Ch. 2, Sec. 2: The Limit of a Function

1. **Quote.**

   "People who haven't tasted bitter things haven't earned sweet things."

   — Gottfried Wilhelm Leibniz.

2. **Learning Objectives.**

3. **Motivating problem.** What is 1 divided by zero?

4. **Problem.** Let $f(x) = \dfrac{x^2 - x - 2}{x - 2}$.

(a) Determine the domain of f, then simplify $f(x)$.

(b) Complete the table

x	$f(x)$	x	$f(x)$
1		3	
1.9		2.1	
1.99		2.01	
1.999		2.001	
1.9999		2.0001	

(c) As x gets closer to 2, where does $f(x)$ approach?

5. **Example.** Is $f(x) = \dfrac{x^3}{x}$ the same function as $g(x) = x^2$?

6. **Problem.** Investigate the animal in the middle without looking at it.

7. **Definition.** We write

$$\lim_{x \to a} f(x) = L$$

and say

"the limit of $\mathrm{f(x)}$, **as** x **approaches** a, **equals** L**"**

if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, with $x \neq a$.

8. **Definition.** (Informal.)

$$\lim_{x \to a} f(x) = L$$

is read: "As x gets really close to a, then y gets really close to L."

9. **Examples.**

(a) Given $g(x) = x^2$, evaluate $\lim_{x \to 7} g(x)$.

(b) What is $\lim_{x \to 0} \frac{x^3}{x}$?

(c) What is $\lim_{x \to 0} \frac{1}{x^2}$?

10. **Example.** What can we say about

$$\lim_{x\to 0} \frac{|x|}{x}\ ?$$

11. **Definition. One-sided limits.** We write

$$\lim_{x \to a^+} f(x) = L$$

and say

"the right-hand limit of $\mathrm{f(x)}$, **as** x **approaches** a, **equals** L**"**

if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, with $x > a$.

Similarly, we write

$$\lim_{x \to a^-} f(x) = L$$

and say

"The left-hand limit of $\mathrm{f(x)}$, **as** x **approaches** a, **equals** L**"**

if we can make the values of $f(x)$ arbitrarily close to L by taking x sufficiently close to a, with $x < a$.

12. **Example.** Compute the one-sided limits of $\dfrac{|x|}{x}$ at $x = 0$.

13. **Definition.** We say $\lim_{x \to a} f(x)$ exists if

    (a) the right-sided limit exists; $\lim_{x \to a^+} f(x) = M$

    (b) the left-sided limit exists; $\lim_{x \to a^-} f(x) = N$

    (c) they're equal; $M = N$

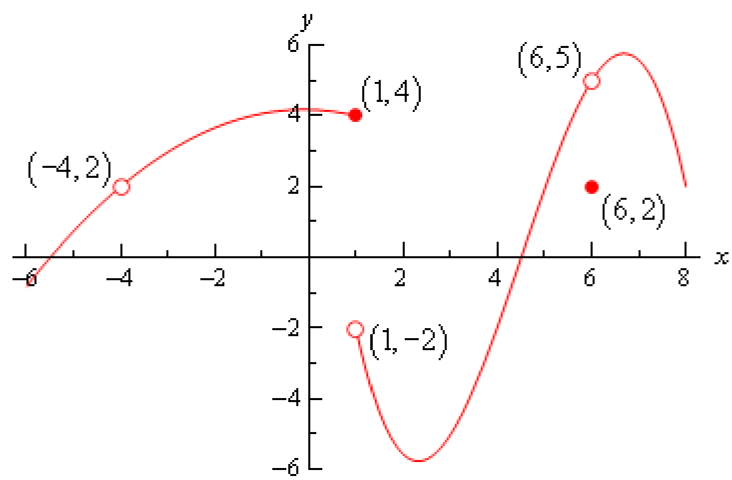
    If the limit exists, we say $\lim_{x \to a} f(x) = L$ (where $L = M = N$).

14. **Example.** Consider the function given by the graph below. Determine the limit at the following values of x. If the limit doesn't exist, find the left- and right-hand limits.

   (a) $x = -4$

   (b) $x = 1$

   (c) $x = 6$



15. **Example.** Sketch $P(x)$. Then determine if the limit exists at $x = -2$ and $x = 1$. As always, justify your answer.

$$P(x) = \begin{cases} 2 & \text{if} \quad x < -2 \\ x^2 - 1 & \text{if} \quad -2 \leq x < 1 \\ x & \text{if} \quad x \geq 1 \end{cases}$$

16. **Homework.** Sketch the graph of the function

$$f(x) = \begin{cases} x+1 & \text{if } x \leq -1 \\ x^2 & \text{if } x \in (-1,0) \\ 1 & \text{if } x = 0 \\ x^2 & \text{if } x \in (0,1] \\ x+1 & \text{if } x > 1 \end{cases}$$

Then, use your graph to find

(a) $\lim_{x \to -1^-} f(x)$

(b) $\lim_{x \to -1^+} f(x)$

(c) $\lim_{x \to -1} f(x)$

(d) $\lim_{x \to 1} f(x)$

(e) $\lim_{x \to 0^-} f(x)$

(f) $\lim_{x \to 0^+} f(x)$

(g) $\lim_{x \to 0} f(x)$

17. **Example.** With the help of a graphing software, investigate

$$\lim_{x \to 0^+} \sin\left(\frac{1}{x}\right)$$

18. **Homework.** Graph $f(x)$, and determine if $\lim_{x\to 1} f(x)$ exists.

(a) $f(x) = \begin{cases} 4 - x^2 & \textbf{if } x < 1 \\ x - 4 & \textbf{if } x \geq 1 \end{cases}$

(b) $f(x) = \begin{cases} -\sqrt{1 - x} & \textbf{if } x < 1 \\ \sqrt{x - 1} & \textbf{if } x \geq 1 \end{cases}$