Homework 1B

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Section 1.4

Exercise 9

Let P(x) be the statement "x can speak Russian" and let Q(x) be the statement "x knows the computer language C++." Express each of these sentences in terms of P(x), Q(x), quantifiers, and logical connectives. The domain for quantifiers consists of all students at your school.

(a) There is a student at your school who can speak Russian and who knows C++.

(b) There is a student at your school who can speak Russian but who doesn't know C++.

(c) Every student at your school either can speak Russian or knows C++.

(d) No student at your school can speak Russian or knows C++.

Solution

We have that $P(x)$: "x can speak Russian" and $Q(x)$: "x knows the computer language C++".

(a) $\exists x(P(x) \land Q(x))$

(b) $\exists x(P(x) \land \neg Q(x))$

(c) $\forall x(P(x) \lor Q(x))$

(d) $\forall x \neg(P(x) \lor Q(x))$

Exercise 10

Let C(x) be the statement "x has a cat," let D(x) be the statement "x has a dog," and let F(x) be the statement "x has a ferret." Express each of these statements in terms of C(x), D(x), F(x), quantifiers, and logical connectives. Let the domain consist of all students in your class.

(a) A student in your class has a cat, a dog, and a ferret.

(b) All students in your class have a cat, a dog, or a ferret.

(c) Some student in your class has a cat and a ferret, but not a dog.

(d) No student in your class has a cat, a dog, and a ferret.

(e) For each of the three animals, cats, dogs, and ferrets, there is a student in your class who has this animal as a pet.

Solution

We have that $C(x)$: "x has a cat", $D(x)$: "x has a dog", $F(x)$: "x has a ferret".

(a) $\exists x(C(x) \wedge D(x) \wedge F(x))$

(b) $\forall x(C(x) \vee D(x) \vee F(x))$

(c) $\exists x(C(x) \wedge F(x) \wedge \neg D(x))$

(d) $\forall x \neg(C(x) \wedge D(x) \wedge F(x))$

(e) Because in this case, each animal will have a different owner and this statement will become true if all of the pets are all have their owners. Therefore, we have that:

$$\exists x C(x) \wedge \exists y D(y) \wedge \exists z F(z)$$

Exercise 33

Express each of these statements using quantifiers. Then form the negation of the statement, so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

(a) Some old dogs can learn new tricks.

(b) No rabbit knows calculus.

(c) Every bird can fly.

(d) There is no dog that can talk.

(e) There is no one in this class who knows French and Russian.

Solution

(a) Let's $P(x)$: "x can learn new tricks." and x will be "old dogs" $\rightarrow \exists x P(x)$. In negation form, it will be: $\forall x \neg P(x)$ and can be written as "No old dog can learn new tricks".

(b) Let's $Q(x)$: "x knows calculus" and x will be "rabbit" $\rightarrow \forall x \neg Q(x)$. In negation form, it will be: $\exists x Q(x)$ and can be written as "Some rabbits know calculus".

(c) Let's $D(x)$: "x can fly" and x will be "bird" $\rightarrow \forall x D(x)$. In negation form, it will be: $\exists x \neg D(x)$ and can be written as "Some birds can not fly".

(d) Let's $F(x)$: "x can talk" and x will be "dog" $\rightarrow \forall x \neg F(x)$. In negation form, it will be: $\forall x F(x)$ and can be written as "All dogs can talk".

(e) Let's $A(x)$: "x knows French", $B(x)$: "x knows Russian" and x will be "student" $\rightarrow \forall x \neg(A(x) \wedge B(x))$. In negation form, it will be $\exists x(A(x) \wedge B(x))$ and can be written as "There is a student who knows French and Russian".

Exercise 34

Express the negation of these propositions using quantifiers, and then express the negation in English.

(a) Some drivers do not obey the speed limit.

(b) All Swedish movies are serious.

(c) No one can keep a secret.

(d) There is someone in this class who does not have a good attitude.

Solution

(a) Let's $P(x)$: "x obeys the speed limit" and x is "driver" $\rightarrow \exists x \neg P(x)$. In negation form, it will be $\forall x P(x)$ and can be translated to "All drivers obey the speed limit".

(b) Let's $Q(x)$: "x are serious" and x is "Swedish movies" $\rightarrow \forall x Q(x)$. In negation form, it will be $\exists x \neg Q(x)$ and can be translated to "There is some Swedish movies are not serious".

(c) Let's $R(x)$: "x can keep a secret" and x is "people" $\rightarrow \forall x \neg R(x)$. In negation form, it will be $\exists x R(x)$ and can be translated to "There is someone who can keep a secret".

(d) Let's $D(x)$: "x has a good attitude" and x is "student" and the domain of x will be the student in the class and the domain will be called W $\rightarrow \exists x \in W \neg D(x)$. In negation form, it will be $\forall x \notin W D(x)$ and can be translated to "All of the students who are not in this class have a good attitude".

Exercise 39

Express each of these statements using predicates and quantifiers.

(a) A passenger on an airline qualifies as an elite flyer if the passenger flies more than 25,000 miles in a year or takes more than 25 flights during that year.

(b) A man qualifies for the marathon if his best previous time is less than 3 hours and a woman qualifies for the marathon if her best previous time is less than 3.5 hours.

(c) A student must take at least 60 course hours, or at least 45 course hours and write a master's thesis, and receive a grade no lower than a B in all required courses, to receive a master's degree.

(d) There is a student who has taken more than 21 credit hours in a semester and received all A's.

Solution

(a) We will have that $P(x)$: "Passenger x on an airline qualifies as an elite flyer", $Q(x, y)$: "Passenger x flies more than y miles in a year", $R(x, y)$: "Passenger x take more than y flights during that year".

$$\exists x((Q(x, 25,000) \vee R(x, 25)) \rightarrow P(x))$$

(b) We will have that $Q(x)$: "x qualifies for the marathon", $R(x, y)$: "x's best previous time is less than y hour", $G(x)$: "x is a man" and x is "people".

$$\exists x((R(G(x), 3) \vee R(\neg G(x), 3.5)) \rightarrow Q(x))$$

(c) We will have that $P(x, y)$: "Student x takes at least y course hours", $Q(x)$: "Student x write a master's thesis", $R(x, y, z)$: "Student x receive a grade no lower than a y in courses z", $C(x)$: "Student x receive a master's degree".

$$\exists x(C(x) \rightarrow (P(x, 60) \vee (P(x, 45) \wedge Q(x) \wedge R(x, B) \wedge \forall z R(x, B, z))))$$

The reason we write $C(x) \rightarrow (...)$ is because the student will get the master's degree only if he satisfy all of conditions given above, which mean that if he has not finished all the conditions given above, he will not receive the master's degree. Therefore, if we write $(...) \rightarrow C(x)$, it means that if the student has not finished all the conditions given above, he will still receive the master's degree.

(d) We will have that $P(x, y)$: "Student x has taken more than y credit hours in a semester", $Q(x, y, z)$: "x receive y grade in z subject"

$$\exists x(P(x, 21) \land \forall z(Q(x, A, z)))$$

Exercise 44

Express each of these system specifications using predicates, quantifiers, and logical connective.

(a) Every user has access to an electronic mailbox.

(b) The system mailbox can be accessed by everyone in the group if the file system is locked.

(c) The firewall is in a diagnostic state only if the proxy server is in a diagnostic state.

(d) At least one router is functioning normally if the throughput is between 100 kbps and 500 kbps and the proxy server is not in diagnostic mode.

Solution

(a) Let's $P(x)$: "User x has access to an electronic mailbox $\rightarrow \forall x P(x)$.

(b) Let's $P(x, y)$: "The system mailbox can be y by group member x in the group" and y will be the "permission", $Q(x, y)$: "x is y" and x will be system's component and y will be the state of the system.

$$Q(filesystem, locked) \rightarrow \forall x P(x, accessed)\text{"}$$

(c) Let's $P(x, y)$: "x is in y state", $Q(x, y)$: "z is in y state" where the domain of x is the system's components and the domain of y is the state of those components.

$$P(firewall, diagnostic) \rightarrow Q(proxy\_server, diagnostic)$$

(d) Let's $P(x, y)$: "x router is functioning with state y" and x is the number of routers, $Q(y)$: "the throughput is larger and equal y kbps", $T(z)$: "The throughput is smaller and equal z kbps, $R(x, y)$: "x is not in state y" where x will be the system's components.

$$(Q(100) \land T(500) \land R(proxy\_server, diagnostic)) \rightarrow \exists x P(x, normal)$$

Exercise 45

Determine whether $\forall x(P(x) \rightarrow Q(x))$ and $\forall x P(x) \rightarrow \forall x Q(x)$ are logically equivalent. Justify your answer.

