

# Near-consistent robust estimations of moments for unimodal distributions

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orderliness | invariant | unimodal | adaptive estimation |  $U$ -statistics

**A. Congruent distribution.** In the realm of nonparametric statistics, the relative differences, or orders, of robust estimators are of primary importance. A key implication of this principle is that when there is a shift in the parameters of the underlying distribution, all nonparametric estimates should asymptotically change in the same direction, if they are estimating the same attribute of the distribution. If, on the other hand, the mean suggests an increase in the location of the distribution while the median indicates a decrease, a contradiction arises. It is worth noting that such contradiction is not possible for any  $LL$ -statistics in a location-scale distribution, as explained in the previous article on semiparametric robust mean. However, it is possible to construct counterexamples to the aforementioned implication in a shape-scale distribution. In the case of the Weibull distribution, its quantile function is  $Q_{Wei}(p) = \lambda(-\ln(1-p))^{1/\alpha}$ , where  $0 \leq p \leq 1$ ,  $\alpha > 0$ ,  $\lambda > 0$ ,  $\lambda$  is a scale parameter,  $\alpha$  is a shape parameter,  $\ln$  is the natural logarithm function. Then,  $m = \lambda \sqrt[1/\alpha]{\ln(2)}$ ,  $\mu = \lambda \Gamma(1 + \frac{1}{\alpha})$ , where  $\Gamma$  is the gamma function. When  $\alpha = 1$ ,  $m = \lambda \ln(2) \approx 0.693\lambda$ ,  $\mu = \lambda$ , when  $\alpha = \frac{1}{2}$ ,  $m = \lambda \ln^2(2) \approx 0.480\lambda$ ,  $\mu = 2\lambda$ , the mean increases as  $\alpha$  changes from 1 to  $\frac{1}{2}$ , but the median decreases. Previously, the fundamental role of quantile average and its relation to nearly all common nonparametric robust location estimates were demonstrated by using the method of classifying distributions through the signs of derivatives. To avoid such scenarios, this method can also be used. Let the quantile average function of a parametric distribution be denoted as  $QA(\epsilon, \gamma, \alpha_1, \dots, \alpha_i, \dots, \alpha_k)$ , where  $\alpha_i$  represent the parameters of the distribution, then, a distribution is  $\gamma$ -congruent if and only if the sign of  $\frac{\partial QA}{\partial \alpha_i}$  remains the same for all  $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ . If  $\frac{\partial QA}{\partial \alpha_i}$  is equal to zero or undefined, it can be considered both positive and negative, and thus does not impact the analysis. A distribution is completely  $\gamma$ -congruent if and only if it is  $\gamma$ -congruent and all its central moment kernel distributions are also  $\gamma$ -congruent. Setting  $\gamma = 1$  constitutes the definitions of congruence and complete congruence. Replacing the QA with  $\gamma mHLM$  gives the definition of  $\gamma$ - $U$ -congruence. Chebyshev's inequality implies that, for any probability distributions with finite second moments, as the parameters change, even if some  $LL$ -statistics change in a direction different from that of the population mean, the magnitude of the changes in the  $LL$ -statistics remains bounded compared to the changes in the population mean. Furthermore, distributions with infinite moments can be  $\gamma$ -congruent, since the definition is based on the quantile average, not the population mean.

The following theorems show the conditions that a distribution is congruent or  $\gamma$ -congruent.

**Theorem A.1.** A  $\gamma$ -symmetric distribution is always  $\gamma$ -congruent and  $\gamma$ - $U$ -congruent.

*Proof.* As shown in RSSM I, Theorem .2 and Theorem .18, for any  $\gamma$ -symmetric distribution, all quantile averages and all  $\gamma mHLM$ s coincide. The conclusion follows immediately.  $\square$

**Theorem A.2.** A positive definite location-scale distribution is always  $\gamma$ -congruent.

*Proof.* As shown in RSSM I, Theorem .2, for a location-scale distribution, any quantile average can be expressed as  $\lambda QA_0(\epsilon, \gamma) + \mu$ . Therefore, the derivatives with respect to the parameters  $\lambda$  or  $\mu$  are always positive. By application of the definition, the desired outcome is obtained.  $\square$

**Theorem A.3.** The second central moment kernel distribution derived from a continuous location-scale unimodal distribution is always  $\gamma$ -congruent.

