Equality

One of the most important relations is **equality** (**identity**).

Elements x and y are equal, written

$$x = y,$$

if they are the same element.

Equality can be expressed as a predicate, e.g., we can choose that $P(x, y)$ represents proposition *"x and y are equal"*.

But $P(x, y)$ can be true in some interpretations even when x and y are distinct elements, and in some interpretations, $P(x, x)$ can be false for some element x.

Equality

Example: We would like to describe relationship *"x is a sibling of y"* using a binary predicate R, where $R(x, y)$ means *"x is a parent of y"*.

An attempt for a possible solution:

"x is a sibling of y"

iff

$$\exists z (R(z, x) \wedge R(z, y))$$

Problem: If for a given x there is an element z such that $R(z, x)$, then it is true that

$$\exists z (R(z, x) \wedge R(z, x)),$$

and so it is also true that *"x is a sibling of x"*.

Equality

Alphabet:

- $\ldots$
- Symbol for equality: "$=$"
- $\ldots$

Atomic formulas (cont.)

- $\ldots$
- If x and y are variables then $x = y$ is a well-formed atomic formula.
- $\ldots$

Symbol "$=$" is interpreted as equality in every interpretation, i.e., in every interpretation $\mathcal{A}$ and valuation v is:

- $\mathcal{A}, v \models x = y \quad$ iff $\quad v(x) = v(y)$.

Equality

Example: The relationship *"x is a sibling of y"* can be expressed by formula

$$\neg(x = y) \ \wedge \ \exists z(R(z, x) \wedge R(z, y)),$$

where $R(x, y)$ means that *"x is a parent of y"*.

Remark: The notation $x \neq y$ is often used instead of $\neg(x = y)$.

Equality

- *"There exists* **exactly one** x *such that* $P(x)$*"*:
$$\exists x(P(x) \ \wedge \ \forall y(P(y) \ \rightarrow \ x = y))$$

- *"There exist* **at least two** *elements* x *such that* $P(x)$*"*:
$$\exists x \exists y(P(x) \ \wedge \ P(y) \ \wedge \ \neg(x = y))$$

- *"There exist* **exactly two** *elements* x *such that* $P(x)$*"*:
$$\exists x \exists y(P(x) \wedge P(y) \wedge \neg(x = y) \ \wedge \ \forall z(P(z) \ \rightarrow \ (z = x \ \vee \ z = y)))$$

- *"There exists* **exactly one** x, *for which* φ *holds"*:
$$\exists x(\varphi \ \wedge \ \forall y(\varphi[y/x] \ \rightarrow \ x = y))$$

Constants

Sometimes we want to talk about some particular element of the universe.

Example: *"There exists at least one x such that John Smith is a parent of x and x is a woman."* (I.e., *"John Smith has at least one daughter."*)

If the value assigned to variable y is *"John Smith"*:

$$\exists x(R(y,x) \wedge S(x))$$

- $R(x,y)$ — *"x is a parent of y"*
- $S(x)$ — *"x is a woman"*

We could introduce an unary predicate N representing property *"to be John Smith"*:

$$\forall y(N(y) \rightarrow \exists x(R(y,x) \wedge S(x)))$$

Constants

If we have some unary predicate N where we are interested only in those interpretations where exists **exactly one** element x, for which $N(x)$ holds, it would be convenient to have some way to name this element and refer to it directly instead of using of the predicate N.

Constant symbols (**constants**) can be used for this purpose.

Alphabet:

- ...
- constant symbols: "a", "b", "c", "d", ...
- ...

Constants

In **atomic formulas**, constants can occur at the same places as variables:

$$P(c,x) \qquad Q(d) \qquad R(a,a) \qquad x = a$$

- Constants **must not** be used in quantifiers — e.g., $\exists c P(x,c)$ **is not** a well-formed formula.

Values assigned to constant symbols are determined by a given **interpretation**:

- A given interpretation $\mathcal{A}$ (with universe A) assigns to every constant symbol c some element of the universe A.

  This element is denoted $c^{\mathcal{A}}$. So $c^{\mathcal{A}} \in A$.

