Ch. 2, Sec. 3: Calculating Limits Using Limit Laws

1. **Quote.**

> *"What has been affirmed without proof can also be denied without proof."*
>
> — Euclid.

2. **Learning Objectives.**

3. **Motivating problem.** Compute

$$\lim_{x \to 3} \frac{x-3}{x^2+x-12}$$

4. **Theorem. Two Special Limit Laws.**

   (a) $\lim_{x \to a} c = c$

   (b) $\lim_{x \to a} x = a$

5. **Question.** In regular arithmetic, does multiplication distribute across addition?

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7. **Theorem. Limit Laws.** Suppose that c is a constant and the limits

$$\lim_{x\to a} f(x) \text{ and } \lim_{x\to a} g(x)$$

exist. Then

(a) $\lim_{x\to a}(f(x)+g(x)) = \lim_{x\to a} f(x) + \lim_{x\to a} g(x)$

(b) $\lim_{x\to a}(f(x)-g(x)) = \lim_{x\to a} f(x) - \lim_{x\to a} g(x)$

(c) $\lim_{x\to a}(c\cdot f(x)) = c\cdot \lim_{x\to a} f(x)$

(d) $\lim_{x\to a}(f(x)\cdot g(x)) = \lim_{x\to a} f(x) \cdot \lim_{x\to a} g(x)$

(e) $\lim_{x\to a} \dfrac{f(x)}{g(x)} = \dfrac{\lim_{x\to a} f(x)}{\lim_{x\to a} g(x)}$ **if** $\lim_{x\to a} g(x) \neq 0$.

8. **Theorem. Exponentiation Limit Laws.**

   (a) $\lim_{x\to a}(f(x))^n = \left(\lim_{x\to a} f(x)\right)^n$, $n \in \mathbb{N}$

   (b) $\lim_{x\to a} x^n = a^n$, $n \in \mathbb{N}$

   (c) $\lim_{x\to a} \sqrt[n]{x} = \sqrt[n]{a}$, $n \in \mathbb{N}$ **(and if** n **is even,** $a > 0$**)**

   (d) $\lim_{x\to a} \sqrt[n]{f(x)} = \sqrt[n]{\lim_{x\to a} f(x)}$, $n \in \mathbb{N}$ **(and if** n **is even,** $a > 0$**)**

9. **Example.** Compute the following limit. Show every step, and at every step, justify which Limit Law you are invoking.

$$\lim_{x\to 2}(x^3 + 3x^2 - 4x + 5)$$

10. **Homework.** Use the Limit Laws to compute the following limit (i.e., do not use Direct Substitution). At every step, state which Limit Law you are invoking.

$$\lim_{x \to -2} \frac{5x^2 - \sqrt{2-x}}{x^3+5}$$

11. **Theorem.** **Direct Substitution Property.** If f is a polynomial or a rational function and a is in the domain of f, then

$$\lim_{x\to a} f(x) = f(a)\ .$$

12. **Example.** Compute $\lim_{x\to 3} \dfrac{x-3}{x^2+x-12}$

13. **Fact.** Suppose $f(x) = g(x)$ whenever $x \neq a$. Then

$$\lim_{x\to a} f(x) = \lim_{x\to a} g(x),$$

provided the limits exist.

14. **Example.** Compute $\lim\limits_{x\to 3} \dfrac{x^2 - 9}{x^2 - 2x - 3}$

15. **Example.** Find $\lim\limits_{t\to 0} \dfrac{\sqrt{t+9}-3}{t}$

16. **Example.** Determine $\lim_{x \to 2} \frac{\frac{1}{4} - \frac{1}{x^2}}{x - 2}$

17. **Homework.** Solve the following limits.

(a) $\displaystyle\lim_{t \to -1} \frac{x-1}{x^2-3x+2}$

(b) $\displaystyle\lim_{x \to 9} \frac{\sqrt{x+16}-5}{x^2-8x-9}$

18. **Homework.** Solve the following limits.

(a) $\displaystyle\lim_{t \to 0} \left(\frac{1}{t} - \frac{1}{t^2+t}\right)$

(b) $\displaystyle\lim_{x \to 9} \frac{3-\sqrt{x}}{9x-x^2}$

19. **Homework.** For the following functions, find the difference quotient $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x)}{h}$

(a) $f(x) = x^2$

(b) $f(x) = \sqrt{x}$

20. **Homework.** For the following functions, find the modified difference quotient $\lim\limits_{h\to 0} \dfrac{f(x+h)-f(x-h)}{2h}$

(a) $f(x) = \dfrac{1}{x}$

(b) $f(x) = \sqrt{x+1}$

21. **Example.** Determine $\lim\limits_{x\to -2} g(x)$, if it exists, where

$$g(x) = \begin{cases} x^2+6x+5 & \textbf{if } x \le -2 \\ \frac{x^2+x-2}{x+2} & \textbf{if } x > -2 \end{cases}$$

Reminder.

$$\lim_{x\to a} f(x) = L \Leftrightarrow \left(\lim_{x\to a^-} f(x) = L \textbf{ and } \lim_{x\to a^+} f(x) = L \right)$$

22. **Example.** Find $\lim_{x \to 0} \frac{x^2}{|x|}$

23. **Theorem.** If $f(x) \leq g(x)$ when x is near a (except possibly at a) and the limits of f and g both exist as x approaches a, then

$$\lim_{x \to a} f(x) \leq \lim_{x \to a} g(x) \ .$$

24. **Theorem.** **Squeeze Theorem.** If $f(x) \leq g(x) \leq h(x)$ when x is near a (except possibly at a) and

$$\lim_{x \to a} f(x) = \lim_{x \to a} h(x) = L$$

then

$$\lim_{x \to a} g(x) = L \ .$$

25. **Example.** Find

$$\lim_{x\to 0^+}\left(x^2\sin\frac{1}{x}\right)$$

26. **Homework.** Show that

$$\lim_{x\to 0}\left[x\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)\right]=0\ .$$

Hint. Recall $-2 \leq \sin\theta + \cos\theta \leq 2$.