Homework 2

Nguyen Tuan Anh - 2252038 - CN01

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Section 2.1

Exercise 11

Determine whether each of these statements is true or false.

(a) $0 \in \emptyset$

(b) $\emptyset \in \{0\}$

(c) $\{0\} \subset \emptyset$

(d) $\emptyset \subset \{0\}$

(e) $\{0\} \in \{0\}$

(f) $\{0\} \subset \{0\}$

(g) $\{\emptyset\} \subseteq \{\emptyset\}$

Solution

(a) $0 \in \emptyset$: This statement is false because $\emptyset$ has no elements so 0 can not be an element of the empty set.

(b) $\emptyset \in \{0\}$: This statement is false because $\emptyset$ is not an element in set $\{0\}$.

(c) $\{0\} \subset \emptyset$: This statement is false because $\emptyset$ has no elements so that $\{0\}$ can not be a subset of $\emptyset$.

(d) $\emptyset \subset \{0\}$: This statement is true because $\emptyset$ is one of the two sets that every nonempty set is guaranteed to have.

(e) $\{0\} \in \{0\}$: This statement is false because $\{0\}$ is an element of $\{\{0\}\}$ not an element of $\{0\}$.

(f) $\{0\} \subset \{0\}$: This statement is false because the two set all have the same elements 0 so that is must be $\subseteq$.

(g) $\{\emptyset\} \subseteq \{\emptyset\}$: This statement is true because both singleton set have the same element $\emptyset$. Therefore, this statement is true.

Exercise 12

Determine whether these statements are true or false.

(a) $\emptyset \in \{\emptyset\}$

(b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$

(c) $\{\emptyset\} \in \{\emptyset\}$

(d) $\{\emptyset\} \in \{\{\emptyset\}\}$

(e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$

(f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$

(g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

Solution

(a) $\emptyset \in \{\emptyset\}$: This statement is true because $\emptyset$ is an element of a singleton set contains element $\emptyset$.

(b) $\emptyset \in \{\emptyset, \{\emptyset\}\}$: This statement is true because $\emptyset$ is an element of the set $\{\emptyset, \{\emptyset\}\}$.

(c) $\{\emptyset\} \in \{\emptyset\}$: This statement is false because $\emptyset$ must be an element of $\{\{\emptyset\}\}$.

(d) $\{\emptyset\} \in \{\{\emptyset\}\}$: This statement is true because the set $\{\{\emptyset\}\}$ contains $\{\emptyset\}$.

(e) $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$: This statement is true because $\emptyset$ is an element is the set $\{\emptyset, \{\emptyset\}\}$ so the set contains $\emptyset$ is a subset of $\{\emptyset, \{\emptyset\}\}$.

(f) $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$: This statement is true and its reason is the same as problem (e).

(g) $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$: We can see that the set $\{\{\emptyset\}, \{\emptyset\}\}$ has two elements which are equal to each other. Therefore, we can simplify it to $\{\{\emptyset\}\}$. Therefore, this statement is false because these sets are equal to each other so it must be $\subseteq$ instead of $\subset$.

Exercise 13

Determine whether each of these statements is true or false.

(a) $x \in \{x\}$

(b) $\{x\} \subseteq \{x\}$

(c) $\{x\} \in \{x\}$

(d) $\{x\} \in \{\{x\}\}$

(e) $\emptyset \subseteq \{x\}$

(f) $\emptyset \in \{x\}$

Solution

(a) $x \in \{x\}$: This statement is true because x is an element in set x.

(b) $\{x\} \subseteq \{x\}$: This statement is true.

(c) $\{x\} \in \{x\}$: This statement is false because x is an element of $\{\{x\}\}$ not $\{x\}$.

(d) $\{x\} \in \{\{x\}\}$: This statement is true due to the reason from problem(c).

(e) $\emptyset \subseteq \{x\}$: This statement is true according to **Theorem 1**.

(f) $\emptyset \in \{x\}$: This statement is false because $\emptyset$ is not an element of set $\{x\}$.

Exercise 26

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

(a) $\emptyset$

(b) $\{\emptyset, \{a\}\}$

(c) $\{\emptyset, \{a\}, \{\emptyset, a\}\}$

(d) $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

Solution

The set(d) is the power set of set {a, b} because the set has two elements a and b so that its power set has $2^2 = 4$ elements, which has the same number of elements of set(d).

