Applied dynamical systems - Week 02

Tuan Pham

problem 1 - finite time blowup

If $\beta = 1$ the equation becomes $\dfrac{dx}{dt} = x$ meaning the solution would be $x_0 e^t$, which would not "blow up" in finite time so we can ignore this case. Let's consider $\beta \neq 1$.

$$\frac{dx}{d\tau} = x^\beta \implies x^{-\beta}dx = d\tau \implies \int_{x(0)}^{x(t)} x^{-\beta}dx = \int_0^t d\tau \implies \left.\frac{x^{-\beta+1}}{-\beta+1}\right|_{x(0)}^{x(t)} = t$$

$$\implies [x(t)]^{1-\beta} = (1-\beta)t + x_0^{1-\beta}$$

If $0 < \beta < 1$ then the right-hand side is positive (because $x_0 > 0$), meaning the real solution exists $\forall\beta$ in such range, which is $x(t) = [\alpha t + x_0^\alpha]^{1/\alpha}$ with $\alpha = 1-\beta \in (0,1)$. This solution also does not "blow up" in finite time as it's bounded by another exponential function that does not "blow up" in finite time either $((t + x_0^\alpha)^{1/\alpha})$.

Lastly, if $\beta > 1$, then the solution below would "blowup" as $(\beta-1)t \to x_0^{1-\beta}$, in other words at $T = \dfrac{x_0^{1-\beta}}{\beta - 1} > 0$.

$$x(t) = \left[\frac{1}{x_0^{1-\beta} - (\beta-1)t}\right]^{1/(\beta-1)}$$

problem 2 - uniqueness

Let's consider first $f(x) = -|x|^\beta$ and $x \in B_b(x_0)$ that only includes $x > 0$ so we can still have differentiability, we can attempt to find the Lipschitz constant K_+. Since $f(x)$ would then be continuous and differentiable, with any $x_1, x_2 \in B_b(x_0)$, per mean value theorem, we could always find a value in between, which would also be in the ball, $x_c \in [x_1, x_2] \in B_b(x_0 0$ such that $f'(x_c) = \dfrac{f(x_1) - f(x_2)}{x_1 - x_2}$. This means the Lipschitz constant could be found by $K_+ = \sup\left|\dfrac{df}{dx}\right| = \sup\beta|x|^{\beta-1}$ (to be found later). This would also apply if the ball only includes $x < 0$ as well (in other words, K_- could be set equal to K_+).

For the case in which $x = 0 \in B_b(x_0)$, we would then just need to worry about cases where $x_1 \times x_2 \leq 0$ since the cases above already concern x_1, x_2 on the same side.

- If $x_1 \neq 0, x_2 = 0$ then $\left||x_1|^\beta - 0\right| \leq K_0|x_1 - 0|$ with $K_0 = (b + |x_0|)^{\beta-1}$ as $|x_1| \leq |x_0| + |x_1 - x_0| \leq |x_0| + b$ as defined by the ball.
- If $x_1 > 0 \wedge x_2 < 0 \implies \left|f(x_1) - f(x_2)\right| = \left|f(x_1) - f(-x_2)\right|$. Assuming we can find K_+ as above then we'd have $\left|f(x_1) - f(-x_2)\right| \leq K_+\left|x_1 - (-x_2)\right| < K_+|x_1 - x_2|$ because $x_2 < 0$.

$\to$ This implies that for this case where $0 \in B_b(x_0)$, the Lipschitz constant could be defined as $\max\{K_0, K_+\}$

Without further ado, let's find K_+, which can be defined as $K_+ = \sup\left|\dfrac{df}{dx}\right| = \sup\beta|x|^{\beta-1}$ for $x \in B_b(x_0)$ containing only $x > 0$. Because $|x|$ is bounded as shown above by $b_0 = |x_0| + b$,

- With $\beta \geq 1$,$|x|^{\beta-1} \leq b_0^{\beta-1}$, meaning K_+ could be found.
- However with $\beta < 1$, as $x \to 0^+$, we would have $|x|^{\beta-1} \to +\infty$. This means that K_+ could not be found.

