

Near-consistent robust estimations of moments for unimodal distributions

Tuban Lee

This manuscript was compiled on May 22, 2023

Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, leveraging the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments with insignificant asymptotic biases for common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U -statistics

The potential inconsistencies between the sample mean (\bar{x}) and robust location estimators with non-zero asymptotic breakdown points in distributions with finite moments on the real line have been noticed for more than two centuries (1), yet remain unsolved. Strictly speaking, it is unsolvable as by trimming, some information about the original distribution is removed, making it impossible to estimate the values of the removed parts without distributional assumptions. Newcomb (1886) provided the first modern approach to robust parametric estimation by developing a class of estimators that gives "less weight to the more discordant observations" (2). In 1964, Huber (3) used the minimax procedure to obtain M -estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M -estimator increases rapidly. This is a common issue in parameter estimations. For example, He and Fung (1999) constructed (4) a robust M -estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another interesting approach is based on L -estimators, such as percentile estimators. Examples of percentile estimators for the Weibull distribution, the reader is referred to Menon (1963) (5), Dubey (1967) (6), Marks (2005) (7), and Boudt, Caliskan, and Croux (2011) (8)'s works. At the outset of the study of percentile estimators, it was known that they arithmetically utilize the invariant structures of probability distributions (5, 6). Maybe such estimators can be named as I -statistics. Formally, an estimator is classified as an I -statistic if it asymptotically satisfies $I(LE_1, \dots, LE_l) = (\theta_1, \dots, \theta_q)$ for the distribution it is consistent, where LE s are calculated with the use of LU -statistics (defined in Subsection B), I is defined using arithmetic operations and constants but may also incorporate transcendental functions and quantile functions, and θ s are the population parameters it estimates. A subclass of I -statistics, arithmetic I -statistics, is defined as LE s are LU -statistics, I is solely defined using arithmetic operations and constants. Since some percentile estimators use the logarithmic function to transform all random variables before

computing the L -estimators, a percentile estimator might not always be an arithmetic I -statistic (6). In this article, two subclasses of I -statistics are introduced, arithmetic I -statistics and quantile I -statistics. Examples of quantile I -statistics will be discussed later. Based on LU -statistics, I -statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L -estimator can be expressed as an integral of the quantile function, I -statistics are often analytically obtainable. However, it is observed that Marks percentile estimator for the Weibull distribution (7) tends to be inferior to the robust M -estimators (3, 4), especially upon violation of the distributional assumption (SI Dataset S1). Even when distributions such as the Weibull and gamma belong to the same larger family, the generalized gamma distribution, a misassumption can still result in substantial biases for central moments, rendering the approach ill-suited (SI Dataset S1).

The majority of robust location estimators commonly used are symmetric owing to the prevalence of symmetric distributions. An asymmetric weighted L -statistic can achieve consistency for a semiparametric class of skewed distributions; but the lack of symmetry makes it suitable only for certain applications. Shifting from semiparametrics to parametrics, consider an estimator with a non-zero asymptotic breakdown point that is consistent simultaneously for both a semiparametric class of distributions and a distinct parametric distribution with finite moments, such a robust location estimator is called an invariant mean. Based on the mean-weighted L -statistic- γ -median inequality, the recombined mean is defined as

$$rm_{d,\epsilon,\gamma,n} := \lim_{c \rightarrow \infty} \left(\frac{(WL_{\epsilon,\gamma,n} + c)^{d+1}}{(\gamma m_n + c)^d} - c \right),$$

where d is the key factor for bias correction, γm_n is the sample γ -median, $WL_{\epsilon,\gamma,n}$ is the weighted L -statistic. If γ is omitted, $\gamma = 1$ is assumed. The subsequent theorem shows the significance of this arithmetic I -statistic.

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. In this article, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

¹To whom correspondence should be addressed. E-mail: tl@biomathematics.org

Theorem .1. Let $BM_{\epsilon,n}$ be the WL, $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ is a consistent mean estimator for the exponential distribution, any symmetric distributions and the Pareto distribution with quantile function $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, when $\alpha \rightarrow \infty$, provided that the second moments are finite.

