# Ch. 2, Sec. 4: Continuity

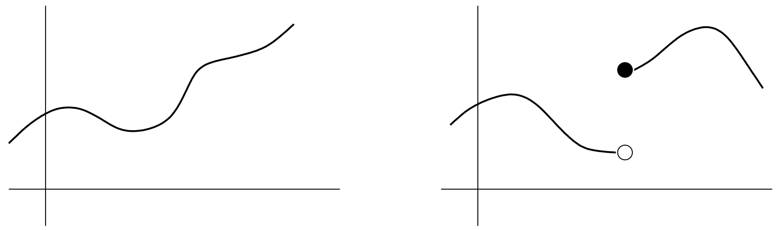
1. **Quote.**

   *"Kind words do not cost much. Yet they accomplish much."*

   — Blaise Pascal.

2. **Learning Objectives.**

3. **Example.** What is the difference between the two graphs?



4. **Definition.** (Informal.) A function is continuous if you can draw its graph without lifting your pen from the page.

5. **Question.** How can a function be *discontinuous* at $x = a$?

6. **Definition.** A function $f$ is **continuous at a number** a if

$$\lim_{x \to a} f(x) = f(a) \; .$$

(Don't use this definition.)

7. **Definition.** (Improved). A function $f$ is **continuous at a number** a if

   (a) $f(a)$ exists

   (b) $\lim_{x \to a} f(x)$ exists

   (c) $\lim_{x \to a} f(x) = f(a)$

8. **Example.** Determine if $f(x) = |x|$ is continuous at $x = 0$.

9. **Example.** Determine if the following functions are continuous or discontinuous at $x = 2$.

(a)

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

(b)

$$f(x) = \begin{cases} \frac{x^2-4}{x-2} & \text{if } x \neq 2 \\ 4 & \text{if } x = 2 \end{cases}$$

(c)

$$g(x) = \begin{cases} \frac{1}{x-2} & \text{if } x \neq 2 \\ 5 & \text{if } x = 2 \end{cases}$$

(d)

$$h(x) = \begin{cases} -1 & \text{if } x \in [1, 2) \\ 5 & \text{if } x \in [2, 3) \end{cases}$$

10. **Homework.** Justify why the following functions are discontinuous at $x = 2$.

(a) $f(x) = \begin{cases} x^3 - 2x^2 + x + 7 & \text{if } x > 2 \\ 9 & \text{if } x < 2 \end{cases}$

(b) $g(x) = \begin{cases} \dfrac{x^2 + 3x - 10}{x - 2} & \text{if } x \geq 2 \\ \dfrac{\sqrt{x + 14} - 4}{x - 2} & \text{if } x < 2 \end{cases}$

11. **Question.** Is the function $f(x) = x^2$ continuous on the domain $[-1, 3]$?

12. **Definition.** If

    (1) $f$ is defined on an open interval containing $a$, except perhaps at $a$

    (2) $f$ is **not** continuous at $a$

    we say that $f$ is **discontinuous** at $a$.

13. **Definition.** A function $f$ is **continuous from the right at a number** a if

$$\lim_{x \to a^+} f(x) = f(a)$$

and $f$ is **continuous from the left at** a if

$$\lim_{x \to a^-} f(x) = f(a) \ .$$

14. **Example.** Revisit Example 9d and determine if $h(x)$ is continuous from the left or from the right at $x = 2$.

$$h(x) = \begin{cases} -1 & \text{if } x \in [1, 2) \\ 5 & \text{if } x \in [2, 3) \end{cases}$$

15. **Homework.** Determine if $f(x)$ is continuous from the left- or right- (or neither) at $x = -2$, where

$$f(x) = \begin{cases} \dfrac{x^2 + x - 2}{x + 2} & \text{if } x > -2 \\ -3 & \text{if } x = -2 \\ x^2 - 3x - 7 & \text{if } x < -2 \end{cases}$$

16. **Definition.** A function $f$ is **continuous on an interval** if it is continuous at every number in that interval. We understand *continuous at the endpoint* to mean *continuous from the right* or *continuous from the left*.

17. **Example.** Find the number $c$ that makes $f(x)$ continuous for every $x$.

$$f(x) = \begin{cases} \dfrac{x^4 - 1}{x^3 - 1} & \text{if } x \neq 1 \\ c & \text{if } x = 1 \end{cases}$$

Hint: $x^{k+1} - 1 = x^k + x^{k-1} + x^{k-2} + \ldots + x^2 + x + 1$, for $k \in \mathbb{N}$

18. **Homework.** Determine the value of $c$ so that $g(x)$ is continuous

$$g(x) = \begin{cases} \dfrac{x^2-9}{x-3} & \text{if } x < 3 \\ cx-5 & \text{if } x \geq 3 \end{cases}$$

19. **Homework.** (Hard!) For which $a, b \in \mathbb{R}$ is the function

$$f(x) = \begin{cases} \dfrac{\sqrt{1-x}-1}{ax} & \text{if } \ x \in (0,1] \\ 1 & \text{if } \ x = 0 \\ \dfrac{bx^4+bx}{x^2+x} & \text{if } \ x \in (-1,0) \end{cases}$$

continuous on $(-1, 1]$?

20. **Theorem.** The following types of functions are continuous on their domains:

    (a) polynomials
    (b) rational functions
    (c) root functions
    (d) trigonometric functions
    (e) inverse trigonometric functions
    (f) exponential functions
    (g) logarithmic functions

21. **Theorem.** If $f$ and $g$ are continuous at $a$ and $c$ is a constant, then the following functions are also continuous at $a$:

$$f+g,\ f-g,\ cf,\ fg,\ \frac{f}{g} \text{ if } g(a) \neq 0\ .$$

22. **Theorem.** If $f$ is continuous at $b$ and $\lim_{x\to a} g(x) = b$ then

$$\lim_{x\to a} f(g(x)) = f(\lim_{x\to a} g(x)) = f(b)\ .$$

23. **Example.** Evaluate

$$\lim_{x\to 0} \exp\left(\frac{\sqrt{1-x}-1}{x}\right)\ .$$

**Hint.** $\exp(u) = e^u$

24. **Theorem.** If $g$ is continuous at $a$ and $f$ is continuous at $g(a)$, then the composite function $f \circ g$ given by $(f \circ g)(x) = f(g(x))$ is continuous at $a$.

25. **Theorem.** **Intermediate Value Theorem.** Suppose that $f$ is continuous on the closed interval $[a, b]$ and let $N$ be any number between $f(a)$ and $f(b)$, where $f(a) \neq f(b)$. Then there exists a number $c$ in $(a, b)$ such that $f(c) = N$.

26. **Example.** Use the Intermediate Value Theorem to prove that $x^2 - 2 = 0$ has a solution.
**Note.** This proves that $\sqrt{2}$ exists.

27. **Example.** Use the Intermediate Value Theorem to show that the equation

$$e^x = 2 - x$$

has at least one real solution.

28. **Homework.** Prove that $x^7 = x^2 - 1$ has a real solution.

29. **Homework.** Prove that $3^x = 5 - x^2$ has two real solutions.

30. **Homework.** Prove that there is a number that is exactly 1 more than its square.