In **conclusion**, for $\beta \geq 1$, the Lipschitz constant could be found for $f(x) = -|x|^\beta$, meaning the solution to $\dot{x} = f(x)$ would be unique for the initial value x_0. However, we cannot conclude whether the $\beta \in (0,1)$ does not have unique solutions based on this (although we have a hunch it might not).

Indeed, for $0 < \beta < 1$, deriving similarly to **problem 1**'s solution, with changing only the sign of t, we would have $x(t) = \left[x_0^{\alpha} - \alpha t\right]^{1/\alpha}$ where $\alpha = 1 - \beta \in (0, 1)$. This solution becomes 0 at $t = \dfrac{x_0^{\alpha}}{\alpha}$. This means that the solution is not unique, as expected, meaning for different initial conditions, with a given β, eventually they all meet at 0, at different *finite* times $T(x_0)$ depending on the initial conditions, with $T(x_0) = \dfrac{x_0^{1-\beta}}{1-\beta}$. The difference between this and the case where $\beta \geq 1$ is that the solutions to the latter reach $x = 0$ *asymptotically* as $t \to \infty$ instead of in finite time.

problem 3 - Lipschitz constant and the existence & uniqueness theorem

The Lipschitz constant K puts an extra constraints on the time interval $J = [-a, a]$ for both the existence and uniqueness of the solution if contraction mapping theorem is to used for proving. As K is particular to the function defining the ODE $f(x)$, a must be chosen such that not only $a \leq b/M$, where b is the radius of the ball $B_b(x_0)$ to which the Lipschitz constant is defined upon and $M = \max\left|f(x)\right|$ with $x \in B_b(x_0)$, **but also** that $a < 1/K$. This is because the constant $c = Ka$ of the contraction mapping must be < 1.

However, not using contraction mapping theoream, **proof2** in the book could prove the existence of solution wihtout relying on the actual value of K, only needing the existence of K to prove the existence of the solution. Then only relying on just the existence of K, **proof2** also shows that the solution is unique.

problem 4 - discontinued system

For my own sanity, I'm using $s = (x, y)^T$ instead of $x = (x_1, x_2)^T$. Just to rephrase the system's description using this notation:

$$\frac{ds}{dt} = \begin{cases} f^+ & \text{if } y > 0 \\ f^- & \text{if } y < 0 \end{cases} \qquad f^+\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 + y \\ y - x \end{pmatrix} \qquad f^-\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} -1 - y \\ y + x \end{pmatrix}$$

Assuming $\exists f^0 = \begin{pmatrix} g_1(x) \\ g_2(x) \end{pmatrix}$ (does not depend on y) to mitigate the discontinuity boundary $\sigma(s) = y = 0$, so that $f(s)$ is Lipschitz with a constant K near the boundary. Let's pick $s_0 = \begin{pmatrix} x \\ 0 \end{pmatrix}, s_1 = \begin{pmatrix} x \\ y \end{pmatrix}, s_2 = \begin{pmatrix} x \\ -y \end{pmatrix}$ with bounded L2-norm of a radius b (as in $s_1, s_2 \in B_b(s_0)$) and $y > 0$. Since these points all have the same x, the only bounded variable by the ball becomes y.

$$\begin{aligned}
&\begin{cases} \left|f(s_1) - f(s_0)\right| & \leq K|s_1 - s_0| = K\sqrt{y} \\ \left|f(s_0) - f(s_2)\right| & \leq K|s_0 - s_2| = K\sqrt{y} \end{cases} \\
\implies & \left|f(s_1) - f(s_2)\right| \leq \left|f(s_1) - f(s_0)\right| + \left|f(s_0) - f(s_2)\right| \leq 2K\sqrt{y} \\
\implies & \left|\begin{pmatrix} -1 + y - (-1 + y) \\ y - x - (-y + x) \end{pmatrix}\right| = \left|\begin{pmatrix} 0 \\ 2y - 2x \end{pmatrix}\right| \leq 2K\sqrt{y} \\
\implies & |y - x| \leq K\sqrt{y}
\end{aligned}$$

