Near-consistent robust estimations of moments for unimodal distributions

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Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, leveraging the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments for common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U-statistics

The potential inconsistencies between the sample mean ($\bar{x}$) and robust location estimators in distributions with finite moments have been noticed for more than two centuries (1), with numerous significant attempts made to address them. In calculating a robust location estimator, the procedure of identifying and downweighting extreme values inherently necessitates the formulation of certain distributional assumptions. Inconsistencies natually arise when these assumptions, parametric or semiparametric, are violated. Previously, it was demonstrated that, due to the presence of infinite-dimensional nuisance shape parameters, the semiparametric approach struggles to consistently address distributions with shapes more intricate than γ-symmetry. Newcomb (1886) provided the first modern approach to robust parametric estimation by developing a class of estimators that gives "less weight to the more discordant observations" (2). In 1964, Huber (3) used the minimax procedure to obtain M-estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parametric robust statistics. For example, He and Fung (1999) constructed (4) a robust M-estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for other parametric distributions, e.g., the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another interesting approach is based on L-estimators, such as percentile estimators. For examples of percentile estimators for the Weibull distribution, the reader is referred to the works of Menon (1963) (5), Dubey (1967) (6), Marks (2005) (7), and Boudt, Caliskan, and Croux (2011) (8). At the outset of the study of percentile estimators, it was known that they arithmetically utilize the invariant structures of probability distributions (5, 6). Maybe such estimators can be named as I-statistics. Formally, an estimator is classified as an I-statistic if it asymptotically satisfies $\mathrm{I}(\mathrm{LE}_1, \dots, \mathrm{LE}_l) = (\theta_1, \dots, \theta_q)$ for the distribution it is consistent, where LEs are calculated with the use of LU-statistics (defined in Subsection ??), I is defined using arithmetic operations and constants but may also incorporate transcendental functions and quantile functions, and θs are the population parameters it estimates. A subclass of I-statistics, arithmetic I-statistics, is defined as LEs are LU-statistics, I is solely defined using arithmetic operations and constants. Since some percentile estimators use the logarithmic function to transform all random variables before computing the L-estimators, a percentile estimator might not always be an arithmetic I-statistic (6). In this article, two subclasses of I-statistics are introduced, arithmetic I-statistics and quantile I-statistics. Examples of quantile I-statistics will be discussed later. Based on LU-statistics, I-statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L-estimator can be expressed as an integral of the quantile function, I-statistics are often analytically obtainable. However, it is observed that even when the sample follows a gamma distribution, which belongs to the same larger family as the Weibull model, the generalized gamma distribution, a misassumption can still lead to substantial biases in Marks percentile estimator (7), rendering the approach ill-suited (SI Dataset S1).

Most robust location estimators commonly used are symmetric owing to the prevalence of symmetric distributions. A γ-weighted Hodges-Lehmann mean ($\mathrm{WHLM}_{k,\epsilon,\gamma,n}$) can achieve consistency for any γ-symmetric distribution. However, it falls considerably short of effectively handling a broad spectrum of other common distributions. Shifting from semiparametrics to parametrics, consider an estimator with a non-sample-dependent breakdown point (defined in Subsection ??) that is consistent simultanously for both a semiparametric class of distributions and a distinct parametric distribution, such a robust estimator is named with the prefix 'invariant' followed by the population parameter it is consistent with. Here, the recombined mean is defined as

$$rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma,n,\mathrm{WL}_1,\mathrm{WL}_2} \coloneqq \lim_{c\to\infty}\left(\frac{(\mathrm{WL}1_{k_1,\epsilon_1,\gamma,n}+c)^{d+1}}{(\mathrm{WL}2_{k_2,\epsilon_2,\gamma,n}+c)^{d}}-c\right),$$

where d is the key factor for bias correction, $\mathrm{WL}_{k,\epsilon,\gamma,n}$ is the

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. In this article, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

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weighted L-statistic. It is assumed in this article that in the subscript of an estimator, if k and γ is omitted, $k=1$ and $\gamma=1$ is assumed, if n is omitted, only the asymptotic behavior is considered, in the absence of subscripts, no assumptions are made. The subsequent theorem shows the significance of this arithmetic I-statistic.

