Ch. 4, Sec. 5: L'Hospital's Rule

1. **Quote.**

   "All birds need to fly are the right-shaped wings, the right pressure and the right angle."

   — Daniel Bernoulli.

2. **Learning Objectives.**

3. **Homework.** Identify what's wrong with the following "proof".

   Let $a = b$.

   $a^2 = ab$ (Multiply both sides by a.)

   $a^2 + a^2 - 2ab = ab + a^2 - 2ab$ (Add $a^2 - 2ab$ to both sides.)

   $2(a^2 - ab) = a^2 - ab$ (Factor the left, and collect like terms on the right.)

   $2 = 1$ (Divide both sides by $a^2 - ab$.)

4. **Motivating problem.** What's zero divided by zero? Or infinity divided by infinity?

5. **Example.** Evaluate $\lim_{x \to 0} \frac{\sin kx}{x}$ for $k \in \mathbb{R}$.

6. **Example.** Assuming that k is a nonzero constant, evaluate

(a) $\lim_{x \to \infty} \frac{kx + 3}{x^2 + 5}$

(b) $\lim_{x \to \infty} \frac{kx^2 + 3}{x + 5}$

(c) $\lim_{x \to \infty} \frac{kx^2 + 3}{x^2 + 5}$

7. **Remark.** Limits approaching $\frac{0}{0}$ or $\frac{\infty}{\infty}$ are known as *indeterminate*.

8. **Example.** Evaluate $\lim_{x \to 0} \dfrac{e^x - 1}{\sin(2x)}$

9. **Example.** Evaluate $\lim_{x \to \infty} \dfrac{(\ln x)^2}{x}$

10. **Discussion.** Suppose $\lim_{x\to\infty} f(x) = \infty$ and $\lim_{x\to\infty} g(x) = \infty$. Try to analyze

$$\lim_{x\to\infty} \frac{f(x)}{g(x)}$$

11. **Theorem.** **L'Hospital's Rule.** Suppose that f and g are differentiable and $g'(x) \neq 0$ near a (except possibly at a.)

   (a) **0/0 Case.** If
   $$\lim_{x\to a} f(x) = 0 \text{ and } \lim_{x\to a} g(x) = 0,$$
   then
   $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
   provided the limit on the right side exists (or is ∞ or $-\infty$).

   (b) ∞/∞ **Case.** If
   $$\lim_{x\to a} f(x) = \pm\infty \text{ and } \lim_{x\to a} g(x) = \pm\infty,$$
   then
   $$\lim_{x\to a} \frac{f(x)}{g(x)} = \lim_{x\to a} \frac{f'(x)}{g'(x)},$$
   provided the limit on the right side exists (or is ∞ or $-\infty$).

12. **Example.** Evaluate $\lim_{x \to 0} \dfrac{e^x - 1}{\sin(2x)}$

13. **Example.** Evaluate $\lim\limits_{x\to\infty} \dfrac{(\ln x)^2}{x}$

14. **Examples.** Solve

(a) $\lim_{x\to\infty} \dfrac{e^x}{x^2+x}$

(b) $\lim_{x\to\infty} \dfrac{\ln x}{\sqrt{x}}$

15. **Discussion.** There are 7 types of indeterminate forms. Discuss what they might be.

(i)

(ii)

(iii)

(iv)

(v)

(vi)

(vii)

16. **Example. Indeterminate Form $0 \cdot \infty$.**

Find $\lim_{x \to -\infty} 3xe^x$.

17. **Example. Indeterminate Form $\infty - \infty$.**

Find $\displaystyle\lim_{x\to 0}\left(\frac{1}{x}-\frac{1}{\sin x}\right)$.

18. **Examples. Indeterminate Forms $0^0, \infty^0, 1^\infty$.**

Compute the following limits.

(a) $\displaystyle\lim_{x\to 0^+} (\cos x)^{1/x^2}$

(b) $\displaystyle\lim_{x\to 0^+} x^x$

(c) $\displaystyle\lim_{x\to \pi/2^-} (\tan x)^{\cos x}$

19. **Caution.** **Use L'Hospital's rule responsibly.** Examples:

(a) $\displaystyle\lim_{x\to\infty} \frac{2x^5+2x^3-1}{(3x^2-2)(2x^3+2x^2-7x)}$

(b) $\displaystyle\lim_{x\to\infty} \frac{x}{\sqrt{x^2+1}}$

(c) $\displaystyle\lim_{x\to\infty} \frac{x+\sin x}{x+1}$

20. **Just for fun.** This is an example to demonstrate the issues with assuming a limit exists when it might not. Consider the infinite sum

$$1 + 2 + 4 + 8 + 16 + \ldots$$

Assume this summation exists and see what happens.

Homework. For the following limits, state what kind of indeterminate form it is. Then evaluate the limit. If the limit does not exist, determine if it goes to ∞, $-\infty$, or neither.

1. $\displaystyle\lim_{x\to 0} \frac{e^x - x - 1}{x^2}$

2. $\displaystyle\lim_{x\to 0} \left(\frac{1}{x} - \frac{1}{\cos x - 1}\right)$.

3. $\displaystyle\lim_{x\to \infty} x \ln\left(\frac{x-1}{x+1}\right)$

4. $\displaystyle\lim_{x\to \infty} \left(1 + \frac{a}{x}\right)^{bx}$

5. $\displaystyle\lim_{x\to 1} \left(\frac{x}{x-1} - \frac{1}{\ln x}\right)$

6. $\displaystyle\lim_{x\to 1^+} (x-1)^{\sin(\pi x)}$