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Representation formulae for Bertrand curves in the Minkowski 3-space

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Abstract In this paper, we study the representation formulae for Bertrand curves in the Minkowski 3-space.

Keywords Bertrand curve, Minkowski space, represantation formulae.

§1. Introduction

Bertrand curves are one of the important and interesting topic of classical spatial curve theory [4, 6, 10]. A Bertand curve is defined as a spatial curve which shares its principal normals with another spatial curve (called *Bertrand mate*). Note that Bertrand mates are particular examples of offset curves used in computer-aided design CADG, (see [5]).

Bertrand curves are characterised as spatial curves whose curvature and torsion are in linear relation. Thus Bertrand curves may be regraded as one-dimensional analogue of Weingarten surfaces [9]. Application of Weingarten surfaces to CADG, (see [8]).

Bertrand curves and their geodesic imbedding in surfaces are recently rediscovered and studied in the context of modern soliton theory by Schief [7].

Straightforward modification of classical theory to spacelike or timelike curves in Minkowski 3-space is easily obtained, (see [1]). Null Bertrand curves in Minkowski 3-space are studied in [2]. Nonnull Bertrand curves in the n-dimensional Lorentzian space are examined in [3].

However in [1]-[2], representation formulae for Bertrand curves are not obtained.

In this paper, we study representation formulae for Bertrand curves in Minkowski 3-space.

§2. Bertrand curves and Representation Formulae in Minkowski 3-space

In this section, we collect classical results on Bertrand curves in Minkowski 3-space $\mathbb{E}_1^3$.

Let $\mathbb{E}_1^3$ be the Minkowski 3-space and γ a regular non-null curve. Then γ can be parametrised by the unit speed parameter s;

$$\langle \gamma'(s), \gamma'(s) \rangle = \varepsilon_1 = \pm 1.$$

If $\gamma(s)$ is spacelike (resp. timelike), s is called the *arclength parameter* (resp.*proper time parameter*). Let us denote by T the tangent vector field of γ;

$$T(s) := \gamma'(s).$$

Hereafter, in case $\varepsilon_1 = 1$ (spacelike curve), we assume that the acceleration vector field T' is nonnull. Then there exist vector fields N and B along γ such that

$$T' = \varepsilon_2 \kappa N, \quad N' = -\varepsilon_1 \kappa T - \varepsilon_3 \tau B, \quad B' = \varepsilon_2 \tau N. \tag{1}$$

Here ε_2 and ε_3 are *second* and *third causal characters* of γ defined by

$$\varepsilon_2 = \langle N, N\rangle, \ \varepsilon_3 = \langle B, B\rangle.$$

The vector field N and B are called the *principal normal* and *binormal vector field* of γ respectively. The functions κ and τ are called the *curvature* and *torsion* of γ respectively.

If there exists a spatial curve $\bar{\gamma}(\bar{s})$ whose principal normal direction coincides with that of original curve, then γ is said to be a *Bertrand curve*. The pair $(\gamma, \bar{\gamma})$ is said to be a *Bertrand mate*.

There are several possibilities for Bertrand mates denoted by$\{\ \overline{\varepsilon}_i\}$, the causal characters of the Bertrand mate $\bar{\gamma}$. Then by definition, $\overline{\varepsilon}_2 = \varepsilon_2$.

1. γ is spacelike with $\varepsilon_2 = 1$. In this case there are two subcases.

(a) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (+1, +1, -1)$: In this case, the mate is also spacelike. Both the rectifying planes of γ and $\bar{\gamma}$ are timelike. Thus the tangent vector fields are related by

$$\bar{T} = \pm(\cosh\theta T + \sinh\theta B)$$

for some function $\theta = \theta(s)$.

(b) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (-1, +1, +1)$: In this case, the mate is timelike. Both the rectifying planes of γ and $\bar{\gamma}$ are timelike. Thus the tangent vector fields are related by

$$\bar{T} = \pm(\sinh\theta T + \cosh\theta B)$$

for some function $\theta = \theta(s)$.

2. γ is spacelike with $\varepsilon_2 = -1$. Then $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (+1, -1, +1)$. Both the rectifying planes are spacelike. Thus

$$\bar{T} = \cos\theta T + \sin\theta B$$

for some function $\theta = \theta(s)$.

