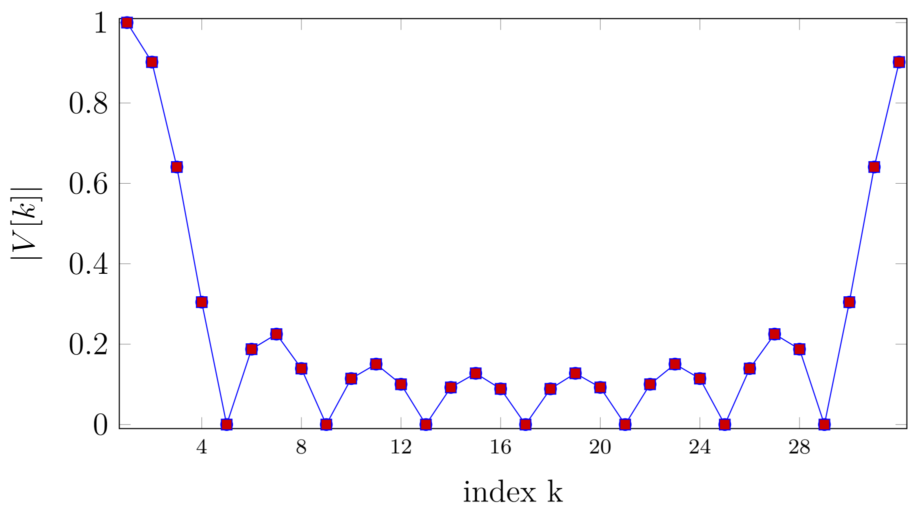
$$\begin{aligned} X[k] &= X\left(\frac{2\pi k}{N}\right) = \sum_{n=0}^{L-1} x[n]e^{-j2\pi kn/N} \\ X[k] &= \sum_{n=0}^{N-1} x[n]e^{-j2\pi kn/N} \qquad k{=}0, 1, \ldots, N-1 \end{aligned} \tag{5.36}$$

where for convenience in the sum has been increased from $L{-}1$ to $N{-}1$ since $x[n]$=0 for $n \geq L$.





(i)



(j)

Figure 5.3: Illustration of discrete Fourier transform

5.5 Efficient Computation of DFT, FFT

Complexity of DFT calculation in (5.1) for $v[n] \in C, V_M[k] \in C$

$$V_M[k] = \underbrace{\sum_{n=0}^{M-1} \underbrace{v[n]W_M^{nk}}_{\text{1 complx multiplication}}}_{\text{M complx mult., M cmplx add.}} \quad \text{for} \quad \underbrace{k : 0, \ldots, M-1}_{\text{M results}} \tag{5.37}$$

$\Rightarrow$ $\underline{M^2}$ overall complex multiplications and $\underline{M^2}$ complex additions.

Remarks:

- 1 complex multiplication $\rightarrow$ 4 real-valued mult.+2 real-valued additions.
- 1 complex addition is 2 real-valued additions.
- A closer evaluation reveals that there are less then M^2 operations:

$-$M values have to be added up$\Rightarrow$ $(M-1)$ additions.

$-$Factors such as $e^{j0}, e^{j\pi\lambda}, e^{\pm j\frac{\pi}{2}\lambda}$ $\Rightarrow$ no real multiplications.

$-$For $k=0$ no multiplications at all.

- Complexity of DFT becomes extremely large for large values of M (i.e. M=1024)

$\Rightarrow$ efficient algorithms necessary for practical implementation. Fast algorithms for DFT, fast Fourier transform (FFT) exploit symmetry and periodicity properties of $W_M^{\mu k}$ such as:

- complex conjugate symmetry: $W_M^{k(M-n)} = W_M^{-kn} = (W_M^{kn})^*$.
- Periodicity in k and n: $W_M^{nk} = W_M^{k(n+M)} = W_M^{(k+M)n}$.

5.6 Decimation-in-time FFT algorithm

Principle:

Decomposing the DFT computation into DFT computations of smaller size by means of decomposing $M-$point input sequence of $v[n]$ into smaller sequences

$\Rightarrow$ *Decimation-in- time*.

Perquisites:

M integer must be a power of 2 i.e.$M = 2^m$, $m = \log_2(M) \in N$ $\Rightarrow$ "*Radix-2*"

Decomposing M-point DFT into two M/2-point transforms

DFT $V[k]$(M is dropped for clarity) can be written as

$$\begin{aligned} V[k] &= \sum_{n=0}^{M-1} v[n] W_M^{nk}, \quad k = 0, \ldots, M-1 \\ &= \sum_{n=0}^{M/2-1} v(2n) W_M^{2nk} + \sum_{n=0}^{M/2-1} v(2n+1) W_M^{\mu(2n+1)} \end{aligned} \tag{5.38}$$

where in the last step $v[n]$ is separated into two $M/2-$point sequences consisting of the even and odd numbered points in $v[n]$.

since

$$W_M^2 = e^{-2j \cdot 2\pi/M} = e^{-j \cdot 2\pi/(M/2)} = W_{M/2}$$

we can write (5.38) as

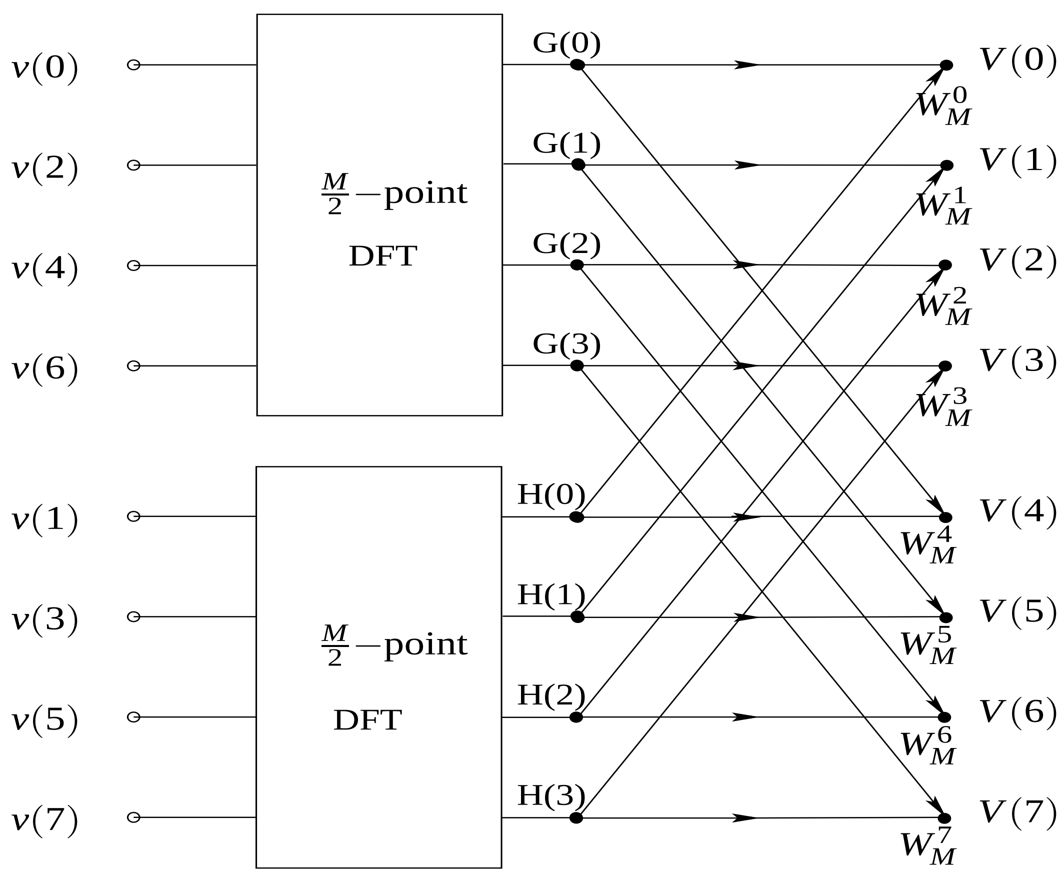
$$= \sum_{n=0}^{M/2-1} v(2n) W_{M/2}^{nk} + W_M^k \sum_{n=0}^{M/2-1} v(2n+1) W_{M/2}^{nk} \quad (5.39)$$

$$= G[k] + W_M^k H[k], \quad k = 0, \ldots, M-1 \quad (5.40)$$

- Each sum in (5.40) represents a $M/2$ point DFT over even and odd-numbered points of $v(n)$, respectively.
- $G[k]$ and $H[k]$ need to be computed over $0, \ldots, M/2-1$ points since both are periodic with period $M/2$.

Further Reading!
Some text goes here just to illustrate the command

signal flowgraph for M=8



Complexity:

2 DFTs of length $M/2 \rightarrow 2 \cdot M^2/4 = M^2/2$ operations + M operations for combining of both DFTs.

$\Rightarrow M + M^2/2$ operations (less than M^2 for $M > 2$).

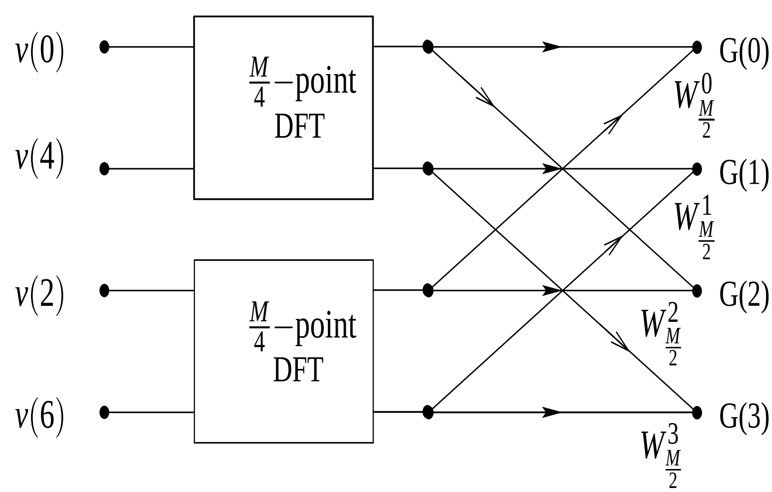
Decomposition in 4 $M/4$-point DFTs

$G[k]$ and $H[k]$ from (5.40) can be written as

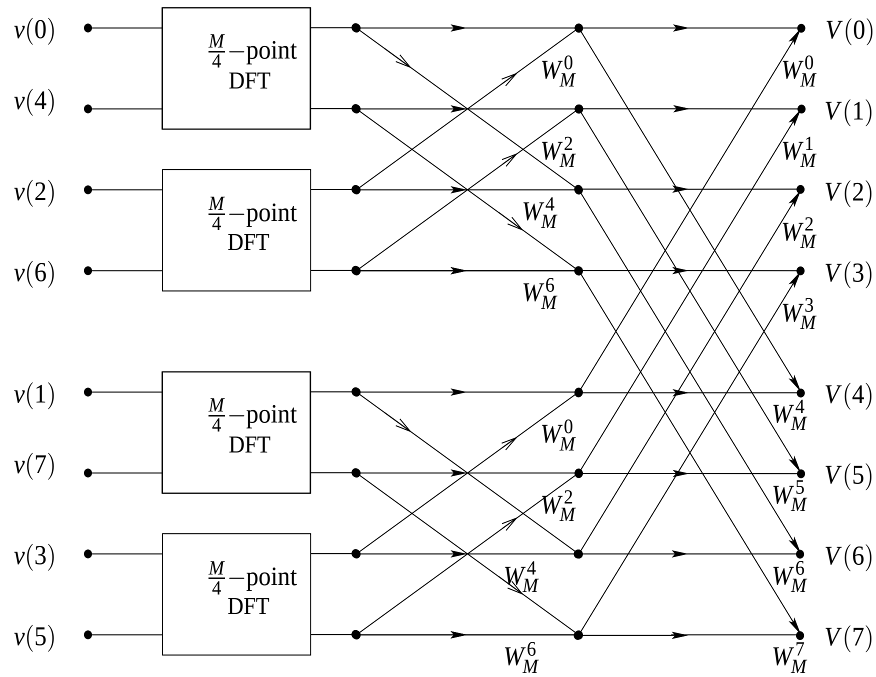
$$G[k] = \sum_{n=0}^{M/4-1} g[2n] W_{M/4}^{nk} + W_{M/2}^{k} \sum_{n=0}^{M/4-1} g[2n+1] W_{M/4}^{nk} \tag{5.41}$$

$$H[k] = \sum_{n=0}^{M/4-1} h[2n] W_{M/4}^{nk} + W_{M/2}^{k} \sum_{n=0}^{M/4-1} h[2n+1] W_{M/4}^{nk} \tag{5.42}$$

where $k = 0, \ldots, M/2 - 1$.