However, since the only bounded variable is y by the ball, it would not make sense for $|y - x|$ to be bounded by the right-hand side by just y, especially when we fix x and choose an extremely small y. Hence it doesn't make sense for $f(s)$ to be Lipschitz everywhere, especially the discontinuity boundary, regardless of the choice of f^0 to mitigate such discontinuity.

Assuming such above statements are correct, I still can't rely on the inexistence of the Lipschitz constant to conclude there doesn't exist a choice of f^0 to allow uniqueness of solution should solutions exist. So at this point, it's a hunch that there's no mitigation to allow for uniqueness.

I tried using the modification proposed with $f^0 = \alpha f^+ + (1 - \alpha) f^-$ at the boundary where $y = 0$. The results are shown in Figure 1. The main summary is that the initial condition value solution is not unique, and the solutions seem to be sensitive to a particular point value of $\alpha = 0.5$.

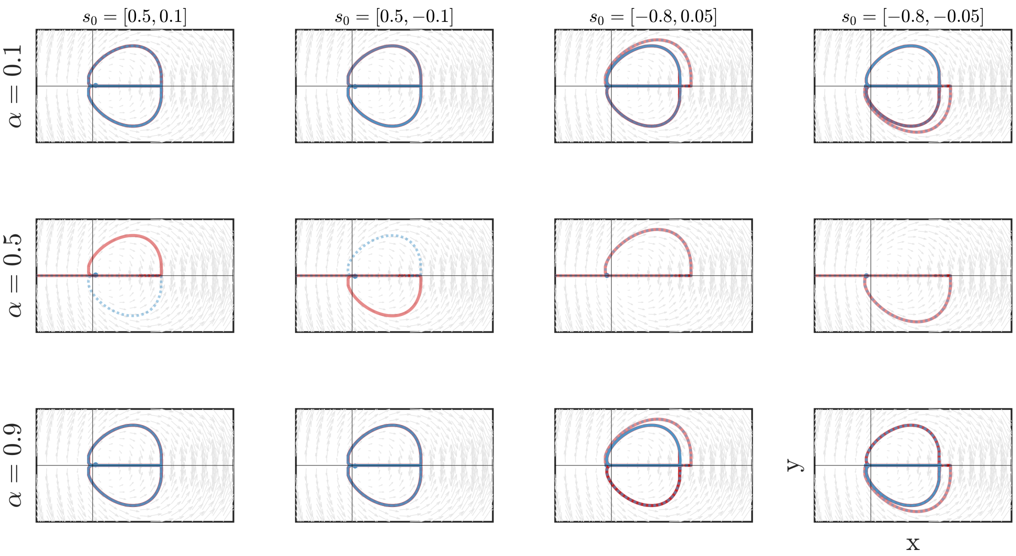

Figure 1: *Looking for uniqueness via* α. Each of the panel shows the simulation using $f^0 = \alpha f^+ + (1-\alpha)f^-$ at the discontinuity condition, along with the vector field. The red line indicates the initial condition s_0 while the blue dotted line corresponds to the perturbed initial condition $s_0 + \delta s$ with $\delta s = [0, 10^{-5}]$ (look for the filled circles near the origin). Darker shades indicate the system has traversed through through such trajectory many times. For all (α, s_0) these trajectories eventually meet up, with the case of $\alpha = 0.5$ the final trajectory is $y = 0, x \to \infty$ (this is simply because $\dot{x} = -1$ once y hits 0)

