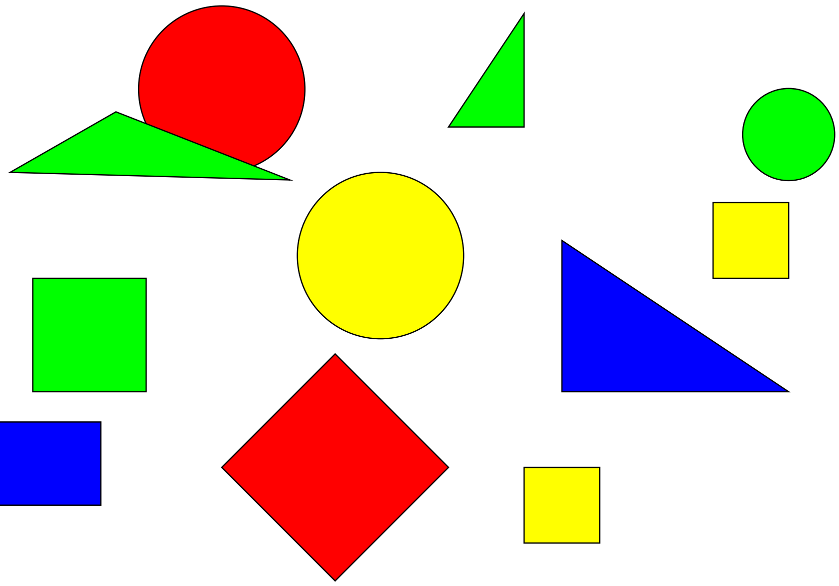
Tutorial 5

Exercise 1: Consider a particular interpretation where the universe is the set of geometric figures depicted in the following picture.



Objects in this universe are situated in a plane, have different colors and can overlap.
Lets assume we have the following predicates:

- S — unary predicate representing property *„to be a square“* (i.e., $S(x)$ means „x *is a square*“)
- G — unary predicate representing property *„to be green“* (i.e., $G(x)$ means „x *is green*“)
- R — unary predicate representing property *„to be red“* (i.e., $R(x)$ means „x *is red*“)
- O — binary predicate representing relation *„to partially overlap“* (i.e., $O(x, y)$ means „x *partially overlaps* y*, resp. „object* y *is partially overlapped with object* x““)

For each of the following formulas, do the following:

- Formulate the proposition expressed by the given formula in a natural language.
- Determine if the given proposition is true in the interpretation described above.

  In some cases, the truth value of the proposition depends on a particular valuation, i.e., on particular values assigned to variables. In such cases, choose some possible valuation and determine the truth value in this chosen valuation.

  If this is possible, try to find an example of valuation where the formula is true, and an example of valuation where the formula is false.

1. $S(x)$
2. $\forall x S(x)$
3. $\exists x S(x)$
4. $R(x) \rightarrow S(x)$
5. $\exists x(G(x) \land S(x))$
6. $\forall x(S(x) \rightarrow R(x))$

7. $\exists x(R(x) \vee G(x))$
8. $\forall x(S(x) \rightarrow \neg G(x) \wedge \neg R(x))$
9. $\exists x \exists y O(x, y)$
10. $\forall x \forall y(O(x, y) \vee O(y, x))$
11. $\exists x \forall y O(y, x)$
12. $\forall x \exists y O(x, y)$
13. $\forall x(\exists y O(y, x) \rightarrow \exists z O(x, z))$
14. $\forall x(S(x) \rightarrow O(x, y))$

Exercise 2: Consider the same predicates as in the previous exercise.
Write down the following propositions as formulas of predicate logic:

a) There exists a green square.

Solution: $\exists x(G(x) \wedge S(x))$

b) Every square is red or green.

Solution: $\forall x(S(x) \rightarrow (R(x) \vee G(x)))$

c) There exists an object that partially overlaps object x.

Solution: $\exists y O(y, x)$

d) For each green object there exists a red object that partially overlaps it.

