Semiparametric robust mean estimation based on the orderliness of quantile averages

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As arguably the most fundamental problem in statistics, nonparametric robust location estimation has many prominent solutions, such as the trimmed mean, Winsorized mean, Hodges–Lehmann estimator, and median of means. Recent research findings suggest that their biases with respect to mean can be quite different in asymmetric distributions. However, the underlying mechanisms remain largely unclear. Here, similar to the mean-median-mode inequality, it is proven that in the context of nearly all common unimodal distributions, there exists an orderliness of symmetric quantile averages with different breakdown points. Further deductions explain why the Winsorized mean and median of means generally have smaller biases compared to the trimmed mean. Building on the U-orderliness, the supremacy of weighted Hodges–Lehmann mean is discussed.

semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

In 1823, Gauss (1) proved that for any unimodal distribution with a finite second moment, $|m-\mu| \leq \sqrt{\frac{3}{4}}\omega$, where μ is the population mean, m is the population median, ω is the root mean square deviation from the mode, M. Bernard, Kazzi, and Vanduffel (2020) (2) derived bias bounds for the ϵ-symmetric quantile average (SQA_ϵ) for unimodal distributions, building on the works of Karlin and Novikoff (1963) and Li, Shao, Wang, and Yang (2018) (3, 4). They showed that the m has the smallest maximum distance to the μ among all symmetric quantile averages. Daniell, in 1920, (5) analyzed a class of estimators, linear combinations of order statistics, and identified that ϵ-symmetric trimmed mean (TM_ϵ) belongs to this class. Another popular choice, the ϵ-symmetric Winsorized mean (WM_ϵ), which was named after Winsor and introduced by Tukey (6) and Dixon (7) in 1960, is also an L-statistic. Without assuming unimodality, Bieniek (2016) derived exact bias upper bounds of the Winsorized mean based on Danielak and Rychlik's work (2003) on the trimmed mean and confirmed that the former is smaller than the latter (8, 9). In 1963, Hodges and Lehmann (10) proposed a class of nonparametric location estimators based on rank tests and, from the Wilcoxon signed-rank statistic (11), deduced the median of pairwise means as a robust location estimator for a symmetric population. The concept of median of means ($\mathrm{MoM}_{k,b}$, k is the number of size in each block, b is the number of blocks) was implicit several times in Nemirovsky and Yudin (1983) (12), Jerrum, Valiant, and Vazirani (1986), (13) and Alon, Matias and Szegedy (1996) (14)'s works. Having good performance even for distributions with infinite second moments, the advantages of MoM have received increasing attention over the past decade (15–22). Devroye, Lerasle, Lugosi, and Oliveira (2016) showed that MoM nears the optimum of nonparametric mean estimation with regards to concentration bounds when the distribution has a heavy tail (20). In fact, asymptotically, the Hodges-Lehmann (H-L) estimator is equivalent to $\mathrm{MoM}_{k=2,b=\frac{n}{k}}$, and it can be seen as the pairwise mean distribution is approximated by the bootstrap and sampling without replacement, respectively (for the asymptotic validity, the reader is referred to the foundational works of Efron (1979) (23), Bickel and Freedman (1981, 1984) (24, 25), and Helmers, Janssen, and Veraverbeke (1990) (26)).

Here, the ϵ,b-stratified mean is defined as

$$\mathrm{SM}_{\epsilon,b,n} \coloneqq \frac{b}{n}\left(\sum_{j=1}^{\frac{b-1}{2b\epsilon}} \sum_{i_j=\frac{(2bj-b-1)n\epsilon}{b-1}+1}^{\frac{(2bj-b+1)n\epsilon}{b-1}} X_{i_j}\right),$$

where $X_1 \leq \ldots \leq X_n$ denote the order statistics of a sample of n independent and identically distributed random variables $X_1, \ldots, X_n$, $\epsilon \bmod \frac{2}{b-1} = 0$, $\frac{1}{\epsilon} \geq 9$. $n \geq \frac{b-1}{2\epsilon}$. If the subscript n is omitted, only the asymptotic behavior is considered. If b is omitted, $b = 3$ is assumed. A solution for $n \bmod \frac{b-1}{2\epsilon} \neq 0$ is sampling without replacement to create several smaller samples that satisfy the equality and then computing the mean of all estimations. It can be seen as sampling smaller samples from the population several times and thus this approach preserves the original distribution. The basic idea of the stratified mean is to distribute the random variables into $\frac{b-1}{2\epsilon}$ blocks according to their order, then further sequentially group these blocks into b strata and compute the mean of the middle stratum, which is the median of means of each stratum. Therefore, the stratified mean is a type of stratum mean in stratified sampling introduced by Neyman in 1934 (27). Although the principle is similar to the median of means, without the random shift, the result is different from $\mathrm{MoM}_{k=\frac{n}{b},b}$. The median of means and stratified mean are consistent mean estimators if their asymptotic breakdown points are zero. However, if $\epsilon = \frac{1}{9}$, the biases of the $\mathrm{SM}_{\frac{1}{9}}$ are nearly identical to those of

Significance Statement

In 1964, van Zwet introduced the convex transformation order for comparing the skewness of two distributions. This paradigm shift played a fundamental role in defining robust measures of distributions, from spread to kurtosis. Here, rather than the stochastic ordering between two distributions, the orderliness of quantile averages within a distribution is investigated. By classifying distributions through inequalities, a series of sophisticated robust mean estimators are deduced. Nearly all common nonparametric robust location estimators are found to be special cases thereof.

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the $\mathrm{WM}_{\frac{1}{9}}$ in asymmetric distributions (Figure 1, if no other subscripts, ϵ is omitted for simplicity), i.e., their robustness to departures from the symmetry assumption is similar in practice. More importantly, the bounds confirm that the worst-case performances of WM_ϵ are better than those of TM_ϵ in terms of bias, due to the complexity, any extensions are difficult. The aim of this paper is to define a series of semiparametric models using inequalities, demonstrate their elegant interrelations and connections to parametric models, and deduce a set of sophisticated robust mean estimators.

