Homework 2

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Section 2.1

Exercise 11

Determine whether each of these statements is true or false.

(a) $0 \in \emptyset$

(b) $\emptyset \in \{0\}$

(c) $\{0\} \subset \emptyset$

(d) $\emptyset \subset \{0\}$

(e) $\{0\} \in \{0\}$

(f) $\{0\} \subset \{0\}$

(g) $\{\emptyset\} \subseteq \{\emptyset\}$

Solution

(a) $0 \in \emptyset$: This statement is false because $\emptyset$ has no elements so 0 can not be an element of the empty set.

(b) $\emptyset \in \{0\}$: This statement is false because $\emptyset$ is not an element in set $\{0\}$.

(c) $\{0\} \subset \emptyset$: This statement is false because $\emptyset$ has no elements so that $\{0\}$ can not be a subset of $\emptyset$.

(d) $\emptyset \subset \{0\}$: This statement is true because $\emptyset$ is one of the two sets that every nonempty set is guaranteed to have.

(e) $\{0\} \in \{0\}$: This statement is false because $\{0\}$ is an element of $\{\{0\}\}$ not an element of $\{0\}$.

(f) $\{0\} \subset \{0\}$: This statement is false because the two set all have the same elements 0 so that is must be $\subseteq$.

(g) $\{\emptyset\} \subseteq \{\emptyset\}$: This statement is true because both singleton set have the same element $\emptyset$. Therefore, this statement is true.

Exercise 12

Determine whether these statements are true or false.

(a) $\emptyset \in \{\emptyset\}$

(b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset\} \in \{\emptyset\}$

(d) $\{\emptyset\} \in \{\{\emptyset\}\}$

(e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

(f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

(g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Solution

(a) $\emptyset \in \{\emptyset\}$: This statement is true because $\emptyset$ is an element of a singleton set contains element $\emptyset$.

(b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$: This statement is true because $\emptyset$ is an element of the set $\{\emptyset, \{\emptyset\}\}$.

(c) $\{\emptyset\} \in \{\emptyset\}$: This statement is false because $\emptyset$ must be an element of $\{\{\emptyset\}\}$.

(d) $\{\emptyset\} \in \{\{\emptyset\}\}$: This statement is true because the set $\{\{\emptyset\}\}$ contains $\{\emptyset\}$.

(e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$: This statement is true because $\emptyset$ is an element is the set $\{\emptyset, \{\emptyset\}\}$ so the set contains $\emptyset$ is a subset of $\{\emptyset, \{\emptyset\}\}$.

(f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$: This statement is true and its reason is the same as problem (e).

(g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$: We can see that the set $\{\{\emptyset\}, \{\emptyset\}\}$ has two elements which are equal to each other. Therefore, we can simplify it to $\{\{\emptyset\}\}$. Therefore, this statement is false because these sets are equal to each other so it must be $\subseteq$ instead of $\subset$.

Exercise 13

Determine whether each of these statements is true or false.

(a) $x \in \{x\}$

(b) $\{x\} \subseteq \{x\}$

(c) $\{x\} \in \{x\}$

(d) $\{x\} \in \{\{x\}\}$

(e) $\emptyset \subseteq \{x\}$

(f) $\emptyset \in \{x\}$

Solution

(a) $x \in \{x\}$: This statement is true because x is an element in set x.

(b) $\{x\} \subseteq \{x\}$: This statement is true.

(c) $\{x\} \in \{x\}$: This statement is false because x is an element of $\{\{x\}\}$ not $\{x\}$.

(d) $\{x\} \in \{\{x\}\}$: This statement is true due to the reason from problem(c).

(e) $\emptyset \subseteq \{x\}$: This statement is true according to **Theorem 1**.

(f) $\emptyset \in \{x\}$: This statement is false because $\emptyset$ is not an element of set $\{x\}$.

Exercise 26

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

(a) $\emptyset$

(b) $\{\emptyset, \{a\}\}$

(c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

(d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution

The set(d) is the power set of set {a, b} because the set has two elements a and b so that its power set has $2^2 = 4$ elements, which has the same number of elements of set(d).

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Exercise 27

Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution

There are two things we need to prove:

$$(\mathcal{P}(A) \subseteq \mathcal{P}(B) \to A \subseteq B) \wedge (A \subseteq B \to \mathcal{P}(A) \subseteq \mathcal{P}(B))$$

- $\mathcal{P}(A) \subseteq \mathcal{P}(B) \to A \subseteq B$:

  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ means that every element of power set A is also an element of power set B. Addtionally, we all know that power set of a set has 2^n elements created from the combinations of all the elements from the original set. Because all every element of power set A is also an element of power set B so we can infer that the element of set A is also an element of set B because they have the same combinations in the power set. Therefore, this case is true.