Solution: $\forall x(G(x) \rightarrow \exists y(R(y) \wedge O(y, x)))$

e) It holds for each pair of objects x and y that x partially overlaps y or y partially overlaps x.

Solution: $\forall x \forall y(O(x, y) \vee O(y, x))$

f) There is no object that partially overlaps itself.

Solution: $\neg \exists x O(x, x)$

g) If some object is red then all objects are red.

Solution: $\exists x R(x) \rightarrow \forall x R(x)$

h) For each square it holds that if it is not green then it is not partially overlaped with any object.

Solution: $\forall x(S(x) \rightarrow (\neg G(x) \rightarrow \neg \exists y O(y, x)))$

i) If x partially overlaps y then there exists z such that z partially overlaps y and x partially overlaps z.

Solution: $O(x, y) \rightarrow \exists z(O(z, y) \wedge O(x, z))$

j) It is not the case that all squares are green.

Solution: $\neg \forall x(S(x) \rightarrow G(x))$

Exercise 3: Consider the following predicates:

- M — unary predicate representing propery *„to be a man (male)“*
- W — unary predicate representing property *„to be a woman (female)“*

- P — binary predicate representing relationship „*to be a parent of*“ (i.e., $\mathsf{P}(x, y)$ means „x *is a parent of* y“)
- Q — binary predicate representing relationship „*to be a sibling of*“ (i.e., $\mathsf{Q}(x, y)$ means „x *is a sibling of* y“)
- R — binary predicate representing relationship „*loves*“ (i.e., $\mathsf{R}(x, y)$ means „x *loves* y“)

Represent the following propositions by formulas of predicate logic:

a) If x is a parent of y then x loves y.

Solution: $\mathsf{P}(x, y) \rightarrow \mathsf{R}(x, y)$

b) For each woman x and each y such that x is a parent of y there exists a man, which is a parent of y.

Solution: $\forall x(W(x) \rightarrow \forall y(\mathsf{P}(x, y) \rightarrow \exists z(\mathsf{M}(z) \wedge \mathsf{P}(z, y))))$

c) If x and y are siblings then they have at least one common parent.

Solution: $\mathsf{Q}(x, y) \rightarrow \exists z(\mathsf{P}(z, x) \wedge \mathsf{P}(z, y))$

d) There exists a woman than has no siblings.

Solution: $\exists x(W(x) \wedge \neg\exists y(\mathsf{Q}(y, x)))$

e) Everybody loves somebody.

Solution: $\forall x \exists y \mathsf{R}(x, y)$

f) There is nobody who loves everybody.

Solution: $\neg\exists x \forall y \mathsf{R}(x, y)$

Exercise 4: Consider predicates M, W, and P from the previous exercise. Add an additional binary predicate D to these predicates such that $\mathsf{D}(x, y)$ expresses that x and y are two different persons (i.e., $\mathsf{D}(x, y)$ is false iff x and y are the same person).

Express the following relationships between family members by formulas of predicate logic. Use only the predicates introduced in the previous exercise (i.e., do not introduce any new predicates). The formulas should express as precisely as possible the standard meaning of the described notions in a natural language (e.g., „x *is a mother of* y“ means that x is a parent of y and x is a woman).

Remark: You can assume that all elements of the universe are people.

a) x is the mother of y

Solution: $\mathsf{P}(x, y) \wedge W(x)$

b) x is the father of y

Solution: $\mathsf{P}(x, y) \wedge \mathsf{M}(x)$

c) x is a son of y

Solution: $\mathsf{P}(y, x) \wedge \mathsf{M}(x)$

d) x is a sibling of y

Solution: $\mathsf{D}(x, y) \wedge \exists z(\mathsf{P}(z, x) \wedge \mathsf{P}(z, y))$

e) x is a sister of y

Solution: $\mathsf{D}(x, y) \wedge W(x) \wedge \exists z(\mathsf{P}(z, x) \wedge \mathsf{P}(z, y))$

f) x is a grandmother of y

Solution: $W(x) \land \exists z(P(x,z) \land P(z,y))$

g) x is a cousin of y

Solution: $\exists v \exists w \exists z(D(v,w) \land P(z,v) \land P(v,x) \land P(z,w) \land P(w,y))$

