

# Homework 2

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## Section 2.1

### Exercise 11

Determine whether each of these statements is true or false.

- (a)  $0 \in \emptyset$
- (b)  $\emptyset \in \{0\}$
- (c)  $\{0\} \subset \emptyset$
- (d)  $\emptyset \subset \{0\}$
- (e)  $\{0\} \in \{0\}$
- (f)  $\{0\} \subset \{0\}$
- (g)  $\{\emptyset\} \subseteq \{\emptyset\}$

### Solution

- (a)  $0 \in \emptyset$ : This statement is false because  $\emptyset$  has no elements so 0 can not be an element of the empty set.
- (b)  $\emptyset \in \{0\}$ : This statement is false because  $\emptyset$  is not an element in set  $\{0\}$ .
- (c)  $\{0\} \subset \emptyset$ : This statement is false because  $\emptyset$  has no elements so that  $\{0\}$  can not be a subset of  $\emptyset$ .
- (d)  $\emptyset \subset \{0\}$ : This statement is true because  $\emptyset$  is one of the two sets that every nonempty set is guaranteed to have.
- (e)  $\{0\} \in \{0\}$ : This statement is false because  $\{0\}$  is an element of  $\{\{0\}\}$  not an element of  $\{0\}$ .
- (f)  $\{0\} \subset \{0\}$ : This statement is false because the two set all have the same elements 0 so that is must be  $\subseteq$ .
- (g)  $\{\emptyset\} \subseteq \{\emptyset\}$ : This statement is true because both singleton set have the same element  $\emptyset$ . Therefore, this statement is true.

## Exercise 12

Determine whether these statements are true or false.

- (a)  $\emptyset \in \{\emptyset\}$
- (b)  $\emptyset \in \{\emptyset, \{\emptyset\}\}$
- (c)  $\{\emptyset\} \in \{\emptyset\}$
- (d)  $\{\emptyset\} \in \{\{\emptyset\}\}$
- (e)  $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$
- (f)  $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$
- (g)  $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$

## Solution

- (a)  $\emptyset \in \{\emptyset\}$ : This statement is true because  $\emptyset$  is an element of a singleton set contains element  $\emptyset$ .
- (b)  $\emptyset \in \{\emptyset, \{\emptyset\}\}$ : This statement is true because  $\emptyset$  is an element of the set  $\{\emptyset, \{\emptyset\}\}$ .
- (c)  $\{\emptyset\} \in \{\emptyset\}$ : This statement is false because  $\emptyset$  must be an element of  $\{\{\emptyset\}\}$ .
- (d)  $\{\emptyset\} \in \{\{\emptyset\}\}$ : This statement is true because the set  $\{\{\emptyset\}\}$  contains  $\{\emptyset\}$ .
- (e)  $\{\emptyset\} \subset \{\emptyset, \{\emptyset\}\}$ : This statement is true because  $\emptyset$  is an element is the set  $\{\emptyset, \{\emptyset\}\}$  so the set contains  $\emptyset$  is a subset of  $\{\emptyset, \{\emptyset\}\}$ .
- (f)  $\{\{\emptyset\}\} \subset \{\emptyset, \{\emptyset\}\}$ : This statement is true and its reason is the same as problem (e).
- (g)  $\{\{\emptyset\}\} \subset \{\{\emptyset\}, \{\emptyset\}\}$ : We can see that the set  $\{\{\emptyset\}, \{\emptyset\}\}$  has two elements which are equal to each other. Therefore, we can simplify it to  $\{\{\emptyset\}\}$ . Therefore, this statement is false because these sets are equal to each other so it must be  $\subseteq$  instead of  $\subset$ .

## Exercise 13

Determine whether each of these statements is true or false.

- (a)  $x \in \{x\}$
- (b)  $\{x\} \subseteq \{x\}$
- (c)  $\{x\} \in \{x\}$
- (d)  $\{x\} \in \{\{x\}\}$
- (e)  $\emptyset \subseteq \{x\}$
- (f)  $\emptyset \in \{x\}$

**Solution**

- (a)  $x \in \{x\}$ : This statement is true because  $x$  is an element in set  $x$ .
- (b)  $\{x\} \subseteq \{x\}$ : This statement is true.
- (c)  $\{x\} \in \{x\}$ : This statement is false because  $x$  is an element of  $\{\{x\}\}$  not  $\{x\}$ .
- (d)  $\{x\} \in \{\{x\}\}$ : This statement is true due to the reason from problem(c).
- (e)  $\emptyset \subseteq \{x\}$ : This statement is true according to **Theorem 1**.
- (f)  $\emptyset \in \{x\}$ : This statement is false because  $\emptyset$  is not an element of set  $\{x\}$ .

