Applied dynamical systems - Week 01

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Bounded solutions of the pendulum

$$
\begin{aligned}
& \frac{d^2\theta}{dt^2} = -\frac{g}{l}\left(1+\alpha\cos\omega t\right)\sin\theta \\
\implies & \frac{d^2\theta}{(g/l)dt^2} = -\left(1+\alpha\cos\omega t\right)\sin\theta \\
\implies & \frac{d^2\theta}{d\left(\sqrt{g/l}t\right)^2} = -\left(1+\alpha\cos\left(\omega/\sqrt{g/l}\times t\sqrt{g/l}\right)\right)\sin\theta \\
\implies & \frac{d^2\theta}{dt^2} = -\left(1+\alpha\cos\omega t\right)\sin\theta && \text{set } t \leftarrow \sqrt{g/l}t \text{ and } \omega \leftarrow \omega/\sqrt{g/l} \\
\implies & \begin{cases} \dfrac{d\theta}{dt} & = \Omega \\ \dfrac{d\Omega}{dt} & = -\left(1+\alpha\cos\omega t\right)\sin\theta \end{cases} && \text{set } \Omega = \frac{d\theta}{dt}
\end{aligned}
$$

Linearize around $(\theta^*, \Omega^*) = (\pi, 0)$ with $\begin{cases} \theta = \theta^* + x = \pi + x \\ \Omega = \Omega^* + y = y \\ |x|, |y| \ll 1 \end{cases}$ and we basically consider $\begin{cases} x(t) = \theta(t) - \theta^* \\ y(t) = \Omega(t) - \Omega^* \end{cases}$

That means $\sin\theta = \sin(\pi + x) \approx sin\pi + (cos\pi)x = -x$, hence.

$$
\implies \begin{cases} \dfrac{dx}{dt} & = y \\ \dfrac{dy}{dt} & = \left(1+\alpha\cos\omega t\right)x \end{cases} \implies \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1+\alpha\cos\omega t & 0 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}
$$

Denote $s(t) = \begin{pmatrix} x(t) \\ y(t) \end{pmatrix}$ and the time-dependent matrix $A(t) = \begin{pmatrix} 0 & 1 \\ 1+\alpha\cos\omega t & 0 \end{pmatrix}$ then we have

$$
\frac{ds}{dt} = A(t)s(t)
$$

According to **Theorem 2.11 (Abel)** from the book (page 63, **eq 2.50**), the determinant of the fundamental matrix $\phi(t, t0)$ ($\frac{d}{dt}\Phi = A(t)\Phi$ with $\Phi(t_0, t_0) = I$) of the system can be determined by the integral of the trace of $A(t)$ as followed:

$$
\det\left(\Phi(t, t_0)\right) = \exp\int_{t_0}^{t} \operatorname{tr}\left(A(\tau)\right) d\tau
$$

Because $\operatorname{tr}(A(\tau)) = 0\ \forall\tau \implies \det\left(\Phi(t, t_0)\right) = 1$.

$A(t)$ is period with a period $T = \frac{2\pi}{\omega}$, in which $A(t+T) = A(t)\ \forall t$ because the only time-depedent component of the matrix of the is $cos(\omega t)$ having a period of T. Hence, we can look at the eigenvalues μ_1, μ_2 of the monodromy matrix defined as $M = \Omega(T, 0)$ to examine whether the system would be stable around that point, particularly how their magnitudes compare to 1 (smaller corresponding to stable and vice versa).

Since $\det\left(\Phi(t, t_0)\right) = 1\ \forall t, t_0 \implies \det(M) = 1$. Also, because $\det(M) = \mu_1\mu_2 \implies \mu_1\mu_2 = 1$. Hence $|\mu_1||\mu_2| = |\mu_1\mu_2| = 1$. This means that we cannot have situation where both of them have magnitudes less than 1 simultaneously, meaning there is no stable solution for $x \to 0$. The solutions for this are either $\mu_{1,2} = \left\{(a \pm bi), (a \mp bi)\right\}$ or $c, 1/c$ with $a, b, c \in \mathbb{R}, a^2 + b^2 = 1$. To obtain bounded solutions, their magnitudes need to be 1, which is the former cases of complex conjugates. In other words, the Floquet multipliers are conjugates on the unit circles.

