Logical Entailment

Definition

Formula ψ **logically follows** from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ if formula ψ is true in every truth valuation v where all these assumptions are true.

The fact the ψ logically follows from $\varphi_1, \varphi_2, \ldots, \varphi_n$ is denoted

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi.$$

- $\varphi_1, \varphi_2, \ldots, \varphi_n$ — assumptions
- ψ — conclusion

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Example: Conclusion $r \rightarrow p$ logically follows from assumption $p \vee (q \wedge \neg r)$, i.e.,

$$p \vee (q \wedge \neg r) \models r \rightarrow p$$

	p	q	r	$p \vee (q \wedge \neg r)$	$r \rightarrow p$
	0	0	0	0	1
	0	0	1	0	0
$*$	0	1	0	1	1
	0	1	1	0	0
$*$	1	0	0	1	1
$*$	1	0	1	1	1
$*$	1	1	0	1	1
$*$	1	1	1	1	1

Logical Entailment

Example:

- *If the train arrives late and there are no taxis at the station, then John is late for his meeting.*
- *John is not late for his meeting.*
- *The train did arrive late.*

- *There were taxis at the station.*

$$(p \wedge \neg q) \rightarrow r,\ \neg r,\ p \ \models\ q$$

Logical Entailment

$$(p \wedge \neg q) \rightarrow r,\ \neg r,\ p \models q$$

	p	q	r	$(p \wedge \neg q) \rightarrow r$	$\neg r$	p	q
	0	0	0	1	1	0	0
	0	0	1	1	0	0	0
	0	1	0	1	1	0	1
	0	1	1	1	0	0	1
	1	0	0	0	1	1	0
	1	0	1	1	0	1	0
*	1	1	0	1	1	1	1
	1	1	1	1	0	1	1

Logical Entailment

To find out whether ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, the **table method** can be used:

- If, in all lines corresponding to valuations where all assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ have value 1, ψ also have value 1 then the conclusion ψ **logically follows** from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$.
- If there exists at least one valuation in the table where $\varphi_1, \varphi_2, \ldots, \varphi_n$ have value 1 and conclusion ψ has value 0 then this conclusion **does not logically follow** from the assumptions.

Remark: Those valuations where at least one of assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ has value 0 are not important with respect to logical entailment — the conclusion ψ can be true or false in these valuations.

Logical Entailment

To find out whether ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, also **semantic contradiction** can be used:

- A directed acyclic graph common for all formular $\varphi_1, \varphi_2, \ldots, \varphi_n$ and ψ is created.
- Values 1 are assigned to nodes corresponding to assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ and value 0 to the node corresponding to conclusion ψ.
- If it is possible to assign values to all remaining nodes of the graph, we have an example of a valuation where the assumptions are true but the conclusion is false — i.e., the conclusion **does not logically follow** from the assumptions.
- If we show that there is no such valuation (because every attempt to assign remaining values to nodes leads to a contradiction), the conclusion **logically follows** from the assumptions.

Logical Entailment

If a formula ψ is a **tautology** then it logically follows from every set of assumptions, i.e., for every set of assumptions $\varphi_1, \varphi_2, \dots, \varphi_n$ it holds that

$$\varphi_1, \varphi_2, \dots, \varphi_n \models \psi$$

In particular, if ψ is a tautology then it follows from the empty set of assumptions:

$$\models \psi$$

Tautologies are the only formulas that logically follow from the empty set of assumptions.