Constants

Example: *"There exists at least one x such that John Smith is a parent of x and x is a woman."*

$$\exists x(R(a,x) \wedge S(x))$$

- $R(x,y)$ — *"x is a parent of y"*
- $S(x)$ — *"x is a woman"*
- a — constant symbol representing *"John Smith"*

Constants

Example: *"Every prime is greater than one."*

$$\forall x(P(x) \rightarrow R(x, e))$$

- $P(x)$ — *"x is a prime"*
- $R(x, y)$ — *"x is greater than y"*
- e — constant symbol representing value 1

Functions

A binary relation R is a (unary) **function** if for each x there exists at most one y such that

$$(x, y) \in R.$$

This function is **total** if for each x there exists exactly one such y.

Example: Binary relation R on the set of natural numbers $\mathbb{N}$ where

$$(x, y) \in R \qquad \text{iff} \qquad y = x + 1$$

We have

$$R = \{(0,1),\ (1,2),\ (2,3),\ (3,4),\ (4,5),\ \ldots\}$$

Functions

Similarly, a ternary relation T is a (binary) function if for every pair of elements x_1 and x_2 there exists at most one (resp., exactly one for total function) y such that

$$(x_1, x_2, y) \in T.$$

Example: Addition on the set of real numbers $\mathbb{R}$ can be viewed as a ternary relation S (i.e., as a set of triples of real numbers) where

$$(x_1, x_2, y) \in S \qquad \text{iff} \qquad x_1 + x_2 = y$$

Functions

In predicate logic, functions can be expressed using predicates representing the corresponding relations — this is not very straightforward nor convenient.

Example: *"For each x and y it holds that $x + y \geq y + x$."*

$$\forall x \forall y \exists z \exists w (S(x, y, z) \ \wedge \ S(y, x, w) \ \wedge \ P(z, w))$$

- $S(x, y, z)$ — *"z is the sum of values x and y"*
- $P(x, y)$ — *"x greater than or equal to y"*

Remark: Moreover, we must assume that for every pair of elements x and y there exists exactly one element z such that $S(x, y, z)$.

Functions

In predicate logic, functions can be represented by **function symbols**.

Alphabet:

- ...
- function symbols: "f", "g", "h", ...
- ...

Every function symbol must have a specified **arity** corresponding to the arity of a function represented by this symbol (i.e., the number of arguments of this function).

Terms

Terms — expressions, consisting of variables, constant symbols, and function symbols; values of terms are elements of the universe

Example:

- Let us say that we have a predicate F where we assume that for every x there exists exactly one y such that

  $$F(x, y).$$

  Instead of binary predicate F, we can use unary function symbol f.

  Term

  $$f(x)$$

  represents this one particular element y, for which $F(x, y)$ holds.

  Instead of $\exists y(F(x, y) \wedge P(y))$, we can write $P(f(x))$.

Example:

- Let us say that we have a ternary predicate G where we assume that for every pair of elements x_1 and x_2 there exists exactly one y such that
$$G(x_1, x_2, y).$$
Instead of ternary predicate G, we can use binary function symbol g.
Term
$$g(x_1, x_2)$$
represents this one particular element y, for which $G(x_1, x_2, y)$ holds.

Terms

Example: *"For each x and y it holds that $x + y \geq y + x$."*

$$\forall x \forall y P(f(x, y), f(y, x))$$

- f — binary function symbol where $f(x, y)$ represents the sum of values x and y
- P — binary predicate symbol where $P(x, y)$ represents relation *"x is greater than or equal to y"*

Terms

Variables, constant symbols and function symbols can be composed in terms in arbitrary way — it is only necessary to comply with the arity of the symbols (to apply each function symbol to a correct number of arguments).

Example:

- c — constant symbol
- f — unary function symbol
- g — binary function symbol
- h — binary function symbol

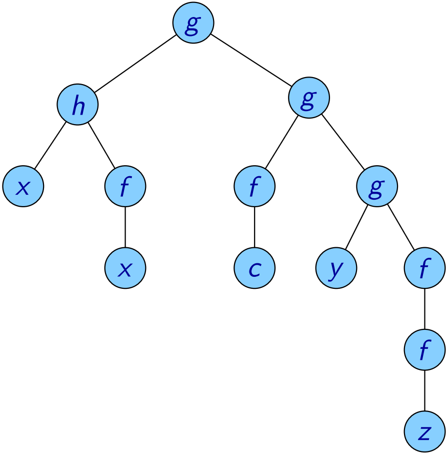
Examples of terms:

$$x \qquad f(y) \qquad g(c,x) \qquad g(h(x,x),f(c))$$

$$g(h(x,f(x)),g(f(c),g(y,f(f(z)))))$$

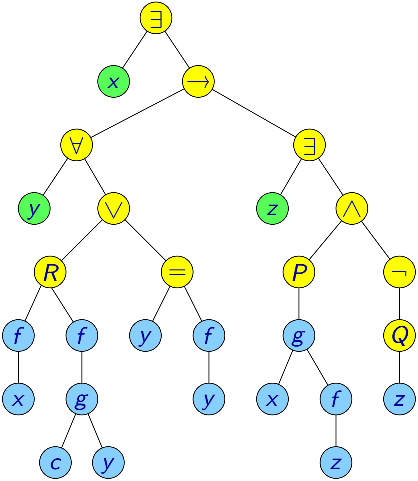
Terms

The syntactic tree of term $g(h(x, f(x)), g(f(c), g(y, f(f(z)))))$



Terms in Formulas

The syntactic tree of formula
$\exists x(\forall y(R(f(x), f(g(c, y))) \vee y = f(y)) \rightarrow \exists z(P(g(x, f(z))) \wedge \neg Q(z)))$



Terms in Formulas

Example:

- For each x, y, and z it holds that $(x+y)+z = x+(y+z)$:
$$\forall x \forall y \forall z (f(f(x,y),z) = f(x, f(y,z)))$$
- For each x it holds that $x+0 = x$ and $0+x = x$:
$$\forall x (f(x,e) = x \ \land \ f(e,x) = x)$$
- For each x there exists y such that $x+y=0$:
$$\forall x \exists y (f(x,y) = e)$$

Constant and function symbols:

- f — binary function symbol representing *"addition"* (operation "$+$")
- e — constant symbol representing element "0"

Terms in Formulas

Example:

- For each x, y, and z it holds that $x \cdot (y + z) = x \cdot y + x \cdot z$:
$$\forall x \forall y \forall z (g(x, f(y, z)) = f(g(x, y), g(x, z)))$$
- For each x and y such that $x \leq y$ it holds that $x + z \leq y + z$:
$$\forall x \forall y (R(x, y) \ \rightarrow \ \forall z R(f(x, y), f(y, z)))$$

Constant and function symbols:

- f — binary function symbol representing *"addition"* (operation "$+$")
- g — binary function symbol representing *"multiplication"* (operation "$\cdot$")
- R — binary predicate symbol representing relation *"less than or equal to"* (relation "$\leq$")

Syntax of Formulas of Predicate Logic

Alphabet:

- **logical connectives** — "$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"
- **quantifiers** — "$\forall$" and "$\exists$"
- **equality** — "$=$"
- **auxiliary symbols** — "$($", "$)$", and "$,$"
- **variables** — "x", "y", "z", ..., "x_0", "x_1", "x_2", ...

- **predicate symbols** — for example symbols "P", "Q", "R", etc. (for each symbols, there must be specified its arity)
- **function symbols** — for example symbols "f", "g", "h", etc. (for each symbol, there must be specified its arity)
- **constant symbols** — for example symbols "a", "b", "c", etc.

Remark: Constant symbols can be viewed as function symbols of arity 0.

Syntax of Formulas of Predicate Logic

Definition

Well-formed **terms** are defined as follows:

1. If x is a variable then x is a well-formed term.
2. If c is a constant symbol then c is a well-formed term.
3. If f is a function symbol of arity n and t_1, t_2, ..., t_n are well-formed terms then
$$f(t_1, t_2, \ldots, t_n)$$
is a well-formed term.
4. There are no other well-formed terms than those constructed according to the previous rules.

Syntax of Formulas of Predicate Logic

Definition

Well-formed **atomic formulas** are defined as follows:

1. If P is a predicate symbol of arity n and t_1, t_2, ..., t_n are well-formed terms then
$$P(t_1, t_2, \ldots, t_n)$$
is a well-formed atomic formula.