The overall flow graph now looks like this



Complexity:

4 DFTs of length $M/4 \rightarrow M^2/4$ operations $+ \; 2 \cdot M/2 + M$ operations for reconstruction.

$\Rightarrow M^2/4 + 2M$ complex multiplications and additions.

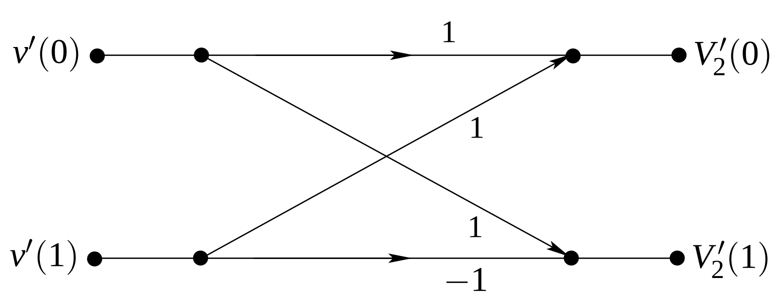
final step: Decomposition of 2-point DFT

DFT of length 2:

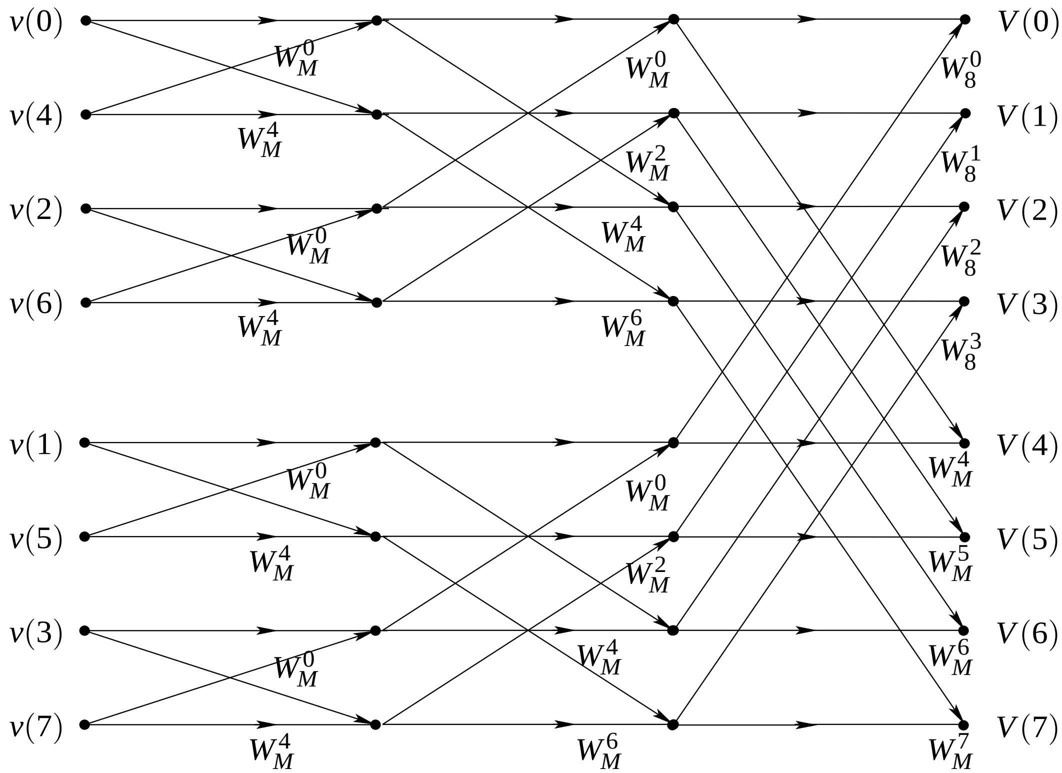
$$\acute{V}_2[0] = \acute{v}[0] + W_2^0 \acute{v}[1] = \acute{v}[0] + \acute{v}[1] \tag{5.43}$$

$$\acute{V}_2[1] = \acute{v}[0] + W_2^1 \acute{v}[1] = \acute{v}[0] - \acute{v}[1] \tag{5.44}$$

Flow graph:



Insert this flow-graph from the last step in the above structure yields overall signal flow graph.



In general, our decomposition requires $m = log_2(M) = ldM$ stages and for $M \gg 1$ we have

$$M \cdot m = M log_2 M \quad \text{complex multiplications \& additions instead of } M^2.$$

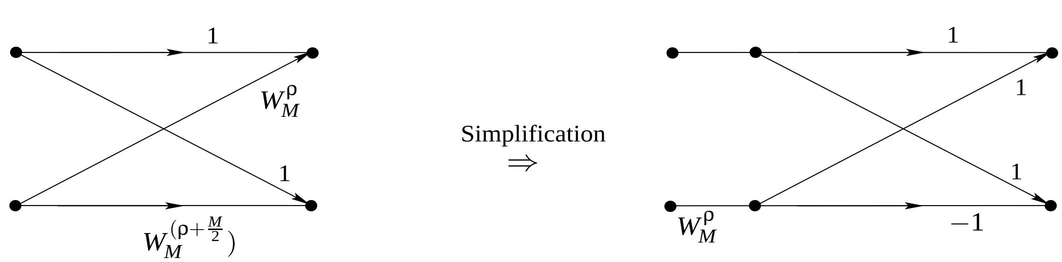
Examples

$M = 32 \rightarrow M^2 \approx 1000, M\,ld\,M \approx 160 \rightarrow$factor of 6

$M = 1024 \rightarrow M^2 \approx 10^6, M\,ld\,M \approx 10^4 \rightarrow$factor of 100

Butterfly computation:

Basic building block of above flow graph is called *butterfly* $\rho \in 0, \ldots, M/2 - 1$



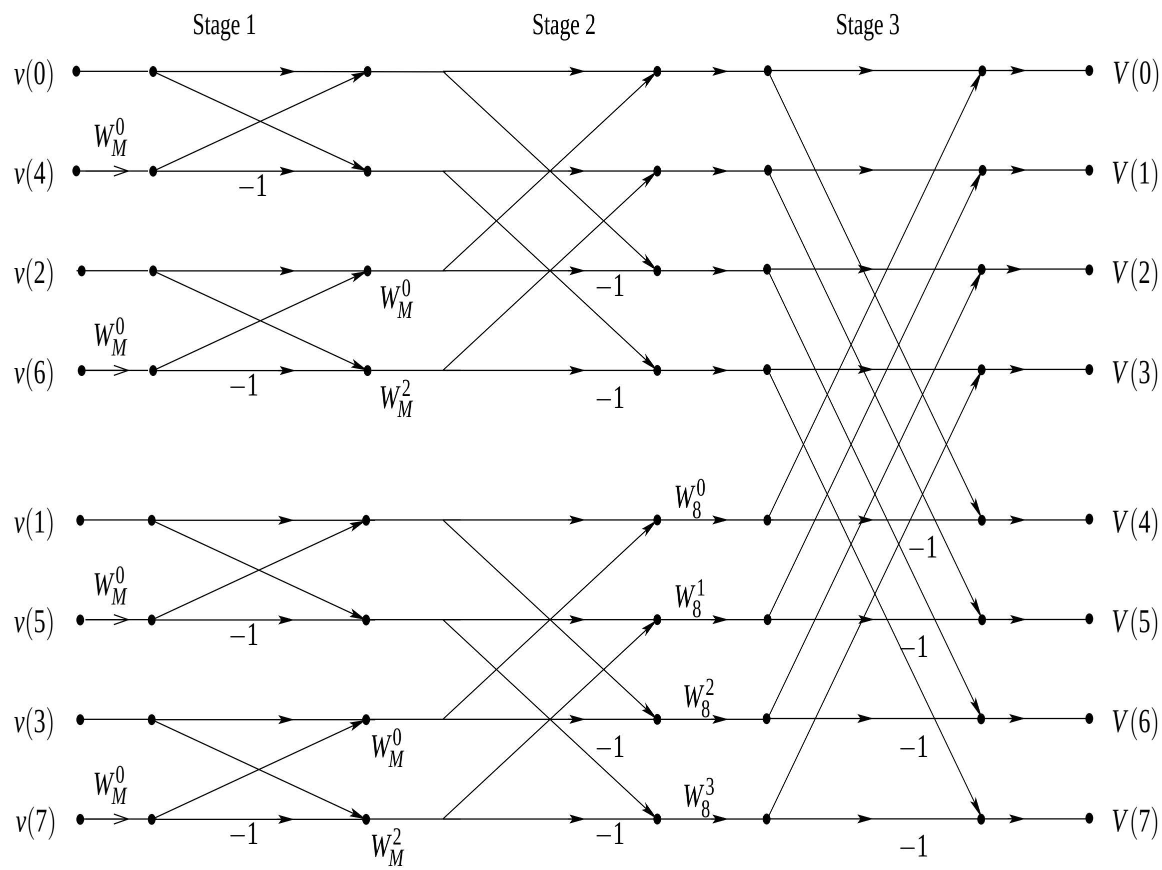
the simplification is due to the fact that
$W_M^{M/2} = e^{-j(2\pi/M)M/2} = e^{-j\pi} = -1$

$$W_M^{\rho+M/2} = W_M^{\rho} W_M^{M/2} = -W_M^{\rho}$$

Using these modifications, the resulting flow graph for $M = 8$ is given as

- $v[n]-$ values at the input of the decimation-time flow graph in permuted order.

$\Rightarrow$ Input data is stored in *bit-reversed* order.



5.7 Inverse FFT

according to (5.2) we have for the IDFT

$$\begin{aligned} v[n] &= \frac{1}{M}\sum_{k=0}^{M-1} v[n] W_M^{-nk} \\ \text{that is} & \\ v[-n] &= \frac{1}{M}\sum_{k=0}^{M-1} v[n] W_M^{nk}, \Leftrightarrow \\ v[M-n] &= \frac{1}{M} DFT\{V_M[k]\} \end{aligned} \tag{5.45}$$

$\Rightarrow$ with additional scaling and permutation the IDFT can be calculated with the same FFT algorithm as DFT.

Other alternatives

- Decimation-in-frequency
- Radix-r and mixed Radix FFT
- Inplace computation

5.8 Transformation of real-valued sequences

$v[n] \in R \rightarrow$FFT program/hardware $v_R[n] + j\underbrace{v_i[n]}_{=0} \in C \Rightarrow$ inefficient due to performing arithmetic calculations with zero values.

In the following we develop a method for efficient usage of a complex FFT for real-valued data.

5.9 FFT of two real-valued sequences

Further Reading!

Some text goes here just to illustrate the command

Given: $v_1[n], v_2[n] \in R, n = 0, \ldots, M-1$

How can we obtain $V_{M_1}(\mu) \bullet\!-\!-\!\circ\, v_1[n], V_{M_2}[k] \bullet\!-\!-\!\circ\, v_2[k]$?