Logical Entailment

The question whether a given conclusion logically follows from given assumptions can be reformulated as a question whether a certain formula is a tautology:

$$\varphi_1, \varphi_2, \varphi_3, \ldots, \varphi_n \models \psi$$

iff

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_3 \rightarrow (\cdots \rightarrow (\varphi_n \rightarrow \psi) \cdots))) \text{ is a tautology}$$

Example:

$$\varphi_1, \varphi_2, \varphi_3, \varphi_4 \models \psi$$

iff

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_3 \rightarrow (\varphi_4 \rightarrow \psi))) \text{ is a tautology}$$

Logical Entailment

The following two formulas are logically equivalent:

- $\varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_3 \rightarrow (\cdots \rightarrow (\varphi_n \rightarrow \psi) \cdots)))$
- $(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$

For example

$$\varphi_1 \rightarrow (\varphi_2 \rightarrow (\varphi_3 \rightarrow (\varphi_4 \rightarrow \psi))) \quad \Leftrightarrow \quad (\varphi_1 \wedge \varphi_2 \wedge \varphi_3 \wedge \varphi_4) \rightarrow \psi$$

(This can be easily checked by equivalent transformations using the following equivalence: $p \rightarrow (q \rightarrow r) \;\Leftrightarrow\; (p \wedge q) \rightarrow r$)

So it also holds that

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$$

iff

$(\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n) \rightarrow \psi$ is a tautology

Logical Entailment

One more possibility how to characterize when a conclusion follows from given assumptions, is provided by the following equivalence:

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \psi$$

iff

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \quad \Leftrightarrow \quad \varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \psi$$

When some conclusion ψ, which logically follows from given assumptions $\varphi_1, \varphi_2, \ldots \varphi_n$, to these assumptions as new additional assumption, it does not change the set of truth valuations where the assutions are true.
(The sets of assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ and $\varphi_1, \varphi_2, \ldots, \varphi_n, \psi$ are true at the same truth valuations.)

Logical Entailment

So when an assumption ψ that logically follows from given assumption is added to these assumptions, the set of all conclusions that follow the assumptions is not affected:

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \psi$$

iff

it holds for each formula χ that:

$$\varphi_1, \varphi_2, \ldots \varphi_n \models \chi \quad \text{iff} \quad \varphi_1, \varphi_2, \ldots \varphi_n, \psi \models \chi$$

Logical Entailment

If some conclutions logically follow from given assumptions and some other conclution follows from these conclutions, then this conclution logically follows also from the original assumptions.

Let us assume that

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \chi_1 \qquad\qquad \varphi_1, \varphi_2, \ldots, \varphi_n \models \chi_2$$

and also $\quad \chi_1, \chi_2 \models \psi.$

Then $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$.

Example:

- If $\varphi_1, \varphi_2, \varphi_3 \models (q \vee \neg p)$ and $\varphi_1, \varphi_2, \varphi_3 \models \neg s$ then

$$\varphi_1, \varphi_2, \varphi_3 \models (q \vee \neg p) \wedge \neg s$$

  because $q \vee \neg p, \neg s \models (q \vee \neg p) \wedge \neg s$.

Logical Entailment

Example:

- If $\varphi_1, \varphi_2, \varphi_3 \models p \to q$ and $\varphi_1, \varphi_2, \varphi_3 \models p$ then

  $$\varphi_1, \varphi_2, \varphi_3 \models q$$

  because $p \to q,\ p \models q$.

Example:

- If $\varphi_1, \varphi_2, \varphi_3, \varphi_4 \models \neg p \to \neg q$ then

  $$\varphi_1, \varphi_2, \varphi_3, \varphi_4 \models q \to p$$

  because $\neg p \to \neg q \models q \to p$.

Logical Entailment

When we try to prove that ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, we can proceed via smaller steps.

We start we the assumptions, for example:

$$\varphi_1,\ \varphi_2,\ \varphi_3$$

Then we gradually add other formulas in such a way that every newly added formula logically follows from the previous formulas. For example:

$$\varphi_1,\ \varphi_2,\ \varphi_3,\ \chi_1,\ \chi_2,\ \chi_3,\ \chi_4,\ \chi_5,\ \chi_6,\ \chi_7,\ \chi_8,\ \psi$$

Logical Entailment

Assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are **inconsistent** (**contradictory**) if there is no truth valuation v, in which all these assumptions would be true.

Assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent iff

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n$$

is a contradiction.

Example: Assumptions $p \rightarrow q$, $r \rightarrow p$, r, $\neg q$ are inconsistent.

Logical Entailment

From inconsistent assumptions, any conclusion logically follows.

If assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent then it holds for every formula ψ that

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi.$$

So for example also the following conclusions follow from inconsistent assumptions:

- $\bot$
- formulas χ and $\neg\chi$, where χ is an arbitrary formula

Logical Entailment

Formula $\bot$ cannot be true and it is also not possible that formulas χ and $\neg\chi$ are both true at some truth valuation.

So when we find out that

- $\bot$ or,
- χ and also $\neg\chi$,

logically follow from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$, this means that the assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are inconsistent and anything follows from them.

Remark: Note that the following formulas are tautologies:

- $\bot \rightarrow \psi$
- $\chi \rightarrow (\neg\chi \rightarrow \psi)$

So $\qquad \bot \models \psi \qquad$ and $\qquad \chi, \neg\chi \models \psi.$

Logical Entailment

The principle of proof by contradiction

$$\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$$

iff

assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi$ are inconsistent

So in a proof by contradiction, justification that a given conclusion follows from given assumptions is transformed to justification that it is not possible that the assumptions and the negation of the conclusion would be true at the same time.

The question whether $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$ can be transformed to the question whether

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \neg\psi$$

is a **contradiction**.

Resolution Method

The **resolution method** is one of algorithms for finding out whether a given conclusion follows from given assumptions.

It solves the following problem:

Input: Formulas φ_1, φ_2, ..., φ_n, ψ.

Question: Is it true that $\varphi_1, \varphi_2, \ldots, \varphi_n \models \psi$?

Remark: The method can be used for finding out whether a given formula is a tautology, a contradiction, or satisfiable.

Different variants of the resolution method are used for example in some systems for automatic theorem proving and also in implementations of logic programming languages such as Prolog.

Resolution Method

- It works with formulas in CNF.
- It constructs a proof that the given conclusion follows from the assumptions.
- It is a proof by contradiction — the algorithm generates successively formulas following from the assumptions

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi$$

- A computation can finish in two different ways:
  - A contradiction is found, i.e., formula $\bot$ is derived — then the conclusion ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$.
  - The algorithm does not succeed in deriving formula $\bot$ and no other new formulas can be added — then the conclusion ψ does not follow from the assumptions.

Resolution Method

- The resolution method works with formulas that have the form of elementary disjunctions, e.g.,

$$(\neg p \vee q \vee \neg s \vee \neg t)$$

  These formulas are called **clauses**.

- A special case of clause is the **empty clause** $\perp$ that represents a found contradiction.

- The algorithm starts the computation by transforming formulas

$$\varphi_1, \varphi_2, \ldots, \varphi_n, \neg\psi$$

  to CNF. Then it takes all clauses from the transformed formulas as the initial set of assumptions

$$\chi_1, \chi_2, \ldots, \chi_m.$$

Resolution Method

For generating other clauses, which are added to already constructed clauses, the algorithm uses so called **resolution rule** (or **resolution principle**):

For each formulas φ, ψ and χ it holds that

$$\varphi \vee \psi, \neg\varphi \vee \chi \models \psi \vee \chi$$

In the resolution method, this principle is used only for clauses.
In the resolution method, $\varphi \vee \psi$, $\neg\varphi \vee \chi$ and $\psi \vee \chi$ are always clauses, and φ is always an atomic proposition.

Example: From clauses

$$p \vee \neg q \vee r \vee s \qquad \text{a} \qquad \neg r \vee t \vee \neg u$$

we can derive the following clause by the resolution rule:
$p \vee \neg q \vee s \vee t \vee \neg u$.

