Predicate Logic

Predicate Logic

- *Fish are vertebrates living in water.*
- *Carps are fish.*
- *There exists at least one carp.*

- *There exists at least one vertebrate living in water.*

- *Triangles are convex polygons.*
- *Equilateral triangles are triangles.*
- *There exists at least one equilateral triangle.*

- *There exists at least one convex polygon.*

Predicate Logic

- *Fish are vertebrates living in water.*
- *Carps are fish.*
- *There exists at least one carp.*

- *There exists at least one vertebrate living in water.*

The use of **variables**:

- *For each x it holds that if x is a fish then x is a vertebrate and x lives in water.*
- *For each x it holds that if x is a carp then x is a fish.*
- *There exists at least one x such that x is a carp.*

- *There exists at least one x such that x is a vertebrate and x lives in water.*

Predicate Logic

- *Triangles are convex polygons.*
- *Equilateral triangles are triangles.*
- *There exists at least one equilateral triangle.*

- *There exists at least one convex polygon.*

The use of **variables**:

- *For each x it holds that if x is a triangle then x is a polygon and x is convex.*
- *For each x it holds that if x is an equilateral triangle then x is a triangle.*
- *There exists at least one x such that x is an equilateral triangle.*

- *There exists at least one x such that x is a polygon and x is convex.*

Predicate Logic

- *For each x it holds that if x has property P then x has property Q and x has property R.*
- *For each x it holds that if x has property S then x has property P.*
- *There exists at least one x such that x has property S.*

- *There exists at least one x such that x has property Q and x has property R.*

P	is a fish	is a triangle
Q	is a vertebrate	is a polygon
R	lives in water	is convex
S	is a carp	is an equilateral triangle

Predicate Logic

- *For each* x *it holds that if* $P(x)$ *then* $Q(x)$ *and* $R(x)$.
- *For each* x *it holds that if* $S(x)$ *then* $P(x)$.
- *There exists* x *such that* $S(x)$.

- *There exists* x *such that* $Q(x)$ *and* $R(x)$.

$P(x)$	x is a fish	x is a triangle
$Q(x)$	x is a vertebrate	x is a polygon
$R(x)$	x lives in water	x is convex
$S(x)$	x is a carp	x is an equilateral triangle

Predicate Logic

- *For each* x, $(P(x) \rightarrow (Q(x) \wedge R(x)))$.
- *For each* x, $(S(x) \rightarrow P(x))$.
- *There exists* x *such that* $S(x)$.

- *There exists* x *such that* $(Q(x) \wedge R(x))$.

$P(x)$	x is a fish	x is a triangle
$Q(x)$	x is a vertebrate	x is a polygon
$R(x)$	x lives in water	x is convex
$S(x)$	x is a carp	x is an equilateral triangle

Predicate Logic

- $\forall x (P(x) \rightarrow (Q(x) \wedge R(x)))$
- $\forall x (S(x) \rightarrow P(x))$
- $\exists x\, S(x)$

- $\exists x (Q(x) \wedge R(x))$

$P(x)$	x is a fish	x is a triangle
$Q(x)$	x is a vertebrate	x is a polygon
$R(x)$	x lives in water	x is convex
$S(x)$	x is a carp	x is an equilateral triangle

- $\forall$ — universal quantifier (*"for all"*)
- $\exists$ — existential quantifier (*"there exists"*)

Predicate Logic

Formulas of propositional logic express propositions about objects with some properties and which can be in some relationships.

Interpretation or **interpretation structure** — a particular set of these objects, their properties and relationships.

Universe — the set of all objects in a given interpretation

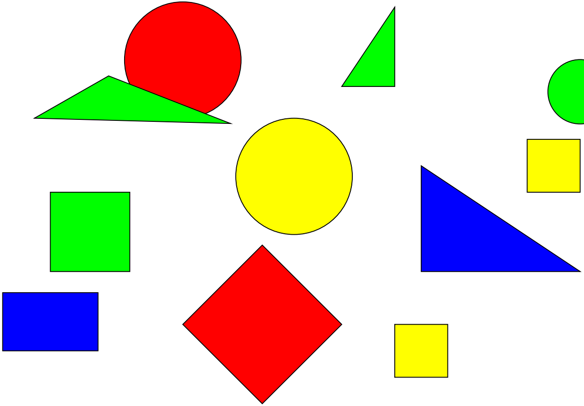
- An arbitrary **non-empty** set can be the universe.
- Objects in a given universe are called the **elements** of the universe.

Valuation — an assignment of elements of the universe to variables

The truth values of formulas depend on a given interpretation and valuation.

Predicate Logic

An example of a universe:



Predicate Logic

Other examples of universes:

- Some precisely specified set of people, for example, the set of people that live in some specified house (*"John Smith"*, *"John Doe"*, . . .)
- The set of all books in a given library.
- The set of natural numbers $\mathbb{N} = \{0, 1, 2, 3, \ldots\}$.
- The set of all points in a plane.
- The set $\{a, b, c, d, e\}$.
- The set $\{a\}$.

Variables

Variables — x, y, z, ..., possibly with indexes — x_0, x_1, x_2, ...

It is assumed that there are infinitely many variables.

Valuation — an assignment of elements of the universe to the variables

Example:

- Universe — a set of people; valuation v, where:

  $v(x) =$ *"John Doe"*
  $v(y) =$ *"Mary Smith"*
  ...

- Universe — the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$; valuation v, where

  $$v(x) = 57 \qquad v(y) = 3 \qquad v(z) = 57 \qquad \ldots$$

Predicates

Predicates — P, Q, R, ...