In order to calculate the monodromy matrix M in my simulations, for a given initial condition s_0 and period T, theoretically $\begin{cases} s_1 = s(1T) = Ms_0 \\ s_2 = s(2T) = Ms(1T) = M^2 s_0 \end{cases} \implies M[s_0, s_1] = [s_1, s_2] \implies M = [s_1, s_2]/[s_0, s_1]$

The results for the (α, ω) parameter-space resulting in bounded solutions for $x(t), y(t)$ are shown in Figure 1. There's a wide V-shaped region Figure 1**a**, along with a discontinued region (Figure 1**a5-6** and take a closer look at Figure 1**b**). And as expected, the bounded solutions show complex conjugates of magnitudes 1 - oscillating within a bounded range, and there are no solutions in which $x \to 0$.

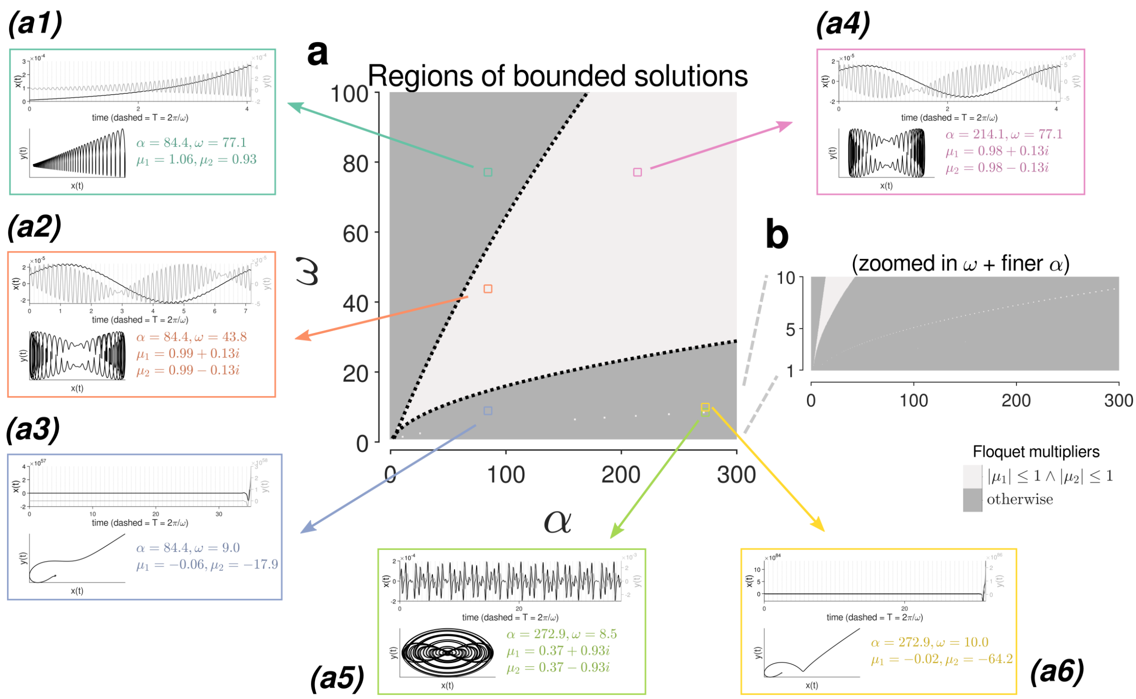

Figure 1: *Regions of bounded solutions with complex conjugate eigen-values of the monodromy matrix.* (**a**) The parameters space of (α, ω), in which the lighter regions indicate where the magnitudes of the eigen-values being ≤ 1 - for examples ***a-2,4,5***. For each of the example panel, the top shows the simulation results of $x(t), y(t)$ through time and the phase space, along with the calculated eigen-values (Floquet multipliers $\mu_{1,2}$). Notice examples ***a-5,6*** show that the bounded regions are not exactly continuous in the parameter space. (**b**) The parameter space with a focus range of ω and finer scale of α to show that there really exists the discontinuity of the "bounded" regime.

My open question - addition of dampening

I was wondering if the equations could be modified so that we can find a stable solutions with the Floquet multipliers of magnitudes < 1. And since $\det(M)$ could be expressed as the exponent of the integral of $\mathrm{tr}(A(t))$, if we can find $|\mu_1| \leq |\mu_2| < 1$ then $\left|\det(M)\right| < 1$, and a way to achieve that is $\mathrm{tr}(A(t)) < 0$ - for example having one negative element on the diagonal line of $A(t)$ while the other could simply be 0. To do that, I introduced dampening force [1], as in:

