



Introduction to **Machine Learning and Data Mining** (Học máy và Khai phá dữ liệu)

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Content

- Introduction to Machine Learning & Data Mining
- Unsupervised learning
- Supervised learning
- **Probabilistic modeling**
 - **Expectation maximization**
- Practical advice

Difficult situations

- No closed-form solution for the learning/inference problem?
(không tìm được ngay công thức nghiệm)
 - The examples before are easy cases, as we can find solutions in a closed form by using gradient.
 - Many models (e.g., GMM) do not admit a closed-form solution
- No explicit expression of the density/mass function?
(không có công thức tường minh để tính toán)
- Intractable inference (bài toán không khả thi)
 - Inference in many probabilistic models is NP-hard
[Sontag & Roy, 2011; Tosh & Dasgupta, 2019]

Expectation maximization

The EM algorithm

GMM revisit

- Consider learning GMM, with K Gaussian distributions, from the training data $\mathbf{D} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M\}$.

- The density function is $p(\mathbf{x} | \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi}) = \sum_{k=1}^K \phi_k \mathcal{N}(\mathbf{x} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$

- $\boldsymbol{\phi} = (\phi_1, \dots, \phi_K)$ represents the weights of the Gaussians, $P(z = k | \boldsymbol{\phi}) = \phi_k$.

- Each multivariate Gaussian has density

$$\mathcal{N}(\mathbf{x} | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k) = \frac{1}{\sqrt{\det(2\pi\boldsymbol{\Sigma}_k)}} \exp \left[-\frac{1}{2} (\mathbf{x} - \boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}_k^{-1} (\mathbf{x} - \boldsymbol{\mu}_k) \right]$$

- MLE tries to maximize the following log-likelihood function

$$L(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi}) = \sum_{i=1}^M \log \sum_{k=1}^K \phi_k \mathcal{N}(\mathbf{x}_i | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

- We cannot find a closed-form solution!**

- Naïve gradient decent:** repeat until convergence

- Optimize $L(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$ w.r.t $\boldsymbol{\phi}$, when fixing $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$.

- Optimize $L(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$ w.r.t $(\boldsymbol{\mu}, \boldsymbol{\Sigma})$, when fixing $\boldsymbol{\phi}$.

Still hard

GMM revisit: K-means

□ **GMM:** we need to know

- Among K gaussian components, which generates an instance \mathbf{x} ?
the index z of the gaussian component
- The parameters of individual gaussian components: $(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k, \phi_k)$

□ **K-means:**

- Among K clusters, to which an instance \mathbf{x} belongs?
the cluster index z
- The parameters of individual clusters: the mean

□ Idea for GMM?

- $P(z|\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$?
(note $\sum_{k=1}^K P(z = k|\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi}) = 1$)
(soft assignment)
- Update the parameters of individual gaussians: $(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k, \phi_k)$

□ **K-means training:**

- Step 1: assign each instance \mathbf{x} to the nearest cluster
(the cluster index z for each \mathbf{x})
(hard assignment)
- Step 2: recompute the means of the clusters

GMM: lower bound

□ Idea for GMM?

- Step 1: compute $P(z|\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$? (note $\sum_{k=1}^K P(z = k|\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi}) = 1$)
- Step 2: Update the parameters of the gaussian components: $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$

■ Consider the log-likelihood function

$$L(\boldsymbol{\theta}) = \log P(\mathbf{D}|\boldsymbol{\theta}) = \sum_{i=1}^M \log \sum_{k=1}^K \phi_k \mathcal{N}(\mathbf{x}_i | \boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$$

- Too complex if directly using gradient

- Note that

$$\begin{aligned} \log P(\mathbf{x}|\boldsymbol{\theta}) &= \log \sum_z P(z, \mathbf{x}|\boldsymbol{\theta}) = \log \sum_z P(z|\mathbf{x}, \boldsymbol{\theta}) P(\mathbf{x}|\boldsymbol{\theta}) = \log \mathbb{E}_{z|\mathbf{x}, \boldsymbol{\theta}} P(\mathbf{x}|\boldsymbol{\theta}) \\ &\geq \mathbb{E}_{z|\mathbf{x}, \boldsymbol{\theta}} \log P(\mathbf{x}|\boldsymbol{\theta}) = \sum_z P(z|\mathbf{x}, \boldsymbol{\theta}) \log P(\mathbf{x}|\boldsymbol{\theta}) \end{aligned}$$

