Ch. 4, Sec. 2: Maximum and Minimum Values

1. **Quote.**

   "Nothing takes place in the world whose meaning is not that of some maximum or minimum."

   — Leonhard Euler.

2. **Learning Objectives.**

3. **Example.** Suppose you have a function $f(x)$ and its domain D_f. Suppose $f(x)$ has a maximum value in D_f, which is attained at $x = c$. What must c satisfy? What if it was the minimum?

4. **Definition.** A function f has an **absolute maximum** at c if

$$f(c) \geq f(x) \text{ for all } x \in D, \text{ the domain of } f\ .$$

The number $f(c)$ is called the **maximum value** of f on D.

A function f has an **absolute minimum** at c if

$$f(c) \leq f(x) \text{ for all } x \in D, \text{ the domain of } f\ .$$

The number $f(c)$ is called the **minimum value** of f on D.

5. **Definition.** A function f has a **local maximum** at c if $f(c) \geq f(x)$ when x is near c

   A function f has a **local minimum** at c if $f(c) \leq f(x)$ when x is near c

6. **Definition.** A function f has a **local maximum** at c if

$$f(c) \geq f(x) \text{ for all } x \text{ in an open interval containing } c\,.$$

A function f has a **local minimum** at c if

$$f(c) \leq f(x) \text{ for all } x \text{ in an open interval containing } c\,.$$

7. **Theorem.** **The Extreme Value Theorem.** If f is continuous on a closed interval $[a, b]$, then f attains an absolute maximum value $f(c)$ and an absolute minimum value $f(d)$ at some numbers $c, d \in [a, b]$.

8. **Theorem. Fermat's Theorem.** If f has a local maximum or minimum at c, and $f'(c)$ exists, then $f'(c) = 0$.

9. **Example.** Find all local extrema of

   (a) $f(x) = 3x^4 - 16x^3 + 18x^2,\ -1 \leq x \leq 4$

   (b) $f(x) = |x|, -1 < x < 1$

10. **Definition.** **Critical Number.** A **critical number** of a function f is a number c in the domain of f such that either $f'(c) = 0$ or $f'(c)$ does not exist.

11. **Remark.** If f has a local maximum or minimum at c, then c is a critical number of f.

12. **Procedure.** **Closed Interval Method.** To find the **absolute** maximum and minimum values of a continuous function f on a closed interval $[a, b]$:

    (a) Find the values of f at the critical numbers of f in (a, b).

    (b) Find the values of f at the endpoints of the interval.

    (c) The largest of the values from Step 1 and Step 2 is the absolute maximum value; the smallest of these values is the absolute minimum value.

13. **Example.** Find the maximum and minimum values of the function

$$f(x) = x^2 + 4x + 7,\ -3 \leq x \leq 0$$

14. **Example.** Find the maximum and minimum values of the given functions on the indicated closed intervals.

   (a) $f(x) = x + \dfrac{4}{x},\ x \in [1, 4]$

   (b) $g(x) = 2 - \sqrt[3]{x},\ x \in [-1, 8]$

   (c) $h(x) = x^{2/3},\ x \in [-8, 8]$

   (d) $p(x) = \ln(x^2 + x + 1),\ x \in [-1, 1]$