*Proof.* Theorem ?? shows that the central moment kernel distribution generated from a location-scale distribution is also a location-scale distribution. Theorem ?? shows that it is positively definite. Implementing Theorem A.2 yields the desired result.  $\square$

For the Pareto distribution,  $\frac{\partial Q}{\partial \alpha} = \frac{x_m(1-p)^{-1/\alpha} \ln(1-p)}{\alpha^2}$ . Since  $\ln(1-p) < 0$  for all  $0 < p < 1$ ,  $(1-p)^{-1/\alpha} > 0$  for all  $0 < p < 1$  and  $\alpha > 0$ , so  $\frac{\partial Q}{\partial \alpha} < 0$ , and therefore  $\frac{\partial QA}{\partial \alpha} < 0$ , the Pareto distribution is  $\gamma$ -congruent. It is also  $\gamma$ - $U$ -congruent, since  $\gamma mHLM$  can also express as a function of  $Q(p)$ . For the lognormal distribution,  $\frac{\partial QA}{\partial \sigma} = \frac{1}{2} \left( \sqrt{2} \operatorname{erfc}^{-1}(2\gamma\epsilon) \left( -e^{\frac{\sqrt{2}\mu - 2\sigma \operatorname{erfc}^{-1}(2\gamma\epsilon)}{\sqrt{2}}} \right) + \left( -\sqrt{2} \right) \operatorname{erfc}^{-1}(2(1-\epsilon)) e^{\frac{\sqrt{2}\mu - 2\sigma \operatorname{erfc}^{-1}(2(1-\epsilon))}{\sqrt{2}}} \right)$ . Since the inverse complementary error function is positive when the input is smaller than 1, and negative when the input is larger than 1, and symmetry around 1, if  $0 \leq \gamma \leq 1$ ,  $\operatorname{erfc}^{-1}(2\gamma\epsilon) \geq -\operatorname{erfc}^{-1}(2-2\epsilon)$ ,  $e^{\mu - \sqrt{2}\sigma \operatorname{erfc}^{-1}(2-2\epsilon)} > e^{\mu - \sqrt{2}\sigma \operatorname{erfc}^{-1}(2\gamma\epsilon)}$ . Therefore, if  $0 \leq \gamma \leq 1$ ,  $\frac{\partial QA}{\partial \sigma} > 0$ , the lognormal distribution is  $\gamma$ -congruent. Theorem A.1 implies that the generalized Gaussian distribution is congruent and  $U$ -congruent. For the Weibull distribution, when  $\alpha$  changes from 1 to  $\frac{1}{2}$ , the average probability density on the left side of the median increases, since  $\frac{\frac{1}{2}}{\lambda \ln(2)} < \frac{\frac{1}{2}}{\lambda \ln^2(2)}$ , but the mean increases, indicating that the distribution is more heavy-tailed, the probability density of large values will also increase. So, the reason for non-congruence of the Weibull distribution lies

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90 in the simultaneous increase of probability densities on two op-  
91 posite sides as the shape parameter changes: one approaching  
92 the bound zero and the other approaching infinity. Note that  
93 the gamma distribution does not have this issue, Numerical  
94 results indicate that it is likely to be congruent.

95 Although some parametric distributions are not congruent,  
96 Theorem A.2 establishes that  $\gamma$ -congruence always holds for a  
97 positive definite location-scale family distribution and thus for  
98 the second central moment kernel distribution generated from  
99 a location-scale unimodal distribution as shown in Theorem  
100 A.3. Theorem ?? demonstrates that all central moment kernel  
101 distributions are unimodal-like with mode and median close  
102 to zero, as long as they are generated from unimodal distri-  
103 butions. Assuming finite moments and constant  $Q(0) - Q(1)$ ,  
104 increasing the mean of a distribution will result in a generally  
105 more heavy-tailed distribution, i.e., the probability density of  
106 the values close to  $Q(1)$  increases, since the total probability  
107 density is 1. In the case of the  $k$ th central moment kernel  
108 distribution,  $k > 2$ , while the total probability density on  
109 either side of zero remains generally constant as the median is  
110 generally close to zero and much less impacted by increasing  
111 the mean, the probability density of the values close to zero  
112 decreases as the mean increases. This transformation will  
113 increase nearly all symmetric weighted averages, in the general  
114 sense. Therefore, except for the median, which is assumed  
115 to be zero, nearly all symmetric weighted averages for all  
116 central moment kernel distributions derived from unimodal  
117 distributions should change in the same direction when the  
118 parameters change.

## 119 B. A shape-scale distribution as the consistent distribution.

120 In the last section, the parametric robust estimation is limited  
121 to a location-scale distribution, with the location parameter  
122 often being omitted for simplicity. For improved fit to observed  
123 skewness or kurtosis, shape-scale distributions with shape pa-  
124 rameter ( $\alpha$ ) and scale parameter ( $\lambda$ ) are commonly utilized.  
125 Weibull, gamma, Pareto, lognormal, and generalized Gaus-  
126 sian distributions (when  $\mu$  is a constant) are all shape-scale  
127 unimodal distributions.

128 **Data Availability.** Data for Table ?? are given in SI Dataset S1.  
129 All codes are attached.

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