3. γ is timelike. In this case there are two subcases.

(a) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (+1, +1, -1)$: In this case, the mate is timelike. The tangent vector fields are related by

$$\bar{T} = \pm(\sinh\theta T + \cosh\theta B)$$

for some function $\theta = \theta(s)$.

(b) $(\overline{\varepsilon}_1, \overline{\varepsilon}_2, \overline{\varepsilon}_3) = (-1, +1, +1)$: In this case, the mate is timelike. The tangent vector fields are related by

$$\bar{T} = \pm(\cosh\theta T + \sinh\theta B)$$

for some function $\theta = \theta(s)$.

One can see that the case 3 is reduced to case 1. Thus we may restrict our study to case 1 and case 2.

Now let consider **case 1-(a)**:

Let $(\gamma, \bar{\gamma})$ be a Bertrand mate, then

$$\bar{\gamma}(\bar{s}) = \gamma(s) + u(s)N(s) \tag{2}$$

for some function $u(s) \neq 0$. Differentiating this, we get

$$\bar{T}(\bar{s})\frac{d\bar{s}}{ds} = (1 - \varepsilon_1 u(s)\kappa(s))T(s) + u'(s)\, N(s) + \varepsilon_3 u(s)\tau(s)\, B(s). \tag{3}$$

Since $\bar{T} \perp N$,

$$\langle \bar{T}, N\rangle \bar{s}_s = u' = 0.$$

Hence u is a nonzero constant. Denote by θ the angle between γ and $\bar{\gamma}$:

$$\bar{T} = \varepsilon(\cosh\theta\, T + \sinh\theta\, B),\ \ \varepsilon = \pm 1 \tag{4}$$

Then computing the inner product of (3) and (4), we have

$$\frac{1 - \varepsilon_1 u\kappa}{\cosh\theta} = \frac{\varepsilon_3 u\tau}{\sinh\theta} = \frac{d\bar{s}}{ds}. \tag{5}$$

Differentiating (4),

$$\varepsilon_2\bar{\kappa}\bar{s}_s\bar{N} = (\varepsilon\varepsilon_2\kappa\cosh\theta + \varepsilon\varepsilon_2\tau\sinh\theta)N + \varepsilon\theta'(\sinh\theta T + \cosh\theta B).$$

By the assumption,

$$\bar{N} = \pm N.$$

Hence

$$\theta' = 0,\ \ \bar{\kappa}\bar{s}_s = \varepsilon(\kappa\cosh\theta + \tau\sinh\theta).$$

Thus θ is a constant. If $\sinh\theta = 0$, then from (5), $\tau = 0$. In this case, γ is a planar curve. Note that planar curves are Bertrand curves. In fact, planar curves together with their parallel curves are Bertrand mates.

Next, if $\sin\theta \neq 0$, then (5) is written in the form:

$$a\kappa + b\tau = 1, \tag{6}$$

for constants a and b.

Conversely, if a spatial curve γ satisfies (6), then define $\bar{\gamma}$ by (2). Then

$$\bar{T} = \varepsilon(\cosh\theta T + \sinh\theta B).$$

Differentiating this by s, we obtain

$$\bar{\kappa}\bar{N}\bar{s}_s = \varepsilon(\kappa\cosh\theta + \tau\sinh\theta)N.$$

Hence γ is a Bertrand curve.

Thus we obatin the following result:

Theorem 1. A spatial curve is a Bertand curve in Minkowski 3-space $\mathbb{E}_1^3$ if and only if its curvature and torsion satisfy $a\kappa + b\tau = 1$ for some constants a and b.

Theorem 2. Let $(\gamma, \bar{\gamma})$ be a Bertrand mate in Minkowski 3-space $\mathbb{E}_1^3$. Then $\tau(s)\bar{\tau}(\bar{s})$ is a constant.