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

Exercise 27

Prove that $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ if and only if $A \subseteq B$.

Solution

There are two things we need to prove:

$$(\mathcal{P}(A) \subseteq \mathcal{P}(B) \to A \subseteq B) \wedge (A \subseteq B \to \mathcal{P}(A) \subseteq \mathcal{P}(B))$$

- $\mathcal{P}(A) \subseteq \mathcal{P}(B) \to A \subseteq B$:

  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$ means that every element of power set A is also an element of power set B. Addtionally, we all know that power set of a set has 2^n elements created from the combinations of all the elements from the original set. Because all every element of power set A is also an element of power set B so we can infer that the element of set A is also an element of set B because they have the same combinations in the power set. Therefore, this case is true.

- $A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

  $A \subseteq B$ means that every element of A is also an element of B. Because A and B have the same element so that there combinations of elements of these two sets will be the same. Therefore, the elements power set of A and B will be the same because both of them contain all subsets of A and B(We have that $A \subseteq B$). Therefore, this case is true.

Because both cases are true so that $(\mathcal{P}(A) \subseteq \mathcal{P}(B) \rightarrow A \subseteq B) \wedge (A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B))$ is true and it is equivalent to $\mathcal{P}(A) \subseteq \mathcal{P}(B) \leftrightarrow A \subseteq B$.

Exercise 28

Show that if $A \subseteq C$ and $B \subseteq D$, then $A \times B \subseteq C \times D$

Solution

Because A is a subset of C and B is a subset of D. Suppose that we have set A, B, C and D:

- $A = \{a, b\}$
- $B = \{c, d\}$
- $C = \{a, b\}$
- $D = \{c, d\}$

We have that:

$$A \times B = \{(a, c), (a, d), (b, c), (b, d)\}$$
$$C \times D = \{(a, c), (a, d), (b, c), (b, d)\}$$

After using **Cartesian Product** to calculate $A \times B$, $C \times D$, we can see that every element $A \times B$ is also the element of $C \times D$. Therefore, we can infer that $A \times B \subseteq C \times D$.

Exercise 41

Explain why $A \times B \times C$ and $(A \times B) \times C$ are not the same.

Solution

Let $A = \{0, 1\}, B = \{1, 2\}, C = \{0, 1, 2\}$

$$\begin{aligned} A \times B \times C = \{&(0,1,0),(0,1,1),(0,1,2),(0,2,0),(0,2,1),\\ &(0,2,2),(1,1,0),(1,1,1),(1,1,2),(1,2,0),(1,2,1),(1,2,2)\} \\ A \times B = \{&(0,1),(0,2),(1,1),(1,2)\} \\ (A \times B) \times C = \{&((0,1),0,),((0,1),1),((0,1),2),((0,2),0),((0,2),1),\\ &((0,2),2),((1,1),0),((1,1),1),((1,1),2),((1,2),0),((1,2),1),((1,2),2)\} \end{aligned}$$

As we can see that $A \times B \times C$ gives us a set of 3-tuples has a form (a, b, c) with $a \in A, b \in B, c \in C$. However, $(A \times B) \times C$ gives us a set of 2-tuples has a form ((a, b), c) with $(a, b) \in A \times B$ and $c \in C$. This is different from $A \times B \times C$ because $A \times B \times C$ is a 3-tuples but $(A \times B) \times C$ is a 2-tuples which has the first element is a ordered pair of $A \times B$. Therefore, $A \times B \times C$ is different from $(A \times B) \times C$

Exercise 42

Explain why$(A \times B) \times (C \times D)$and $A \times (B \times C) \times D$ are not the same.

Solution

Let a, b, c, d are elements of set A, B, C, D respectively. Therefore, we have that $a \in A, b \in B, c \in C, d \in D$. We get that $A \times B$ will be a set consists ordered pairs (a, b) and $C \times D$ also consists ordered pair (c, d). Because (a,b) and (c,d) is an element of set $A \times B$ and $C \times D$. Therefore, if we do the Cartesian product $(A \times B) \times (C \times D)$. The result will be a new set consists ordered pair ((a,b),(c,d)) which has the two elements are two ordered pairs from $A \times B$ and $C \times D$.

