Formal Languages

Alphabet and Word

Definition

Alphabet is a nonempty finite set of **symbols**.

Remark: An alphabet is often denoted by the symbol Σ (upper case sigma) of the Greek alphabet.

Definition

A **word** over a given alphabet is a finite sequence of symbols from this alphabet.

Example 1:

$\Sigma = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}, \mathtt{I}, \mathtt{J}, \mathtt{K}, \mathtt{L}, \mathtt{M}, \mathtt{N}, \mathtt{O}, \mathtt{P}, \mathtt{Q}, \mathtt{R}, \mathtt{S}, \mathtt{T}, \mathtt{U}, \mathtt{V}, \mathtt{W}, \mathtt{X}, \mathtt{Y}, \mathtt{Z}\}$

Words over alphabet Σ: `HELLO` `XYZZY` `COMPUTER`

Alphabet and Word

Example 2:

$\Sigma_2 = \{\mathtt{A}, \mathtt{B}, \mathtt{C}, \mathtt{D}, \mathtt{E}, \mathtt{F}, \mathtt{G}, \mathtt{H}, \mathtt{I}, \mathtt{J}, \mathtt{K}, \mathtt{L}, \mathtt{M}, \mathtt{N}, \mathtt{O}, \mathtt{P}, \mathtt{Q}, \mathtt{R}, \mathtt{S}, \mathtt{T}, \mathtt{U}, \mathtt{V}, \mathtt{W}, \mathtt{X}, \mathtt{Y}, \mathtt{Z}, \llcorner\!\lrcorner\}$

A word over alphabet Σ_2: $\mathtt{HELLO}\llcorner\!\lrcorner\mathtt{WORLD}$

Example 3:

$\Sigma_3 = \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9\}$

Words over alphabet Σ_3: 0, 31415926536, 65536

Example 4:

Words over alphabet $\Sigma_4 = \{0, 1\}$: 011010001, 111, 1010101010101010

Example 5:

Words over alphabet $\Sigma_5 = \{a, b\}$: *aababb*, *abbabbba*, *aaab*

Alphabet and Word

Example 6:

Alphabet Σ_6 is the set of all ASCII characters.

Example of a word:

```
class HelloWorld {
    public static void main(String[] args) {
        System.out.println("Hello, world!");
    }
}
```

class␣HelloWorld␣{ ↩ ␣␣␣␣public␣static␣void␣main(Str ···

Theory of Formal Languages – Motivation

Language — a set of (some) words of symbols from a given alphabet

Examples of problem types, where theory of formal languages is useful:

- Construction of compilers:
  - Lexical analysis
  - Syntactic analysis
- Searching in text:
  - Searching for a given text pattern
  - Seaching for a part of text specified by a regular expression

Representation of Formal Languages

To describe a language, there are several possibilities:

- We can enumerate all words of the language (however, this is possible only for small finite languages).

  Example: $L = \{aab, babba, aaaaaa\}$

- We can specify a property of the words of the language:

  Example: The language over alphabet $\{0, 1\}$ containing all words with even number of occurrences of symbol 1.

Representation of Formal Languages

In particular, the following two approaches are used in the theory of formal languages:

- To describe an (idealized) machine, device, algorithm, that recognizes words of the given language – approaches based on **automata**.
- To describe some mechanism that allows to generate all words of the given language – approaches based on **grammars** or **regular expressions**.

Some Basic Concepts

The **set of all words** over alphabet Σ is denoted Σ^*.

The **length of a word** is the number of symbols of the word.
For example, the length of word $abaab$ is 5.

The length of a word w is denoted $|w|$.
For example, if $w = abaab$ then $|w| = 5$.

We denote the number of occurrences of a symbol a in a word w by $|w|_a$.
For word $w = ababb$ we have $|w|_a = 2$ and $|w|_b = 3$.

An **empty word** is a word of length 0, i.e., the word containing no symbols.
The empty word is denoted by the letter ε (epsilon) of the Greek alphabet.

$$|\varepsilon| = 0$$

Concatenation of Words

One of operations we can do on words is the operation of **concatenation**:

For example, the concatenation of words `cabc` and `bba` is the word `cabcbba`.