[1] *Disclaimer*: I am absolutely unsure how valid this is

$$\frac{d^2\theta}{dt^2} = -\left(1 + \alpha\cos\omega t\right)\sin\theta - \beta\frac{d\theta}{dt}$$

$$\Longrightarrow \begin{cases} \dfrac{d\theta}{dt} & = \Omega \\ \dfrac{d\Omega}{dt} & = -\left(1 + \alpha\cos\omega t\right)\sin\theta - \beta\Omega \end{cases} \qquad \text{set } \Omega = \frac{d\theta}{dt}$$

Again, linearize around $(\theta^*, \Omega^*) = (\pi, 0)$ with $\begin{cases} \theta = \theta^* + x = \pi + x \\ \Omega = \Omega^* + y = y \\ |x|, |y| \ll 1 \end{cases}$ and with $sin\theta \approx -x$

$$\Longrightarrow \begin{cases} \dfrac{dx}{dt} & = y \\ \dfrac{dy}{dt} & = \left(1 + \alpha\cos\omega t\right)x - \beta y \end{cases} \Longrightarrow \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 1 + \alpha\cos\omega t & -\beta \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

We still have a periodic $A(t)$ but now $\mathrm{tr}(A(t)) = -\beta\ \forall t$. Hence we could find parameters with stable solutions. Indeed, the examples shown in Figure 2**a-b** show such soluions where $x \to 0$ for both complex and real eigen-value pairs. A 3D parameter scan is shown in Figure 2**c**. With $\beta = 0$ there are no regions with strict inequality to represent stable solutions. But as β increases, the regions start to appear with a V-shaped boundary again, and qualitatively "move" to the right. In other words, higher β generally also requires higher ω to obtain stable solutions.

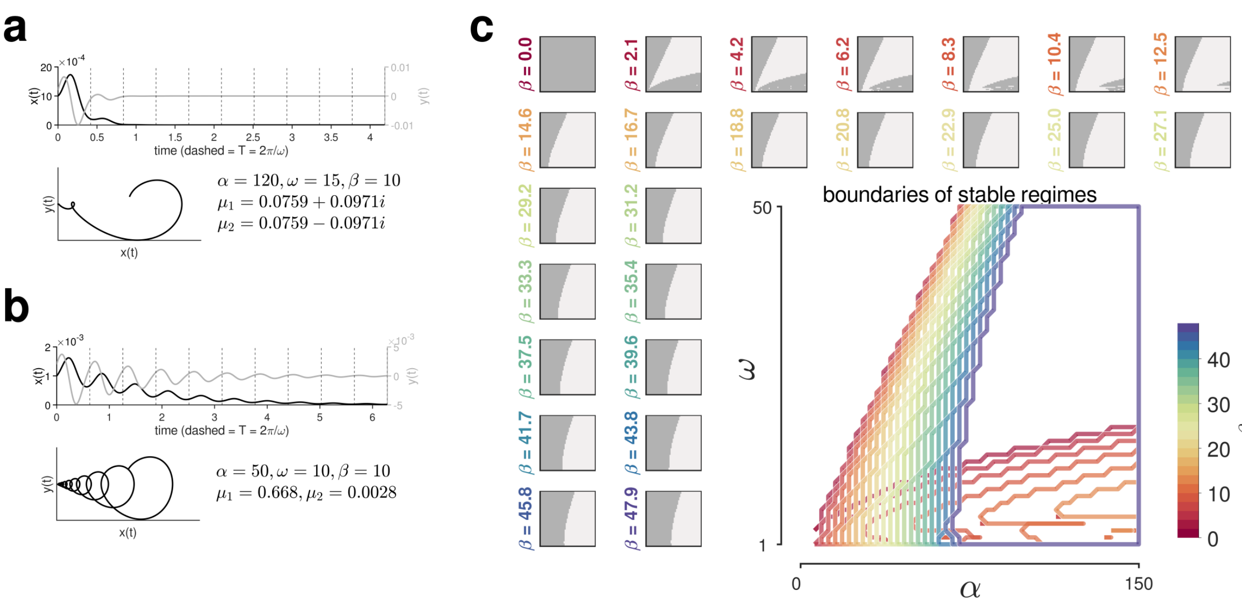

Figure 2: *Stability could be achieved with appropriate dampening* (**a, b**) Examples of parameter sets where stability could be achieved ($x \to 0$), with resulting complex conjugate and real solutions respectively. (**c**) The stability regions are shown for a 2D parameter space (α, ω) for each values of β. Lighter shades indicate $|\mu_1| < 1 \wedge |\mu_2| < 1$ (strict inequality). The big bottom right shows the rough estimates of the stable boundaries for different values of β represented by the colors, ignoring the discontinuities.