Robust measures of semiparametric models II: Moments

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A. Invariant Moments. All popular robust location estimators, such as the symmetric trimmed mean, symmetric Winsorized mean, Hodges-Lehmann estimator, Huber M-estimator, and median of means, are symmetric. As shown in RSSM I, a γ-weighted Hodges-Lehmann mean ($\mathrm{WHLM}_{k,\epsilon,\gamma}$) can achieve consistency for the population mean in any γ-symmetric distribution with a finite mean. However, it falls considerably short of consistently handling other parametric distributions that are not γ-symmetric. Shifting from semiparametrics to parametrics, consider a robust estimator with a non-sample-dependent breakdown point (defined in Subsection **??**) which is consistent simultaneously for both a semiparametric distribution and a parametric distribution that does not belong to that semiparametric distribution, it is named with the prefix 'invariant' followed by the name of the population parameter it is consistent with. Here, the recombined I-statistic is defined as

$$\mathrm{RI}_{d,h_{\mathbf{k}},\mathbf{k}_1,\mathbf{k}_2,k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,n,LU_1,LU_2} :=$$
$$\lim_{c\to\infty}\left(\frac{\left(LU_{1_{h_{\mathbf{k}},\mathbf{k}_1,k_1,\epsilon_1,\gamma_1,n}}+c\right)^{d+1}}{\left(LU_{2_{h_{\mathbf{k}},\mathbf{k}_2,k_2,\epsilon_2,\gamma_2,n}}+c\right)^{d}}-c\right),$$

1 where d is the key factor for bias correction, $LU_{h_{\mathbf{k}},\mathbf{k},k,\epsilon,\gamma,n}$ is
2 the LU-statistic, $\mathbf{k}$ is the degree of the U-statistic, k is the
3 degree of the LL-statistic, ϵ is the upper asymptotic breakdown
4 point of the LU-statistic. It is assumed in this series that in
5 the subscript of an estimator, if $\mathbf{k}$, k and γ are omitted, $\mathbf{k}=1$,
6 $k=1$, $\gamma=1$ are assumed, if just one γ is indicated, $\gamma_1=\gamma_2$,
7 if n is omitted, only the asymptotic behavior is considered,
8 in the absence of subscripts, no assumptions are made. The
9 subsequent theorem shows the significance of a recombined
10 I-statistic.

11 **Theorem A.1.** *Define the recombined*
12 *mean as* $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,n,WL_1,WL_2} :=$
13 $RI_{d,h_{\mathbf{k}}=x,\mathbf{k}_1=1,\mathbf{k}_2=1,k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,n,LU_1=WL_1,LU_2=WL_2}$. *As-*
14 *suming finite means,* $rm_{d=\frac{\mu-WL_{1_{k_1,\epsilon_1,\gamma_1}}}{WL_{1_{k_1,\epsilon_1,\gamma_1}}-WL_{2_{k_2,\epsilon_2,\gamma_2}}},k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,W\ldots}$
15 *is a consistent mean estimator for a location-scale distribution,*
16 *where* μ, $WL_{1_{k_1,\epsilon_1,\gamma_1}}$, *and* $WL_{2_{k_2,\epsilon_2,\gamma_2}}$ *are different location*
17 *parameters from that location-scale distribution. If* $\gamma_1=\gamma_2$,
18 $WL = WHLM$, rm *is also consistent for any* γ-*symmetric*
19 *distributions.*

20 *Proof.* Finding d that make $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,WL_1,WL_2}$ a con-
21 sistent mean estimator is equivalent to finding the solution of
22 $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,WL_1,WL_2} = \mu$. First consider the location-
23 scale distribution. Since $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,WL_1,WL_2} =$
24 $\lim_{c\to\infty}\left(\frac{\left(WL_{1_{k_1,\epsilon_1,\gamma_1}}+c\right)^{d+1}}{\left(WL_{2_{k_2,\epsilon_2,\gamma_2}}+c\right)^{d}}-c\right) = (d+1)\,WL_{1_{k_1,\epsilon_1,\gamma}}-$