The operation of concatenation is denoted by symbol $\cdot$ (it is similar to multiplication). This symbol can be omitted.

So, for $u, v \in \Sigma^*$, the concatenation of words u and v is written as $u \cdot v$ or just uv.

Example: If $u = \texttt{cabc}$ and $v = \texttt{bba}$, then

$$uv = \texttt{cabcbba}$$

Remark: Formally, the concatenation of words over alphabet Σ is a fuction of type

$$\Sigma^* \times \Sigma^* \to \Sigma^*$$

Concatenation of Words

Concatenation is **associative**, i.e., for every three words u, v, and w we have

$$(u \cdot v) \cdot w = u \cdot (v \cdot w)$$

which means that we can omit parenthesis when we write multiple concatenations. For example, we can write $w_1 \cdot w_2 \cdot w_3 \cdot w_4 \cdot w_5$ instead of $(w_1 \cdot (w_2 \cdot w_3)) \cdot (w_4 \cdot w_5)$.

Word ε is a neutral element for the operation of concatenation, so for every word w we also have:

$$\varepsilon \cdot w = w \cdot \varepsilon = w$$

Remark: It is obvious that if the given alphabet contains at least two different symbols, the operation of concatenation is not commutative, e.g.,

$$\mathtt{a} \cdot \mathtt{b} \neq \mathtt{b} \cdot \mathtt{a}$$

Prefixes, Suffixes, and Subwords

Definition

A word x is a **prefix** of a word y, if there exists a word v such that $y = xv$.

A word x is a **suffix** of a word y, if there exists a word u such that $y = ux$.

A word x is a **subword** of a word y, if there exist words u and v such that $y = uxv$.

Example:

- Prefixes of the word $\mathtt{abaab}$ are ε, $\mathtt{a}$, $\mathtt{ab}$, $\mathtt{aba}$, $\mathtt{abaa}$, $\mathtt{abaab}$.
- Suffixes of the word $\mathtt{abaab}$ are ε, $\mathtt{b}$, $\mathtt{ab}$, $\mathtt{aab}$, $\mathtt{baab}$, $\mathtt{abaab}$.
- Subwords of the word $\mathtt{abaab}$ are ε, $\mathtt{a}$, $\mathtt{b}$, $\mathtt{ab}$, $\mathtt{ba}$, $\mathtt{aa}$, $\mathtt{aba}$, $\mathtt{baa}$, $\mathtt{aab}$, $\mathtt{abaa}$, $\mathtt{baab}$, $\mathtt{abaab}$.

Language

Definition

A **(formal) language** L over an alphabet Σ is a subset of Σ^*, i.e., $L \subseteq \Sigma^*$.

Example 1: The set $\{00, 01001, 1101\}$ is a language over alphabet $\{0, 1\}$.

Example 2: The set of all syntactically correct programs in the C programming language is a language over the alphabet consisting of all ASCII characters.

Example 3: The set of all texts containing the sequence `hello` is a language over alphabet consisting of all ASCII characters.

Set Operations on Languages

Since languages are sets, we can apply any set operations to them:

Union – $L_1 \cup L_2$ is the language consisting of the words belonging to language L_1 or to language L_2 (or to both of them).

Intersection – $L_1 \cap L_2$ is the language consisting of the words belonging to language L_1 and also to language L_2.

Complement – $\overline{L_1}$ is the language containing those words from Σ^* that do not belong to L_1.

Difference – $L_1 - L_2$ is the language containing those words of L_1 that do not belong to L_2.

Remark: It is assumed the languages involved in these operations use the same alphabet Σ.

Set Operations on Languages

Formally:

Union: $L_1 \cup L_2 = \{w \in \Sigma^* \mid w \in L_1 \vee w \in L_2\}$

Intersection: $L_1 \cap L_2 = \{w \in \Sigma^* \mid w \in L_1 \wedge w \in L_2\}$

Complement: $\overline{L_1} = \{w \in \Sigma^* \mid w \notin L_1\}$

Difference: $L_1 - L_2 = \{w \in \Sigma^* \mid w \in L_1 \wedge w \notin L_2\}$

Remark: We assume that $L_1, L_2 \subseteq \Sigma^*$ for some given alphabet Σ.

