Tutorial 6

Exercise 1: For each of the following sequences of symbols, determine if it is a) a term, b) a formula of first-order predicate logic (use the standard conventions for omitting parentheses). If it is a formula of first-order predicate logic, determine if it is i) atomic, ii) closed; if the formula is not closed, specify the set of free variables occurring in the formula.

For each sequence of symbols, which is a term or a formula, draw the corresponding abstract syntax tree.

You can assume that:

- P, Q, and R are predicate symbols — P is unary and Q and R are binary,
- f is a unary function symbol and g is a binary function symbol,
- c and d are constant symbols.

1. $(\neg((\neg p) \to (\neg(\neg r))))$
2. $\forall x \in A : P$
3. $f(c)$
4. $R(c, d)$
5. $\forall x \exists y P(c)$
6. $\forall x \exists y f(R(x, y))$
7. $\forall x \exists y P(g(x, y))$
8. $\forall x \exists y f(g(x, y))$
9. $\forall x \exists y P(g(f(f(x)), c))$
10. $\forall x(P(d) \land \exists y Q(y, c))$
11. $P(d) \land \exists y Q(y, c)$
12. $P(x) \land \exists y Q(d, c)$
13. $\forall x \exists y(R(x, f(y)) \leftrightarrow \exists z Q(z, c))$
14. $\forall x P(g(x))$
15. $\forall x R(f(x))$
16. $\forall x R(f(x), f(x), f(x))$
17. $\forall x P(f(x, x))$
18. $\forall x P(g(x, x))$
19. $f(f(g(c, d)))$
20. $P(f(g(c, d)))$
21. $P(f(d)) \to \forall x P(x)$
22. $P(f(g(f, f)))$
23. $P(f(g(c, x)))$
24. $\forall x(f(x) \to g(c, x))$
25. $\forall x P(f(x) \to g(c, x))$
26. $\forall x P(\neg f(x))$
27. $\forall x \neg P(f(x))$
28. $\neg(P(f(x)) \lor Q(y, z))$

Exercise 2: Write down formally as formulas of predicate logic the following propositions formulated in a natural language.

Remarks:

- At first, determine what predicate, function, and constant symbols will be used in the formula, what these symbols will stand for, and what will be their arities.
- As an intermediate step in the creation of a resulting formula, start by forming a "formula", where you can use standard mathematical symbols as $>$, $+$, $\cap$, $\in$, $\subseteq$, where numbers are represented by digits, etc., and where you can use standard conventions such as the infix notation for binary function and predicate symbols.
- Based on the "formula" created in the previous step, create a formula formed according to the formal definion of the syntax of formulas of predicate logic (where you can use standard conventions for omitting parentheses), and where you use only letters of Latin alphabet as predicate, function, and constant symbols.

a) For every natural number there exists a prime number greater than this number.
b) Some natural number is not divisible by number 5 nor by number 7.
c) It holds for each real number greater than 10 that if we subtract number 9 from it, we obtain a positive number.
d) Empty set is a subset of every set.
e) It holds for each pair of sets that both of them are subsets of their union.
f) The intersection of two sets is a subset of both these sets.

Exercise 3: Let us assume that

- P and Q are unary predicate symbols and R is a binary predicate symbol,
- f is a binary function symbol and g is a unary function symbol,
- c and d are constant symbols.

For each of the following formulas, and each of the following interpretations with valuations, determine the truth value of the the given formula in the given interpretation and valuation. Formulas:

1. $R(c, d)$
2. $R(c, d) \rightarrow R(c, x)$
3. $\forall x \forall y \forall z (R(x, y) \wedge R(y, z) \rightarrow R(x, z))$
4. $\exists x (Q(x) \wedge \forall y R(y, g(x)))$
5. $\exists x \neg P(f(x, y))$
6. $\forall x \exists y \neg R(x, g(g(y)))$

Interpretations:

a) Interpretation $\mathcal{S}$ where the universe is the set $A = \{\alpha, \beta, \gamma\}$.