Exercise 5: For each of the following sequences of symbols, decide whether it is a well-formed formula of predicate logic (use the standard conventions for omitting parentheses). If it is a well-formed formula, do the following:

- Draw an abstract syntax tree of a given formula.
- Determine the predicate symbols that occur in the given formula and for each of these symbols, determine what is the arity of the symbol.
- For each occurrence of a variable, determine whether this occurrence is free or bound. If this is a bound occurrence, determine the quantifier that bounds this occurrence of a variable.

Remark: P, Q, R, S are predicate symbols

1. P
2. x
3. $P(x)$
4. $x(P)$
5. Pxy
6. $\forall P(x)$
7. $P(x) \land \exists x$
8. $\exists x R(x,y)$
9. $\forall z(R(x,y) \to R(y,x))$
10. $\neg \exists x(\neg P(x,y) \to R(y,x))$
11. $P(P(x,y),z)$
12. $\neg P(x) \land \exists x Q(x)$
13. $\forall x \forall y(P(x) \lor P(x,y))$
14. $\forall x \exists y P(x)$
15. $\forall x \exists x P(x)$
16. $\forall x \exists y P(R(x,y))$
17. $\forall x \exists y P(\neg x, y)$
18. $\exists x(P(y,z) \land \forall y(\neg Q(y,x) \lor P(y,z)))$
19. $\forall x \forall x \forall x \forall x \forall x \forall x \forall x\, S(x,x,x)$
20. $\exists x(\forall y(\neg P(x,y)) \to \forall y Q(y)) \leftrightarrow R(x,y)$

Exercise 6: Let us assume that P and Q are unary predicate symbols, and R is a binary predicate symbol.

For each of the following formulas and each of the following interpretations with valuations, determine whether the truth value of the given formula in the given interpretation and valuation.

Formulas:

1. $R(x,y)$
2. $R(x,y) \to R(z,x)$
3. $\forall x \forall y \forall z(R(x,y) \land R(y,z) \to R(x,z))$
4. $\exists x(Q(x) \land \forall y R(y,x))$
5. $\exists x(\neg P(x))$
6. $\forall x \exists y \neg R(x,y)$
7. $\forall x \forall y(Q(x) \land Q(y) \to \exists z(R(x,z) \land R(z,y) \land Q(z)))$

Interpretations:

a) The interpretation $\mathcal{A}$ where the universe is the set $A = \{a, b, c\}$. The following relations are assigned to predicates P, Q, R:

- $P^{\mathcal{A}} = \{a, c\}$
- $Q^{\mathcal{A}} = \emptyset$
- $R^{\mathcal{A}} = \{(a, c),\ (b, b),\ (b, c),\ (c, a)\}$

Let us assume a valuation ν where $\nu(x) = c$, $\nu(y) = a$, and $\nu(z) = a$.

Solution:

3. False: For example, under valuation $\nu(x) = a$, $\nu(y) = c$, $\nu(z) = a$, the formula $R(x, y) \wedge R(y, z) \to R(x, z)$ is false.
7. True: Since $Q^{\mathcal{A}} = \emptyset$, the formula $Q(x) \wedge Q(y)$will be false under every valuation, and so the implication $Q(x) \wedge Q(y) \to \exists z(R(x, z) \wedge R(z, y) \wedge Q(z))$ will be true under every valuation.

b) Interpretation $\mathcal{B}$ where the universe is the set of real numbers $\mathbb{R}$. The following relations are assigned to predicates P, Q, R:

- $P^{\mathcal{B}}$ is the set of all non-negative real numbers
- $Q^{\mathcal{B}}$ is the set of all rational numbers
- $R^{\mathcal{B}}$ is the set of all pairs of real numbers (x, y) where $x < y$.

