

Near-consistent robust estimations of moments for unimodal distributions

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This manuscript was compiled on May 20, 2023

Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, leveraging the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U -statistics

The potential inconsistencies between the sample mean (\bar{x}) and robust location estimators with non-zero asymptotic breakdown points in distributions with finite moments on the real line have been noticed for more than two centuries (1), yet remain unsolved. Strictly speaking, it is unsolvable as by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886) provided the first modern approach to robust parametric estimation by developing a class of estimators that gives "less weight to the more discordant observations" (2). In 1964, Huber (3) used the minimax procedure to obtain M -estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M -estimator increases rapidly. This is a common issue in parameter estimations. For example, He and Fung (1999) constructed (4) a robust M -estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another interesting approach is based on L -estimators, such as percentile estimators. Examples of percentile estimators for the Weibull distribution, the reader is referred to Menon (1963) (5), Dubey (1967) (6), Marks (2005) (7), and Boudt, Caliskan, and Croux (2011) (8)'s works. At the outset of the study of percentile estimators, it was known that they arithmetically utilize the invariant structures of probability distributions (5, 6). Maybe such estimators can be named as I -statistics. Formally, an estimator is classified as an I -statistic if it asymptotically satisfies $I(LE_1, \dots, LE_l) = (\theta_1, \dots, \theta_q)$ for the distribution it is consistent, where LE s are calculated with the use of LU -statistics (defined in Subsection B), I is defined using arithmetic operations and constants but may also incorporate transcendental functions and quantile functions, and θ s are the population parameters it estimates. A subclass of I -statistics, arithmetic I -statistics, is defined as LE s are LU -statistics, I is solely defined using arithmetic operations and constants. Since some percentile estimators use the logarithmic function to transform all random variables before

computing the L -estimators, a percentile estimator might not always be an arithmetic I -statistic (6). In this article, two subclasses of I -statistics are introduced, arithmetic I -statistics and quantile I -statistics. Examples of quantile I -statistics will be discussed later. Based on LU -statistics, I -statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L -estimator can be expressed as an integral of the quantile function, I -statistics are often analytically obtainable. However, it is observed that Marks percentile estimator for the Weibull distribution (7) tends to be inferior to the robust M -estimators (3, 4), especially upon violation of the distributional assumption (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases for central moments, rendering the approach ill-suited (SI Dataset S1).

The majority of robust location estimators commonly used are symmetric owing to the prevalence of symmetric distributions. An asymmetric weighted L -statistic can achieve consistency for a semiparametric class of skewed distributions; but the lack of symmetry makes it suitable only for certain applications. Shifting from semiparametrics to parametrics, consider an estimator with a non-zero asymptotic breakdown point that is consistent simultaneously for both a semiparametric class of distributions and a distinct parametric distribution with finite moments, such a robust location estimator is called an invariant mean. Based on the mean-weighted L -statistic- γ -median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,\gamma,n} := \lim_{c \rightarrow \infty} \left(\frac{(WL_{\epsilon,\gamma,n} + c)^{d+1}}{(\gamma m_n + c)^d} - c \right),$$

where d is the key factor for bias correction, γm_n is the sample γ -median, $WL_{\epsilon,\gamma,n}$ is the weighted L -statistic. If γ is omitted, $\gamma = 1$ is assumed. The subsequent theorem shows the significance of this arithmetic I -statistic.

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. In this article, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

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Theorem .1. Let $BM_{\epsilon,n}$ be the WL, $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ is a consistent mean estimator for the exponential distribution, any symmetric distributions and the Pareto distribution with quantile function $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, when $\alpha \rightarrow \infty$, provided that the second moments are finite.