25 $dWL_{2_{k_2,\epsilon_2,\gamma}} = \mu$. So, $d = \frac{\mu-WL_{1_{k_1,\epsilon_1,\gamma_1}}}{WL_{1_{k_1,\epsilon_1,\gamma_1}}-WL_{2_{k_2,\epsilon_2,\gamma_2}}}$. In
26 RSSM I, it was established that any $WL(k,\epsilon,\gamma)$ can be ex-
27 pressed as $\lambda WL_0(k,\epsilon,\gamma)+\mu$ for a location-scale distribution
28 parameterized by a location parameter μ and a scale pa-
29 rameter λ, where $WL_0(k,\epsilon,\gamma)$ is a function of $Q_0(p)$, the
30 quantile function of a standard distribution without any
31 shifts or scaling, according to the definition of the weighted
32 L-statistic. The simultaneous cancellation of μ and λ in
33 $\frac{(\lambda\mu_0+\mu)-(\lambda WL_{10}(k_1,\epsilon_1,\gamma_1)+\mu)}{(\lambda WL_{10}(k_1,\epsilon_1,\gamma_1)+\mu)-(\lambda WL_{20}(k_2,\epsilon_2,\gamma_2)+\mu)}$ assures that the d in
34 rm is always a constant for a location-scale distribution. The
35 proof of the second assertion follows directly from the coinci-
36 dence property. According to Theorem 18 in RSSM I, for any
37 γ-symmetric distribution with a finite mean, $WHLM_{1_{k_1,\epsilon_1,\gamma}} =$
38 $WHLM_{2_{k_2,\epsilon_2,\gamma}} = \mu$. Then $rm_{d,k_1,k_2,\epsilon_1,\epsilon_2,\gamma,WHLM_1,WHLM_2} =$
39 $\lim_{c\to\infty}\left(\frac{(\mu+c)^{d+1}}{(\mu+c)^{d}}-c\right) = \mu$. This completes the demonstra-
40 tion. $\square$

41 For example, the Pareto distribution has a quantile function
42 $Q_{Par}(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, where x_m is the minimum possible
43 value that a random variable following the Pareto distribution
44 can take, serving a scale parameter, α is a shape parameter.
45 The mean of the Pareto distribution is given by $\frac{\alpha x_m}{\alpha-1}$. As
46 $WL(k,\epsilon,\gamma)$ can be expressed as a function of $Q(p)$, one can
47 set the two $WL_{k,\epsilon,\gamma}$s in the d value of rm as two arbitrary
48 quantiles $Q_{Par}(p_1)$ and $Q_{Par}(p_2)$. For the Pareto distribution,
49 $d_{Per,rm} = \frac{\mu_{Per}-Q_{Par}(p_1)}{Q_{Par}(p_1)-Q_{Par}(p_2)} = \frac{\frac{\alpha x_m}{\alpha-1}-x_m(1-p_1)^{-\frac{1}{\alpha}}}{x_m(1-p_1)^{-\frac{1}{\alpha}}-x_m(1-p_2)^{-\frac{1}{\alpha}}}$.
50 x_m can be canceled out. Intriguingly, the quantile func-
51 tion of exponential distribution is $Q_{exp}(p) = \ln\left(\frac{1}{1-p}\right)\lambda$,
52 $\lambda \geq 0$. $\mu_{exp} = \lambda$. Then, $d_{exp,rm} = \frac{\mu_{exp}-Q_{exp}(p_1)}{Q_{exp}(p_1)-Q_{exp}(p_2)} =$
53 $\frac{\lambda-\ln\left(\frac{1}{1-p_1}\right)\lambda}{\ln\left(\frac{1}{1-p_1}\right)\lambda-\ln\left(\frac{1}{1-p_2}\right)\lambda} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$. Since
54 $\lim_{\alpha\to\infty}\frac{\frac{\alpha}{\alpha-1}-(1-p_1)^{-1/\alpha}}{(1-p_1)^{-1/\alpha}-(1-p_2)^{-1/\alpha}} = -\frac{\ln(1-p_1)+1}{\ln(1-p_1)-\ln(1-p_2)}$,
55 $d_{Per,rm}$ approaches $d_{exp,rm}$, as $\alpha \to \infty$, regard-
56 less of the type of weighted L-statistic used. That

