# Ch. 3, Sec. 9: Linear Approximations and Differentials

1. **Quote.**

   *"Be approximately right rather than exactly wrong."*

   — John Tukey.

2. **Learning Objectives.**

3. **Motivating problem.** On a certain beach in the Maldives, the coral's height on June 1 was measured to be 302.5cm high, while on July 1 it was 302.7cm high. Estimate the height of the coral on

   (a) August 1.

   (b) June 1, next year.

4. **Question.** Suppose you are currently 23km north of a certain gas station. You are driving a Prius northward, and currently you are going 30km/h. Guess how far you'll be from the gas station in

   (a) an hour.

   (b) half an hour.

   (c) 10 minutes.

   Which of these guesses do you think is the most accurate?

5. **Example.** If $f(5) = 4$ and $f'(5) = \frac{3}{2}$, do we know anything about $f(6)$? Or $f(5.5)$? Or $f(5.01)$?

6. **Definition. Linear Approximation.** The linear function

$$L(x) = f(a) + f'(a)(x - a)$$

is called the **linearization** or the of $f$ at $a$, or the **linear approximation** of $f(x)$ at $x = a$.

For $x$ *close* to $a$ we have that

$$f(x) \approx L(x) = f(a) + f'(a)(x - a)$$

and this approximation is called the **linear approximation** of $f$ at $a$.

7. **Example.** If $f(1) = 4$ and $f'(1) = \frac{1}{2}$ use the linear approximation to $f(x)$ at $x = 1$ to approximate $f(2)$.

8. **Example.** Find the linearization of $f(x) = \dfrac{1}{x}$ at $x = 10$. Then, use your linearization to estimate $1/11$.

9. **Example.** Use linear approximation to approximate $\sqrt{37}$. What is the accuracy of this approximation?

10. **Example.** Find the linear approximation of $f(x) = \sqrt{1-x}$ at $a = 0$. Use this to approximate the number $\sqrt{0.9}$.

11. **Definition.** **Differential.** Let $f$ be a function differentiable at $x \in \mathbb{R}$. Let $\Delta x = dx$ be a (small) given number. The **differential** $dy$ is defined as

$$dy = f'(x)\Delta x \ .$$

12. **Example.** Explain why the approximation $(1.01)^6 \approx 1.06$ is reasonable.

13. **Example.** A spherical balloon filled with helium currently has a volume of $\frac{4000\pi}{3}$ cm$^3$. If we increase the radius by $0.1$ cm, use differentials to estimate much more helium we need.
*Hint:* The volume of a sphere is $\frac{4}{3}\pi r^3$.

14. **Example.** The radius of a circular disk is given as 24 cm with a maximum error in measurement of 0.2cm. Use differentials to estimate the maximum error in the calculated area of the disk. What is the relative error? Percentage error?

1. **Homework.** Use linear approximation or differentials to estimate $\tan(0.03)$.
2. **Homework.** Use linear approximation or differentials to estimate $\sqrt[3]{63}$.
3. **Homework.** Consider the function $f(x) = \sqrt{5x+4}$.
   (a) Find the linearization of $f(x)$ at $x = 12$.
   (b) Use the linearization to compute above to estimate $\sqrt{59}$.
   (c) What is the absolute error of this estimate?
4. **Homework.** The equatorial radius of the earth is approximately 3960 mi. Suppose that a wire is wrapped tightly around the earth at the equator. Approximately how much must this wire be lengthened if it is to be strung all the way around the earth on poles 10 ft above the ground. (1 mi = 1760 yards = $1760 \cdot 3$ ft.)
5. **Homework.** An ice cube as an initial volume of 27mL. Use differentials to estimate how much its side length shrinks if 1mL of ice melts away (assume that the shape of the melting ice remains as a cube).