Define:

$$v[n] = v_1[n] + jv_2[n] \tag{5.46}$$

leading to the DFT

$$V_M[k] = DFT\{v[n]\} = \underbrace{V_{M_1}[k]}_{\in C} + j\underbrace{V_{M_2}[k]}_{\in C} \tag{5.47}$$

$$v_1[n] = \frac{v[n] + v^*[n]}{2} \tag{5.48}$$

$$v_2[n] = \frac{v[n] - v^*[n]}{j2} \tag{5.49}$$

Hence the DFT of these sequences is:

$$V_1[k] = \frac{1}{2}[DFT\{v[n]\} + DFT\{v^*[n]\}] \tag{5.50}$$

$$V_2[k] = \frac{1}{2j}[DFT\{v[n]\} - DFT\{v^*[n]\}] \tag{5.51}$$

$$\tag{5.52}$$

Separation of $V_M[k]$ into $V_{M_1}[k]$ and $V_{M_2}[k]$?

$$v[n] = \underbrace{v_{Re}[n] + v_{Ro}[n]}_{v_1[n]} + j\underbrace{(v_{Ie}[n] + v_{Io}[n]}_{v_2[n]} \tag{5.53}$$

DFT components

$$v[n] = v_{Re}[n] + v_{Ro}[n] + jv_{Ie}[n] + jv_{Io}[n]$$

$$V_M[k] = V_{M_{Re}}[k] + V_{M_{Ro}}[k] + jV_{M_{Ie}}[k] + jV_{M_{Io}}[k] \tag{5.54}$$

Thus we have,

$$V_{M_1}[k] = \frac{1}{2}\Big(V_{M_R}[k] + V_{M_R}(M-k)\Big) + \frac{j}{2}\Big(V_{M_I}[k] - V_{M_I}(M-k)\Big) \tag{5.55}$$

where

$$\begin{aligned}
V_{M_{Re}}[k] &= \frac{1}{2}\Big(V_{M_R}[k] + V_{M_R}(M-k)\Big) \\
V_{M_{Io}}[k] &= \frac{1}{2}\Big((V_{M_I}[k] - V_{M_I}(M-k))\Big)
\end{aligned}$$

likewise, we have for $V_{M_2}[k]$ the relation

$$V_{M_2}[k] = \frac{1}{2}\Big(V_{M_I}[k] + V_{M_I}(M-k)\Big) + \frac{j}{2}\Big(V_{M_R}[k] - V_{M_R}(M-k)\Big) \tag{5.56}$$

with

$$\begin{aligned}
V_{M_{Ie}}[k] &= \frac{1}{2}\Big(V_{M_I}[k] + V_{M_I}(M-k)\Big) \\
V_{M_{Ro}}[k] &= \frac{1}{2}\Big((V_{M_R}[k] - V_{M_R}(M-k))\Big)
\end{aligned} \tag{5.57}$$

Rearranging the results (5.55) and (5.56)

$$\begin{aligned}
V_{M_1}[k] &= \frac{1}{2}\Big(V_M[k] + V_M^*(k-M)\Big) \\
V_{M_2}[k] &= -\frac{j}{2}\Big(V_M[k] - V_M^*(k-M)\Big)
\end{aligned} \tag{5.58}$$

Due to Hermitian symmetry of real-valued sequences

$$V_{M_{(1,2)}}[k] = V^*_{M_{(1,2)}}[M-k] \tag{5.59}$$

The values $V_{M_{(1,2)}}[k]$ for $\mu \in \{\frac{M}{2}+1, \ldots, M-1\}$ can be obtained from those for $k \in \{0, \ldots, \frac{M}{2}\}$such that only calculation of $\frac{M}{2}+1$ values in necessary. Furthermore, DFT of $2M$ point real-valued sequences possible with a M point DFT.

Further Reading!

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$\rightarrow$ Fast convolution of two real-valued sequences using DFT/FFT.

Chapter 6

Digital filter

6.1 FIR filters structures

Digital filter is a linear time-invariant (LTI) causal system with a rational transfer function.

$$H(z) = \frac{\sum_{i=0}^{N} b_{N-i} z^i}{\sum_{i=0}^{N} a_{N-i} z^i} \tag{6.1}$$

Further Reading!

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Where $a_0 = 1$ without loss of generality. a_i, b_i: parameters of LTI system ($\Rightarrow$ coefficients of the digital filter), N is the order of the filter.

6.1 in product notation.

$$H(z) = b_0 \frac{\prod_{i=0}^{N} (z - z_{0_i})}{\prod_{i=0}^{N} (z - z_{\infty_i})} \tag{6.2}$$

where the z_{0_1} are the zeros, and z_{∞_i} are the poles of the transfer function. In a filter poles are responsible for stability. Equation 6.1 can also be expressed as a difference equation

$$y[n] = \sum_{i=0}^{N} b_i v[n-i] - \sum_{i=1}^{N} a_i y[n-i] \tag{6.3}$$

with $v(k)$ denoting the input signal and y(k) denoting the resulting signal after filtering.

- Generally 6.3 describes the infinite impulse response (IIR) filter.

$y(k)$ is calculated from $v(k),\ v(k-1), \ldots,\ v(k-N)$ and recursively from $y(k-1)\ , y(k-2)\ , \ldots, y(k-N)$.

- The calculation of $y(k)$ requires memory elements in order to store $v(k-1)\ , \ldots, v(k-N)$ and $y(k-1)\ , \ldots, y(k-N) \rightarrow$ we have a dynamical system.

- $b_i \equiv 0$ for $\forall i \neq 0$.

$$H(z) = \frac{b_0 z^N}{\sum\limits_{i=0}^{N} a_{N-i} z^i} = \frac{b_0 z^N}{\prod\limits_{i=1}^{N} (z - z_{\infty_i})} \tag{6.4}$$

$\Rightarrow$ this filter has no zeros$\rightarrow$ All-pole or auto-regressive (AR)-filter. The transfer function is purely recursive.

$$y[n] = b_0 v[n] - \sum_{i=1}^{N} a_i y[n-i] \tag{6.5}$$

- $a_i \equiv 0$ for all $i \neq 0$, $a_0 = 1$(causal filter required): Difference equation is purely non-recursive.

$$y[n] = \sum_{i=0}^{N} b_i v[n-i] \tag{6.6}$$

$\Rightarrow$ Non-recursive filter.

Transfer function:

$$H(z) = \frac{1}{z^N} \sum_{i=0}^{N} b_{N-i} z^i = \sum_{i=0}^{N} b_i z^{-i} \tag{6.7}$$

Remarks:

$-$Poles $z_{\infty_i} = 0, \quad i = 1, \cdots, N$, but not relevant for stability $\Rightarrow$ all-zero filter

$-$According to (6.6), $y[n]$ obtained by weighted average of the last $N+1$ input values $\Rightarrow$ Moving average (MA) filter, as opposite to AR filter from above.

$-$ From (6.7) it can be seen that the impulse response has a finite length

$\Rightarrow$ Finite impulse response (FIR) filter of length $L = N + 1$ and order N.

6.1.1 Structures of FIR filters

- Difference equation given by (6.6).
- Transfer function given by (6.7).

Further Reading!

Some text goes here just to illustrate the command

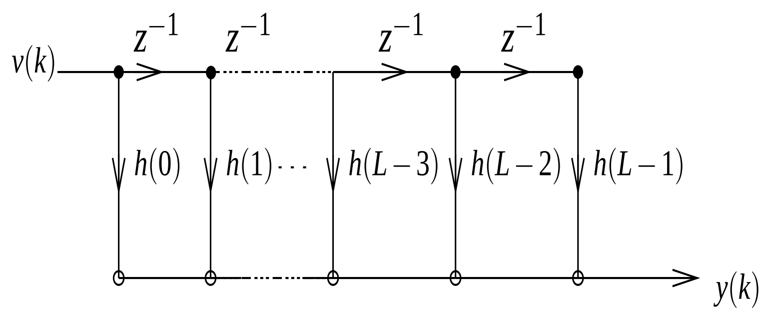
- Unit impulse response is equal to the coefficients b_i.

$$h(k) = \begin{cases} b_k & 0 \leq k \leq L-1 \\ 0 & otherwise, \end{cases}$$

Direct form Structure

The direct form structure follows immediately from non recursive difference equation given in (6.6), which is equivalent to the linear convolution sum

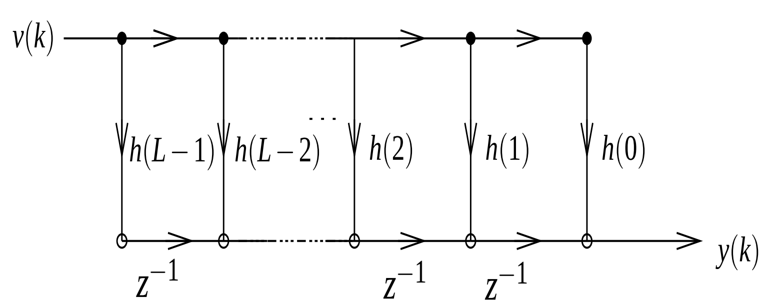
$$y[k] = \sum_{n=0}^{L-1} h[n]v[k-n] \tag{6.8}$$



$\Rightarrow$ tapped-delay-line or transversal filter in first-direct form.

If the unit impulse $v(k) = \gamma_0(k)$ is chosen as input signal, all the samples of the impulse response $h(k)$ appears successively at the output of the structure. Second direct-form can be obtained by transposing the flow graphs. i.e.

- Reversing the direction of all branches.
- Exchanging the input and output of the flow-graph.
- Exchanging the summation points with the branching points and vice versa.

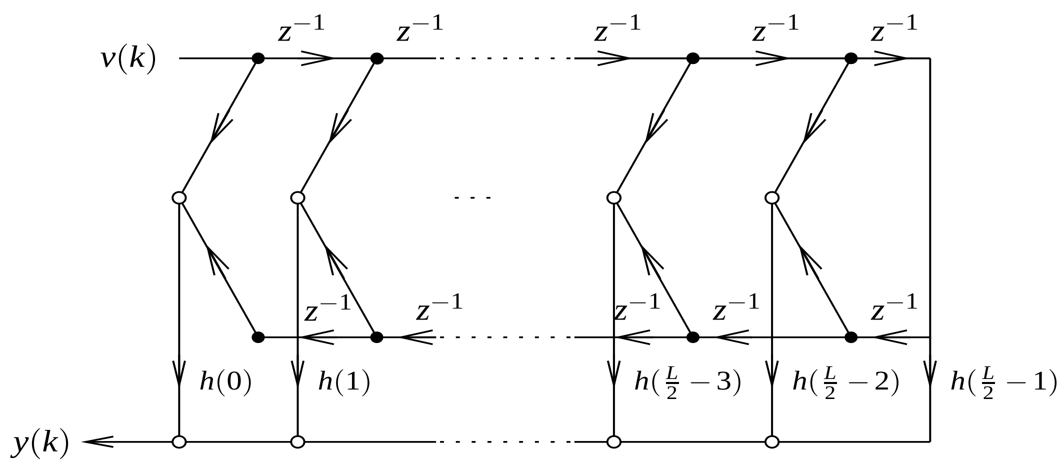


Linear-phase filters

If the impulse response of a causal FIR filter satisfies either the symmetry or asymmetry condition.

$$h(k) = \pm h(L - 1 - k) \tag{6.9}$$

$\rightarrow$ Linear-Phase filter. Therefore in cases of symmetry/asymmetry in impulse response the number of multiplications required reduce from L to $L/2$ if L is even and $(L + 1)/2$ for L is odd.



