

Near-consistent robust estimations of moments for unimodal distributions

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Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, leveraging the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments for common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U -statistics

The potential biases of robust location estimators in estimating the population mean have been noticed for more than two centuries (1), with numerous significant attempts made to address them. In calculating a robust location estimator, the procedure of identifying and downweighting extreme values inherently necessitates the formulation of certain distributional assumptions. Inconsistencies naturally arise when these assumptions, parametric or semiparametric, are violated. Previously, it was demonstrated that, due to the presence of infinite-dimensional nuisance shape parameters, the semiparametric approach struggles to consistently address distributions with shapes more intricate than γ -symmetry. Newcomb (1886) provided the first modern approach to robust parametric estimation by developing a class of estimators that gives "less weight to the more discordant observations" (2). In 1964, Huber (3) used the minimax procedure to obtain M -estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M -estimator increases rapidly. This is a common issue in parametric robust statistics. For example, He and Fung (1999) constructed (4) a robust M -estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for other parametric distributions, e.g., the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another interesting approach is based on L -estimators, such as percentile estimators. For examples of percentile estimators for the Weibull distribution, the reader is referred to the works of Menon (1963) (5), Dubey (1967) (6), Marks (2005) (7), and Boudt, Caliskan, and Croux (2011) (8). At the outset of the study of percentile estimators, it was known that they arithmetically utilize the invariant structures of probability distributions (5, 6). An estimator is classified as an I -statistic if it asymptotically satisfies $I(LE_1, \dots, LE_l) = (\theta_1, \dots, \theta_q)$ for the distribution it is consistent, where LE s are calculated with the use of LU -statistics (defined in Subsection ??), I is defined using arithmetic operations and constants but may also incorporate transcendental functions and quantile functions, and θ s are the population parameters it estimates. In this article, two subclasses of I -statistics are introduced, recombined

I -statistics and quantile I -statistics. Based on LU -statistics, I -statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L -estimator can be expressed as an integral of the quantile function, I -statistics are often analytically obtainable. However, it is observed that even when the sample follows a gamma distribution, which belongs to the same larger family as the Weibull model, the generalized gamma distribution, a misassumption can still lead to substantial biases in Marks percentile estimator (7), rendering the approach ill-suited (SI Dataset S1).

On the other hand, while robust estimation of scale has also been intensively studied with established methods (9, 10), the development of robust measures of asymmetry and kurtosis lags behind, despite the availability of several approaches (11–15). The purpose of this paper is to demonstrate that, in light of previous works, the estimation of central moments can be transformed into a location estimation problem by using U -statistics, the central moment kernel distributions possess desirable properties, and by utilizing the invariant structures of unimodal distributions, a suite of robust estimators can be constructed whose biases are typically smaller than the variances (as seen in Table ?? for $n = 4096$).

Proof. □

Data Availability. Data for Table ?? are given in SI Dataset S1. All codes have been deposited in [GitHub](#).

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Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. In this article, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

T.L. designed research, performed research, analyzed data, and wrote the paper.

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