Solution

Let $P(x)$ be "x is an even number", $Q(x)$ be "x is divisible by 4". Firstly, $\forall x(P(x) \rightarrow Q(x))$ means that **For every x, if x is an even number, x is divisible by 4**. This statement is always give **FALSE** because not all of the value x is divisible by 4 such as 2, 6, 14, ... However, $\forall xP(x) \rightarrow \forall xQ(x)$ is different, $\forall xP(x)$ will give us **FALSE** value not all the value of x are even number and $\forall xQ(x)$ will give us **FALSE** value too because not all of the value of x are divisible by 4. Therefore, **FALSE** $\rightarrow$ **FALSE** will give us **TRUE** value.

$\Rightarrow$ $\forall x(P(x) \rightarrow Q(x))$ always give **FALSE** and $\forall xP(x) \rightarrow \forall xQ(x)$ always give **TRUE** so they are not logically equivalent.

Exercise 46

Determine whether $\forall x(P(x) \leftrightarrow Q(x))$ and $\forall xP(x) \leftrightarrow \forall xQ(x)$ are logically equivalent. Justify your answer.

Solution

Let $P(x)$ be "x is divisible by 3", $Q(x)$ be "x is an odd number". We can see that $\forall x(P(x) \leftrightarrow Q(x))$ will not always **TRUE** with all of the value of x such as $x = 6$ so that $\forall x(P(x) \leftrightarrow Q(x))$ will be **FALSE**. However, $\forall xP(x) \leftrightarrow \forall xQ(x)$ is different because $\forall xP(x)$ is **FALSE** because not every x is divisible by 3 and $\forall xQ(x)$ is also **FALSE** because not every x is an odd number. Therefore, **FALSE** $\leftrightarrow$ **FALSE** will be **TRUE**.

$\Rightarrow$ $\forall x(P(x) \leftrightarrow Q(x))$ always give **FALSE** and $\forall xP(x) \leftrightarrow \forall xQ(x)$ always give **TRUE** so they are not logically equivalent.

Exercise 47

Show that $\exists x(P(x) \vee Q(x))$ and $\exists xP(x) \vee \exists xQ(x)$ are logically equivalent.

Solution

Let $P(x)$: "$x > 0$", $Q(x)$: "$x < 0$. Firstly, $\exists x(P(x) \vee Q(x))$ if $x = 1$ so that $P(x)$ will be **true** and $Q(x)$ will be **false** and $P(x) \vee Q(x)$ will be **true** and there are at least **1** x value to make $P(x) \vee Q(x)$ be **true** so $\exists x(P(x) \vee Q(x))$ is **true**. Let move to $\exists xP(x) \vee \exists xQ(x)$, since they have two different quantifiers so that the value of x in each can be different. if we all give $x = 1$ then $\exists xP(x)$ will be **true** and $\exists xQ(x)$ will be **false** but $\exists xP(x) \vee \exists xQ(x)$ still be **true** because **true** $\vee$ **false** will be **true**.

$\Rightarrow$ $\exists x(P(x) \vee Q(x))$ is **true** and $\exists xP(x) \vee \exists xQ(x)$ is also **true** so they are logically equivalent.

Exercise 63

Let $P(x), Q(x), R(x), S(x)$ be the statements "x is a baby," "x is logical," "x is able to manage a crocodile," and "x is despised," respectively. Suppose that the domain consists of all people. Express each of these statements using quantifiers; logical connectives; and $P(x), Q(x), R(x), S(x)$.

(a) Babies are illogical.

(b) Nobody is despised who can manage a crocodile.

(c) Illogical persons are despised.

(d) Babies cannot manage crocodiles.

(e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

Solution

(a) $\forall x(P(x) \to \neg Q(x))$

(b) $\forall x(R(x) \to \neg S(x))$

(c) $\forall x(\neg Q(x) \to S(x))$

(d) $\forall x(P(x) \to \neg R(x))$

(e) (d) follow from (a), (b), and (c) because (b) state that "Nobody is despised who can manage a crocodile." so that people is despised if they can not manage a crocodile. Moreover, illogical persons are despised and babies are illogical. Therefore, babies are despised and because babies are despised so that babies can not manage a crocodile.

Exercise 64

Let $P(x)$, $Q(x)$, $R(x)$, $S(x)$ be the statements "x is a duck," "x is one of my poultry," "x is an officer," and "x is willing to waltz," respectively. Express each of these statements using quantifiers; logical connectives; and $P(x)$, $Q(x)$, $R(x)$, $S(x)$.

(a) No ducks are willing to waltz.

(b) No officers ever decline to waltz.

(c) All my poultry are ducks.

(d) My poultry are not officers.

(e) Does (d) follow from (a), (b), and (c)? If not, is there a correct conclusion?

Solution

(a) $\forall x(P(x) \to \neg S(x))$

(b) $\forall x(R(x) \to S(x))$

(c) $\forall x(Q(x) \to P(x))$

(d) $\forall x(Q(x) \to \neg R(x))$

(e) (d) follow from (a), (b), and (c) because on (c) states that "All of my poultry are ducks" and they (a) states that ducks are not willing to waltz. Therefore, if my poultry are officers, they are willing to waltz and will make the contradiction of (c) so that we can conclude that my poultry are not officers from (a), (b), and (c).

Section 1.5

Exercise 17

Express each of these system specifications using predicates, quantifiers, and logical connectives, if necessary.

(a) Every user has access to exactly one mailbox.

(b) There is a process that continues to run during all error conditions only if the kernel is working correctly.

(c) All users on the campus network can access all websites whose url has a .edu extension.

(d) There are exactly two systems that monitor every remote server.

Solution

(a) Let $P(x, y)$: "User x has access to y mailbox" $\to \forall x \exists y P(x, y)$.

(b) Let $P(x, y)$: "Process x continues to run during error conditions y", $Q(x, y)$: "x is working with status y".

$$\exists x \forall y P(x, y) \to Q(kernel, correct)$$

(c) Let $P(x, y)$: "User x on the campus network can access website y", $Q(y, z)$: "Website y has the .z extension" $\to \forall x \forall y(P(x, y) \to Q(y, edu))$

(d) The statement "There are exactly two systems that monitor every remote server" also can be written as "There are system x can monitor every remote server and system y can monitor every remote server and no more

system can monitor every remote server.
Let $P(x, y)$: "System a can monitor system b"

$$\exists x \exists y(x \neq y \land \forall z P(x, z) \land \forall z P(y, z) \land \neg \exists w(w \neq x \land w \neq z \land \forall z P(w, z)))$$

In English, the equation above mean that there are two systems x and y(x $\neq$ y) that can control every remote server z and there is no system w that different from x and y can control every remote server z.

Exercise 18

(a) At least one console must be accessible during every fault condition.

(b) The e-mail address of every user can be retrieved whenever the archive contains at least one message sent by every user on the system.

(c) For every security breach there is at least one mechanism that can detect that breach if and only if there is a process that has not been compromised.

(d) There are at least two paths connecting every two distinct endpoints on the network.

(e) No one knows the password of every user on the system except for the system administrator, who knows all passwords.

Solution

(a) Let $P(x)$: "Console x must be accessible", $Q(y)$: "fault condition y".

$$\forall y(Q(y) \rightarrow \exists x P(x))$$

(b) Let $P(x)$: "The e-mail address of user x can be retrieved", $Q(x, y)$: "The archive contains message y sent by user x on the system".

$$\forall x \exists y Q(x, y)) \rightarrow \forall x P(x)$$

(c) Let $P(x, y)$: "Mechanism x can detect breach y", $Q(z, t)$: "Status t of process z".

$$\forall y \exists x P(x, y) \leftrightarrow \exists z Q(z, \textit{not compromise})$$

(d) Let $P(x, y, z)$: "There is path x connects endpoint y to endpoints z"
The statement means that there is path x connects every two different endpoints and there is path y connects every two different endpoints on the network. Means that if we have to check that if the two endpoints are different first then these two different endpoints will be connected by path.

$$\forall y \forall z(y \neq z \rightarrow \exists p \exists q(p \neq q \land P(p, y, z) \land P(q, y, z)))$$

(e) Let $P(x, y)$: "User x knows password of user y

$$\exists! x(\forall y P(x, y) \leftrightarrow (x = Administrator))$$

Exercise 34

Find a common domain for the variables x, y, and z for which the statement $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ is true and another domain for which it is false.

Solution

As we have that $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ means for every two variable x and y, there is a number z which is equal to x or equal to y for every z. Therefore, we will have the domain of x, y, z will be $D = \{0, 1\}$ because if $(x \neq y) \rightarrow \forall z((z = x) \vee (z = y))$ **true**, there will be 2 cases:

- $(x \neq y)$ have **false** value. When $(x \neq y)$ has false value, it is when $(x = y)$, in that case, because $(x \neq y)$ is the clause stand before "$\rightarrow$" so whole statement will become **true** because it does not care about the conclusion of the statement.

- $(x \neq y)$ have **true** value. When $(x \neq y)$ has true value, it means that x and y has the different value in the domain $D = \{0, 1\}$, $x = 0$ and $y = 1$ or vice versa. When $(x \neq y)$ true, because z has the same domain as x and y, so it will have the value 0 or 1 and z will be equal to x or y so it will make $\forall z((z = x) \vee (z = y))$ **true**.

