Ch. 2, Sec. 7: The Derivative as a Function

1. **Quote.**

   "The greatest strategy is doomed if it's implemented badly."

   — Bernhard Riemann.

2. **Learning Objectives.**

3. **Motivating problem.** Find the derivative of the function $f(x) = x^2$ at

   (a) $x = 1$

   (b) $x = 2$

   (c) $x = 10$.

Hint. The derivative of a function f at $x = a$, denoted $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

4. **Definition.** **The derivative as a function.** The derivative of a function $f(x)$, denoted $f'(x)$, is the function

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h},$$

provided this limit exists. If $f'(x)$ exists, we say $f(x)$ is **differentiable**.

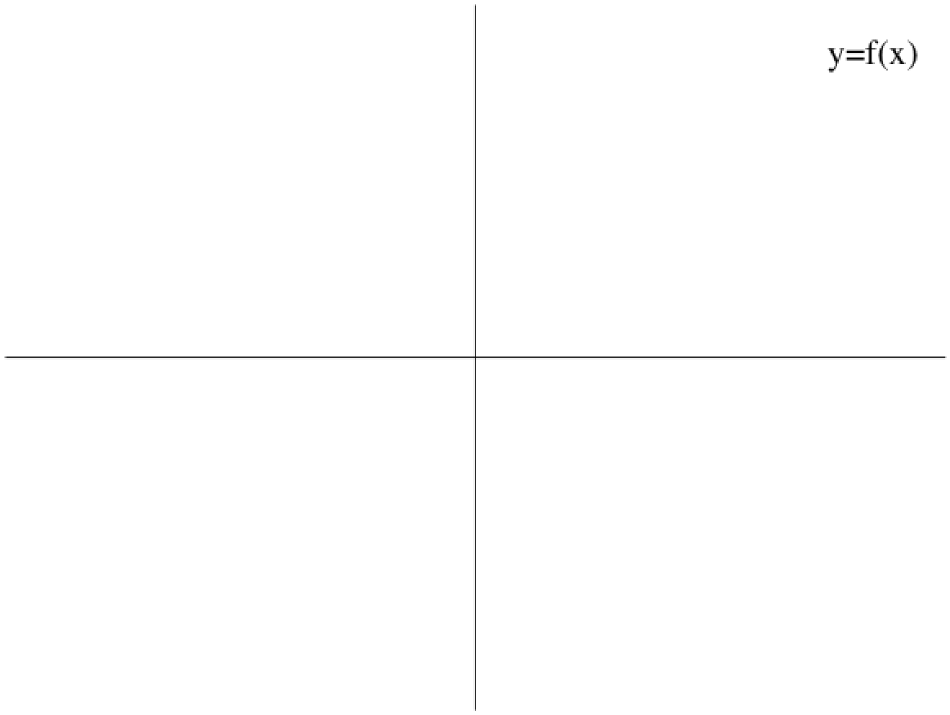
5. **Example.** Let

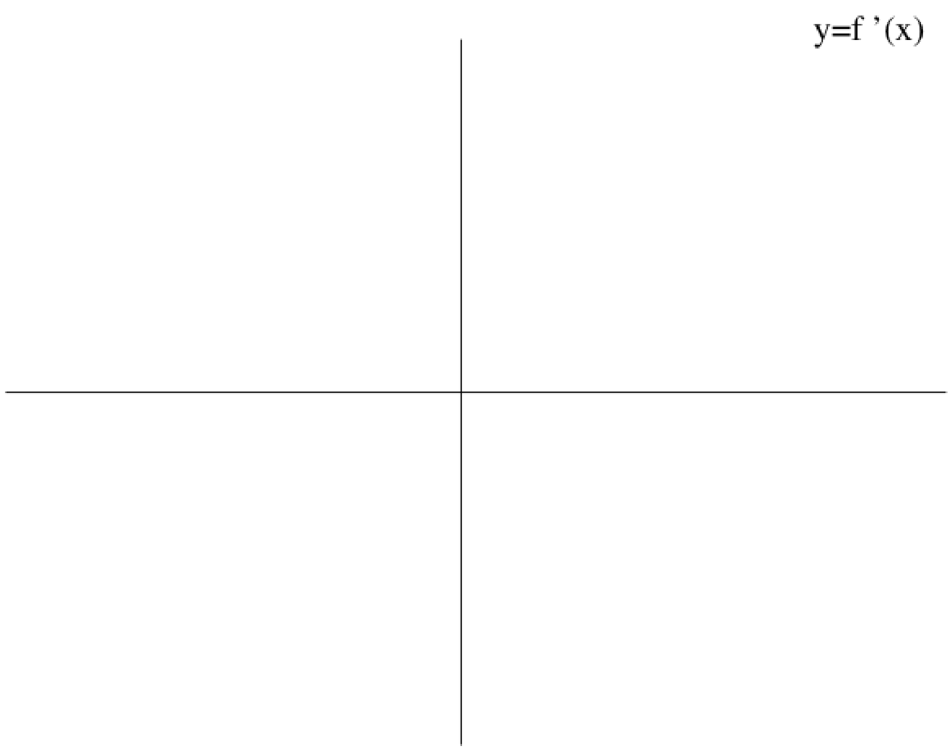
$$f(x) = \sqrt{x}.$$

(a) Determine the domain of f.

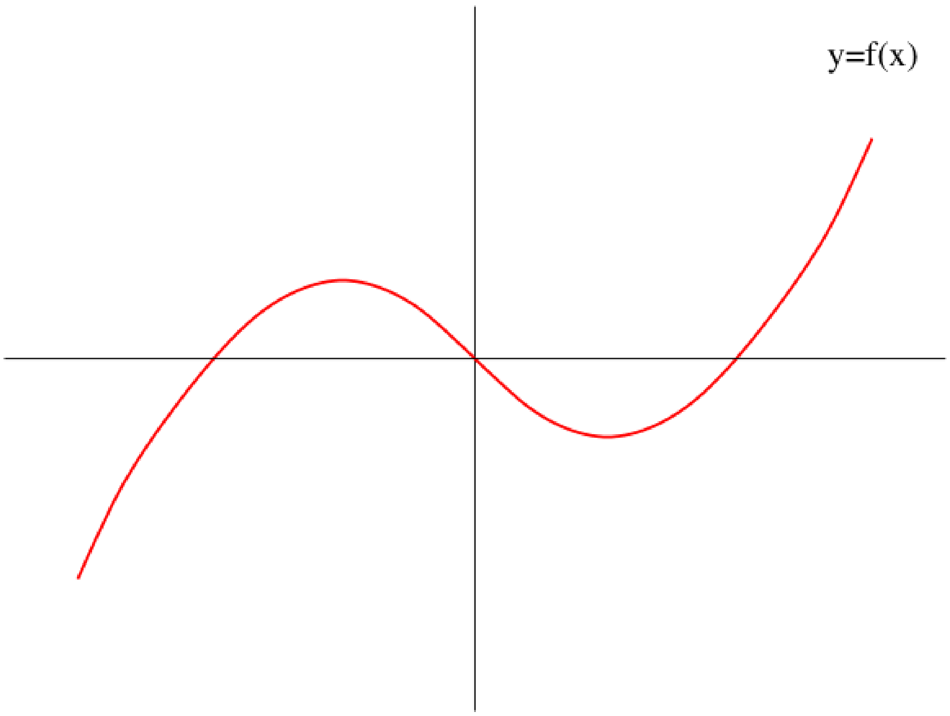
(b) Find $f'(x)$. What is the domain of f'?