Construct the diagram for odd L.

we define $s = (L - 1)/2$.

$$
\begin{aligned}
H(z) &= \sum_{k=0}^{L-1} h(k) z^{-k} \\
&= \sum_{k=0}^{s-1} h(k) z^{-k} \pm \sum_{k=0}^{s-1} h(k) z^{k-2s} \\
&= z^{-s} \sum_{k=0}^{s-1} h(k) z^{s-k} \pm \sum_{k=0}^{s-1} h(k) z^{k-s} \\
H(z) &= z^{-s} \sum_{k=0}^{s-1} h(k) \left(z^{(s-k)} \pm z^{(-s-k)} \right)
\end{aligned}
$$

→ The frequency response

$$
H(e^{j\Omega}) = e^{-js\Omega} \left[2 \sum_{k=0}^{L/2-1} h(k) cos\left((s-k)\Omega\right) \right] \quad \text{For even L, even sym}
$$

$$
H(e^{j\Omega}) = j \cdot e^{-js\Omega} \left[2 \sum_{k=0}^{L/2-1} h(k) sin\left((s-k)\Omega\right) \right] \quad \text{For even L, odd s}
$$

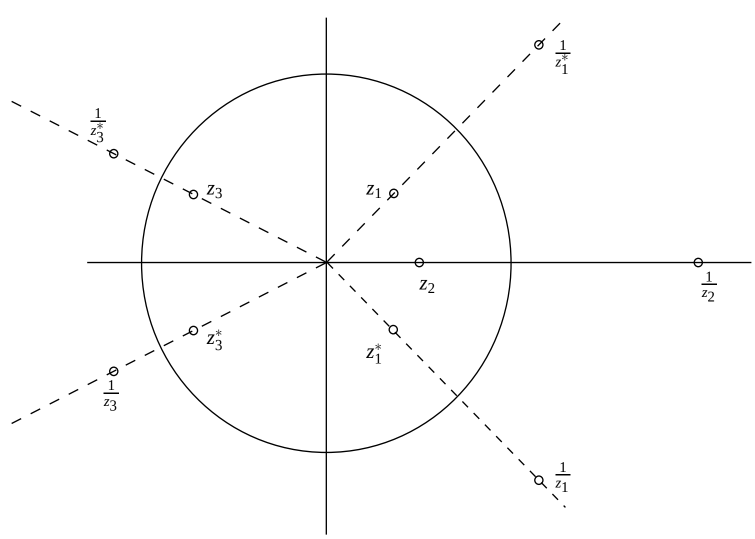
The responses for Odd length sequences can be calculated similarly.

The Z-transform of equation (6.9) is

$$
z^{-(L-1)} H(z^{-1}) = \pm H(z) \tag{6.13}
$$

Equation (6.13) implies that if z_o is root of the polynomial $H(z)$ then z^{-1}, z^* and $1/z^*$ are also roots of the polynomial.

⇒ The roots of linear phase filter occur in quadripples.



6.1.2 Cascade form Structure

By factorizing the transfer function

$$H(z) = H_0 \prod_{c=1}^{C} H_c(z) \tag{6.14}$$

we obtain a cascade realization. The $H_c(z)$ are normally second-order, since in order to obtain real coefficients, conjugate complex zeros z_{0_i} and $z_{0_i}^*$ have to be grouped.

$$\begin{aligned} H_c(z) &= (1 - z_{0_i} z^{-1})(1 - z_{0_i}^* z^{-1}) \\ &= 1 + b_1 z^{-1} + b_2 z^{-2} \end{aligned} \tag{6.15}$$

for linear phase filters due to special symmetry (6.9) the zeros appear in quadripples.

here must be a pole-zero plot

6.2 Windowing functions for FIR filter design

Further Reading!
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The desired frequency response specfiction $H_d(\Omega)$ can be used to determine the required impulse response $h_d[n]$

$$H_d(\Omega) = \sum_{n=0}^{\infty} h_d[n] e^{-j\Omega n} \tag{6.16}$$

where

$$h_d[n] = \int_{-\pi}^{\pi} H_d(\Omega) e^{j\Omega n} d\Omega \tag{6.17}$$

The unit impulse response obtained from (6.16) is infinite in duration and must be truncated at some point n=$M-1$, to yield an FIR filter length M. Truncation of $h_d[n]$ to a length $M-1$ is equivalent to multiplying $h_d[n]$ by a rectangular window defined as

$$w[n] = \begin{cases} 1, & n = 0, 1, \cdots, M-1 \\ 0, & \text{otherwise} \end{cases} \tag{6.18}$$

Thus the unit impulse response of the FIR filter becomes

$$h[n] = \begin{cases} h_d[n], & n = 0, 1, \cdots, M-1 \\ 0, & \text{otherwise} \end{cases} \tag{6.19}$$

The characteristics of rectangular window play a significant role in determining the resulting frequency response of the FIR filter obtained by truncating $h_d[n]$ to length M. Specifically convolution of $H_d(\Omega)$ and $W(\Omega)$ has the effect of smoothing the $H_d(\Omega)$. As M is increased, $W(\Omega)$ becomes narrows, and the smoothing provided by $W(\Omega)$ is reduced. The large sidelobes of $W(\Omega)$ result in some undesirable ringing effects in FIR frequency response $H(\Omega)$. These undesirable effects are best alleviated by the use of windows that do not contain abrupt discontinuity in their time domain characteristics and have correspondingly low sidelobes in their frequency domain characteristics.

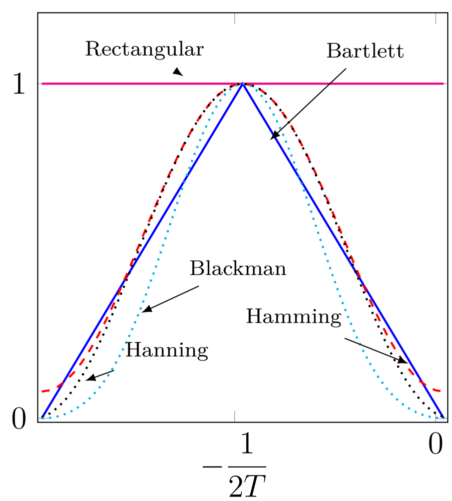


Figure 6.1: Illustration of (selected few) windowing functions.

Name of window	Time domain sequence $h[n]$ $0 < n < M-1$
Bartlett	$1 - \dfrac{2\left\lvert n - \frac{M-1}{2} \right\rvert}{M-1}$
Blackman	$0.42 - 0.5\cos\dfrac{2\pi n}{M-1} + 0.08\cos\dfrac{4\pi}{M-1}$
Hamming	$0.54 - 0.46\cos\dfrac{2\pi n}{M-1}$
Hanning	$\dfrac{1}{2}\left(1 - \cos\dfrac{2\pi n}{M-1}\right)$
Kaiser	$\dfrac{I_0\left[\alpha\sqrt{\left(\frac{M-1}{2}\right)^2 - \left(n - \frac{M-1}{2}\right)^2}\right]}{I_0\left[\alpha\left(\frac{M-1}{2}\right)\right]}$

Table 6.1: Expression of (selected few) windowing functions.

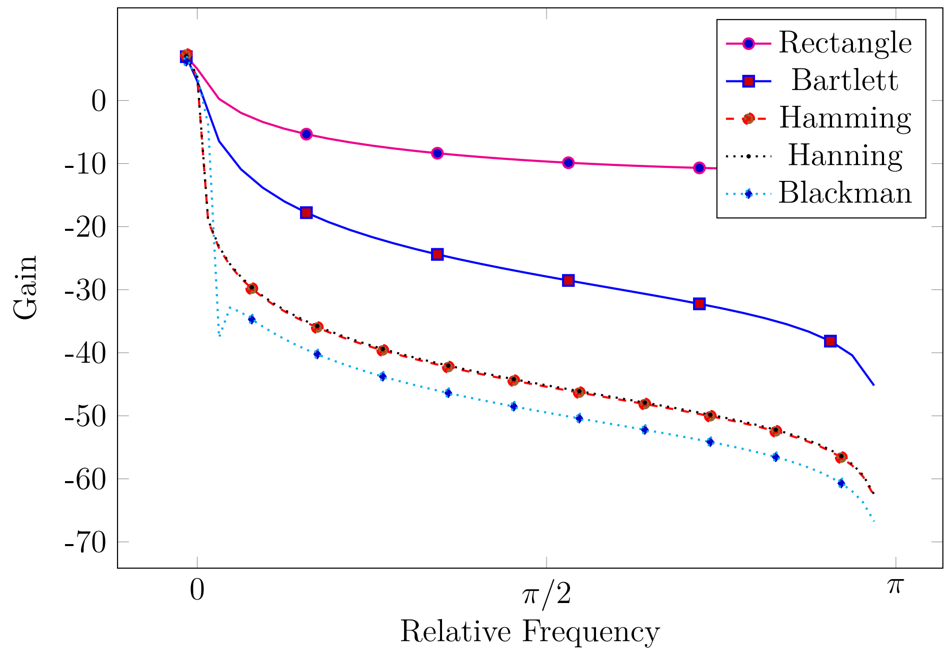


Figure 6.2: Spectral representation of the Windowing functions.

Table 6.2: Frequency domain characteristics of some window functions

Type of window	Approximate transition with of main lobe	Peak side lobe [dB]
Rectangle	$4\pi/M$	-13
Bartlett	$8\pi/M$	-25
Hamming	$8\pi/M$	-31
Hanning	$8\pi/M$	-41
Blackman	$8\pi/M$	-57

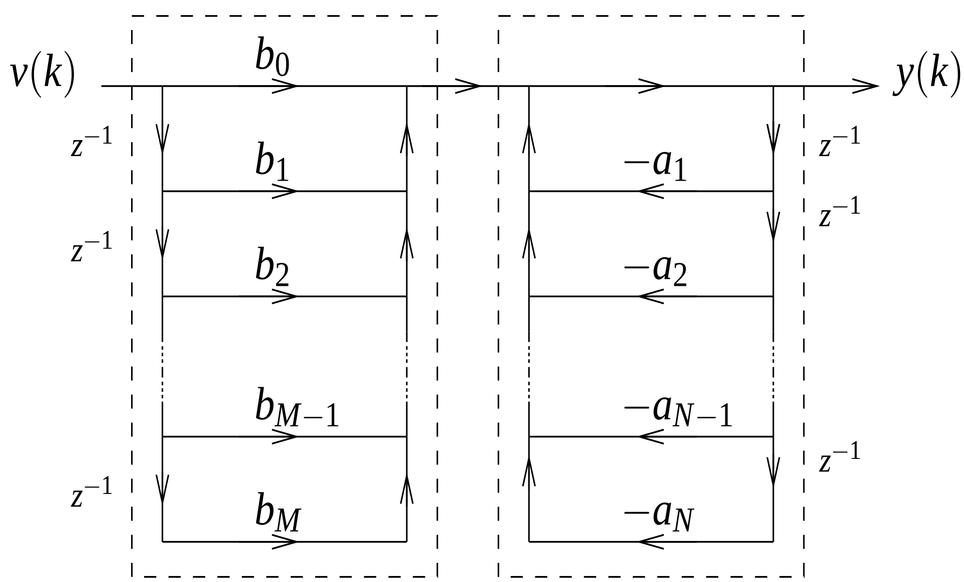
6.3 IIR filter structures

Direct form structure

Rational system function (6.1) can be viewed as two systems in cascade $H(z) = N(z)/D(z) = H_1(z) \cdot H_2(z)$ with

$$H_1(z) = \sum_{i=0}^{N} b_i z^{-i}, \quad H_2(z) = \frac{1}{1 + \sum_{i=1}^{N} a_i z^{-i}}$$

The all-zero $H_1(z)$ can be realized with the direct form structure (6.1.1) by attaching the all-pole system $H_2(z)$ in cascade, we obtain direct form I realization.



Further Reading!

