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HARMONIC 1-TYPE CURVES AND WEAK BIHARMONIC CURVES IN LORENTZIAN 3-SPACE

BY

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Abstract. In this paper, we give definitions and characterizations of harmonic 1-type and weak biharmonic curves by using the mean curvature vector field of a Frenet curve in the Lorentzian 3-space L^3. We also study weak biharmonic curves whose mean curvature vector fields are in the kernel of normal Laplacian $\nabla^{\perp}$. We give some theorems for them in L^3. Moreover, we give some characterizations and results for a Frenet curve in the same space.

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1. Introduction and preliminaries

Chen and Ishikawa have classified the Biharmonic curves in pseudo-Euclidean space E^n_v. They have obtained that every biharmonic curve lies in a 3-dimensional totally geodesic subspace. Thus, it suffices to classify biharmonic curves in a semi-Euclidean 3-space (see CHEN and ISHIKAWA [2]). KOCAYIĞIT and HACISALIHOĞLU [13] have given the characterizations of harmonic and biharmonic curves in Euclidean 3-space. Also, they have studied the characterizations of 1-type and biharmonic curves in Lorentzian 3-space L^3 and found the relationships between the curvature and the torsion of these curves (see KOCAYIĞIT and HACISALIHOĞLU [12]).

In this paper, we give the characterizations of harmonic 1-type and weak biharmonic curves in semi-Euclidean 3-space in terms of curvature and torsion. Firstly, according to normal Levi-Civita connection, we point out the general differential equations characterizing Frenet curves (both non-null and null) in Lorentzian 3-space L^3. Moreover, we classify the special curves from the differential equations. More precisely, we show that a circular helix is a harmonic 1-type curve in L^3 and a pseudo circle (degenerate helix) is a weak biharmonic curve in the same space. In the final section, we give some theorems, corollaries and propositions.

Let now introduce a brief summary of the Lorentzian 3-space L^3.

Let (M, g) be a time-oriented Lorentz 3-manifold. Let $\gamma : \mathrm{I} \to M$ be a unit speed curve on M. Namely, the velocity vector filed γ' satisfies $g(\gamma', \gamma') = \varepsilon_1 = \pm 1$, where g shows the Lorentzian metric given by $g(a, b) = -a_1 b_1 + a_2 b_2 + a_3 b_3$ for the vectors $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3) \in L^3$. The constant ε_1 is called the causal character of γ. Then, a unit speed curve γ is said to be spacelike or timelike if its causal character is 1 or -1, respectively. A unit speed curve γ is called geodesic if $\nabla_{\gamma'}\gamma' = 0$, where ∇ is the Levi-Civita connection of (M, g). Moreover, the curve γ is also said to be a geodesic if $\nabla^{\perp}_{\gamma'}\gamma' = 0$, where $\nabla^{\perp}$ is the normal Levi-Civita connection of (M, g). A unit speed curve γ is said to be a Frenet curve if $g(\gamma'', \gamma'') \neq 0$. Like Euclidean geometry, every Frenet curve γ on (M, g) admits an othonormal Frenet frame field $\{V_1, V_2, V_3\}$ along γ such that $V_1 = \gamma'(s)$ and $\{V_1, V_2, V_3\}$ satisfies the following Frenet-Serret formula (see Izumiya and Takiyama [8,9] and Kobayashi [11]):

$$\begin{bmatrix} \nabla_{\gamma'} V_1 \\ \nabla_{\gamma'} V_2 \\ \nabla_{\gamma'} V_3 \end{bmatrix} = \begin{bmatrix} 0 & \varepsilon_2 \kappa & 0 \\ -\varepsilon_1 \kappa & 0 & -\varepsilon_3 \tau \\ 0 & \varepsilon_2 \tau & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}. \tag{1.1}$$

The functions $\kappa \geq 0$ and τ are called the curvature and torsion, respectively. The vector fields V_1, V_2, V_3 are called tangent vector field, principle normal vector field and binormal vector field of γ, respectively. The constants ε_2 and ε_3 are defined by

$$\varepsilon_i = g(V_i, V_i), \quad i = 2, 3, \tag{1.2}$$

and called second causal character and third causal character of γ, respectively. Note that

$$\varepsilon_3 = -\varepsilon_1 \varepsilon_2. \tag{1.3}$$

As in the case of Riemannian geometry, a Frenet curve γ is a geodesic if and only if $\kappa = 0$. A circular helix is a Frenet curve whose curvature and torsion are constants. A Frenet curve with constant curvature and zero torsion is called a pseudo circle. Pseudo circles are regarded as degenerate helices. Helices, which are not circles, are frequently called proper helices.

Here we use the normal Laplacian.