However, $A \times (B \times C) \times D$ is completely different. We get $B \times C$ will be a set consists ordered pair (b, c) and continue to calculate $A \times (B \times C) \times D$, we will get a new set consists 3-tuples (a,(b,c),d) with the fist element is an element $\in A$, the second element is the element $\in (B \times C)$, and the last element is an element $\in D$.

Therefore, we can infer that $(A \times B) \times (C \times D)$ and $A \times (B \times C) \times D$ are different.

Exercise 43

Prove or disprove that if A and B are sets, then $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

Solution

Let $A = \{0, 1\}$ and $B = 1, 2$. Therefore, we get that $|A| = |B| = 2$ and $|A \times B| = |A| \times |B| = 2 \times 2 = 4$. Because $A \times B$ has 4 elements so that $\mathcal{P}(A)$ will have $2^{|A| \times |B|} = 2^4 = 16$. However, we get that $|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$ and $|\mathcal{P}(B)| = 2^{|B|} = 2^2 = 4$ and

$$|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{|A|} \times 2^{|B|} = 2^{|A|+|B|} = 2^4 = 16$$

Although the result of $|\mathcal{P}(A \times B)|$ and $|\mathcal{P}(A) \times \mathcal{P}(B)|$ are both equal to 16 but the way they give the result 16 are completely different.

$$|\mathcal{P}(A \times B)| = 2^{|A| \times |B|} = 2^4 = 16$$
$$|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{|A|+|B|} = 16$$

Because $2^{|A| \times |B|} \neq 2^{|A|+|B|}$ so we can conclude that $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$

Exercise 44

Prove or disprove that if A, B, and C are nonempty sets and $A \times B = A \times C$, then $B = C$.

Solution

We know that $A \times B$ and $A \times C$ will create a set whose element is ordered pairs. Let $a \in A, b \in B, c \in C$ so that $A \times B$ will be (a, b) and $A \times C$ will be (a, c). Because we have that $A \times B = A \times C$ so that $(a, b) = (a, c)$. $(a, b) = (a, c)$ if and only if a = a and b = c. Addtionally, $b \in B, c \in C$ so that $B = C$.

Exercise 49

The defining property of an ordered pair is that two ordered pairs are equal if and only if their first elements are equal and their second elements are equal. Surprisingly, instead of taking the ordered pair as a primitive concept, we can construct ordered pairs using basic notions from set theory. Show that if we define the ordered pair (a, b) to be {{a}, {a, b}}, then (a, b) = (c, d) if and only if a = c and b = d.

Solution

Define the ordered pairs (a,b) and (c,d) to be{{a},{a,b}} and {{c},{c,d}}. We need to prove that $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\} \leftrightarrow a = b \text{ } and \text{ } c = d$

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Firstly, we prove that $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\} \rightarrow a = b \text{ } and \text{ } c = d$, suppose that $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$ so that $a = c$ and we can infer that $a = c$. For $\{a,b\} = \{c,d\}$, we have that $a = c$ so that $b = d$.

Secondly, we prove that $a = b \text{ } and \text{ } c = d \rightarrow \{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$, suppose that $a = c$ and $b = d$ so that we can infer $\{a\} \subseteq \{c\}$ and $\{c\} \subseteq \{a\}$ so that $\{a\} = \{c\}$. Because $a = c$ and $b = d$ it result in $\{a,b\} \subseteq \{c,d\}$ and $\{c,d\} \subseteq \{a,b\}$, which means that $\{a,b\} = \{c,d\}$. We have $\{a\} = \{c\}$ and $\{a,b\} = \{c,d\}$, we infer that $\{\{a\},\{a,b\}\} \subseteq \{\{c\},\{c,d\}\}$ and $\{\{c\},\{c,d\}\} \subseteq \{\{a\},\{a,b\}\}$. Therefore, $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\}$.

Because two cases above are both true so we can conclude that $\{\{a\},\{a,b\}\} = \{\{c\},\{c,d\}\} \leftrightarrow a = b \text{ } and \text{ } c = d$. Moreover, because the ordered pair is defined in {{a},{a, b}} so that it can be transfered to $(a,b) = (c,d) \leftrightarrow a = b \text{ } and \text{ } c = d$.