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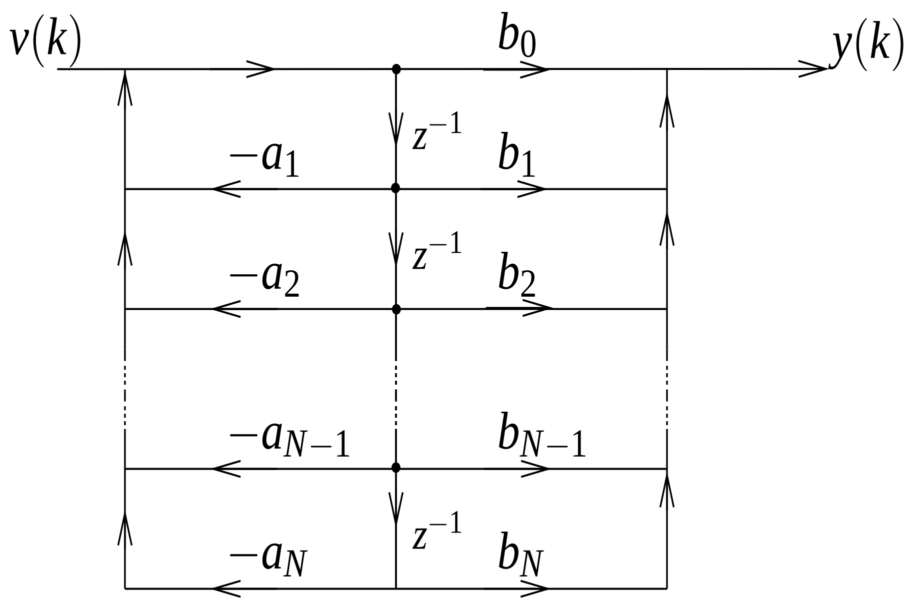
This structure requires $M + N + 1$ multiplications, $M + N$ additions and $M + N + 1$ storage locations. Where M and N represent the order of numerator and denominator polynomials. Another realization can be obtained by exchanging the order of the all-pole and all-zero filter. Then, the difference equation for the all-pole section is

$$w(k) = -\sum_{i=1}^{N} a_i w(k-i) + v(k) \cdot b_0$$

where $w(k)$ is an intermediate result and represents the input to the all-zero section:

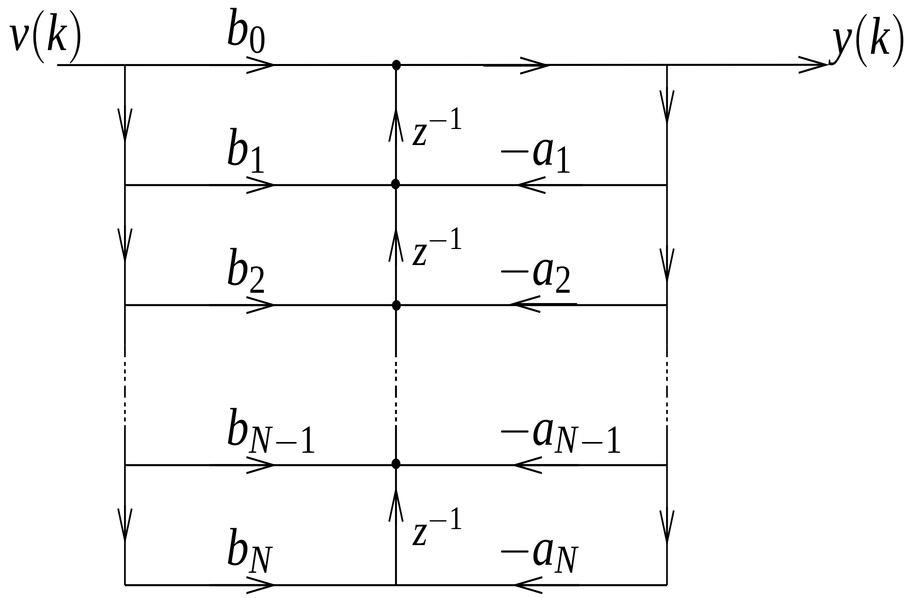
$$y(k) = \sum_{i=0}^{N} b_k w(k-i)$$

The resulting structure is given as follows



$\Rightarrow$ only a single delay line is required for storing the delayed versions of the sequence $w(k)$. The resulting structure is called direct form II realization. Furthermore, it is said to be canonic, since it minimizes the number of memory locations(among other structures).

Transposing the direct form II realization leads to the following structure, which requires the same number of multiplications, additions and memory locations.



Cascade form structure

Like section(6.1.2) we can also factor an IIR filter system $H(z)$ into first and second order subsystems $H_p(z)$ according to

Further Reading!
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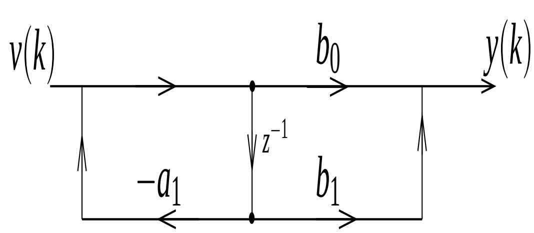
$$H(z) = \prod_{p=1}^{N} H_p(z) \tag{6.20}$$

$\Rightarrow$ Degree of freedom in grouping poles and zeros.

- **First order system**

canonical direct form for $N = 1$

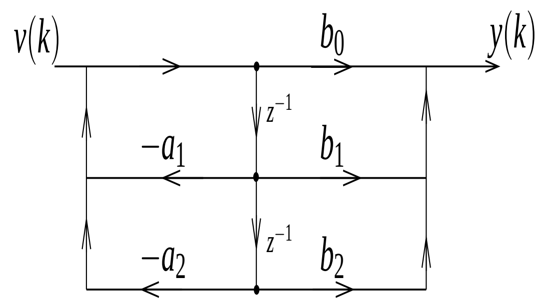


$$H(z) = \frac{Y(z)}{V(z)} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1}} \tag{6.21}$$

every first order transfer function can be realized with the above flow-graph. As explained before a_0 can be set to 1 without loss of generality.

- **Second order system**

canonical direct form for $N = 2$



$$H(z) = \frac{Y(z)}{V(z)} = \frac{b_0 + b_1 z^{-1} + b_2 z^{-2}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{6.22}$$

Parallel form structure

Further Reading!
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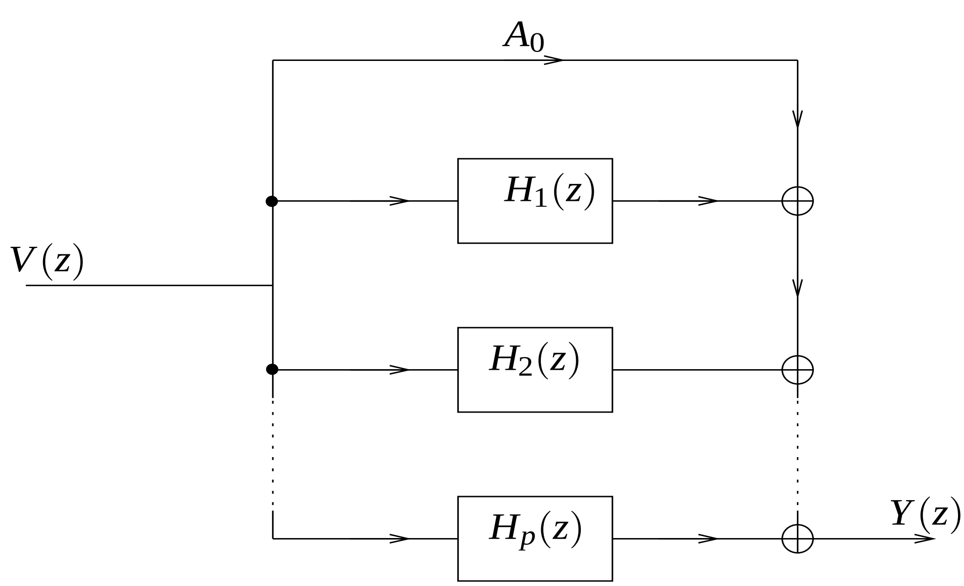
⇒ an alternative to the factorization of a general transfer function is to use a partial fraction expansion, which leads to a parallel form structure.

- In the following we assume that we have only distinct poles. The partial fraction expansion of a transfer function $H(z)$ with the numerator and denominator degree N is given as:

$$H(z) = A_0 + \sum_{i=1}^{N} \frac{A_i}{1 - z_{\infty_i} z^{-1}} \tag{6.23}$$



The A_i, $i \in 1, \ldots, N$ are the co-efficients(residues) in the partial fraction expansion, $A_0 = b_n/a_n$.

- Further more we assume that we have only real-valued coefficients, such that we can combine pairs of complex conjugate poles to form a second order subsystem $i \in \{1, .., N\}$.

$$= \frac{A_i}{1 - z_{\infty i} z^{-1}} + \frac{A_i^*}{1 - z_{\infty i}^* z^{-1}}$$

$$= \frac{2\Re\{A_i\} - 2\Re\{A_i z_{\infty i}^*\} z^{-1}}{1 - \Re\{z_{\infty i}\} z^{-1} + |z_{\infty i}|^2 z^{-2}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} \tag{6.24}$$

- Two real-valued poles can also be combined to a second order transfer function $i, j \in \{1, \ldots, N\}$

$$\begin{aligned} &= \frac{A_i}{1 - z_{\infty i} z^{-1}} + \frac{A_j}{1 - z_{\infty j}^* z^{-1}} \\ &= \frac{(A_i + A_j) - (A_i z_{\infty j} + A_j z_{\infty i}) z^{-1}}{1 - (z_{\infty j} + z_{\infty i}) z^{-1} + (z_{\infty i} z_{\infty j}) z^{-2}} = \frac{b_0 + b_1 z^{-1}}{1 + a_1 z^{-1} + a_2 z^{-2}} \end{aligned} \tag{6.25}$$

- If N is odd, there is real-valued pole left, which leads to one first order partial fraction

Example:

Determine the cascade and parallel realization for the system described by the system function.

$$H(z) = \frac{10(1 - \frac{1}{2}z^{-1})(1 - \frac{2}{3}z^{-1})(1 + 2z^{-1})}{(1 - \frac{3}{4}z^{-1})(1 - \frac{1}{8}z^{-1})[1 - (\frac{1}{2} + j\frac{1}{2})z^{-1}][1 - (\frac{1}{2} - j\frac{1}{2})z^{-1}]}$$

Solution in cascade form can be easily obtained by pairing the poles and zeros together, one possible solution is:

$$\begin{aligned} H_1(z) &= \frac{1 - \frac{2}{3}z^{-1}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}} \\ H_2(z) &= \frac{1 + \frac{3}{2}z^{-1} - z^{-2}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \end{aligned}$$

the overall filter is

$$H(z) = 10 H_1(z) H_2(z)$$

To obtain the parallel form representation $H(z)$ must be expanded in partial fraction. We have

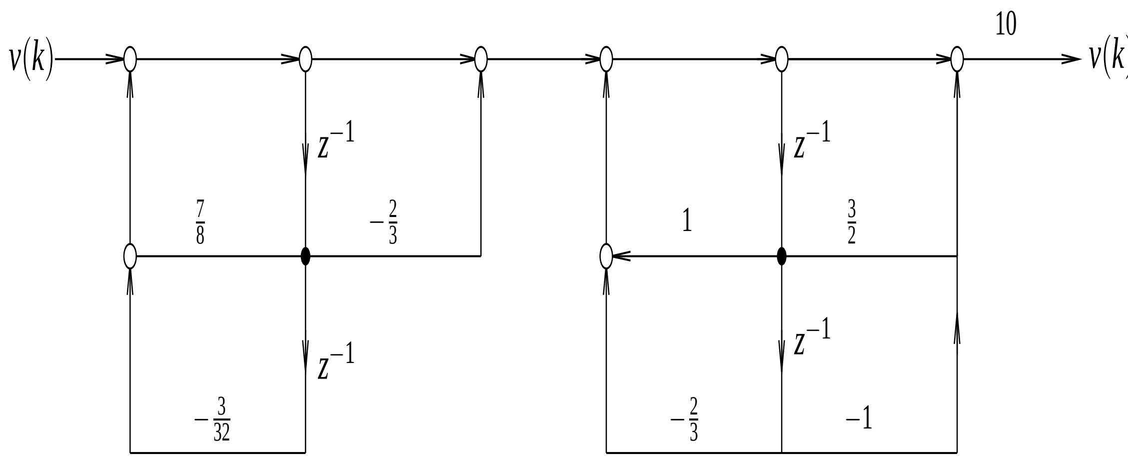
$$H(z) = \frac{A_1}{1 - \frac{3}{4}z^{-1}} + \frac{A_2}{1 - \frac{1}{8}z^{-1}} + \frac{A_3}{1 - (\frac{1}{2} + j\frac{1}{2})z^{-1}} + \frac{A_3^*}{1 - (\frac{1}{2} - j\frac{1}{2})z^{-1}}$$