Significance Statement

Bias, variance, and contamination are the three main errors in statistics. Consistent robust estimation is unattainable without parametric assumptions. In this article, invariant moments are proposed as a means of achieving near-consistent and robust estimations of moments, even in scenarios where moderate violations of distributional assumptions occur, while the variances are sometimes smaller than those of the sample moments.

means, for the Weibull, gamma, Pareto, log-normal and generalized Gaussian distribution, $rm_{d=\frac{\mu-\text{WHLM}_{1k_1,\epsilon_1,\gamma}}{\text{WHLM}_{1k_1,\epsilon_1,\gamma}-\text{WHLM}_{2k_2,\epsilon_2,\gamma}},k_1,k_2,\epsilon_1,\epsilon_2,\gamma,\text{WHLM}_1,\text{WHLM}_2}$ is consistent for at least one particular case, where μ, $\text{WHLM}_{1k_1,\epsilon_1,\gamma}$, and $\text{WHLM}_{2k_2,\epsilon_2,\gamma}$ are different location parameters from an exponential distribution. Let $\text{WHLM}_{1k_1,\epsilon_1,\gamma} = \text{BM}_{\nu=3,\epsilon=\frac{1}{24}}$, $\text{WHLM}_{2k_2,\epsilon_2,\gamma} = m$, then $\mu = \lambda$, $m = Q\left(\frac{1}{2}\right) = \ln 2\lambda$, $\text{BM}_{\nu=3,\epsilon=\frac{1}{24}} = \lambda\left(1+\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)$, the detailed formula is given in the SI Text. So, $d = \frac{\mu-\text{BM}_{\nu=3,\epsilon=\frac{1}{24}}}{\text{BM}_{\nu=3,\epsilon=\frac{1}{24}}-m} = \frac{\lambda-\lambda\left(1+\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)}{\lambda\left(1+\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)-\ln 2\lambda} = $

$-\frac{\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)}{1-\ln(2)+\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)} \approx 0.103$. The biases of $rm_{d\approx0.103,\nu=3,\epsilon_1=\frac{1}{24},\epsilon_2=\frac{1}{2},\text{BM},m}$ for distributions with skewness between those of the exponential and symmetric distributions are tiny (SI Dataset S1). $rm_{d\approx0.103,\nu=3,\epsilon_1=\frac{1}{24},\epsilon_2=\frac{1}{2},\text{BM},m}$ exhibits excellent performance for all these common unimodal distributions (SI Dataset S1).

The recombined mean is an recombined I-statistic. Consider an I-statistic whose LEs are percentiles of a distribution obtained by plugging LU-statistics into a cumulative distribution function, I is defined with arithmetic operations, constants and quantile functions, such an estimator is classified as a quantile I-statistic. One version of the quantile I-statistic can be defined as $\text{QI}_{d,h_{\mathbf{k}},\mathbf{k}_1,\mathbf{k}_2,k_1,k_2,\epsilon_1,\epsilon_2,\gamma_1,\gamma_2,n,LU_1,LU_2} :=$