**Exercise 26**

Determine whether each of these sets is the power set of a set, where a and b are distinct elements.

- (a)  $\emptyset$
- (b)  $\{\emptyset, \{a\}\}$
- (c)  $\{\emptyset, \{a\}, \{\emptyset, a\}\}$
- (d)  $\{\emptyset, \{a\}, \{b\}, \{a, b\}\}$

**Solution**

The set(d) is the power set of set  $\{a, b\}$  because the set has two elements a and b so that its power set has  $2^2 = 4$  elements, which has the same number of elements of set(d).

$$\mathcal{P}(\{a, b\}) = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$$

**Exercise 27**

Prove that  $\mathcal{P}(A) \subseteq \mathcal{P}(B)$  if and only if  $A \subseteq B$ .

**Solution**

There are two things we need to prove:

$$(\mathcal{P}(A) \subseteq \mathcal{P}(B) \rightarrow A \subseteq B) \wedge (A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B))$$

- $\mathcal{P}(A) \subseteq \mathcal{P}(B) \rightarrow A \subseteq B$ :

$\mathcal{P}(A) \subseteq \mathcal{P}(B)$  means that every element of power set A is also an element of power set B. Additionally, we all know that power set of a set has  $2^n$  elements created from the combinations of all the elements from the original set. Because all every element of power set A is also an element of power set B so we can infer that the element of set A is also an element of set B because they have the same combinations in the power set. Therefore, this case is true.

- $A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B)$

$A \subseteq B$  means that every element of A is also an element of B. Because A and B have the same element so that there combinations of elements of these two sets will be the same. Therefore, the elements power set of A and B will be the same because both of them contain all subsets of A and B (We have that  $A \subseteq B$ ). Therefore, this case is true.

Because both cases are true so that  $(\mathcal{P}(A) \subseteq \mathcal{P}(B) \rightarrow A \subseteq B) \wedge (A \subseteq B \rightarrow \mathcal{P}(A) \subseteq \mathcal{P}(B))$  is true and it is equivalent to  $\mathcal{P}(A) \subseteq \mathcal{P}(B) \leftrightarrow A \subseteq B$ .

## Exercise 28

Show that if  $A \subseteq C$  and  $B \subseteq D$ , then  $A \times B \subseteq C \times D$

### Solution

Because A is a subset of C and B is a subset of D. Suppose that we have set A, B, C and D:

- $A = \{a, b\}$
- $B = \{c, d\}$
- $C = \{a, b\}$
- $D = \{c, d\}$

We have that:

$$\begin{aligned} A \times B &= \{(a, c), (a, d), (b, c), (b, d)\} \\ C \times D &= \{(a, c), (a, d), (b, c), (b, d)\} \end{aligned}$$

After using **Cartesian Product** to calculate  $A \times B$ ,  $C \times D$ , we can see that every element  $A \times B$  is also the element of  $C \times D$ . Therefore, we can infer that  $A \times B \subseteq C \times D$ .

## Exercise 41

Explain why  $A \times B \times C$  and  $(A \times B) \times C$  are not the same.

### Solution

Let  $A = \{0, 1\}, B = \{1, 2\}, C = \{0, 1, 2\}$

$$\begin{aligned} A \times B \times C &= \{(0, 1, 0), (0, 1, 1), (0, 1, 2), (0, 2, 0), (0, 2, 1), \\ &\quad (0, 2, 2), (1, 1, 0), (1, 1, 1), (1, 1, 2), (1, 2, 0), (1, 2, 1), (1, 2, 2)\} \end{aligned}$$

$$A \times B = \{(0, 1), (0, 2), (1, 1), (1, 2)\}$$

$$\begin{aligned} (A \times B) \times C &= \{((0, 1), 0), ((0, 1), 1), ((0, 1), 2), ((0, 2), 0), ((0, 2), 1), \\ &\quad ((0, 2), 2), ((1, 1), 0), ((1, 1), 1), ((1, 1), 2), ((1, 2), 0), ((1, 2), 1), ((1, 2), 2)\} \end{aligned}$$

As we can see that  $A \times B \times C$  gives us a set of 3-tuples has a form  $(a, b, c)$  with  $a \in A, b \in B, c \in C$ . However,  $(A \times B) \times C$  gives us a set of 2-tuples has a form  $((a, b), c)$  with  $(a, b) \in A \times B$  and  $c \in C$ . This is different from  $A \times B \times C$  because  $A \times B \times C$  is a 3-tuples but  $(A \times B) \times C$  is a 2-tuples which has the first element is a ordered pair of  $A \times B$ . Therefore,  $A \times B \times C$  is different from  $(A \times B) \times C$

### Exercise 42

Explain why  $(A \times B) \times (C \times D)$  and  $A \times (B \times C) \times D$  are not the same.