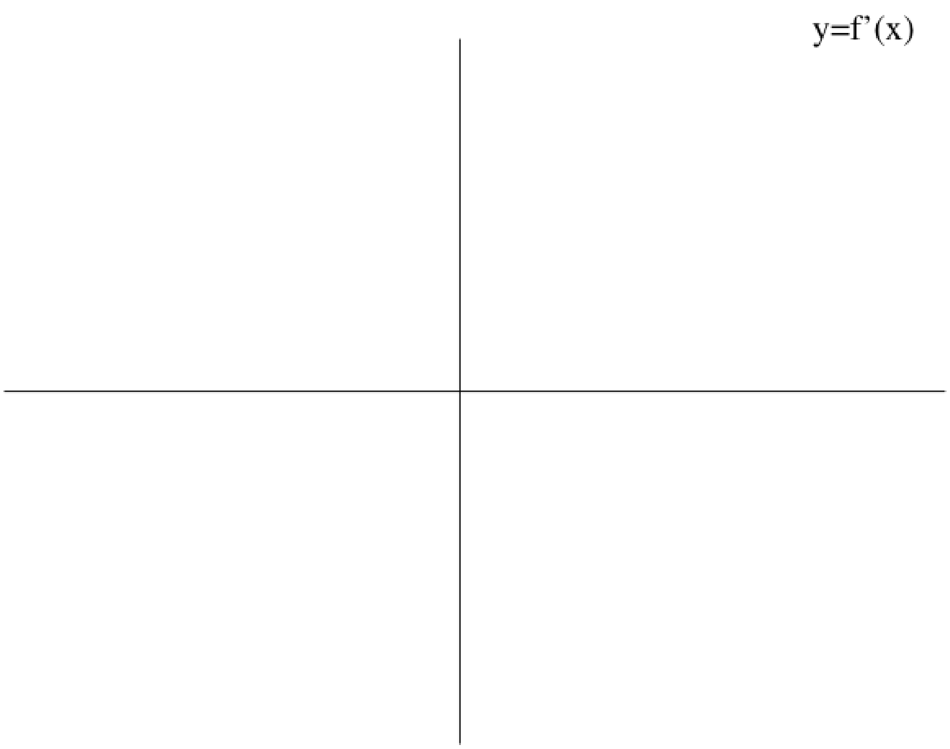
(c) Sketch graphs of $f = \sqrt{x}$ and f'.



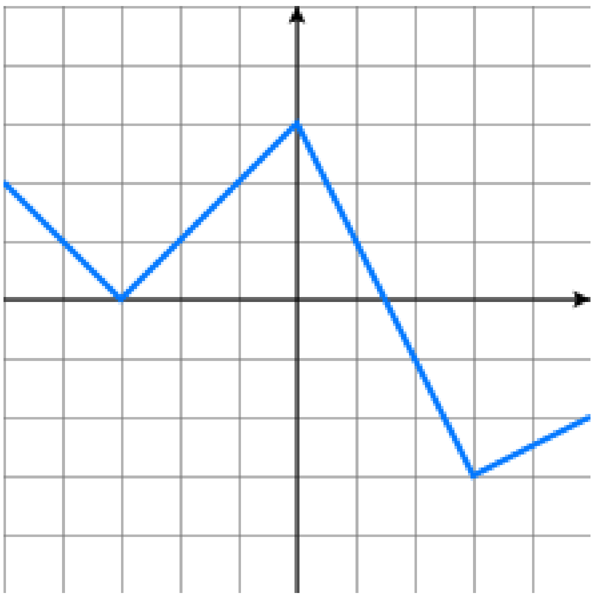


6. **Example.** The graph of f is given. Sketch the graph of f'.





7. **Homework.** The graph of $f(x)$ is given. Sketch f', then state an analytic formula for f'.



8. **Notation.** For $y = f(x)$ it is common to write:

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}f(x) = Df(x) = D_x f(x)$$

Also,

$$f'(a) = \left.\frac{dy}{dx}\right|_{x=a}.$$

9. **Homework.** Find the derivatives of the following functions using the definition of the derivative.

   (a) $f(x) = ax^2 + bx + c$, where a, b, and c are constants

   (b) $g(t) = \frac{t}{t-5}$

   (c) $p(x) = \sqrt{3x+1} + x^2$

10. **Definition.** A function is **differentiable at** a if $f'(a)$ exists. It is **differentiable on an open interval** if it is differentiable at every number in the interval.

11. **Questions.**

    (a) Is every continuous function differentiable?

    (b) Is every differentiable function continuous?

12. **Questions.** How might a function be non-differentiable?

    Hint. There are three answers.

13. **Definition.** **Higher Derivatives.** Suppose that f is a differentiable function. The **second derivative** of f is the derivative of f'.

Notation.