The following relations are assigned to predicates P, Q, and R:

- $P^{\mathcal{A}} = \{\alpha, \gamma\}$
- $Q^{\mathcal{A}} = \emptyset$
- $R^{\mathcal{A}} = \{(\alpha, \beta), (\beta, \gamma), (\alpha, \gamma)\}$

Functions $f^{\mathcal{A}}$ and $g^{\mathcal{A}}$ described by the following tables are assigned to function symbols f and g:

$f^{\mathcal{A}}$	α	β	γ
α	β	α	γ
β	β	β	β
γ	α	γ	β

x	$g^{\mathcal{A}}(x)$
α	β
β	γ
γ	γ

Elements α and β are assigned to constant symbols c and d, i.e., $c^{\mathcal{A}} = \alpha$ a $d^{\mathcal{A}} = \beta$.

Let us assume valuation v where $v(x) = \gamma$, $v(y) = \alpha$, and $v(z) = \alpha$.

b) Interpretation $\mathcal{B}$ where the universe is the set of natural numbers $\mathbb{N} = \{0, 1, 2, \ldots\}$.

- Relation $P^{\mathcal{B}} = \{x \in \mathbb{N} \mid x \bmod 2 = 0\}$ is assigned to predicate symbol P.
- Relation $Q^{\mathcal{B}} = \{x \in \mathbb{N} \mid x = y^2 \text{ for some } y \in \mathbb{N}\}$ is assigned to predicate symbol Q.
- Relation $R^{\mathcal{B}} = \{(x, y) \in \mathbb{N} \times \mathbb{N} \mid x < y\}$ is assigned to predicate symbol R.

- Function $f^{\mathcal{B}} : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$, where $f^{\mathcal{B}}(x, y) = x + y$, is assigned to function symbol f.
- Function $g^{\mathcal{B}} : \mathbb{N} \to \mathbb{N}$, where $g^{\mathcal{B}}(x) = x + 1$, is assigned to function symbol g.
- Elements 0 and 2 are assigned to constants c and d, i.e., $c^{\mathcal{B}} = 0$ and $d^{\mathcal{B}} = 2$.

Let us assume valuation ν where $\nu(x) = 7$, $\nu(y) = 2$, $\nu(z) = 9$.

Exercise 4: For each of the following formulas, find some interpretation, which is a model of this formula, and some interpretation, which is not its model.

1. $\forall x\big((P(x) \wedge Q(x, a)) \to R(x)\big)$
2. $\forall x\big(P(a, x) \to Q(x)\big)$
3. $\exists x\big(P(x, f(x))\big)$

Exercise 5: Give an example of an interpretation where all four following formulas are true:

- $\neg\exists x R(x, x)$
- $\forall x \exists y R(x, y)$
- $\exists x \forall y (\neg R(y, x))$
- $\forall x \forall y \forall z (R(x, y) \to (R(y, z) \to R(x, z)))$

Exercise 6: Find out whether the given conclusion logically follows from the given assumption. Justify your answers.

a) assumption: $\forall x \exists y P(x, y)$, conclusion: $\exists y \forall x P(x, y)$
b) assumption: $\exists x \forall y R(x, y)$, conclusion: $\forall y \exists x R(x, y)$
c) assumption: $\forall x(P(x) \to Q(x))$, conclusion: $\exists x P(x) \to \forall x Q(x)$
d) assumption: $\exists x P(x) \wedge \exists x Q(x)$, conclusion: $\exists x(P(x) \wedge Q(x))$
e) assumption: $\exists x P(x) \vee \exists x Q(x)$, conclusion: $\exists x(P(x) \vee Q(x))$
f) assumption: $\exists x(P(x) \vee Q(x))$, conclusion: $\exists x P(x) \vee \exists x Q(x)$
g) assumption: $\forall x(P(x) \wedge Q(x))$, conclusion: $\forall x P(x) \wedge \forall x Q(x)$
h) assumption: $\forall x P(x) \wedge \forall x Q(x)$, conclusion: $\forall x(P(x) \wedge Q(x))$
i) assumption: $\forall x(P(x) \vee Q(x))$, conclusion: $\forall x P(x) \vee \forall x Q(x)$
j) assumption: $\forall x P(x) \to \forall x Q(x)$, conclusion: $\forall x(P(x) \to Q(x))$
k) assumption: $\forall x(\forall y P(y) \to Q(x))$, conclusion: $\forall y P(y) \to \forall x Q(x)$
l) assumption: $\forall x(P(x) \to \forall y Q(y))$, conclusion: $\exists x P(x) \to \forall y Q(y)$

