Ch. 2, Sec. 5: Limits Involving Infinity

1. **Quote.**

   "Rest satisfied with doing well, and leave others to talk of you as they please."

   — Pythagoras.

2. **Learning Objectives.**

3. **Examples.** Sketch the graphs of the following functions

(a) $f(x) = \dfrac{1}{x}$

(b) $g(x) = e^x$

4. **Definition.** The meaning of

$$\lim_{x \to a^+} f(x) = \infty$$

means that the values of $f(x)$ can be made as big as we want by taking $x > a$ closer to a (but not equal to a).

5. **Definition.** The notation

$$\lim_{x \to a} f(x) = \infty$$

means that the values of $f(x)$ can be made as big as we want by taking x closer to a (but not equal to a).

6. **Example.** Suppose
$$f(x) = \frac{1}{7 - x}.$$
Analyze the left- and right- hand limits as x approaches 7.

7. **Homework.** Suppose
$$g(x) = \frac{x+1}{x+2}.$$
Analyze the left- and right- hand limits as x approaches -2.

8. **Example.** Consider

$$f(x) = \frac{x^2 - 9}{x^2 + 2x - 3}.$$

Investigate the following limits (split into one-sided limits as needed).

(a) $\lim_{x \to 3} f(x)$

(b) $\lim_{x \to -3} f(x)$

(c) $\lim_{x \to 1} f(x)$

9. **Definition.** The line $x = a$ is a **vertical asymptote** of the curve $y = f(x)$ if at least one of the following is true:

$$\lim_{x\to a^+} f(x) = \infty \qquad \lim_{x\to a^-} f(x) = \infty \qquad \lim_{x\to a} f(x) = \infty$$
$$\lim_{x\to a^+} f(x) = -\infty \qquad \lim_{x\to a^-} f(x) = -\infty \qquad \lim_{x\to a} f(x) = -\infty$$

10. **Note.** Vertical asymptotes typically occur at two places:

11. **Example.** Find the domain of $f(x)$, then find its vertical asymptotes.

$$f(x) = \frac{x^3 + x^2 - 6x}{x^4 - 9x^3 + 24x^2 - 20x} = \frac{x(x+3)(x-2)}{x(x-5)(x-2)^2}$$

12. **Definition.** Let f be a function defined on some interval (a, ∞). Then

$$\lim_{x \to \infty} f(x) = L$$

means that the values of $f(x)$ can be made arbitrarily close to L by taking x sufficiently large.

13. **Definition.** The line $y = L$ is called a **horizontal asymptote** of the curve $y = f(x)$ if either

$$\lim_{x \to \infty} f(x) = L \text{ or } \lim_{x \to -\infty} f(x) = L$$

14. **Question.** Can a function cross it's horizontal asymptote?

15. **Example.** Find the horizontal asymptotes of the following functions:

   (a) $f(x) = \frac{1}{x}$

   (b) $g(x) = e^x$

   (c) $h(x) = \arctan(x)$

16. **Question.** Using regular numbers and infinity, what arithmetical operation can you meaningfully calculate something?

17. **Remark.** If $r > 0$ is a rational number, then

$$\lim_{x\to\infty} \frac{1}{x^r} = 0 \ .$$

If $r > 0$ is a rational number such that x^r is defined for all x then

$$\lim_{x\to-\infty} \frac{1}{x^r} = 0 \ .$$

18. **Example.** Evaluate

$$\lim_{x\to\infty} \frac{3x^3 - 4x^2 - 1}{6x^3 + x + 2}$$

19. **Example.** Evaluate the following:

(a)

$$\lim_{x\to\infty} \frac{3x^3 - 4x^2 - 1}{6x^4 + x + 2}$$

(b)

$$\lim_{x\to-\infty} \frac{3x^3 - 4x^2 - 1}{6x^4 + x + 2}$$

(c)

$$\lim_{x\to\infty} \frac{3x^4 - 4x^2 - 1}{6x^3 + x + 2}$$

(d)

$$\lim_{x\to-\infty} \frac{3x^4 - 4x^2 - 1}{6x^3 + x + 2}$$

20. **Example.** Solve

$$\lim_{x \to -\infty} \frac{\sqrt{3x^2 - 5} - 1}{2x + 5}$$

21. **Homework.** Solve

$$\lim_{x\to -\infty}(\sqrt{x^2+ax}-\sqrt{x^2+bx})$$

22. **Procedure.** Find the following limits.