- $A \subseteq B \to \mathcal{P}(A) \subseteq \mathcal{P}(B)$

  $A \subseteq B$ means that every element of A is also an element of B. Because A and B have the same element so that there combinations of elements of these two sets will be the same. Therefore, the elements power set of A and B will be the same because both of them contain all subsets of A and B(We have that $A \subseteq B$). Therefore, this case is true.

Because both cases are true so that $(\mathcal{P}(A) \subseteq \mathcal{P}(B) \to A \subseteq B) \wedge (A \subseteq B \to \mathcal{P}(A) \subseteq \mathcal{P}(B))$ is true and it is equivalent to $\mathcal{P}(A) \subseteq \mathcal{P}(B) \leftrightarrow A \subseteq B$.

Exercise 28

Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

Solution

Because A is a subset of C and B is a subset of D. Suppose that we have set A, B, C and D:

- $A = \{a, b\}$
- $B = \{c, d\}$
- $C = \{a, b\}$
- $D = \{c, d\}$

We have that:

$$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$$
$$C \times D = \{(a, c), (a, d), (b, c), (b, d)\}$$

After using **Cartesian Product** to calculate $A \times B$, $C \times D$, we can see that every element $A \times B$ is also the element of $C \times D$. Therefore, we can infer that $A \times B \subseteq C \times D$.

Exercise 41

Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

Solution

Let $A = \{0, 1\}, B = \{1, 2\}, C = \{0, 1, 2\}$

$$\begin{aligned}
A \times B \times C = \{&(0,1,0), (0,1,1), (0,1,2), (0,2,0), (0,2,1), \\
&(0,2,2), (1,1,0), (1,1,1), (1,1,2), (1,2,0), (1,2,1), (1,2,2)\} \\
A \times B = \{&(0,1), (0,2), (1,1), (1,2)\} \\
(A \times B) \times C = \{&((0,1),0,), ((0,1),1), ((0,1),2), ((0,2),0), ((0,2),1), \\
&((0,2),2), ((1,1),0), ((1,1),1), ((1,1),2), ((1,2),0), ((1,2),1), ((1,2),2)\}
\end{aligned}$$

As we can see that $A \times B \times C$ gives us a set of 3-tuples has a form (a, b, c) with $a \in A, b \in B, c \in C$. However, $(A \times B) \times C$ gives us a set of 2-tuples has a form ((a, b), c) with $(a, b) \in A \times B$ and $c \in C$. This is different from $A \times B \times C$ because $A \times B \times C$ is a 3-tuples but $(A \times B) \times C$ is a 2-tuples which has the first element is a ordered pair of $A \times B$. Therefore, $A \times B \times C$ is different from $(A \times B) \times C$

Exercise 42

Explain why$(A \times B) \times (C \times D)$and $A \times (B \times C) \times D$ are not the same.

Solution

Let a, b, c, d are elements of set A, B, C, D respectively. Therefore, we have that $a \in A, b \in B, c \in C, d \in D$. We get that $A \times B$ will be a set consists ordered pairs (a, b) and $C \times D$ also consists ordered pair (c, d). Because (a,b) and (c,d) is an element of set $A \times B$ and $C \times D$. Therefore, if we do the Cartesian product $(A \times B) \times (C \times D)$. The result will be a new set consists ordered pair ((a,b),(c,d)) which has the two elements are two ordered pairs from $A \times B$ and $C \times D$.

However, $A \times (B \times C) \times D$ is completely different. We get $B \times C$ will be a set consists ordered pair (b, c) and continue to calculate $A \times (B \times C) \times D$, we will get a new set consists 3-tuples (a,(b,c),d) with the fist element is an element $\in A$, the second element is the element $\in (B \times C)$, and the last element is an element $\in D$.

Therefore, we can infer that $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are different.

Exercise 43

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

Solution

Let $A = \{0, 1\}$ and $B = 1, 2$. Therefore, we get that $|A| = |B| = 2$ and $|A \times B| = |A| \times |B| = 2 \times 2 = 4$. Because $A \times B$ has 4 elements so that $\mathcal{P}(A)$ will have $2^{|A| \times |B|} = 2^4 = 16$. However, we get that $|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$ and $|\mathcal{P}(B)| = 2^{|B|} = 2^2 = 4$ and

$$|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{|A|} \times 2^{|B|} = 2^{|A|+|B|} = 2^4 = 16$$

Although the result of $|\mathcal{P}(A \times B)|$ and $|\mathcal{P}(A) \times \mathcal{P}(B)|$ are both equal to 16 but the way they give the result 16 are completely different.

$$|\mathcal{P}(A \times B)| = 2^{|A| \times |B|} = 2^4 = 16$$
$$|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{|A|+|B|} = 16$$

Because $2^{|A| \times |B|} \neq 2^{|A|+|B|}$ so we can conclude that $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$

Exercise 44

Prove or disprove that if A, B, and C are nonempty sets and $A \times B = A \times C$, then $B = C$.