Quantile average and weighted average

ϵ-symmetric trimmed mean, ϵ-symmetric Winsorized mean, and ϵ-stratified mean are all L-statistics. More specifically, they are symmetric weighted averages, which is defined as

$$\mathrm{SWA}_{\epsilon,n} := \frac{\sum_{i=1}^{\frac{n}{2}} \frac{X_i + X_{n-i+1}}{2} w_i}{\sum_{i=1}^{\frac{n}{2}} w_i},$$

where w_is are the weights applied to the symmetric quantile average according to the definition of the corresponding L-statistic. For example, for the ϵ-symmetric trimmed mean, $w_i = \begin{cases} 0, & i < n\epsilon \\ 1, & i \geq n\epsilon \end{cases}$. Mean ($\lim_{\epsilon \to 0} \mathrm{TM}_\epsilon$) and median ($\mathrm{TM}_{\frac{1}{2}}$) are two special cases of symmetric trimmed mean. In 1974, Hogg investigated asymmetric trimmed mean and found its advantages for some special applications (28). To extend to the asymmetric case, the quantile average can be defined as

$$\mathrm{QA}(\epsilon, \gamma, n) := \frac{1}{2}(\hat{Q}_n(\epsilon) + \hat{Q}_n(1 - \gamma\epsilon)).$$

where $\gamma > 0$ and $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\hat{Q}_n(p)$ is the empirical quantile function. For example, $\mathrm{QA}(\epsilon = 0.2, \gamma = 2, n) = \frac{1}{2}(\hat{Q}_n(0.2) + \hat{Q}_n(0.6))$. Symmetric quantile average is a special case of quantile average when $\gamma = 1$.

Analogously, weighted average can be defined as

$$\mathrm{WA}_{\epsilon,\gamma} := \frac{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} \mathrm{QA}\left(\epsilon_0, \gamma\right) w_{\epsilon_0}}{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} w_{\epsilon_0}}.$$

Converting this asymptotic definition to finite sample definition requires rounding the $n\epsilon_0$, for simplicity, only asymptotic definition is considered here. For example, the ϵ,γ-asymmetric trimmed mean ($\mathrm{TM}_{\epsilon,\gamma}$) is a weighted average that trims the left side ϵ and trims the right side $\gamma\epsilon$, where $w_{\epsilon_0} = \begin{cases} 0, & \epsilon_0 < \epsilon \\ 1, & \epsilon_0 \geq \epsilon \end{cases}$. Noted that a weighted average is an L-statistic, but an L-statistic might not be a weighted average, because in a weighted average, all quantiles are paired with the same γ. For the sake of brevity, in the following text, if γ is not indicated, symmetry will be assumed.

Classifying distributions through inequalities

Let $\mathcal{P}_k$ denote the set of all distributions over $\mathbb{R}$ whose moments, from the first to the kth, are all finite. Without loss of generality, all classes discussed in the following are subclasses of the nonparametric class of distributions such that $\mathcal{P}_\Upsilon^k := \{P \text{ is continuous } \wedge \text{ all } P \in \mathcal{P}_k\}$. Besides fully and smoothly parameterizing by a Euclidean parameter, or just assuming regularity conditions, there are many ways to classify distributions. In 1956, Stein initiated the problem of estimating parameters in the presence of an infinite dimensional nuisance shape parameter (29). A notable example discussed in his groundbreaking work was the estimation of the center of symmetry for an unknown symmetric distribution. In 1993, Bickel, Klaassen, Ritov, and Wellner published an influential semiparametrics textbook (30). They systematically classified nearly all common models into three classes: parametric, nonparametric, and semiparametric. However, there is another old and commonly encountered class of distributions that receives little attention in semiparametric literature: the unimodal distribution. It is a very unique semiparametric model because its definition is based on inequalities, i.e., assuming P is continuous, $(f'(x) > 0 \text{ for } x \leq M) \wedge (f'(x) < 0 \text{ for } x \geq M)$. Let $\mathcal{P}_U$ denote the set of all unimodal distributions. Five parametric distributions in $\mathcal{P}_U$ are detailed as examples here: Weibull, gamma, Pareto, lognormal and generalized Gaussian.

There was a widespread misbelief that the median is always located between the mean and the mode for an arbitrary unimodal distribution until Runnenburg (1978) and van Zwet (1979) (31, 32) endeavored to determine sufficient conditions under which the inequality holds, thus implying the possibility of its violation (counterexamples see Dharmadhikari and Joag-Dev (1988), Basu and DasGupta (1997), and Abadir (2005)'s papers) (33–35). The class of distributions satisfying the mean-median-mode inequality constitutes a subclass of $\mathcal{P}_U$. Analogously, a right-skewed distribution is called γ-ordered, if and only if

$$\forall \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \mathrm{QA}_{\epsilon_1,\gamma} \geq \mathrm{QA}_{\epsilon_2,\gamma}.$$

It is reasonable, although not necessary, to further assume $\gamma \geq 1$ since the gross errors of a right-skewed distribution, often, are mainly from the right side. The left-skewed case is just reversing the inequality and, if needed, assuming $\gamma \leq 1$; for simplicity, it will be completely omitted in the following discussion. If $\gamma = 1$, it is referred to as ordered. This nomenclature will be assumed in the following text. Let $\mathcal{P}_O$ denote the set of all ordered distributions. Nearly all common unimodal distributions, including Weibull, gamma, Pareto, lognormal and generalized Gaussian, are in $\mathcal{P}_U \cap \mathcal{P}_O$ (proven in the following discussion and SI Text). The only minor exceptions occur when the Weibull and gamma distribution are near-symmetric (shown in the SI Text). Unlike the mean-median-mode inequality, whose sufficient conditions are very cumbersome, a necessary and sufficient condition of the γ-orderliness is the monotonic property of the bias function of $\mathrm{QA}_{\epsilon,\gamma}$ with respect to ϵ (proven in the SI Text). The following necessary and sufficient condition hints at the relation between the mean-median-mode inequality and the orderliness.