Exercise 50

This exercise presents Russell's paradox. Let S be the that contains a set x if the set x does not belong to itself, so that $S = \{x | x \notin x\}$.

(a) Show the assumption that S is a member of S leads to a contradiction.

(b) Show the assumption that S is not a member of S leads to a contradiction.

Solution

(a) If S is a member of S it means that S belongs to itself($S \in S$). However, we define that **S be the that contains a set x if the set x does not belong to itself** so that we get $S \notin S$ but the assumption is $S \in S$, therefore, it leads to a contradiction.

(b) If S not a member of S, it satisfies with the hypothesis **S be the that contains a set x if the set x does not belong to itself** so that we get $S \in S$. However, the assumption is $S \notin S$ so it leads to a contradiction.

Exercise 51

Describe a procedure for listing all the subsets of a finite set.

Solution

To list all the subsets of a finite set, we will use the binary to represent it. Let X is a set contains $\{a_1, a_2, a_3, ..., a_n\}$. We will represent these elements of the set in binary, with n elements, it will be n digits of the binary and the number of subset will be $2^n - 1$. The order of the digits in binary is the same as the order of the elements in the subset. If digits *i(i⩽ n)* is 1 so it in a subset and if it is 0, it will be remove from the subset.

For example, $X = \{1, 2, 3\}$. Because there are 3 elements in the set X so that it will be $2^3 - 1 = 7$ subsets and 3 binary digits.

$$\begin{aligned}
000 &= \{\} \\
001 &= \{3\} \\
010 &= \{2\} \\
011 &= \{2, 3\} \\
100 &= \{1\} \\
101 &= \{1, 3\} \\
110 &= \{1, 2\} \\
111 &= \{1, 2, 3\}
\end{aligned}$$

Section 2.2

Exercise 15

Prove the second De Morgan law in Table 1 by showing that if A and B are sets, then $\overline{A \cap B} = \overline{A} \cup \overline{B}$

(a) by showing each side is a subset of the other side.

(b) using membership table.

Solution

(a) To show each side is a subset of other side, we will need to prove that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ and $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$

- $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$: Suppose x is an element in $\overline{A \cap B}$ so $x \in \overline{A \cap B}$. Therefore, $x \notin A \cap B$ and we can transfer it into $\neg((x \in A) \wedge (x \in B))$. Applying De Morgan law, it will be $\neg(x \in A) \vee \neg(x \in B)$. Using the definition of negation of propositions, we have $x \notin A$ or $x \notin B$ and can be transfered to $x \in \overline{A}$ or $x \in \overline{B}$. From this, we can infer that $x \in \overline{A} \cup \overline{B}$. Because $x \in \overline{A \cap B}$ and also $x \in \overline{A} \cup \overline{B}$ so that $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$.
- $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$: Suppose x is an element in $\overline{A} \cup \overline{B}$ so $x \in \overline{A} \cup \overline{B}$. Therefore, $x \notin A$ or $x \notin B$ and we can transfer it to $\neg(x \in A) \vee \neg(x \in B)$. Applying De Morgan law, it will be $\neg((x \in A) \wedge (x \in B))$. By the definition of intersection, it follows that $\neg(x \in A \cap B)$, which means that $x \notin A \cap B$ and we can infer that $x \in \overline{A \cap B}$. Because $x \in \overline{A} \cup \overline{B}$ and also $x \in \overline{A \cap B}$ so that $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$.

Because we have shown that each set is a subset of the other, so $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

(b) Using membership table:

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	1	1

From the membership table, we see that the value of $\overline{A \cap B}$ and $\overline{A} \cup \overline{B}$ are the same so that we can conclude that $\overline{A \cap B} = \overline{A} \cup \overline{B}$.

Exercise 16

Let A and B be sets. Show that

(a) $(A \cap B) \subseteq A$

(b) $A \subseteq (A \cup B)$

(c) $A - B \subseteq A$

(d) $A \cap (B - A) = \emptyset$

(e) $A \cup (B - A) = A \cup B$

Solution

(a) We transfer $A \cap B$ in propositions, it will be $(x \in A) \wedge (x \in B)$. Because in $A \cap B$, we have $x \in A$ so that we can infer that $A \cap B$ is a subset of A.