Proof. Finding d and ϵ that make $rm_{d,\epsilon}$ a consistent mean estimator is equivalent to finding the solution of $E[rm_{d,\epsilon,n}] = E[X]$. The quantile function of the exponential distribution is $Q(p) = \ln\left(\frac{1}{1-p}\right)\lambda$. $E[X] = \lambda$. $E[m_n] = Q\left(\frac{1}{2}\right) = \ln 2\lambda$. For the exponential distribution, $E\left[BM_{\epsilon=\frac{1}{24},n}\right] = \lambda\left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)$, the detailed formula is given in the SI Text. Since $rm_{d,\epsilon} = \lim_{c \rightarrow \infty} \left(\frac{(BM_{\epsilon}+c)^{d+1}}{(m+c)^d} - c\right) = (d+1)BM_{\epsilon} - dm = \mu$. So, $d = \frac{\mu - BM_{\epsilon}}{BM_{\epsilon} - m} = \frac{\lambda - \lambda\left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)}{\lambda\left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right) - \ln 2\lambda} = \frac{\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)}{1 - \ln(2) + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)} \approx 0.103$. The proof of the second assertion follows directly from the coincidence property. For any symmetric distribution with a finite second moment, $E[BM_{\epsilon,n}] = E[m_n] = E[X]$. Then $E[rm_{d,\epsilon,n}] = \lim_{c \rightarrow \infty} \left(\frac{(E[X]+c)^{d+1}}{(E[X]+c)^d} - c\right) = E[X]$. The proof for the Pareto distribution is more general. The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha - 1}$. Since any weighted L -statistic can be expressed as an integral of the quantile function as shown in Theorem ??, the γ -median is also a percentile, replacing the WL and γm in the d value with two arbitrary percentiles p_1 and p_2 , for the Pareto distribution, $d_{Pareto} = \frac{\mu - Q(p_1)}{Q(p_1) - Q(p_2)} = \frac{\frac{\alpha x_m}{\alpha - 1} - x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}} - x_m(1-p_2)^{-\frac{1}{\alpha}}}$. x_m can be canceled out. For the exponential distribution, $d_{exp} = \frac{\mu - Q(p_1)}{Q(p_1) - Q(p_2)} = \frac{\lambda - \ln\left(\frac{1}{1-p_1}\right)\lambda}{\ln\left(\frac{1}{1-p_1}\right)\lambda - \ln\left(\frac{1}{1-p_2}\right)\lambda} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$. Since $\lim_{\alpha \rightarrow \infty} \frac{\frac{\alpha}{\alpha-1} - (1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha} - (1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$, the d value for the Pareto distribution approaches that of the exponential distribution, as $\alpha \rightarrow \infty$, regardless of the type of weighted L -statistic used. This completes the demonstration. \square

Theorem .1 implies that for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ is consistent for at least one particular case. The biases of $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ exhibits excellent performance for all these common unimodal distributions (SI Dataset S1).

Besides introducing the concept of invariant mean, the purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U -statistics, the central moment kernel distributions possess desirable properties, and a series of sophisticated yet efficient robust estimators can be constructed whose biases are typically smaller than

the variances (as seen in Table ?? for $n = 4096$) for unimodal distributions.

Background and Main Results

A. Invariant mean. Previously, it was established that for any location-scale distribution parametrized by a location parameter μ and a scale parameter λ , a weighted L -statistic can be expressed as $\lambda WL_0(\epsilon, \gamma) + \mu$, where WL_0 denote the expected value of the weighted L -statistic for a standard distribution without any shifts or scaling. The simultaneous cancellation of μ and λ in $\frac{(\lambda\mu_0 + \mu) - (\lambda WL_0(\epsilon, \gamma) + \mu)}{(\lambda WL_0(\epsilon, \gamma) + \mu) - (\lambda\gamma m_0 + \mu)}$ assures that the d in rm is always a constant for a location-scale distribution.