Theorem .1. *Let P_Υ^k denote an arbitrary distribution in the set $\mathcal{P}_\Upsilon^k$. $P_\Upsilon^k \in \mathcal{P}_O$ if and only if the pdf satisfies the inequality $f(Q(\epsilon)) \geq f(Q(1-\epsilon))$, where $0 \leq \epsilon \leq \frac{1}{2}$ (also assumed in the following discussions for ordered distributions), $Q(\epsilon)$ is the quantile function.*

Proof. From the definition of ordered distribution, deducing $\frac{Q(\epsilon-\delta)+Q(1-\epsilon+\delta)}{2} \geq \frac{Q(\epsilon)+Q(1-\epsilon)}{2} \Leftrightarrow Q(\epsilon - \delta) - Q(\epsilon) \geq Q(1 - \epsilon) - Q(1-\epsilon+\delta) \Leftrightarrow Q'(1-\epsilon) \geq Q'(\epsilon)$, where δ is an infinitesimal quantity. Since the quantile function is the inverse function of the cumulative distribution function (cdf), $Q'(1 - \epsilon) \geq$

$Q'(\epsilon) \Leftrightarrow F'(Q(\epsilon)) \geq F'(Q(1-\epsilon))$, the proof is complete by noticing that the derivative of cdf is pdf. $\square$

The mean-median difference $|\mu - m|$ was proposed to measure skewness by Pearson (1895) (36). Bowley (1926) proposed a robust skewness based on the SQA-median difference $|\mathrm{SQA}_\epsilon - m|$ (37). Groeneveld and Meeden (1984) (38) generalized these measures of skewness based on van Zwet's convex transformation (39) and investigated their properties. Suppose P_Υ^k follows the mean-median-mode inequality. Then, the probability density $f(Q(\epsilon))$ on the left side of the median, on average, is greater than the corresponding $f(Q(1-\epsilon))$, since $m < \frac{Q(0)+Q(1)}{2} \Leftrightarrow m - Q(0) < Q(1) - m$. If $Q(\epsilon) > M$, the inequality $f(Q(\epsilon)) \geq f(Q(1-\epsilon))$ holds. The principle can be extended to unimodal-like distributions. Suppose there is a right-skewed continuous multimodal distribution following the mean-median-first mode inequality with many small modes on the right side, the first mode, M, has the greatest probability density and the median is within the first dominant mode, i.e., if $x > m$, $f(m) \geq f(x)$, then, if $Q(\epsilon) > M$, the inequality $f(Q(\epsilon)) \geq f(Q(1-\epsilon))$ will also holds.

Furthermore, most common right-skewed distributions are partial bounded. This implies the convex decreasing behavior of the QA function when $\epsilon \to 0$. If assuming convexity further, the second γ-orderliness can be defined as the following for a right-skewed distribution plus the γ-orderliness,

$$\forall \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \frac{1}{1+\gamma}, \frac{\mathrm{QA}_{\epsilon_1,\gamma} - \mathrm{QA}_{\epsilon_2,\gamma}}{\epsilon_2 - \epsilon_1} \geq \frac{\mathrm{QA}_{\epsilon_2,\gamma} - \mathrm{QA}_{\epsilon_3,\gamma}}{\epsilon_3 - \epsilon_2}.$$

An equivalent expression is $\frac{d^2\mathrm{QA}}{d\epsilon^2} \geq 0 \wedge \frac{d\mathrm{QA}}{d\epsilon} \leq 0$. Analogously, the νth γ-orderliness can be defined as $(-1)^\nu \frac{d^\nu \mathrm{QA}}{d\epsilon^\nu} \geq 0 \wedge \ldots \wedge -\frac{d\mathrm{QA}}{d\epsilon} \geq 0$. The definition of νth orderliness is the same, just setting $\gamma = 1$. Common unimodal distributions are also second and third ordered (shown in the SI Text). Let $\mathcal{P}_{O_\nu}$ and $\mathcal{P}_{\gamma O_\nu}$ denote the sets of all distributions which are νth ordered and νth γ-ordered. The following theorems can be used to quickly identify parametric distributions in $\mathcal{P}_{O_\nu}$ and $\mathcal{P}_{\gamma O_\nu}$ without solving the exact derivative.

Theorem .2. *Any symmetric distribution with a finite second moment is νth ordered.*

Proof. The assertion follows from the fact that for any symmetric distribution with a finite second moment, all symmetric quantile averages coincide. Therefore, the SQA function is always a horizontal line; the νth order derivative is zero. $\square$

As a consequence of Theorem .2 and the fact that generalized Gaussian distribution is symmetric around the median, it is νth ordered.

Theorem .3. *Any continuous right skewed distribution whose Q satisfies $Q^{(\nu)}(p) \geq 0 \wedge \ldots Q^{(i)}(p) \geq 0 \ldots \wedge Q^{(2)}(p) \geq 0$, i mod $2 = 0$, is νth γ-ordered, provided that $\gamma \geq 1$.*

Proof. Let $\mathrm{QA}(\epsilon) = \frac{1}{2}(Q(\epsilon) + Q(1-\gamma\epsilon))$, where $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ (also assumed in the following discussions for γ-ordered distributions), then $(-1)^j \frac{d^j \mathrm{QA}}{d\epsilon^j} = \frac{1}{2}((-1)^j Q^j(\epsilon) + \gamma^j Q^j(1-\gamma\epsilon))$, $\nu \geq j \geq 1$, when j mod $2 = 0$, $(-1)^j \frac{d^j \mathrm{QA}}{d\epsilon^j} \geq 0$, when j mod $2 = 1$, the strict positivity is uncertain. If assuming $\gamma \geq 1$, $(-1)^j \frac{d^j \mathrm{QA}}{d\epsilon^j} \geq 0$, since $Q^{(j+1)}(\epsilon) \geq 0$. $\square$

It is now trivial to prove that the Pareto distribution follows the νth γ-orderliness, provided that $\gamma \geq 1$, since the quantile function of the Pareto distribution is $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, $x_m > 0$, $\alpha > 0$, $Q^{(\nu)}(p) \geq 0$ according to the chain rule.