Proof of Theorem 2. From (2)–(6),

$$\tau = -\frac{\sinh\theta}{u}\frac{d\bar{s}}{ds}, \quad \bar{\tau} = -\frac{\sinh\bar{\theta}}{\bar{u}}\frac{ds}{d\bar{s}}, \quad \bar{u} = \pm u.$$

Hence

$$\tau\bar{\tau} = \frac{\sinh\theta\sinh\bar{\theta}}{u\bar{u}} = \text{constant}.$$

Corollary 1. Let γ be a Bertrand curve with $a\kappa + b\tau = 1$ and $\bar{\gamma}$ a Bertrand mate. Then the fundamental quantities of the Bertrand mate are given by

$$\bar{T} = \varepsilon\frac{-bT + aB}{\sqrt{b^2 - a^2}}, \quad \bar{N} = \pm N, \quad \bar{B} = \varepsilon\frac{aT - bB}{\sqrt{b^2 - a^2}},$$

$$\bar{\kappa} = -\varepsilon\frac{(-b\kappa + a\tau)}{(b^2 - a^2)\tau}, \quad \bar{\tau} = -\varepsilon\frac{a\kappa - b\tau}{(b^2 - a^2)\tau}, \quad d\bar{s} = -\sqrt{b^2 - a^2}\,\tau(s)ds.$$

Similarly, if we consider **case 1-(b)** we have following :

Theorem 3. A spatial curve is a Bertand curve in Minkowski 3-space $\mathbb{E}_1^3$ if and only if its curvature and torsion satisfy $a\kappa + b\tau = 1$ for some constants a and b.

Theorem 4. Let $(\gamma, \bar{\gamma})$ be a Bertrand mate in Minkowski 3-space $\mathbb{E}_1^3$. Then $\tau(s)\bar{\tau}(\bar{s})$ is a constant.

Corollary 2. Let γ be a Bertrand curve with $a\kappa + b\tau = 1$ and $\bar{\gamma}$ a Bertrand mate. Then the fundamental quantities of the Bertrand mate are given by

$$\bar{T} = \varepsilon\frac{-bT + aB}{\sqrt{a^2 - b^2}}, \quad \bar{N} = \pm N, \quad \bar{B} = \varepsilon\frac{aT - bB}{\sqrt{a^2 - b^2}},$$

$$\bar{\kappa} = -\varepsilon\frac{(-b\kappa + a\tau)}{(a^2 - b^2)\tau}, \quad \bar{\tau} = -\varepsilon\frac{a\kappa - b\tau}{(a^2 - b^2)\tau}, \quad d\bar{s} = -\sqrt{a^2 - b^2}\,\tau(s)ds.$$

In **case 2**, We obtained the classical results in Euclidean space.

Lemma 1. Let $e(t)$ be a unit vector field which is not parallel to a fixed plane. Take a nonzero constant a. Then

$$\alpha(t) := -\varepsilon_2 a \int e(t) \times \dot{e}(t)dt$$

is a spatial curve of constant torsion $-\varepsilon_2/a$ and binormal vector field $\pm e(t)$.

Proof of Lemma 1. Direct computations show

$$\dot{\alpha} \times \ddot{\alpha} = a^2((e \times \dot{e}) \times (e \times \ddot{e})) = a^2\{\det(e, e, \ddot{e})\dot{e} - \det(\dot{e}, e, \ddot{e})e\} = a^2\det(e, \dot{e}, \ddot{e})e.$$

Here we used the following formula in Minkowski 3-space $\mathbb{E}_1^3$:

$$(x \times y) \times (z \times w) = \det(x, z, w)y - \det(y, z, w)x$$

By the assumption, $\det(e,\dot{e},\ddot{e})\neq 0$, the binormal vector field of α is $B_\alpha=\pm e$.

Next, since $\det(\dot{\alpha},\ddot{\alpha},\dddot{\alpha})=-\varepsilon_2 a^3\det(e,\dot{e},\ddot{e})^2$. Hence the torsion of α is

$$\tau_\alpha=\frac{\det(\dot{\alpha},\ddot{\alpha},\dddot{\alpha})}{|\dot{\alpha}\times\ddot{\alpha}|^2}=\frac{-\varepsilon_2 a^3\ \det(e,\dot{e},\ddot{e})^2}{a^4\ \det(e,\dot{e},\ddot{e})^2}=\frac{-\varepsilon_2}{a}.$$

Conversely, let $\alpha(s)$ be a curve of constant torsion $-\varepsilon_2/a$. Here s is the arclength parameter. Then put $e(s)=B(s)$. Then the Frenet-Serret formula implies

$$e\times\dot{e}=\varepsilon_2\tau\ B\times N=\frac{-\varepsilon_2}{a}\alpha'.$$

Hence $\alpha(s)=-\varepsilon_2 a\int^s e\times\dot{e}ds$.