Therefore, we can conclude that $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ will only **true** when the domain of x, y, z has exactly two elements. Otherwise, $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ will be **false**. For example, if the domain of x, y, z are all real numbers, $\forall x \forall y((x \neq y) \rightarrow \forall z((z = x) \vee (z = y)))$ will completely false because z will have a lot of values different from x and y.

Exercise 35

Find a common domain for the variables x, y, z, and w for which the statement $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ is true and another common domain for these variables for which it is false.

Solution

As we have that $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ means for every x, y, z there is a w that w are different from x, y, z . Therefore, because w must be completely different from x, y, z so we will take the domain of x, y, z, w is $D = \{1, 2, 3, 4\}$. In this case, x, y, z can have the same value to each other but also they can also have different values such as $x = 1, y = 2, z = 3$ or many many cases. Because x, y, z just only take maximum 3 out of 4 elements of the domain D so that there will one element left and that element obviously different from the rest and that element will be w and the domain D will satisfy with the condition of $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ and make it become **true**.

Therefore, we can conclude that $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ will become true when the domain of x, y, z, w has the number of elements greater or equal than 4, which means that we can take the domain of all real numbers for x, y, z, w. Moreover, we can also conclude that $\forall x \forall y \forall z \exists w((w \neq x) \wedge (w \neq y) \wedge (w \neq z))$ will be **false** when the domain of x, y, z, w has the number of elements smaller than 4. For example, with $D = \{1\}$, $x = y = z = w$ so that it will be **false**.

Exercise 36

Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

(a) No one has lost more than one thousand dollars playing the lottery.

(b) There is a student in this class who has chatted with exactly one other student.

(c) No student in this class has sent e-mail to exactly two other students in this class.

(d) Some student has solved every exercise in this book.

(e) No student has solved at least one exercise in every section of this book.

Solution

(a) No one has lost more than one thousand dollars playing the lottery means that "Everyone loses at most one thousand dollars when playing the lottery". Let $Q(x, y)$: "Player x loses y thousand dollars playing the lottery".

$$Statement : \forall x \forall y(Q(x, y) \wedge (y <= 1))$$
$$Negation : \exists x \exists y(Q(x, y) \wedge (y >= 1))$$

In Negation form, it means that "There are someone has lost more than one thousand dollars when playing the lottery".

(b) Let $Q(x, y)$: "x chats with y" and the domain of x, y will be the students in this class.

$$Statement : \exists x \exists y(x \neq y \wedge Q(x, y) \wedge \forall z((z \neq y) \rightarrow \neg Q(x, z)))$$
$$Negation : \forall x \forall y(x = y \vee \neg Q(x, y) \vee \exists z(z \neq y \wedge Q(x, z)))$$

In Negation form, it means that "Every student in this class chats with no one or more than one student".

(c) Let $P(x)$: "x sends email to y"

$$Statement: \neg\exists x\exists y\exists z(x \neq y \land x \neq z \land y \neq z \land P(x,y) \land P(x,z) \land \forall w((w \neq y \land w \neq z) \to \neg P(x,w)))$$
$$Negation: \exists x\exists y\exists z(x \neq y \land x \neq z \land y \neq z \land P(x,y) \land P(x,z) \land \forall w((w \neq y \land w \neq z) \to \neg P(x,w)))$$

In Negation form, it means that "There is a student has sent e-mail to exactly to other students in this class.

(d) Let $P(x, y)$: Student x has solved exercise y in this book.

$$Statement: \exists x\forall y P(x,y)$$
$$Negation: \forall x\exists y\neg P(x,y)$$

In Negation form, it means that "For every student, there are some exercises that has not solved.

(e) Let $P(x, y)$: "Student x has solved exercise y", $Q(y, z)$: "Exercise y in section z of the book"

$$Statement: \neg\exists x\forall z\exists y(P(x,y) \land Q(y,z))$$
$$Negation: \exists x\forall z\exists y(P(x,y) \land Q(y,z))$$

In Negation form, it means that "There is a student who has solved at least one exercise in every section of this book."

Exercise 37

Express each of these statements using quantifiers. Then form the negation of the statement so that no negation is to the left of a quantifier. Next, express the negation in simple English. (Do not simply use the phrase "It is not the case that.")

(a) Every student in this class has taken exactly two mathematics classes at this school.

(b) Someone has visited every country in the world except Libya

(c) No one has climbed every mountain in the Himalayas.

(d) Every movie actor has either been in a movie with Kevin Bacon or has been in a movie with someone who has been in a movie with Kevin Bacon.

Solution

(a) Let $P(x, y)$: "Student x in this class has taken mathematics classes y at this school".

$$Statement: \forall x\exists y\exists z(y \neq z \land P(x,y) \land P(x,z) \land \forall w(P(x,w) \to (w = z \lor w = y)))$$
$$Negation: \exists x\forall y\forall z(y = z \lor \neg P(x,y) \lor \neg P(x,z) \lor \exists w(P(x,w) \land (w \neq z \land w \neq y))$$

In Negation form, it means "There is a student in this class has taken not only these two course, which means that he can get more courses".

(b) Let $P(x, y)$: x has visited country y

$$Statement : \exists x \forall y (P(x, y) \leftrightarrow (x \neq Libya))$$
$$Negation : \forall x \exists y (P(x, y) \wedge (x = Libya) \vee (x \neq Libya) \wedge \neg P(x, y))$$

In Negation form, it means "Everyone has visited Libya or has not visited some countries that not Libya".

(c) Let $P(x, y)$: x has climbed mountain y in Himalayas.

$$Statement : \neg \exists x \forall y (P(x, y))$$
$$Negation : \exists x \forall y (P(x, y))$$

In Negation form, it means "There is someone has climbed every mountain in Himalayas".

(d) Let $P(x, y)$: Movie actor x has been in a movie with y.

$$Statement : \forall x (P(x, KevinBacon) \vee \exists z ((P(z, KevinBacon) \wedge x \neq z) \rightarrow P(x, z)))$$
$$Negation : \exists x (\neg P(x, KevinBacon) \wedge \forall z ((P(z, KevinBacon) \wedge x \neq z) \wedge \neg P(x, z))$$

In Negation form, it means that "There is a movie actor who has not been in a movie with Kevin Bacon and has not been in a movie with people who has been in a movie with Kevin Bacon".

Exercise 48

Show that $\forall x P(x) \vee \forall x Q(x)$ and $\forall x \forall y (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain, are logically equivalent. (The new variable y is used to combine the quantifications correctly.)

Solution

We see that in $\forall x \forall y (P(x) \vee Q(y))$, $P(x)$ is not bounded by variable y and $Q(y)$ is not bounded by variable x so we have that

$$\forall x \forall y (P(x) \vee Q(y)) \equiv \forall x P(x) \vee \forall y Q(y)$$

$\forall x P(x) \vee \forall y Q(y)$ will equivalent to $\forall x P(x) \vee \forall x Q(x)$ because in $\forall x P(x) \vee \forall x Q(x)$, the value x of $Q(x)$ and $P(x)$ and two different variables and the are not taking the same value. Therefore, we can change the x in $\forall x Q(x)$ to $y, z, ...$ or whatever we want because these two variables are complete different and have same value

$$\Rightarrow \forall x \forall y (P(x) \vee Q(y)) \equiv \forall x P(x) \vee \forall x Q(x)$$

Exercise 49

(a) Show that $\forall x P(x) \wedge \exists x Q(x)$ is logically equivalent to $\forall x \exists y (P(x) \wedge Q(y))$, where all quantifiers have the same nonempty domain.

(b) Show that $\forall x P(x) \vee \exists x Q(x)$ is equivalent to $\forall x \exists y (P(x) \vee Q(y))$, where all quantifiers have the same nonempty domain.

Solution

(a) According to **Binding Variable** mentioned above we can transfer $\forall x P(x) \wedge \exists x Q(x)$ into $\forall x P(x) \wedge \exists y Q(y)$

- Suppose that $\forall x P(x) \wedge \exists y Q(y)$ is **true**. Therefore, the whole statement will only **true** when $P(x)$ is **true** for every x and $Q(y)$ is **true** for at least one value of y and these 2 quantifiers is completely different and do not get the same value. So that we can also infer that $\forall x \exists y (P(x) \wedge Q(y))$ also **true** because it has the same meaning that $P(x)$ is all **true** for every x and $Q(y)$ is **true** for at least one value of y.
- Suppose that $\forall x \exists y (P(x) \wedge Q(y))$ is **true**. Therefore, for each x makes $P(x)$ **true**, there is a y makes $Q(y)$ **true**, too. We also know that these x, y have the same domain and there just need one value of y to make $Q(y)$ be **true**. So, we can infer that $\forall x P(x) \wedge \exists y Q(y)$ **true**, too.