Theorem .4. *A right-skewed continuous distribution with a monotonic decreasing pdf is γ-ordered, if $\gamma \geq 1$.*

Proof. A monotonic decreasing pdf means $f'(x) = F^{(2)}(x) \leq 0$. Since $Q'(p) \geq 0$, let $x = Q(F(x))$, then by differentiating both sides of the equation twice, one can obtain $0 = Q^{(2)}(F(x))(F'(x))^2 + Q'(F(x))F^{(2)}(x) \Leftrightarrow Q^{(2)}(F(x)) = -\frac{Q'(F(x))F^{(2)}(x)}{(F'(x))^2} \geq 0$. The desired result is derived from Theorem .3. $\square$

Theorem .4 gives an interesting insight into the relation between modality and γ-orderliness. According to the conventional definition, a distribution with a monotonic pdf is still a unimodal distribution. However, within the interval supported, its mode number is zero. In fact, the number of modes and their magnitudes are closely related to the possibility of the validity of orderliness, even though counterexamples can always be constructed. A proof of γ-orderliness, if $\gamma \geq 1$, can be easily done for the gamma distributions when $\alpha \leq 1$ since the pdf of the gamma distribution is $f(x) = \frac{\lambda^{-\alpha} x^{\alpha-1} e^{-\frac{x}{\lambda}}}{\Gamma(\alpha)}$, $x \geq 0$, $\lambda > 0$, $\alpha > 0$, which is a product of two monotonic decreasing functions under constraints. For $\alpha > 1$, the proof is hard, numerical results show that the orderliness is valid until $\alpha > 140$ (SI Text), but it is instructive to consider that when $\alpha \to \infty$ the gamma distribution converges to a Gaussian distribution with mean $\mu = \alpha\lambda$ and variance $\sigma = \alpha\lambda^2$.

Theorem .5. *If transforming a symmetric unimodal random variable X with a function $\phi(x)$ such that $\frac{d^2\phi}{dx^2} \geq 0 \wedge \frac{d\phi}{dx} \geq 0$ over the interval supported, then the convex transformed distribution is ordered. If the quantile function of X satifies $Q^{(2)}(\epsilon) \leq 0$, the convex transformed distribution is second ordered.*

Proof. Let $\phi\mathrm{SQA}(\epsilon) = \frac{1}{2}(\phi(Q(\epsilon)) + \phi(Q(1-\epsilon)))$, then, $\frac{d\phi\mathrm{SQA}}{d\epsilon} = \frac{1}{2}(\phi'(Q(\epsilon))Q'(\epsilon) - \phi'(Q(1-\epsilon))Q'(1-\epsilon)) = \frac{1}{2}Q'(\epsilon)(\phi'(Q(\epsilon)) - \phi'(Q(1-\epsilon))) \leq 0$, since for a symmetric distribution, $m - Q(\epsilon) = Q(1-\epsilon) - m$, differentiating both sides, $-Q'(\epsilon) = -Q'(1-\epsilon)$, $Q'(\epsilon) \geq 0, \phi^{(2)} \geq 0$. Notably, differentiating twice, $Q^{(2)}(\epsilon) = -Q^{(2)}(1-\epsilon)$, $\frac{d^{(2)}\phi\mathrm{SQA}}{d\epsilon^{(2)}} = \frac{1}{2}\left(\left(\phi^{(2)}(Q(\epsilon)) + \phi^{(2)}(Q(1-\epsilon))\right)(Q'(\epsilon))^2\right) + \frac{1}{2}\left((\phi'(Q(\epsilon)) - \phi'(Q(1-\epsilon)))Q^{(2)}(\epsilon)\right)$. The sign of $\frac{d^{(2)}\phi\mathrm{SQA}}{d\epsilon^{(2)}}$ depends on $Q^{(2)}(\epsilon)$. $\square$

The mean-median-mode inequality for distributions of the powers and roots of the variates of a given distribution was investigated by Henry Rietz in 1927 (40), but the most trivial solution is the exponential transformation since the derivatives are always positive. An application of Theorem .5 is that the lognormal distribution is ordered as it is exponentially transformed from the Gaussian distribution whose $Q^{(2)}(\epsilon) = -2\sqrt{2}\pi\sigma e^{2\mathrm{erfc}^{-1}(2\epsilon)^2}\mathrm{erfc}^{-1}(2\epsilon) \leq 0$ (so, it is also second ordered).

Theorem .5 also reveals a relation between convex transformation and orderliness, since ϕ is the non-decreasing convex function in van Zwet's trailblazing work *Convex transformations of random variables* (39). Consider there is a near-symmetric distribution S such that SQA_ϵ as a function of ϵ is fluctuating from 0 to $\frac{1}{2}$, and $\mu = m$. Based on the definition, S is not ordered. Let s be the pdf of S. Transforming S with $\phi(x)$ will decrease $s(Q_S(\epsilon))$, and the decrease rate, due to the order, is much smaller than $s(Q_S(1-\epsilon))$. That means, as the second derivative of $\phi(x)$ increases, eventually, after a point, $s(Q_S(\epsilon))$ will always be greater than $s(Q_S(1-\epsilon))$ even previously not, i.e., the SQA_ϵ function will be monotonic decreasing and S will be ordered. Accordingly, in a family of distributions that differ by a skewness-increasing transformation in van Zwet's sense, violations of orderliness typically occur only when the distribution is near-symmetric.

Remarkably, Bernard et al. (2020) (2) derived the bias bound of the symmetric quantile average for $\mathcal{P}_U$,

$$B_{\mathrm{SQAB}}(\epsilon) = \begin{cases} \frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1}+\sqrt{\frac{3\epsilon}{4-3\epsilon}}\right) & \frac{1}{6} \geq \epsilon \geq 0 \\ \frac{1}{2}\left(\sqrt{\frac{1-\epsilon}{\epsilon+\frac{1}{3}}}+\sqrt{\frac{3\epsilon}{4-3\epsilon}}\right) & \frac{1}{2} \geq \epsilon > \frac{1}{6}. \end{cases}$$

Theorem .6. *The above bias bound function, $B_{SQAB}(\epsilon)$, is monotonic decreasing over the interval $(0, \frac{1}{2})$.*

