Near-consistent robust estimations of moments for unimodal distributions

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Descriptive statistics for parametric models currently heavily rely on the accuracy of distributional assumptions. Here, leveraging the invariant structures of unimodal distributions, a series of sophisticated yet efficient estimators, robust to both gross errors and departures from parametric assumptions, are proposed for estimating mean and central moments for common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

orderliness | invariant | unimodal | adaptive estimation | U-statistics

The potential inconsistencies between the sample mean ($\bar{x}$) and robust location estimators in distributions with finite moments have been noticed for more than two centuries (1), with numerous significant attempts made to address them. In calculating the sample mean, extreme values are incorporated; however, they are much less weighted in robust location estimators. The procedure of identifying and downweighting extreme values inherently necessitates the formulation of certain distributional assumptions. Inconsistencies may arise when these assumptions, parametric or semiparametric, are violated. Due to the presence of infinite dimensional nuisance shape parameters, the semiparametric approach struggles to adequately address distributions with more intricate shapes. Newcomb (1886) provided the first modern approach to robust parametric estimation by developing a class of estimators that gives "less weight to the more discordant observations" (2). In 1964, Huber (3) used the minimax procedure to obtain M-estimator for the contaminated normal distribution, which has played a pre-eminent role in the later development of robust statistics. However, as previously demonstrated, under growing asymmetric departures from normality, the bias of the Huber M-estimator increases rapidly. This is a common issue in parametric robust statistics. For example, He and Fung (1999) constructed (4) a robust M-estimator for the two-parameter Weibull distribution, from which all moments can be calculated. Nonetheless, it is inadequate for other parametric distributions, e.g., the gamma, Perato, lognormal, and the generalized Gaussian distributions (SI Dataset S1). Another interesting approach is based on L-estimators, such as percentile estimators. For examples of percentile estimators for the Weibull distribution, the reader is referred to the works of Menon (1963) (5), Dubey (1967) (6), Marks (2005) (7), and Boudt, Caliskan, and Croux (2011) (8). At the outset of the study of percentile estimators, it was known that they arithmetically utilize the invariant structures of probability distributions (5, 6). Maybe such estimators can be named as I-statistics. Formally, an estimator is classified as an I-statistic if it asymptotically satisfies $\mathrm{I}\left(\mathrm{LE}_1, \ldots, \mathrm{LE}_l\right) = \left(\theta_1, \ldots, \theta_q\right)$ for the distribution it is consistent, where LEs are calculated with the use of LU-statistics (defined in Subsection **??**), I is defined using arithmetic operations and constants but may also incorporate transcendental functions and quantile functions, and θs are the population parameters it estimates. A subclass of I-statistics, arithmetic I-statistics, is defined as LEs are LU-statistics, I is solely defined using arithmetic operations and constants. Since some percentile estimators use the logarithmic function to transform all random variables before computing the L-estimators, a percentile estimator might not always be an arithmetic I-statistic (6). In this article, two subclasses of I-statistics are introduced, arithmetic I-statistics and quantile I-statistics. Examples of quantile I-statistics will be discussed later. Based on LU-statistics, I-statistics are naturally robust. Compared to probability density functions (pdfs) and cumulative distribution functions (cdfs), the quantile functions of many parametric distributions are more elegant. Since the expectation of an L-estimator can be expressed as an integral of the quantile function, I-statistics are often analytically obtainable. However, it is observed that Marks percentile estimator for the Weibull distribution (7) tends to be inferior to the robust M-estimators (3, 4), especially upon violation of the parametric assumptions (SI Dataset S1). Even when the sample follows a gamma distribution, which belongs to the same larger family as the Weibull model, the generalized gamma distribution, a misassumption can still lead to substantial biases in central moments, rendering the approach ill-suited (SI Dataset S1).

The majority of robust location estimators commonly used are symmetric owing to the prevalence of symmetric distributions. An asymmetric weighted L-statistic can achieve consistency for a semiparametric class of skewed distributions; but the lack of symmetry makes it suitable only for certain applications. Shifting from semiparametrics to parametrics, consider an estimator with a non-zero asymptotic breakdown point that is consistent for simultanously for both a semiparametric class of distributions and a distinct parametric distribution, such a robust mean estimator is called an invariant mean.

A. Invariant mean.

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. In this article, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

T.L. designed research, performed research, analyzed data, and wrote the paper.

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Data Availability. Data for Table **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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