$\begin{cases} \hat{Q}_{n,h_{\mathbf{k}}}\left(\left(\hat{F}_{n,h_{\mathbf{k}}}\left(LU\right)-\frac{\gamma}{1+\gamma}\right)d+\hat{F}_{n,h_{\mathbf{k}}}\left(LU\right)\right) & \hat{F}_{n,h_{\mathbf{k}}}\left(LU\right)\geq\frac{\gamma}{1+\gamma} \\ \hat{Q}_{n,h_{\mathbf{k}}}\left(\hat{F}_{n,h_{\mathbf{k}}}\left(LU\right)-\left(\frac{\gamma}{1+\gamma}-\hat{F}_{n,h_{\mathbf{k}}}\left(LU\right)\right)d\right) & \hat{F}_{n,h_{\mathbf{k}}}\left(LU\right)<\frac{\gamma}{1+\gamma} \end{cases}$

where LU is $LU_{\mathbf{k},k,\epsilon,\gamma,n}$, $\hat{F}_{n,h_{\mathbf{k}}}\left(x\right)$ is the empirical cumulative distribution function of the $h_{\mathbf{k}}$ kernel distribution, $\hat{Q}_{n,h_{\mathbf{k}}}$ is the quantile function of the $h_{\mathbf{k}}$ kernel distribution.

Similarly, the quantile mean can be defined as $qm_{d,k,\epsilon,\gamma,n,\text{WL}} := \text{QI}_{d,h_{\mathbf{k}}=x,\mathbf{k}=1,k,\epsilon,\gamma,n,LU=\text{WL}}$. Moreover, in extreme right-skewed heavy-tailed distributions, if the calculated percentile exceeds $1-\epsilon$, it will be adjusted to $1-\epsilon$. In a left-skewed distribution, if the obtained percentile is smaller than $\gamma\epsilon$, it will also be adjusted to $\gamma\epsilon$. Without loss of generality, in the following discussion, only the case where $\hat{F}_n\left(\text{WL}_{k,\epsilon,\gamma,n}\right)\geq\frac{\gamma}{1+\gamma}$ is considered. A widely used method for calculating the sample quantile function involves employing linear interpolation of modes corresponding to the order statistics of the uniform distribution on the interval $[0,1]$, i.e., $\hat{Q}_n\left(p\right) = X_{\lfloor h\rfloor}+\left(h-\lfloor h\rfloor\right)\left(X_{\lceil h\rceil}-X_{\lfloor h\rfloor}\right)$, $h = (n-1)p+1$. To minimize the finite sample bias, here, the inverse function of $\hat{Q}_n$ is deduced as $\hat{F}_n\left(x\right) := \frac{1}{n-1}\left(cf-1+\frac{x-X_{cf}}{X_{cf+1}-X_{cf}}\right)$, where $cf = \sum_{i=1}^n\mathbf{1}_{X_i\leq x}$, $\mathbf{1}_A$ is the indicator of event A. The quantile mean uses the location-scale invariant in a different way, as shown in the subsequent proof.

Theorem A.2. $qm_{d=\frac{F(\mu)-F(WL_{k,\epsilon,\gamma})}{F(WL_{k,\epsilon,\gamma})-\frac{\gamma}{1+\gamma}},k,\epsilon,\gamma,\text{WL}}$ *is a consistent mean estimator for a location-scale distribution provided that the means are finite and $F(\mu)$, $F\left(WL_{k,\epsilon,\gamma}\right)$ and $\frac{\gamma}{1+\gamma}$ are all within the range of $[\gamma\epsilon,1-\epsilon]$, where μ and $WL_{k,\epsilon,\gamma}$ are location parameters from that location-scale distribution. If*