Solution

We know that $A \times B$ and $A \times C$ will create a set whose element is ordered pairs. Let $a \in A, b \in B, c \in C$ so that $A \times B$ will be (a, b) and $A \times C$ will be (a, c). Because we have that $A \times B = A \times C$ so that $(a, b) = (a, c)$. $(a, b) = (a, c)$ if and only if a = a and b = c. Addtionally, $b \in B, c \in C$ so that $B = C$.

Section 2.2

Exercise 15

Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(a) by showing each side is a subset of the other side.

(b) using membership table.

Solution

(a) To show each side is a subset of other side, we will need to prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

- $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$: Suppose x is an element in $\overline{A \cap B}$ so $x \in \overline{A \cap B}$. Therefore, $x \notin A \cap B$ and we can transfer it into $\neg((x \in A) \wedge (x \in B))$. Applying De Morgan law, it will be $\neg(x \in A) \vee \neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$ and can be transfered to $x \in \overline{A}$ or $x \in \overline{B}$. From this, we can infer that $x \in \overline{A} \cup \overline{B}$. Because $x \in \overline{A \cap B}$ and also $x \in \overline{A} \cup \overline{B}$ so that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.
- $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$: Suppose x is an element in $\overline{A} \cup \overline{B}$ so $x \in \overline{A} \cup \overline{B}$. Therefore, $x \notin A$ or $x \notin B$ and we can transfer it to $\neg(x \in A) \vee \neg(x \in B)$. Applying De Morgan law, it will be $\neg((x \in A) \wedge (x \in B))$. By the definition of intersection, it follows that $\neg(x \in A \cap B)$, which means that $x \notin A \cap B$ and we can infer that $x \in \overline{A \cap B}$. Because $x \in \overline{A} \cup \overline{B}$ and also $x \in \overline{A \cap B}$ so that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Because we have shown that each set is a subset of the other, so $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

(b) Using membership table:

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	1	1

From the membership table, we see that the value of $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are the same so that we can conclude that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Exercise 16

Let A and B be sets. Show that

(a) $(A \cap B) \subseteq A$

(b) $A \subseteq (A \cup B)$

(c) $A - B \subseteq A$

(d) $A \cap (B - A) = \emptyset$

(e) $A \cup (B - A) = A \cup B$

Solution

(a) We transfer $A \cap B$ in propositions, it will be $(x \in A) \wedge (x \in B)$. Because in $A \cap B$, we have $x \in A$ so that we can infer that $A \cap B$ is a subset of A.

(b) We transfer $A \cup B$ in propositions, it will be $(x \in A) \vee (x \in B)$. Because x is an element of A and in $A \cup B$, x also an element of A so A is a subset of B.

(c) We transfer $A - B = A \cap \overline{B}$ in propositions, it will be $(x \in A) \wedge (x \notin B)$. In $A \cap \overline{B}$ we have $x \in A$ so that x is an element in $A \cap \overline{B}$ and also an element of A. So $A \cap \overline{B}$ is a subset of A.

(d) Because $B - A = B \cap \overline{A}$. Transfer it into propositions, it will be $(x \in B) \wedge (x \notin A)$. x not an element of $B - A$ so that x can not be an elemnent of A. Therefore, if we use intersection operation, it will be $(x \in A) \wedge (x \in B) \wedge (x \notin A)$. We see that it is a contradiction because x can not be both an element of A and do not belong to A. Thus, it will be an $\emptyset$.

(e) Transfer $B - A = B \cap \overline{A}$. Then, it will be $A \cup (B \cap \overline{A}) = (A \cup B) \cap (A \cup \overline{A}) = (A \cup B) \cap U = A \cup B$.

Exercise 17

Show that if A and B are sets in a universe U then $A \subseteq B$ if and only if $\overline{A} \cup B = U$.

Solution

- $A \subseteq B \rightarrow \overline{A} \cup B = U$: $A \subseteq B$ means x is an element of A and x also an element of B. We get that $\overline{A} \cup B$ is $x \notin A \vee x \in B$. Because $x \in B$ and x also in A due to $A \subseteq B$. Moreover, A and B are sets in universe U so that $\overline{A} \cup B$ still in be in universe.

- $\overline{A} \cup B = U \rightarrow A \subseteq B$: From the complement laws, we have that $\overline{A} \cup A = U$. Therefore, if $\overline{A} \cup B = U$ then x must be an elemnent of A and also an element of B. Therefore, we have that $(x \in A) \wedge (x \in B)$, which means that $A \subseteq B$.

Because both cases above are all make sense so we can conclude that if A and B are sets in a universe U then

$$A \subseteq B \leftrightarrow \overline{A} \cup B = U$$