Ch. 3, Sec. 5: Implicit Differentiation

1. **Quote.**

   "[Mathematics] is the work of the human mind, which is destined rather to study than to know, to seek the truth rather than to find it."

   — Évariste Galois.

2. **Learning Objectives.**

3. **Motivating problem.** The curve

$$x^3 + y^3 = 3xy$$

is called the **folium of Descartes**. Find the equation of the tangent line at $x = 3/2$.

4. **Definition. Implicit.**

   1. implied: not stated, but understood in what is expressed

   Asking us when we would like to start was an implicit acceptance of our terms.

   2. contained: present as a necessary part of something

   Confidentiality is implicit in the relationship between doctor and patient.

5. **Definition. Implicitly Defined Function.** An equation in two variables x and y may have one or more solutions for y in terms of x or for x in terms of y. These solutions are functions that are said to be **implicitly defined** by the equation.

6. **Example.** In the following equation, is y a function of x?

$$x^2 + y^2 = 1.$$

7. **Reminder:** The graph of an equation are points in the Cartesian plane that satisfy the equation.

8. **Example.** Is $(1, 1)$ included in the graph of $x^2 + y^2 = 1$? How about $(1/2, 1/2)$? Find some points that are in the graph.

9. **Example.** What is $\frac{d}{dx}[y^5]$?

10. **Examples.** Differentiate the following.

(a) $(x + \sin x)^5$

(b) $(x + y)^5$

(c) $(y + 7)^5$

(d) $(y + 0)^5$

11. **Procedure. Implicit Differentiation.**

    (a) Use the chain rule to differentiate both sides of the given equation, thinking of x as the independent variable.

    (b) Solve the resulting equation for $\dfrac{dy}{dx}$.

12. **Example.** The curve
$$x^3 + y^3 = 3xy$$
is called the **folium of Descartes**. Find the equation of the tangent line at the point $\left(\frac{3}{2}, \frac{3}{2}\right)$.

13. **Example.** Determine the points on the circle

$$(x-1)^2 + (y-2)^2 = 4$$

where the tangent line is

(a) horizontal;

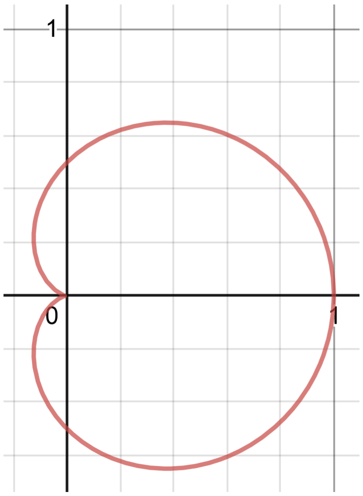
(b) vertical.

14. **Example.** Determine where the curve $sinx = \cos^2 y$ has a vertical tangent.

15. **Example.** Consider the cardioid, which is the curve of the following equation:

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2.$$

Find the y-coordinates corresponding to $x = 0$, and at each of these places, find the equation of the tangent line there.



1. **Homework.** Find y' using implicit differentiation for the following equations.

   (a) $x^3 + y^3 = 1$

   (b) $e^{x/y} = x - y$

   (c) $\cos(x + y) + x^2 y = y^2 \sin(y)$

2. **Homework.** For the curve

$$x^2 + y^2 = 5$$

   find y'' by implicit differentiation.

3. **Homework.** Determine the points on the circle

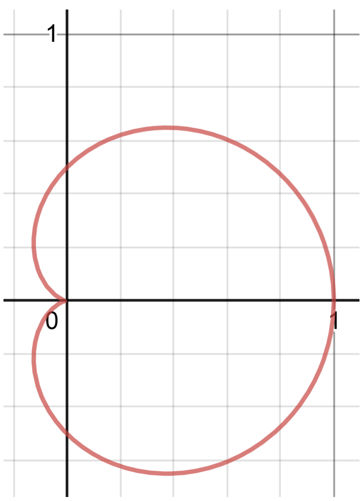
$$(x + 3)^2 + (y - 2)^2 = 9$$

   where the tangent line has a slope of $m = 1$.

4. **Homework.** Consider the cardioid, which is the curve of the following equation:

$$x^2 + y^2 = (2x^2 + 2y^2 - x)^2.$$

   Determine where the tangent line is horizontal, and where it is vertical.



5. **Homework.** **Challenging problem.** **Orthogonal Trajectories.** Two curves are called **orthogonal** if at each point of intersection their tangent lines are perpendicular. Show that the curves

$$x^2 + y^2 = ax \text{ and } x^2 + y^2 = ay\ .$$

are orthogonal for any value of $a \in \mathbb{R}$.

Hint. Steps to solve the problem:

(a) Determine where the two curves intersect.

(b) If $x^2 + y^2 = ax$, find $m_1 = \frac{dy}{dx}$.

(c) If $x^2 + y^2 = ay$, find $m_2 = \frac{dy}{dx}$.

(d) Finally, show that the two curves are orthogonal.