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Supporting Information for

Semiparametric robust mean estimation based on the orderliness of quantile averages

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Supporting text
Figs. S1 to S2
Legend for Dataset S1
SI References

Other supporting materials for this manuscript include the following:

Dataset S1

Supporting Information Text

Methods

The asymptotic biases of all robust location estimators proposed in this article for the Weibull, gamma, Pareto, and lognormal distributions were computed within the specified range of kurtosis and standardized by the standard deviations of the respective distributions. For the Weibull, gamma, and lognormal distributions, the kurtosis range is from 3 to 15 (there are two shape parameter solutions for the Weibull distribution, the lower one is used here). For the Pareto distribution, the range is from 9 to 21.

The asymptotic results were approximated by a quasi-Monte Carlo study (1, 2) based on generating a large quasi-random sample with sample size 1.8 million for corresponding distributions and quasi-subsampling the sample $1.8k$ million times (3) to approximate the distributions of the kernels of the weighted Hodges-Lehmann mean. The accuracies were checked by comparing the results to the asymptotic values for the exponential distribution (errors), showing that the errors were all smaller than 0.001σ.

Orderliness and weighted average inequality

Unlike the mean-median-mode inequality, whose necessary and sufficient condition is often hard to compute. The following result highlights another advantage of trimming inequality.

Theorem 0.1. *A necessary and sufficient condition of the γ-trimming inequality is the monotonic behavior of the bias of trimmed mean as a function of the breakdown point ϵ.*

Proof. Just considering the right-skewed distributions without loss of generality, from the definition of γ-trimming inequality, since $\forall\epsilon_1 \leq \epsilon_2 \leq \frac{1}{2}, \mathrm{TM}_{\epsilon_1,\gamma} \geq \mathrm{TM}_{\epsilon_2,\gamma}$, therefore

$$\frac{\mathrm{TM}_{\epsilon_1,\gamma} - \mu}{\sigma} \geq \frac{\mathrm{TM}_{\epsilon_2,\gamma} - \mu}{\sigma} \iff B_{\mathrm{TM}_{\epsilon_1,\gamma}} \geq B_{\mathrm{TM}_{\epsilon_2,\gamma}} \iff B_{\mathrm{TM}}\left(\epsilon_1, \gamma\right) \geq B_{\mathrm{TM}}\left(\epsilon_2, \gamma\right),$$

which implies the monotonicity of $B_{\mathrm{TM}}(\epsilon, \gamma)$ with respect to ϵ. □

The bias function is free of scale parameter, so the derivatives are much easier to compute. A useful sufficient condition is the monotonicity of symmetric quantile average with respect to ϵ.

Theorem 0.2. *A sufficient condition of the trimming inequality for a right-skewed distribution is the monotonicity of the symmetric quantile average with respect to the breakdown point ϵ.*

Proof. The trimming inequality is equivalent to, $\forall 0 < \epsilon < \frac{1}{2}$, $\frac{1}{1-2\epsilon+2\delta}\int_{\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right)du \geq \frac{1}{1-2\epsilon}\int_{\epsilon}^{1-\epsilon} Q\left(u\right)du$, where δ is an infinitesimal quantity.