Set Operations on Languages

Example:

Consider languages over alphabet $\{\mathtt{a}, \mathtt{b}\}$.

- L_1 — the set of all words containing subword baa
- L_2 — the set of all words with an even number of occurrences of symbol b

Then

- $L_1 \cup L_2$ — the set of all words containing subword baa or an even number of occurrences of b
- $L_1 \cap L_2$ — the set of all words containing subword baa and an even number of occurrences of b
- $\overline{L_1}$ — the set of all words that do not contain subword baa
- $L_1 - L_2$ — the set of all words that contain subword baa but do not contain an even number of occurrences of b

Concatenation of Languages

Definition

Concatenation of languages L_1 and L_2, where $L_1, L_2 \subseteq \Sigma^*$, is the language $L \subseteq \Sigma^*$ such that for each $w \in \Sigma^*$ it holds that

$$w \in L \;\leftrightarrow\; (\exists u \in L_1)(\exists v \in L_2)(w = u \cdot v)$$

The concatenation of languages L_1 and L_2 is denoted $L_1 \cdot L_2$.

Example:

$$\begin{aligned} L_1 &= \{abb, ba\} \\ L_2 &= \{a, ab, bbb\} \end{aligned}$$

The language $L_1 \cdot L_2$ contains the following words:

$abba \quad abbab \quad abbbbb \quad baa \quad baab \quad babbb$

Remark: Note that the concatenation of languages is associative.

Iteration of a Language

Definition

The **iteration (Kleene star) of language** L, denoted L^*, is the language consisting of words created by concatenation of some arbitrary number of words from language L.
I.e. $w \in L^*$ iff

$$\exists n \in \mathbb{N} : \exists w_1, w_2, \ldots, w_n \in L : w = w_1 w_2 \cdots w_n$$

Example: $L = \{aa, b\}$

$$L^* = \{\varepsilon, aa, b, aaaa, aab, baa, bb, aaaaaa, aaaab, aabaa, aabb, \ldots\}$$

Remark: The number of concatenated words can be 0, which means that $\varepsilon \in L^*$ always holds (it does not matter if $\varepsilon \in L$ or not).

Iteration of a Language – Alternative Definition

At first, for a language L and a number $k \in \mathbb{N}$ we define the language L^k:

$$L^0 = \{\varepsilon\}, \qquad L^k = L^{k-1} \cdot L \quad \text{for } k \geq 1$$

This means

$$\begin{array}{rcl}
L^0 & = & \{\varepsilon\} \\
L^1 & = & L \\
L^2 & = & L \cdot L \\
L^3 & = & L \cdot L \cdot L \\
L^4 & = & L \cdot L \cdot L \cdot L \\
L^5 & = & L \cdot L \cdot L \cdot L \cdot L \\
 & & \cdots
\end{array}$$

Example: For $L = \{aa, b\}$, the language L^3 contains the following words:

$$aaaaaa \quad aaaab \quad aabaa \quad aabb \quad baaaa \quad baab \quad bbaa \quad bbb$$

Iteration of a Language – Alternative Definition

Alternative definition

The **iteration (Kleene star) of language** L is the language

$$L^* = \bigcup_{k \geq 0} L^k$$

Remark:

$$\bigcup_{k \geq 0} L^k = L^0 \cup L^1 \cup L^2 \cup L^3 \cup \cdots$$

Iteration of a Language

Remark: Sometimes, notation L^+ is used as an abbreviation for $L \cdot L^*$, i.e.,

$$L^+ = \bigcup_{k \geq 1} L^k$$

Reverse

The **reverse** of a word w is the word w written from backwards (in the opposite order).

The reverse of a word w is denoted w^R.

Example: $w = \mathtt{HELLO}$ $w^R = \mathtt{OLLEH}$

Formally, for $w = a_1 a_2 \cdots a_n$ (where $a_i \in \Sigma$) is $w^R = a_n a_{n-1} \cdots a_1$.

Reverse

The **reverse** of a language L is the language consisting of reverses of all words of L.