Jensen's
inequality

- Maximizing $L(\boldsymbol{\theta})$ can be done by *maximizing the lower bound*
 $\mathbb{E}_{z|\mathbf{D}, \boldsymbol{\theta}} \log P(\mathbf{D}|\boldsymbol{\theta})$

GMM: maximize the lower bound

- Step 1: compute $P(z|\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$? (note $\sum_{k=1}^K P(z = k|\mathbf{x}, \boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi}) = 1$)
- Step 2: Update the parameters of the gaussian components: $\boldsymbol{\theta} = (\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$

- Bayes' rule: $P(z|\mathbf{x}, \boldsymbol{\theta}) = P(\mathbf{x}|z, \boldsymbol{\theta})P(z|\boldsymbol{\phi})/P(\mathbf{x}) = \phi_z \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_z, \boldsymbol{\Sigma}_z)/C$, where $C = \sum_k \phi_k \mathcal{N}(\mathbf{x}|\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$ is the normalizing constant.

- Meaning that one can compute $P(z|\mathbf{x}, \boldsymbol{\theta})$ if $\boldsymbol{\theta}$ is known
- Denoting $T_{ki} = P(z = k|\mathbf{x}_i, \boldsymbol{\theta})$ for any index $k = \overline{1, K}, i = \overline{1, M}$

- How about $\boldsymbol{\phi}$?

- $\phi_z = P(z|\boldsymbol{\phi}) = P(z|\boldsymbol{\theta}) = \int P(z, \mathbf{x}|\boldsymbol{\theta})d\mathbf{x} = \int P(z|\mathbf{x}, \boldsymbol{\theta})P(\mathbf{x}|\boldsymbol{\theta})d\mathbf{x} = \mathbb{E}_{\mathbf{x}}(P(z|\mathbf{x}, \boldsymbol{\theta})) \approx \frac{1}{M} \sum_{\mathbf{x} \in D} P(z|\mathbf{x}, \boldsymbol{\theta}) = \frac{1}{M} \sum_{i=1}^M T_{zi}$

- Then the lower bound can be maximized w.r.t individual $(\boldsymbol{\mu}_k, \boldsymbol{\Sigma}_k)$:

$$\begin{aligned} \mathbb{E}_{z|\mathbf{D}, \boldsymbol{\theta}} \log P(\mathbf{D}|\boldsymbol{\theta}) &= \sum_{\mathbf{x} \in D} \sum_z P(z|\mathbf{x}, \boldsymbol{\theta}) \log P(\mathbf{x}|\boldsymbol{\theta}) \\ &= \sum_{i=1}^M \sum_{k=1}^K T_{ki} \left[-\frac{1}{2} (\mathbf{x}_i - \boldsymbol{\mu}_k)^T \boldsymbol{\Sigma}_k^{-1} (\mathbf{x}_i - \boldsymbol{\mu}_k) - \log \sqrt{\det(2\pi \boldsymbol{\Sigma}_k)} \right] \end{aligned}$$

GMM: EM algorithm

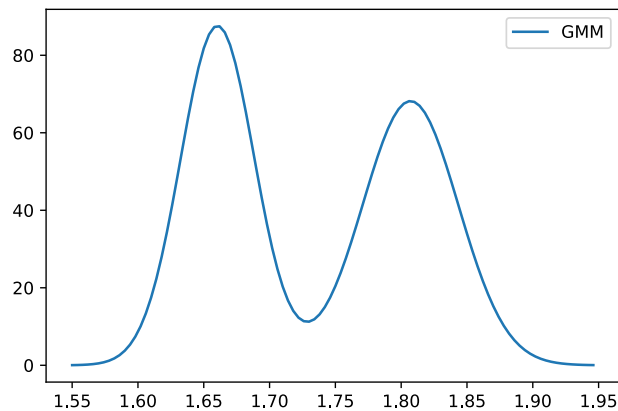
- **Input:** training data $\mathbf{D} = \{\mathbf{x}_1, \mathbf{x}_2, \dots, \mathbf{x}_M\}$, $K > 0$
- **Output:** model parameter $(\boldsymbol{\mu}, \boldsymbol{\Sigma}, \boldsymbol{\phi})$
- Initialize $(\boldsymbol{\mu}^{(0)}, \boldsymbol{\Sigma}^{(0)}, \boldsymbol{\phi}^{(0)})$ randomly
 - $\boldsymbol{\phi}^{(0)}$ must be non-negative and sum to 1.
- At iteration t :
 - **E step:** compute $T_{ki} = P(z = k | \mathbf{x}_i, \boldsymbol{\theta}^{(t)}) = \phi_k^{(t)} \mathcal{N}(\mathbf{x}_i | \boldsymbol{\mu}_k^{(t)}, \boldsymbol{\Sigma}_k^{(t)}) / C$ for any index $k = \overline{1, K}, i = \overline{1, M}$
 - **M step:** update for any k ,

$$\phi_k^{(t+1)} = \frac{1}{M} \sum_{i=1}^M T_{ki}; \boldsymbol{\mu}_k^{(t+1)} = \frac{1}{M\phi_k} \sum_{i=1}^M T_{ki} \mathbf{x}_i;$$

