Near-consistent robust estimations of moments for unimodal distributions

Tuban Lee

This manuscript was compiled on June 7, 2023

Descriptive statistics for parametric models currently rely heavily on the accuracy of distributional assumptions. Here, leveraging the structures of parametric distributions and their central moment kernel distributions, a class of estimators, consistent simultanously for both a semiparametric distribution and a distinct parametric distribution, is proposed. These efficient estimators are robust to both gross errors and departures from parametric assumptions, making them ideal for estimating the mean and central moments of common unimodal distributions. This article also illuminates the understanding of the common nature of probability distributions and the measures of them.

A. Robust Estimations of the Central Moments. In 1979, Bickel and Lehmann (1), in their final paper of the landmark series *Descriptive Statistics for Nonparametric Models*, generalized a class of estimators called measures of spread, which "do not require the assumption of symmetry." From this, a popular efficient scale estimator, the Rousseeuw-Croux scale estimator (2), was derived in 1993, but the importance of tackling the symmetry assumption has been greatly underestimated. While they had already considered one version of the trimmed standard deviation, which is a measures of dispersion, in the third paper of that series (3); in the final section of that paper (1), they explored another two versions of the trimmed standard deviation based on pairwise differences, one is modified here for comparison,

$$\left[\binom{n}{2}(1-\epsilon-\gamma\epsilon)\right]^{-\frac{1}{2}}\left[\sum_{i=\binom{n}{2}\gamma\epsilon}^{\binom{n}{2}(1-\epsilon)}\left(X-X'\right)_i^2\right]^{\frac{1}{2}}, \quad [1]$$

where $(X-X')_1 \leq \ldots \leq (X-X')_{\binom{n}{2}}$ are the order statistics of the pseudo-sample, $X_i - X_j$, $i<j$. Let $\Delta = X_i - X_j$. They showed that, when $\epsilon = 0$, [1] is $\sqrt{2}$ times the standard deviation. The paper ended with, “We do not know a fortiori which of the measures is preferable and leave these interesting questions open.”

To address their open question, the nomenclature used in this paper is introduced as follows:

Nomenclature. Given a robust estimator $\hat{\theta}$, which has an adjustable breakdown point that can approach zero asymptotically, the name of $\hat{\theta}$ comprises two parts: the first part denotes the type of estimator, and the second part is the name of the population parameter θ that the estimator approaches as $\epsilon \to 0$. The abbreviation of the estimator combines the initial letters of the first part and the population parameter. If the estimator is symmetric, the upper asymptotic breakdown point, ϵ (defined in Subsection **??**, or $\epsilon_{U_{\mathbf{k}}}$ for LU-statistic), is indicated in the subscript of the abbreviation of the estimator, with the exception of the median. For an asymmetric estimator based on quantile average, the associated γ follows ϵ.

In the previous article on semiparametric robust mean estimation, it was shown that the bias of a robust estimator with an adjustable breakdown point is often monotonic with respect to the breakdown point in a semiparametric distribution. Naturally, the estimator's name should reflect the population parameter that it approaches as $\epsilon \to 0$. If multiplying all pseudo-samples by a factor of $\frac{1}{\sqrt{2}}$, then [1] is the trimmed standard deviation adhering to this nomenclature, since $\psi_2(x_1, x_2) = \frac{1}{2}(x_1 - x_2)^2$ is the kernel function of the unbiased estimation of the second central moment by using U-statistic. It should be preferable, not only because it is the square root of a trimmed U-statistic, which is closely related to the minimum-variance unbiased estimator (MVUE), but also because the second γ-orderliness of the second central moment kernel distribution is ensured by the next exciting theorem.

Theorem A.1. *The second central moment kernel distribution generated from any unimodal distribution is second γ-ordered.*

Proof. In 1954, Hodges and Lehmann established that if X and Y are independently drawn from the same unimodal distribution, $X-Y$ will be a symmetric unimodal distribution peaking at zero (4). Given the constraint in the pseudo-sample that $X_i < X_j$, $i<j$, it directly follows from Theorem 1 in (4) that the pairwise difference distribution (Ξ_Δ) generated from any unimodal distribution is always monotonic increasing with a mode at zero. The transformation of the pairwise difference distribution via squaring and multiplication by $\frac{1}{2}$ does not change the monotonicity, making the pdf to become monotonically decreasing with a mode at zero. In the previous article, it was proven that a right-skewed distribution with a monotonic decreasing pdf is always second γ-ordered, which gives the desired result. □

Theorem A.2.

Proof. □

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. In this article, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

[1] To whom correspondence should be addressed. E-mail: tl@biomathematics.org

1. PJ Bickel, EL Lehmann, Descriptive statistics for nonparametric models iv. spread in *Selected Works of EL Lehmann*. (Springer), pp. 519–526 (2012).
2. PJ Rousseeuw, C Croux, Alternatives to the median absolute deviation. *J. Am. Stat. association* **88**, 1273–1283 (1993).
3. PJ Bickel, EL Lehmann, Descriptive statistics for nonparametric models. iii. dispersion in *Selected works of EL Lehmann*. (Springer), pp. 499–518 (2012).
4. J Hodges, E Lehmann, Matching in paired comparisons. *The Annals Math. Stat*. **25**, 787–791 (1954).