$WL = WHLM$, *qm is also consistent for any γ-symmetric distributions.*

Proof. When $F\left(\text{WL}_{k,\epsilon,\gamma}\right) \geq \frac{\gamma}{1+\gamma}$, the solution of $\left(F\left(\text{WL}_{k,\epsilon,\gamma}\right)-\frac{\gamma}{1+\gamma}\right)d + F\left(\text{WL}_{k,\epsilon,\gamma}\right) = F\left(\mu\right)$ is $d = \frac{F(\mu)-F(\text{WL}_{k,\epsilon,\gamma})}{F(\text{WL}_{k,\epsilon,\gamma})-\frac{\gamma}{1+\gamma}}$. The d value for the case where $F\left(\text{WL}_{k,\epsilon,\gamma,n}\right) < \frac{\gamma}{1+\gamma}$ is the same. The definitions of the location and scale parameters are such that they must satisfy $F(x;\lambda,\mu) = F(\frac{x-\mu}{\lambda};1,0)$, then $F(\text{WL}(k,\epsilon,\gamma);\lambda,\mu) = F(\frac{\lambda\text{WL}_0(k,\epsilon,\gamma)+\mu-\mu}{\lambda};1,0) = F(\text{WL}_0(k,\epsilon,\gamma);1,0)$. It follows that the percentile of any weighted L-statistic is free of λ and μ for a location-scale distribution. Therefore d in qm is also invariably a constant. For the γ-symmetric case, $F\left(\text{WHLM}_{k,\epsilon,\gamma}\right) = F\left(\mu\right) = F(Q(\frac{\gamma}{1+\gamma})) = \frac{\gamma}{1+\gamma}$ is valid for any γ-symmetric distribution with a finite second moment, as the same values correspond to same percentiles. Then, $qm_{d,k,\epsilon,\gamma,\text{WHLM}} = F^{-1}\left(\left(F\left(\text{WHLM}_{k,\epsilon,\gamma}\right)-\frac{\gamma}{1+\gamma}\right)d+F\left(\mu\right)\right) = F^{-1}\left(0+F\left(\mu\right)\right) = \mu$. To avoid inconsistency due to post-adjustment, $F(\mu)$, $F\left(\text{WL}_{k,\epsilon,\gamma}\right)$ and $\frac{\gamma}{1+\gamma}$ must reside within the range of $[\gamma\epsilon,1-\epsilon]$. All results are now proven. $\square$