- **Unary predicates** — they represent **properties** of elements of the universe

  Example: Predicate P representing the property *"to be blue"*:

  $$P(x) \quad — \quad \text{"}x \textit{ is blue"}$$

  A unary predicate assigns truth values to the elements of the universe.

  E.g., the value of $P(x)$ can be:

  - 1 — the element assigned to variable x has property P (i.e., it is blue)
  - 0 — the element assigned to variable x does not have this property P (i.e., it is not blue)

Predicates

- **Binary predicates** — they represent **relationships** between pairs of elements of the universe

  Example: Predicate R representing the relationship *"to be a parent of"*:

  $$R(x, y) \quad — \quad \text{“}x \text{ is a parent of } y\text{”}$$

  A binary predicate assigns truth values to pair of elements of the universe.

  E.g., the value of $R(x, y)$ can be:

  - 1 — when x and y are in the given relationship (i.e., when x is a parent of y)
  - 0 — when x and y are not in the given relationship (i.e., when x is not a parent of y)

Predicates

We can consider predicates of arbitrary arities.

For example:

- **Ternary** predicate T (i.e., predicate of arity 3) representing the relationship between parents and their child:

  $$T(x, y, z)$$

  — x and y are parents of child z, and x is his/her mother and y is his/her father

- **Nulary** predicates (i.e., precates of arity 0) can be viewed as atomic propositions, not related to the elements of the universe.

Formulas of Predicate Logic

Atomic formula — a predicate applied on some variables

Example:

- P — a unary predicate representing property *"to be blue"*
- Q — a unary predicate representing propery *"to be a square"*
- R — a binary predicate representing relationships *"overlaps"*

$$
\begin{array}{ccl}
P(x) & — & \text{“}x\text{ is blue”} \\
P(y) & — & \text{“}y\text{ is blue”} \\
Q(y) & — & \text{“}y\text{ is a square”} \\
R(z,x) & — & \text{“}z\text{ overlaps }x\text{”} \\
R(y,y) & — & \text{“}y\text{ overlaps itself”}
\end{array}
$$

Remark: Later, we will extend the notion of an atomic formula a little bit.

Formulas of Predicate Logic

Using **logical connectives** ("$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"), more complicated formulas can be created from simpler formulas, similarly as in propositional logic.

Example:

- P — unary predicate representing property *"is blue"*
- Q — unary predicate representing property *"is a square"*
- R — binary predicate representing relationship *"overlaps"*

"If x is a blue square or y does not overlap x, then z is not a square."

$$\big((P(x) \wedge Q(x)) \vee \neg R(y,x)\big) \rightarrow \neg Q(z)$$

Formulas of Predicate Logic

Using **logical connectives** ("$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"), more complicated formulas can be created from simpler formulas, similarly as in propositional logic.

Example:

- P — unary predicate representing property *"is a woman"*
- Q — unary predicate representing property *"has dark hair"*
- R — binary predicate representing relationship *"is a parent of"*

"If x is a woman with dark hair or y is not a parent of x, then z does not have dark hair."

$$\big((P(x) \wedge Q(x)) \vee \neg R(y,x)\big) \rightarrow \neg Q(z)$$

Formulas of Predicate Logic

Using **logical connectives** ("$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\leftrightarrow$"), more complicated formulas can be created from simpler formulas, similarly as in propositional logic.

Example:

- P — unary predicate representing property *"is even"*
- Q — unary predicate representing property *"is a prime"*
- R — binary predicate representing relationship *"is greater than"*

"If x is an even prime or y is not greater than x, then z is not a prime."

$$\big((P(x) \wedge Q(x)) \vee \neg R(y,x)\big) \rightarrow \neg Q(z)$$

Quantifiers

Universal quantifier — symbol "$\forall$"

If φ is a formula representing some proposition then

$$\forall x \varphi$$

is a formula representing proposition

"for every x φ holds".

Example: P — *"to be a square"*

$$\forall x\, P(x)$$

- *"For every x it holds that x is a square."*
- *"Every x is a square."*
- *"All elements are squares."*

Quantifiers

Example:

- *"For every* x *it holds that if* x *is a square then* x *is green."*
- *"For each* x *it holds that if* x *is a square then* x *is green."*
- *"For all* x *it holds that if* x *is a square then* x *is green."*
- *"All squares are green."*

$$\forall x(P(x) \rightarrow Q(x))$$

- P — *"to be a square"* (arity 1)
- Q — *"to be green"* (arity 1)

Quantifiers

Example:

- *"If it holds for all x that x is a square or x is green then it holds for all y that y is a triangle."*
- *"If every object is a square or is green then all objects are triangles."*

$$\forall x(P(x) \vee Q(x)) \rightarrow \forall y T(y)$$

- P — *"to be a square"* (arity 1)
- Q — *"to be green"* (arity 1)
- T — *"to be a triangle"* (arity 1)

Quantifiers

There is a big difference between the following formulas:

- $P(x)$ — *"x is a square"*

  It claims something about **one** particular element assigned to variable x.

  The truth value of this claim depends on the particular element assigned to variable x, i.e., on the particular valuation.

- $\forall x P(x)$ — *"every x is a square"* (i.e., *"all elements are squares"*)

  It claims something about **all** elements of the universe.

  The truth value of this claim does not depend on a valuation.