Reverse of a language L is denoted L^R.

$$L^R = \{w^R \mid w \in L\}$$

Example: $L = \{ab, baaba, aaab\}$

$L^R = \{ba, abaab, baaa\}$

Order on Words

Let us assume some (linear) order $<$ on the symbols of alphabet Σ, i.e., if $\Sigma = \{a_1, a_2, \ldots, a_n\}$ then

$$a_1 < a_2 < \ldots < a_n .$$

Example: $\Sigma = \{a, b, c\}$ with $a < b < c$.

The following (linear) order $<_L$ can be defined on Σ^*:
$x <_L y$ iff:

- $|x| < |y|$, or
- $|x| = |y|$ there exist words $u, v, w \in \Sigma^*$ and symbols $a, b \in \Sigma$ such that

$$x = uav \qquad y = ubw \qquad a < b$$

Informally, we can say that in order $<_L$ we order words according to their length, and in case of the same length we order them lexicographically.

Order on Words

All words over alphabet Σ can be ordered by $<_L$ into a sequence

$$w_0, w_1, w_2, \ldots$$

where every word $w \in \Sigma^*$ occurs exactly once, and where for each $i, j \in \mathbb{N}$ it holds that $w_i <_L w_j$ iff $i < j$.

Example: For alphabet $\Sigma = \{a, b, c\}$ (where $a < b < c$) , the initial part of the sequence looks as follows:

$$\varepsilon, a, b, c, aa, ab, ac, ba, bb, bc, ca, cb, cc, aaa, aab, aac, aba, abb, abc, \ldots$$

For example, when we talk about the first ten words of a language $L \subseteq \Sigma^*$, we mean ten words that belong to language L and that are smallest of all words of L according to order $<_L$.

Regular Expressions

Regular Expressions

Regular expressions describing languages over an alphabet Σ:

- $\emptyset$, ε, a (where $a \in \Sigma$) are regular expressions:

  $\emptyset$ ... denotes the empty language

  ε ... denotes the language $\{\varepsilon\}$

  a ... denotes the language $\{a\}$

- If α, β are regular expressions then also $(\alpha + \beta)$, $(\alpha \cdot \beta)$, (α^*) are regular expressions:

  $(\alpha + \beta)$ ... denotes the union of languages denoted α and β

  $(\alpha \cdot \beta)$ ... denotes the concatenation of languages denoted α and β

  (α^*) ... denotes the iteration of a language denoted α

- There are no other regular expressions except those defined in the two points mentioned above.

Regular Expressions

Example: alphabet $\Sigma = \{0, 1\}$

- According to the definition, $\mathtt{0}$ and $\mathtt{1}$ are regular expressions.

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- Since 0 and 1 are regular expression, $(0 + 1)$ is also a regular expression.
- Since 0 is a regular expression, (0^*) is also a regular expression.
- Since $(0 + 1)$ and (0^*) are regular expressions, $((0 + 1) \cdot (0^*))$ is also a regular expression.

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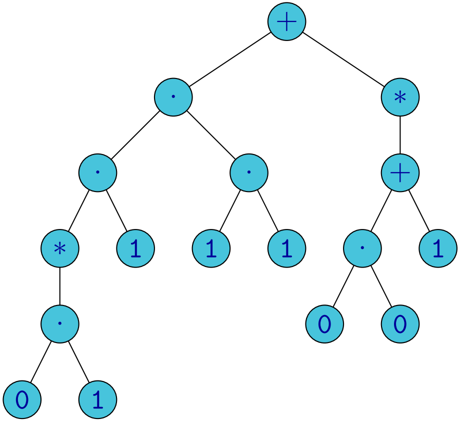
- Since 0 and 1 are regular expression, $(0 + 1)$ is also a regular expression.
- Since 0 is a regular expression, (0^*) is also a regular expression.
- Since $(0 + 1)$ and (0^*) are regular expressions, $((0 + 1) \cdot (0^*))$ is also a regular expression.

