Ch. 2, Sec. 6: Derivatives and Rates of Change

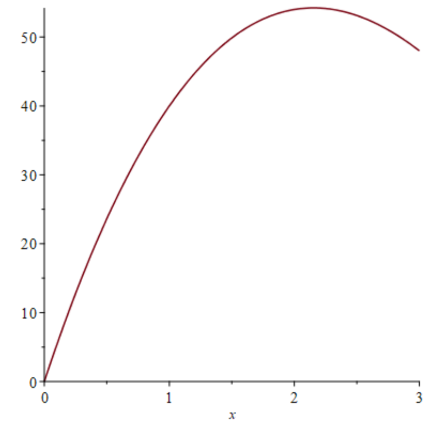
1. **Quote.**

   "Reserve your right to think, for even to think wrongly is better than not to think at all."

   — Hypatia.

2. **Learning Objectives.**

3. **Motivating problem.** Consider a function with graph $y = f(x)$. Approximate the slope of its tangent line at a point P. How do we improve the approximation?



4. **Definition.** The **tangent line** to the curve $y = f(x)$ at the point $P = (a, f(a))$ is the line through P with slope

$$m = \lim_{x \to a} \frac{f(x) - f(a)}{x - a}$$

provided that this limit exists.

5. **Remark.** If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists then

$$\lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a + h) - f(a)}{h}$$

6. **Example.** Find the slope of the tangent line to the graph of $f(x) = x^2+3x$ at the point $x = 1$.

7. **Homework.** Find the slope of the tangent line to the graph of $f(x) = 3x^2 - 7x + 5$ at the point

   (a) $x = 2$

   (b) $x = a$

   Hint: Solutions are (a) $y = 5(x-2)+3$
   (b) $y = (6a-7)(x-a) + (3a^2 - 7a + 5)$

8. **Example.**

(a) Find the slope of the tangent to the curve

$$y = \frac{1}{\sqrt{x}}$$

at the point where $x = a$.

(b) Find the equation of the tangent line at the point $x = 4$.

9. **Definition.** **The Most Important Definition in This Course.** The **derivative of a function** f **at a number** a, denoted by $f'(a)$, is

$$f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

if this limit exists.

10. **Reminder.** If $\lim_{x \to a} \frac{f(x) - f(a)}{x - a}$ exists then

$$f'(a) = \lim_{x \to a} \frac{f(x) - f(a)}{x - a} = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}$$

11. **Example.** Find the derivative of the function

$$y = \frac{1}{x-1}$$

at the point where $x = 3$.

12. **Example.** The following limit represents the derivative of some function f at some number a. State f and a.

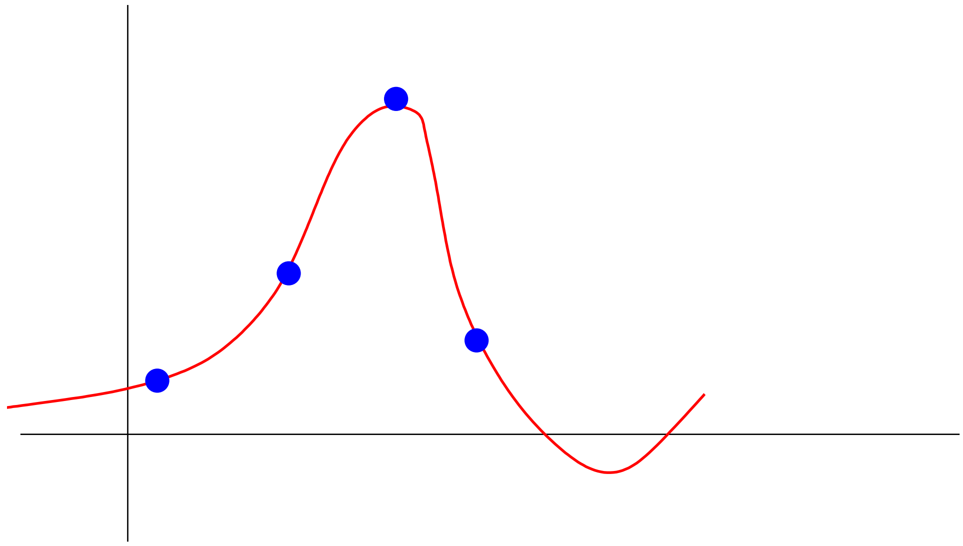
$$\lim_{h\to 0}\frac{2^{h+3}-8}{h}$$

13. **Example.** Let $f(x)=|x|$. Does $f'(0)$ exist?

14. **Reminder.** An equation of the tangent line to $y = f(x)$ at $(a, f(a))$ is given by
$$y - f(a) = f'(a)(x - a) \ .$$

15. **Example.** Find the equation of the tangent line to $f(x) = \dfrac{1}{x-1}$ at the point where $x = 3$.

16. **Example.** Compare the derivatives at each of the points on the graph.



17. **Reminder.** By definition

$$\text{average velocity} = \frac{\text{displacement}}{\text{time}}$$

18. **More Precisely...** Suppose an object moves along a straight line according to an equation of motion $s = f(t)$, where s is the **displacement** of the object from the origin at **time** t.

The average velocity of the object in the time interval from $t = a$ to $t = a + h$ is given by

$$\text{average velocity} = \frac{f(a+h) - f(a)}{h}\ .$$

19. **Definition.** We define the **velocity** (or **instantaneous velocity**) $v(a)$ at time $t = a$ as

$$v(a) = \lim_{h\to 0} \frac{f(a+h) - f(a)}{h}\ .$$

20. **Example.** If an arrow is shot upward on the moon with a velocity of 58 m/s, its height (in meters) after t seconds is given by

$$H = 58t - 0.83t^2 \ .$$

(a) Find the velocity of the arrow when $t = a$.

(b) When will the arrow hit the moon?

(c) With what velocity will the arrow hit the moon?

21. **Notation. Rate of Change.** Let f be a function defined on an interval I and let $x_1, x_2 \in I$. Then the **increment** of x is defined as

$$\Delta x = x_2 - x_1$$

and the corresponding change in y is

$$\Delta y = f(x_2) - f(x_1)\ .$$

The **average rate of change of** y **with respect to** x over the interval $[x_1, x_2]$ is defined as

$$\frac{\Delta y}{\Delta x} = \frac{f(x_2) - f(x_1)}{x_2 - x_1}\ .$$

The **instantaneous rate of change of** y **with respect to** x is defined as

$$\lim_{\Delta x \to 0} \frac{\Delta y}{\Delta x} = \lim_{x_2 \to x_1} \frac{f(x_2) - f(x_1)}{x_2 - x_1}\ .$$

22. **Homework.** If a cylindrical tank holds 100,000 gallons of water, which can be drained from the bottom of the tank in an hour, then Torricelli's Law gives the volume V of water remaining in the tank after t minutes as

$$V = 100,000\left(1 - \frac{t}{60}\right)^2 \quad 0 \leq t \leq 60\ .$$

Find the rate at which the water is flowing out of the tank (the instantaneous rate of change of V with respect to t) at $t = t_1$. What are the units?

Hint: The solution is $-\frac{500}{9}(60 - t_1)$, in gallons /min. Note that the arithmetic gets messy.

23. **Example.** The quantity (in pounds) of a gourmet ground coffee that is sold by a coffee company at a price of p dollars per pound is $Q = f(p)$.

    (a) What is the meaning of the derivative $f'(8)$? What are the units?

    (b) Is $f'(8)$ positive or negative? Explain.