Let $\gamma : \mathrm{I} \to M$ be a curve parametrized by the arclength on Lorentz 3-manifold M. Denote by $\{V_1, V_2, V_3\}$ the Frenet frame of γ as before. The normal bundle $\chi^{\perp}(\gamma(I))$ of the curve γ is given by

$$\chi^{\perp}(\gamma(I)) = Sp\{V_2(s), V_3(s)\}.$$

The normal connection $\nabla^{\perp}_{\gamma'}$ of γ is

$$\nabla^{\perp}_{\gamma'} : \chi^{\perp}(\gamma(I)) \to \chi^{\perp}(\gamma(I))$$

$$\nabla^{\perp}_{\gamma'}\xi = \nabla_{\gamma'}\xi - g\left(\nabla_{\gamma'}\xi, V_1\right)V_1, \quad \forall \xi \in \chi^{\perp}(\gamma(I)). \tag{1.4}$$

$\nabla^{\perp}_{\gamma'}\xi$ is called normal component of $\nabla_{\gamma'}\xi$. According to (1.4), the Frenet-Serret formulas are given by

$$\nabla^{\perp}_{\gamma'}V_2 = -\varepsilon_3\tau V_3, \quad \nabla^{\perp}_{\gamma'}V_3 = \varepsilon_2\tau V_2. \tag{1.5}$$

Let us denote the normal Laplace-Beltrami operator by $\Delta^{\perp}$of γ and mean curvature vector field along γ by H. The mean curvature vector field H of a unit speed curve γ is $\mathrm{H} = \varepsilon_1\nabla_{\gamma'}\gamma'$. If γ is a Frenet curve, then H is given by

$$\mathrm{H} = -\varepsilon_3\kappa V_2, \tag{1.6}$$

where κ is the curvature of γ.

The normal Laplacian operator of γ is defined by

$$\Delta^{\perp} = -\varepsilon_1\left(\nabla^{\perp}_{\gamma'}\right)^2 = -\varepsilon_1\nabla^{\perp}_{\gamma'}\nabla^{\perp}_{\gamma'} \tag{1.7}$$

(see Chen and Ishikawa [3], Ferrandez, Lucas and Merono [4] and Ikawa [7]).

To close this section, we recall the notion of harmonic 1-type and weak biharmonicity for unit speed curves.

Definition 1.1. A unit speed curve $\gamma : I \to M$ on a Lorentz 3-manifold M is said to be harmonic 1-type if

$$\Delta^{\perp}\mathrm{H} = \lambda\mathrm{H} \tag{1.8}$$

holds (Kiliç [10]).

Definition 1.2. A unit speed curve $\gamma : I \to M$ on a Lorentz 3-manifold M is said to be weak biharmonic if

$$\Delta^{\perp}\mathrm{H} = 0, \tag{1.9}$$

holds (Kılıç [10]).

Definition 1.3. A unit speed curve $\gamma : I \to M$ on a Lorentz 3-manifold M is said to be a general helix if $\frac{\kappa}{\tau}$ is constant but κ and τ are not constant (Hacısalihoğlu and Öztürk [5,6]).

2. Harmonic 1-type non-null curves and weak biharmonic non-null curves on Lorentzian 3-manifolds

In this section, we give the characterizations of harmonic 1-type and weak biharmonic non-null curves in Lorentzian 3-space.

Theorem 2.1. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). Then, along the curve γ, $\Delta^{\perp}\mathrm{H} = \lambda\mathrm{H}$ holds if and only if*

$$\varepsilon_1\varepsilon_2\kappa\tau^2 - \varepsilon_1\varepsilon_3\kappa'' = \varepsilon_3\lambda\kappa, \;\; 2\kappa'\tau + \kappa\tau' = 0. \tag{2.1}$$

Proof. From (1.5), (1.6) and (1.7) we get

$$\Delta^{\perp}\mathrm{H} = (-\varepsilon_1\varepsilon_2\kappa\tau^2 + \varepsilon_1\varepsilon_3\kappa'')V_2 - (2\kappa'\tau + \kappa\tau')V_3. \tag{2.2}$$

Then, from (2.2) and (1.8) we have

$$(-\varepsilon_1\varepsilon_2\kappa\tau^2 + \varepsilon_1\varepsilon_3\kappa'')V_2 - (2\kappa'\tau + \kappa\tau')V_3 = \varepsilon_3\lambda\kappa V_2. \tag{2.3}$$

According to (2.3), the equations (2.1) are obtained.
Conversely, the equations (2.1) satisfy the equation (1.8). □

Theorem 2.2. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). Then, along the curve γ, $\Delta^{\perp}\mathrm{H} = \lambda\mathrm{H}$ holds if γ is a circular helix, where*

$$\lambda = -\tau^2. \tag{2.4}$$

Proof. Since γ is a circular helix, the curvatures κ and τ are constants. Then (2.1) holds and we have $\lambda = -\tau^2$. From that $\Delta^{\perp}\mathrm{H} = \lambda\mathrm{H}$ is satisfied.□

Corollary 2.1. *Let γ be a non-null Frenet curve on Lorentzian 3-manifold (M, g). Then γ is harmonic 1-type if γ is a circular helix.*