$\Rightarrow \forall x P(x) \wedge \exists x Q(x) \equiv \forall x \exists y (P(x) \wedge Q(y))$

(b) According to **Binding Variable** mentioned above we can transfer $\forall x P(x) \vee \exists x Q(x)$ to $\forall x P(x) \vee \exists y Q(y)$

- Suppose that $\forall x P(x) \vee \exists y Q(y)$ **true**, which means $P(x)$ is **true** for every x or there is at least a y to make $Q(x)$ **true**. If $\forall x P(x)$ **true**, it means that for every x $P(x) \vee Q(y)$ is **true**. If $\exists y Q(y)$ **true**, it means that there is a y to make $P(x) \vee Q(y)$ **true**. Therefore, we can infer that $\forall x \exists y (P(x) \vee Q(y))$ **true**.
- Suppose that $\forall x \exists y (P(x) \vee Q(y))$ **true**. If only $P(x)$ **true**, which means that $P(x)$ is true for every x and $\forall x \exists y (P(x) \vee Q(y))$ will still be true. If only $Q(y)$ **true**, which means that there is a y that makes $Q(y)$ **true** and $\forall x \exists y (P(x) \vee Q(y))$ will still be true. Therefore, if we separate these statement into $\forall x P(x) \vee \exists y Q(y)$ nothing change because they just need one of them to have true value to make the whole statement to be **true**.

$\Rightarrow \forall x P(x) \vee \exists x Q(x) \equiv \forall x \exists y (P(x) \vee Q(y))$

Section 1.6

Exercise 11

Show that the argument form with premises $p_1, p_2, \ldots, p_n$ and conclusion $q \to r$ is valid if the argument form with premises $p_1, p_2, \ldots, p_n, q$, and conclusion r is valid.

Solution

We already have that

$$\begin{array}{l} p_1 \\ p_2 \\ ... \\ p_n \\ q \\ \hline \therefore \quad r \end{array}$$

is valid. Therefore, it means that $(p_1 \wedge p_2 \wedge ... \wedge p_n \wedge q) \rightarrow r$ is tautology and all the premises $p_1, p_2, ..., p_n, q$ must be all **true**. So, the argument

$$\begin{array}{l} p_1 \\ p_2 \\ ... \\ p_n \\ \hline \therefore q \rightarrow r \end{array}$$

will be valid if and only if r has **true** value because we know that $p_1, p_2, ..., p_n$ and q is **true** due to the argument above. Therefore, $q \rightarrow r$ is **true** when both q and r are **true**. Since q is already **true**, so r must be **true**.

Exercise 12

Show that the argument form with premises $(p \wedge t) \rightarrow (r \vee s), q \rightarrow (u \wedge t), u \rightarrow p, and \neg s$ and conclusion $q \rightarrow r$ is valid by first using Exercise 11 and then using rules of inference from Table 1.

Solution

Applying Exercise 11, $(p \wedge t) \rightarrow (r \vee s), q \rightarrow (u \wedge t), u \rightarrow p, and \neg s$ and conclusion $q \rightarrow r$ is valid if $(p \wedge t) \rightarrow (r \vee s), q \rightarrow (u \wedge t), u \rightarrow p, \neg s, q$ and conclusion r is valid

$$\begin{array}{l} (p \wedge t) \rightarrow (r \vee s) \\ q \rightarrow (u \wedge t) \\ u \rightarrow p \\ \neg s \\ q \\ \hline \therefore \qquad\qquad\qquad r \end{array}$$

Step	Reason
1. $q \to (u \wedge t)$	Premise
2. q	Premise
3. $u \wedge t$	Modus ponens using (1) and (2)
4. u	Simplification using (3)
5. t	Simplification using (3)
6. $u \to p$	Premise
7. p	Modus ponens using (4) and (5)
8. $(p \wedge t) \to (r \vee s)$	Premise
9. $p \wedge t$	Conjunction using (5) and (7)
10. $r \vee s$	Using (8) and (9)
11. $\neg s$	Premise
12. r	Disjunctive syllogism using (10) and (11)

Therefore, the argument above is valid so we can infer that $(p \wedge t) \to (r \vee s), q \to (u \wedge t), u \to p, and \neg s$ and conclusion $q \to r$ is valid due to **Exercise 11**

Exercise 23

Identify the error or errors in this argument that supposedly shows that if $\exists x P(x) \wedge \exists x Q(x)$ is true then $\exists x(P(x) \wedge Q(x))$ is true.

1	$\exists x P(x) \wedge \exists x Q(x)$	Premise
2	$\exists x P(x)$	Simplification from (1)
3	$P(c)$	Existential instantiation from (2)
4	$\exists x Q(x)$	Simplification from (1)
5	$Q(c)$	Existential instantiation from (4)
6	$P(c) \wedge Q(c)$	Conjunction from (3) and (5)
7	$\exists x(P(x) \wedge Q(x))$	Existential generalization

Solution

The error of this proof is in step (5) because we can not assume that the value (c) is **true** for both $P(x)$ and $Q(x)$ because as we knew that (c) is an arbitrary element so we can not make the assumption that (c) make both $P(x)$ and $Q(x)$ **true**.

Exercise 24

Identify the error or errors in this argument that supposedly shows that if $\forall x(P(x) \vee Q(x))$ is true then $\forall x P(x) \vee \forall x Q(x)$ is true.

1	$\forall x(P(x) \vee Q(x))$	Premise
2	$P(c) \vee Q(c)$	Universal instantiation from (1)
3	$P(c)$	Simplification from (2)
4	$\forall x P(x)$	Universal generalization from (3)
5	$Q(c)$	Simplification from (2)
6	$\forall x Q(x)$	Universal generalization from (5)
7	$\forall x P(x) \vee \forall x Q(x)$	Existential generalization

Solution

The error of this proof is in step (2) because the operator is disjunction($\vee$) not conjunction($\wedge$), therefore, we can not use **Simplification** in step (3) and step (5). So, this error result in fallacies.

Exercise 34

The Logic Problem, taken from WFF'N PROOF, The Game of Logic, has these two assumptions:

1. "Logic is difficult or not many students like logic."
2. "If mathematics is easy, then logic is not difficult."

By translating these assumptions into statements involving propositional variables and logical connectives, determine whether each of the following are valid conclusions of these assumptions:

(a) That mathematics is not easy, if many students like logic.

(b) That not many students like logic, if mathematics is not easy.

(c) That mathematics is not easy or logic is difficult.

(d) That logic is not difficult or mathematics is not easy.

(e) That if not many students like logic, then either mathematics is not easy or logic is not difficult.

Solution

Let p: Logic is difficult, q: Many students like logic, r: Mathematics is easy. We can transfer these two assumptions into:

1. $p \vee \neg q$
2. $r \to \neg p$

(a) $q \to \neg r$

$$\begin{array}{l} p \lor \neg q \\ r \to \neg p \\ \hline \therefore q \to \neg r \end{array}$$

1	$r \to \neg p$	Premise
2	$\neg r \lor \neg p$	Logical equivalence from (1)
3	$p \lor \neg q$	Premise
4	$\neg q \lor \neg r$	Resolution from using (2) and (3)
5	$q \to \neg r$	Logical equivalence from (4)

Therefore, conclusion (a) is **valid**.

(b) $\neg r \to \neg q$

$$\begin{array}{l} p \lor \neg q \\ r \to \neg p \\ \hline \therefore \neg r \to \neg q \end{array}$$

1	$r \to \neg p$	Premise
2	$\neg r \lor \neg p$	Logical equivalence from (1)
3	$p \lor \neg q$	Premise
4	$\neg r \lor \neg q$	Resolution from using (2) and (3)
5	$r \to \neg q$	Logical equivalence from (4)

As we can see that, the final conclusion is $r \to \neg q$ not $\neg r \to \neg q$. Therefore, conclusion (b) is **invalid**.

(c) $\neg r \lor p$

As we can see from the premise $r \to \neg p$, its logical equivalence is $\neg r \lor \neg p$ not $\neg r \lor p$. Therefore, conclusion (c) is **invalid**.

(d) $\neg p \lor \neg r$

As we can see from the premise $r \to \neg p$, its logical equivalence is $\neg r \lor \neg p$ and it is definitely the same as $\neg p \lor \neg r$. Therefore, conclusion (d) is **valid**.

(e) $\neg q \to (\neg r \lor \neg p)$

We can see that $\neg r \lor \neg p$ is a premise so it is definitely a tautology, therefore, the conclusion can be converted to $\neg q \to \mathbf{T}$. Because the consequent is always **true** value so it does not care about the value of the antecedent and always **true**, or also known as a tautology. Finally, we can conclude that conclusion (e) is **valid**.

Exercise 35

Determine whether this argument, taken from Kalish and Montague [KaMo64], is valid.

> If Superman were able and willing to prevent evil, he would do so. If Superman were unable to prevent evil, he would be impotent; if he were unwilling to prevent evil, he would be malevolent. Superman does not prevent evil. If Superman exists, he is neither impotent nor malevolent. Therefore, Superman does not exist.