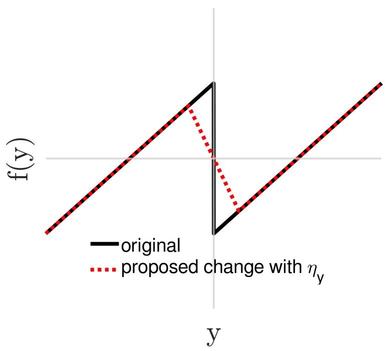

Figure 2: *Proposal of* η_y, f^η

However, I noticed that $\dot{x}$ is continuous but $\dot{y}$ is not when fixing x. This is shown in Figure 2, in which $\eta_y > 0$ is introduced to mend that discontinuity near $y = 0$ (Equation 1). Additionally, the reason I introduced η_y was because numerically checking for $y = 0$ at the discontinuity might not be as accurate - and such inaccuracy may introduce bias in interpreting whether there could be a unique solution.

$$\frac{ds}{dt} = \begin{cases} f^+ & \text{if } y > \eta_y \\ f^- & \text{if } y < -\eta_y \\ f^\eta & \text{otherwise} \end{cases} \quad \text{where } f^\eta = \begin{pmatrix} -1 + |y| \\ y\left(1 - \dfrac{x}{\eta_y}\right) \end{pmatrix} \tag{1}$$

The results are shown in Figure 3. When η_y is large it seems the solution is not unique, and the phase portraits look quite different to Figure 2. However, unexpected, there seems to be more "oscillation" with small η_y (first row Figure 2). At first I thought as $\eta_y \to 0$, the phase portraits would look more like in Figure 2 and there would not be unique solutions there. However, it is possible that it's quite numerically unstable when η_y is quite small, as $\left|\dfrac{x}{\eta_y}\right|$ might be large enough to introduce such instability in numerical simulation. But at least it hints towards a possibility [1] to uniqueness with the introduction of η_y and f^η in a discontinued system.

[1]does it really?

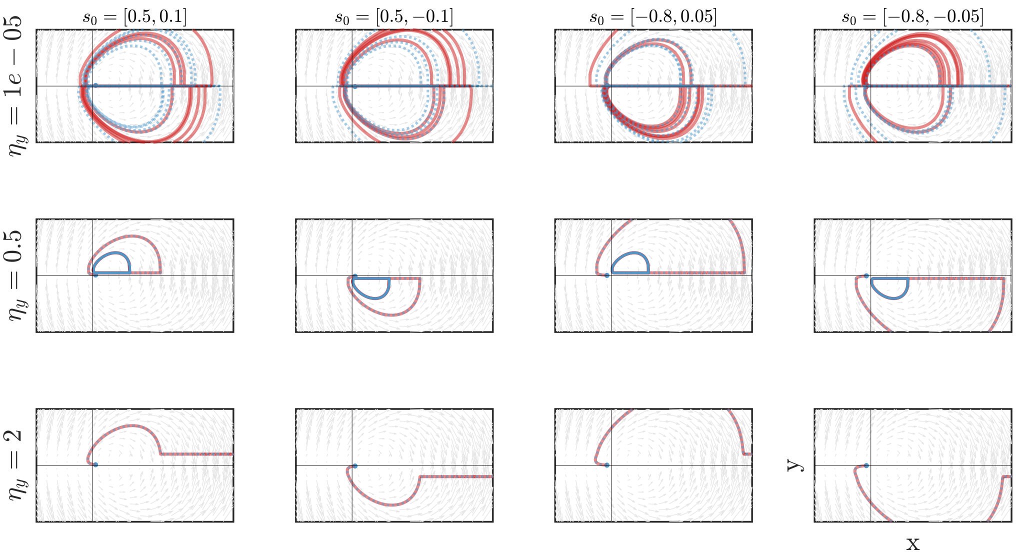

Figure 3: *Looking for uniqueness via* η, f^{η}. The layout of the panels are similar to Figure 1. These show simulations and phase portraits of the system when η_y and f^{η} are introduced in Equation 1.