Theorem .1. *Assuming finite second moments, $rm_{d=\frac{\mu-WHLM_{1_{k_1,\epsilon_1,\gamma}}}{WHLM_{1_{k_1,\epsilon_1,\gamma}}-WHLM_{2_{k_2,\epsilon_2,\gamma}}},k_1,k_2,\epsilon_1,\epsilon_2,\gamma,WHLM_1,WHLM_2}$ is a consistent mean estimator for a location-scale distribution and any γ-symmetric distributions, where μ, $WHLM_{1_{k_1,\epsilon_1,\gamma}}$, and $WHLM_{2_{k_2,\epsilon_2,\gamma}}$ are different location parameters from that location-scale distribution.*

Proof. Finding d that make $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma,\mathrm{WHLM}_1,\mathrm{WHLM}_2}$ a consistent mean estimator is equivalent to finding the solution of $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma,\mathrm{WHLM}_1,\mathrm{WHLM}_2}=\mu$. First consider the location-scale distribution. Since $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma,\mathrm{WHLM}_1,\mathrm{WHLM}_2} = \lim_{c\to\infty}\left(\frac{\left(\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}}+c\right)^{d+1}}{\left(\mathrm{WHLM}_{2_{k_2,\epsilon_2,\gamma}}+c\right)^{d}}-c\right) = (d+1)\,\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}} - d\mathrm{WHLM}_{2_{k_2,\epsilon_2,\gamma}} = \mu$. So, $d=\frac{\mu-\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}}}{\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}}-\mathrm{WHLM}_{2_{k_2,\epsilon_2,\gamma}}}$. Previously, it was established that any $\mathrm{WL}(k,\epsilon,\gamma)$ can be expressed as $\lambda\mathrm{WL}_0(k,\epsilon,\gamma)+\mu$ for a location-scale distribution parameterized by a location parameter μ and a scale parameter λ, where $\mathrm{WL}_0(k,\epsilon,\gamma)$ is a function of $Q_0(p)$, the quantile function of a standard distribution without any shifts or scaling, according to the definition of the weighted L-statistic. The simultaneous cancellation of μ and λ in $\frac{(\lambda\mu_0+\mu)-(\lambda\mathrm{WL}_{1_0}(k_1,\epsilon_1,\gamma)+\mu)}{(\lambda\mathrm{WL}_{1_0}(k_1,\epsilon_1,\gamma)+\mu)-(\lambda\mathrm{WL}_{2_0}(k_2,\epsilon_2,\gamma)+\mu)}$ assures that the d in rm is always a constant for a location-scale distribution. The proof of the second assertion follows directly from the coincidence property. According to Theorem 18 in the previous article, for any γ-symmetric distribution with a finite second moment, $\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}}=\mathrm{WHLM}_{2_{k_2,\epsilon_2,\gamma}}=\mu$. Then $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma,\mathrm{WHLM}_1,\mathrm{WHLM}_2}=\lim_{c\to\infty}\left(\frac{(\mu+c)^{d+1}}{(\mu+c)^{d}}-c\right)=\mu$. This completes the demonstration. □

For example, the Pareto distribution has a quantile function $Q_{Par}(p)=x_m(1-p)^{-\frac{1}{\alpha}}$, where x_m is the minimum possible value that a random variable following the Pareto distribution can take, serving a scale parameter, α is a shape parameter. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha-1}$. As $\mathrm{WL}(k,\epsilon,\gamma)$ can be expressed as a function of $Q(p)$, one can set the two $\mathrm{WL}_{k,\epsilon,\gamma}$s in the d value as two arbitrary quantiles $Q_{Par}(p_1)$ and $Q_{Par}(p_2)$. For the Pareto distribution, $d_{Per}=\frac{\mu_{Per}-Q_{Par}(p_1)}{Q_{Par}(p_1)-Q_{Par}(p_2)}=\frac{\frac{\alpha x_m}{\alpha-1}-x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}}-x_m(1-p_2)^{-\frac{1}{\alpha}}}$. x_m can be canceled out. Intriguingly, the quantile function of exponential distribution is $Q_{exp}(p)=\ln\left(\frac{1}{1-p}\right)\lambda$, $\lambda\geq 0$. $\mu_{exp}=\lambda$. Then, $d_{exp}=\frac{\mu_{exp}-Q_{exp}(p_1)}{Q_{exp}(p_1)-Q_{exp}(p_2)}=\frac{\lambda-\ln\left(\frac{1}{1-p_1}\right)\lambda}{\ln\left(\frac{1}{1-p_1}\right)\lambda-\ln\left(\frac{1}{1-p_2}\right)\lambda}=-\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$. Since $\lim_{\alpha\to\infty}\frac{\frac{\alpha}{\alpha-1}-(1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha}-(1-p_2)^{-1/\alpha}}=-\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution, as $\alpha\to\infty$, regardless of the type of weighted L-statistic used. That means, for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d=\frac{\mu-\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}}}{\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}}-\mathrm{WHLM}_{2_{k_2,\epsilon_2,\gamma}}},k_1,k_2,\epsilon_1,\epsilon_2,\gamma,\mathrm{WHLM}_1,\mathrm{WHLM}_2}$ is consistent for at least one particular case, where μ, $\mathrm{WHLM}_{1_{k_1,\epsilon_1,\gamma}}$, and $\mathrm{WHLM}_{2_{k_2,\epsilon_2,\gamma}}$ are different location parameters from an exponential distribution.

Data Availability. Data for Table **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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