Solution

We have that:

- p: Superman was able to prevent evil.
- q: Superman was willing to prevent evil.
- r: Superman prevents evil.
- z: Superman is impotent.
- y: Superman is malevolent.
- t: Superman exists.

$$
\begin{array}{ll}
 & (p \wedge q) \to r \\
 & \neg p \to z \\
 & \neg q \to y \\
 & \neg r \\
 & t \to (\neg z \wedge \neg y) \\
\hline
\therefore & \neg t
\end{array}
$$

1	$(p \wedge q) \to r$	Premise
2	$\neg r$	Premise
3	$\neg(p \wedge q)$	Modus tollens from (1) and (2)
4	$\neg p \vee \neg q$	Logical equivalence from (3)
5	$\neg p \to z$	Premise
6	$\neg q \to y$	Premise
7	$p \vee z$	Logical equivalence from (5)
8	$\neg q \vee z$	Resolution from (4) and (7)
9	$q \vee y$	Logical equivalence from (6)
10	$z \vee y$	Resolution from (8) and (9)
11	$t \to (\neg z \wedge \neg y)$	Premise
12	$\neg t \vee (\neg z \wedge \neg y)$	Logical equivalence from (11)
13	$\neg t \vee \neg(z \vee y)$	Logical equivalence from (12)
14	$\neg t$	Disjunctive syllogism from (13) and (10)

Therefore, the argument taken from Kalish and Montague [KaMo64], is **valid**.

Section 1.7

Exercise 26

Show that at least three of any 25 days chosen must fall in the same month of the year.

Solution

To proof that "There are at least three of any 25 days chosen must fall in the same month of the year", we will use **proofs by contradiction**. Let p be the proposition "There are at least three of any 25 days chosen must fall in the same month of the year". Suppose that $\neg p$ is **true**, which means that "There are less than three of any 25 days chosen must fall in the same month of the year". Because there are 12 months a year, so if we picks each 2 days in each month, there are at most 24 days fall in the same month of the year. However, there are 25 days in the premise. Let r be the proposition "It is 25 days" and we have that:

$$\neg p \to (r \land \neg r)$$

$(r \land \neg r)$ is a contradiction and we suppose that $\neg p$ is **true** so that the compound proposition above will give result **false**. Therefore, it is only **true** when $\neg p$ is **false**, which means that p is **true**. Because p is **true** so we can prove that "There are at least three of any 25 days chosen must fall in the same month of the year".

Exercise 27

Use a proof by contradiction to show that there is no rational number r for which $r^3 + r + 1 = 0$.

Solution

By using proof by contradiction, we will show that "There is a rational number r for which $r^3 + r + 1 = 0$". Assume that $r = \frac{a}{b}$ is a root with a and b are integer and $\frac{a}{b}$ is in lowest terms. Next, we will substitute the root in to the equation $r^3 + r + 1 = 0$ so the equation will be:

$$\frac{a^3}{b^3} + \frac{a}{b} + 1 = 0 \tag{1}$$

We will multiply (1) for b^3 and we will get:

$$a^3 + ab^2 + b^3 = 0 \tag{2}$$

Because the result of this equation is equal to **0**, which means that the result of this equation is an even number. Therefore, we need to check that whether a or b is an even or odd number. We will have three cases:

- a is an **even number** and b is an **odd number**:

  If a is an even number and b is an odd number, we can infer that a^3 is also an even number and b^3 is also an odd number. Also, we can get that ab^2 is an even number since a is an even number and b^2 is an odd number so their product will be an even number. Therefore, we have that:

  $$even + even + odd = odd$$

  However, the result of the equation (2) is equal to 0 and 0 is an even number so this case is **invalid**.

- a is an **odd number** and b is an **even number**:

  If a is an odd number and b is an even number, we can infer that a^3 is also an odd number and b^3 is also an even number. Also, we can get that ab^2 is an even number since a is an odd number and b^2 is an even number so their product will be an even number. Therefore, we have that:

  $$odd + even + even = odd$$

  However, the result of the equation (2) is equal to 0 and 0 is an even number so this case is **invalid**.

- Both a and b is **odd number**:

  If both a and b are odd number, it means that a^3, b^3, ab^2 are odd number, too. Therefore, we have that:

  $$odd + odd + odd = odd$$

  However, the result of the equation (2) is equal to 0 and 0 is an even number so this case is **invalid**.

- We do not have the case that both a and b are even number because $\frac{a}{b}$ is fully simplified fraction as we assumed above.

From three cases, we can conclude that there is no root of the equation $r^3 + r + 1 = 0$ that are ration numbers because they can not represent as fraction. Therefore, the root r of $r^3 + r + 1 = 0$ is a irrational number.

Exercise 39

Show that the propositions $p_1, p_2, p_3, p_4, and p_5$ can be shown to be equivalent by proving that the conditional statements $p_1 \to p_4,\ p_3 \to p_1,\ p_4 \to p_2,\ p_2 \to p_5,\ and\ p_5 \to p_3$ are true.

Solution

To prove that $p_1, p_2, p_3, p_4,\ and\ p_5$ can be shown to be equivalent, we need to show that

$$\begin{array}{l} p_1 \to p_4 \\ p_4 \to p_2 \\ p_2 \to p_5 \\ p_5 \to p_3 \\ p_3 \to p_1 \\ \hline \therefore p_1 \to p_1 \end{array}$$

1	$p_1 \to p_4$	Premise
2	$p_4 \to p_2$	Premise
3	$p_1 \to p_2$	Hypothetical syllogism from (1) and (2)
4	$p_2 \to p_5$	Premise
5	$p_1 \to p_5$	Hypothetical syllogism from (3) and (4)
6	$p_5 \to p_3$	Premise
7	$p_1 \to p_3$	Hypothetical syllogism from (5) and (6)
8	$p_3 \to p_1$	Premise
9	$p_1 \to p_1$	Hypothetical syllogism from (7) and (8)

From using Rule of inference in **Section 1.6**, we can have the final conclusion from those premises that $p_1 \to p_1$, which is a tautology and the conclusion is only **true** when all of the premises have **true** value also
$\Rightarrow p_1 \to p_4,\ p_3 \to p_1,\ p_4 \to p_2,\ p_2 \to p_5,\ and\ p_5 \to p_3$ are **true**.
$\Rightarrow p_1 \to p_4 \to p_2 \to p_5 \to p_3 \to p_1$. Therefore, the propositions p_1, p_2, p_3, p_4, p_5 can be shown to be equivalent

Exercise 40

Find a counterexample to the statement that every positive integer can be written as the sum of the squares of three integers.

Solution

We have that sums of the squares of three integers mean that there are three number a, b, c and $a^2 + b^2 + c^2 = d$. And all of d which is positive integer can always be represented in $a^2 + b^2 + c^2$. However, this is not completely true. For example, if we give $d = 31$, d can not be created from $a^2 + b^2 + c^2$. So we will take a sequence of integer start from 0, 1, 2, 3, 4, 5 and their square will be 0, 1, 4, 9, 16, 25 respectively.

If we take $a = 5$, which means that $b^2 + c^2 = 6$ but there are only 0, 1, 4 and they can not create a sum of $b^2 + c^2 = 6$. If we take $a = 4, b = 3$, which means that $c^2 = 6$ but square root of c will not be a integer. Moreover, we can not create $a^2+b^2+c^2=31$ with small value such as 0, 1, 4, 9 and big value such as $6^2=36,\ 7^2=49, \dots$. Therefore, $d=31$ is a counterexample of the statement **"Every positive integer can be written as the sum of the squares of three integers"**.

Extra

Show that $\sqrt{2023} \notin \mathbb{Q}$(means that $\sqrt{2023}$ is a irrational number).

Solution

Euclid's lemma:

"If a prime p divides the product ab of two integers a and b, then p must divide at least one of those integers a or b."

We have that: $\sqrt{2023}=\sqrt{7\times 17^2}=17\times\sqrt{7}$. Since 17 is a integer so we just need to prove that $\sqrt{7}$ is a irrational number. To prove this, we will use proof by contradiction. Suppose that $\sqrt{7}$ is a rational number. If $\sqrt{7}$ is a rational number, $\sqrt{7}$ can be represented in $\frac{a}{b}$, where a and b are coprime(integer that is a divisor of both of them is 1), and $\frac{a}{b}$ is in lowest terms.

$\sqrt{7}=\frac{a}{b} \Rightarrow \frac{a^2}{b^2}=7 \leftrightarrow a^2=7b^2$. Because b^2 is an positive integer multiply with 7, a^2 is divisible by 7. a^2 is divisible by 7 so a is also divisible by 7 because 7 is a prime number and according to **Euclid's lemma**, a is divisible by 7.

We will give $a=7c$ with c is an integer $\Rightarrow 49c^2=7b^2 \leftrightarrow 7c^2=b^2$. From this, we can infer that b is also divisible by 7. However, a and b are coprime so that they can not both divisible by 7. Therefore, $\sqrt{7}$ must not be a rational number so we can conclude that $\sqrt{7}$ is a irrational number.

Section 1.8

Exercise 15

Prove or disprove that there is a rational number x and an irrational number y such that x^y is irrational.

Solution

In this exercise, we will use **Existence Proofs**. Therefore, we will give an existence proof that is nonconstructive.