Then, by deducing $\frac{1}{1-2\epsilon+2\delta}\int_{\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right)du = \frac{1}{\frac{1}{2}-\epsilon+\delta}\int_{\epsilon-\delta}^{\frac{1}{2}} \mathrm{SQA}\left(u\right)du = \lim_{n\to\infty}\left(\frac{1}{\left(\frac{1}{2}n-n\epsilon+1\right)}\sum_{i=n\epsilon-1}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2}\right)$
$> \lim_{n\to\infty}\left(\frac{1}{\left(\frac{1}{2}n-n\epsilon+1\right)}\left(\left(\sum_{i=n\epsilon}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2}\right)+\frac{X_{n\epsilon}+X_{n-n\epsilon}}{2}\right)\right)$
$= \lim_{n\to\infty}\left(\frac{1}{\left(\frac{1}{2}n-n\epsilon\right)}\left(\frac{\left(\frac{1}{2}n-n\epsilon\right)}{\left(\frac{1}{2}n-n\epsilon+1\right)}\left(\sum_{i=n\epsilon+1}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2}\right)+\frac{2\left(\frac{1}{2}n-n\epsilon\right)}{\left(\frac{1}{2}n-n\epsilon+1\right)}\frac{X_{n\epsilon}+X_{n-n\epsilon}}{2}\right)\right)$
$= \lim_{n\to\infty}\left(\frac{1}{\left(\frac{1}{2}n-n\epsilon\right)}\left(\frac{\left(\frac{1}{2}n-n\epsilon\right)}{\left(\frac{1}{2}n-n\epsilon+1\right)}\left(\sum_{i=n\epsilon+1}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2}\right)+\frac{X_{n\epsilon}+X_{n-n\epsilon}}{2}+\frac{\left(\frac{1}{2}n-n\epsilon-1\right)}{\left(\frac{1}{2}n-n\epsilon+1\right)}\frac{X_{n\epsilon}+X_{n-n\epsilon}}{2}\right)\right)$
$> \lim_{n\to\infty}\left(\frac{1}{\frac{1}{2}n-n\epsilon}\left(\frac{\frac{1}{2}n-n\epsilon}{\frac{1}{2}n-n\epsilon+1}\sum_{i=n\epsilon+1}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2}+\frac{X_{n\epsilon}+X_{n-n\epsilon}}{2}+\frac{1}{\frac{1}{2}n-n\epsilon+1}\left(\frac{X_{n\epsilon+1}+X_{n-n\epsilon-1}}{2}+\cdots+\frac{X_{\frac{1}{2}n}+X_{\frac{1}{2}n}}{2}\right)\right)\right)$
$= \lim_{n\to\infty}\left(\frac{1}{\left(\frac{1}{2}n-n\epsilon\right)}\left(\left(\sum_{i=n\epsilon}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2}\right)\right)\right) = \frac{1}{1-2\epsilon}\int_{\epsilon}^{1-\epsilon} Q\left(u\right)du$, the proof is complete. □

Theorem 0.3. *A necessary and sufficient condition of the γ-orderliness is the monotonic behavior of the bias of the quantile average as a function of the breakdown point ϵ.*

Proof. The proof is analogous to Theorem 0.1, just replacing the TM as QA. □

Orderliness is more fundamental than trimming inequality; for example, orderliness also implies Winsorization inequality.

Definition 0.1 (Winsorization inequality). A distribution follows the Winsorization inequality if and only if $\forall\epsilon_1 \leq \epsilon_2 \leq \frac{1}{2}, \mathrm{WM}_{\epsilon_1} \geq \mathrm{WM}_{\epsilon_2}$, or $\forall\epsilon_1 \leq \epsilon_2 \leq \frac{1}{2}, \mathrm{WM}_{\epsilon_1} \leq \mathrm{WM}_{\epsilon_2}$.

Theorem 0.4. *A sufficient condition of the Winsorization inequality for a right-skewed distribution is the monotonicity of the symmetric quantile average function with respect to the breakdown point ϵ.*

Proof. The Winsorization inequality in Definition 0.1 is equivalent to,

$\forall 0 < \epsilon < \frac{1}{2}$, $\int_{\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du + (\epsilon-\delta)\left(Q(\epsilon-\delta)+Q(1-\epsilon+\delta)\right) \geq \int_{\epsilon}^{1-\epsilon} Q(u)\,du + \epsilon\left(Q(\epsilon)+Q(1-\epsilon)\right)$, where δ is an infinitesimal quantity.

Then, by deducing $\int_{\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du + (\epsilon-\delta)\left(Q(\epsilon-\delta)+Q(1-\epsilon+\delta)\right) = \int_{\epsilon-\delta}^{\frac{1}{2}} SQA(u)\,du + 2(\epsilon-\delta)\,SQA(\epsilon-\delta)$

$= \lim_{n\to\infty}\left(\frac{2}{n}\left(\sum_{i=n\epsilon-1}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2} + (n\epsilon-1)\left(\frac{X_{n\epsilon-1}+X_{n-n\epsilon+1}}{2}\right)\right)\right)$

$> \lim_{n\to\infty}\left(\frac{2}{n}\left(\sum_{i=n\epsilon}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2} + \frac{X_{n\epsilon}+X_{n-n\epsilon}}{2} + (n\epsilon-1)\left(\frac{X_{n\epsilon-1}+X_{n-(n\epsilon-1)}}{2}\right)\right)\right)$

$= \lim_{n\to\infty}\left(\frac{2}{n}\left(\sum_{i=n\epsilon}^{\frac{1}{2}n}\frac{X_i+X_{n-i}}{2} + (n\epsilon)\left(\frac{X_{n\epsilon}+X_{n-n\epsilon}}{2}\right)\right)\right)$

$= \int_{\epsilon}^{1-\epsilon} Q(u)\,du + (\epsilon)\left(Q(\epsilon)+Q(1-\epsilon)\right)$, the proof is complete. □

Theorem 0.5. *A necessary and sufficient condition of the second orderliness is the monotonicity and convexity of the bias function of the symmetric quantile average with respect to the breakdown point ϵ.*

Proof. The proof is analogous to Theorem 0.1. □

Then, the orderliness for parametric distributions will be discussed. For simplicity, $0 < \epsilon < \frac{1}{2}$ is assumed in the following proofs unless otherwise specified.