Remark: If α is a regular expression, by $\mathcal{L}(\alpha)$ we denote the language defined by the regular expression α.

$$\mathcal{L}((0 + 1) \cdot (0^*)) = \{0,\ 1,\ 00,\ 10,\ 000,\ 100,\ 0000,\ 1000,\ 00000,\ \dots\}$$

The structure of a regular expression can be represented by an abstract syntax tree:



$$(((((0 \cdot 1)^*) \cdot 1) \cdot (1 \cdot 1)) + (((0 \cdot 0) + 1)^*))$$

Regular Expressions

The formal definition of semantics of regular expressions:

- $\mathcal{L}(\emptyset) = \emptyset$
- $\mathcal{L}(\varepsilon) = \{\varepsilon\}$
- $\mathcal{L}(a) = \{a\}$
- $\mathcal{L}(\alpha^*) = \mathcal{L}(\alpha)^*$
- $\mathcal{L}(\alpha \cdot \beta) = \mathcal{L}(\alpha) \cdot \mathcal{L}(\beta)$
- $\mathcal{L}(\alpha + \beta) = \mathcal{L}(\alpha) \cup \mathcal{L}(\beta)$

Regular Expressions

To make regular expressions more lucid and succinct, we use the following conventions:

- The outward pair of parentheses can be omitted.
- We can omit parentheses that are superflous due to associativity of operations of union ($+$) and concatenation ($\cdot$).
- We can omit parentheses that are superflous due to the defined priority of operators (iteration (*) has the highest priority, concatenation ($\cdot$) has lower priority, and union ($+$) has the lowest priority).
- A dot denoting concatenation can be omitted.

Example: Instead of

$$(((((0 \cdot 1)^*) \cdot 1) \cdot (1 \cdot 1)) + (((0 \cdot 0) + 1)^*))$$

we usually write

$$(01)^* 111 + (00 + 1)^*$$

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

$\mathtt{0}$ ... the language containing the only word $\mathtt{0}$

$\mathtt{01}$ ... the language containing the only word $\mathtt{01}$

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

01 ... the language containing the only word 01

$0 + 1$ ... the language containing two words 0 and 1

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

01 ... the language containing the only word 01

$0 + 1$ ... the language containing two words 0 and 1

0^* ... the language containing words ε, 0, 00, 000, ...

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

01 ... the language containing the only word 01

$0 + 1$ ... the language containing two words 0 and 1

0^* ... the language containing words ε, 0, 00, 000, ...

$(01)^*$ ... the language containing words ε, 01, 0101, 010101, ...

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

01 ... the language containing the only word 01

$0 + 1$ ... the language containing two words 0 and 1

0^* ... the language containing words ε, 0, 00, 000, ...

$(01)^*$ ... the language containing words ε, 01, 0101, 010101, ...

$(0 + 1)^*$ ... the language containing all words over the alphabet $\{0, 1\}$

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0	... the language containing the only word 0
01	... the language containing the only word 01
$0 + 1$	... the language containing two words 0 and 1
0^*	... the language containing words ε, 0, 00, 000, ...
$(01)^*$	... the language containing words ε, 01, 0101, 010101, ...
$(0 + 1)^*$	... the language containing all words over the alphabet $\{0, 1\}$
$(0 + 1)^*00$	... the language containing all words ending with 00

Regular Expressions

Examples: In all examples $\Sigma = \{0, 1\}$.

0 ... the language containing the only word 0

01 ... the language containing the only word 01

$0 + 1$ ... the language containing two words 0 and 1

0^* ... the language containing words ε, 0, 00, 000, ...

$(01)^*$ ... the language containing words ε, 01, 0101, 010101, ...

$(0 + 1)^*$ ... the language containing all words over the alphabet $\{0, 1\}$

$(0 + 1)^*00$ ... the language containing all words ending with 00

$(01)^*111(01)^*$ ... the language containing all words that contain a subword 111 preceded and followed by an arbitrary number of copies of the word 01

Regular Expressions

$(0+1)^*00 + (01)^*111(01)^*$ ... the language containing all words that either end with 00 or contain a subwords 111 preceded and followed with some arbitrary number of words 01

Regular Expressions

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$(0+1)^*1(0+1)^*$ ... the language of all words that contain at least one occurrence of symbol 1

Regular Expressions

$(0+1)^*00 + (01)^*111(01)^*$ ... the language containing all words that either end with 00 or contain a subwords 111 preceded and followed with some arbitrary number of words 01

$(0+1)^*1(0+1)^*$ ... the language of all words that contain at least one occurrence of symbol 1

$0^*(10^*10^*)^*$ ... the language containg all words with an even number of occurrences of symbol 1