- Case 1: Suppose that $x=2$ and $y=\sqrt{2}$ so $x^y=2^{\sqrt{2}}$. If $2^{\sqrt{2}}$ is an irrational number, then we are done.

- Case 2: If $2^{\sqrt{2}}$ is not an irrational number, then we will suppose that $x = 2^{\sqrt{2}}$ and $y = \frac{\sqrt{2}}{4}$. So we have x^y:

$$x^y = 2^{\sqrt{2}^{\frac{\sqrt{2}}{4}}} = 2^{\sqrt{2}\cdot\frac{\sqrt{2}}{4}} = 2^{\frac{1}{2}} = \sqrt{2}$$

  Because $\sqrt{2}$ is an irrational number. Therefore, x^y is an irrational number.

We have proved that there is a rational number x and an irrational number x such that x^y is irrational by giving two cases and if **Cases 1** is **true**, we know that it exists a number x^y is irrational number where there is a rational number x and an irrational number y. If **Case 1** is not **true**, we will have **Case 2** to prove that a^b is an irrational number depend on **Case 1**.

Exercise 16

Prove or disprove that if a and b are rational numbers, then a^b is also rational.

Solution

In this exercise the statement we need to prove or disprove means that for every a and b are rational numbers, then a^b is also rational. This statement **must be true** for all value of a and b are rational numbers. However, there is a counterexample of this statement.

If we give $a = \frac{2}{1} = 2$ and $b = \frac{1}{2}$, then $a^b = \sqrt{2}$. $\sqrt{2}$ is an irrational number, therefore, it exists two values a and b that make a^b not a rational number so this statement is **false**.

Exercise 17

Show that each of these statements can be used to express the fact that there is a unique element x such that $P(x)$ is true.

(a) $\exists x \forall y (P(y) \leftrightarrow (x = y))$

(b) $\exists x P(x) \land \forall x \forall y (P(x) \land P(y) \rightarrow x = y)$

(c) $\exists x (P(x) \land \forall y (P(y) \rightarrow x = y))$

Solution

(a) $\exists x \forall y (P(y) \leftrightarrow (x = y))$
This statement mean that exists an x that for every y, $P(y)$ is **true** if and only if $x = y$. Therefore, we can infer that if $x \neq y$, $P(y)$ will be **false**. Hence, $P(y)$ only true if y is x and y can not be a different value from x. $\Rightarrow x$ is a unique element that $P(x)$ is **true**.

(b) $\exists x P(x) \wedge \forall x \forall y (P(x) \wedge P(y) \to x = y)$
So we will split it into two parts, the first part is $\exists x P(x)$, the second part is $\forall x \forall y (P(x) \wedge P(y) \to x = y)$:

- $\exists x P(x)$ means that exist an x in a domain that make $P(x)$ **true**.
- $\forall x \forall y (P(x) \wedge P(y) \to x = y)$ means that for any value of x and y in the domain, if both $P(x)$ and $P(y)$ are both true, then $x = y$.

Because we are using $\wedge$ operator to connect $\exists x P(x)$, $\forall x \forall y (P(x) \wedge P(y) \to x = y)$, therefore, we need to find the common point between these two statements to make the whole statement **true**. So that means that there must be a value x to make both $\exists x P(x)$ and $\forall x \forall y (P(x) \wedge P(y) \to x = y)$ **true**.
$\Rightarrow$ x is an unique element such that $P(x)$ is true.

(c) $\exists x (P(x) \wedge \forall y (P(y) \to x = y))$
This statement means that exists a value x that make $P(x)$ **true** and if there is any element that make P **true**, that element must be x. Hence, it means that P is **true** only when has the value x.
$\Rightarrow$ x is an unique element such that $P(x)$ is true.

Exercise 27

Write the numbers 1, 2, ... , 2n on a blackboard, where n is an odd integer. Pick any two of the numbers, j and k, write $|j - k|$ on the board and erase j and k. Continue this process until only one integer is written on the board. Prove that this integer must be odd.

Solution

We know that because n is an odd integer so that in the sequence 1,2, ..., 2n there will be n odd integers and n even integers.

WLOG, we will have three cases of picking number j and k from the board:

- j and k are both **even** numbers. If both of them are even number, then $|j - k|$ is an even number and the number of even numbers in the sequence will be decreased by 1 and the number of odd numbers remain unchanged.
- j and k are both **odd** numbers. If both of them are odd number, then $|j - k|$ is an even number and the number of even numbers in the sequence will be increased by 1 and the number of odd numbers are decreased by 2.
- j is an odd number and k is an even number. If j is an odd number and k is an even number, then $|j - k|$ is an odd number and the number of even numbers in the sequence will be decreased by 1 and the number of odd numbers remain unchanged.

If we continue these process of picking j and k, no matter which case we choose to pick, there will be remain an odd number in the sequence because there are n odd numbers and the number of odds integers can be decreased by 2 in each turn. If it happens like that, there will always remain an odd number in the sequence and the rest are even numbers. Because $odd - even$ always give odd so when there is only one number remain on the board, it will be an odd number.

Exercise 28

Suppose that five ones and four zeros are arranged around a circle. Between any two equal bits you insert a 0 and between any two unequal bits you insert a 1 to produce nine new bits. Then you erase the nine original bits. Show that when you iterate this procedure, you can never get nine zeros.

Solution

Suppose that we are in the final step that we have nine zeros. To have nine zeros, the previous step must be all nine ones or nine zeros. We have two possible cases:

- Case 1: The circle have nine ones. If the circle contains nine ones, it means that the previous circle is nine numbers and those numbers 0, 1 are interspersed. This is impossible for the circle contains odd elements because there will be 2 same number are lying next to each other. Therefore, we can not go to the step where there is nine ones.
- Case 2: The circle have nine zeros. If the circle contains nine zeros, it means that the previous circle is nine ones and it is impossible case due to case 1.

Therefore, we can infer that we can not create a circle contains nine zeros because we can not have the circle which have nine numbers and those numbers 0, 1 are interspersed.

Exercise 32

Prove that there are no solutions in integers x and y to the equation

$$2x^2 + 5y^2 = 14$$

Solution

We have that:

$$\begin{aligned} 2x^2 + 5y^2 &= 14 \\ 2x^2 &= 14 - 5y^2 \\ x^2 &= 7 - 2.5y^2 \end{aligned} \tag{3}$$

As we can see that from the equation (3), x^2 only ¿ 0 when y has the value 0 or ± 1. If $|y|$ is ≥ 2 then $7 - 2.5y^2 < 0$. However, x^2 is a positive integer so that if $7 - 2.5y^2 < 0$, it is a contradiction so the case $|y| \geq 2$ is impossible. Now we will substitute the value $y = 0$ *or* $y = \pm 1$ to (3) to check if x is an integer.

- $y = 0$

$$x^2 = 7 - 2.5 \times 0^2 = 7$$
$$\Rightarrow x = \sqrt{7} \textit{ or } x = -\sqrt{7}$$

- $y = \pm 1$

$$x^2 = 7 - 2.5 \times 1^2 = 4.5$$

We can see that in two cases $y = 0$ or $y = \pm 1$, they all give the result of x is an irrational number. Therefore, we can conclude that there are no solutions in integers x and y satisfies with the equation $2x^2 + 5y^2 = 14$.

Exercise 33

Prove that there are no solutions in positive integers x and y to the equation $x^4 + y^4 = 625$.

Solution

We have that:

$$x^4 + y^4 = 625$$
$$x^4 = 625 - y^4$$

Because both x^4 and y^4 are positive number so x^4 only valid when $|y| \leq 5$. If $|y| > 5$, x^4 will be a negative number. So the case when $|y| > 5$ is impossible. Now let check when $|y| <= 5$, is x an integer:

- $y = \pm 1$

$$x^4 = 625 - 1 = 624 \Rightarrow x = 4.997...$$

- $y = \pm 2$

$$x^4 = 625 - 2^4 = 625 - 16 = 609 \Rightarrow x = 4.967..$$

- $y = \pm 3$

$$x^4 = 625 - 3^4 = 625 - 81 = 544 \Rightarrow x = 4.829..$$

- $y = \pm 4$

$$x^4 = 625 - 4^4 = 625 - 256 = 369 \Rightarrow x = 4.3828..$$

- $y = \pm 5$

$$x^4 = 625 - 5^4 = 625 - 625 = 0 \Rightarrow x = 0$$

We can see that, from 5 cases that x is a positive number but from $|y| = 1$ to $|y| = 4$, x not a integer and when $|y| = 5$, x not a positive integer so we can infer that there is no possible for x to be a positive integer when $|y| \leq 5$. $\Rightarrow$ There are no solutions in positive integers x and y to the equation

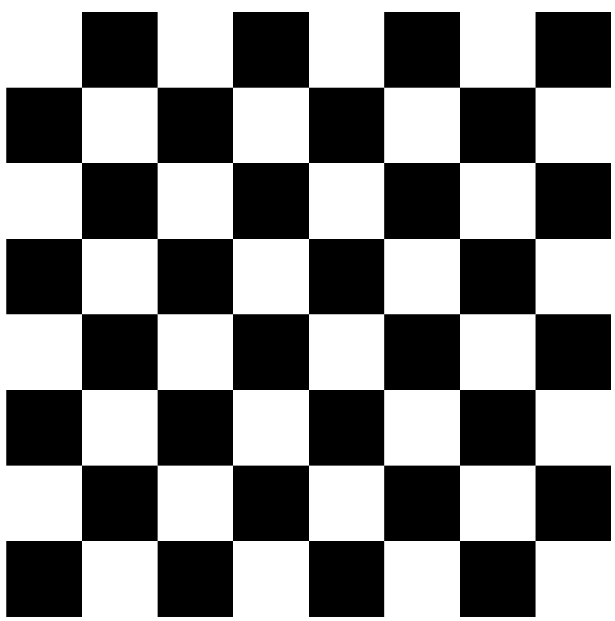
$$x^4 + y^4 = 625$$

Exercise 45

Prove that you can use dominoes to tile a rectangular checkerboard with an even number of squares.