Theorem 2.3. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). Then, γ is weak biharmonic curve if and only if*

$$\varepsilon_1\varepsilon_2\kappa\tau^2 - \varepsilon_1\varepsilon_3\kappa'' = 0, \;\; 2\kappa'\tau + \kappa\tau' = 0. \tag{2.5}$$

Proof. Since γ is weak biharmonic, $\Delta^{\perp}\mathrm{H} = 0$ holds. From Theorem 2.1, we have (2.5). Conversely, the Equations (2.5) satisfy the equation $\Delta^{\perp}\mathrm{H} = 0$. Thus, γ is weak biharmonic curve. □

Corollary 2.2. *Let γ be a non-null Frenet curve on Lorentzian 3-manifold (M, g). Then γ is weak biharmonic if γ is a pseudo circle (degenerate helix).*

Example 2.1. Let consider the spacelike curve $\gamma(s)$ given by the parametrization

$$\gamma(s) = \left(a\cosh\frac{s}{\sqrt{a^2+b^2}},\; a\sinh\frac{s}{\sqrt{a^2+b^2}},\; \frac{bs}{\sqrt{a^2+b^2}}\right); \;\; a, b > 0.$$

Frenet vectors of $\gamma(s)$ are found as follows

$$\begin{aligned}
V_1(s) &= \frac{1}{\sqrt{a^2+b^2}}\left(a\sinh\frac{s}{\sqrt{a^2+b^2}},\; a\cosh\frac{s}{\sqrt{a^2+b^2}},\; b\right),\\
V_2(s) &= \left(\cosh\frac{s}{\sqrt{a^2+b^2}},\; \sinh\frac{s}{\sqrt{a^2+b^2}},\; 0\right),\\
V_3(s) &= \frac{1}{\sqrt{a^2+b^2}}\left(b\sinh\frac{s}{\sqrt{a^2+b^2}},\; b\cosh\frac{s}{\sqrt{a^2+b^2}},\; -a\right).
\end{aligned}$$

Then, curvature and torsion of $\gamma(s)$ are

$$\kappa = \frac{a}{a^2+b^2}, \quad \tau = -\frac{b}{a^2+b^2},$$

respectively. From (1.6) the mean curvature vector field of $\gamma(s)$ is

$$H(s) = -\frac{a}{a^2+b^2}\left(\cosh\frac{s}{\sqrt{a^2+b^2}},\; \sinh\frac{s}{\sqrt{a^2+b^2}},\; 0\right).$$

Then by using (1.7) we have

$$\Delta^{\perp}\mathrm{H} = \frac{ab^2}{(a^2+b^2)^3}\left(\cosh\frac{s}{\sqrt{a^2+b^2}},\ \sinh\frac{s}{\sqrt{a^2+b^2}},\ 0\right),$$

which gives us $\Delta^{\perp}\mathrm{H}(s) = \frac{-b^2}{(a^2+b^2)^2}H(s)$. By Definition 1.1 we have that $\gamma(s)$ is harmonic 1-type curve with $\lambda = \frac{-b^2}{(a^2+b^2)^2}$. Moreover, it is easily seen that (2.1) holds.

Furthermore, since κ and τ are constants, $\gamma(s)$ is a spacelike circular helix. Then by Theorem 2.2 we have $\lambda = -\tau^2$ which can be seen from the obtained equalities of λ and τ.

Example 2.2. Let now consider timelike curve $\gamma(s)$ given by $\gamma(s) = (\sinh s, \cosh s, 0)$ which is a pseudo-circle. Then Frenet vectors of $\gamma(s)$ are

$$V_1(s) = (\cosh s, \sinh s, 0), \quad V_2(s) = (\sinh s, \cosh s, 0), \quad V_3(s) = (0,0,1)$$

and curvature and torsion are $\kappa = 1$, $\tau = 0$ respectively. From (1.6) the mean curvature vector field of $\gamma(s)$ is $H(s) = -(\sinh s, \cosh s, 0)$. Then by using (1.7) we have $\Delta^{\perp}\mathrm{H} = 0$ which gives us that $\gamma(s)$ is weak-biharmonic timelike circle(degenerate timelike helix) and it is easily seen that (2.5) holds.

3. General characterizations for a non-null Frenet curve and some results

In this section, we give the general characterizations of a Frenet curve by means of the mean curvature vector H and Frenet vectors.

