Tutorial 3

Exercise 1: Determine for the following formulas whether they are tautologies. If this is the case, prove it by finding a semantic contradiction, if not, give an example of a truth valuation, in which the given formula does not hold (to find such valuation, use the same approach as for finding a semantic contradiction).

1. $\neg(p \vee q) \rightarrow p$
2. $((p \rightarrow q) \rightarrow p) \rightarrow p$
3. $(p \rightarrow q \vee p) \wedge (p \rightarrow (\neg p \rightarrow r))$
4. $(p \vee \neg(q \wedge r)) \rightarrow ((p \leftrightarrow r) \vee q)$
5. $(p \wedge q) \leftrightarrow ((p \rightarrow q) \leftrightarrow p)$

Exercise 2:

a) Let us assume that φ is a formula such that for every formula ψ, formula $\varphi \vee \psi$ is always true. What can be said about truth values of formula φ?

b) Let us assume that φ is a formula such that for every formula ψ , formula $\varphi \wedge \psi$ is always false. What can be said about truth values of formula φ?

Exercise 3: For each of the following formulas, give an example of such formulas φ and ψ, so that for these formulas the given formula is a tautology:

a) $\varphi \wedge \psi$
b) $\varphi \vee (\varphi \wedge \neg\psi)$
c) $\varphi \rightarrow \varphi \wedge \neg\psi$
d) $\varphi \rightarrow \neg\varphi$

Exercise 4: Is there any formula φ, for which the formula $\varphi \wedge \neg\varphi$ is a tautology?

Exercise 5: Recall what it means that formulas of propositional logic are logically equivalent. For which of the following formulas it holds that they are logically equivalent to p?

- If equivalence $\varphi \Leftrightarrow p$ holds for the given formula φ, prove it using the table method or by a semantic contradiction.
- If equivalence $\varphi \Leftrightarrow p$ does not hold, give an example of a truth valuation, where one of formulas φ and p holds and the other does not hold.

a) $p \vee p$
b) $p \vee q$
c) $p \vee \neg p$
d) $p \wedge p$
e) $p \wedge q$
f) $p \rightarrow p$
g) $\neg p \rightarrow p$
h) $p \rightarrow \neg p$
i) $q \vee \neg q \rightarrow p$
j) $p \leftrightarrow p$

Exercise 6: Which of the following equivalences between pairs of formulas are valid? Justify your answers:

- If equivalence $\varphi \Leftrightarrow \psi$ holds, prove it using the table method or by a semantic contradiction.
- If equivalence $\varphi \Leftrightarrow \psi$ does not hold, give an example of a truth valuation, where one of formulas φ and ψ holds and the other does not hold.

1. $p \Leftrightarrow p$
2. $p \Leftrightarrow \neg p$
3. $p \Leftrightarrow \neg\neg p$
4. $\neg p \Leftrightarrow \neg\neg p$
5. $p \wedge q \Leftrightarrow q \wedge p$
6. $p \vee q \Leftrightarrow q \vee p$
7. $p \to q \Leftrightarrow q \to p$
8. $p \leftrightarrow q \Leftrightarrow q \leftrightarrow p$
9. $(p \wedge q) \wedge r \Leftrightarrow p \wedge (q \wedge r)$
10. $(p \vee q) \vee r \Leftrightarrow p \vee (q \vee r)$
11. $(p \to q) \to r \Leftrightarrow p \to (q \to r)$
12. $(p \leftrightarrow q) \leftrightarrow r \Leftrightarrow p \leftrightarrow (q \leftrightarrow r)$
13. $p \wedge (q \vee r) \Leftrightarrow (p \wedge q) \vee r$
14. $p \vee (q \wedge r) \Leftrightarrow (p \vee q) \wedge r$
15. $\neg(p \wedge q) \Leftrightarrow \neg p \wedge \neg q$
16. $\neg(p \wedge q) \Leftrightarrow \neg p \vee \neg q$
17. $\neg(p \wedge q) \Leftrightarrow p \vee q$
18. $\neg(p \vee q) \Leftrightarrow \neg p \vee \neg q$
19. $\neg(p \vee q) \Leftrightarrow p \wedge q$
20. $\neg(p \vee q) \Leftrightarrow \neg p \wedge \neg q$
21. $(p \to q) \Leftrightarrow \neg p \wedge q$
22. $(p \to q) \Leftrightarrow p \wedge \neg q$
23. $(p \to q) \Leftrightarrow \neg p \vee q$
24. $(p \to q) \Leftrightarrow p \vee \neg q$
25. $\neg(p \to q) \Leftrightarrow \neg p \vee q$
26. $\neg(p \to q) \Leftrightarrow p \wedge \neg q$
27. $\neg(p \to q) \Leftrightarrow \neg p \to \neg q$
28. $(p \leftrightarrow q) \Leftrightarrow (p \vee \neg q) \wedge (\neg p \vee q)$
29. $(p \leftrightarrow q) \Leftrightarrow (p \wedge \neg q) \vee (\neg p \wedge q)$
30. $(p \to q) \Leftrightarrow (p \leftrightarrow q) \vee (q \leftrightarrow p)$
31. $(p \leftrightarrow q) \Leftrightarrow (p \to q) \vee (q \to p)$
32. $(p \leftrightarrow q) \Leftrightarrow (p \to q) \wedge (q \to p)$