Lemma 2. If a spatial curve α is of constant nonzero torsion τ_α, then the curve

$$\beta(s)=-\frac{1}{\tau_\alpha}N(s)-\varepsilon_3\int B(s)ds$$

has constant curvature $|\tau_\alpha|$.

Proof of Lemma 2. We use the subscript $\cdot_\alpha$ for expressing geometric objects of α. By the Frenet-Serret formula for α, we have

$$\frac{d\beta}{ds}=-\frac{1}{\tau_\alpha}(-\varepsilon_1\kappa_\alpha T_\alpha-\varepsilon_3\tau_\alpha B_\alpha)-\varepsilon_3 B_\alpha=\varepsilon_1\frac{\kappa_\alpha}{\tau_\alpha}T_\alpha.$$

Hence the unit tangent vectot field of β is $T_\beta=\varepsilon T,\ \varepsilon=\mathrm{sgn}(\tau)$. Hence the arclength parameter s_β of β is

$$s_\beta=\int^s\varepsilon_1\frac{\kappa_\alpha}{|\tau_\alpha|}ds.$$

Thus

$$\frac{dT_\beta}{ds_\beta}=\varepsilon\frac{dT}{ds}\frac{ds}{ds_\beta}=\varepsilon\varepsilon_1\varepsilon_2\ |\tau_\alpha|N,\quad N_\beta=\varepsilon N,\quad \kappa_\beta=|\tau_\alpha|=\text{constant}.$$

Lemma 3. If a spatial curve α is of constant nonzero torsion τ_α, then the curve

$$\beta(s)=a\alpha(s)+b\left(-\frac{1}{\tau_\alpha}N(s)-\varepsilon_3\int B(s)ds\right)$$

is a Bertand curve.

Proof of Lemma 3. Direct computations show that

$$\beta'=(a+\frac{\varepsilon_1 b\kappa_\alpha}{\tau_\alpha})T,\quad \beta''=\frac{\varepsilon_1 b\kappa_\alpha'}{\tau_\alpha}T+\left(a\varepsilon_2\kappa_\alpha+\frac{\varepsilon_1\varepsilon_2 b\kappa_\alpha^2}{\tau_\alpha}\right)N,$$

$$\begin{aligned}\beta'''\ &=\ (b\varepsilon_1\frac{\kappa_\alpha''}{\tau_\alpha}-a\varepsilon_1\varepsilon_2\kappa_\alpha^2-b\varepsilon_2\frac{\kappa_\alpha^3}{\tau_\alpha})T+(a\varepsilon_2\kappa_\alpha'+\frac{3\varepsilon_1\varepsilon_2 b\kappa_\alpha'\kappa_\alpha}{\tau_\alpha})N\\ &\quad+\left(-a\varepsilon_2\varepsilon_3\kappa_\alpha\tau_\alpha-b\varepsilon_1\varepsilon_2\varepsilon_3\kappa_\alpha^2\right)B.\end{aligned}$$

From these

$$\kappa_\beta=\frac{|\beta'\times\beta''|}{|\beta'|^{3/2}}=\frac{-\varepsilon_2\kappa_\alpha}{\left|a+\varepsilon_1 b\frac{\kappa_\alpha}{\tau_\alpha}\right|},$$

$$\tau_\beta = \frac{\det(\beta', \beta'', \beta''')}{|\beta' \times \beta''|^2} = \frac{-\varepsilon_3 \tau_\alpha}{a + \varepsilon_1 b \frac{\kappa_\alpha}{\tau_\alpha}}.$$

Put $\varepsilon = \mathrm{sgn}\{\varepsilon_1 \varepsilon_2 \varepsilon_3 (a + (b\varepsilon_1 \kappa_\alpha / \tau_\alpha)\}$. Then $\varepsilon b \kappa_\beta + a\tau_\beta = -\varepsilon_3 \tau_\alpha =$ constant.

