Ch. 3, Sec. 1: Derivatives of Polynomials and Exponential Functions

1. **Quote.**

   "Mathematics is the art of giving the same name to different things."

   — Henri Poincaré.

2. **Learning Objectives.**

3. **Reminder.** The **derivative of a function** f is the function f' defined by
$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$
for all x for which this limit exists.

4. **Theorem. Derivative of a Constant.**

$$\frac{d}{dx}[c] = 0$$

5. **Example.** Find x', $(x^2)'$, and $(x^3)'$.

6. **Theorem. The (General) Power Rule.** If n is any real number, then

$$\frac{d}{dx}(x^n) = nx^{n-1}$$

7. **Theorem. Constant Multiple Rule.** If c is a constant and f is a differentiable function, then

$$\frac{d}{dx}[cf(x)] = c \cdot \frac{d}{dx}[f(x)]$$

8. **Theorem. Sum Rule.** If f and g are differentiable functions, then

$$\frac{d}{dx}[f(x) + g(x)] = \frac{d}{dx}f(x) + \frac{d}{dx}g(x)$$

9. **Problem.** Prove that if

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$$

where n is a nonnegative integer and $a_n \neq 0$ then

$$p'(x) = n a_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \ldots + a_1 \ .$$

Explicitly mention which theorem you use at each step.

10. **Example.** Find an equation of the tangent line to the curve

$$f(x) = 2x^3 - 7x^2 + 3x + 4$$

at the point $(1, 2)$.

11. **Challenging problem.** Find an equation for the straight line that passes through the point $(0, 3)$ and is tangent to the curve $y = 2\sqrt{x} + 1$.

12. **Theorem.** If $f(x) = a^x$, $a > 0$, $a \neq 1$, is an exponential function then

$$f'(0) = \lim_{h \to 0} \frac{a^h - 1}{h}$$

exists.

13. **Problem.** Prove that if $f(x) = a^x$, then

$$f'(x) = f'(0) \cdot a^x \ .$$

14. **Definition.** e is is the number such that

$$\lim_{h \to 0} \frac{e^h - 1}{h} = 1 \ .$$

$e \approx 2.71828$

15. **Theorem. Derivative of the Natural Exponential Function.** If $f(x) = e^x$ is the natural exponential function then

$$f'(x) = f(x) \ .$$

Thus

$$\frac{d}{dx}[e^x] = e^x \ .$$

16. **Example.** Differentiate the function

$$f(x) = 2x^3 + 3x^{\frac{2}{3}} - e^{x+2}\ .$$

17. **Example.** At what point on the curve $y = e^x$ is the tangent line parallel to the line $y = 2x$?

1. **Homework.** Differentiate the following functions.
   (a) $f(x) = x^3 + 7x^2 - 1$
   (b) $y = \sqrt[3]{x} + e^x$
   (c) $h(x) = 2 + \sqrt{2}$
2. **Homework.** Differentiate the following functions. Make sure you only use differentiation rules presented in this section (i.e., do not use the product / quotient / chain rule; instead, manipulate the function using algebra before differentiating).
   (a) $y = \sqrt{x}(x^2 - 7)$
   (b) $g(x) = (x^3 - 3)^2$
   (c) $z = \dfrac{2x^2 + 5x - 7}{x}$
3. **Homework.** Find the equation of the tangent line and normal line to the curve at the given point.
   (a) $f_1(x) = x^2 - 3x + 5$, at $x = -2$
   (b) $f_2(x) = e^x + x^3$, at $x = 1$.
4. **Homework.** Find the equation of the tangent line and normal line to the curve at the given point.
   (a) $g_1(x) = \sqrt{3x} + \sqrt[3]{x}$, at $x = 27$
   (b) $g_2(x) = x^3 - 3x^2 + 1$, at $x = a$.
5. **Homework.** On the curve $y = x^5 - 4x^3 + 8$, where is the tangent line horizontal?
6. **Homework.** For what values of x does the graph of $y = x^3 - 6x^2 + 7x - 5$ have a tangent line with slope 1?
7. **Homework.** Find equations of both lines that are tangent to the curve $y = 3 + x^3$ and parallel to the line $6x - y = 7$.

8. **Homework. Challenging problem.** Find an equation for the straight line that passes through the point $(1, 5)$ and it is tangent to the curve $y = x^3$.

9. **Homework. Challenging problem.** Suppose a is a constant and

$$f(x) = ax^3 + x + 7.$$

For what values of a does $f(x)$ have a horizontal tangent?

10. **Homework. Challenging problem.** Suppose the function

$$G(x) = x^4 + ax^3 + bx^2 + cx + d$$

satisfies the following:

(a) at $x = 0$, it's tangent line is the equation $y = 1 - x$

(b) at $x = -1$, it's tangent line is $y = 5 - 2x$

Determine the values a, b, c, and d.