Proof. When $\frac{1}{6} \geq \epsilon \geq 0$, $B'_{\mathrm{SQAB}}(\epsilon) = \frac{1}{(4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}}} - \frac{1}{3\sqrt{\frac{4}{\epsilon}-9\epsilon^2}}$. To prove $B'_{\mathrm{SQAB}} < 0$, it is equivalent to proving $(4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}} > 3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2$. Let $L(\epsilon) = (4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}}$, $R(\epsilon) = 3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2$, then $\frac{L(\epsilon)}{\epsilon^2} = \frac{(4-3\epsilon)^2}{\epsilon^2}\sqrt{\frac{\epsilon}{12-9\epsilon}} = \left(\frac{4}{\epsilon}-3\right)^2\sqrt{\frac{1}{\frac{12}{\epsilon}-9}}$, $\frac{R(\epsilon)}{\epsilon^2} = 3\sqrt{\frac{4}{\epsilon}-9}$. Assuming, $\frac{1}{\epsilon} \in \left(\frac{9}{4}, \infty\right)$, $\frac{L(\epsilon)}{\epsilon^2} > \frac{R(\epsilon)}{\epsilon^2} \iff \left(\frac{4}{\epsilon}-3\right)^2\sqrt{\frac{1}{\frac{12}{\epsilon}-9}} > 3\sqrt{\frac{4}{\epsilon}-9} \iff \left(\frac{4}{\epsilon}-3\right)^2 > 3\sqrt{\frac{4}{\epsilon}-9}\sqrt{\frac{12}{\epsilon}-9}$. Let $LmR\left(\frac{1}{\epsilon}\right) = \left(\frac{4}{\epsilon}-3\right)^4 - 9\left(\frac{4}{\epsilon}-9\right)\left(\frac{12}{\epsilon}-9\right)$, $\frac{dLmR(1/\epsilon)}{d(1/\epsilon)} = 32\left(32\left(\frac{1}{\epsilon}\right)^3 - 72\left(\frac{1}{\epsilon}\right)^2 + 27\frac{1}{\epsilon} + 27\right)$, $\frac{d^2LmR(1/\epsilon)}{d^2(1/\epsilon)} = 32\left(96\left(\frac{1}{\epsilon}\right)^2 - 144\left(\frac{1}{\epsilon}\right) + 27\right) > 0$, let $\frac{1}{\epsilon} = \frac{9}{4}$, $\frac{dLmR(1/\epsilon)}{d(1/\epsilon)} > 0$, therefore, $\frac{dLmR(1/\epsilon)}{d(1/\epsilon)} > 0$, for $\frac{1}{\epsilon} \in \left(\frac{9}{4}, \infty\right)$. Also, $LmR\left(\frac{9}{4}\right) > 0$, so, $LmR\left(\frac{1}{\epsilon}\right) > 0$ for $\epsilon \in \left(0, \frac{4}{9}\right)$. The first part is finished.

When $\frac{1}{2} \geq \epsilon > \frac{1}{6}$, $B'_{\mathrm{SQAB}}(\epsilon) = \frac{1}{(4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}}} - \frac{1}{(3\epsilon+1)^2\sqrt{\frac{1-\epsilon}{9\epsilon+3}}}$. To check whether $B'_{\mathrm{SQAB}}(\epsilon) < 0$, first using the two identities $\sqrt{\frac{1}{12-9\epsilon}} = \sqrt{\frac{1}{3(4-3\epsilon)}}$ and $\sqrt{\frac{1}{3+9\epsilon}} = \sqrt{\frac{1}{3(1+3\epsilon)}}$ to simplify the expression, and then the inequality becomes, $(4-3\epsilon)^{\frac{3}{2}}\sqrt{\epsilon} > (3\epsilon+1)^{\frac{3}{2}}\sqrt{1-\epsilon}\sqrt{\frac{1}{3}} \iff (4-3\epsilon)^{\frac{3}{2}}\sqrt{\epsilon} > (3\epsilon+1)^{\frac{3}{2}}\sqrt{1-\epsilon}\sqrt{\frac{1}{3}} \iff 3(4-3\epsilon)^3\epsilon > (3\epsilon+1)^3(1-\epsilon) \iff -54\epsilon^4 + 324\epsilon^3 - 450\epsilon^2 + 184\epsilon - 1 > 0$. Since when $\epsilon < 1$, $-54\epsilon^4 + 54\epsilon^3 > 0$, just consider the condition that $270\epsilon^3 - 450\epsilon^2 + 184\epsilon - 1 > 0 \iff \epsilon\left(270\epsilon^2 - 450\epsilon + 174\right) + 10\epsilon - 1 > 0$. Since $270\epsilon^2 - 450\epsilon + 174 > 0$ is valid for $\epsilon < \frac{1}{30}\left(25 - 3\sqrt{5}\right)$, so just need $10\epsilon - 1 > 0$, $10\epsilon > 1$, $\epsilon > \frac{1}{10}$. So, the inequality is valid for $\frac{1}{30}\left(25 - 3\sqrt{5}\right) \approx 0.610 > \epsilon > \frac{1}{10}$, within the range of $\frac{1}{2} \geq \epsilon > \frac{1}{6}$, therefore, $B'_{\mathrm{SQAB}} < 0$ for $\frac{1}{2} \geq \epsilon > \frac{1}{6}$. The first and second formula, when $\epsilon = \frac{1}{6}$, are all equal to $\frac{1}{2}\left(\sqrt{\frac{5}{3}} + \frac{1}{\sqrt{7}}\right)$. It follows that $B_{\mathrm{SQAB}}(\epsilon)$ is continuous over $\left(0, \frac{1}{2}\right)$. Hence, $B'_{\mathrm{SQAB}}(\epsilon) < 0$ is valid for $0 < \epsilon < \frac{1}{2}$, which leads to the assertion of this theorem. □

The proof is given in the SI Text. This monotonicity indicates that the extent of any violations of the orderliness is bounded for a unimodal distribution, e.g., for a right-skewed unimodal distribution, if $\exists\epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \frac{1}{2}$, $\mathrm{SQA}_{\epsilon_2} \geq \mathrm{SQA}_{\epsilon_3} \geq \mathrm{SQA}_{\epsilon_1}$, $\mathrm{SQA}_{\epsilon_2}$ will not be too far away from $\mathrm{SQA}_{\epsilon_1}$, since $\sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon}(\mathrm{SQA}_{\epsilon_1}) > \sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon}(\mathrm{SQA}_{\epsilon_2}) > \sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon}(\mathrm{SQA}_{\epsilon_3})$.

Inequalities related to weighted averages

The bias bound of the ϵ-symmetric trimmed mean is also monotonic for $\mathcal{P}_U \cap \mathcal{P}_2$ (proven in the SI Text using the formula provided in Bernard et al.'s paper) (2). So far, it appears clear that the average bias of an estimator is closely related to its degree of robustness. For a right-skewed unimodal distribution, often, the mean, ϵ-symmetric trimmed mean, and median occur in reverse alphabetical order, then analogous to the orderliness, $\forall\epsilon_1 \leq \epsilon_2 \leq \frac{1}{2}, \mathrm{TM}_{\epsilon_1} \geq \mathrm{TM}_{\epsilon_2}$. It is called trimming inequality. γ-trimming inequality is the same, just adding the term γ and replacing $\frac{1}{2}$ with $\frac{1}{1+\gamma}$. A necessary and sufficient condition of the γ-trimming inequality is also the monotonic behavior of the bias of the ϵ,γ-trimmed mean as a function of the breakdown point ϵ (proven in the SI Text). Orderliness is a sufficient condition for the trimming inequality (proven in the SI Text), but it is not necessary.