Exercise 7: Using Venn diagrams, find out whether the given conclusion logically follows from the given assumptions. If the conclusion does not follow from the assumptions, give an example of an interpretation where the assumptions are true and the conclusion is false.

a) *All animals are mortal.*
All human beings are animals.
___
All humans beings are mortal.

b) *All people need oxygen to keep alive.*
All people are living organisms.
___
All living organisms need oxygen to keep alive.

c) *Some men are liars.*
Adam is a man.
___
Adam is a liar.

d) *Between the paintings are valuable exactly those, which are portraits.*
All portraits are painted in oil.
Some of the paintings are not painted in oil.
___
Those paintings that are not painted in oil are not valuable.

Remark: For simplicity assume that all elements of the universe are paintings.

e) *All integers are rational.*
There exists at least one rational number, which is not an integer.
Every real number either is rational or is not rational.
___
There exists a real number, which is rational.

Remark: For simplicity assume that all elements of the universe are numbers.

Exercise 8: Which of the following pairs of formulas are logically equivalent? Justify your answers.

1. Is $\exists x P(x) \Leftrightarrow P(x)$?
2. Is $\exists y \exists x P(x) \Leftrightarrow \exists x P(x)$?
3. Is $\exists y \exists x P(x) \Leftrightarrow \exists y P(y)$?
4. Is $\exists x P(x,y) \Leftrightarrow \exists y P(y,y)$?
5. Is $\forall x P(x,y) \Leftrightarrow \forall y P(y,y)$?
6. Is $\forall x \forall y P(x,y) \Leftrightarrow \forall y \forall y P(y,y)$?

Exercise 9: Using equivalent transformations, prove the following equivalences:

a) $\neg \forall y \exists x P(x,y) \Leftrightarrow \exists y \forall x \neg P(x,y)$
b) $\exists x \forall y Q(y) \Leftrightarrow \forall y \forall x Q(y)$
c) $\forall x P(x) \rightarrow \exists z (\neg \forall y (Q(y) \vee R(z,y))) \Leftrightarrow \exists x \neg P(x) \vee \exists z \exists y (\neg R(z,y) \wedge \neg Q(y))$
d) $\neg \forall x (P(x) \rightarrow Q(x)) \Leftrightarrow \exists x (P(x) \wedge \neg Q(x))$
e) $\neg \exists x (P(x) \wedge Q(x)) \Leftrightarrow \forall x (P(x) \rightarrow \neg Q(x))$
f) $\forall x (P(x) \rightarrow (Q(x) \wedge R(x))) \Leftrightarrow \forall x (P(x) \rightarrow Q(x)) \wedge \forall x (P(x) \rightarrow R(x))$
g) $\exists x (P(x) \wedge (Q(x) \vee R(x))) \Leftrightarrow \exists x (P(x) \wedge Q(x)) \vee \exists x (P(x) \wedge R(x))$

Exercise 10: Recall that the symbol "=" denotes equality (identity). Explain in a natural language what the following formula claims:

$$\exists x \exists y (\neg(x = y) \ \land \ \forall z (z = x \ \lor \ z = y))$$

How models of this formula look?

Exercise 11: Let us assume that P is an unary predicate. Express the following propositions by formulas of predicate logic (you can use symbol "="):

a) There exist at least three elements with property P (i.e., $P(x)$ holds for at least three distinct elements x).
b) There exist at most two elements with property P (i.e., $P(x)$ holds for at most two distinct elements x).