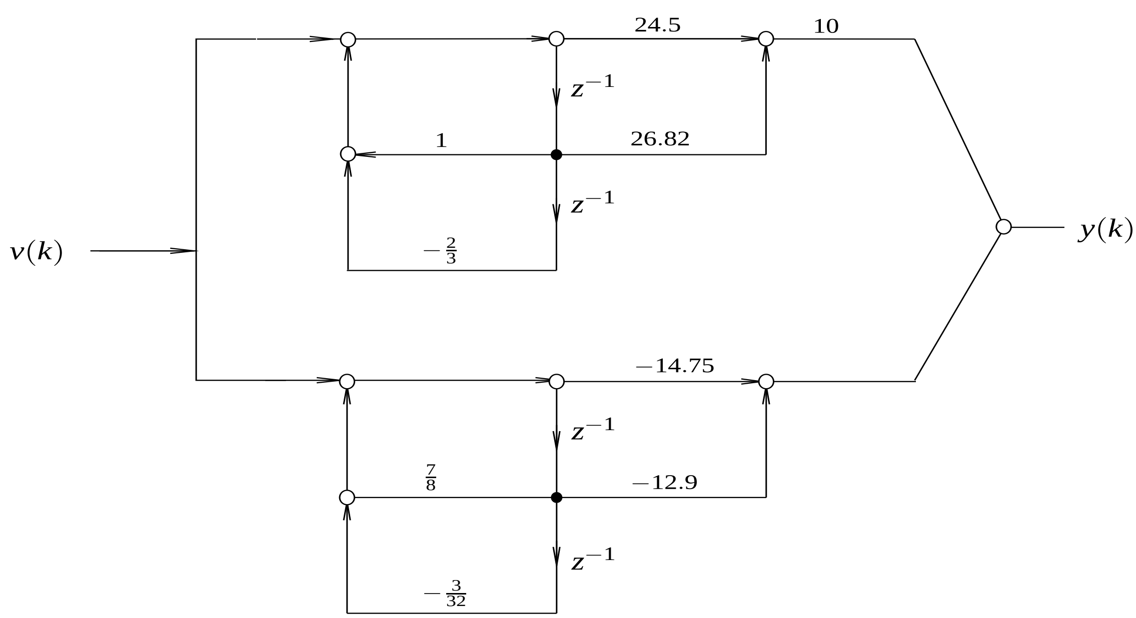
Where A_i are partial fraction coefficients to be calculated.

$A_1 = 2.93, \quad A_2 = -17.68, \quad A_3 = 12.25 - j14.57, \quad A_3^* = 12.25 + j14.57.$

Combining the pairs of poles we have

$$H(z) = \frac{-14.75 + 12.90z^{-1}}{1 - \frac{7}{8}z^{-1} + \frac{3}{32}z^{-2}} + \frac{24.5 + 26.82z^{-1}}{1 - z^{-1} + \frac{1}{2}z^{-2}} \tag{6.26}$$





if $H(e^{j\Omega})$ exists it means the filter is stable and frequency response can be expressed as,

$$H(e^{j\Omega}) = H_R(e^{j\Omega}) + H_I(e^{j\Omega}) \tag{6.27}$$

If $h(k)$ is a real-valued and causal filter then from the symmetry properties of the Fourier transform in (3.13) imply that

$$h_e(k) \circ - - - \bullet H_R(e^{j\Omega}) \tag{6.28}$$

$$h_o(k) \circ - - - \bullet H_I(e^{j\Omega}) \tag{6.29}$$

So if $h(k)$ is completely specified by $h_e(k)$ then it follows that $H(e^{j\Omega})$ is completely specified by $H_R(e^{j\Omega})$ and if $h(k)$ can be completely expressed in terms of $h_o(k)$ then $H(e^{j\Omega})$ can be completely found in terms of $H_I(e^{j\Omega})$.

$\Rightarrow$ The magnitude and phase response of a causal filter are interdependent on each other and hence can not be specified independently.

6.4 Computation of frequency response

In evaluating the magnitude response and the phase response as a function of frequency. It's convenient to express $H(e^{j\Omega})$ in terms of its poles and zeros.

$$H(e^{j\Omega}) = b_0 \frac{\prod_{i=0}^{M}(1 - z_{0_i} e^{-j\Omega})}{\prod_{i=1}^{N}(1 - z_{\infty_i} e^{-j\Omega})}$$

equivalently

$$H(e^{j\Omega}) = b_0 e^{j\Omega(N-M)} \frac{\prod_{i=0}^{M}(e^{j\Omega} - z_{0_i})}{\prod_{i=1}^{N}(e^{j\Omega} - z_{\infty_i})} \tag{6.30}$$

$$\tag{6.31}$$

expressing the complex-valued factors in (6.30)

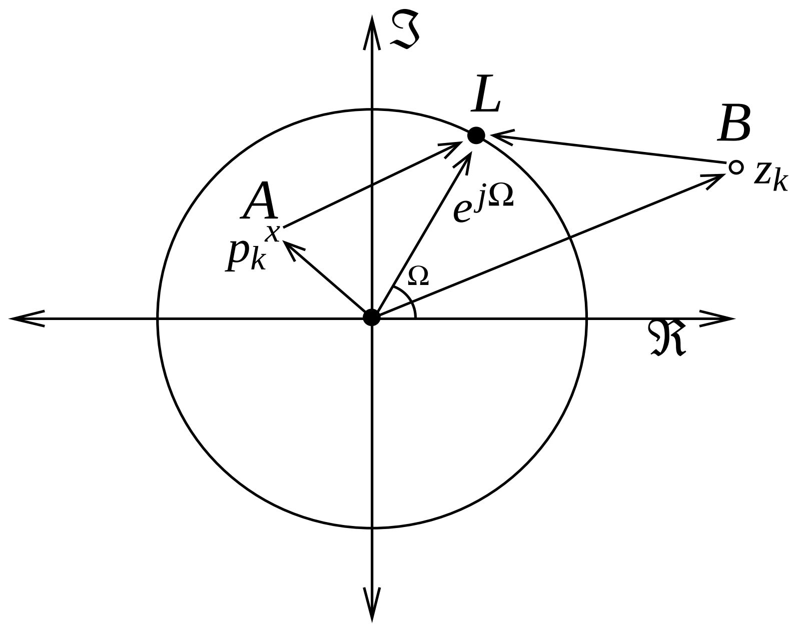
$$
\begin{aligned}
e^{j\Omega} - z_{0_i} &= V_i(e^{j\Omega})e^{j\theta_i(\Omega)} \\
\text{and} \quad e^{j\Omega} - z_{\infty_i} &= U_i(e^{j\Omega})e^{j\phi_i(\Omega)} \\
\text{here} \quad V_i(e^{j\Omega}) &= |e^{j\Omega} - z_{0_i}| \qquad \text{and} \qquad \theta_i(e^{j\Omega}) = \angle(e^{j\Omega} - z_{0_i}) \\
U_i(e^{j\Omega}) &= |e^{j\Omega} - z_{\infty_i}| \qquad \text{and} \qquad \phi_i(e^{j\Omega}) = \angle(e^{j\Omega} - z_{\infty_i})
\end{aligned}
$$

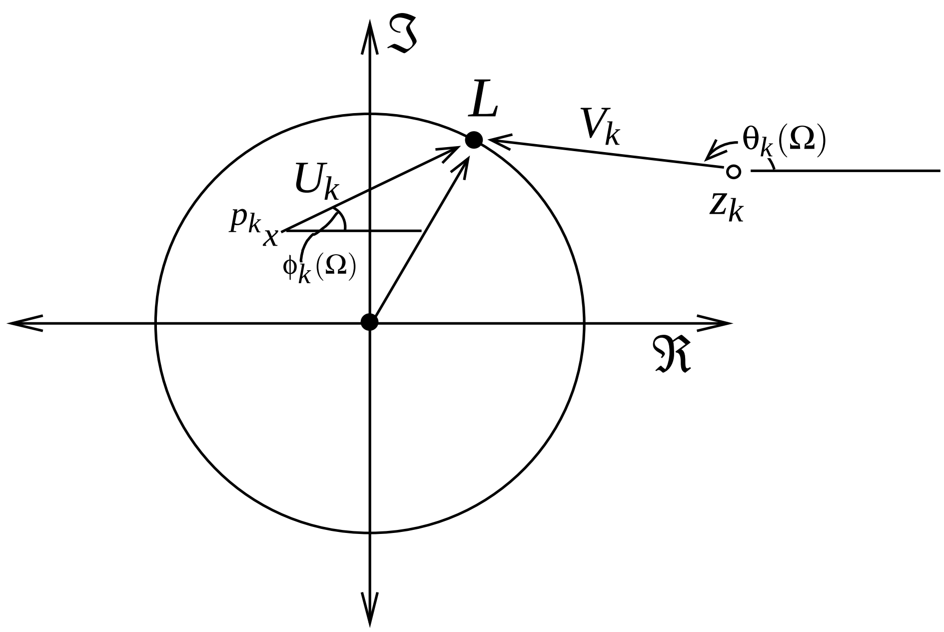
The magnitude response of $H(e^{j\Omega})$ is equal to the product of magnitudes of all terms in (6.30)

$$
|H(e^{j\Omega})| = |b_0|\frac{V_1(e^{j\Omega})\ldots V_N(e^{j\Omega})}{U_1(e^{j\Omega})\ldots U_M(e^{j\Omega})} \tag{6.32}
$$

the phase response

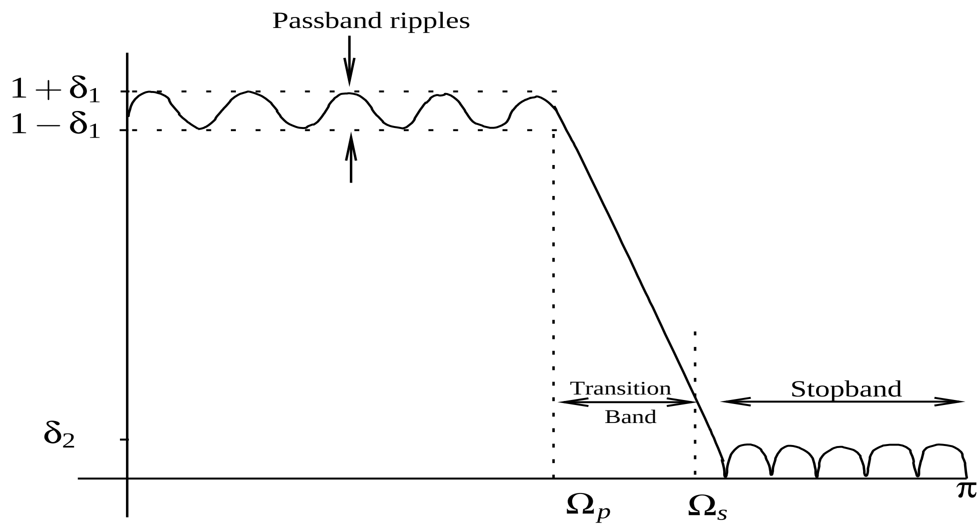
$$
\angle H(e^{j\Omega}) = \angle b_0+\Omega(N-M)+\theta_1(\Omega)+\theta_2(\Omega)+\cdots+\theta_N(\Omega)-\Big[\phi_1(\Omega)+\phi_2(\Omega)+\cdots+\phi_1(\Omega)\Big] \tag{6.33}
$$





6.5 Characteristics of practical frequency selective filters

As it is well-known that ideal filters are non-causal and hence practically unrealizable for real-time signal processing. Causality implies that the filter response characteristic $H(e^{j\Omega})$ can not be zero except for a finite set of points in the frequency range. In addition $H(e^{j\Omega})$ can not have infinitely sharp cutoff from pass-band to stop-band. Although ideal filter responses are desirable but they are not really required in practice. In particular its not necessary to insist on constant $|H(e^{j\Omega})|$ in the pass-band and similarly it is not necessary to have $H(e^{j\Omega})$ as exactly zero in the stop-band. Small nonzero ripples in stop and pass band are tolerable.