Solution

In this exercise, we will use Existence Proof to prove this problem. We will take a checkerboard has the size 8×8 to be an example. Since the number of square on this checkerboard is 64 squares and 64 is an even number. And we will use the two dominoes to tile it. It has total 64 squares so that we need about 32 dominoes to tile the whole checkerboard.



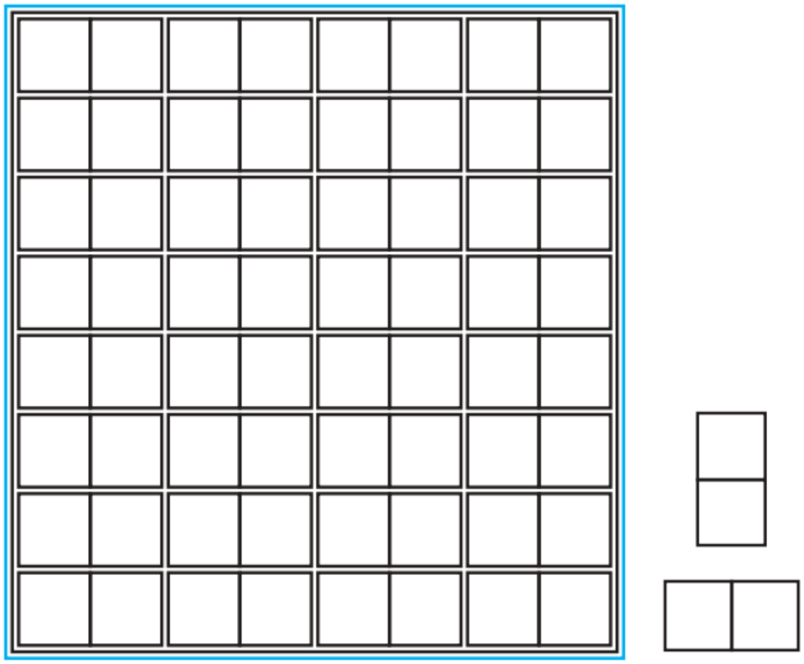


Figure 1: 8×8 Checkerboard tiled with two dominoes

As we can see from **Figure 1**, all the squares on the checkerboard are filled with 32 two dominoes horizontally. Moreover, this is also a constructive existence proof that show we can use dominoes to tile rectangular checkerboard with an even number of squares. Additionally, there are still many ways to tile a checkerboard not only place it horizontally.

Exercise 46

Prove or disprove that you can use dominoes to tile a 5 × 5 checkerboard with three corners removed.

Solution

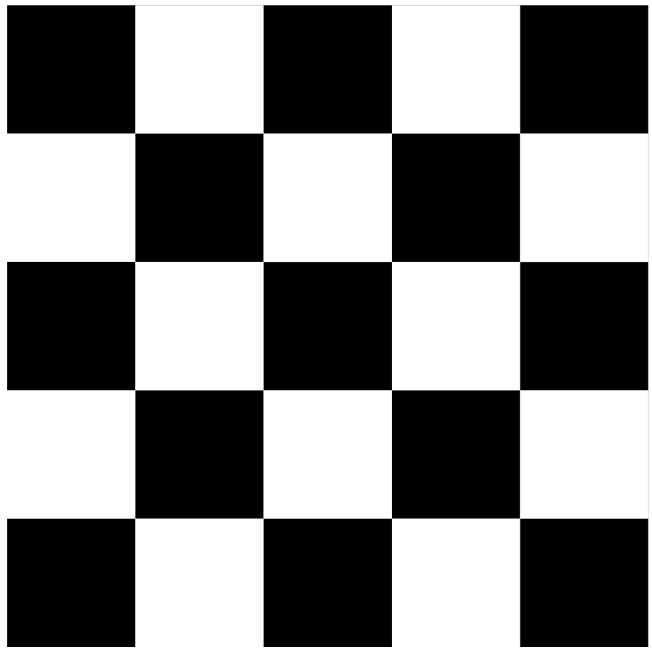


Figure 2: 5×5 Checkerboard

As we can see from the 5×5 checkerboard on **Figure 2**, there are 13 black squares and 12 white squares. If we remove 3 black squares at three corner of the checkerboard, there will be remain 10 black squares and 12 white squares.

From **Exercise 45**, we can tile the checkerboard with even squares but it requires that the number of white squares and the number of black squares must be equal so that they can tile the checkerboard. For example, the 8×8 checkerboard has 64 squares so that there will be 32 black squares and 32 white squares. However, in this Exercise, it remains 10 black squares and 12 white squares so it has together 22 squares. 22 squares mean that there are 11 black squares and 11 white squares. This create a contradiction that the checkerboard only contains 10 black squares and 12 white squares but it requires 11 black squares and 11 white squares to tile the checkerboard. So we can infer that we can not tile the 5×5 checkerboard with three corners removed.

Exercise 47

Use a proof by exhaustion to show that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist.

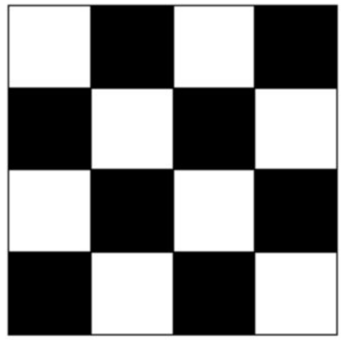


Figure 3: 4×4 Checkerboard

Solution

Firstly, assume that the squares in the upper left and lower right corners removed. Number of the squares of the original checkerboard from 1 to 16, starting in the first row, moving right in this row, then starting in the leftmost square in the second row and moving right, and so on. We will remove squares 1 and 16.

To prove this by using proof by exhaustion, we will divide the problem into 2 cases:

- The first dominoes is laid horizontally start from square 2. So the first dominoes will cover square 2-3. After that, the other squares covered by dominoes in this order: 4-8, 6-7, 5-9, 10-11, 13-14. Therefore, there are 2 square left that are not covered are squares 12 and 15. So this case is impossible.
- The first dominoes is laid vertically start from square 2. So the first dominoes will cover square 2-6. After that, the other squares covered by dominoes in this order: 3-4, 4-8, 5-9, 10-11, 13-14. Therefore, there are 2 square left that are not covered are squares 12 and 15. So this case is impossible.

Because two cases above are all possible so we can infer that a tiling using dominoes of a 4×4 checkerboard with opposite corners removed does not exist.

Exercise 48

Prove that when a white square and a black square are removed from an 8 $\times$ 8 checkerboard (colored as in the text) you can tile the remaining squares of the checker- board using dominoes.

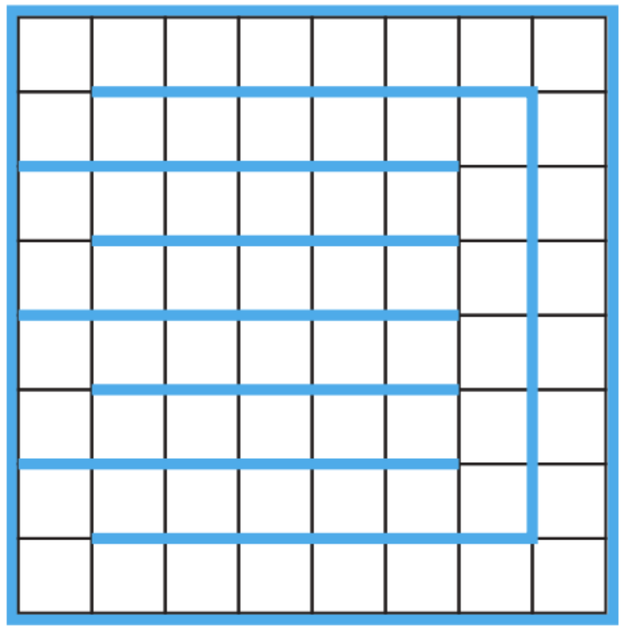


Figure 4: Checkerboard with barrier

Solution

As we can see from the **Figure 4**, within the barrier the checkerboard is become a 64 continuos squares on the checkerboard. It starts at the upper left corner, go all the way to the right, then all the way down, then all the way to the left, and then weave your way back up to the starting point. Therefore, the checkerboard will start at white square and end at black square. Moreover, this rule is **true** for every single path on the checkerboard which has these barriers. If it starts with white and ends with black, these paths can be tiled with dominoes. Moreover, because path starts with white and end with black, therefore, its length will be an even number.