The performance in heavy-tailed distributions can be further improved by defining the quantile mean as

$$qm_{d,\epsilon,\gamma,n} := \hat{Q}_n \left(\left(\hat{F}_n(WL_{\epsilon,\gamma,n}) - \frac{1}{1+\gamma} \right) d + \hat{F}_n(WL_{\epsilon,\gamma,n}) \right),$$

provided that $\hat{F}_n(WL_{\epsilon,\gamma,n}) \geq \frac{1}{1+\gamma}$, where $\hat{F}_n(x)$ is the empirical cumulative distribution function of the sample, \hat{Q}_n is the sample quantile function. When $\hat{F}_n(WL_{\epsilon,\gamma,n}) < \frac{1}{1+\gamma}$, $qm_{d,\epsilon,\gamma,n}$ is defined as $\hat{Q}_n \left(\hat{F}_n(WL_{\epsilon,\gamma,n}) - \left(\frac{1}{1+\gamma} - \hat{F}_n(WL_{\epsilon,\gamma,n}) \right) d \right)$. Without loss of generality, in the following discussion, only the case where $\hat{F}_n(WL_{\epsilon,\gamma,n}) \geq \frac{1}{1+\gamma}$ is considered. Moreover, in extreme right-skewed heavy-tailed distributions, if the calculated percentile surpasses $1 - \epsilon$, it will be modified to $1 - \epsilon$. A widely used method for calculating the sample quantile function involves employing linear interpolation of modes corresponding to the order statistics of the uniform distribution on the interval $[0, 1]$, i.e., $\hat{Q}_n(p) = X_{[h]} + (h - [h])(X_{[h]} - X_{[h]})$, $h = (n-1)p + 1$. To minimize the finite sample bias, here, the inverse function of \hat{Q}_n is deduced as $\hat{F}_n(x) := \frac{1}{n-1} \left(cf - 1 + \frac{x - X_{cf}}{X_{cf+1} - X_{cf}} \right)$, where $cf = \sum_{i=1}^n \mathbf{1}_{X_i \leq x}$, $\mathbf{1}_A$ is the indicator of event A . The quantile mean uses the location-scale invariant in a different way as shown in the subsequent proof.

Theorem A.1. Let $BM_{\epsilon,n}$ be the WL, $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ is a consistent mean estimator for the exponential, Pareto ($\alpha \rightarrow \infty$) and any symmetric distributions provided that the second moments are finite.

Proof. The cdf of the exponential distribution is $F(x) = 1 - e^{-\lambda^{-1}x}$, $\lambda \geq 0$, $x \geq 0$. Recall that the expectation of $BM_{\epsilon,n}$ can be expressed as $\lambda BM_0(\epsilon)$, so $F(BM_{\epsilon})$ is free of λ , as are $F(\mu)$ and $F(m)$. When $\epsilon = \frac{1}{24}$, $d = \frac{F(\mu) - F(BM_{\epsilon})}{F(BM_{\epsilon}) - \frac{1}{2}} = \frac{-e^{-1+\epsilon} - \left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)}{-\left(1 + \ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right) - \frac{1}{2} - e} = \frac{101898752449325\sqrt{5}\sqrt[6]{\frac{7}{247}}391^{5/6} - e}{26068394603446272\sqrt[3]{11}e - \frac{101898752449325\sqrt{5}\sqrt[6]{\frac{7}{247}}391^{5/6}}{2} - \frac{26068394603446272\sqrt[3]{11}e}{26068394603446272\sqrt[3]{11}e}} \approx 0.088$. The proof of the symmetric case: since for any symmetric distribution with a finite second moment, $F(E[BM_{\epsilon,n}]) = F(\mu) = \frac{1}{2}$. Then, the expectation of the quantile mean is $qm_{d,\epsilon} = F^{-1} \left(\left(F(\mu) - \frac{1}{2} \right) d + F(\mu) \right) = F^{-1} \left(0 + F(\mu) \right) = \mu$.

For the assertion related to the Pareto distribution, the cdf of it is $1 - \left(\frac{x_m}{x}\right)^\alpha$. Similar to Theorem .1, replacing the $F(WL_{\epsilon,\gamma})$ and $\frac{1}{1+\gamma}$ in the

160 d value with two arbitrary percentiles p_1 and p_2 ,
161
$$d_{Pareto} = \frac{1 - \left(\frac{x_m}{\frac{\alpha x_m}{\alpha - 1}} \right)^\alpha - \left(1 - \left(\frac{x_m}{x_m(1-p_1) - \frac{1}{\alpha}} \right)^\alpha \right)}{\left(1 - \left(\frac{x_m}{x_m(1-p_1) - \frac{1}{\alpha}} \right)^\alpha \right) - \left(1 - \left(\frac{x_m}{x_m(1-p_2) - \frac{1}{\alpha}} \right)^\alpha \right)} =$$