Exercise 7: Using equivalent transformations, prove the equivalences given in items (a)–(c). Use only the following equivalences in individual steps:

- in item (a), use only equivalences of the form $\varphi \wedge (\psi \wedge \chi) \Leftrightarrow (\varphi \wedge \psi) \wedge \chi$,
- in item (b), you can use also equivalences of the form $\varphi \wedge \psi \Leftrightarrow \psi \wedge \varphi$,
- in item (c), you can use also equivalences of the form $\varphi \wedge \varphi \Leftrightarrow \varphi$.

(All these equivalences can be used in both directions and they can be applied on subformulas.)

a) $p \wedge ((q \wedge r) \wedge (s \wedge t)) \Leftrightarrow (p \wedge q) \wedge ((r \wedge s) \wedge t)$
b) $(r \wedge q) \wedge (s \wedge p) \Leftrightarrow p \wedge (q \wedge (r \wedge s))$
c) $(p \wedge q) \wedge p \Leftrightarrow q \wedge (p \wedge q)$

Exercise 8: Using equivalent transformations, prove the following equivalences:

1. $(p \to q) \wedge p \Leftrightarrow p \wedge q$
2. $(p \to q) \to q \Leftrightarrow p \vee q$
3. $p \wedge (p \vee q) \Leftrightarrow p$

4. $(p \lor q) \leftrightarrow q \;\Leftrightarrow\; p \rightarrow q$
5. $(p \land q) \leftrightarrow p \;\Leftrightarrow\; p \rightarrow q$
6. $(p \rightarrow q) \leftrightarrow p \;\Leftrightarrow\; p \land q$
7. $((p \lor q) \leftrightarrow q) \leftrightarrow p \;\Leftrightarrow\; p \land q$

Exercise 9: Using equivalent transformations, decide whether a given formula is a tautology, a contradiction, or a satisfiable formula.

1. $((p \land \neg q) \rightarrow (\neg p \rightarrow (q \lor p)))$
2. $\big((p \lor \neg q) \land \neg(p \land q)\big) \rightarrow (\neg p \lor q)$
3. $\neg\big((q \land p) \rightarrow ((p \rightarrow q) \land (\neg p \lor q))\big)$
4. $\big((p \lor \neg(p \land q)) \rightarrow (\neg p \lor q \lor p)\big) \rightarrow (p \leftrightarrow \neg q)$

Exercise 10: Transform the following formulas to CNF and to DNF:

1. $\neg(p \land \neg r \land s)$
2. $(p \land q \land \neg r) \lor (r \land q)$
3. $p \rightarrow (q \land r)$
4. $p \leftrightarrow q$
5. $((p \rightarrow \neg q) \rightarrow r) \land \neg p$
6. $((p \land (q \lor r)) \lor (q \rightarrow \neg r)$
7. $\neg(\neg(p \rightarrow \neg q) \land (r \leftrightarrow \neg p))$

Exercise 11: Find a CCNF and CDNF of the following formulas by the table method or by equivalent transformations.

1. $(p \leftrightarrow \neg q)$
2. $(p \land \neg q) \rightarrow (\neg p \rightarrow (q \lor p))$
3. $\big((p \rightarrow q) \land (\neg r \rightarrow \neg q)\big) \land \neg r \land p$