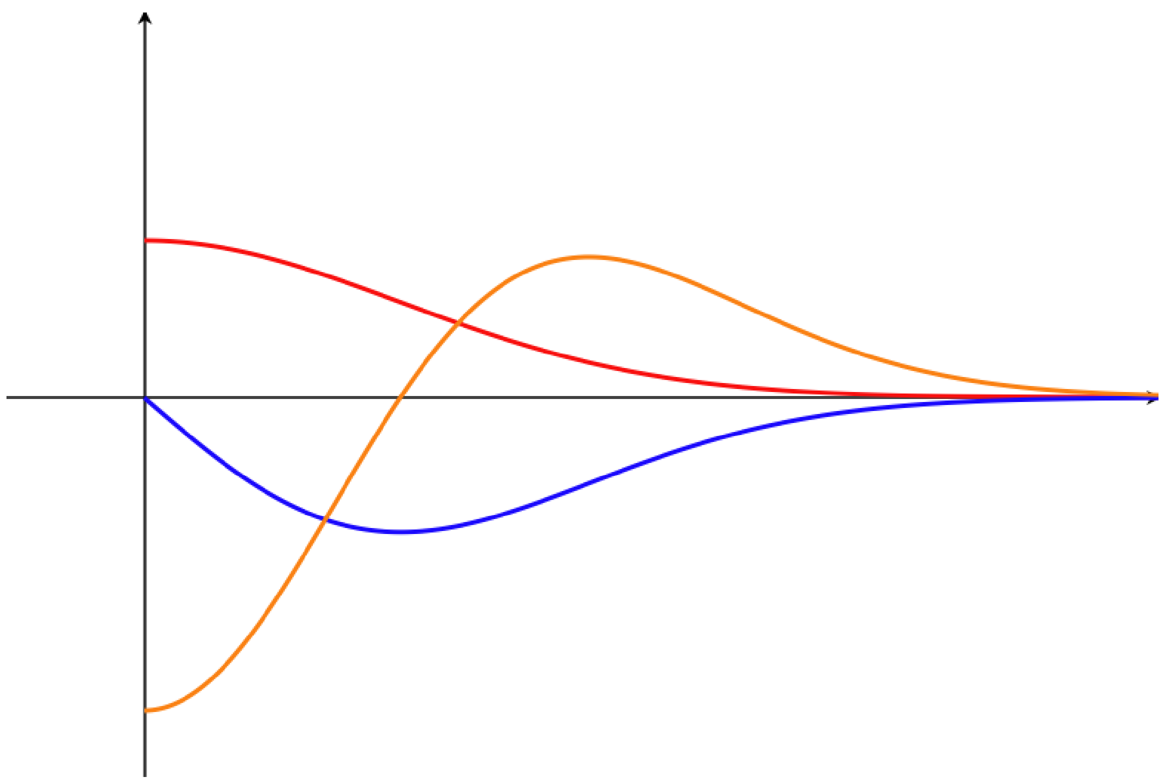
$$(f')' = f''$$
$$(y')' = y''$$
$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

14. **Example.** Find $f''(x)$ if $f(x) = x^2$.

15. **Definition. Acceleration.** The instantaneous rate of change of velocity with respect to time is called the **acceleration** of the object.

$$a(t) = v'(t) = s''(t).$$

16. **Example.** The figure shows the graphs of three functions. One is the position function of a particle, one is its velocity, and one is its acceleration. Identify each curve.



Homework. If you're struggling with the limit definition of the derivative, I suggest you try Ch2.6's homework first, which features some easier questions.

1. **Homework.** Find the derivatives of the following functions using the definition of the derivative.

   (a) $f(x) = ax^2 + bx + c$, where a, b, and c are constants

   (b) $g(t) = \frac{t}{t-5}$

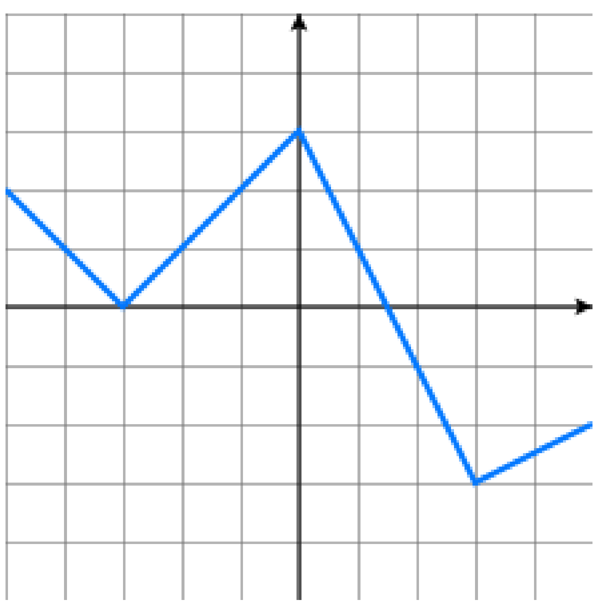
   (c) $p(x) = \sqrt{3x+1} + x^2$

2. **Homework.** Use the limit definition of the derivative to find $f'(x)$.

