Ch. 4, Sec. 3: Derivatives and the Shapes of Curves

1. **Quote.**

"I much prefer the sharpest criticism of a single intelligent man to the thoughtless approval of the masses."

— Johannes Kepler.

2. **Learning Objectives.**

3. **Example.** Suppose $f(x)$ is *increasing* on an interval. What does this mean intuitively? Mathematically? What if $f(x)$ is *decreasing*?

4. **Definition.** Suppose $f(x)$ is defined on an interval I. We say $f(x)$ is **increasing** on I if for all $x_1, x_2 \in I$

$$x_1 < x_2 \implies f(x_1) < f(x_2).$$

We say $f(x)$ is **decreasing** on I if for all $x_1, x_2 \in I$

$$x_1 < x_2 \implies f(x_1) > f(x_2).$$

5. **Theorem. Increasing/Decreasing Test.**

(a) If $f'(x) > 0$ on an interval, then f is increasing on that interval.

(b) If $f'(x) < 0$ on an interval, then f is decreasing on that interval.

6. **Example.** Find the open intervals on which the function

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5$$

is increasing and those on which is decreasing.

7. **Theorem. The First Derivative Test.** Suppose that c is a critical number of a continuous function f.

   (a) If f' changes from positive to negative at c, then f has a local maximum at c.

   (b) If f' changes from negative to positive at c, then f has a local minimum at c.

   (c) If f' does not change sign at c, then f has no local minimum or maximum at c.

8. **Example.** Find and classify all local extrema of the following functions.

   (a) $f(x) = 3x^4 - 4x^3 - 12x^2 + 5$.

   (b) $g(x) = |x|$.

9. **Definition.** If the graph of f lies above all of its tangent lines on an interval I, then it is called **concave upward** on I. If the graph of f lies below all of its tangents on I, it is called **concave downward** on I.

10. **Theorem. Concavity Test.**

    (a) If $f''(x) > 0$ for all $x \in I$, then the graph of f is concave upward on I.

    (b) If $f''(x) < 0$ for all $x \in I$, then the graph of f is concave downward on I.

11. **Definition.** A point P on a curve $y = f(x)$ is called an **inflection point** if f is continuous there and the curve changes from concave upward to concave downward or from concave downward to concave upward at P.

12. **Remark.** In practice, when finding inflection points, check for where the second derivative changes sign. These typically occur when:

    (1)

    (2)

13. **Example.** Find the intervals of concavity, and the inflection points, of the following functions.

    (a) $g(x) = x^2 e^{-x}$

    (b) $h(x) = \sqrt[3]{x-1}$

    (c) $L(x) = \log_2(x^2 + 3)$

14. **Theorem.** **The Second Derivative Test.** Suppose f'' is continuous near c.

(a) If $f'(c) = 0$ and $f''(c) > 0$ then f has a local minimum at c.

(b) If $f'(c) = 0$ and $f''(c) < 0$ then f has a local maximum at c.

15. **Example.** Find and classify all local extrema from (13); i.e.,

(a) $g(x) = x^2 e^{-x}$

(b) $h(x) = \sqrt[3]{x-1}$

(c) $L(x) = \log_2(x^2+3)$

16. **Example.** Find the inflection points and the intervals of concavity of

$$f(x) = 3x^4 - 4x^3 - 12x^2 + 5.$$

Use these to classify the local extrema.

17. **Theorem.** **Rolle's Theorem.** (Michel Rolle, French mathematician, 1652-1719) Let f be a function that satisfies the following three hypotheses:

   (a) f is continuous on the closed interval $[a, b]$.

   (b) f is differentiable on the open interval (a, b).

   (c) $f(a) = f(b)$.

   Then there is a number c in (a, b) such that $f'(c) = 0$.

18. **Theorem. The Mean Value Theorem.** Suppose $f(x)$ is differentiable on $[a, b]$. Then there is a number c in (a, b) such that

$$f'(c) = \frac{f(b) - f(a)}{b - a}$$

or, equivalently,

$$f(b) - f(a) = f'(c)(b - a).$$

19. **Example.** A car is driving along a rural road where the speed limit is 70 km/h. At 3:00 pm its odometer reads 18075 km. At 3:18 its reads 18100 km. Prove that the driver violated the speed limit at some instant between 3:00 and 3:18 pm.