With the rule we established above, we can infer that if we remove one random white square and black square, it will break this 64 continuos squares path into 2 different path start with white square and end with black square(their length will be an even number) and all two paths can be tiled with dominoes. There will be 2 examples below:

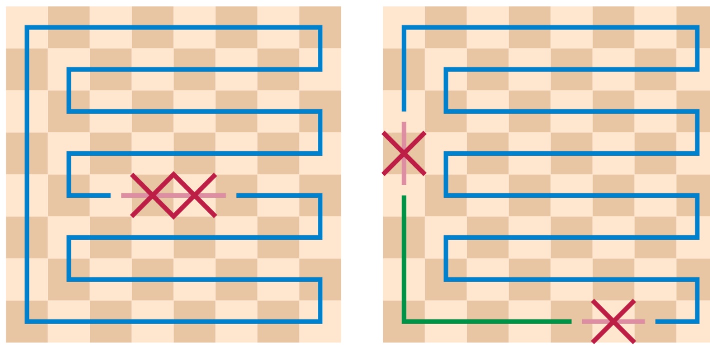


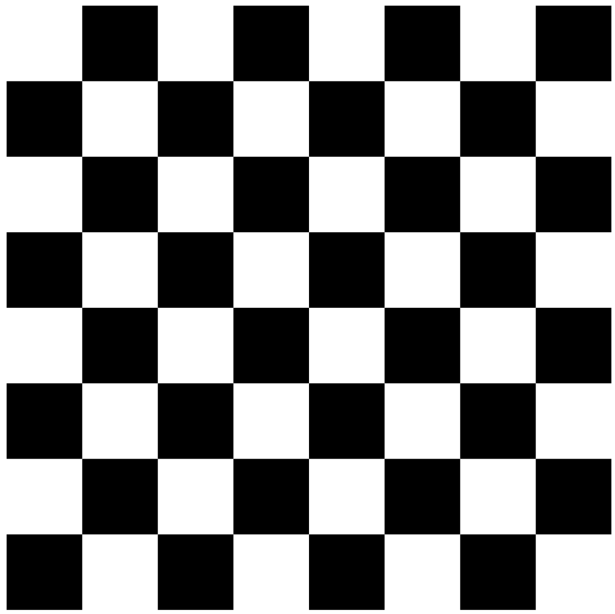
Figure 5: Example of Exercise 48

As you can see from **Figure 5**, if we remove two opposite color square, we still can tile the whole checkerboard by tiling two paths created by removing two opposite color square with even length.

Exercise 49

Show that by removing two white squares and two black squares from an 8 $\times$ 8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

Solution



So to prove that by removing two white squares and two black squares from an 8 $\times$ 8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes, we will use existence proof, which means that we just need to work out a example that can make the 8 $\times$ 8 checkerboard can not be tiled by removing two white squares and two black squares. Therefore, we have an example:

- WLOG, if we first remove two black squares which is adjacent to the white corner square. Next, we will continue remove to white squares which is adjecent to the black corner square. Therefore, there obviously impossible to tile that checkerboard because the domino can not cover two diagonal squares.

Because it exists a case that when we remove two black squares and two white square from an 8 $\times$ 8 checkerboard that make the checkerboard can not be tiled. Therefore, we can prove that by removing two white squares and two black squares from an 8 $\times$ 8 checkerboard (colored as in the text) you can make it impossible to tile the remaining squares using dominoes.

Exercise 50

Find all squares, if they exist, on an 8 $\times$ 8 checkerboard such that the board obtained by removing one of these squares can be tiled using straight triominoes.

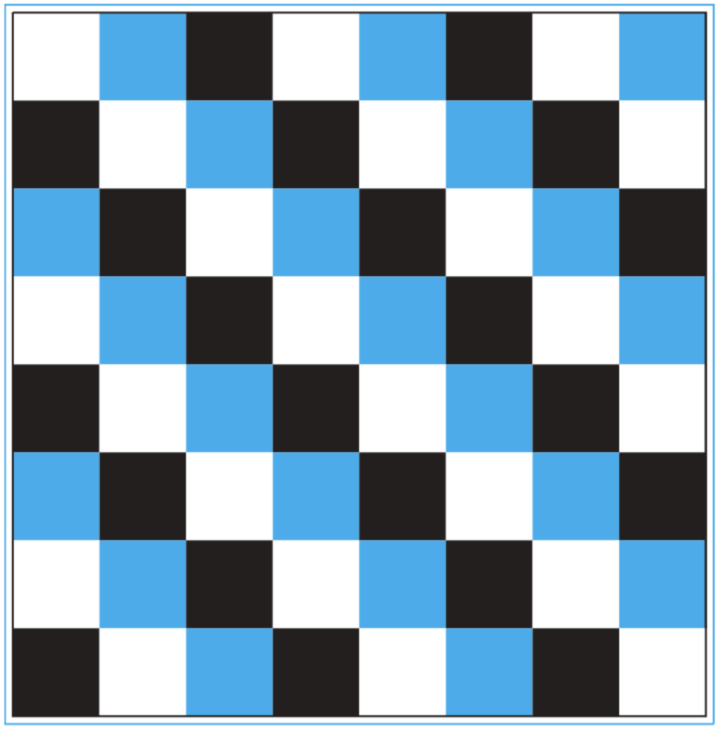


Figure 6: 8×8 checkerboard with 3 colors

Solution

As we can see from **Figure 6**, there are 21 black squares, 21 blue squares and 22 white squares (64 total). Next, we will check about three cases for the removed square. When we remove one square, there will be 63 squares left:

- The square removed is black square:
  If we remove one black square, there will be 20 black, 21 blue and 22 white left. However $63/3 = 21$, which means that there will be 21 black, 21 blue and 21 white but we only have 20 black, 21 blue and 22 white so it is a contradiction. Therefore, this case is impossible.

- The square removed is blue square: If we remove one blue square, there will be 21 black, 20 blue and 22 white left. However $63/3 = 21$, which means that there will be 21 black, 21 blue and 21 white but we only have 21 black, 20 blue and 22 white so it is a contradiction. Therefore, this case is impossible

- The square removed is white square: If we remove one white square, there will be 21 black, 21 blue and 21 white left. Moreover, $63/3 = 21$ so there

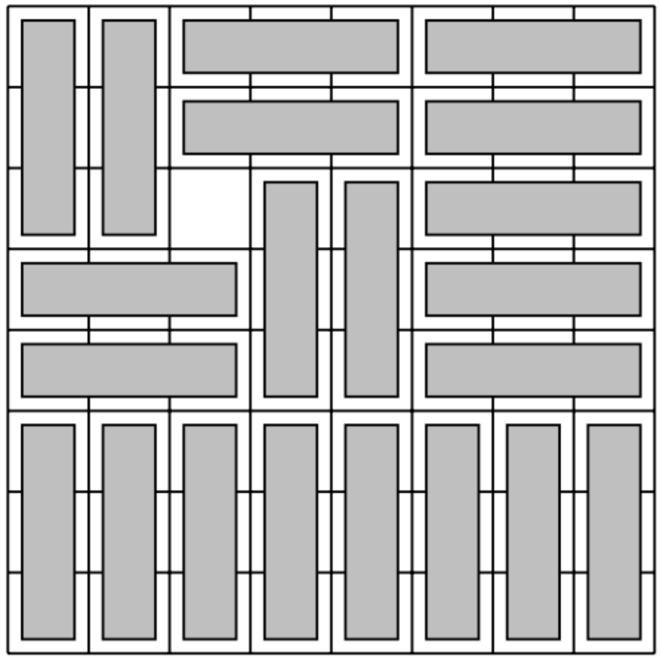


Figure 7: 8×8 checkerboard with one white square removed

will be exist a checkerboard with one white square removed can be tiled using string triominoes. Additionally, we can also see that on the checkerboard, the coordinate (3,3), (3,6), (6,3), (6,6) is always keep white color no matter we rotate the checkerboard. Because only these four squares keep the color in all condition so that if we remove on of these square, we are able to tile the checkerboard within one square missing.

As we can see from **Figure 7**, the coordinate (3, 3) square is the only square which is not filled by triominoes so if we remove this square and so on the 3 other coordinates above. Therefore, it exist the board can be tiled with one square removed.

Exercise 51

(a) Draw each of the five different tetrominoes, where a tetromino is a polyomino consisting of four squares.

(b) For each of the five different tetrominoes, prove or disprove that you can tile a standard checkerboard us- ing these tetrominoes.

Solution

Exercise 52

Prove or disprove that you can tile a 10×10 checker- board using straight tetrominoes.

Solution