The transition from passband to stopband defines the <u>transition band</u>

- δ_1 Passband ripple.
- δ_2 Stopband ripple.
- Ω_p Passband edge frequency.
- Ω_s Stopband edge frequency.

Filter design problem:

- Specify δ_1,δ_2 ,Ω_p and Ω_s corresponding to the desired application.
- Select coefficients a_i and b_i as free parameters such that the resulting frequency response $H(e^{j\Omega})$ best satisfies the requirement of δ_1,δ_2 ,Ω_p and Ω_s .
- The degree to which $H(e^{j\Omega})$ approximates the specification depends on the criterion used for selecting a_i and b_i and also the order of polynomials, i.e the number of coefficients.

The impulse response of FIR filters is defined in (6.6) and (6.7). The specification of a filter can be defined in two ways absolute specification which provides a set of requirements on the magnitude response function $|H(e^{j\Omega})|$. The second approach is called relative specifications, they provide the requirements in *decibel*[dB]

$$\text{dB scale} = -20log_{10}\frac{H(e^{j\Omega})|_{max}}{H(e^{j\Omega})} \leq 0$$

The typical specifications for lowpass filter are shown in figure(). The band of frequencies that is allowed to pass through the filter is called the passband and is given by $0 \leq |\Omega| \leq \Omega_p$. The band of frequencies that is suppressed by the filter is called stopband and is given by $\Omega_s \leq \Omega \leq \pi$. The band $\Omega_p \leq \Omega \leq \Omega_s$ is called transition band. In absolute specifications, we require that the passband and the stopband ripples δ_1 and δ_2 respectively must satisfy

Further Reading!

Some text goes here just to illustrate the command

$$\begin{aligned}
\text{Passband:} &\quad 1-\delta_1 \leq |H(e^{j\Omega})| \leq 1+\delta_1, \quad \text{for} \quad |\Omega| \leq \Omega_p \\
\text{Stopband:} &\quad |H(e^{j\Omega})| \leq \delta_2 \quad \text{for} \quad \Omega_s \leq |\Omega| \leq \pi
\end{aligned}$$

In the relative specifications, we require that the passband ripple R_p and the stopband attenuation A_s must satisfy

$$\begin{aligned}
\text{Passband:} &\quad 0 \leq \text{dB scale} \leq R_p, \quad \text{for} \quad |\Omega| \leq \Omega_p \\
\text{Stopband:} &\quad \text{dB scale} > A_s \quad \text{for} \quad \Omega_s \leq |\Omega| \leq \pi
\end{aligned}$$

The relationship between the two sets of parameters is given by

$$R_p = -20log_{10}\frac{1-\delta_1}{1+\delta_1} \tag{6.34}$$

$$A_s = -20log_{10}\frac{\delta_2}{1+\delta_1} \tag{6.35}$$

6.5.1 Design of FIR filters

Windowing functions

The natural approach to designing FIR filters via the window design is to choose a proper ideal (or desired) frequency selective filter (which is always noncausal,infinite duration impulse response $h_d(k)$) and truncate it to obtain a linear-phase and causal FIR filter. The impulse response of designed filter is given by:

$$h(k) = h_d(k).w(k) \tag{6.36}$$

where $w(k)$ is some symmetrical function over $0 \leq k \leq L-1$ The approach of this method is to select suitable frequency response and windowing function. The frequency response of designed filter is

$$H(e^{j\Omega}) = H(e^{j\Omega}) \circledast W(e^{j\Omega}) = \frac{1}{2\pi}\int_{-\pi}^{\pi} H_d(e^{j\eta})W(e^{j(\Omega-\eta)})d\eta \tag{6.37}$$

$\Rightarrow$ (6.37) produces a smeared version of ideal filter.

Example:1 An ideal lowpass filter with cutoff frequency $0 < \Omega_c < \pi$ is defined as

$$H_{LP}(e^{j\Omega}) = \begin{cases} 1 \cdot e^{-j\Omega} & |\Omega| \leq \Omega_c \\ 0 & , \Omega_c \leq \Omega \leq \pi \end{cases}$$

The impulse response of this filter is noncausal and of infinite duration . Which can be truncated and shifted to yield a realizable filter.

$$h_{LP}(k) = \frac{sin(\Omega_c(k - \frac{L-1}{2}))}{\pi(k - \frac{L-1}{2})} \tag{6.38}$$

6.5.2 Design of IIR filters

To design analog filters the magnitude-squared response should be provided. Three designs all widely used for prototyping filters, namely Butterworth, Chebyshev and

Further Reading!

Some text goes here just to illustrate the command

Elliptic filters. The filter design requirements are defined as follows

$$\frac{1}{1+\epsilon^2} \leq |H(j\omega)|^2 \leq 1, \quad |\omega| \leq \omega_p$$
$$0 \leq |H(j\omega)|^2 \leq \frac{1}{A^2}, \quad \omega_s \leq |\omega| \tag{6.39}$$

where ϵ is the passband ripple parameter,ω_p is the passband cutoff frequency, A is the stopband attenuation parameter and ω_s is stopband cutoff frequency.The parameters are associated with relative parameters R_p and A by

$$R_p = -10log_{10}\frac{1}{1+\epsilon^2} \Rightarrow \epsilon = \sqrt{10^{R_p/10}-1} \tag{6.40}$$
$$A_s = -10log_{10}\frac{1}{A^2} \Rightarrow A = 10^{A_s/20} \tag{6.41}$$

The squared magnitude response $|H(j\omega)|^2$ can be rewritten as

$$|H(j\omega)|^2|_{s=j\omega} = H(s)H(-s) \tag{6.42}$$

From here we can obtain the system function $H_a(s)$ of analog filter. To obtain a stable and causal filter the left half poles and zeros of $H(s)H(-s)$ are assigned to $H_a(s)$.

Butterworth filters have monotone behavior both passband and stopband. The magnitude-squared response of an N-th order Butterworth filter is

$$|H(j\omega)|^2 = \frac{1}{1+(\frac{\omega}{\omega_c})^{2N}} \tag{6.43}$$

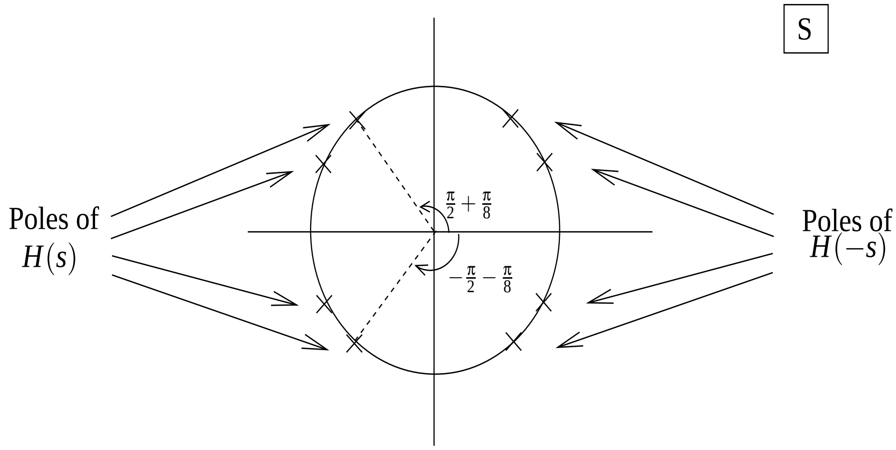
Where N is the order of the filter and ω_c is the -3dB frequency(cutoff frequency). Using (6.42) in (6.43)

$$H(s) \cdot H(-s) = \frac{1}{1+(\frac{-s^2}{\omega_c^2})^N} \tag{6.44}$$

$\Rightarrow$ Poles of $H(s) \cdot H(-s)$:

$$\frac{-s^2}{\omega_c^2} = (-1)^{1/N} = e^{j(2n+1)\pi/N}$$
$$\rightarrow \quad s = \omega_c e^{j\pi/2} \cdot e^{j(2n+1)\pi/2N}, \quad n = 0, \ldots, 2N-1 \tag{6.45}$$

please specify the diagram features



- The $2N$ poles of $H(s)H(-s)$ occur on a circle of radius ω_c at equally spaced points in the S-plane.

Further Reading!
Some text goes here just to illustrate the command

- N poles for $n = 0\ldots, N-1$ in (6.45) are located in the left half of the s-plane and belong to $H(s)$.
- The remaining poles lie in the right half of the s-plane and belong to $H(-s)$.
- Butterworth filter has N zeros at $\omega \to \infty$.

Estimation of required filter order N.

At the stopband edge frequency (6.43) can be written as

$$\delta^2 = \frac{1}{1+(\frac{\omega}{\omega_c})^{2N}}$$

which can be resolved to

$$N = \frac{log(\frac{1}{\delta^2})-1}{2log(\frac{\omega}{\omega_c})} \tag{6.46}$$

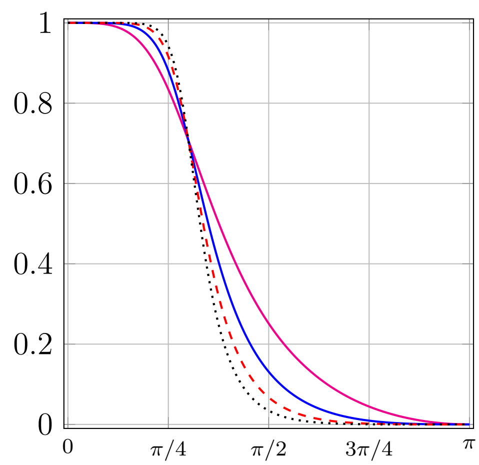
insert the magnitude, pole-zero plot and phase response here

6.5.3 Chebyshev Filter

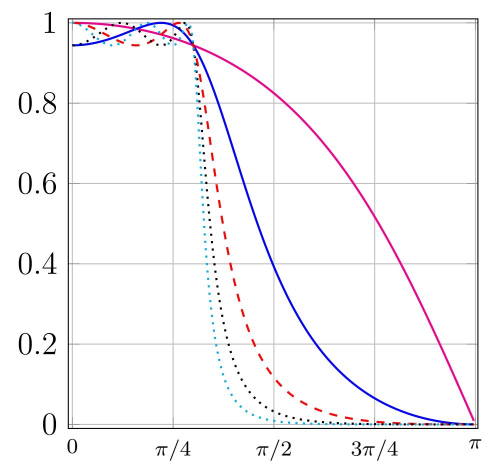
Two type of Chebyshev filters,

- Type 1 filters are all pole filters with equi-ripple behaviour in passband and monotonic characteristic (similar to Butterworth filter) in the stopband.

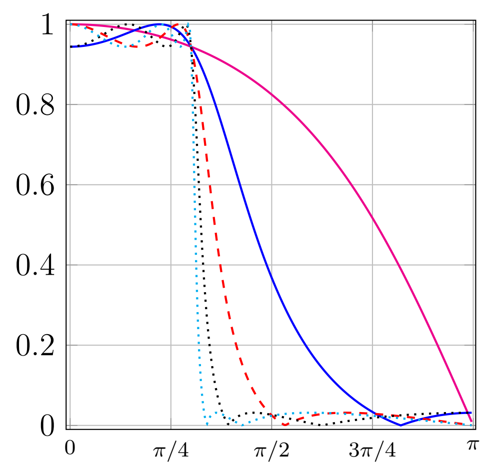
$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\omega/\omega_p)} \tag{6.47}$$



(a) Frequency response of Butterworth filter with 'n'



(b) Frequency response of Chebyshev Type 1 filter with 'n'



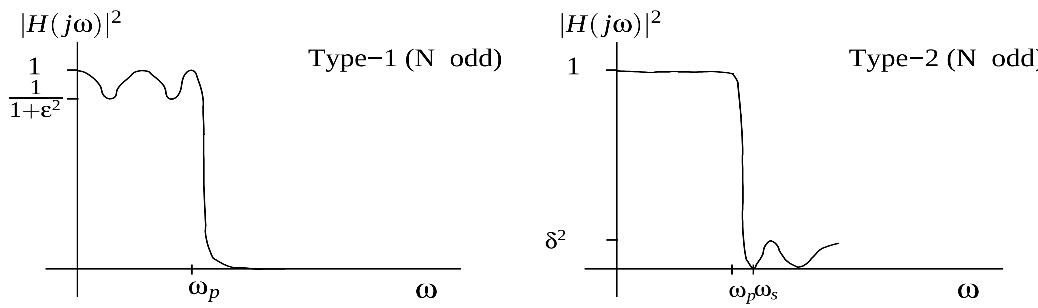
(c) Elliptic-Cauer Filter performance as a function of filter order.

