2 Ill-posedness in Inverse Problems

- **Read:** Chapter 3.1, 3.2, 3.3, 3.5, 3.6.
- **Exercises:**
  1. [MS] P39, Exercise 3.2.2.
  2. [MS] P47, Exercise 3.3.5.
  3. Consider a kernel in (3.8) defined as $K(s,t) = s + 2t$. First, show that the SVE of the kernel K is given by
  $$\mu_1 = \frac{4}{\sqrt{3}},\ \mu_2 = \frac{2}{\sqrt{3}},\ \mu_3 = \mu_4 = \cdots = 0$$
  and
  $$u_1(s) = \frac{1}{\sqrt{2}},\ u_2(s) = \sqrt{\frac{3}{2}}s,\ v_1(t) = \sqrt{\frac{3}{2}}t,\ v_2(t) = \frac{1}{\sqrt{2}}.$$
  Show that this integral equation has a solution only if the right-hand side is a linear function. (Hint: evaluate the left side of (3.8) for an arbitrary function $f(t)$)
  4. Consider a kernel in (3.8) defined as
  $$K(s,t) = \begin{cases} s(t-1), & s < t, \\ t(s-1), & s \geq t. \end{cases}$$
  This equation is associated with the computation of the second derivative of a function. An alternative expression for the kernel is given by
  $$K(s,t) = -\frac{2}{\pi^2}\sum_{i=1}^{\infty}\frac{\sin(i\pi s)\sin(i\pi t)}{i^2}.$$
  The SVE of the kernel K is given by
  $$\mu_i = \frac{1}{(i\pi)^2},\ u_i(s) = \sqrt{2}\sin(i\pi s),\ v_i(t) = -\sqrt{2}\sin(i\pi t),\ i = 1, 2, \cdots.$$
  Consider the two function g in (3.8) and their Fourier series:
  $$\begin{aligned} g_1(s) =& s && (1)\\ =& \frac{2}{\pi}\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\sin(i\pi s)}{i}, && \text{Not} \quad (2)\\ g_2(s) =& s(1+s)(1-s) && (3)\\ =& \frac{12}{\pi^3}\sum_{i=1}^{\infty}(-1)^{i+1}\frac{\sin(i\pi s)}{i^3}, && \text{Yes} \quad (4)\end{aligned}$$
  which of these two functions satisfies the Picard condition?

5. [MS] P52, Exercise 3.6.2.

6. Use the one-dimensional reconstruction test problems, which is implemented in *Regularization Tools* as function "shaw". (*Regularization Tools* can be downloaded from `http://www.mathworks.com/matlabcentral/fileexchange/52-regtools`, and the complete manuel can be found in `http://www.imm.dtu.dk/~pcha/Regutools/RTv4manual.pdf`). The kernel in this problem is given by

$$K(s,t) = (\cos(s) + \cos(t))^2 \left(\frac{\sin(\pi(\sin(s) + \sin(t)))}{\pi(\sin(s) + \sin(t))} \right)^2 \quad \text{with} \ -\frac{\pi}{2} \leq s, t \leq \frac{\pi}{2},$$

while the solution is

$$f(t) = 2\exp(-6(t - 0.8)^2) + \exp(-2(t + 0.5)^2).$$

This integral equation models a situation where light passes through an infinitely long slit, and the function $f(t)$ is the incoming light intensity as a function of the incidence angle t. The problem is discretized by means of the midpoint quadrature rule to produce A and f^{exact}, after which the exact right-hand side is computed as $m^{exact} = Af^{exact}$. The elements of m^{exact} represent the outgoing light intensity on the other side of the slit.

Choose $n = 24$ and generate the problem. Then compute the SVD of A, and plot and inspect the left and right singular vectors. What can be said about the number of sign changes in these vectors?

Use the function "picard" from *Regularization Tools* to inspect the singular values σ_i and the SVD coefficients $u_i^\top m^{exact}$, as well as the corresponding solution coefficients $u_i^\top m^{exact}/\sigma_i$.

Add a very small amount of noise e to the right-hand side m^{exact} with $\|e\|_2/\|m^{exact}\|_2 = 10^{-10}$. Inspect the singular values and SVD coefficients again. What happens to the SVD coefficients $u_i^\top m$ corresponding to the small singular values?