162 $\frac{1 - \left(\frac{\alpha-1}{p_1-p_2} \right)^\alpha - p_1}{p_1-p_2}$. When $\alpha \rightarrow \infty$, $\left(\frac{\alpha-1}{\alpha} \right)^\alpha = \frac{1}{e}$, so in this
163 case, d_{Pareto} is identical to that of the exponential distribution, since $d_{exp} = \frac{(1-e^{-1}) - \left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)} \right)}{\left(1 - e^{-\ln\left(\frac{1}{1-p_1}\right)} \right) - \left(1 - e^{-\ln\left(\frac{1}{1-p_2}\right)} \right)} =$
164 $\frac{1 - \frac{1}{e} - p_1}{p_1 - p_2}$. Therefore, same logic as in Theorem .1, their d values
165 are always identical, regardless of the type of weighted
166 L -statistic used. All results are now proven. \square

168 The definitions of location and scale parameters are such
169 that they must satisfy $F(x; \lambda, \mu) = F\left(\frac{x-\mu}{\lambda}; 1, 0\right)$. By recalling
170 $WL = \lambda WL_0(\epsilon, \gamma) + \mu$, it follows that the percentile of any
171 weighted L -statistic is free of λ and μ , which guarantees the
172 validity of the quantile mean. The quantile mean is a quantile
173 I -statistic. Specifically, an estimator is classified as a quantile
174 I -statistic if LEs are percentiles of a distribution obtained by
175 plugging LU -statistics into a cumulative distribution function
176 and I is defined with arithmetic operations, constants and
177 quantile functions. $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ works better in the fat-tail
178 scenarios (SI Dataset S1). Theorem .1 and A.1 show that
179 $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ and $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ are both consistent mean
180 estimators for any symmetric distribution and a skewed distribution
181 with finite second moments. The breakdown points of
182 $rm_{d \approx 0.103, \epsilon = \frac{1}{24}}$ and $qm_{d \approx 0.088, \epsilon = \frac{1}{24}}$ are both $\frac{1}{24}$. Therefore
183 they are all invariant means.

184 To study the impact of the choice of WLs in rm and qm ,
185 it is constructive to recall that a weighted L -statistic is a
186 combination of order statistics. While using a less-biased
187 weighted L -statistic can generally enhance performance (SI
188 Dataset S1), there is a greater risk of violation in the semi-
189 parametric framework. However, the mean- $WA_{\epsilon, \gamma}$ -median
190 inequality is robust to slight fluctuations of the QA function
191 of the underlying distribution when $0 \leq \gamma \leq 1$. Suppose the
192 QA function is generally decreasing in $[0, u]$, but increasing
193 in $[u, \frac{1}{1+\gamma}]$, since all quantile averages with breakdown points
194 from ϵ to $\frac{1}{1+\gamma}$ will be included in the computation of $WA_{\epsilon, \gamma}$,
195 as long as $\frac{1}{1+\gamma} - u \ll \frac{1}{1+\gamma} - \gamma\epsilon$, and other portions of the QA
196 function satisfy the inequality constraints that define the ν th
197 γ -orderliness on which the $WA_{\epsilon, \gamma}$ is based, the mean- $WA_{\epsilon, \gamma}$ - γ -
198 median inequality still holds. This is due to the violation being
199 bounded (9) when $0 \leq \gamma \leq 1$ and therefore cannot be extreme
200 for unimodal distributions with finite second moments. For
201 instance, the SQA function is non-monotonic when the shape
202 parameter of the Weibull distribution $\alpha > \frac{1}{1-\ln(2)} \approx 3.259$ as
203 shown in the previous article, the violation of the third order-
204 liness starts near this parameter as well, yet the mean-BM $\frac{1}{24}$ -
205 median inequality retains valid when $\alpha \leq 3.387$. Another
206 key factor in determining the risk of violation is the skewness
207 of the distribution. Previously, it was demonstrated that in
208 a family of distributions differing by a skewness-increasing
209 transformation in van Zwet's sense, the violation of orderli-
210 ness, if it happens, often only occurs as the distribution nears
211 symmetry (10). When $\gamma = 1$, the over-corrections in rm and

212 qm are dependent on the SWA_ϵ -median difference, which can
213 be a reasonable measure of skewness after standardization
214 (11, 12), implying that the over-correction is often tiny with
215 moderate d . This qualitative analysis suggests the general
216 reliability of rm and qm based on the mean- $WA_{\epsilon, \gamma}$ -median
217 inequality for unimodal distributions with finite second mo-
218 ments. Extending this rationale to other weighted L -statistics
219 is possible, since the U -orderliness can also be bounded with
220 certain assumptions, as discussed previously.

