

Ch. 3, Sec. 3: Derivatives of Trigonometric Functions

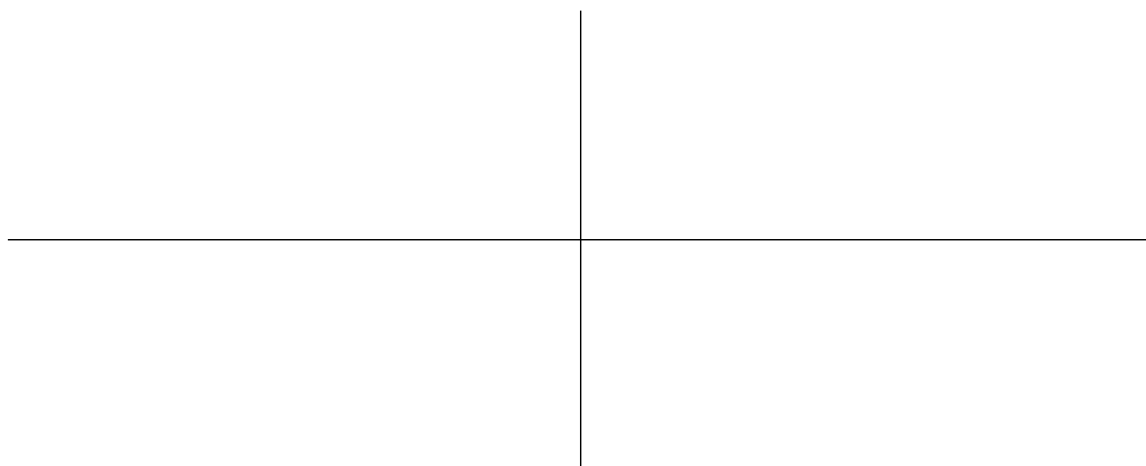
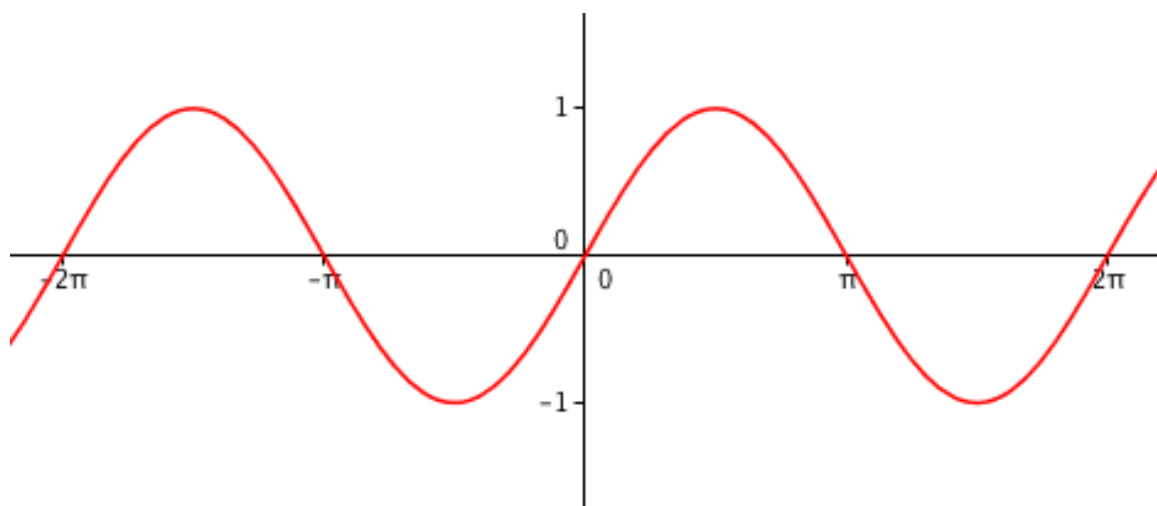
1. Quote.

“Begin with the simplest examples.”

— David Hilbert.

2. Learning Objectives.

3. **Motivating problem.** What is the derivative of $\sin x$?



4. **Reminder.** The unit circle interpretation of sine and cosine.

5. Theorem.

(a)

$$\frac{d}{dx}(\sin x) = \cos x$$

(b)

$$\frac{d}{dx}(\cos x) = -\sin x$$

(c)

$$\frac{d}{dx}(\tan x) = \sec^2 x$$

(d)

$$\frac{d}{dx}(\csc x) = -\csc x \cot x$$

(e)

$$\frac{d}{dx}(\sec x) = \sec x \tan x$$

(f)

$$\frac{d}{dx}(\cot x) = -\csc^2 x$$

6. **Example.** Prove $\frac{d}{dx}(\tan x) = \sec^2 x$

7. **Problem.** Prove that

$$\frac{d}{dx}(\sin x) = \cos x .$$

Hint:

$$\sin(\alpha + \beta) = \sin \alpha \cos \beta + \cos \alpha \sin \beta$$

$$\cos(\alpha + \beta) = \cos \alpha \cos \beta - \sin \alpha \sin \beta$$

Lemma.

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0.$$

8. **Theorem. Trigonometric Limit Lemmas.** Above we used the very important results

$$\lim_{\theta \rightarrow 0} \frac{\sin \theta}{\theta} = 1 \quad \text{and} \quad \lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\theta} = 0.$$

We now prove these results.

9. **Example.** Differentiate $y = \frac{1 + \tan x}{x - \sin x}$.

10. **Example.** Find the points on the curve

$$y = \frac{\cos x}{2 + \sin x}$$

at which the tangent is horizontal.

11. **Homework.** A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?

12. **Example.** Evaluate

(a) $\lim_{x \rightarrow 0} \frac{\sin 7x}{x}$

(b) $\lim_{\theta \rightarrow 0} \frac{\cos \theta - 1}{\sin 7\theta}$

1. **Homework.** Prove that $(\cot x)' = -\csc^2 x$ using the quotient rule.
2. **Homework.** If $f(x) = \cos x$, find a formula for $f^{(k)}(x)$, the k -th derivative of $f(x)$.
3. **Homework.** Differentiate the following.

(a) $f(\theta) = \cot(\theta) + \theta \sin \theta$

(b) $y = \frac{5 + \tan x}{\cos x - \sin x + 3}$

4. **Homework.** A ladder 10 ft long rests against a vertical wall. Let θ be the angle between the top of the ladder and the wall and let x be the distance from the bottom of the ladder to the wall. If the bottom of the ladder slides away from the wall, how fast does x change with respect to θ when $\theta = \pi/3$?
5. **Homework.** For which values of x does the curve

$$y = x - \sqrt{2} \cos x$$

have a horizontal tangent line?

6. **Homework.** Find $\lim_{x \rightarrow \infty} \frac{\cos x}{x}$.
7. **Homework.** Evaluate the following limit

$$\lim_{x \rightarrow 0} \frac{1 - \cos x}{x^2}.$$

Hint: Use the Pythagorean Trigonometric Identity, $\sin^2 x + \cos^2 x = 1$.