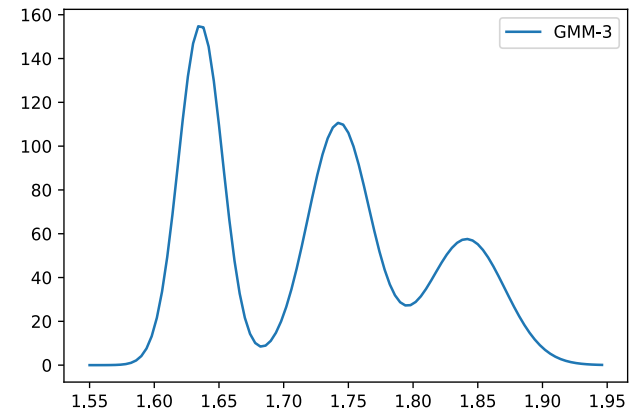
$$\boldsymbol{\Sigma}_k^{(t+1)} = \frac{1}{M\phi_k} \sum_{i=1}^M T_{ki} (\mathbf{x}_i - \boldsymbol{\mu}_k^{(t+1)}) (\mathbf{x}_i - \boldsymbol{\mu}_k^{(t+1)})^T$$
- If not convergence, go to iteration $t + 1$.

GMM: example 1

- We wish to model the height of a person
 - We had collected a dataset from 10 people in Hanoi + 10 people in Sydney
 $D = \{1.6, 1.7, 1.65, 1.63, 1.75, 1.71, 1.68, 1.72, 1.77, 1.62, 1.75, 1.80, 1.85, 1.65, 1.91, 1.78, 1.88, 1.79, 1.82, 1.81\}$