Theorem 3.1. *Let γ be a unit speed non-null Frenet curve on Lorentzian 3-manifold (M,g). According to the normal connection, the following equation holds*

$$\Delta^{\perp}\mathrm{H} + \lambda\nabla^{\perp}_{\gamma'}\mathrm{H} + \mu\mathrm{H} = 0, \tag{3.1}$$

where

$$\lambda = \frac{2\kappa'\tau + \kappa\tau'}{\kappa\tau}, \quad \mu = \frac{\kappa\tau^2 + \varepsilon_1\kappa''}{\kappa} - \lambda\frac{\kappa'}{\kappa}. \tag{3.2}$$

Proof. By (2.2), (1.6) and Frenet formulas (1.5) we get

$$\Delta^{\perp}\mathrm{H} = (-\varepsilon_1\varepsilon_2\kappa\tau^2 + \varepsilon_1\varepsilon_3\kappa'')V_2 - (2\kappa'\tau + \kappa\tau')V_3 \tag{3.3}$$

and

$$\nabla^{\perp}_{\gamma'}H = -\varepsilon_3\kappa' V_2 + \kappa\tau V_3. \tag{3.4}$$

Furthermore, from (1.5) and (1.6) we have

$$V_2 = -\varepsilon_3\frac{1}{\kappa}H, \quad V_3 = \frac{1}{\tau}\left[\left(\frac{1}{\kappa}\right)' H + \frac{1}{\kappa}\nabla^{\perp}H\right]. \tag{3.5}$$

Substituting the equations (3.5) into the (3.4) we have $\Delta^{\perp}\mathrm{H}+\lambda\nabla^{\perp}_{\gamma'}\mathrm{H}+\mu\mathrm{H} = 0$ where $\lambda = \frac{2\kappa'\tau+\kappa\tau'}{\kappa\tau}$, $\mu = \frac{\kappa\tau^2+\varepsilon_1\kappa''}{\kappa} - \lambda\frac{\kappa'}{\kappa}$. □

From Theorem 3.1 we have the following corollaries:

Corollary 3.1. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). If γ is a circular helix, then according to the normal connection, $\Delta^{\perp}\mathrm{H} + \tau^2\mathrm{H} = 0$ holds.*

Proof. The proof is clear from (3.1) and (3.2). □

Corollary 3.2. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). If γ is a general helix, then according to the normal connection, $\Delta^{\perp}\mathrm{H} + \lambda\nabla^{\perp}_{\gamma'}\mathrm{H} + \mu\mathrm{H} = 0$ holds, where*

$$\lambda = 3\frac{\kappa'}{\kappa}, \quad \mu = \frac{\kappa\tau^2 + \varepsilon_1\kappa''}{\kappa} - 3\left(\frac{\kappa'}{\kappa}\right)^2.$$

Proof. The proof can be easily seen from (3.1) and (3.2). □

From Corollary 3.2 we can find the following result.

Result 3.1. Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). Then according to the normal connection

i) If γ is a timelike general helix, then

$$\Delta^{\perp}\mathrm{H} + 3\frac{\kappa'}{\kappa}\nabla^{\perp}_{\gamma'}\mathrm{H} + \left[\frac{\kappa\tau^2 - \kappa''}{\kappa} - 3\left(\frac{\kappa'}{\kappa}\right)^2\right]\mathrm{H} = 0$$

holds.

ii) If γ is a spacelike general helix, then

$$\Delta^{\perp}\mathrm{H} + 3\frac{\kappa'}{\kappa}\nabla^{\perp}_{\gamma'}\mathrm{H} + \left[\frac{\kappa\tau^2 + \kappa''}{\kappa} - 3\left(\frac{\kappa'}{\kappa}\right)^2\right]\mathrm{H} = 0.$$

When the curve γ is a geodesic we have Theorem 2.3 again, i.e., γ is a weak biharmonic curve.

Theorem 3.2. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M,g). According to the normal connection, γ satisfies the following differential equation*

$$\tau\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 - \tau'\,\nabla^{\perp}_{\gamma'}V_2 - \varepsilon_1\tau^3 V_2 = 0. \tag{3.6}$$

Proof. From the Frenet formulas (1.5) we have

$$\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 = \varepsilon_1\tau^2 V_2 - \varepsilon_3\tau'\,V_3 \tag{3.7}$$

and since $V_3 = -\varepsilon_3\frac{1}{\tau}\nabla^{\perp}_{\gamma'}V_2$, from (3.7) we have the equation (3.6). □

From Theorem 3.2 we have the following corollary.

Corollary 3.3. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M,g). If γ is a circular helix then the following equation holds* $\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 - \varepsilon_1\tau^2 V_2 = 0$.

Proof. The proof is clear from (3.6). □

Result 3.2. Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M,g).

i) If γ is a timelike circular helix then $\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 + \tau^2 V_2 = 0$ holds.

ii) If γ is a spacelike circular helix then $\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 - \tau^2 V_2 = 0$ holds.

Corollary 3.4. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M,g). If γ is a general helix then according to the normal connection the following equality holds* $\kappa\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 - \kappa'\,\nabla^{\perp}_{\gamma'}V_2 - \varepsilon_1\kappa\tau^2 V_2 = 0$.