#### Solution

Let  $a, b, c, d$  are elements of set  $A, B, C, D$  respectively. Therefore, we have that  $a \in A, b \in B, c \in C, d \in D$ . We get that  $A \times B$  will be a set consists ordered pairs  $(a, b)$  and  $C \times D$  also consists ordered pair  $(c, d)$ . Because  $(a, b)$  and  $(c, d)$  is an element of set  $A \times B$  and  $C \times D$ . Therefore, if we do the Cartesian product  $(A \times B) \times (C \times D)$ . The result will be a new set consists ordered pair  $((a, b), (c, d))$  which has the two elements are two ordered pairs from  $A \times B$  and  $C \times D$ .

However,  $A \times (B \times C) \times D$  is completely different. We get  $B \times C$  will be a set consists ordered pair  $(b, c)$  and continue to calculate  $A \times (B \times C) \times D$ , we will get a new set consists 3-tuples  $(a, (b, c), d)$  with the first element is an element  $\in A$ , the second element is the element  $\in (B \times C)$ , and the last element is an element  $\in D$ .

Therefore, we can infer that  $(A \times B) \times (C \times D)$  and  $A \times (B \times C) \times D$  are different.

### Exercise 43

Prove or disprove that if  $A$  and  $B$  are sets, then  $\mathcal{P}(A \times B) = \mathcal{P}(A) \times \mathcal{P}(B)$

#### Solution

Let  $A = \{0, 1\}$  and  $B = 1, 2$ . Therefore, we get that  $|A| = |B| = 2$  and  $|A \times B| = |A| \times |B| = 2 \times 2 = 4$ . Because  $A \times B$  has 4 elements so that  $\mathcal{P}(A \times B)$  will have  $2^{|A \times B|} = 2^4 = 16$ . However, we get that  $|\mathcal{P}(A)| = 2^{|A|} = 2^2 = 4$  and  $|\mathcal{P}(B)| = 2^{|B|} = 2^2 = 4$  and

$$|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{|A|} \times 2^{|B|} = 2^{|A|+|B|} = 2^4 = 16$$

Although the result of  $|\mathcal{P}(A \times B)|$  and  $|\mathcal{P}(A) \times \mathcal{P}(B)|$  are both equal to 16 but the way they give the result 16 are completely different.

$$|\mathcal{P}(A \times B)| = 2^{|A| \times |B|} = 2^4 = 16$$

$$|\mathcal{P}(A) \times \mathcal{P}(B)| = 2^{|A|+|B|} = 16$$

Because  $2^{|A| \times |B|} \neq 2^{|A|+|B|}$  so we can conclude that  $\mathcal{P}(A \times B) \neq \mathcal{P}(A) \times \mathcal{P}(B)$

### Exercise 44

Prove or disprove that if  $A, B$ , and  $C$  are nonempty sets and  $A \times B = A \times C$ , then  $B = C$ .

#### Solution

We know that  $A \times B$  and  $A \times C$  will create a set whose element is ordered pairs. Let  $a \in A, b \in B, c \in C$  so that  $A \times B$  will be  $(a, b)$  and  $A \times C$  will be  $(a, c)$ . Because we have that  $A \times B = A \times C$  so that  $(a, b) = (a, c)$ .  $(a, b) = (a, c)$  if and only if  $a = a$  and  $b = c$ . Additionally,  $b \in B, c \in C$  so that  $B = C$ .

## Section 2.2

### Exercise 15

Prove the second De Morgan law in Table 1 by showing that if  $A$  and  $B$  are sets, then  $\overline{A \cap B} = \overline{A} \cup \overline{B}$

- (a) by showing each side is a subset of the other side.
- (b) using membership table.