Let us assume a valuation ν where $\nu(x) = 7$, $\nu(y) = 2.3$, $\nu(z) = 9$.

Solution:

3. True: It holds for all real numbers x, y, z that if $x < y$ and $y < z$, then $x < z$.
7. False: In the given interpretation, the formula claims that for each pair of rational numbers x and y there is some rational number z such that $x < z$ and $z < y$. But this does not hold in those cases where $x \geq y$ (for example if $x = 1$ and $y = 0$).

Exercise 7: For each of the following formulas, find some interpretation, which is a model of this formula, and some interpretation, which is not its model.

1. $\exists x\big(P(x) \wedge \neg Q(x)\big)$
2. $\forall x\big(P(x) \to \neg Q(x)\big)$
3. $\exists x\big(P(x) \wedge Q(x)\big)$
4. $\forall x \exists y\big(P(x) \to \neg R(x, y)\big)$

Exercise 8: Let us assume that R is a binary predicate. For each of the following properties, give a formula of predicate logic expressing that a binary relation represented by the predicate R has the corresponding property:

a) relation R is reflexive,

Solution: $\forall x(\mathsf{R}(x,x))$

b) relation R is ireflexive,

Solution: $\forall x(\neg\mathsf{R}(x,x))$

c) relation R is symmetric,

Solution: $\forall x\forall y(\mathsf{R}(x,y) \rightarrow \mathsf{R}(y,x))$

d) relation R is asymmetric,

Solution: $\forall x\forall y(\mathsf{R}(x,y) \rightarrow \neg\mathsf{R}(y,x))$

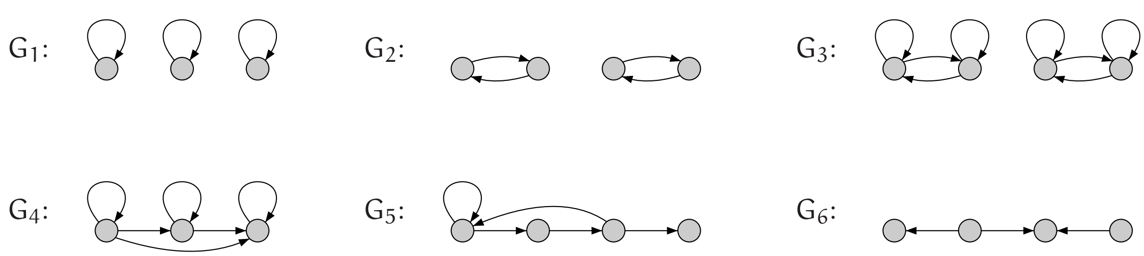
e) relation R is antisymmetric,

Solution: $\forall x\forall y(\mathsf{R}(x,y) \wedge \mathsf{R}(y,x) \rightarrow x = y)$

f) relation R is transitive.

Solution:
$\forall x\forall y\forall z(\mathsf{R}(x,y) \wedge \mathsf{R}(y,z) \rightarrow \mathsf{R}(x,z))$

Consider the following six interpretations represented by graphs G_1–G_6, where nodes of a graph represent the universe and its edges correspond to the binary relation assigned to the predicate R (i.e., in a given interpretation, $\mathsf{R}(x,y)$ is true iff there exists an edge from node x to node y in the given graph).
For each of formulas constructed in points (a)–(f), decide in which of these interpretations the given formula holds and in which not.



Solution:

a) relation R is reflexive: G_1, G_3, G_4

b) relation R is ireflexive: G_2, G_6

c) relation R is symmetric: G_1, G_2, G_3

d) relation R is asymmetric: G_6

e) relation R is antisymmetric: G_1, G_4, G_5, G_6

f) relation R is transitive: G_1, G_3, G_4, G_6