Proof. Since γ is a general helix, by taking $\frac{\kappa}{\tau} = const.$, i.e., $\tau' = \frac{\kappa'}{\kappa}\tau$, from (3.6) the proof follows immediately. □

Result 3.3. Let γ *be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold* (M,g). Then

i) If γ is a timelike general helix, then $\kappa\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 - \kappa'\,\nabla^{\perp}_{\gamma'} V_2 + \kappa\tau^2 V_2 = 0$, holds.

ii) If γ is a spacelike general helix, then $\kappa\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2 - \kappa'\,\nabla^{\perp}_{\gamma'} V_2 - \kappa\tau^2 V_2 = 0$, holds.

Theorem 3.3. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). According to the normal connection, γ satisfies the following differential equation*

$$\tau\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 - \tau'\,\nabla^{\perp}_{\gamma'} V_3 - \varepsilon_1\tau^3 V_3 = 0. \tag{3.8}$$

Proof. From the Frenet formulas (1.5) we have

$$\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 = \varepsilon_2\tau'\,V_2 - \varepsilon_2\varepsilon_3\tau^2 V_3 \tag{3.9}$$

and since $V_2 = \varepsilon_2\frac{1}{\tau}\nabla^{\perp}_{\gamma'} V_3$, from (3.9) we have the equation (3.8). □

From Theorem 3.3 we have the following corollary.

Corollary 3.5. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). If γ is a circular helix then the following equation holds,* $\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 - \varepsilon_1\tau^2 V_3 = 0$.

Proof. By (3.8), the proof can be easily seen. □

Result 3.4. Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g).

i) If γ is a timelike circular helix then $\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 + \tau^2 V_3 = 0$ holds.

ii) If γ is a spacelike circular helix then $\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 - \tau^2 V_3 = 0$ holds.

Corollary 3.6. *Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M, g). If γ is a general helix then according to the normal connection the following equality holds*

$$\kappa\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 + \kappa'\,\nabla^{\perp}_{\gamma'} V_3 - \varepsilon_1\kappa\tau^2 V_3 = 0.$$

Proof. Since γ is a general helix, by taking $\frac{\kappa}{\tau} = const.$, i.e., $\tau' = \frac{\kappa'}{\kappa}\tau$ from (3.8) the proof is straightforward. □

Result 3.5. Let γ be an arclength parametrized non-null Frenet curve on Lorentzian 3-manifold (M,g). Then

i) If γ is a timelike general helix, then $\kappa\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3+\kappa'\,\nabla^{\perp}_{\gamma'}V_3+\kappa\tau^2V_3=0$ holds.

ii) If γ is a spacelike general helix, then $\kappa(\nabla^{\perp}_{\gamma'})^2V_3+\kappa'\,\nabla^{\perp}_{\gamma'}V_3-\kappa\tau^2V_3=0$ holds.

Example 3.1. For the timelike curve $\gamma(s)$ given by the parametrization

$$\gamma(s)=\left(\frac{s^2}{2}+s,\ \frac{s^2}{2},\ \frac{2\sqrt{2}}{3}s^{3/2}\right);\ \ (s\neq 0),$$

the Frenet vectors are obtained as follows,

$$V_1(s)=(s+1,\ s,\ \sqrt{2s}),\ \ V_2(s)=(\sqrt{2s},\ \sqrt{2s},\ 1),\ \ V_3(s)=\left(1,\ 1,\ \frac{1}{\sqrt{2s}}\right),$$

and, curvature and torsion functions are $\kappa=\frac{\sqrt{2}}{2\sqrt{s}}$, $\tau=\frac{\sqrt{2}}{2\sqrt{s}}$, respectively. From (1.6) the mean curvature vector field of $\gamma(s)$ is

$$H(s)=\left(-\frac{\sqrt{2}}{s},\ -\frac{\sqrt{2}}{s},\ -\frac{1}{s^{3/2}}\right).$$

Then we have the followings

$$\Delta^{\perp}\mathrm{H}=\left(\frac{2s-3}{4s^2},\ \frac{2s-6}{4s^2},\ \frac{\sqrt{2}(2s-3)}{8s^{5/2}}\right),\ \ \nabla^{\perp}_{\gamma'}\mathrm{H}=\left(\frac{1}{2s},\ \frac{1}{2s},\ \frac{\sqrt{2}}{4s^{3/2}}\right).$$

Thus it is obtained that (3.1) holds for $\lambda=-\frac{3}{2s}$, $\mu=\frac{\sqrt{2}(2s-3)-6\sqrt{s}}{8s^{5/2}}$. Similarly, Frenet vector V_2 satisfies equation (3.6) and equation (3.8) holds for the Frenet vector V_3.

Moreover, we have that $\frac{\kappa}{\tau}$ is constant i.e., $\gamma(s)$ is a general timelike helix. Then it is obvious that $\gamma(s)$ satisfies the conditions of Corollary 3.2, Result 3.1 (i), Corollary 3.4, Result 3.3 (i) and Result 3.5 (i).