B. Robust estimations of the central moments. In 1979, Bickel
221 and Lehmann (13), in their final paper of the landmark series
222 *Descriptive Statistics for Nonparametric Models*, generalized a
223 class of estimators called measures of spread, which "do not
224 require the assumption of symmetry." From this, a popular
225 efficient scale estimator, the Rousseeuw-Croux scale estimator
226 (14), was derived in 1993, but the importance of tackling
227 the symmetry assumption has been greatly underestimated.
228 While they had already considered one version of the trimmed
229 standard deviation, which is a measures of dispersion, in
230 the third paper of that series (15); in the final section of
231 that paper (13), they explored another two versions of the
232 trimmed standard deviation based on pairwise differences, one
233 is modified here for comparison, 234

$$\left[\binom{n}{2} (1 - \epsilon - \gamma\epsilon) \right]^{-\frac{1}{2}} \left[\sum_{i=\binom{n}{2}\gamma\epsilon}^{\binom{n}{2}(1-\epsilon)} (X - X')_i^2 \right]^{\frac{1}{2}}, \quad [1] \quad 235$$

236 where $(X - X')_1 \leq \dots \leq (X - X')_{\binom{n}{2}}$ are the order statistics
237 of the pseudo-sample, $X_i - X_j$, $i < j$. They showed that,
238 when $\epsilon = 0$, [1] is $\sqrt{2}$ times the standard deviation. The paper
239 ended with, "We do not know a fortiori which of the measures
240 is preferable and leave these interesting questions open." To
241 address their open question, the nomenclature used in this
242 paper is introduced as follows:

Nomenclature. Given a robust estimator $\hat{\theta}$, which has an
243 adjustable breakdown point that can approach zero asymptoti-
244 cally, the name of $\hat{\theta}$ comprises two parts: the first part denotes
245 the type of estimator, and the second part is the name of the
246 population parameter θ , with which the estimator is consistent
247 as $\epsilon \rightarrow 0$. The abbreviation of the estimator combines the
248 initial letters of the first part and the population parameter. If
249 the estimator is symmetric, the upper asymptotic breakdown
250 point, ϵ (defined in Subsection ??, or ϵ_{U_k} for LU -statistic),
251 is indicated in the subscript of the abbreviation of the esti-
252 mator, with the exception of the median. For an asymmetric
253 estimator based on quantile average, the associated γ follows
254 ϵ . 255

256 In the previous article on semiparametric robust mean es-
257 timation, it was shown that the bias of a robust estimator
258 with an adjustable breakdown point is often monotonic with
259 respect to the breakdown point in a semiparametric distri-
260 bution. Naturally, the estimator's name should reflect the
261 population parameter with which it is consistent as $\epsilon \rightarrow 0$. If
262 multiplying all pseudo-samples by a factor of $\frac{1}{\sqrt{2}}$, then [1] is
263 the trimmed standard deviation adhering to this nomenclature,
264 since $\psi_2(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$ is the kernel function of the
265 unbiased estimation of the second central moment by using
266 U -statistic. It should be preferable, not only because it is the
267 square root of a trimmed U -statistic, which is closely related

to the minimum-variance unbiased estimator (MVUE), but also because the second γ -orderliness of the second central moment kernel distribution is ensured by the next exciting theorem.