The cdf of the Pareto distribution is $F_{Par}(x) = 1-\left(\frac{x_m}{x}\right)^\alpha$. So, set the d value in qm with two arbitrary percentiles p_1 and p_2, $d_{Par,qm} = \frac{1-\left(\frac{x_m}{\frac{\alpha x_m}{\alpha-1}}\right)^\alpha-\left(1-\left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right)}{\left(1-\left(\frac{x_m}{x_m(1-p_1)^{-\frac{1}{\alpha}}}\right)^\alpha\right)-\left(1-\left(\frac{x_m}{x_m(1-p_2)^{-\frac{1}{\alpha}}}\right)^\alpha\right)} = \frac{1-\left(\frac{\alpha-1}{\alpha}\right)^\alpha-p_1}{p_1-p_2}$. The d value in qm for the exponential distribution is always identical to $d_{Par,qm}$ as $\alpha \to \infty$, since $\lim_{\alpha\to\infty}\left(\frac{\alpha-1}{\alpha}\right)^\alpha = \frac{1}{e}$ and the cdf of the exponential distribution is $F_{exp}\left(x\right) = 1-e^{-\lambda^{-1}x}$, then $d_{exp,qm} = \frac{(1-e^{-1})-\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)}{\left(1-e^{-\ln\left(\frac{1}{1-p_1}\right)}\right)-\left(1-e^{-\ln\left(\frac{1}{1-p_2}\right)}\right)} = \frac{1-\frac{1}{e}-p_1}{p_1-p_2}$. So, for the Weibull, gamma, Pareto, lognormal and generalized Gaussian distribution, $qm_{d=\frac{F_{exp}(\mu)-F_{exp}(\text{WHLM}_{k,\epsilon,\gamma})}{F_{exp}(\text{WHLM}_{k,\epsilon,\gamma})-\frac{\gamma}{1+\gamma}},k,\epsilon,\gamma,\text{WHLM}}$ is also consistent for at least one particular case, provided that μ and $\text{WHLM}_{k,\epsilon,\gamma}$ are different location parameters from an exponential distribution and $F(\mu)$, $F(\text{WHLM}_{k,\epsilon,\gamma})$ and $\frac{\gamma}{1+\gamma}$ are all within the range of $[\gamma\epsilon,1-\epsilon]$. Also let $\text{WHLM}_{k,\epsilon,\gamma} = \text{BM}_{\nu=3,\epsilon=\frac{1}{24}}$ and $\mu = \lambda$, then $d = \frac{F_{exp}(\mu)-F_{exp}(\text{BM}_{\nu=3,\epsilon=\frac{1}{24}})}{F_{exp}(\text{BM}_{\nu=3,\epsilon=\frac{1}{24}})-\frac{1}{2}} = \frac{-e^{-1}+e^{-\left(1+\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)}}{\frac{1}{2}-e^{-\left(1+\ln\left(\frac{26068394603446272\sqrt[6]{\frac{7}{247}}\sqrt[3]{11}}{391^{5/6}101898752449325\sqrt{5}}\right)\right)}} = \frac{\frac{101898752449325\sqrt{5}\sqrt[6]{\frac{247}{7}}391^{5/6}}{26068394603446272\sqrt[3]{11}e}-\frac{1}{e}}{\frac{1}{2}-\frac{101898752449325\sqrt{5}\sqrt[6]{\frac{247}{7}}391^{5/6}}{26068394603446272\sqrt[3]{11}e}} \approx 0.088$. $F_{exp}(\mu)$, $F_{exp}(\text{BM}_{\nu=3,\epsilon=\frac{1}{24}})$ and $\frac{1}{2}$ are all within the range of $[\frac{1}{24},\frac{23}{24}]$. $qm_{d\approx0.088,\nu=3,\epsilon=\frac{1}{24},\text{BM}}$ works better in the fat-tail scenarios (SI Dataset S1). Theorem A.1 and A.2 show that $rm_{d\approx0.103,\nu=3,\epsilon_1=\frac{1}{24},\epsilon_2=\frac{1}{2},\text{BM},m}$ and $qm_{d\approx0.088,\nu=3,\epsilon=\frac{1}{24},\text{BM}}$

are both consistent mean estimators for any symmetric distribution and the exponential distribution with finite second moments. It's obvious that the asymptotic breakdown points of $rm_{d\approx 0.103,\nu=3,\epsilon_1=\frac{1}{24},\epsilon_2=\frac{1}{2},\mathrm{BM},m}$ and $qm_{d\approx 0.088,\nu=3,\epsilon=\frac{1}{24},\mathrm{BM}}$ are both $\frac{1}{24}$. Therefore they are all invariant means.

To study the impact of the choice of WLs in rm and qm, it is constructive to recall that a weighted L-statistic is a combination of order statistics. While using a less-biased weighted L-statistic can generally enhance performance (SI Dataset S1), there is a greater risk of violation in the semiparametric framework. However, the mean-WA$_{\epsilon,\gamma}$-γ-median inequality is robust to slight fluctuations of the QA function of the underlying distribution. Suppose for a right-skewed distribution, the QA function is generally decreasing with respect to ϵ in $[0, u]$, but increasing in $[u, \frac{1}{1+\gamma}]$, since all quantile averages with breakdown points from ϵ to $\frac{1}{1+\gamma}$ will be included in the computation of WA$_{\epsilon,\gamma}$, as long as $\frac{1}{1+\gamma} - u \ll \frac{1}{1+\gamma} - \gamma\epsilon$, and other portions of the QA function satisfy the inequality constraints that define the νth γ-orderliness on which the WA$_{\epsilon,\gamma}$ is based, if $0 \leq \gamma \leq 1$, the mean-WA$_{\epsilon,\gamma}$-γ-median inequality still holds. This is due to the violation of νth γ-orderliness being bounded, when $0 \leq \gamma \leq 1$, as shown in RMSM I and therefore cannot be extreme for unimodal distributions with finite second moments.

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