Theorem .7. *For a right-skewed continuous distribution following the γ-trimming inequality, the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ.*

Proof. By deducing $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\epsilon-\delta}^{1-\gamma\epsilon+\delta} Q(u)\,du \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du \Leftrightarrow \int_{\epsilon-\delta}^{1-\gamma\epsilon+\delta} Q(u)\,du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon}\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du \geq 0 \Leftrightarrow \int_{1-\gamma\epsilon}^{1-\gamma\epsilon+\delta} Q(u)\,du + \int_{\epsilon-\delta}^{\epsilon} Q(u)\,du - \frac{2\delta}{1-\epsilon-\gamma\epsilon}\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du \geq 0 \Leftrightarrow \frac{1}{2\delta}\left(\int_{1-\gamma\epsilon}^{1-\gamma\epsilon+\delta} Q(u)\,du + \int_{\epsilon-\delta}^{\epsilon} Q(u)\,du\right) \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du$ and noting that $\lim_{\delta\to 0}\frac{1}{2\delta}\left(\int_{1-\gamma\epsilon}^{1-\gamma\epsilon+\delta} Q(u)\,du + \int_{\epsilon-\delta}^{\epsilon} Q(u)\,du\right) = \frac{Q(\epsilon)+Q(1-\gamma\epsilon)}{2}$, the proof is complete. □

Additionally, the orderliness is a sufficient condition of the Winsorization inequality (proven in the SI Text). A similar result can be obtained in the following theorem.

Theorem .8. *For a right-skewed continuous distribution following the γ-trimming inequality, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, provided that $\gamma \geq 1$.*

Proof. Continue the above deduction, $\frac{Q(\epsilon)+Q(1-\gamma\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du \Leftrightarrow \epsilon\left(Q(\epsilon)+Q(1-\gamma\epsilon)\right) \geq \left(\frac{2\epsilon}{1-\epsilon-\gamma\epsilon}\right)\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du \Leftrightarrow \left(1-\frac{1}{1-\epsilon-\gamma\epsilon}\right)\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du + \epsilon\left(Q(\epsilon)+Q(1-\gamma\epsilon)\right) \geq 0 \Leftrightarrow \int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du + \epsilon\left(Q(\epsilon)+Q(1-\gamma\epsilon)\right) \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du$. Then, if $\gamma \geq 1$, $\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du + \epsilon Q(\epsilon) + \gamma\epsilon Q(1-\gamma\epsilon) \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\epsilon}^{1-\gamma\epsilon} Q(u)\,du$, the proof is complete. $\square$

Theorem .8 unequivocally establishes that, for a distribution that follows the γ-trimming inequality, the Winsorized mean is always less biased than the trimmed mean, as long as $\gamma \geq 1$. For simplicity, just consider the symmetric case, if one wants to construct a robust SWA following the mean-symmetric weighted average-median inequality. According to the definition of the orderliness, for a right-skewed or symmetric ordered distribution, SWA $\geq m$. So, the condition that SWA is between m and μ is $\mu \geq$ SWA. Let $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} \mathrm{SQA}(u)\,du$, $ka = k\epsilon + b$, from the orderliness, $-\frac{d\mathrm{SQA}}{d\epsilon} \geq 0 \Rightarrow \forall a \leq 2a \leq \frac{1}{2}, -\frac{(\mathrm{SQA}(2a)-\mathrm{SQA}(a))}{a} \geq 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$. Let $\mathcal{B}_i = \mathcal{B}_0$, then, based the orderliness, another version of ϵ-symmetric Winsorized mean, block Winsorized mean, is defined here for comparison in the SI Dataset S1 as

$$\mathrm{BWM}_{\epsilon,n} \coloneqq \frac{1}{n}\left(\sum_{i=n\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\epsilon+1}^{2n\epsilon} X_i + \sum_{i=(1-2\epsilon)n+1}^{(1-\epsilon)n} X_i\right),$$

which is double weighting the leftest and rightest blocks having the size ϵn. From the second orderliness $\frac{d^2\mathrm{SQA}}{d^2\epsilon} \geq 0 \Rightarrow \forall a \leq 2a \leq 3a \leq \frac{1}{2}, \frac{1}{a}\left(\frac{(\mathrm{SQA}(3a)-\mathrm{SQA}(2a))}{a} - \frac{(\mathrm{SQA}(2a)-\mathrm{SQA}(a))}{a}\right) \geq 0 \Rightarrow \mathcal{B}_i - 2\mathcal{B}_{i+1} + \mathcal{B}_{i+2} \geq 0$. So, based the second orderliness, SM_ϵ can be seen as replacing the two blocks, $\mathcal{B}_i + \mathcal{B}_{i+2}$ with one block $2\mathcal{B}_{i+1}$. From the νth orderliness, the recurrence relation of the derivatives naturally produces the alternating binomial coefficients,

$$(-1)^{\nu}\frac{d^{\nu}\mathrm{SQA}}{d\epsilon^{\nu}} \geq 0 \Rightarrow \forall a \leq \ldots \leq (\nu+1)a \leq \frac{1}{2},$$

$$\frac{(-1)^{\nu}}{a}\left(\frac{\frac{\mathrm{SQA}(\nu a+a)^{\cdot^{\cdot^{\cdot}}}}{a} - \frac{\cdot^{\cdot^{\cdot}}\mathrm{SQA}(2a)}{a}}{a} - \frac{\frac{\mathrm{SQA}(\nu a)^{\cdot^{\cdot^{\cdot}}}}{a} - \frac{\cdot^{\cdot^{\cdot}}\mathrm{SQA}(a)}{a}}{a}\right)$$

$$\geq 0 \Leftrightarrow \frac{(-1)^{\nu}}{a^{\nu}}\left(\sum_{j=0}^{\nu}(-1)^{j}\binom{\nu}{j}\mathrm{SQA}\left((\nu-j+1)\,a\right)\right) \geq 0$$