Quantifiers

Example:

- *"If x is a prime then x is odd."*

$$P(x) \rightarrow L(x)$$

- *"For every x it holds that if x is a prime then it is odd"*. (I.e., *"all primes are odd"*.)

$$\forall x(P(x) \rightarrow L(x))$$

Predicates:

- P — *"to be a prime"* (arity 1)
- L — *"to be odd"* (arity 1)

Quantifiers

Example:

- *"It holds for every y that if y is green then x overlaps y."*
- *"Object x overlaps all green objects."*

$$\forall y(G(y) \rightarrow R(x, y))$$

Predicates:

- R — *"overlaps"* (arity 2)
- G — *"to be green"* (arity 1)

Quantifiers

Example:

- *"It holds for every x that it holds for every y that if x is a parent of y then x loves y."*
- *"It holds for each x and y that if x is a parent of y then x loves y."*
- *"For every pair of elements x and y it holds that if x is a parent of y then x loves y."*

$$\forall x \forall y (R(x, y) \rightarrow S(x, y))$$

Predicates:

- R — *"is a parent"* (arity 2)
- S — *"loves"* (arity 2)

Quantifiers

Existential quantifier — symbol "$\exists$"

If φ is a formula representing some proposition then

$$\exists x \varphi$$

is a formula representing proposition

"there exists x, for which φ holds".

Example: P — *"to be a square"*

$$\exists x\, P(x)$$

- *"There exists x, for which it holds that x is a square."*
- *"There is x such that x is a square."*
- *"There exists at least one square."*

Quantifiers

Example:

- *"There exists* x*, for which it holds that* x *is a square and* x *is green."*
- *"There is* x *such that* x *is a square and* x *is green."*
- *"For some* x *it holds that* x *is a square and* x *is green."*
- *"There exists a green square."*
- *"Some squares are green."*
- *"At least one* x *is a green square."*

$$\exists x(P(x) \land Q(x))$$

Predicates:

- P — *"to be a square"* (arity 1)
- Q — *"to be green"* (arity 1)

Quantifiers

Example:

- *"There exists* x *such that for each* y *it holds that* x *is greater than* y*."*

$$\exists x \forall y\, P(x, y)$$

- *"For each* y *there is* x *such that* x *is greater than* y*."*

$$\forall y \exists x\, P(x, y)$$

P — *"to be greater than"* (arity 2)

Syntax of Formulas of Predicate Logic

Alphabet:

- **logical connectives** — "$\neg$", "$\wedge$", "$\vee$" "$\rightarrow$", "$\leftrightarrow$"
- **quantifiers** — "$\forall$", "$\exists$"
- **auxiliary symbols** — "$($", "$)$", "$,$"
- **variables** — "x", "y", "z", ..., "x_0", "x_1", "x_2", ...
- **predicate symbols** — for example symbols "P", "Q", "R", etc. (for each symbol, its arity must be specified)
- ...

Remark: Other types of symbols will be described later.

Syntax of Formulas of Predicate Logic

Definition

Well-formed **atomic formulas** of predicate logic are formulas of the form:

- $P(x_1, x_2, \ldots, x_n)$, where P is a predicate symbol of arity n and $x_1, x_2, \ldots, x_n$ are (not necessarily different) variables.
- $\ldots$

Remark: This is not the whole definition. Later, it will be generalized a little bit and some additional items will be added.

Example:

$$P(x, y) \qquad\qquad R(z, z, z) \qquad\qquad S(y)$$

Syntax of Formulas of Predicate Logic

Definition

Well-formed **formulas of predicate logic** are sequences of symbols constructed according to the following rules:

1. Well-formed atomic formulas are well-formed formulas.
2. If φ and ψ are well-formed formulas, then also $(\neg\varphi)$, $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$ a $(\varphi \leftrightarrow \psi)$ are well-formed formulas.
3. If φ is a well-formed formula and x is a variable, then $\forall x \varphi$ and $\exists x \varphi$ are well-formed formulas.
4. There are no other well-formed formulas than those constructed according to the previous rules.

Syntax of Formulas of Predicate Logic

Notions like

- subformulas
- an abstract syntax tree

are introduced in a similar way like in propositional logic (they are only extended with the additional constructions not present in propositional logic).

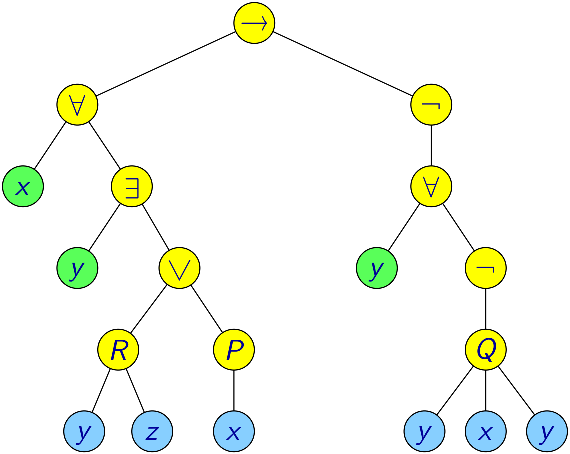
Convention for omitting parentheses:

- The same conventions as in propositional logic.
- Quantifiers ("$\forall$" and "$\exists$") have the same priority as negation ("$\neg$"), i.e., the highest priority.

Syntax of Formulas of Predicate Logic

An abstract syntax tree of formula

$$\forall x \exists y (R(y,z) \vee P(x)) \rightarrow \neg \forall y \neg Q(y,x,y)$$



Free and Bound Occurrences of Variables

Every occurrence of variable x in a subformula of the form $\exists x \varphi$ or $\forall x \varphi$ is **bound**.

An occurrence of a variable, which is not bound, is **free**.

