

## Ch. 3, Sec. 4: The Chain Rule

### 1. Quote.

*“If people do not believe that mathematics is simple, it is only because they do not realize how complicated life is.”*

— John von Neumann.

### 2. Learning Objectives.

3. **Reminder.** The **composition** of the functions  $f$  and  $g$  is defined by

$$(f \circ g)(x) = f(g(x)) .$$

4. **Example.** Let  $f(u) = \sin u$  and  $g(x) = 1 + x^2$ . Find  $(f \circ g)(x)$ , then find  $(g \circ f)(x)$ .

5. **Theorem. Chain Rule** If  $f$  and  $g$  are both differentiable, then the derivative of  $h(x) = (f \circ g)(x) = f(g(x))$  is

$$h'(x) = f'(g(x)) \cdot g'(x) .$$

In Leibniz notation, if  $y = f(u)$  and  $u = g(x)$  are both differentiable functions, then

$$\frac{dy}{dx} = \frac{dy}{du} \cdot \frac{du}{dx} .$$

6. **Example.** Let  $f(u) = \sin u$  and  $g(x) = 1 + x^2$  and let  $F = f \circ g$ . Find the derivative of  $F$ .

7. **Example.** Find  $y' = \frac{dy}{dx}$ , given  $y = (2 - 5x)^3$

8. **Example.** Differentiate  $y = (x + \sin x)^5(1 + e^x)^2$

9. **Example.** Compute  $\frac{d}{dx} \left[ \left( \frac{(4x^2 - 7x + 3)}{x} \right)^5 \right]$

10. **Example.** Differentiate

$$y = \sqrt[3]{\cos(x^2 + \sin x)}$$



**11. Must Know!**

$$\frac{d}{dx}(a^x) = a^x \ln a .$$

**12. Examples.**

- (a) If  $y_1 = (g(x))^b$ , where  $b$  is a positive constant, find  $y'_1$ .
- (b) If  $y_2 = b^{g(x)}$ , where  $b$  is a positive constant, find  $y'_2$ .

13. **Example.** A pebble dropped into a lake creates an expanding circular ripple. Suppose that the radius of the circle is increasing at the rate of 2 in./s. At what rate is its area increasing when its radius is 10 in.?

14. **Example.** Suppose  $F(x) = f(g(h(x)))$ . Find  $F'(x)$ .

1. **Homework.** Find  $f'$ .

(a)

$$f(x) = \sqrt{2 + 5x^2}$$

(b)

$$f(x) = e^{\cos x}$$

(c)

$$f(x) = (\tan(x^2))^3$$

2. **Homework.** Let  $f(x) = \sqrt{x}$  and  $g(x) = \cos(x)$ , and  $C(x) = (f \circ g)(x)$ .

(a) Compute the derivative of  $C(x)$  using Newtonian prime notation.

(b) Compute the derivative of  $C(x)$  using Leibniz notation.

3. **Homework.** Find an equation of the tangent line to the curve at the given point.

(a)  $y = \sqrt{5x + 9}$ , at  $x = 10$ .

(b)  $y = \cos(x) + \cos^3(x)$ , at  $x = \pi/6$ .

4. **Homework.** Find the 50th derivative of  $y = 2^{kx}$ , where  $k$  is a constant.

5. **Homework.** Find the 50th derivative of  $y = \cos(kx)$ , where  $k$  is a constant.

6. **Homework.** Under certain circumstances a rumor spreads according to the equation

$$p(t) = \frac{1}{1 + ae^{-kt}}$$

where  $p(t)$  is the proportion of the population that knows the rumor at time  $t$  and  $a$  and  $k$  are positive constants.

(a) Find the rate of spread of the rumour.

(b) Find  $\lim_{t \rightarrow \infty} p(t)$ , then interpret your result physically.