Near-consistent robust estimations of moments for unimodal distributions

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A. Congruent distribution. In the realm of nonparametric statistics, the relative differences, or orders, of robust estimators are of primary importance. A key implication of this principle is that when there is a shift in the parameters of the underlying distribution, all nonparametric estimates should asymptotically change in the same direction, if they are estimating the same attribute of the distribution. If, on the other hand, the mean suggests an increase in the location of the distribution while the median indicates a decrease, a contradiction arises. It is worth noting that such contradiction is not possible for any LL-statistics in a location-scale distribution, as explained in the previous article on semiparametric robust mean. However, it is possible to construct counterexamples to the aforementioned implication in a shape-scale distribution. In the case of the Weibull distribution, its quantile function is $Q_{Wei}(p) = \lambda(-\ln(1-p))^{1/\alpha}$, where $0 \leq p \leq 1$, $\alpha > 0$, $\lambda > 0$, λ is a scale parameter, α is a shape parameter, ln is the natural logarithm function. Then, $m = \lambda \sqrt[\alpha]{\ln(2)}$, $\mu = \lambda\Gamma\left(1 + \frac{1}{\alpha}\right)$, where Γ is the gamma function. When $\alpha = 1$, $m = \lambda \ln(2) \approx 0.693\lambda$, $\mu = \lambda$, when $\alpha = \frac{1}{2}$, $m = \lambda \ln^2(2) \approx 0.480\lambda$, $\mu = 2\lambda$, the mean increases as α changes from 1 to $\frac{1}{2}$, but the median decreases. Previously, the fundamental role of quantile average and its relation to nearly all common nonparametric robust location estimates were demonstrated by using the method of classifying distributions through the signs of derivatives. To avoid such scenarios, this method can also be used. Let the quantile average function of a parametric distribution be denoted as $\mathrm{QA}(\epsilon, \gamma, \alpha_1, \cdots, \alpha_i, \cdots, \alpha_k)$, where α_i represent the parameters of the distribution, then, a distribution is γ-congruent if and only if the sign of $\frac{\partial \mathrm{QA}}{\partial \alpha_i}$ remains the same for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$. If $\frac{\partial \mathrm{QA}}{\partial \alpha_i}$ is equal to zero or undefined, it can be considered both positive and negative, and thus does not impact the analysis. A distribution is completely γ-congruent if and only if it is γ-congruent and all its central moment kernel distributions are also γ-congruent. Setting $\gamma = 1$ constitutes the definitions of congruence and complete congruence. Replacing the QA with γmHLM gives the definition of γ-U-congruence. Chebyshev's inequality implies that, for any probability distributions with finite second moments, as the parameters change, even if some LL-statistics change in a direction different from that of the population mean, the magnitude of the changes in the LL-statistics remains bounded compared to the changes in the population mean. Furthermore, distributions with infinite moments can be γ-congruent, since the definition is based on the quantile average, not the population mean.

The following theorems show the conditions that a distribution is congruent or γ-congruent.

Theorem A.1. *A γ-symmetric distribution is always γ-congruent and γ-U-congruent.*

Proof. As shown in RSSM I, Theorem .2 and Theorem .18, for any γ-symmetric distribution, all quantile averages and all γmHLMs conincide. The conclusion follows immediately. □

Theorem A.2. *A positive definite location-scale distribution is always γ-congruent.*

Proof. As shown in RSSM I, Theorem .2, for a location-scale distribution, any quantile average can be expressed as $\lambda \mathrm{QA}_0(\epsilon, \gamma) + \mu$. Therefore, the derivatives with respect to the parameters λ or μ are always positive. By application of the definition, the desired outcome is obtained. □

Theorem A.3. *The second central moment kernal distribution derived from a continuous location-scale unimodal distribution is always γ-congruent.*

Proof. Theorem **??** shows that the central moment kernel distribution generated from a location-scale distribution is also a location-scale distribution. Theorem **??** shows that it is positively definite. Implementing Theorem A.2 yields the desired result. □

For the Pareto distribution, $\frac{\partial Q}{\partial \alpha} = \frac{x_m(1-p)^{-1/\alpha}\ln(1-p)}{\alpha^2}$. Since $\ln(1-p) < 0$ for all $0 < p < 1$, $(1-p)^{-1/\alpha} > 0$ for all $0 < p < 1$ and $\alpha > 0$, so $\frac{\partial Q}{\partial \alpha} < 0$, and therefore $\frac{\partial \mathrm{QA}}{\partial \alpha} < 0$, the Pareto distribution is γ-congruent. It is also γ-U-congruent, since γmHLM can also express as a function of $Q(p)$. For the lognormal distribution, $\frac{\partial \mathrm{QA}}{\partial \sigma} = \frac{1}{2}\left(\sqrt{2}\mathrm{erfc}^{-1}(2\gamma\epsilon)\left(-e^{\frac{\sqrt{2}\mu - 2\sigma \mathrm{erfc}^{-1}(2\gamma\epsilon)}{\sqrt{2}}}\right) + \left(-\sqrt{2}\right)\mathrm{erfc}^{-1}(2(1-\epsilon))e^{\frac{\sqrt{2}\mu - 2\sigma\mathrm{erfc}^{-1}(2(1-\epsilon))}{\sqrt{2}}}\right)$. Since the inverse complementary error function is positive when the input is smaller than 1, and negative when the input is larger than 1, and symmetry around 1, if $0 \leq \gamma \leq 1$, $\mathrm{erfc}^{-1}(2\gamma\epsilon) \geq -\mathrm{erfc}^{-1}(2-2\epsilon)$, $e^{\mu - \sqrt{2}\sigma\mathrm{erfc}^{-1}(2-2\epsilon)} > e^{\mu-\sqrt{2}\sigma\mathrm{erfc}^{-1}(2\gamma\epsilon)}$. Therefore, if $0 \leq \gamma \leq 1$, $\frac{\partial \mathrm{QA}}{\partial \sigma} > 0$, the lognormal distribution is γ-congruent. Theorem A.1 implies that the generalized Gaussian distribution is congruent and U-congruent. For the Weibull distribution, when α changes from 1 to $\frac{1}{2}$, the average probability density on the left side of the median increases, since $\frac{\frac{1}{2}}{\lambda\ln(2)} < \frac{\frac{1}{2}}{\lambda\ln^2(2)}$,

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Data Availability. Data for Table **??** are given in SI Dataset S1. All codes are attached.

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