Proof. Finding d and ϵ that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E[rm_{d,\epsilon,n}] = E[X]$. The quantile function of the exponential distribution is $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$. $E[X] = \lambda$. $E[m_n] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$. For the exponential distribution, $E\left[BM_{\epsilon=\frac{1}{24},n}\right] = \lambda\left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)$, the detailed formula is given in the SI Text. Since $rm_{d,\epsilon} = \lim_{c \rightarrow \infty} \left(\frac{(BM_{\epsilon}+c)^{d+1}}{(m+c)^d} - c\right) = (d+1)BM_{\epsilon} - dm = \mu$. So,

$$d = \frac{\mu - BM_{\epsilon}}{BM_{\epsilon} - m} = \frac{\lambda - \lambda\left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)}{\lambda\left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right) - \ln 2\lambda} =$$

$$-\frac{\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)}{1 - \ln(2) + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)} \approx 0.103.$$

The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment, $E[BM_{\epsilon,n}] = E[m_n] = E[X]$. Then $E[rm_{d,\epsilon,n}] = \lim_{c \rightarrow \infty} \left(\frac{(E[X]+c)^{d+1}}{(E[X]+c)^d} - c\right) = E[X]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha-1}$. Since any weighted L -statistic can be expressed as an integral of the quantile function as shown in Theorem A.1, the γ -median is also a percentile, replacing the WL and γm in the d value with two arbitrary percentiles p_1 and p_2 , for the Pareto distribution,

$$d_{Pareto} = \frac{\mu - Q(p_1)}{Q(p_1) - Q(p_2)} = \frac{\frac{\alpha x_m}{\alpha-1} - x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}} - x_m(1-p_2)^{-\frac{1}{\alpha}}}. \quad x_m \text{ can be canceled out.}$$

For the exponential distribution, $d_{exp} = \frac{\mu - Q(p_1)}{Q(p_1) - Q(p_2)} = \frac{\lambda - \ln\left(\frac{1}{1-p_1}\right)\lambda}{\ln\left(\frac{1}{1-p_1}\right)\lambda - \ln\left(\frac{1}{1-p_2}\right)\lambda} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}.$

Since $\lim_{\alpha \rightarrow \infty} \frac{\frac{\alpha}{\alpha-1} - (1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha} - (1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution, as $\alpha \rightarrow \infty$, regardless of the type of weighted L -statistic used. This completes the demonstration. \square

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ is consistent for at least one particular case. The biases of $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ exhibits excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U -statistics, the central moment kernel distributions possess desirable properties, and a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than

the variances (as seen in Table ?? for $n = 4096$) for unimodal distributions.

Background and Main Results

A. Invariant mean. It is well established that a theoretical model can be adjusted to fit the first two moments of the observed data. A continuous distribution belonging to a location-scale family, parametrized by a location parameter μ and a scale parameter λ , takes the form $F(x) = F_0\left(\frac{x-\mu}{\lambda}\right)$, where F_0 is a standard distribution without any shifts or scaling. Consequently, $F(x) = Q^{-1}(x) \rightarrow x = Q(p) = \lambda Q_0(p) + \mu$. Thus, for a location-scale distribution, any $WA(\epsilon, \gamma)$ can be expressed as $\lambda WA_0(\epsilon, \gamma) + \mu$, where $WA_0(\epsilon, \gamma)$ is an integral of $Q_0(p)$ according to the definition of the weighted average. The succeeding theorem shows that the whl_k kernel distribution is invariably a location-scale distribution if the original distribution belongs to a location-scale family with the same location and scale parameters. The proof is given in the SI Text.

Theorem A.1. $whl_k(x_1 = \lambda x_1 + \mu, \dots, x_k = \lambda x_k + \mu) = \lambda whl_k(x_1, \dots, x_k) + \mu$.

Let $WeHLM_0(\epsilon, \gamma)$ denote the expected value of a weighted Hodges-Lehmann mean for the standard distribution, then for a location-scale family of distributions parametrized by a location parameter μ and a scale parameter λ , the $WeHLM$ can also be expressed as $\lambda WeHLM_0(\epsilon, \gamma) + \mu$. As Theorem A.1 proved the $w_i \neq 1$ case, this form is valid for all weighted L -statistics. The simultaneous cancellation of μ and λ in $\frac{(\lambda\mu_0 + \mu) - (\lambda WeL_0(\epsilon, \gamma) + \mu)}{(\lambda WeL_0(\epsilon, \gamma) + \mu) - (\lambda\gamma m_0 + \mu)}$ assures that d is always a constant for a location-scale distribution.

