Ch. 1, Sec. 7: Parametric Curves

1. **Quote.**

> *"A happy man is too satisfied with the present to dwell too much on the future."*
>
> — Albert Einstein.

2. **Learning Objectives.**

3. **Motivating problem.** Benji is running around on a field. We have a drone overhead to capture his movements. Is Benji's position a function?

4. **Example.** In the xy-plane draw the set $P = \{(t^2, t) : t \in \mathbb{R}\}$.

5. **Definitions.** Let I be an interval and let f and g be continuous on I.
   (a) The set of points $C = \{(f(t), g(t)) : t \in I\}$ is called a **parametric curve**.
   (b) The variable t is called a **parameter**.
   (c) We say that the curve C is defined by **parametric equations**
   $$x = f(t), \quad y = g(t).$$
   (d) We say that $x = f(t)$, $y = g(t)$ is a **parametrization** of C.
   (e) If $I = [a, b]$ then $(f(a), g(a))$ is called the **initial point** of C and $(f(b), g(b))$ is called the **terminal point** of C.

6. **Remark.** When sketching parametric curves, be sure to label the following important information about the curve:
   (a) Initial point.
   (b) Terminal point.
   (c) Direction.

7. **Remark.** When sketching parametric curves, try the following methods:
   (a) Eliminating t. This typically means either
      i. Isolate t, if possible.
      ii. **The** trigonometric identity.
   (b) Table of values.

8. **Example.** Sketch and identify the curve defined by the parametric equations

$$x = t^2 - 2t,\ y = t - 1,\ \ -2 < t < 3$$

9. **Example.** Sketch and identify the curve defined by the parametric equations

$$x = \cos t,\ y = \sin t,\ t \in [0, 2\pi)$$

10. **Example.** Sketch the graph of the curve defined by parametric equations

$$x = \sin t, \quad y = \sin^2 t$$

on the following intervals:

(a) $t \in [0, \pi/2]$

(b) $t \in [\pi/2, \pi]$

(c) $t \in [\pi, 2\pi]$

(d) $t \in (-\infty, \infty)$

11. **Theorem.** **Derivatives of Parametric Curves:** The derivative to the parametric curve $x = f(t)$, $y = g(t)$ is given by

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} = \frac{g'(t)}{f'(t)}.$$

12. **Example.** Find the slope of the tangent line to the curve

$$x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi.$$

at the point corresponding to $t = \pi/4$.

13. **Example.** Consider the parametric curve defined by the parametric equations
$$x = \sin t,\ y = t^2,\ t \in \mathbb{R}.$$

(a) Sketch the curve.

(b) Determine when the tangent line is horizontal.

(c) Try to determine when the tangent line has slope 1 (you will not be able to solve this explicitly).

14. **Example.** Sketch the following parametric curve, then determine the equation of the tangent line at $t = t_1$

$$x = t + 1,\ y = e^{2t},\ t \in \mathbb{R}$$

15. **Example.** Consider the parametric curve defined by the parametric equations

$$x = t^2 - t,\ y = t^3 - t^2 - 2t,\ t \in \mathbb{R}.$$

(a) When does the curve cross the origin?

(b) Where does the curve intersect the x-axis?

(c) The curve intersects the y-axis twice. Determine the equation of the tangent line at both of these times.