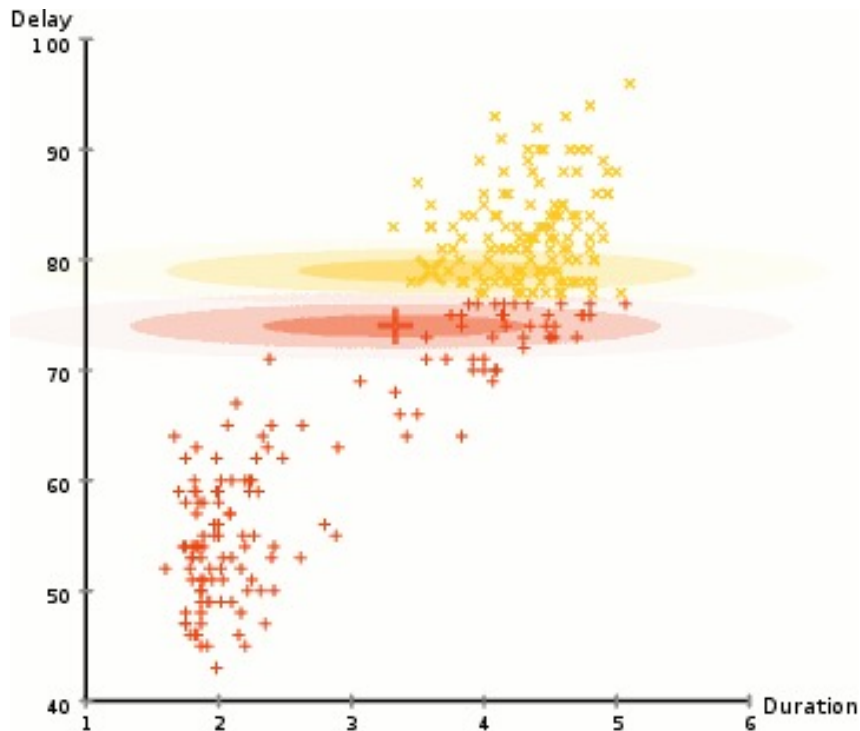
GMM with
2 components



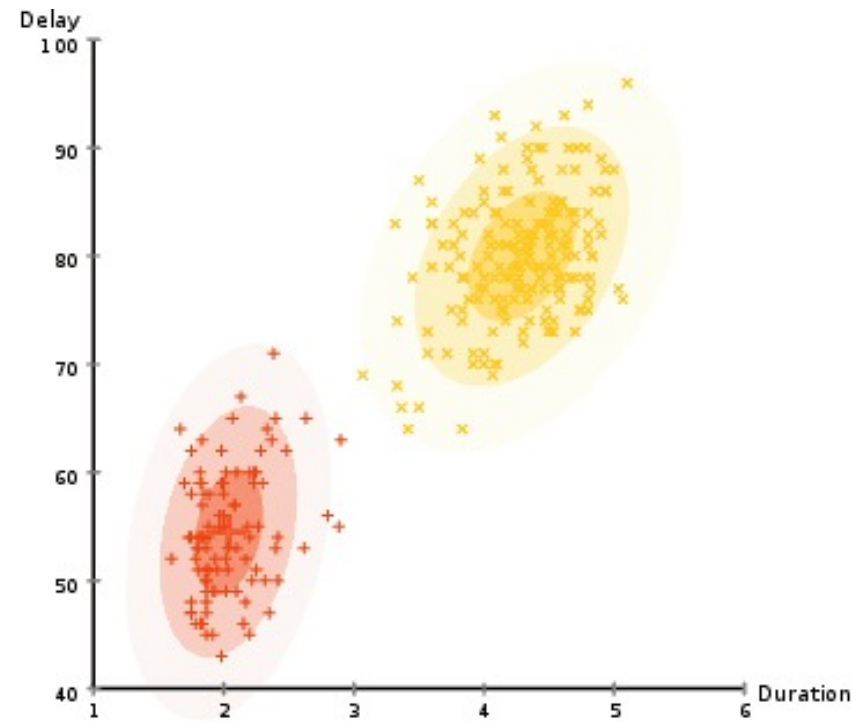
GMM with
3 components

GMM: example 2

- A GMM is fitted in a 2-dimensional dataset to do clustering.



From initialization



To convergence

GMM: comparison with K-means

□ K-means:

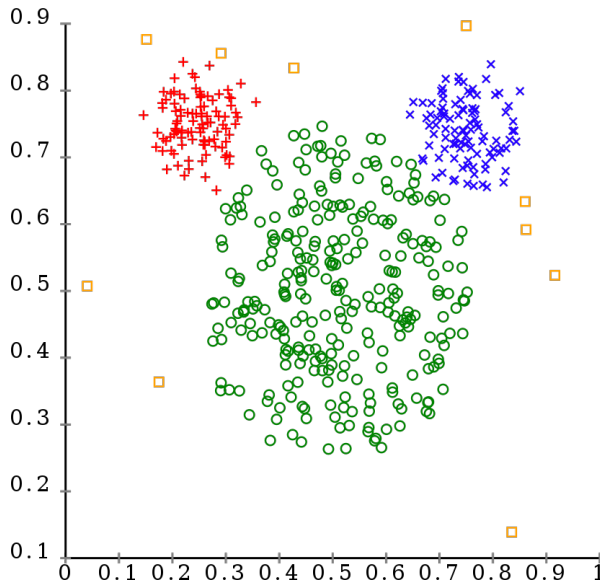
- Step 1: hard assignment
- Step 2: the means
→ similar shape for the clusters?

□ GMM clustering

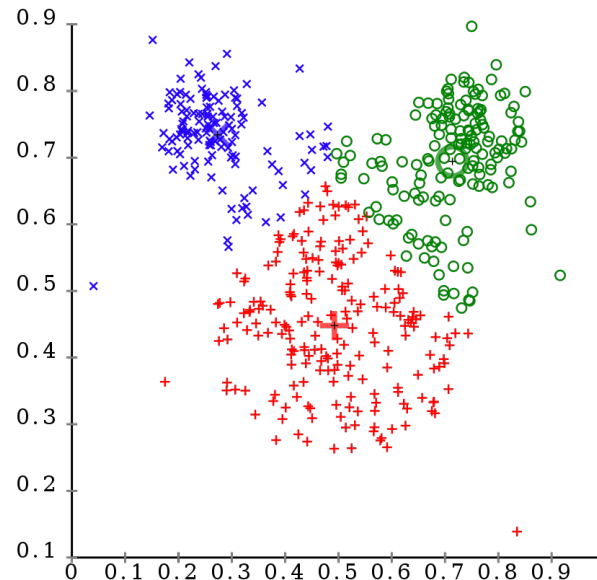
- Soft assignment of data to the clusters
- Parameters (μ_k, Σ_k, ϕ_k)
→ different shapes for the clusters