The performance in heavy-tailed distributions can be further improved by formulating the quantile mean as

$$qm_{d,\epsilon,\gamma,n} := \hat{Q}_n \left(\left(\hat{F}_n(WL_{\epsilon,\gamma,n}) - \frac{1}{1+\gamma} \right) d + \hat{F}_n(WL_{\epsilon,\gamma,n}) \right),$$

provided that $\hat{F}_n(WL_{\epsilon,\gamma,n}) \geq \frac{1}{1+\gamma}$, where $\hat{F}_n(x)$ is the empirical cumulative distribution function of the sample, \hat{Q}_n is the sample quantile function. When $\hat{F}_n(WL_{\epsilon,\gamma,n}) < \frac{1}{1+\gamma}$, $qm_{d,\epsilon,\gamma,n}$ is defined as $\hat{Q}_n\left(\hat{F}_n(WL_{\epsilon,\gamma,n}) - \left(\frac{1}{1+\gamma} - \hat{F}_n(WL_{\epsilon,\gamma,n})\right)d\right)$. Without loss of generality, in the following discussion, only the case where $\hat{F}_n(WL_{\epsilon,\gamma,n}) \geq \frac{1}{1+\gamma}$ is considered. Moreover, in extreme right-skewed heavy-tailed distributions, if the calculated percentile surpasses $1 - \epsilon$, it will be modified to $1 - \epsilon$. A widely used method for calculating the sample quantile function involves employing linear interpolation of modes corresponding to the order statistics of the uniform distribution on the interval $[0, 1]$, i.e., $\hat{Q}_n(p) = X_{[h]} + (h - [h])(X_{[h]} - X_{[h]})$, $h = (n-1)p + 1$. To minimize the finite sample bias, here, the inverse function of \hat{Q}_n is deduced as $\hat{F}_n(x) := \frac{1}{n-1} \left(cf - 1 + \frac{x - X_{cf}}{X_{cf+1} - X_{cf}} \right)$, where $cf = \sum_{i=1}^n \mathbf{1}_{X_i \leq x}$, $\mathbf{1}_A$ is the indicator of event A . The quantile mean uses the location-scale invariant in a different way as depicted in the subsequent proof.

Theorem A.2. Let $BM_{\epsilon,n}$ be the WL, $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ is a consistent mean estimator for the exponential, Pareto ($\alpha \rightarrow \infty$) and any symmetric distributions provided that the second moments are finite.

165 *Proof.* The cdf of the exponential distribution is $F(x) =$
166 $1 - e^{-\lambda^{-1}x}$, $\lambda \geq 0$, $x \geq 0$. Recall that the expecta-
167 tion of $\text{BM}_{\epsilon,n}$ can be expressed as $\lambda \text{BM}_0(\epsilon)$, so $F(\text{BM}_\epsilon)$ is
168 free of λ , as are $F(\mu)$ and $F(m)$. When $\epsilon = \frac{1}{24}$, $d =$
169
$$\frac{F(\mu) - F(\text{BM}_\epsilon)}{F(\text{BM}_\epsilon) - \frac{1}{2}} = \frac{-e^{-1} + e^{-\left(1 + \ln\left(\frac{26068394603446272 \sqrt[6]{\frac{7}{247}} \sqrt[3]{\frac{11}{11}}\right)}\right)}}{-\left(1 + \ln\left(\frac{26068394603446272 \sqrt[6]{\frac{7}{247}} \sqrt[3]{\frac{11}{11}}\right)}\right)}} =$$