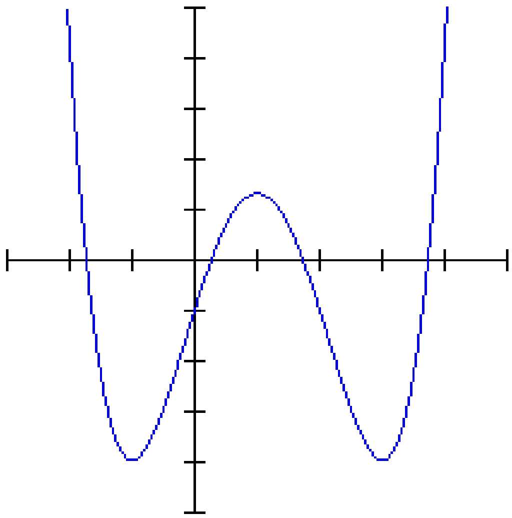
   (a) $f(x) = \sqrt{3x-5}$

   (b) $f(x) = \frac{1}{\sqrt{x}}$

3. **Homework.** The graph of f is given. Sketch a labelled graph of f', and state an analytic formula for f' (i.e., write $f'(x)$ as a piecewise function).



4. **Homework.** The graph of $y = q(x)$ is given. Sketch a labelled graph of

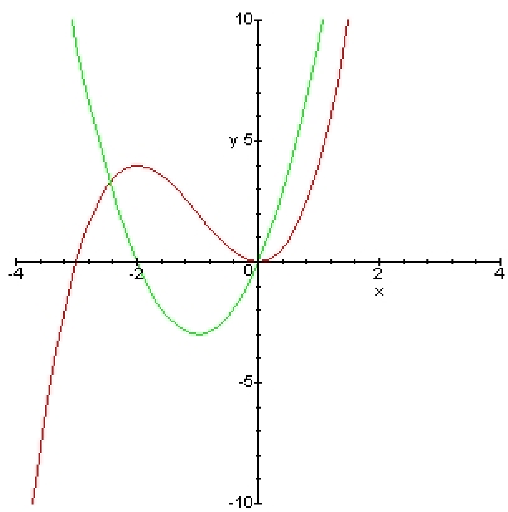


$q'(x)$.

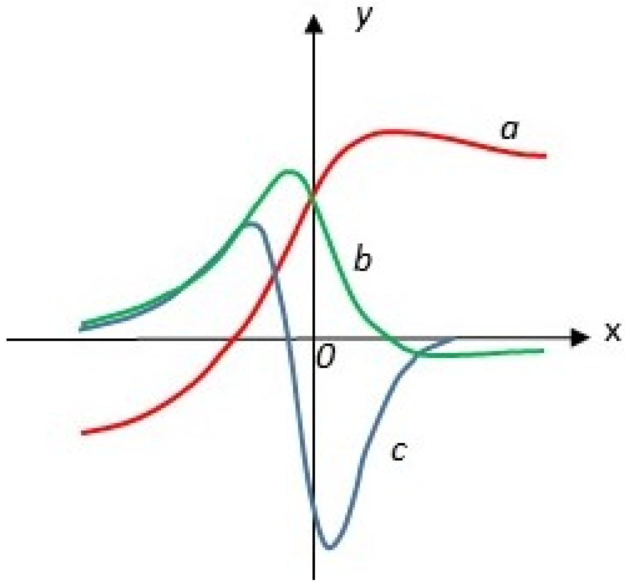
5. **Homework.** A particle's displacement can be described by $s(t) = t^3 - 2t^2 + t - 7$. Find it's velocity $v(t)$ and acceleration $a(t)$ using the limit definition of the derivative.

6. **Homework.** A particle's displacement can be described by $s(t) = \frac{1}{\sqrt{t+1}}$. Find it's velocity $v(t)$ and acceleration $a(t)$ using the limit definition of the derivative.

7. **Homework.** The graph of $f(x)$ and $f'(x)$ is given below. Discuss which graph is which. Be sure to justify your answer extensively.



8. **Homework.** The graph of $f(x)$, $f'(x)$, and $f''(x)$ is given below. Discuss which graph is which. Be sure to justify your answer extensively.


y
a
b
x
O
c