Different cluster analysis results on "mouse" data set:

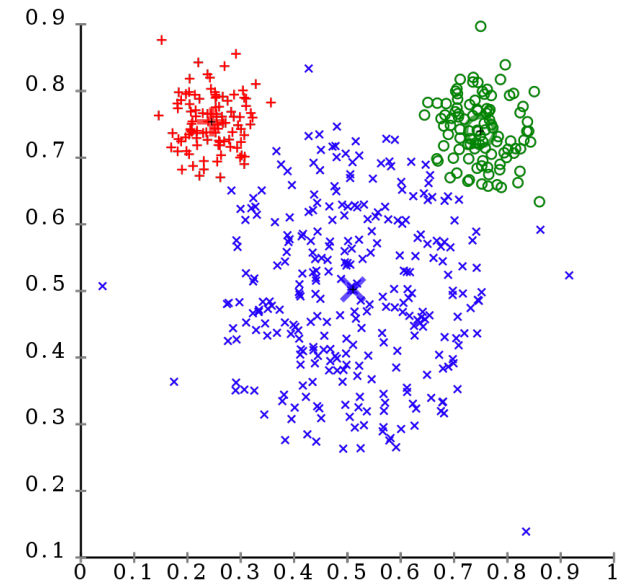
Original Data



k-Means Clustering



EM Clustering



General models

- We can make the EM algorithm in more general cases.
- Consider a model $B(\mathbf{x}, \mathbf{z}; \boldsymbol{\theta})$ with observed variable \mathbf{x} , hidden variable \mathbf{z} , and parameterized by $\boldsymbol{\theta}$
(mô hình có một biến \mathbf{x} quan sát được, biến ẩn \mathbf{z} , và tham số $\boldsymbol{\theta}$)
 - \mathbf{x} depends on \mathbf{z} and $\boldsymbol{\theta}$, while \mathbf{z} may depend on $\boldsymbol{\theta}$
 - Mixture models: each observed data point has a corresponding latent variable, specifying the mixture component which generated the data point
- The learning task is to find a specific model, from the model family parameterized by $\boldsymbol{\theta}$, that maximizes the log-likelihood of training data \mathbf{D} :
$$\boldsymbol{\theta}^* = \operatorname{argmax}_{\boldsymbol{\theta}} \log P(\mathbf{D}|\boldsymbol{\theta})$$
- We assume \mathbf{D} consists of i.i.d samples of \mathbf{x} , the the log-likelihood function can be expressed analytically, $\mathbb{E}_{\mathbf{z}|\mathbf{D},\boldsymbol{\theta}} \log P(\mathbf{D}|\boldsymbol{\theta})$ can be computed easily (hàm log-likelihood có thể viết một cách tường minh)
 - Since there is a latent variable, MLE may not have a close form solution

The Expectation Maximization algorithm

- The Expectation maximization (EM) algorithm was introduced in 1977 by Arthur Dempster, Nan Laird, and Donald Rubin.

- The EM algorithm maximizes the lower bound of the log-likelihood

$$L(\boldsymbol{\theta}; \mathbf{D}) = \log P(\mathbf{D}|\boldsymbol{\theta}) \geq \mathbb{E}_{z|\mathbf{D},\boldsymbol{\theta}} \log P(\mathbf{D}|\boldsymbol{\theta}) = \sum_z P(z|\mathbf{D}, \boldsymbol{\theta}) \log P(\mathbf{D}|\boldsymbol{\theta})$$

- *Initialization:* $\boldsymbol{\theta}^{(0)}, t = 0$

- *At iteration t:*

- **E step:** compute the expectation $Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)}) = \mathbb{E}_{z|\mathbf{D},\boldsymbol{\theta}^{(t)}} \log P(\mathbf{D}|\boldsymbol{\theta}^{(t)})$

(tính hàm kỳ vọng Q khi cố định giá trị $\boldsymbol{\theta}^{(t)}$ đã biết ở bước trước)

- **M step:** find $\boldsymbol{\theta}^{(t+1)} = \operatorname{argmax}_{\boldsymbol{\theta}} Q(\boldsymbol{\theta}|\boldsymbol{\theta}^{(t)})$

(tìm điểm $\boldsymbol{\theta}^{(t+1)}$ mà làm cho hàm Q đạt cực đại)