(b) We transfer $A \cup B$ in propositions, it will be $(x \in A) \vee (x \in B)$. Because x is an element of A and in $A \cup B$, x also an element of A so A is a subset of B.

(c) We transfer $A - B = A \cap \overline{B}$ in propositions, it will be $(x \in A) \wedge (x \notin B)$. In $A \cap \overline{B}$ we have $x \in A$ so that x is an element in $A \cap \overline{B}$ and also an element of A. So $A \cap \overline{B}$ is a subset of A.

(d) Because $B - A = B \cap \overline{A}$. Transfer it into propositions, it will be $(x \in B) \wedge (x \notin A)$. x not an element of $B - A$ so that x can not be an elemnent of A. Therefore, if we use intersection operation, it will be $(x \in A) \wedge (x \in B) \wedge (x \notin A)$. We see that it is a contradiction because x can not be both an element of A and do not belong to A. Thus, it will be an $\emptyset$.

(e) Transfer $B - A = B \cap \overline{A}$. Then, it will be $A \cup (B \cap \overline{A}) = (A \cup B) \cap (A \cup \overline{A}) = (A \cup B) \cap U = A \cup B$.

Exercise 17

Show that if A and B are sets in a universe U then $A \subseteq B$ if and only if $\overline{A} \cup B = U$.

Solution

- $A \subseteq B \to \overline{A} \cup B = U$: $A \subseteq B$ means x is an element of A and x also an element of B. We get that $\overline{A} \cup B$ is $x \notin A \vee x \in B$. Because $x \in B$ and x also in A due to $A \subseteq B$. Moreover, A and B are sets in universe U so that $\overline{A} \cup B$ still in be in universe.
- $\overline{A} \cup B = U \to A \subseteq B$: From the complement laws, we have that $\overline{A} \cup A = U$. Therefore, if $\overline{A} \cup B = U$ then x must be an elemnent of A and also an element of B. Therefore, we have that $(x \in A) \wedge (x \in B)$, which means that $A \subseteq B$.

Because both cases above are all make sense so we can conclude that if A and B are sets in a universe U then

$$A \subseteq B \leftrightarrow \overline{A} \cup B = U$$

Exercise 19

Show that if A, B, and C are sets, then $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$

(a) by showing each side is a subset of other side.

(b) using a membership table.

Solution

(a) We need to prove that $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$ and conversely.

Firstly, suppose x is an element of $\overline{A \cap B \cap C}$, we transfer $\overline{A \cap B \cap C}$ in propositions, it will be $\neg((x \in A) \wedge (x \in B) \wedge (x \in C))$. Applying the De Morgan law in propositions, we transfer it to $\neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)$. Moreover, it is also equivalent to $(x \notin A) \vee (x \notin B) \vee (x \notin C) \equiv (x \in \overline{A}) \vee (x \in \overline{B}) \vee (x \in \overline{C})$. According to the definition of *union*, $(x \in \overline{A}) \vee (x \in \overline{B}) \vee (x \in \overline{C}) = \overline{A} \cup \overline{B} \cup \overline{C}$. Therefore, $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$.

Secondly, suppose x is an element of $\overline{A} \cup \overline{B} \cup \overline{C}$we transfer $\overline{A} \cup \overline{B} \cup \overline{C}$ in propositions, it will be $(x \in \overline{A}) \vee (x \in \overline{B}) \vee (x \in \overline{C}) \equiv (x \notin A) \vee (x \notin B) \vee (x \notin C) \equiv \neg(x \in A) \vee \neg(x \in B) \vee \neg(x \in C)$. Applying De Morgan law in propositions, we transfer it to $\neg((x \in A) \wedge (x \in B) \wedge (x \in C))$. By the definition of intersection, it will be $\neg(A \cap B \cap C)$, which means that $x \notin A \cap B \cap C$. Therefore, we can infer that $x \in \overline{A \cap B \cap C}$. Thus, $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$.

Because we have proved that $\overline{A \cap B \cap C} \subseteq \overline{A} \cup \overline{B} \cup \overline{C}$ and $\overline{A} \cup \overline{B} \cup \overline{C} \subseteq \overline{A \cap B \cap C}$ so that we can infer $\overline{A \cap B \cap C} = \overline{A} \cup \overline{B} \cup \overline{C}$.