Example: Formula

$$\forall x \exists y (R(y, z) \vee P(x)) \rightarrow \neg \forall y \neg Q(y, x, y)$$

- y in subformula $R(y, z)$ — the bound occurrence ($\exists y$)
- z in subformula $R(y, z)$ — the free occurrence
- x in subformula $P(x)$ — the bound occurrence ($\forall x$)
- both occurrences of y in subformula $Q(y, x, y)$ — the bound occurrences ($\forall y$)
- x in subformula $Q(y, x, y)$ — the free occurrence

Free and Bound Occurrences of Variables

The set of those variables, which occur as **free** variables in formula φ, will be denoted $\mathit{free}(\varphi)$.

Example:

- If φ is formula $P(x, y)$, then $\mathit{free}(\varphi) = \{x, y\}$.
- If ψ is formula $\exists x \exists y P(x, y)$, then $\mathit{free}(\psi) = \emptyset$.
- If χ is formula

$$\forall x \exists y (R(y, z) \vee P(x)) \rightarrow \neg \forall y \neg Q(y, x, y)$$

  then $\mathit{free}(\chi) = \{x, z\}$.

Free and Bound Occurrences of Variables

The set of free variables $\mathit{free}(\varphi)$ can be described by the following inductive definition:

- $\mathit{free}(P(x_1, x_2, \ldots, x_n)) = \{x_1, x_2, \ldots, x_n\}$
  (where P is a predicate symbol)
- $\mathit{free}(\neg\varphi) = \mathit{free}(\varphi)$
- $\mathit{free}(\varphi \wedge \psi) = \mathit{free}(\varphi) \cup \mathit{free}(\psi)$
  (it is similar for formulas of the form $\varphi \vee \psi$, $\varphi \rightarrow \psi$, and $\varphi \leftrightarrow \psi$)
- $\mathit{free}(\forall x \varphi) = \mathit{free}(\varphi) - \{x\}$ (where x is a variable)
- $\mathit{free}(\exists x \varphi) = \mathit{free}(\varphi) - \{x\}$ (where x is a variable)

Free and Bound Occurrences of Variables

A formula φ is **closed** if it contains no free occurrences of variables (i.e., when $\mathit{free}(\varphi) = \emptyset$).

A formula φ is **open** if it is not closed (i.e., when $\mathit{free}(\varphi) \neq \emptyset$).

Remark: Closed formulas are sometimes also called **sentences**.

Example:

- Formula $\exists x \exists y P(x, y)$ is closed.
- Formula $\forall x \exists y (R(y, z) \vee P(x)) \rightarrow \neg \forall y \neg Q(y, x, y)$ is open (because it contains free occurrences of variables z and x).

Truth values of closed formulas do not depend on a valuation, only on an interpretation.

Semantics of Predicate Logic

Formulas are evaluated in a given **interpretation** (**interpretation structure**) and **valuation**.

The fact that formula φ holds (i.e., it has truth value 1) in interpretation $\mathcal{A}$ and valuation v, is denoted

$$\mathcal{A}, v \models \varphi$$

The fact that formula φ does not hold (i.e., it has truth value 0) in interpretation $\mathcal{A}$ and valuation v, is denoted $\mathcal{A}, v \not\models \varphi$.

Semantics of Predicate Logic

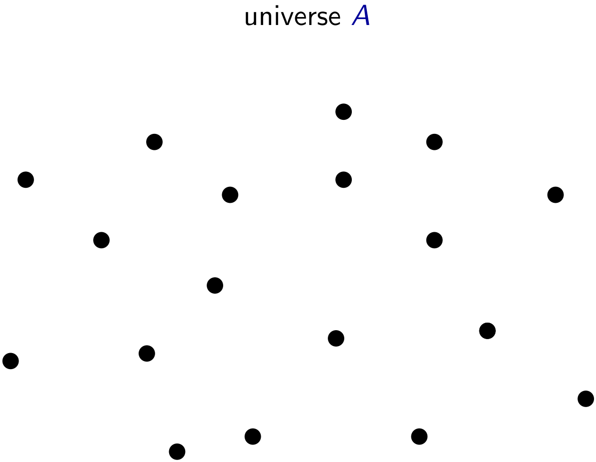
An **interpretation** $\mathcal{A}$ is a structure consisting of the following items:

- **Universe** A — an arbitraty non-empty set
- Some subset of the set A is assigned to every unary predicate symbol P — it is denoted $P^{\mathcal{A}}$.
  (And so $P^{\mathcal{A}} \subseteq A$.)
- Some binary relation on A is assigned to every binary predicate symbol Q — it is denoted $Q^{\mathcal{A}}$.
  (And so $Q^{\mathcal{A}} \subseteq A \times A$.)
- It is similar for predicate symbols with other arities (3, 4, 5, ...).

Remark: This definition is not complete yet, and it will be later extended by other items.