From these Lemma, one can deduce the following:

Theorem 5. (Representation formula) Let $u(\sigma)$ be a curve in the H^2 parametrised by arclength. Then define three spatial curves α, β and γ by

$$\alpha := a \int u(\sigma) d\sigma, \quad \beta := a \tanh\theta \int u(\sigma) \times du,$$

$$\gamma := \alpha - \beta.$$

Then α is a constant curvature curve, β is a constant torsion curve and γ is a Bertrand curve. Conversely, every Bertrand curve can be represented in this form.

Proof of Theorem 5. Here we give a detailed proof.

Let $u = u(\sigma)$ be a timelike curve in H^2 parametrised by the arclength σ. Then $\{\xi = u', \eta = u \times u', u\}$ is a positive orthonormal frame field along u. Hence,

$$u'' = u + \lambda\eta, \quad u \times u'' = -\lambda\xi.$$

for some function λ. From the definition of γ, we get

$$\gamma' = a(u + \tanh\theta\eta), \quad \gamma'' = a(1 - \lambda\tanh\theta)\xi, \ \gamma''' = a(1 - \lambda\tanh\theta)(u + \lambda\eta)$$

The arclength parameter s of γ is determined by

$$\langle\gamma', \gamma'\rangle = -\frac{a^2}{\cosh^2\theta}\left(\frac{ds}{d\sigma}\right)^2.$$

Moreover we have

$$\gamma' \times \gamma'' = a^2(1 - \lambda\tanh\theta)(\eta + \tanh\theta u),$$

$$\det(\gamma', \gamma'', \gamma''') = -\langle\gamma' \times \gamma'', \gamma'''\rangle = a^3(1 - \lambda\tanh\theta)^2(\tanh\theta - \lambda).$$

Using these,

$$\kappa = \frac{|\gamma' \times \gamma''|}{|\gamma'|^3} = \frac{(1 - \lambda\tanh\theta)\cosh^2\theta}{a},$$

$$\tau = \frac{\det(\gamma', \gamma'', \gamma''')}{|\gamma' \times \gamma''|^2} = \frac{\cosh^2\theta(\tanh\theta - \lambda)}{a}.$$

Hence we have

$$a(\kappa - \tanh\theta\tau) = 1.$$

Thus γ is a Bertrand curve.

Next, we compute the curvature of α and torsion of β.

Direct computation shows that

$$\alpha' = au, \quad \alpha'' = a\xi, \quad \alpha' \times \alpha'' = a^2\eta,$$

$$\beta' = a\tanh\eta, \ \beta'' = -a\lambda\tanh\xi, \ \beta' \times \beta'' = -a^2\lambda\tanh^2\theta u,$$

$$\det(\beta', \beta'', \beta''') = a^3\lambda^2 \tanh^3\theta.$$

Hence

$$\kappa_\alpha = \frac{1}{a}, \quad \tau_\beta = \frac{1}{a\tanh\theta}.$$

Conversely. let $\gamma(s)$ be a timelike Bertrand curve with relation:

$$a(\kappa - \tanh\theta\tau) = 1.$$

Denote by σ the arclength parameter of the spherical curve:

$$u = \cosh\theta T - \sinh\theta B.$$

Then

$$u_\sigma = \frac{\cosh\theta}{a}N.$$

Hence $d\sigma/ds = \cosh\theta/a$. Thus

$$au\sigma_s = \cosh\theta(\cosh\theta T - \sinh\theta B),$$

$$a\tanh\theta u \times u_\sigma\sigma_s = -\sinh\theta(\cosh\theta B - \sinh\theta T).$$

Henceforth,

$$a\int ud\sigma - a\tanh\theta\int u \times \frac{du}{d\sigma}d\sigma = \int T(s)ds = \gamma(s).$$

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Conclusion

In this paper, we gave some characterizations for Bertrand curves and spatial curves in Minkowski 3-space We obtained representation formulae for Bertrand curves in $\mathbb{E}^3_1$. We hope these results will be helpful to mathematicians who are specialized on mathematical modeling.

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