(b) We have the membership table:

A	B	C	$\overline{A}$	$\overline{B}$	$\overline{C}$	$\overline{A \cap B \cap C}$	$\overline{A} \cup \overline{B} \cup \overline{C}$
1	1	1	0	0	0	0	0
1	1	0	0	0	1	1	1
1	0	1	0	1	0	1	1
1	0	0	0	1	1	1	1
0	1	1	1	0	0	1	1
0	1	0	1	0	1	1	1
0	0	1	1	0	0	1	1
0	0	0	1	1	1	1	1

Exercise 20

Let A, B, and C be sets. Show that:

(a) $(A \cup B) \subseteq (A \cup B \cup C)$

(b) $(A \cap B \cap C) \subseteq (A \cap B)$

(c) $(A - B) - C \subseteq A - C$

(d) $(A - C) \cap (C - B) = \emptyset$

(e) $(B - A) \cup (C - A) = (B \cup C) - A$

Solution

(a) Suppose that $x \in (A \cup B)$, in propositions it is $(x \in A) \vee (x \in B)$ and $A \cup B \cup C$ in propositions will be $(x \in A) \vee (x \in B) \vee (x \in C)$. Because in $A \cup B \cup C$, we also have that $x \in A$ and $x \in B$. Therefore, $(A \cup B) \subseteq (A \cup B \cup C)$.

(b) Suppose that $x \in (A \cap B \cap C)$ so x is an element that A, B, C all have. In $A \cap B$, x is the elements that both A and B have. Therefore, the element in $A \cap B \cap C$ will also the element in $A \cap B$. Thus, $(A \cap B \cap C) \subseteq (A \cap B)$.

(c) $(A - B) - C = (A \cap \overline{B}) \cap \overline{C} = A \cap \overline{B} \cap \overline{C}$. Suppose that $x \in (A \cap \overline{B} \cap \overline{C})$ so x is an element that $A, \overline{B}, \overline{C}$. In $A - C = A \cap \overline{C}$, x is the element that both $A, \overline{C}$ have. Therefore, the element in $(A - B) - C$ will also the element in $A - C$. Thus, $(A - B) - C \subseteq A - C$.

(d) $(A - C) \cap (B - C) = A \cap \overline{C} \cap C \cap \overline{B}$. According to **Complement laws**, $C \cap \overline{C} = \emptyset$. Due to **Identity laws**, $A \cap \overline{B} \cap \emptyset = \emptyset$.

(e) We have that:

$$\begin{aligned}
(B - A) \cup (C - A) &= (B \cap \overline{A}) \cup (C \cap \overline{A}) \\
(B \cup C) - A &= (B \cup C) \cap \overline{A} \\
&= (B \cap \overline{A}) \cup (C \cap \overline{A})
\end{aligned}$$

Because both of them are equal to $(B \cap \overline{A}) \cup (C \cap \overline{A})$ so that $(B - A) \cup (C - A) = (B \cup C) - A$.

Exercise 21

Show that if A and B are sets, then

(a) $A - B = A \cap \overline{B}$

(b) $(A \cap B) \cup (A \cap \overline{B}) = A$

Solution

(a) $A - B = A \cap \overline{B}$: We got that $A - B$ is the difference of A and B. According to the definition of *difference*, transfer in propositions $(x \in A) \wedge (x \notin B)$, which is the same as $A \cap \overline{B}$ because in propositions, $A \cap \overline{B}$ is $(x \in A) \cap (x \in \overline{B})$. $x \in \overline{B}$ means x is the element not in B. Therefore, $A - B = A \cap \overline{B}$.

(b) $(A \cap B) \cup (A \cap \overline{B}) = A$

$$\begin{aligned}
(A \cap B) \cup (A \cap \overline{B}) &= ((A \cap B) \cup A) \cap ((A \cap B) \cup \overline{B}) \\
&= (A \cup A) \cap (B \cup A) \cap (A \cup \overline{B}) \\
&= A \cap (B \cup A) \cap (A \cup \overline{B}) \\
&= A \cap ((B \cap A) \cup (B \cap \overline{B}) \cup (A \cap A) \cup (A \cap \overline{B})) \\
&= A \cap ((B \cap A) \cup \emptyset \cup A \cup (A \cap \overline{B})) \\
&= A \cap ((A \cap (B \cup \overline{B})) \cup A) \\
&= A \cap A = A
\end{aligned}$$