Theorem B.1. *The second central moment kernel distribution generated from any unimodal distribution is second γ -ordered, if $\gamma \geq 0$.*

Proof. In 1954, Hodges and Lehmann established that if X and Y are independently drawn from the same unimodal distribution, $X - Y$ will be a symmetric unimodal distribution peaking at zero (16). Given the constraint in the pseudo-sample that $X_i < X_j$, $i < j$, it directly follows from Theorem 1 in (16) that the pairwise difference distribution (Ξ) generated from any unimodal distribution is always monotonic increasing with a mode at zero. The transformation of the pairwise difference distribution via squaring and multiplication by $\frac{1}{2}$ does not change the monotonicity, making the pdf to become monotonically decreasing with a mode at zero. In the previous article, it was proven that a right-skewed distribution with a monotonic decreasing pdf is always second γ -ordered for non-negative γ , which gives the desired result. \square

Previously, it was shown that any symmetric distribution with a finite second moment is ν th ordered, suggesting that ν th orderliness does not require unimodality, e.g., a symmetric bimodal distribution is also ν th ordered. An analysis of the Weibull distribution showed that unimodality does not assure orderliness. Theorem B.1 uncovers a profound relationship between unimodality and second γ -orderliness, which is sufficient for γ -trimming inequality and γ -orderliness.

In 1928, Fisher constructed k -statistics as unbiased estimators of cumulants (17). Halmos (1946) proved that a functional θ admits an unbiased estimator if and only if it is a regular statistical functional of degree k and showed a relation of symmetry, unbiasedness and minimum variance (18). Hoeffding, in 1948, generalized U -statistics (19) which enable the derivation of a minimum-variance unbiased estimator from each unbiased estimator of an estimable parameter. In 1984, Serfling pointed out the speciality of Hodges-Lehmann estimator, which is neither a simple L -statistic nor a U -statistic, and considered the generalized L -statistics and trimmed U -statistics (20). Replacing the trimmed mean in the trimmed U -statistics with LL -statistic forms the definition of LU -statistics.

In 1997, Heffernan (21) obtained an unbiased estimator of the k th central moment by using U -statistics and demonstrated that it is the minimum variance unbiased estimator for distributions with the finite first k moments. The weighted k th central moment ($2 \leq k \leq n$) is thus defined as,

$$Wkm_{\epsilon_{U_k}, \gamma, n} := WL_{\epsilon, \gamma, n} \left((\psi_k(X_{N_1}, \dots, X_{N_k}))_{N=1}^{\binom{n}{k}} \right),$$

where $\epsilon_{U_k} = 1 - (1 - \epsilon)^{\frac{1}{k}}$, X_{N_1}, \dots, X_{N_k} are the n choose k elements from the sample, $\psi_k(x_1, \dots, x_k) = \sum_{j=0}^{k-2} (-1)^j \left(\frac{1}{k-j} \right) \sum (x_{i_1}^{k-j} x_{i_2} \dots x_{i_{j+1}}) + (-1)^{k-1} (k-1) x_1 \dots x_k$, the second summation is over $i_1, \dots, i_{j+1} = 1$ to k with $i_1 \neq i_2 \neq \dots \neq i_{j+1}$ and $i_2 < i_3 < \dots < i_{j+1}$ (21). Despite the complexity, the following theorem offers an approach to infer the general structure of such kernel distributions.

Theorem B.2. *Consider each k -tuple $(Q(p_1), Q(p_2), \dots, Q(p_{k-1}), Q(p_k))$ from the original distribution such that $Q(p_1) < Q(p_2) < \dots < Q(p_{k-1}) < Q(p_k)$, let the k -tuple be the inputs to the kernel function $\psi_k(x_1, \dots, x_k)$. Define $\Delta = Q(p_1) - Q(p_k)$ and denote ξ_Δ be the component quasi-distribution which depends on Δ , as detailed in the proof. The k th, where $k > 2$, central moment kernel distribution, labeled Ξ_k , can be seen as a quasi-mixture distribution with an infinite number of components. Each component possesses the support $(-\left(\frac{k}{3+(-1)^k}\right)^{-1}(-\Delta)^k, \frac{1}{k}(-\Delta)^k)$.*

Proof. Without loss of generality, under continuity, the probability density of the k -tuple is $f_{X, \dots, X}(Q(p_1), \dots, Q(p_k)) = k! f(Q(p_1)) \dots f(Q(p_k))$ (a result after a modification of the Jacobian density theorem). Transforming the distribution of the k -tuples by the function $\psi_k(x_1, \dots, x_k)$. The probability $f_{\Xi_k}(\Delta) = \sum_{\Delta = \psi_k(Q(p_1), \dots, Q(p_k))} f_{X, \dots, X}(Q(p_1), \dots, Q(p_k))$ is the summation of the probabilities of all k -tuples that satisfy $\Delta = \psi_k(Q(p_1), \dots, Q(p_k))$. The following Ξ_k is equivalent.