#### Solution

- (a) To show each side is a subset of other side, we will need to prove that  $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$  and  $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ 
  - $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ : Suppose  $x$  is an element in  $\overline{A \cap B}$  so  $x \in \overline{A \cap B}$ . Therefore,  $x \notin A \cap B$  and we can transfer it into  $\neg((x \in A) \wedge (x \in B))$ . Applying De Morgan law, it will be  $\neg(x \in A) \vee \neg(x \in B)$ . Using the definition of negation of propositions, we have  $x \notin A$  or  $x \notin B$  and can be transferred to  $x \in \overline{A}$  or  $x \in \overline{B}$ . From this, we can infer that  $x \in \overline{A} \cup \overline{B}$ . Because  $x \in \overline{A \cap B}$  and also  $x \in \overline{A} \cup \overline{B}$  so that  $\overline{A \cap B} \subseteq \overline{A} \cup \overline{B}$ .
  - $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ : Suppose  $x$  is an element in  $\overline{A} \cup \overline{B}$  so  $x \in \overline{A} \cup \overline{B}$ . Therefore,  $x \notin A$  or  $x \notin B$  and we can transfer it to  $\neg(x \in A) \vee \neg(x \in B)$ . Applying De Morgan law, it will be  $\neg((x \in A) \wedge (x \in B))$ . By the definition of intersection, it follows that  $\neg(x \in A \cap B)$ , which means that  $x \notin A \cap B$  and we can infer that  $x \in \overline{A \cap B}$ . Because  $x \in \overline{A} \cup \overline{B}$  and also  $x \in \overline{A \cap B}$  so that  $\overline{A} \cup \overline{B} \subseteq \overline{A \cap B}$ .

Because we have shown that each set is a subset of the other, so  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

- (b) Using membership table:

A	B	$\overline{A}$	$\overline{B}$	$A \cap B$	$\overline{A \cap B}$	$\overline{A} \cup \overline{B}$
1	1	0	0	1	0	0
1	0	0	1	0	1	1
0	1	1	0	0	1	1
0	0	1	1	1	1	1

From the membership table, we see that the value of  $\overline{A \cap B}$  and  $\overline{A} \cup \overline{B}$  are the same so that we can conclude that  $\overline{A \cap B} = \overline{A} \cup \overline{B}$ .

### Exercise 16

Let  $A$  and  $B$  be sets. Show that

- (a)  $(A \cap B) \subseteq A$
- (b)  $A \subseteq (A \cup B)$
- (c)  $A - B \subseteq A$
- (d)  $A \cap (B - A) = \emptyset$
- (e)  $A \cup (B - A) = A \cup B$

**Solution**

- (a) We transfer  $A \cap B$  in propositions, it will be  $(x \in A) \wedge (x \in B)$ . Because in  $A \cap B$ , we have  $x \in A$  so that we can infer that  $A \cap B$  is a subset of  $A$ .
- (b) We transfer  $A \cup B$  in propositions, it will be  $(x \in A) \vee (x \in B)$ . Because  $x$  is an element of  $A$  and in  $A \cup B$ ,  $x$  also an element of  $A$  so  $A$  is a subset of  $B$ .
- (c) We transfer  $A - B = A \cap \overline{B}$  in propositions, it will be  $(x \in A) \wedge (x \notin B)$ . In  $A \cap \overline{B}$  we have  $x \in A$  so that  $x$  is an element in  $A \cap \overline{B}$  and also an element of  $A$ . So  $A \cap \overline{B}$  is a subset of  $A$ .
- (d) Because  $B - A = B \cap \overline{A}$ . Transfer it into propositions, it will be  $(x \in B) \wedge (x \notin A)$ .  $x$  not an element of  $B - A$  so that  $x$  can not be an element of  $A$ . Therefore, if we use intersection operation, it will be  $(x \in A) \wedge (x \in B) \wedge (x \notin A)$ . We see that it is a contradiction because  $x$  can not be both an element of  $A$  and do not belong to  $A$ . Thus, it will be an  $\emptyset$ .
- (e) Transfer  $B - A = B \cap \overline{A}$ . Then, it will be  $A \cup (B \cap \overline{A}) = (A \cup B) \cap (A \cup \overline{A}) = (A \cup B) \cap U = A \cup B$ .

**Exercise 17**

Show that if  $A$  and  $B$  are sets in a universe  $U$  then  $A \subseteq B$  if and only if  $\overline{A} \cup B = U$ .

- $A \subseteq B \rightarrow \overline{A} \cup B = U$ :  $A \subseteq B$  means  $x$  is an element of  $A$  and  $x$  also an element of  $B$ . We get that  $\overline{A} \cup B$

**Solution**