Ch. 3, Sec. 6: Inverse Trigonometric Functions and Their Derivatives

1. **Quote.**

> *"In mathematics the art of asking questions is more valuable than solving problems. "*
>
> — Georg Cantor.

2. **Learning Objectives.**

3. **Definition.** A function $f(x)$ is one-to-one if $f(a) = f(b) \implies a = b$.

4. **Theorem.** A function is one-to-one if its graph passes the horizontal line test.

5. **Definition.** Suppose $f(x)$ is a one-to-one function. Then the inverse function, denoted $f^{-1}(x)$, is the function that satisfies the following:
   (a) $f^{-1}(f(x)) = x$
   (b) $f(f^{-1}(x)) = x$
   (c) $\text{Dom}_{f^{-1}} = \text{Ran}_f$ and $\text{Ran}_{f^{-1}} = \text{Dom}_f$

6. **Warning.** $f^{-1}(x) \neq (f(x))^{-1}$

7. **Procedure.** Steps for finding an inverse function.
   (a) Write $y = f(x)$.
   (b) Interchange (swap) x and y to obtain $x = f(y)$.
   (c) Isolate for y. This will be $f^{-1}(x)$.

8. **Homework.** Find the inverse of the following one-to-one functions. State it's domain and range.
   (a) $f(x) = \dfrac{x}{x+1}$
   (b) $g(x) = \frac{1}{e^x+1}$.

9. **Question.** Is $f(x) = \sin(x)$ an invertible function?

10. **Reminder. Summary of Inverse Trigonometric Functions:**

Function	Domain	Range
$\arcsin(x)$	$x \in [-1, 1]$	$\theta \in [-\pi/2, \pi/2]$
$\arccos(x)$	$x \in [-1, 1]$	$\theta \in [0, \pi]$
$\arctan(x)$	$x \in (-\infty, \infty)$	$\theta \in (-\pi/2, \pi/2)$

11. **Theorem.** Derivatives of inverse trigonometric functions.

(a)
$$\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$$

(b)
$$\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$$

(c)
$$\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$$

(d)
$$\frac{d}{dx}(\cot^{-1}(x)) = -\frac{1}{1+x^2}$$

12. **Example.** Prove $\frac{d}{dx}(\sin^{-1}(x)) = \frac{1}{\sqrt{1-x^2}}$.

13. **Problem. Derivative of an inverse function.** Suppose f is a one-to-one differentiable function and its inverse function f^{-1} is also differentiable. Use implicit differentiation to show that

$$\frac{d}{dx}f^{-1}(x) = \frac{1}{f'(f^{-1}(x))}$$

provided that the denominator is not 0.

1. **Homework.** Prove $\frac{d}{dx}(\cos^{-1}(x)) = -\frac{1}{\sqrt{1-x^2}}$.

2. **Homework.** Prove $\frac{d}{dx}(\tan^{-1}(x)) = \frac{1}{1+x^2}$.

3. **Homework.** $\frac{d}{dx}(\cot^{-1}(x)) = \frac{-1}{1+x^2}$