Semantics of Predicate Logic

An example of an interpretation $\mathcal{A}$:



Semantics of Predicate Logic

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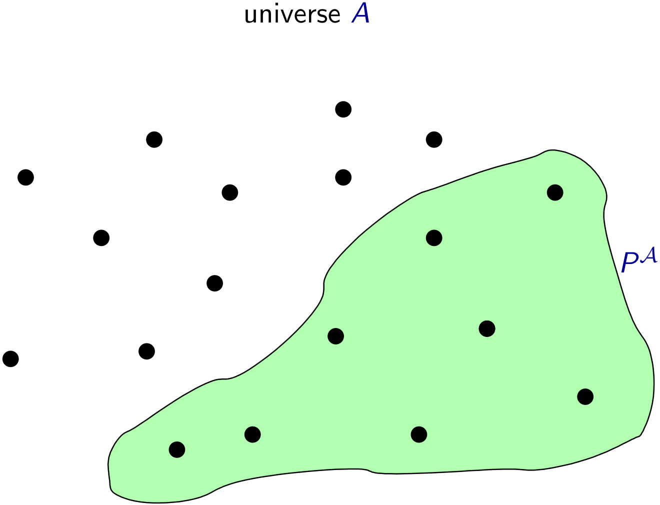
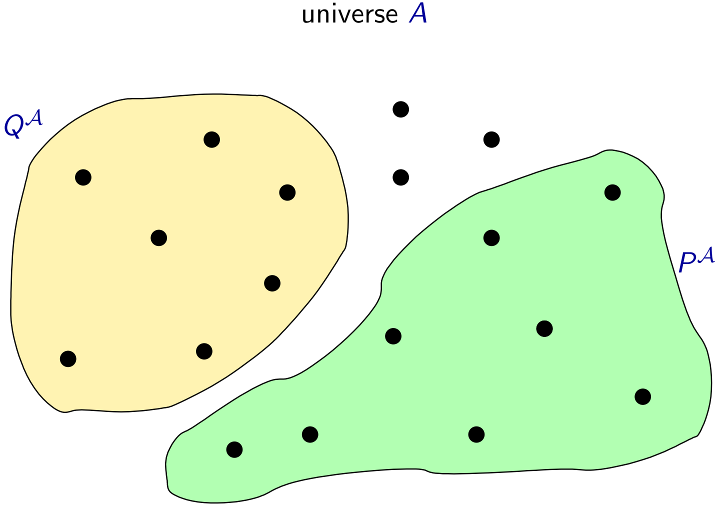


Semantics of Predicate Logic

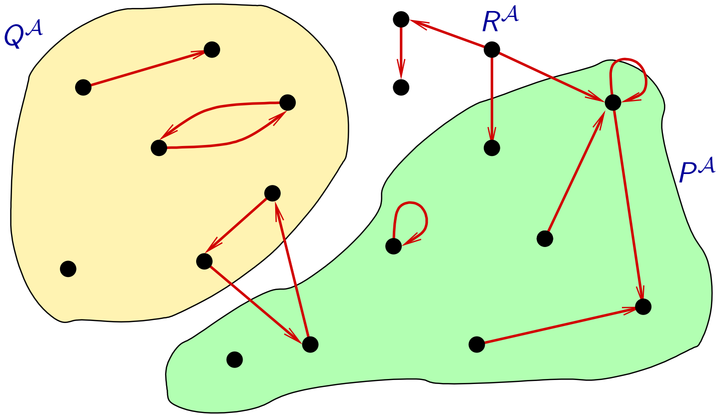
An example of an interpretation $\mathcal{A}$:



Semantics of Predicate Logic

An example of an interpretation $\mathcal{A}$:

universe A



Semantics of Predicate Logic

Other example of an interpretation $\mathcal{A}$:

- universe $A = \{a, b, c, d, e, f, g\}$
- $P^{\mathcal{A}} = \{b, d, e\}$
- $Q^{\mathcal{A}} = \{a, b, e, g\}$
- $R^{\mathcal{A}} = \{(a, b), (a, e), (a, g), (b, b), (c, e), (f, c), (f, g), (g, a), (g, g)\}$

Semantics of Predicate Logic

Let Var be the set of all variables, i.e.,

$$\mathit{Var} = \{x, y, z, \ldots, x_0, x_1, x_2, \ldots\}$$

For a given interpretation $\mathcal{A}$ with a universe A, a **valuation** v is an arbitrary function

$$v : \mathit{Var} \to A$$

that assignes elements of the universe to the variables.

Remark: As we will see, in fact, only values assigned by the valuation v to variables in $\mathit{free}(\varphi)$ are important for determining the truth value of formula φ.

Values assigned by valuation v to the other variables are not important from this point of view.

Semantics of Predicate Logic

Let us consider an interpretation $\mathcal{A}$ with universe A and a valuation v.

Lets assume that (i.e., $x \in \mathit{Var}$) and a is an element of the universe (i.e., $a \in A$).

Notation

$$v[x \mapsto a]$$

denotes the valuation $v' : \mathit{Var} \to A$, which assignes to every variable the same value as valuation v, except that it assignes value a to variable x.

I.e., for every variable y (where $y \in \mathit{Var}$) is

$$v'(y) = \begin{cases} a & \text{if } y = x \\ v(y) & \text{otherwise} \end{cases}$$

Example:

- universe $A = \{a, b, c, d, e, f, g, \ldots\}$

  valuation v:

  $$v(x_0) = c \qquad v(x_1) = e \qquad v(x_2) = b \qquad v(x_3) = e \qquad \ldots$$

  valuation $v[x_2 \mapsto g]$:

  $$v(x_0) = c \qquad v(x_1) = e \qquad v(x_2) = g \qquad v(x_3) = e \qquad \ldots$$

Semantics of Predicate Logic

Definition

Let us assume an interpretation $\mathcal{A}$ with universe A and a valuation v, assigning elements of the universe A to the variables.