4. Harmonic 1-type null curves and weak biharmonic null curves on Lorentzian 3-manifolds

Let γ be a curve on Lorentzian 3-manifold (M,g). Then, the curve γ is called a null curve if $g(V_1, V_1) = 0$. By a Cartan frame $\{V_1, V_2, V_3\}$ of γ we mean a family of vector fields $V_1 = V_1(s)$, $V_2 = V_2(s)$, $V_3 = V_3(s)$ along the curve γ satisfying the following conditions:

$$\gamma'(s) = V_1, \quad g(V_1, V_1) = g(V_2, V_2) = 0, \quad g(V_1, V_2) = -1,$$
$$g(V_1, V_3) = g(V_2, V_3) = 0, \quad g(V_3, V_3) = 1,$$

$$\begin{bmatrix} \nabla_{\gamma'} V_1 \\ \nabla_{\gamma'} V_2 \\ \nabla_{\gamma'} V_3 \end{bmatrix} = \begin{bmatrix} 0 & 0 & \kappa \\ 0 & 0 & \tau \\ \tau & \kappa & 0 \end{bmatrix} \begin{bmatrix} V_1 \\ V_2 \\ V_3 \end{bmatrix}, \tag{4.1}$$

where κ and τ are called the curvature and torsion of γ, respectively (Ikawa [7]). Here V_1 and V_2 are null vectors and V_3 is a unit spacelike vector.

According to (1.4)

$$\nabla^{\perp}_{\gamma'} V_2 = \tau V_3, \quad \nabla^{\perp}_{\gamma'} V_3 = \kappa V_2. \tag{4.2}$$

Let us denote by H the mean curvature vector field of γ and by the normal Laplace-Beltrami operator $\Delta^{\perp}$ of γ. Then the mean curvature vector field H is defined by

$$\mathrm{H} = \nabla_{\gamma'} V_1 = \kappa V_3 \tag{4.3}$$

and the normal Laplacian operator along γ is given by

$$\Delta^{\perp} = -\left(\nabla^{\perp}_{\gamma'}\right)^2 = -\nabla^{\perp}_{\gamma'} \nabla^{\perp}_{\gamma'} \tag{4.4}$$

(see Chen and Ishikawa [2,3]).

Theorem 4.1. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M, g). Then, along the curve $\gamma, \Delta^{\perp}\mathrm{H} = \lambda \mathrm{H}$ holds if and only if*

$$\kappa^2 \tau + \kappa'' = -\lambda \kappa, \quad \kappa' \kappa = 0. \tag{4.5}$$

Proof. From (4.2), (4.3) and (4.4) we get

$$\Delta^{\perp}\mathrm{H} = -3\kappa\kappa' V_2 - (\kappa^2 \tau + \kappa'') V_3. \tag{4.6}$$

By (4.3) and (1.8), we have

$$-3\kappa\kappa' V_2 - (\kappa^2 \tau + \kappa'') V_3 = \lambda \kappa V_3. \tag{4.7}$$

According to (4.7), the equations (4.5) are obtained. □

Conversely, the equations (4.5) satisfy the equation $\Delta^{\perp}\mathrm{H} = \lambda \mathrm{H}$.

Theorem 4.2. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). Then, along the curve $\gamma, \Delta^{\perp}\mathrm{H} = \lambda\mathrm{H}$ holds if and only if γ is a null curve with constant curvature, where $\lambda = -\kappa\tau$.*

Proof. From Theorem 4.1, we have (4.5). Since γ is a Frenet curve, $\kappa \neq 0$. Thus (4.5) shows that κ is a constant and $\lambda = -\kappa\tau$. Thus γ is a null curve with constant curvature.

Conversely, since γ is a null curve with constant curvature and $\lambda = -\kappa\tau$, $\Delta^{\perp}\mathrm{H} = \lambda\mathrm{H}$ is satisfied. □

Corollary 4.1. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). Then, γ is harmonic 1-type if and only if γ is a null curve with constant curvature.*

Theorem 4.3. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). Then, along the curve $\gamma, \Delta^{\perp}\mathrm{H} = 0$ holds if and only if*

$$\kappa^2\tau + \kappa'' = 0, \quad \kappa'\kappa = 0. \tag{4.8}$$

Proof. From Theorem 4.1 we have (4.8). Conversely, (4.8) satisfy the equation $\Delta^{\perp}\mathrm{H} = 0$. □

Theorem 4.4. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). Then, along the curve $\gamma, \Delta^{\perp}\mathrm{H} = 0$ holds if and only if γ is a pseudo null circle(degenerate null helix).*

Proof. From Theorem 4.3, we have (4.8). Since γ is a Frenet curve, $\kappa \neq 0$. Thus, the equations (4.8) show that γ is e degenerate null helix.