2. If t_1 and t_2 are well-formed terms then
$$t_1 = t_2$$
is a well-formed atomic formula.

3. There are no other well-formed atomic formulas than those constructed according to the previous rules.

Syntax of Formulas of Predicate Logic

Definition (a previously stated definition repeated)

Well-formed **formulas of predicate logic** are sequences of symbols constructed according to the following rules:

1. Well-formed atomic formulas are well-formed formulas.
2. If φ and ψ are well-formed formulas, then also $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ a $(\varphi \leftrightarrow \psi)$ are well-formed formulas.
3. If φ is a well-formed formula and x is a variable, then $\forall x \varphi$ and $\exists x \varphi$ are well-formed formulas.
4. There are no other well-formed formulas than those constructed according to the previous rules.

Semantics of Predicate Logic

Interpretation $\mathcal{A}$:

- universe A
- to every predicate symbol P of arity n, an n-ary relation $P^{\mathcal{A}}$ is assigned, where $P^{\mathcal{A}} \subseteq A \times A \times \cdots \times A$
- to every function symbol f of arity n, an n-ary function $f^{\mathcal{A}}$ is assigned, where $f^{\mathcal{A}} : A \times A \times \cdots \times A \to A$
- to every constant symbol c, an element of the universe $c^{\mathcal{A}}$ is assigned, i.e., $c^{\mathcal{A}} \in A$

Remark: In interpretations, only **total** functions, i.e., functions whose values are defined for all possible values of arguments, are assigned to function symbols.

Semantics of Predicate Logic

The **value of a term** in interpretation $\mathcal{A}$ and valuation v:

- Term x, where x is a variable — the value of this term is an element $a \in A$ such that $v(x) = a$.
- Term c, where c is a constant symbol — the value of this term is the element $c^{\mathcal{A}} \in A$.
- Term $f(t_1, t_2, \ldots, t_n)$, where f is a function symbol with arity n and $t_1, t_2, \ldots, t_n$ are terms — the value of this term is the element $b \in A$ such that

  $$b = f^{\mathcal{A}}(a_1, a_2, \ldots, a_n),$$

  where $a_1, a_2, \ldots, a_n$ are values of terms $t_1, t_2, \ldots, t_n$ in interpretation $\mathcal{A}$ and valuation v.

Semantics of Predicate Logic

Example: Interpretation $\mathcal{A}$ where the universe is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

- $a^{\mathcal{A}} = 0$
- $f^{\mathcal{A}}$ is the function *"successor"*, i.e., $f^{\mathcal{A}}(x) = x + 1$
- $g^{\mathcal{A}}$ is the function *"sum"*, i.e., $g^{\mathcal{A}}(x, y) = x + y$

Valuation v where $v(x) = 5$, $v(y) = 13$, $v(z) = 2$, ...

Values of terms in interpretation $\mathcal{A}$ and valuation v:

- Term x — value 5
- Term a — value 0
- Term $f(a)$ — value 1 $(0 + 1 = 1)$
- Term $f(f(a))$ — value 2 $(1 + 1 = 2)$
- Term $g(x, f(f(a)))$ — value 7 $(5 + 2 = 7)$
- Term $g(z, y)$ — value 15 $(2 + 13 = 15)$
- Term $f(g(z, y))$ — value 16 $(15 + 1 = 16)$

Semantics of Predicate Logic

Truth values of atomic formulas in interpretation $\mathcal{A}$ and valuation v:

- $\mathcal{A}, v \models P(t_1, t_2, \ldots, t_n)$, where P is a predicate symbol of arity n and where $t_1, t_2, \ldots, t_n$ are terms, holds iff

$$(a_1, a_2, \ldots, a_n) \in P^{\mathcal{A}},$$

  where $a_1, a_2, \ldots, a_n$ are values of terms $t_1, t_2, \ldots, t_n$ in interpretation $\mathcal{A}$ and valuation v.

- $\mathcal{A}, v \models t_1 = t_2$, where t_1 and t_2 are terms, holds iff

$$a_1 = a_2,$$

  where a_1 and a_2 are values of terms t_1 and t_2 in interpretation $\mathcal{A}$ and valuation v.

Semantics of Predicate Logic

Example: Interpretation $\mathcal{A}$ where the universe is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

- $f^{\mathcal{A}}$ is the function *"successor"*, i.e., $f^{\mathcal{A}}(x) = x + 1$
- $g^{\mathcal{A}}$ is the function *"sum"*, i.e., $g^{\mathcal{A}}(x, y) = x + y$
- $P^{\mathcal{A}}$ is the set of all primes
- $Q^{\mathcal{A}}$ is the binary relation "$<$" i.e., $(x, y) \in Q^{\mathcal{A}}$ iff $x < y$

Valuation v where $v(x) = 5$, $v(y) = 13$, $v(z) = 2$, ...