Exercise 35

Let A, B, C be sets. Use the identities in Table 1 to show that $\overline{A \cup B} \cap \overline{B \cup C} \cap \overline{A \cup C} = \overline{A} \cap \overline{B} \cap \overline{C}$

Solution

Applying De Morgan law, we have that:

$$\begin{aligned}
\overline{A \cup B} &= \overline{A} \cap \overline{B} \\
\overline{B \cup C} &= \overline{B} \cap \overline{C} \\
\overline{A \cup C} &= \overline{A} \cap \overline{C} \\
\overline{A \cup B} \cap \overline{B \cup C} \cap \overline{A \cup C} &= \overline{A} \cap \overline{B} \cap \overline{B} \cap \overline{C} \cap \overline{A} \cap \overline{C} \\
&= \overline{A} \cap \overline{B} \cap \overline{C}
\end{aligned}$$

Exercise 36

Prove or disprove that for all sets A, B, and C, we have

(a) $A \times (B \cup C) = (A \times B) \cup (A \times C)$

(b) $A \times (B \cap C) = (A \times B) \cap (A \times C)$

Solution

(a) Let x, y an ordered pair of a Cartesian Product of 2 sets. $A \times (B \cup C)$ create ordered pair (x, y) with $x \in A$ and $y \in B \vee y \in C$. In $(A \times B) \cup (A \times C)$, we have that $A \times B$ create a ordered pair $x \in A \wedge y \in B$, $A \times C$ create a ordered pair $x \in A \wedge y \in C$. Represent in propositions, we have that:

$$\begin{aligned}
(x \in A) \wedge (y \in C) \vee (x \in A) \wedge (y \in C) \\
= (x \in A) \wedge (y \in C \vee y \in B)
\end{aligned}$$

Which is the same propositions as $A \times (B \cup C)$.

(b) Let x, y an ordered pair of a Cartesian Product of 2 sets. $A \times (B \cap C)$ create (x, y) with $x \in A \wedge (y \in B \wedge y \in C)$. In $(A \times B) \cap (A \times C)$, we have that $A \times B$ create an ordered pair $x \in A \wedge y \in B$, $A \times C$ create an ordered pair (x, y) with $x \in A \wedge x \in C$. Represent in propositions, we have that:

$$\begin{aligned} x \in A \wedge y \in B \wedge x \in A \wedge y \in C \\ = (x \in A) \wedge (y \in B \wedge y \in C) \end{aligned}$$

Which is the same propositions as $A \times (B \cap C)$.

Exercise 37

Prove or disprove that for all sets A, B, and C, we have

(a) $A \times (B - C) = (A \times B) - (A \times C)$

(b) $\overline{A} \times \overline{B \cup C} = \overline{A \times (B \cup C)}$

Solution

(a) In propositions, $A \times (B - C)$ can be represented in $(x \in A) \wedge (y \in B \wedge y \notin C)$. In $A \times B$, it is $x \in A \wedge y \in B$, $A \times C$ is $x \in A \wedge y \in C$. Therefore, if we take the *difference*, it will be:

$$\begin{aligned} &(x \in A \wedge y \in B) \wedge \neg(x \in A \wedge y \in C) \\ =&(x \in A \wedge y \in B) \wedge (x \notin A \vee y \notin C) \\ =&(x \in A \wedge x \notin A \wedge y \in B) \vee (x \in A \wedge y \in B \wedge y \notin C) \\ =&\emptyset \vee (x \in A \wedge y \in B \wedge y \notin C) \\ =&(x \in A) \wedge (y \in B \wedge y \notin C). \end{aligned}$$

This propositions is the same as $A \times (B - C)$.

(b) We have that:

$$\begin{aligned} \overline{A} \times \overline{B \cup C} &= \overline{A} \times (\overline{B} \cap \overline{C}) \\ &= x \notin A \wedge (y \notin B \wedge y \notin C) \\ \overline{A \times (B \cup C)} &= \neg(x \in A \wedge (y \in B \vee y \in C)) \\ &= x \notin A \vee (y \notin B \wedge y \notin C) \end{aligned}$$

We see that the difference is the $\wedge$ and $\vee$ operator behind $x \notin A$. Therefore,

$$\overline{A} \times \overline{B \cup C} \neq \overline{A \times (B \cup C)}$$

Section 2.3

Exercise 25

Let $f : \mathbb{R} \to \mathbb{R}$ and let $f(x) > 0$ for all $x \in R$. Show that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.