- *If not convergence, go to iteration $t + 1$.*

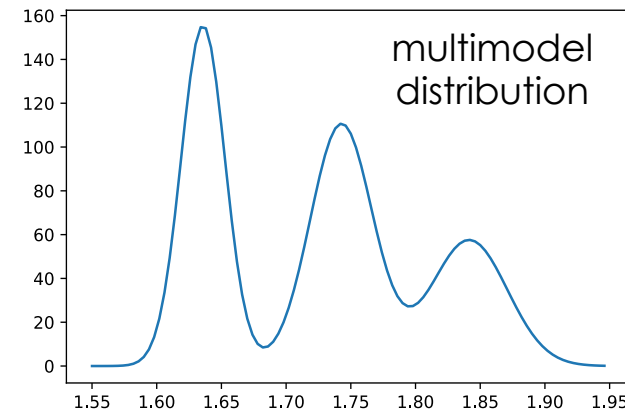
Integration
if z is
continuous

EM: convergence condition

- Different conditions can be used to check convergence
 - $\mathbb{E}_{z|\mathbf{D},\theta} \log P(\mathbf{D}|\theta)$ does not change much between two consecutive iterations
 - θ does not change much between two consecutive iterations
- In practice, we sometimes need to limit the maximum number of iterations

EM: some properties

- The EM algorithm is guaranteed to return a stationary point of the lower bound $\mathbb{E}_{\mathbf{z}|\mathbf{D},\boldsymbol{\theta}} \log P(\mathbf{D}|\boldsymbol{\theta})$
(thuật toán EM đảm bảo sẽ hội tụ về một điểm dừng của hàm cận dưới)
 - It may be the local maximum
- Due to maximizing the lower bound, EM does not necessarily returns the maximizer of the log-likelihood function
(EM chưa chắc trả về điểm cực đại của hàm log-likelihood)
 - No guarantee exists
 - It can be seen in cases of multimodel, where the log-likelihood function is non-concave
- The Baum-Welch algorithm is the a special case of EM for hidden Markov models



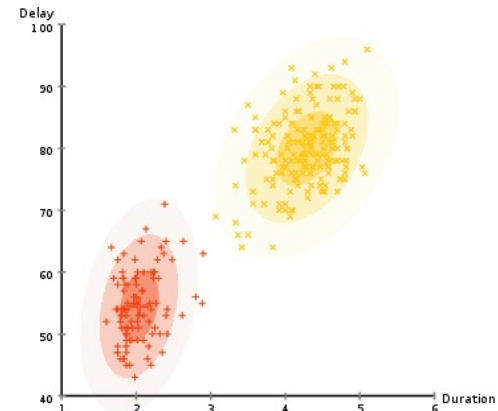
EM, mixture model, and clustering

- **Mixture model:** we assume the data population is composed of K different components (distributions), and each data point is generated from one of those components

- E.g., Gaussian mixture model, categorical mixture model, Bernoulli mixture model,...
- The mixture density function can be written as

$$f(\mathbf{x}; \boldsymbol{\theta}, \boldsymbol{\phi}) = \sum_{k=1}^K \phi_k f_k(\mathbf{x} | \boldsymbol{\theta}_k)$$

where $f_k(\mathbf{x} | \boldsymbol{\theta}_k)$ is the density of the k -th component



- We can interpret that a mixture distribution partitions the data space into different regions, each associates with a component
(Một phân bố hỗn hợp tạo ra một cách chia không gian dữ liệu ra thành các vùng khác nhau, mà mỗi vùng tương ứng với 1 thành phần trong hỗn hợp đó)
- Hence, mixture models provide solutions for clustering
- The EM algorithm provides a natural way to learn mixture models

EM: limitation

- When the lower bound $\mathbb{E}_{z|\mathbf{D},\theta} \log P(\mathbf{D}|\theta)$ does not admit easy computation of the expectation or maximization steps
 - Admixture models, Bayesian mixture models
 - Hierarchical probabilistic models
 - Nonparametric models
- EM finds a point estimate, hence easily gets stuck at a local maximum
- In practice, EM is sensitive with initialization
 - Is it good to use the idea of K-means++ for initialization?
- Sometimes EM converges slowly in practice

Further?

- Variational inference
 - Inference for more general models
- Deep generative models
 - Neural networks + probability theory
- Bayesian neural networks
 - Neural networks + Bayesian inference
- Amortized inference
 - Neural networks for doing Bayesian inference
 - Learning to do inference

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