Resolution Method

Remarks:

- An order of literals in a clause is not important.
- Multiple occurrences of the same literals in one clause can be eliminated.
- If a currently generated clause is the same as some previously generated clause (and differs only in the order of literals), it makes no sense to add it.
- Clauses containing both literals p and $\neg p$ are equivalent to $\top$ and can be eliminated.
- Clauses can be used for the application of the resolution rule repeatedly (with other clauses).

Resolution Method

Some special cases of the use of the resolution rule:

- One of clauses contains just one literal and the other more than one literal:

  From clauses

  $$\neg q \qquad\qquad p \vee q \vee \neg t$$

  we can derive clause $p \vee \neg t$.

- Both clauses contain just one literal:

  From clauses

  $$p \qquad\qquad \neg p$$

  we can derive the empty clause $\perp$, i.e., the contradiction.

Resolution Method

We want to check validity of the following deduction:

- *It is not true that Jane is at school and Peter is not at home.*
- *Jane is not at school or it's a working day or it's raining.*
- *If it's a working day then Peter is not at home.*

- *If Jane is at school then it's raining.*

At first, we formalize the individual propositions by formulae of propositional logic:

$$\begin{array}{l} \neg(j \wedge \neg p) \\ \neg j \vee d \vee r \\ d \rightarrow \neg p \\ \hline j \rightarrow r \end{array}$$

j – Jane is at school
p – Peter is at home
d – it's a working day
r – it's raining

$$\begin{array}{l} \neg(j \wedge \neg p) \\ \neg j \vee d \vee r \\ d \rightarrow \neg p \\ \hline j \rightarrow r \end{array}$$

We transform the individual assumptions into CNF:

- $\neg(j \wedge \neg p) \;\Leftrightarrow\; \neg j \vee p$
- $\neg j \vee d \vee r$
- $d \rightarrow \neg p \;\Leftrightarrow\; \neg d \vee \neg p$

We negate the conclusion and transform it into CNF:

- $\neg(j \rightarrow r) \;\Leftrightarrow\; j \wedge \neg r$

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4
7.	$d \vee r$	– resolution: 2,4

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4
7.	$d \vee r$	– resolution: 2,4
8.	$\neg d$	– resolution: 3,6

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4
7.	$d \vee r$	– resolution: 2,4
8.	$\neg d$	– resolution: 3,6
9.	r	– resolution: 7,8

Resolution Method

Let us write down the individual clauses:

1.	$\neg j \vee p$	– assumption 1
2.	$\neg j \vee d \vee r$	– assumption 2
3.	$\neg d \vee \neg p$	– assumption 3
4.	j	– clause 1 of the negated conclusion
5.	$\neg r$	– clause 2 of the negated conclusion
6.	p	– resolution: 1,4
7.	$d \vee r$	– resolution: 2,4
8.	$\neg d$	– resolution: 3,6
9.	r	– resolution: 7,8
10.	$\perp$	– resolution: 5,9

A contradition was derived, so the conclusion really follows from the given assumptions.

Resolution Method

Remarks:

- The resolution method can be viewed as a construction of one "big" formula in CNF, which is equivalent to

$$\varphi_1 \wedge \varphi_2 \wedge \cdots \wedge \varphi_n \wedge \neg\psi,$$

  and which is constructed by a successive addition of clauses.

- If a contradiction can not be generated, then the derived clauses can be used for finding a truth valuation v where the assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$ are true and the conclusion ψ is false.

Resolution Method

- It is also possible to proceed by a direct method, where the algorithm starts only with assumptions

  $$\varphi_1, \varphi_2, \ldots, \varphi_n$$

  and tries to generate all clauses of the conclusion ψ.

  In this approach, it is not guaranteed that the algorithm succeeds in all cases when a conclusion ψ logically follows from assumptions $\varphi_1, \varphi_2, \ldots, \varphi_n$.

  Example: Clause $p \vee q$ cannot be generated this way from the assumption p, although it holds that

  $$p \models p \vee q.$$