The **truth values of formulas of predicate logic** in interpretation $\mathcal{A}$ and valuation v are defined as follows:

- For a predicate P of arity n, $\mathcal{A}, v \models P(x_1, x_2, \ldots, x_n)$ iff $(v(x_1), v(x_2), \ldots, v(x_n)) \in P^{\mathcal{A}}$.
- $\mathcal{A}, v \models \neg\varphi$ iff $\mathcal{A}, v \not\models \varphi$.
- $\mathcal{A}, v \models \varphi \wedge \psi$ iff $\mathcal{A}, v \models \varphi$ and $\mathcal{A}, v \models \psi$.
- $\mathcal{A}, v \models \varphi \vee \psi$ iff $\mathcal{A}, v \models \varphi$ or $\mathcal{A}, v \models \psi$.
- $\mathcal{A}, v \models \varphi \rightarrow \psi$ iff $\mathcal{A}, v \not\models \varphi$ or $\mathcal{A}, v \models \psi$.
- $\mathcal{A}, v \models \varphi \leftrightarrow \psi$ iff $\mathcal{A}, v \models \varphi$ and $\mathcal{A}, v \models \psi$, or $\mathcal{A}, v \not\models \varphi$ and $\mathcal{A}, v \not\models \psi$.
- $\ldots$

Semantics of Predicate Logic

Definition (cont.)

- . . .
- $\mathcal{A}, v \models \forall x \varphi$ iff for **every** $a \in A$ it holds that $\mathcal{A}, v[x \mapsto a] \models \varphi$.
- $\mathcal{A}, v \models \exists x \varphi$ iff there **exists** some $a \in A$ such that $\mathcal{A}, v[x \mapsto a] \models \varphi$.

Semantics of Predicate Logic

A closed formula φ is **true** (i.e., it has truth value 1) in interpretation $\mathcal{A}$ if it holds for each valuation v that $\mathcal{A}, v \models \varphi$.

The fact that formula φ is true in interpretation $\mathcal{A}$ is denoted

$$\mathcal{A} \models \varphi\,.$$

Remark: A truth value of a closed formula in a given interpretation does not depend on a valuation.

Consider a closed formula φ.
A **model** of the formula φ is an arbitrary interpretation $\mathcal{A}$ such that $\mathcal{A} \models \varphi$.

Evaluation of Truth of Formulas as a Game

Let us consider a formula of the form

$$\exists x_1 \forall x_2 \exists x_3 \forall x_4 \cdots \exists x_{n-1} \forall x_n \varphi,$$

where quantifiers alternate in some arbitrary way, and where φ does not contain quantifiers.

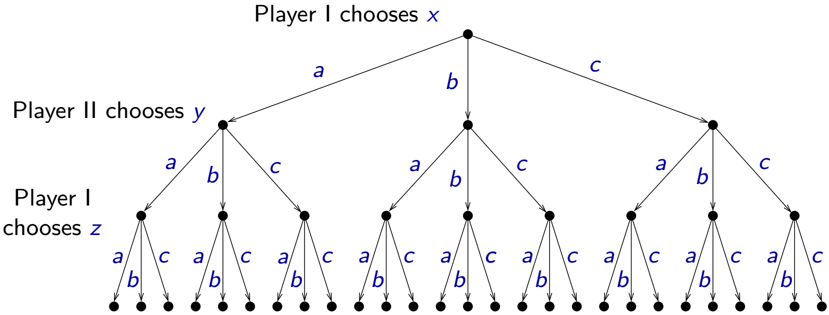
The evaluation of truth values of formulas of this form (in a given interpretation $\mathcal{A}$ and a valuation v) can be viewed as a game:

- It is played by a pair of players — **Player I** and **Player II**.
- Player I wants to show that the formula is true.
- Player II wants to show that the formula is false.
- Player I chooses values of those variables, which are bound by an existential quantifier ($\exists$).
- Player II chooses values of those variables, which are bound by an universal quantifier ($\forall$).

Evaluation of Truth of Formulas as a Game

Example: Formula $\exists x \forall y \exists z (P(x,y) \rightarrow Q(y,z))$

universe $A = \{a, b, c\}$



Evaluation of Truth of Formulas as a Game

- Formula φ is true iff Player I has a **winning strategy** in this game.
- Formula φ is false iff Player II has a winning strategy.

Strategy — determines how a player should play in every situation, i.e., it determines moves of the player for all possible moves of the other player.

Winning strategy — a strategy that guarantees a win of the given player in every play, not matter what the other player does.

Evaluation of Truth of Formulas as a Game

Example: Interpretation where universe is the set of real numbers $\mathbb{R}$ and binary predicate symbol R represents relation *"greater or equal"* (i.e., $R(x, y)$ iff $x \geq y$).

Formula $\exists x \forall y R(x, y)$ — a winning strategy of Player II:

- Player I chooses number x.
- Player II chooses number $y = x + 1$ — Player II wins since it is obviously not true that $x \geq x + 1$.

Formula $\forall y \exists x R(x, y)$ — a winning strategy of Player I:

- Player II chooses number y.
- Player I chooses number $x = y$ — Player I wins since it is obviously true that $x \geq x$.

Logically Valid Formulas

A formula φ is **logically valid** if it has truth value 1 in every interpretation and valuation, i.e., if for every interpretation $\mathcal{A}$ and valuation v is

$$\mathcal{A}, v \models \varphi.$$

Example:

- $\exists x P(x) \rightarrow \exists y P(y)$
- $\forall x P(x) \wedge \neg \exists y Q(y) \rightarrow \forall z (P(z) \wedge \neg Q(z))$
- $\forall x P(x) \rightarrow \exists x P(x)$

Logically Valid Formulas

If we take an arbitrary tautology of propositional logic and replace in it all atomic propositions with arbitrary formulas of predicate logic, we obtain a logically valid formula.