Conversely, since γ is e degenerate null helix, κ is a constant and $\tau = 0$. Thus the equations (4.8) satisfy the equation $\Delta^{\perp}\mathrm{H} = 0$. □

Corollary 4.2. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). Then γ is a weak biharmonic if and only if γ is a pseudo null circle (degenerate null helix).*

Example 4.1. Let consider the parametrized null curve $\gamma(s)$ given by $\gamma(s) = \left(\frac{1}{2}\sinh 2s,\ \frac{1}{2}\cosh 2s,\ s\right)$, (see Balgetir, Bektaş and Inoguchi [1]). Then, Frenet vectors of $\gamma(s)$ are obtained as follows

$$\begin{aligned} V_1(s) &= \gamma'(s) = (\cosh 2s,\ \sinh 2s,\ 1), \\ V_2(s) &= \left(\frac{1}{2}\cosh 2s,\ \frac{1}{2}\sinh 2s,\ -\frac{1}{2}\right), \\ V_3(s) &= (\sinh 2s,\ \cosh 2s,\ 0). \end{aligned}$$

The curvature and torsion of $\gamma(s)$ are $\kappa = 2$, $\tau = -1$ respectively. From (4.3) mean curvature vector field of $\gamma(s)$ is $H(s) = (2\sinh 2s,\ \cosh 2s,\ 0)$. Then, by considering (4.4) we have $\Delta^{\perp}\mathrm{H} = 2\mathrm{H}$ which gives that $\gamma(s)$ is harmonic 1-type null curve and it is clear that $\gamma(s)$ satisfies the condition of Theorem 4.1, Theorem 4.2 and Corollary 4.1.

5. General characterizations for a null Frenet curve and some results

In this section, we give the general characterizations of a null Frenet curve by means of the mean curvature vector H and Frenet vectors according to the normal Laplacian operator $\Delta^{\perp}$ and the normal connection $\nabla^{\perp}$.

Theorem 5.1. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M, g). According to the normal connection, the following equation holds*

$$\Delta^{\perp}\mathrm{H} + \lambda\nabla^{\perp}_{\gamma'}\mathrm{H} + \mu\mathrm{H} = 0, \tag{5.1}$$

where

$$\lambda = 3\frac{\kappa'}{\kappa}, \quad \mu = \frac{\kappa^2\tau + \kappa''}{\kappa} - 3\left(\frac{\kappa'}{\kappa}\right)^2. \tag{5.2}$$

Proof. By (4.6), (4.3) and Frenet formulas (4.1)-(4.2), we get

$$\Delta^{\perp}\mathrm{H} = -3\kappa\kappa' V_2 - (\kappa^2\tau + \kappa'')V_3, \tag{5.3}$$

$$\nabla^{\perp}_{\gamma'} H = \kappa^2 V_2 + \kappa' V_3. \tag{5.4}$$

Furthermore, from (4.2) and (4.3) we have

$$V_3 = \frac{1}{\kappa}H, \quad V_2 = \frac{1}{\tau}\left[\left(\frac{1}{\kappa}\right)' H + \frac{1}{\kappa}\nabla^{\perp} H\right]. \tag{5.5}$$

Substituting the equations (5.5) into the (5.4) we have $\Delta^{\perp}\mathrm{H} + \lambda\nabla^{\perp}_{\gamma'}\mathrm{H} + \mu\mathrm{H} = 0$, where $\lambda = 3\frac{\kappa'}{\kappa}$, $\mu = \frac{\kappa^2\tau+\kappa''}{\kappa} - 3(\frac{\kappa'}{\kappa})^2$. □

From Theorem 5.1 we have the following corollaries:

Corollary 5.1. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M, g). If γ is a circular null helix, then according to the normal connection $\Delta^{\perp}\mathrm{H} + \kappa\tau\mathrm{H} = 0$ holds.*

Proof. The proof is clear from (5.1) and (5.2). □

Corollary 5.2. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). If γ is a general null helix then according to the normal connection the following equality holds $\Delta^{\perp}\mathrm{H}+\lambda\nabla^{\perp}_{\gamma'}\mathrm{H}+\mu\mathrm{H}=0$, where*

$$\lambda=3\frac{\kappa'}{\kappa},\quad \mu=\frac{\kappa^2}{\kappa'}\tau'+\frac{\kappa''}{\kappa}-3\left(\frac{\kappa'}{\kappa}\right)^2.$$

Proof. By taking $\tau=\frac{\kappa}{\kappa'}\tau'$, the proof can be easily seen from (5.1) and (5.2). □

Theorem 5.2. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). According to the normal connection, γ satisfies the following differential equation*

$$\tau\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2-\tau'\,\nabla^{\perp}_{\gamma'}V_2-\kappa\tau^2V_2=0. \tag{5.6}$$

Proof. From the Frenet formulas (4.2) we have

$$\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2=\kappa\tau V_2+\tau'\,V_3 \tag{5.7}$$

and since $V_3=\frac{1}{\tau}\nabla^{\perp}_{\gamma'}V_2$, from (5.7) we have the equation (5.6).

