Double exponential synaptic implementation note

Double-exponential implementation for synapse with rise time constant as τ_r and decay time constant as τ_d, and the equation for synaptic current is $I = g(E - V)$ where the equilibrium potential E is also constant. x sort of represents an "activation" variable of the synapse. The problem with this is: if the network is connected more than once with the synapses of the same dynamics and time constant, the number of equations would scale by a factor of $O(n^2)$, in which n is the number of neurons. And the x term would also be sort of complicated to integrate into the equation as well.

$$\frac{dg}{dt} = z \qquad \alpha = k_n \left(\frac{1}{\tau_r} - \frac{1}{\tau_d} \right) \qquad k_n = \frac{1}{h(t_{peak})} \quad \text{(normz factor)}$$

$$\frac{dz}{dt} = \alpha x + \beta z + \gamma g \qquad \beta = -\left(\frac{1}{\tau_r} + \frac{1}{\tau_d} \right) \qquad h(t) = e^{-t/\tau_d} - e^{-t/\tau_r}$$

$$x = g_{max} \sum_k \delta(t - t_k) \qquad \gamma = -\frac{1}{\tau_r \tau_d} \qquad t_{peak} = \tau_r \tau_d \cdot \frac{\ln \tau_d - \ln \tau_r}{\tau_d - \tau_r}$$

So during updating the ODE of the synaptic conductance for the **same type of synapse** (meaning the same time constants and equilibrium potential, hence same constants in dz/dt), here's what we could do that can result in only keeping exactly n variables (actually $2n$ counting g and z) and hence can generalize for any connectivity pattern of this type of synapse in the network.

Consider knowing what the effects of the network to neuron i, we just need to know the summation of each conductance (since $E - V$ is constant). And we consider all conductances sum linearly, in other words, $I_{syn}(t, i) = \sum_j g_{ij}(t) \times (E - V_i)$, where $g_{ij}(t)$ is the conductance of the synapse from neuron j to neuron i at time t, and $g_{ii} = 0$ (doesn't actually matter). The equations for each synapse become:

$$\frac{dg_{ij}}{dt} = z_{ij} \qquad G_{ij} : \text{max conductance of synapse from j to i}$$

$$\frac{dz_{ij}}{dt} = \alpha x_{ij} + \beta z_{ij} + \gamma g_{ij} \qquad t^{(j)} : \text{spike (or delayed activation) from neuron j}$$

$$x_{ij} = G_{ij} \sum_k \delta(t - t_k^{(j)})$$

However, for simplicity and for the sake of implementation, we wouldn't need to keep track of all the spikes, just whether it spikes (or activates the synapse with a delay) of the last time step, so for the current time step:

$$x_{ij} = G_{ij} \delta_j \qquad \delta_j : \text{whether j sends out activation in the last time step}$$

Again, since (we assume) the conductances add linearly, we can define:

$$g_i = \sum_j g_{ij} \qquad I_{syn}(t, i) = \sum_j g_{ij}(t) \times (E - V_i)$$

$$z_i = \sum_j z_{ij} \qquad = g_i(t) \times (E - V_i)$$

Effectively that makes:
(the derivative notation from here on assumses for numerical integration like $a[m] = a[m-1] + da \times dt$, instead of "continuous" derivative)

$$\frac{dg_i}{dt} = z_i$$

$$\frac{dz_i}{dt} = \alpha X_i + \beta z_i + \gamma g_i$$

$$X_i = \sum_j G_{ij} \delta_j$$

Now we consider for a vector of received synaptic conductance for the network (below for implementation in MATLAB using matrix/vector notation):

$$\frac{d\vec{g}}{dt} = \vec{z} \qquad \vec{g}_{n\times 1} = [g_1, g_2, ..., g_n]^T$$

$$\frac{d\vec{z}}{dt} = \alpha\mathbf{X} + \beta\vec{z} + \gamma\vec{g} \qquad \vec{z}_{n\times 1} = [z_1, z_2, ..., z_n]^T$$

$$\mathbf{X} = \mathbf{G} \times \vec{\delta} \quad \text{(matrix multiplication)} \qquad \mathbf{G}_{n\times n} : \text{conductance matrix, (i,j): from j to i}$$

$$\vec{\delta}_{n\times 1} : \text{column vector of network activation in last time bin}$$

Below are notes for the equations describing double exponential synapse. The impulse function should take a form of $h(t) = k_n * (e^{-t/\tau_d} - e^{-t/\tau_r})$, where $\tau_d > \tau_r$ with some normalization factor k_n. To find k_n, we can just simply set $\frac{dh}{dt} = 0$, in which t_{peak} could be found (above), hence k_n.
The Laplace transform of $h(t)$ hence is

$$H(s) = k_n \left(\frac{1}{s + \frac{1}{\tau_d}} - \frac{1}{s + \frac{1}{\tau_r}} \right)$$

And for the activation term $x(t), X(s)$ and synaptic conductance term $g(t), G(s)$ we have $G(s) = H(s)X(s)$. That makes:

$$\tau_d \tau_r \ddot{g} + (\tau_d + \tau_r)\dot{g} + g = k_n(\tau_d - \tau_r)x$$

When we decouple the 2nd derivative with $z = \dot{g}$, we have the first set of equations.