170
$$\frac{\frac{1}{2} - e^{-\left(1 + \ln\left(\frac{26068394603446272 \sqrt[6]{\frac{7}{247}} \sqrt[3]{\frac{11}{11}}\right)}\right)}}{\frac{1}{2} - e^{-\left(1 + \ln\left(\frac{26068394603446272 \sqrt[6]{\frac{7}{247}} \sqrt[3]{\frac{11}{11}}\right)}\right)}} \approx 0.088.$$
 The proof of the
171 symmetric case: since for any symmetric distribution with
172 a finite second moment, $F(E[\text{BM}_{\epsilon,n}]) = F(\mu) = \frac{1}{2}$.
173 Then, the expectation of the quantile mean is $qm_{d,\epsilon} =$
174 $F^{-1}\left(\left(F(\mu) - \frac{1}{2}\right)d + F(\mu)\right) = F^{-1}\left(0 + F(\mu)\right) = \mu$.
175 For the assertion related to the Pareto distribu-
176 tion, the cdf of it is $1 - \left(\frac{xm}{x}\right)^\alpha$. Similar to The-
177 orem .1, replacing the $F(\text{WL}_{\epsilon,\gamma})$ and $\frac{1}{1+\gamma}$ in the
178 d value with two arbitrary percentiles p_1 and p_2 ,
179
$$d_{\text{Pareto}} = \frac{1 - \left(\frac{xm}{x}\right)^\alpha - \left(1 - \left(\frac{xm}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right)}{\left(1 - \left(\frac{xm}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right) - \left(1 - \left(\frac{xm}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^\alpha\right)} =$$

180
$$\frac{1 - \left(\frac{\alpha-1}{\alpha}\right)^\alpha - p_1}{p_1 - p_2}.$$
 When $\alpha \rightarrow \infty$, $\left(\frac{\alpha-1}{\alpha}\right)^\alpha = \frac{1}{e}$, so in this
181 case, d_{Pareto} is identical to that of the exponential distri-
182 bution, since $d_{\text{exp}} = \frac{(1-e^{-1}) - \left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)}\right) - \left(1 - e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)} =$
183
$$\frac{1 - \frac{1}{e} - p_1}{p_1 - p_2}.$$
 Therefore, same logic as in Theorem .1, their d val-
184 ues are always identical, regardless of the type of weighted
185 L -statistic used. All results are now proven. \square

186 The definitions of location and scale parameters are such
187 that they must satisfy $F(x; \lambda, \mu) = F\left(\frac{x-\mu}{\lambda}; 1, 0\right)$. By recalling
188 $x = \lambda Q_0(p) + \mu$, it follows that the percentile of any weighted
189 L -statistic is free of λ and μ , which guarantees the validity
190 of the quantile mean. The quantile mean is a quantile I -
191 statistic. Specifically, an estimator is classified as a quantile
192 I -statistic if LEs are percentiles of a distribution obtained by
193 plugging LU -statistics into a cumulative distribution function and
194 I is defined with arithmetic operations, constants and
195 quantile functions. $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ works better in the fat-tail
196 scenarios (SI Dataset S1). Theorem .1 and A.2 show that
197 $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ and $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ are both consistent mean
198 estimators for any symmetric distribution and a skewed dis-
199 tribution with finite second moments. The breakdown points
200 of $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ and $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ are both $\frac{1}{24}$. Therefore
201 they are all invariant means.

202 To study the influence of the choice of WLs in rm and qm ,
203 it is constructive to recall that a weighted L -statistic is a com-
204 bination of order statistics. While using a less-biased weighted
205 L -statistic can generally enhance performance (SI Dataset S1),
206 there is a greater risk of violation in the semiparametric frame-
207 work. However, the mean- $\text{WL}_{\epsilon,\gamma}$ - γ -median inequality is robust
208 to slight fluctuations of the QA or QHLM function of the un-
209 derlying distribution. Suppose the QA function is generally
210 decreasing in $[0, u]$, but increasing in $[u, \frac{1}{1+\gamma}]$, since all quantile
211 averages with breakdown points from ϵ to $\frac{1}{1+\gamma}$ will be included

