Problem #2

Due: Wednesday, April 22, turn in to canvas, as described in the syllabus.

The first 3 problems are meant to be practice problems, and the 4th is the open-ended problem.

1. Determine for which values of $\beta > 0$, there is "finite-time blow-up" of solutions (i.e. there exists a finite $T > 0$ such that $\lim_{t\to T} |x(t)| = +\infty$) for the following scalar ode:

$$\dot{x} = x^\beta, \quad x(0) = x_0 \in \mathbf{R}^+.$$

Specify the domain of existence of the solution, i.e. determine T as a function of x_0 and β.

2. For which values of $\beta > 0$ is there guaranteed to be a unique solution to the scalar initial value problem

$$\dot{x} = -x^\beta, \quad x(0) = x_0 > 0.$$

For the values of β where there is no such guarantee, explicitly construct a family of solutions that are identical for $t \geq T$, for some time $T > 0$, but are not identical solutions for $0 < t < T$, demonstrating the lack of uniqueness for solutions in a neighborhood of $t = T$.

3. Short response: what role if any does the Lipschitz constant K play in proving (local) existence and uniqueness of solutions of initial value problems of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}), \quad \mathbf{x}(0) = \mathbf{x}_0 \in \mathbf{R}^n, \quad \mathbf{f} : \mathbf{R}^n \to \mathbf{R}^n,$$

where $\mathbf{f}$ is Lipschitz continuous with constant K.

4. Often in applications, the dynamics of a system being modeled, may be governed by quite different processes in different regions of the state space. And this might lead to a dynamical system that is set-wise defined, e.g.

$$\dot{\mathbf{x}} = \begin{cases} \mathbf{f}^+(\mathbf{x}), & \sigma(\mathbf{x}) > 0, \\ \mathbf{f}^-(\mathbf{x}), & \sigma(\mathbf{x}) < 0, \end{cases} \tag{1}$$

where the scalar equation $\sigma(\mathbf{x}) = 0$ defines the discontinuity boundary between different regions of state space governed by different dynamics. Note that I didn't say anything about the dynamics for points **in** the discontinuity boundary.

In the following open-ended problem, you explore how the definitions of the dynamics in the discontinuity boundary might impact the behavior of the system. Specifically, consider (1) with $\mathbf{f}^\pm(\mathbf{x})$ given as a planar, linear vector field

$$\mathbf{f}^\pm(\mathbf{x}) = \begin{pmatrix} -1 \pm x_2 \\ x_2 \mp x_1 \end{pmatrix},$$

and let the discontinuity boundary be defined by $\sigma(\mathbf{x}) = x_2$.

Examine the phase plane associated with $\dot{\mathbf{x}} = \mathbf{f}^+(\mathbf{x})$ and $\dot{\mathbf{x}} = \mathbf{f}^-(\mathbf{x})$, separately in all of $\mathbf{R}^2$ with a focus in particular on the vector fields near the discontinuity $x_2 = 0$. Is there a way to extend the system (1) to include dynamics on the discontinuity boundary $\sigma(\mathbf{x}) = x_2 = 0$, so that we have existence and uniqueness of solutions for initial conditions everywhere in the plane for this example?

One natural way that the dynamics in the discontinuity boundary might be defined is as a convex combination of $\mathbf{f}^+$ and $\mathbf{f}^-$, i.e. let

$$\dot{\mathbf{x}} = \alpha \mathbf{f}^+(\mathbf{x}) + (1-\alpha)\mathbf{f}^-(\mathbf{x}), \alpha \in [0,1], \text{ for } \mathbf{x} \text{ satisfying } \sigma(\mathbf{x}) = 0.$$

Here we'd want the results to not be sensitive to choice of α. Is this the case in some or all regions of the phase plane? Could we define α to obtain unique solutions to the dynamical system? What might you expect to happen if you just code this up without worrying too much about that boundary, figuring it's a zero measure set and you won't ever hit it with your numerical scheme? Try it to see if any of your hunches are born out.