$$\Rightarrow \sum_{j=0}^{\nu}(-1)^{j}\binom{\nu}{j}\mathcal{B}_{i+j} \geq 0.$$

Based on the νth orderliness, the ϵ-binomial mean is introduced as

$$\mathrm{BM}_{\nu,\epsilon,n} \coloneqq \frac{1}{n}\left(\sum_{i=1}^{\frac{1}{2}\epsilon^{-1}(\nu+1)^{-1}}\sum_{j=0}^{\nu}\left(1-(-1)^{j}\binom{\nu}{j}\right)\mathfrak{B}_{i_j}\right),$$

where $\mathfrak{B}_{i_j} = \sum_{l=n\epsilon(j+(i-1)(\nu+1))+1}^{n\epsilon(j+(i-1)(\nu+1)+1)}(X_l + X_{n-l+1})$, $\frac{1}{2}\epsilon^{-1}(\nu+1)^{-1} \in \mathbb{N}$. If ν is not indicated, it is default as $\frac{1}{2}\epsilon^{-1} - 1$. The solution for $n \bmod \epsilon^{-1} \neq 0$ is the same as that in the stratified mean. $\mathrm{BM}_{\nu=2,\epsilon}$ is identical to $\mathrm{SM}_{\epsilon,b=3}$, if the two ϵs are the same. The reason that $\mathrm{SM}_{\frac{1}{9}}$ has similar biases as the corresponding $\mathrm{WM}_{\frac{1}{9}}$ is the Winsorized mean is using two single quantiles to replace the trimmed parts, not two blocks. If using single quantiles, based on the second orderliness, the stratified quantile mean can be defined as

$$\mathrm{SQM}_{\epsilon,n} \coloneqq 4\epsilon\sum_{i=1}^{\frac{1}{4\epsilon}}\frac{1}{2}(\hat{Q}_n\left((2i-1)\epsilon\right) + \hat{Q}_n\left(1-(2i-1)\epsilon\right)),$$

where $\frac{1}{\epsilon} \bmod 4 = 0$, $\hat{Q}_n(p)$ is the empirical quantile function. $\mathrm{SQM}_{\frac{1}{4}}$ is the Tukey's midhinge (41). The biases of $\mathrm{SQM}_{\frac{1}{8}}$ are also very close to those of $\mathrm{BM}_{\frac{1}{8}}$ (Figure 1), which is based on the third orderliness.

Hodges–Lehmann inequality and U-orderliness

The H-L estimator is a very unique robust location estimator due to its definition being substantially dissimilar from conventional symmetric weighted averages. Hodges and Lehmann (10) in the landmark paper *Estimates of location based on rank tests* proposed two methods to compute the H-L estimator, Wilcoxon score and median of pairwise means, whose time complexities are $O(nlog(n))$ and $O(n^2)$, respectively. The Wilcoxon score is an estimator based on signed-rank test, or R-statistic (10). However, the median of pairwise means is a generalized L-statistic (classified by Serfling in 1984) (42) and a trimmed U-statistic (classified by Janssen, Serfling, and Veraverbeke in 1987) (43). By using the hl_k kernel pointed by Janssen, Serfling and Veraverbeke in 1987 (43) and weighted average generalized here, it is clear now that the H-L estimator is a weighted H-L mean, the definition of which is provided as follows,

$$\mathrm{WHLM}_{k,\epsilon,\gamma,n} \coloneqq \mathrm{WA}_{\epsilon,\gamma,n}\left(\left(hl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right),$$

where $hl_k = \frac{1}{k}\sum_{i=1}^{k} x_i$, $\mathrm{WA}_{\epsilon,\gamma,n}(Y)$ denotes the ϵ,γ-weighted average with the sequence $\left(hl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}$ as an input. Set the WA as TM_ϵ, the $k=2$ case is a version of the Wilcoxon one-sample statistic investigated by Saleh in 1976 (44), which is named as trimmed H-L mean here (Figure 1). The hl_2 kernel distribution has a probability density function $f_{hl_2}(x) = \int_0^{2x} 2f(t)\,f(2x-t)\,dt$ (a result after the transformation of variables), the support of the original distribution is assumed to be $[0,\infty)$ for simplicity. The expected value is the solution of $\int_0^{\text{H-L}}(f_{hl_2}(s))\,ds = \frac{1}{2}$. Due to the complexity of the equation, analytically proving the validity of the mean-H-L-median inequality for a distribution is hard. As an example, for the exponential distribution, $f_{hl_2}(x) = 4\lambda^{-2}xe^{-2\lambda^{-1}x}$, $E[\text{H-L}] = \frac{-W_{-1}\left(-\frac{1}{2e}\right)-1}{2}\lambda \approx 0.839\lambda$, where W is the Lambert W function.

Analogous to the trimming inequality, the Hodges-Lehmann inequality can be defined as $\forall n \geq k_2 \geq k_1 \geq 1, m\mathrm{HLM}_{k_2} \geq m\mathrm{HLM}_{k_1}$, where $m\mathrm{HLM}_k$ is setting the WA as median. Since $m\mathrm{HLM}_{k=1} = m$, $m\mathrm{HLM}_{k=2} =$H-L, $m\mathrm{HLM}_{k=\infty} = \mu$, if a distribution follows the H-L inequality, it also follows the mean-H-L-median inequality. Furthermore, the Hodges-Lehmann inequality is a special case of the U-orderliness, i.e.,

$$(\forall n \geq k_2 \geq k_1 \geq 1, \mathrm{SQHLM}_{k_2,\epsilon} \geq \mathrm{SQHLM}_{k_1,\epsilon}) \wedge$$
$$(\forall n \geq k_2 \geq k_1 \geq 1, \mathrm{SQHLM}_{k_2,\epsilon} \leq \mathrm{SQHLM}_{k_1,\epsilon}),$$

where SQHLM_k is setting the WA as SQA. The direction of the inequality depends on SQA_ϵ and μ, since $\mathrm{SQHLM}_{k=1,\epsilon} = \mathrm{SQA}_\epsilon$. γ-U-orderliness is just replacing SQA with QA.

