Ch. 3, Sec. 2: The Product Rule and Quotient Rule

1. **Quote.**

> *"I believe one can say without speaking metaphorically that arithmetic and geometry are the foundation and essence of all the sciences which deal with quantity... whenever a result has been arrived at, in order to use that result, it is necessary to translate it into numbers or into lines. To translate it into numbers requires the aid of arithmetic; to translate it into lines necessitates the use of geometry."*
>
> — Joseph-Louis de Lagrange.

2. **Learning Objectives.**

3. **Motivating problem.** Suppose we have two functions $f(x) = \sqrt[3]{x^2}$ and $g(x) = e^x$ and we want to compute the derivative of their product

$$\frac{d}{dx}(\sqrt[3]{x^2}e^x).$$

How do we do this?

4. **Question.** Is $(f(x)g(x))' = f'(x)g'(x)$?

5. **Theorem. The Product Rule.** If f and g are both differentiable, then

$$\frac{d}{dx}[f(x)g(x)] = \frac{d}{dx}[f(x)] \cdot g(x) + f(x) \cdot \frac{d}{dx}[g(x)].$$

In Newtonian notation this is written as $(fg)' = f'g + fg'$.

6. **Example.** Differentiate $h(x) = \sqrt[3]{x^2} \cdot e^x$.

7. **Example.** Differentiate $g(x) = (x+1)(2x^2 - x + 1)$.

8. **Theorem. The Quotient Rule.** If f and g are differentiable and $g(x) \neq 0$, then

$$\frac{d}{dx}\left[\frac{f(x)}{g(x)}\right] = \frac{\frac{d}{dx}[f(x)] \cdot g(x) - f(x) \cdot \frac{d}{dx}[g(x)]}{(g(x))^2}.$$

In Newton's notation this is written as $\left(\frac{f}{g}\right)' = \frac{f'g - fg'}{g^2}$.

9. **Examples.**

(a) Differentiate $y = \dfrac{2t^2 - 1}{t^3 + 1}$.

(b) Differentiate $f(x) = e^{-x}$.

10. **Homework.** If $f(3) = 4$, $g(3) = 2$, $f'(3) = -6$, and $g'(3) = 5$, find the following numbers.

    (a) $(f+g)'(3)$

    (b) $(fg)'(3)$

    (c) $\left(\dfrac{f}{g}\right)'(3)$

    (d) $\left(\dfrac{f}{f-g}\right)'(3)$

11. **Problem.** Using the product rule, prove the quotient rule.

1. **Homework.** Assuming that $f_i(x)$ is differentiable (where $1 \leq i \leq 4$), find y' in terms of $f_i(x)$ and $f_i'(x)$, where
$$y = \frac{f_1(x)f_2(x)}{f_3(x) + f_4(x)}.$$

2. **Homework.** If $f(x) = x/e^x$, find $f^{(k)}(x)$, the k-th derivative of $f(x)$. **Hint:** Compute the first few derivatives and try to observe a pattern.

3. **Homework.** Find the equation of tangent line to the given curve at the given point.

   (a) **Witch of Maria Agnesi.** $y = 1/(1 + x^2)$ at $(2, 1/5)$.

   (b) **Serpentine.** $y = x/(1 + x^2)$ at $(-2, -2/5)$.