(a) $\lim\limits_{x \to \infty} (x^2 - x)$

(b) $\lim\limits_{x \to \infty} \dfrac{x^2}{e^x}$

(c) $\lim\limits_{x \to \infty} \dfrac{e^{4x} + 5e^{3x} - 1}{1 + e^{2x} - 7e^{4x}}$

23. **Homework.** Analyze the left- and right- hand limits at $x = 1$ for the following functions. Then, determine if

$$\lim_{x \to 1} f(x)$$

is ∞, $-\infty$, or neither.

(a) $f(x) = \dfrac{x+2}{x-1}$

(b) $f(x) = \dfrac{x-2}{1-x}$

(c) $f(x) = \dfrac{1}{(x-1)^2}$

24. **Homework.** Consider

$$f(x) = \frac{x^2-9}{x^2+2x-3}.$$

Determine whether or not the following limits exist. If the limit does not exist, find both one-sided limits, and finally determine if the full limit goes to ∞, $-\infty$, or neither.

(a) $\lim_{x \to 3} f(x)$

(b) $\lim_{x \to -3} f(x)$

(c) $\lim_{x \to 1} f(x)$

25. **Homework.** State the domain of the following functions. At all points outside the domain, determine whether or not the limit exists. If the limit does not exist, find both one-sided limits, and finally determine if the full limit goes to ∞, $-\infty$, or neither.

(a) $f_1(x) = \dfrac{-5}{(x+5)}$

(b) $f_2(x) = \dfrac{-5}{(x+5)^2}$

26. **Homework.** State the domain of the following functions. At all points outside the domain, determine whether or not the limit exists. If the limit does not exist, find both one-sided limits, and finally determine if the full limit goes to ∞, $-\infty$, or neither.

(a) $g_1(x) = \dfrac{x^2+3x-10}{x^2+2x-15}$

(b) $g_2(x) = \dfrac{x^2-x}{x^4-x^3}$

27. **Homework.** Given $f(x)$ below, find $\lim\limits_{x\to\infty} f(x)$ and $\lim\limits_{x\to-\infty} f(x)$.

(a) $f(x) = \dfrac{3x^2+1}{-6x+1}$

(b) $f(x) = \dfrac{3x^2+1}{-6x^2+1}$

(c) $f(x) = \dfrac{3x^2+1}{-6x^3+1}$

(d) $f(x) = \dfrac{3x^2+1}{-6x^4+1}$

28. **Homework.** Find the equation of all horizontal asymptotes of the following functions.

(a) $f(x) = \dfrac{x^2 - 2x + 7}{5x^2 - 3x + 100}$

(b) $g(x) = \dfrac{x^{100} + 3}{3 - x^{100}}$

29. **Homework.** Find the equation of all horizontal asymptotes of the following functions.

(a) $G_1(x) = \dfrac{\sqrt{9x^2 - 7x + 11} + 1}{2x + 3}$

(b) $G_2(x) = \dfrac{\sqrt{5x^2 + 3} + x - 4}{x + 7}$

30. **Homework.** Given the following functions, find the equation of all asymptotes.

(a) $f(x) = \dfrac{(x+3)(2x-1)}{(x-3)(2x+1)}$

(b) $g(x) = \dfrac{x^2 - 4x - 5}{x^3 - x}$

31. **Homework.** Find the equation of all asymptotes of the following functions.

(a) $h_1(x) = \dfrac{5^{2x} - 5^x - 7}{5^{2x} + 5^x + 11}$

(b) $h_2(x) = \dfrac{5e^{3x} - e^{2x} + 1}{-e^{3x} + e^x - 2}$