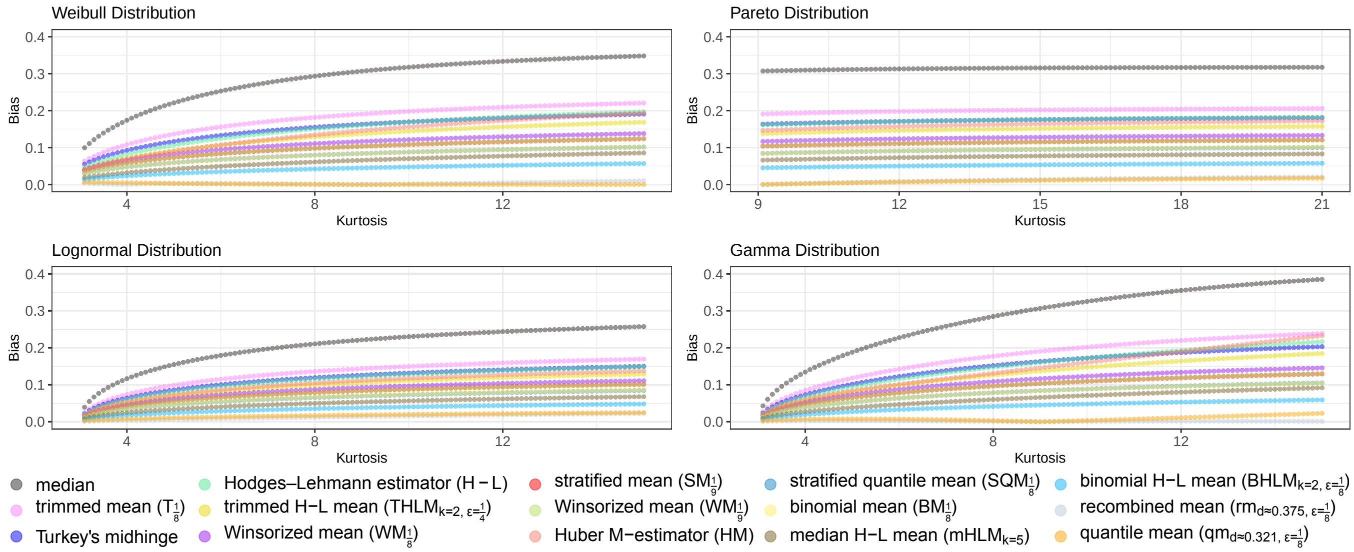

Fig. 1. Standardized asymptotic biases (with respect to μ) of fifteen robust location estimators (including three that detailed in another relevant paper for better comparison) in four two-parameter skewed unimodal distributions as a function of the kurtosis. The methods were described in the SI Text.

Theorem .9. *U-orderliness implies orderliness.*

Proof. Suppose $n \bmod 2 = 0$, $n \to \infty$, $\frac{1}{2}(X_1 + X_n) \geq \ldots \geq \frac{1}{2}(X_i + X_{n-i+1}) \geq \ldots \geq \frac{1}{2}\left(X_{\frac{n}{2}} + X_{\frac{n}{2}+1}\right)$ is valid for a sample from an ordered distribution. Let $\tilde{\epsilon} = \frac{i}{n}$, when $\tilde{\epsilon} \to 0$, $\mathrm{SQHLM}_{k_2,\tilde{\epsilon}} \leq \mathrm{SQHLM}_{k_1,\tilde{\epsilon}}$ is equivalent to the orderliness, since $\mathrm{SQHLM}_{k=j,\tilde{\epsilon}\to 0,n} = \frac{1}{2j}\left(\sum_{i=1}^{j}(X_i + X_{n-i+1})\right)$ and Theorem .7 implies that $\mu \leq \mathrm{SQA}_{\epsilon}$. □

Be aware that the U-orderliness itself does not assume any orderliness within the hl_k kernel distribution. The $hl_{k=n-1}$ kernel distribution has n elements, and their order is the same as the original distribution, so it is ordered if and only if the original distribution is ordered. If assuming symmetry, the result is trivial since the k-fold convolutions of a symmetric distribution is also symmetric (proved by Laha in 1961)(45). Proving other cases is challenging. For example, $f'_{hl_2}(x) = 4f(2x)f(0) + \int_0^{2x} 4f(t)f'(2x-t)\,dt$, the strict negative of $f'_{hl_2}(x)$ is not guaranteed if just assuming $f'(x) < 0$, so, even the original distribution is monotonic, the hl_2 kernel distribution might be non-monotonic. Also, unlike the pairwise difference distribution, if the original distribution is unimodal, the pairwise mean distribution might be non-unimodal, as a counterexample was given by Chung in 1953 and mentioned by Hodges and Lehmann in 1954 (46, 47). If all hl_k kernel distributions, $n \geq k \geq 1$, is νth ordered, the distribution is νth U-ordered. From that, binomial H-L mean (set the WA as BM) is the bias-optimum choice in the semiparametric framework (Figure 1), although its maximum breakdown point is ≈ 0.065. A comparison of the biases of $m\mathrm{HLM}_{k=5}$ and $\mathrm{BM}_{\frac{1}{8}}$ is appropriate (Figure 1), given their comparable breakdown points, with $m\mathrm{HLM}_{k=5}$ exhibiting smaller biases. This result, combined with Theorem .9, aligns with Devroye et al. (2016)'s seminal work that MoM is nearly optimal with regards to concentration bounds (20), since when k is much smaller than n, the difference between sampling with replacement and without replacement is negligible, $m\mathrm{HLM}_{k,n}$ is asymptotically equivalent to $\mathrm{MoM}_{k,b=\frac{n}{k}}$ if assuming $k \ll n$, therefore, it is also based on U-orderliness.

In 1958, Richtmyer introduced the concept of quasi-Monte Carlo simulation that utilizes low-discrepancy sequences, resulting in a significant reduction in computational expenses for large sample simulation (48). Among various numerical sets, Sobol sequences are often favored in quasi-Monte Carlo methods (49). Building upon this principle, In 1991, Do and Hall extended it to bootstrap and found that the quasi-random approach resulted in lower variance compared to other bootstrap Monte Carlo procedures, indicating its superiority (50). If with a deterministic approach, the variance of $m\mathrm{HLM}_{k,n}$ is much lower than that of $\mathrm{MoM}_{k,b=\frac{n}{k}}$ (SI Dataset S1). The median Hodges-Lehmann mean is better than the median of means, not only because it can provide an exact estimate when the sample size is moderate, but also because it allows the use of quasi-bootstrap and the bootstrap size can be set to an arbitrary value in usage.

Data Availability. Data for Figure 1 are given in SI Dataset S1. All codes have been deposited in GitHub.

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