Solution

We need to prove that $f(x)$ is strictly decreasing $\rightarrow g(x) = 1/f(x)$ is strictly increasing and conversely.

- Suppose $f(x)$ is strictly decreasing. When it strictly decreasing, we know that $f(x) > f(y)$, whenever $x < y$. And we take these 2 x and y to be $f(x)$ and $f(y)$ in $g(x)$. $g(x) = 1/f(x)$, $g(y) = 1/f(y)$. Because $f(x) > f(y)$ so that $g(x) = 1/f(x) < g(y) = 1/f(y)$ so that $g(x)$ is stricly increasing.
- Suppose $g(x)$ is strictly increasing. When it strictly increasing, we know that $g(x) < g(y)$, whenever $x < y$. We have that $g(x) = 1/f(x) < g(y) = 1/f(y)$. Because $1/f(x) < 1/f(y)$ so that $f(x) > f(y)$ and $x > y$. Therefore, $f(x)$ is strictly decreasing.

Because both cases are **true** so that $f(x)$ is strictly decreasing if and only if the function $g(x) = 1/f(x)$ is strictly increasing.

Exercise 26

(a) Prove that a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.

(b) Give an example of an increasing function from $\mathbb{R}$ to itself is not one-to-one.

Solution

(a) In a strictly increasing function, we have that $f(x) < f(y)$, whenever $x < y$. From this clue, we know that $f(x) \neq f(y)$ and $x \neq y$. According to definition of one-to-one function, the function is one-to-one if and only if $f(a) \notin f(b)$ whenever $a \neq b$, which is the same as the inference got from the strictly increasing function. Therefore, a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.

(b) The function $f(x) = x^2$ is not one-to-one function because there are 2 values x and $-x$ have the same image.

Exercise 27

(a) Prove that a strictly decreasing function from $\mathbb{R}$ to itself is one-to-one.

(b) Give an example of a decreasing function from $\mathbb{R}$ to itselft that is not one-to-one.

Solution

(a) In a strictly increasing function, we have that $f(x) > f(y)$, whenever $x < y$. From this clue, we know that $f(x) \neq f(y)$ and $x \neq y$. According to definition of one-to-one function, the function is one-to-one if and only if $f(a) \notin f(b)$ whenever $a \neq b$, which is the same as the inference got from the strictly increasing function. Therefore, a strictly increasing function from $\mathbb{R}$ to itself is one-to-one.

(b) The function $f(x) = x^2 - x$ is not one-to-one function because there are 2 values $x = 1$ and $x = 0$ have the same image $y = 0$.

Exercise 33

Suppose that g is a function from A to B and f is a function from B to C

(a) Show that if both f and g are one-to-one functions, then $f \circ g$ is also one-to-one.

(b) Show that if both f and g are onto functions, then $f \circ g$ is also onto.

Solution

(a) g is an one-to-one function so that $g(a) \neq g(b)(g(a), g(b) \in B)$ whenever $a \neq b(a, b \in A)$. f is an one-to-one function so that $f(a) \neq f(b)(f(a), f(b) \in C)$ whenever $a \neq b(a, b \in B)$. Therefore, $f \circ g(a) \neq f \circ g(b) = f(g(a)) \neq f(g(b))$ because f is one-to-one function so that $f(g(a)) \neq f(g(b))$ whenever $g(a) \neq g(b)$ and because g is one-to-one function that $g(a) \neq g(b)$ whenever $a \neq b$ that $f \circ g$ is one-to-one function($f(g(a)) \neq f(g(b))$ whenever $a \neq b$) when f and g are both one-to-one functions.

(b) f is an onto function so that for any $c \in C$, there is a $b \in B$ to $f(b) = c$. Similar to g, for any $b \in B$, there is an $a \in A$ to $g(a) = b$. Therefore, $(f \circ g)(a) = f(g(a)) = f(b) = c$. So for every $c \in C$, there is an $a \in A$ to $f(g(a)) = f(b) = c$ so that $f \circ g$ is onto.