Example: Tautology $p \rightarrow (q \vee p)$

- p is replaced with $\forall z(P(x,z) \leftrightarrow \neg Q(z,y))$
- q is replaced with $R(x)$

We obtain a logically valid formula

$$\forall z(P(x,z) \leftrightarrow \neg Q(z,y)) \ \rightarrow \ (R(x) \vee \forall z(P(x,z) \leftrightarrow \neg Q(z,y)))$$

Logically Equivalent Formulas

Formulas φ and ψ are **logically equivalent** if they have the same truth values in every interpretation and valuation, i.e., if for every interpretation $\mathcal{A}$ and valuation v is

$$\mathcal{A}, v \models \varphi \qquad \text{iff} \qquad \mathcal{A}, v \models \psi.$$

The fact that φ and ψ are logically equivalent is denoted

$$\varphi \Leftrightarrow \psi.$$

- Similarly as in propositional logic, we can do equivalent transformations in predicate logic.
- All equivalences that hold in propositional logic also hold in predicate logic.

Logically Equivalent Formulas

- There are other equivalences in predicate logic that have no analogy in propositional logic.

Examples of some important equivalences:

$$\neg\forall x\varphi \;\Leftrightarrow\; \exists x\neg\varphi$$
$$\neg\exists x\varphi \;\Leftrightarrow\; \forall x\neg\varphi$$

$$\forall x\forall y\varphi \;\Leftrightarrow\; \forall y\forall x\varphi$$
$$\exists x\exists y\varphi \;\Leftrightarrow\; \exists y\exists x\varphi$$

When $x \notin \mathit{free}(\varphi)$:

$$\forall x\varphi \;\Leftrightarrow\; \varphi$$
$$\exists x\varphi \;\Leftrightarrow\; \varphi$$

Logically Equivalent Formulas

Some other important equivalences:

$$(\forall x \varphi) \wedge (\forall x \psi) \ \Leftrightarrow\ \forall x(\varphi \wedge \psi)$$
$$(\exists x \varphi) \vee (\exists x \psi) \ \Leftrightarrow\ \exists x(\varphi \vee \psi)$$

When $x \notin \mathit{free}(\psi)$:

$$(\forall x \varphi) \wedge \psi \ \Leftrightarrow\ \forall x(\varphi \wedge \psi)$$
$$(\forall x \varphi) \vee \psi \ \Leftrightarrow\ \forall x(\varphi \vee \psi)$$
$$(\exists x \varphi) \wedge \psi \ \Leftrightarrow\ \exists x(\varphi \wedge \psi)$$
$$(\exists x \varphi) \vee \psi \ \Leftrightarrow\ \exists x(\varphi \vee \psi)$$

Renaming of Bound Variables

If we rename a bound variable in a formula, we obtain an equvalent formula.

Example: $\forall x P(x, y) \;\Leftrightarrow\; \forall z P(z, y)$

- If we rename for example x to y in formula $\forall x \varphi$ or $\exists x \varphi$, the variable y **must not** occur in formula φ as a free variable.

  $\exists x P(x, y)$ is not equivalent to $\exists y P(y, y)$

- Free occurrences of variables in a subformula **must not** become bound after renaming. E.g.,

  $\exists x \forall y P(x, y)$ is not equivalent to $\exists y \forall y P(y, y)$

Substitution

Let us say that we want replace **free** occurrences of variable x with variable y (i.e., we want to substitute y for x).

This operation on formulas is called **substitution** and the resulting formula is denoted

$$\varphi[y/x].$$

Remark: In general, formulas φ and $\varphi[y/x]$ are **not** equivalent.

Example:

$P(x,z)$ is not equivalent to $P(y,z)$

Renaming of Bound Variables

With the operation of substitution, the renaming of bound variables can be described by the following equivalences.

When $y \notin \mathit{free}(\forall x \varphi)$:

$$\forall x \varphi \quad \Leftrightarrow \quad \forall y (\varphi[y/x])$$

When $y \notin \mathit{free}(\exists x \varphi)$:

$$\exists x \varphi \quad \Leftrightarrow \quad \exists y (\varphi[y/x])$$

Example:

$$\exists x \forall y P(x, y) \quad \Leftrightarrow \quad \exists x \forall z P(x, z) \quad \Leftrightarrow \quad \exists y \forall z P(y, z) \quad \Leftrightarrow \quad \exists y \forall x P(y, x)$$

Logical Entailment

Definition

Conclusion ψ **logically follows** from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, which is denoted

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi,$$

if in every interpretation $\mathcal{A}$ and valuation v where assumption $\varphi_1, \varphi_2, \ldots, \varphi_n$ are true, also the conclusion ψ is true.

- All, what was said about the logical entailment in propositional logic, holds all analogously in predicate logic.

Logical Entailment

If we want to show that a given conclusion ψ **does not** follow from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, it is sufficient to find an example of one particular interpretation $\mathcal{A}$ and valuation v, where the assumptions are true and the conclusion ψ is false.

Example:

- *There exists an aquatic animal, which is meat-eating.*
- *All fish are aquatic animals.*

- *There exists a meat-eating fish.*

$$\frac{\begin{array}{c}\exists x(P(x) \wedge Q(x)) \\ \forall x(R(x) \rightarrow P(x))\end{array}}{\exists x(R(x) \wedge Q(x))}$$

$P(x)$ — "*x is an aquatic animal*"
$Q(x)$ — "*x is meat-eating*"
$R(x)$ — "*x is a fish*"

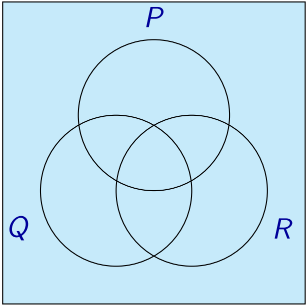
An interpretation $\mathcal{A}$ with universe $A = \{a, b\}$

$$P^{\mathcal{A}} = \{a, b\} \qquad Q^{\mathcal{A}} = \{a\} \qquad R^{\mathcal{A}} = \{b\}$$

Venn Diagrams

In general, it is difficult to find out whether a conclusion does or does not follow from given assumptions.