- $\mathcal{A}, v \models P(x)$ (5 is a prime)
- $\mathcal{A}, v \not\models Q(y, z)$ (it is not the case that $13 < 2$)
- $\mathcal{A}, v \models Q(f(f(z)), g(x, y))$ ($(2 + 1) + 1 < 5 + 13$)
- $\mathcal{A}, v \not\models P(f(g(z, x)))$ ($(2 + 5) + 1 = 8$ and 8 is not a prime)

Predicate Logic — Additional Comments

- The logic described here is **the first order** predicate logic — it is possible to quantify only over the elements of the universe (in the second order predicate logic, it is possible to quantify over relations).

Predicate Logic — Additional Comments

As commonly used in mathematics, it is ofter the case that formulas are not written according to the precise syntax of predicate logic but many kinds of conventions and abbreviations are used.

- For **binary** function and predicate symbols, it is common to use **infix** notation:

  For example, $f(x, y)$ and $R(x, y)$ can be written as

  $$x\ f\ y \qquad\qquad x\ R\ y$$

- To denote predicate, function and constant symbols, many different kinds of symbols are used:

  For example, $R(f(x, y), g(z))$ can be written as

  $$x + \delta \leq |\varepsilon| \qquad \text{or for example as} \qquad x \circ y \sqsupset G(z)$$

Predicate Logic — Additional Comments

Examples of formulas representing propositions from set theory:

- x is an elements of set A:

$$x \in A$$

  "$\in$" — binary predicate symbol representing the relation *"to be an element of"*
  "x", "A" — variables

  If we would do the following changes:

  - instead of symbol "$\in$", we would use binary predicate symbol E,
  - instead of variable A we would use variable y,

  then the formula would look as follows:

$$E(x, y)$$

- Two sets are equal iff they contain the same elements:

$$A = B \leftrightarrow \forall x (x \in A \leftrightarrow x \in B)$$

  If we use predicate E instead of "$\in$", and y and z instead of A and B, the formula would look as follows:

$$y = z \leftrightarrow \forall x (E(x, y) \leftrightarrow E(x, z))$$

- The definition of relation *"to be a subset"* (denoted by symbol "$\subseteq$"):

$$A \subseteq B \leftrightarrow \forall x (x \in A \rightarrow x \in B)$$

  If we use binary predicate symbol S instead of "$\subseteq$":

$$S(y, z) \leftrightarrow \forall x (E(x, y) \rightarrow E(x, z))$$

Predicate Logic — Additional Comments

- The definition of operation *"union"* (denoted by symbol "$\cup$"):

  $$\forall x(x \in A \cup B \ \leftrightarrow\ (x \in A \ \vee\ x \in B))$$

  If we use binary function symbol f instead of "$\cup$":

  $$\forall x(E(x, f(y, z)) \ \leftrightarrow\ (E(x, y) \ \vee\ E(x, z)))$$

Predicate Logic — Additional Comments

- Sometimes

$$\exists x (x \in A \;\wedge\; \ldots\,)$$

is written in an abbreviated way as

$$(\exists x \in A)(\,\ldots\,)$$

I.e., instead of

"there exists x such that $x \in A$ and $\ldots$"

we can say

"there exists $x \in A$ such that $\ldots$"

- Similarly, $\exists x (x \geq 1 \;\wedge\; \ldots)$ can be written in an abbreviated way as

$$(\exists x \geq 1)(\,\ldots\,)$$

Predicate Logic — Additional Comments

- Sometimes

$$\forall x (x \in A \ \rightarrow \ \ldots\ )$$

is written in an abbreviated way as

$$(\forall x \in A)(\ \ldots\ )$$

I.e., instead of

"for each x, for which $x \in A$ holds, we have ..."

we can say

"for each $x \in A$ we have ..."

- Similarly, $\forall x (x \geq 1 \ \rightarrow \ \ldots)$ can be written in an abbreviated way as

$$(\forall x \geq 1)(\ \ldots\ )$$