Figure 6.3: Spectral representation of the well-known IIR Filter design techniques.

Where $T_N(x)$ is the Chebyshev polynomial of $N-$th order.

- Type 2 Filters have poles and zeros, equi-ripple behaviour in the stopband and monotonic characteristics in passband.

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 T_N^2(\omega_s/\omega_p)/T_N^2(\omega_s/\omega)} \tag{6.48}$$



Estimation of filter order:

Chebyshev filter only depends on the parameters N,ϵ,δ_2 and the ratio ω_s/ω_p. It can be shown that the required order can be estimated as

$$N = \frac{log\left[\sqrt{1-\delta_2^2}+\sqrt{1-\delta_2^2(1-\epsilon^2)}/(\epsilon\delta_2)\right]}{log\left[\omega_s/\omega_p+\sqrt{(\omega_s/\omega_p)^2-1}\right]} \tag{6.49}$$

Elliptic Filter

Squared magnitude response:

$$|H(j\omega)|^2 = \frac{1}{1+\epsilon^2 U_N^2(\omega/\omega_c)} \tag{6.50}$$

Where N is the order of Jacobian polynomial $U_N(\cdot)$ and ϵ is the passband ripple. Elliptic filters provide very sharp magnitude response.

Butterwoth filters exhibit fairly linear phase response. Elliptic filters have highly nonlinear phase response and the phase characteristics of chebyshev filters lie in between.

6.6 Analog to digital transformation methods

Impulse invariance methods

The goal is to design a IIR filter with an impulse response $h(k)$ being a sampled version of the impulse $h_a(t)$ of the analog filter:

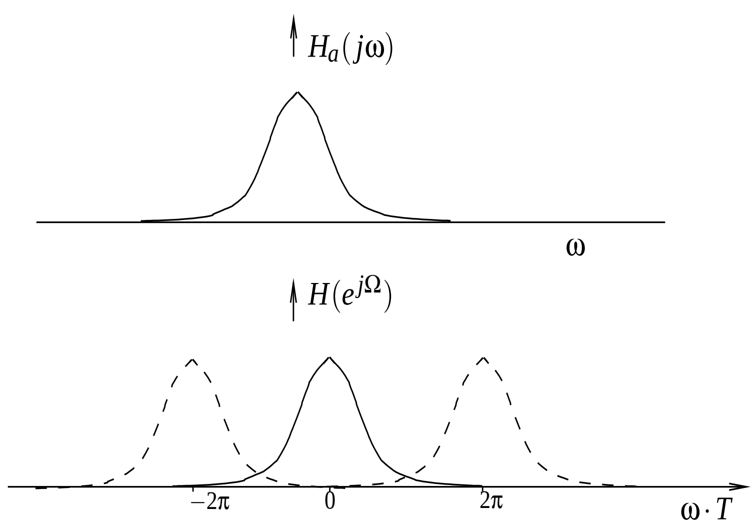
$$h[n] = h_a(nT), \quad n = 0, 1, 2, \ldots$$

The frequency response of the ideally sampled signal from (2.16)

$$H(j\omega) = \frac{1}{T} \sum_{k=-\infty}^{\infty} H_a(j\omega - j\frac{2\pi k}{T}) \tag{6.51}$$

- T Should be chosen sufficiently small to avoid aliasing.
- Method is not suitable for design of highpass filters due to large amount of possible aliasing.

insert the other diagram of mapping also



Suppose that the poles of the analog filter are distinct. Then the partial fraction expansion of $H_a(s)$ writes

$$H_a(s) = \sum_{i=1}^{N} \frac{A_i}{s - s_{\infty i}} \tag{6.52}$$

Where A_i are the coefficient of partial fraction expansion, and $s_{\infty i}$ denote the poles of the analog filter. Inverse Laplace transform of (6.52) yields

$$h_a(t) = \sum_{i=1}^{N} A_i e^{s_{\infty i} t}, \quad t \geq 0$$

periodic sampling of $h_a(t)$

$$h[n] = h_a(nT) = \sum_{i=1}^{N} A_i e^{s_{\infty i} nT}$$

taking $z-$transform of $h[n]$

$$H(z) = \sum_{n=0}^{\infty} h[n] z^{-n} = \sum_{n=0}^{\infty} \Big(\sum_{i=1}^{N} A_i e^{s_{\infty i} nT} \Big) z^{-n} \tag{6.53}$$

Then we have

$$H(z) = \sum_{i=1}^{N} A_i \sum_{k=0}^{\infty} (e^{s_{\infty i} T} z^{-1})^k = \sum_{i=1}^{N} \frac{A_i}{1 - e^{s_{\infty i} T} z^{-1}}$$

Thus given an analog filter $H_a(s)$ with poles $s_{\infty i}$, the transfer function of the corresponding digital filter using impulse invariance transform is

$$H(z) = \sum_{i=1}^{N} \frac{A_i}{1 - e^{s_{\infty i} T} z^{-1}} \tag{6.54}$$

Example: Convert the analog bandpass filter with system function.

$$H_a(s) = \frac{s + 0.1}{(s + 0.1)^2 + 9}$$

into digital IIR filter.

We note that the analog filter has zero at $s = -0.1$ and a pair of complex conjugate poles at $s = -0.1 \pm j3$. We do not have to find the impulse response of $h_a(t)$ to sample it, instead we can directly determine $H(z)$ using (6.54)

$$\begin{aligned} H(s) &= \frac{\frac{1}{2}}{s + 0.1 - j3} + \frac{\frac{1}{2}}{s + 0.1 + j3} \\ H(s) &= \frac{\frac{1}{2}}{1 - e^{-0.1T + j3T} z^{-1}} + \frac{\frac{1}{2}}{1 - e^{-0.1T - j3T} z^{-1}} \end{aligned}$$

further simplification using trigonometric relations yields

$$H(z) = \frac{1 - (e^{-0.1T} cos3T) z^{-1}}{1 - 2e^{-0.1T} cos3T z^{-1} + e^{-0.2T} Z^{-1}}$$

⇒ The magnitude of the frequency response of this filter is clearly associated with sampling interval T.

Bi-linear transforms

Algebraic transformation between variables s and z, mapping of entire $j\omega$ axis of s-plane to one revolution of unit circle in the $z-$plane. Definition:

$$s = \frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right) \tag{6.55}$$

T denotes the sampling interval.

The transfer function of the corresponding digital filter can be obtained from the transfer function of the analog filter $H_a(s)$ according to

$$H(z) = H_a\left[\frac{2}{T}\left(\frac{1-z^{-1}}{1+z^{-1}}\right)\right] \tag{6.56}$$

Properties: solving (6.55) for z yields

$$z = \frac{1+(T/2)s}{1-(T/2)s} \tag{6.57}$$

substituting $s = \sigma + j\omega$ we obtain

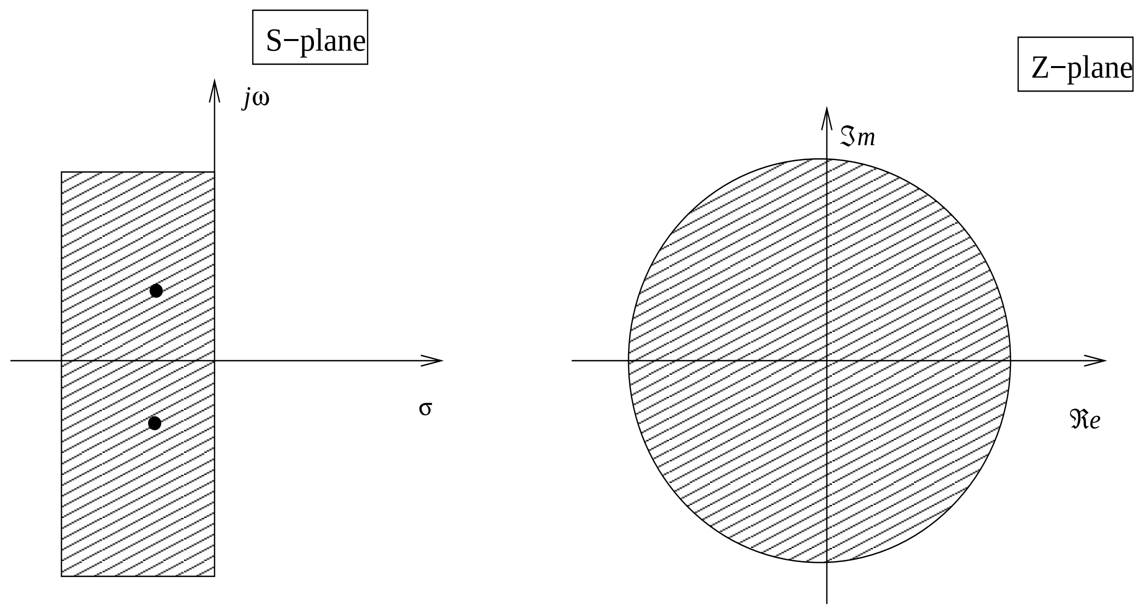
$$z = \frac{1+\sigma T/2 + j\omega T/2}{1-\sigma T/2 - j\omega T/2} \tag{6.58}$$

$\sigma < 0 \rightarrow |z| < 1, \sigma > 0 \rightarrow |z| > 1$ for all ω

⇒ causal stable continuous time filters map in causal stable discrete-time filters.

- By inserting $s = j\omega$ in (6.57) it can be seen that $|z| = 1$ for all values of s on the $j\omega$ axis.

$\Rightarrow$ $j\omega$-axis maps onto the unit circle.



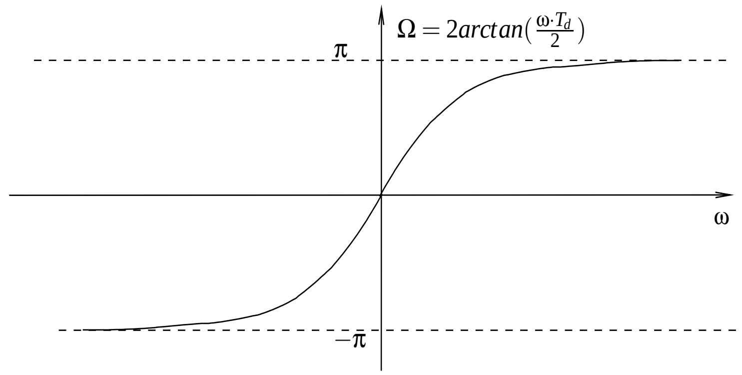
- Relationship between Ω and *omega* can be obtained from (6.55) with $s = j\omega$ and $z = e^{j\Omega}$

$$
\begin{aligned}
j\omega &= \frac{2}{T}\left(\frac{1-e^{-j\Omega}}{1+e^{-j\Omega}}\right) \\
&= \frac{2}{T}\left(\frac{jsin(\Omega/2)}{cos(\Omega/2)}\right) = \frac{2}{T}tan(\Omega/2)
\end{aligned}
$$

$\Rightarrow$ Nonlinear mapping between Ω and ω

$$
\omega = \frac{2}{T}tan(\Omega/2), \quad \Omega = 2tan^{-1}(\omega T/2) \tag{6.59}
$$

Design of a digital filter begins with frequency specification, which are converted into analog domain via (6.59). The analog filter is then designed and converted back into digital domain using the bilinear transforms.



insert the example here as well