In cases when we have only unary predicates and there is only a small number of them (e.g., 3), we use so called **Venn diagrams** as an aid for the reasoning.



Venn Diagrams

Example:

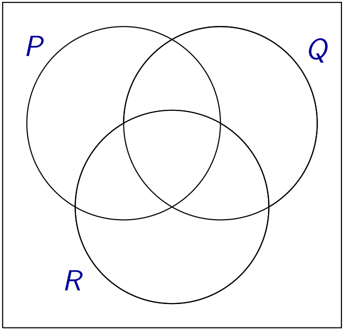
- *Fish are vertebrates.*
- *Fish live in water.*
- *There exists at least one fish.*

- *There exists a vertebrate living in water.*

$$\forall x(P(x) \rightarrow Q(x))$$
$$\forall x(P(x) \rightarrow R(x))$$
$$\exists x P(x)$$

$$\exists x(Q(x) \wedge R(x))$$

$P(x)$ — *"x is a fish"*
$Q(x)$ — *"x is a vertebrate"*
$R(x)$ — *"x lives in water"*



Venn Diagrams

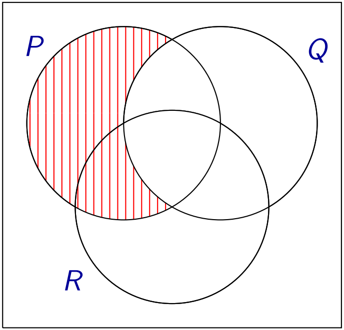
Example:

- *Fish are vertebrates.*
- *Fish live in water.*
- *There exists at least one fish.*

- *There exists a vertebrate living in water.*

$$\forall x(P(x) \rightarrow Q(x))$$
$$\forall x(P(x) \rightarrow R(x))$$
$$\exists x P(x)$$
$$\overline{\qquad\qquad\qquad}$$
$$\exists x(Q(x) \wedge R(x))$$

$P(x)$ — "x *is a fish*"
$Q(x)$ — "x *is a vertebrate*"
$R(x)$ — "x *lives in water*"



Venn Diagrams

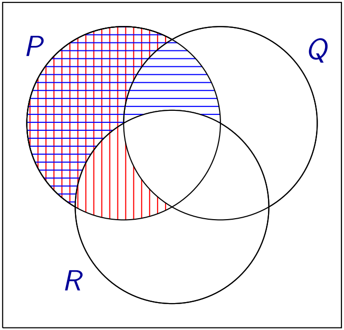
Example:

- *Fish are vertebrates.*
- *Fish live in water.*
- *There exists at least one fish.*

- *There exists a vertebrate living in water.*

$$\begin{array}{l} \forall x(P(x) \rightarrow Q(x)) \\ \forall x(P(x) \rightarrow R(x)) \\ \exists x P(x) \\ \hline \exists x(Q(x) \wedge R(x)) \end{array}$$

$P(x)$ — "*x is a fish*"
$Q(x)$ — "*x is a vertebrate*"
$R(x)$ — "*x lives in water*"



Venn Diagrams

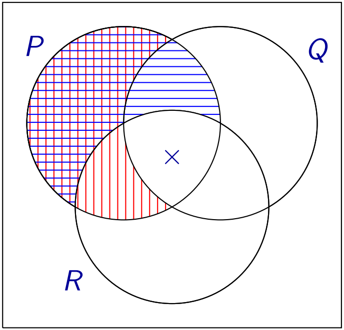
Example:

- *Fish are vertebrates.*
- *Fish live in water.*
- *There exists at least one fish.*

- *There exists a vertebrate living in water.*

$$\forall x(P(x) \rightarrow Q(x))$$
$$\forall x(P(x) \rightarrow R(x))$$
$$\exists x P(x)$$
$$\overline{\exists x(Q(x) \wedge R(x))}$$

$P(x)$ — *"x is a fish"*
$Q(x)$ — *"x is a vertebrate"*
$R(x)$ — *"x lives in water"*



An Example of a Proof

$$\frac{\begin{array}{l}\forall x(\neg R(x,x)) \\ \forall x \forall y \forall z(R(x,y) \wedge R(y,z) \rightarrow R(x,z))\end{array}}{\forall x \forall y(R(x,y) \rightarrow \neg R(y,x))}$$

1.	$\forall x(\neg R(x,x))$	- assumption 1
2.	$\forall x \forall y \forall z(R(x,y) \wedge R(y,z) \rightarrow R(x,z))$	- assumption 2
3.	Lets assume arbitrary elements x and y:	
4.	Lets assume $R(x,y)$:	
5.	Lets assume $R(y,x)$:	
6.	$R(x,y) \wedge R(y,x) \rightarrow R(x,x)$	- from 2.
7.	$R(x,x)$	- from 4., 5., 6.
8.	$\neg R(x,x)$	- from 1.
9.	$\neg R(y,x)$	- contradiction of 7. and 8., so 5. does not hold
10.	$R(x,y) \rightarrow \neg R(y,x)$	- from 4., 9.
11.	$\forall x \forall y(R(x,y) \rightarrow \neg R(y,x))$	- from 3., 10.