Ξ_k : Denote the set of all k -tuples from the original distribution as T . Now consider T_Δ , a subset of T , that includes all k -tuples for which the difference between $Q(p_1)$ and $Q(p_k)$ is a constant, Δ . A quasi-distribution, denoted by ξ_Δ , is derived by applying the function $\psi_k(x_1, \dots, x_k)$ on every k -tuple within T_Δ and associating the result, $\psi_k(Q(p_1), \dots, Q(p_k))$, with the corresponding probability density $f_{X, \dots, X}(Q(p_1), \dots, Q(p_k))$. The integral of this quasi-distribution does not sum to 1, and is thus labeled a "quasi-distribution". Ξ_k is a quasi-mixture distribution that has ξ_Δ s as its component quasi-distributions over all possible values of Δ , ranging from $Q(0) - Q(1)$ to 0. \square

Data Availability. Data for Table ?? are given in SI Dataset S1. All codes have been deposited in [GitHub](#).

ACKNOWLEDGMENTS. I gratefully acknowledge the constructive comments made by the editor which substantially improved the clarity and quality of this paper.

- CF Gauss, *Theoria combinationis observationum erroribus minimis obnoxiae*. (Henricus Dieterich), (1823).
- S Newcomb, A generalized theory of the combination of observations so as to obtain the best result. *Am. Journal Math.* **8**, 343–366 (1886).
- PJ Huber, Robust estimation of a location parameter. *Ann. Math. Stat.* **35**, 73–101 (1964).
- X He, WK Fung, Method of medians for lifetime data with weibull models. *Stat. medicine* **18**, 1993–2009 (1999).
- M Menon, Estimation of the shape and scale parameters of the weibull distribution. *Technometrics* **5**, 175–182 (1963).
- SD Dubey, Some percentile estimators for weibull parameters. *Technometrics* **9**, 119–129 (1967).
- NB Marks, Estimation of weibull parameters from common percentiles. *J. applied Stat.* **32**, 17–24 (2005).
- K Boudt, D Caliskan, C Croux, Robust explicit estimators of weibull parameters. *Metrika* **73**, 187–209 (2011).
- C Bernard, R Kazzi, S Vanduffel, Range value-at-risk bounds for unimodal distributions under partial information. *Insur. Math. Econ.* **94**, 9–24 (2020).
- WR van Zwet, *Convex Transformations of Random Variables: Nebst Stellingen*. (1964).
- AL Bowley, *Elements of statistics*. (King) No. 8, (1926).
- RA Groeneveld, G Meeden, Measuring skewness and kurtosis. *J. Royal Stat. Soc. Ser. D (The Stat.)* **33**, 391–399 (1984).
- PJ Bickel, EL Lehmann, Descriptive statistics for nonparametric models iv. spread in *Selected Works of EL Lehmann*. (Springer), pp. 519–526 (2012).
- PJ Rousseeuw, C Croux, Alternatives to the median absolute deviation. *J. Am. Stat. association* **88**, 1273–1283 (1993).
- PJ Bickel, EL Lehmann, Descriptive statistics for nonparametric models. iii. dispersion in *Selected works of EL Lehmann*. (Springer), pp. 499–518 (2012).
- J Hodges, E Lehmann, Matching in paired comparisons. *The Annals Math. Stat.* **25**, 787–791 (1954).
- RA Fisher, Moments and product moments of sampling distributions. *Proc. Lond. Math. Soc.* **2**, 199–238 (1930).
- PR Halmos, The theory of unbiased estimation. *The Annals Math. Stat.* **17**, 34–43 (1946).

- 385 19. W Hoeffding, A class of statistics with asymptotically normal distribution. *The Annals Math.*
386 *Stat.* **19**, 293–325 (1948).
- 387 20. RJ Serfling, Generalized l-, m-, and r-statistics. *The Annals Stat.* **12**, 76–86 (1984).
- 388 21. PM Heffernan, Unbiased estimation of central moments by using u-statistics. *J. Royal Stat.*
389 *Soc. Ser. B (Statistical Methodol.* **59**, 861–863 (1997).

DRAFT