□

From Theorem 5.2 we have the following corollary.

Corollary 5.3. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). If γ is a circular null helix then the following equation holds $(\nabla^{\perp}_{\gamma'})^2V_2-\kappa\tau V_2=0$.*

Proof. The proof is clear from (5.6). □

Corollary 5.4. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M,g). If γ is a general null helix then according to the normal connection the following equality holds $\kappa\left(\nabla^{\perp}_{\gamma'}\right)^2 V_2-\kappa'\,\nabla^{\perp}_{\gamma'}V_2-\kappa^2\tau V_2=0$.*

Proof. Since γ is a general helix, by taking $\frac{\kappa}{\tau}=const.$ i.e.,$\tau'=\frac{\kappa'}{\kappa}\tau$, from (5.6) the proof follows immediately. □

Theorem 5.3. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M, g). According to the normal connection, γ satisfies the following differential equation*

$$\kappa \left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 - \kappa' \nabla^{\perp}_{\gamma'} V_3 - \kappa^2 \tau V_3 = 0. \tag{5.8}$$

Proof. From the Frenet formulas (4.2) we have

$$\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 = \kappa' V_2 + \kappa\tau V_3 \tag{5.9}$$

and since $V_2 = \frac{1}{\kappa}\nabla^{\perp}_{\gamma'} V_3$, from (5.9) we have the equation (5.8). □

From Theorem 5.3 we have the following corollary.

Corollary 5.5. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M, g). If γ is a circular null helix then the following equation holds, $\left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 - \kappa\tau V_3 = 0$.*

Proof. By (5.8), the proof can be easily seen. □

Corollary 5.6. *Let γ be a null Frenet curve on Lorentzian 3-manifold (M, g). If γ is a general null helix then according to the normal connection the following equality holds $\tau \left(\nabla^{\perp}_{\gamma'}\right)^2 V_3 - \tau' \nabla^{\perp}_{\gamma'} V_3 - \kappa\tau^2 V_3 = 0$.*

Proof. Since γ is a general helix, by taking $\frac{\kappa}{\tau} = const.$ i.e., $\tau' = \frac{\kappa'}{\kappa}\tau$ from (5.8) the proof is straightforward.

□

The null curve $\gamma(s)$ given in Example 4.1 can be taken as an example of Section 5. It is easily seen that $\gamma(s)$ satisfies all conditions given in theorems and corollaries of this section.

REFERENCES

1. Balgetir, H.; Bektaş, M.; Inoguchi, J. – *Null Bertrand curves in Minkowski 3-space and their characterizations*, Note Mat., 23 (2004/05), 7–13.

2. Chen, B.-Y.; Ishikawa, S. – *Biharmonic surfaces in pseudo-Euclidean spaces*, Mem. Fac. Sci. Kyushu Univ. Ser. A, 45 (1991), 323–347.

3. CHEN, B.-Y.; ISHIKAWA, S. – *Biharmonic pseudo-Riemannian submanifolds in pseudo-Euclidean spaces*, Kyushu J. Math., 52 (1998), 167–185.

4. FERRÁNDEZ, A.; LUCAS, P.; MEROÑO, M.A. – *Biharmonic Hopf cylinders*, Rocky Mountain J. Math., 28 (1998), 957–975.

5. HACISALIHOGLU, H.H.; ÖZTÜRK, R. – *On the characterization of inclined curves in E^n I.*, Tensor (N.S.), 64 (2003), 157–162.

6. HACISALIHOGLU, H.H.; ÖZTÜRK, R. – *On the characterization of inclined curves in E^n II.*, Tensor (N.S.), 64 (2003), 163–169.

7. IKAWA, T. – *On curves and submanifolds in an indefinite-Riemannian manifold*, Tsukuba J. Math., 9 (1985), 353–371.

8. IZUMIYA, S.; TAKIYAMA, A. – *A time-like surface in Minkowski 3-space which contains pseudocircles*, Proc. Edinburgh Math. Soc., 40 (1997), 127–136.

9. IZUMIYA, S.; TAKIYAMA, A. – *A time-like surface in Minkowski 3-space which contains light-like lines*, J. Geom., 64 (1999), 95–101.

10. KILIÇ, B. – *Finite type curves and surfaces,* Ph. Thesis, University of Hacettepe, 2002.

11. KOBAYASHI, O. – *Maximal surfaces in the 3-dimensional Minkowski space L^3*, Tokyo J. Math., 6 (1983), 297–309.

12. KOCAYIGIT, H.; HACISALIHOĞLU, H.H. – *1-type and biharmonic Frenet curves in Lorentzian 3-space*, Iran. J. Sci. Technol. Trans. A Sci., 33 (2009), 159–168.

13. KOCAYIGIT, H.; HACISALIHOĞLU, H.H. – *1-type curves and biharmonic curves in Euclidean 3-space*, Int. Electron. J. Geom., 4 (2011), 97–101.

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