in the computation of $\text{WA}_{\epsilon,\gamma}$, as long as $\frac{1}{1+\gamma} - u \ll \frac{1}{1+\gamma} - \gamma\epsilon$,
and other portions of the QA function satisfy the inequality
constraints that define the ν th γ -orderliness on which the
 $\text{WA}_{\epsilon,\gamma}$ is based, the mean- $\text{WA}_{\epsilon,\gamma}$ - γ -median inequality still hold.
This is due to the violation being bounded (9) and therefore
cannot be extreme for unimodal distributions. For instance,
the SQA function is non-monotonic when the shape parameter
of the Weibull distribution $\alpha > \frac{1}{1-\ln(2)} \approx 3.259$ as shown
in the previous article, the violation of the third orderliness
starts near this parameter as well, yet the mean- $\text{BM}_{\frac{1}{24}}$ -median
inequality retains valid when $\alpha \leq 3.387$. This rationale can be
extended to other weighted L -statistics based on U -orderliness,
since the orderliness is necessary for U -orderliness, so its vio-
lation is also bounded. Another key factor in determining the
risk of violation is the skewness of the distribution. Previously,
it was demonstrated that in a family of distributions differing
by a skewness-increasing transformation in van Zwet's sense,
the violation of orderliness, if it happens, often only occurs as
the distribution nears symmetry (10). When $\gamma = 1$, the over-
corrections in rm and qm are dependent on the SWL_ϵ -median
difference, which can be a reasonable measure of skewness
(11, 12), implying that the over-correction is often tiny with
moderate d . This qualitative analysis, in conjunction with
bias bounds (9), denotes the general reliability of rm and qm
based on the mean- $\text{WL}_{\epsilon,\gamma}$ - γ -median inequality for unimodal
distributions.

B. Robust estimations of the central moments. In 1979, Bickel
and Lehmann, in their final paper of the landmark series *De-*
scriptive Statistics for Nonparametric Models (13), generalized
a class of estimators called "measures of spread," which "do
not require the assumption of symmetry." In the final section
of that paper (13), they explored two possible versions of
the trimmed standard deviation based on pairwise differences,
which are modified here for comparison,

Data Availability. Data for Table ?? are given in SI Dataset S1.
All codes have been deposited in [GitHub](#).

ACKNOWLEDGMENTS. I gratefully acknowledge the construc-
tive comments made by the editor which substantially improved
the clarity and quality of this paper.

1. CF Gauss, *Theoria combinationis observationum erroribus minimis obnoxiae*. (Henricus Dieterich), (1823).
2. S Newcomb, A generalized theory of the combination of observations so as to obtain the best result. *Am. journal Math.* **8**, 343–366 (1886).
3. PJ Huber, Robust estimation of a location parameter. *Ann. Math. Stat.* **35**, 73–101 (1964).
4. X He, WK Fung, Method of medians for lifetime data with weibull models. *Stat. medicine* **18**, 1993–2009 (1999).
5. M Menon, Estimation of the shape and scale parameters of the weibull distribution. *Technometrics* **5**, 175–182 (1963).
6. SD Dubey, Some percentile estimators for weibull parameters. *Technometrics* **9**, 119–129 (1967).
7. NB Marks, Estimation of weibull parameters from common percentiles. *J. applied Stat.* **32**, 17–24 (2005).
8. K Boudt, D Caliskan, C Croux, Robust explicit estimators of weibull parameters. *Metrika* **73**, 187–209 (2011).
9. C Bernard, R Kazzi, S Vanduffel, Range value-at-risk bounds for unimodal distributions under partial information. *Insur. Math. Econ.* **94**, 9–24 (2020).
10. WR van Zwet, *Convex Transformations of Random Variables: Nebst Stellingen*. (1964).
11. AL Bowley, *Elements of statistics*. (King) No. 8, (1926).
12. RA Groeneveld, G Meeden, Measuring skewness and kurtosis. *J. Royal Stat. Soc. Ser. D (The Stat.)* **33**, 391–399 (1984).
13. PJ Bickel, EL Lehmann, Descriptive statistics for nonparametric models iv. spread in *Selected Works of EL Lehmann*. (Springer), pp. 519–526 (2012).