Theorem 0.6. *The Weibull distribution is ordered if the shape parameter $\alpha \leqslant \frac{1}{1-\ln(2)} \approx 3.259$.*

Proof. The pdf of the Weibull distribution is $f(x) = \frac{\alpha e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\left(\frac{x}{\lambda}\right)^{\alpha-1}}{\lambda}$, $x \geq 0$, the quantile function is $F^{-1}(p) = \lambda(-\ln(1-p))^{1/\alpha}$, $1 \geq p \geq 0$, $\alpha > 0, \lambda > 0$. Then, the standardized bias of a symmetric quantile average with a breakdown point ϵ, is

$$B_{\mathrm{SQA}}(\epsilon,\alpha) = \frac{\mathrm{SQA}_{\epsilon}-\mu}{\sigma} = \frac{\frac{1}{2}\left(\lambda(-\ln(1-\epsilon))^{1/\alpha}+\lambda(-\ln(\epsilon))^{1/\alpha}\right)-\lambda\Gamma\left(1+\frac{1}{\alpha}\right)}{\sqrt{\lambda^2\left(\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2\right)}}.$$

$\frac{\partial B_{\mathrm{SQA}}}{\partial\epsilon} = \frac{\frac{(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1}}{\alpha-\alpha\epsilon}+\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon\ln(\epsilon)}}{2\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$. Let $g(\epsilon,\alpha) = \frac{(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1}}{\alpha-\alpha\epsilon}+\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon\ln(\epsilon)} = -\left(-\ln(1-\epsilon)\right)^{\frac{1}{\alpha}}\left((1-\epsilon)\left(\ln(1-\epsilon)\right)\right)^{-1} + \left(-\ln(\epsilon)\right)^{\frac{1}{\alpha}}\left(\epsilon\ln(\epsilon)\right)^{-1}$. Arranging the equation $g(\epsilon,\alpha)=0$, it can be shown that $\frac{\epsilon}{(1-\epsilon)} = \left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)^{\frac{1}{\alpha}-1}$. Let $L(\epsilon) = \frac{\epsilon}{(1-\epsilon)}$, $R(\epsilon,\alpha) = \left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)^{\frac{1}{\alpha}-1}$, $LmR(\epsilon,\alpha) = L(\epsilon)-R(\epsilon,\alpha)$, then $\frac{\partial LmR}{\partial\alpha} = \frac{\ln\left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)\left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)^{\frac{1}{\alpha}-1}}{\alpha^2}$. For $0<\epsilon<\frac{1}{2}$, $\frac{\partial LmR}{\partial\alpha} > 0$, so $LmR(\epsilon,\alpha)$ is monotonic with respect to α. When $\alpha = \frac{1}{1-\ln(2)}$, $g(\epsilon) = -\frac{1}{\epsilon(-\ln(\epsilon))^{\ln(2)}} + \frac{1}{(1-\epsilon)(-\ln(1-\epsilon))^{\ln(2)}}$ Let $h(\epsilon) = \epsilon\left(-\ln(\epsilon)\right)^{\ln(2)}$, $h'(\epsilon) = \frac{(-\ln(\epsilon))^{\ln(2)}\ln(2\epsilon)}{\ln(\epsilon)}$, for $0 < \epsilon < e^{-ln(2)} = \frac{1}{2}$, $h'(\epsilon) > 0$. As a result, $h(\epsilon)$ is monotonic increasing, $-h(1-\epsilon)$ is monotonic increasing, $h(\epsilon)-h(1-\epsilon)$ is also monotonic increasing. So, if $0<\epsilon<\frac{1}{2}$, $h(\epsilon)-h(1-\epsilon) < h\left(\frac{1}{2}\right)-h\left(1-\frac{1}{2}\right) = 0$, $g(\epsilon,\alpha) < 0$. So, $\frac{\partial B_{\mathrm{SQA}}}{\partial\epsilon} < 0$, $B_{\mathrm{SQA}}(\epsilon,\alpha)$ is monotonic decreasing in ϵ when $\alpha \leqslant \frac{1}{1-\ln(2)}$. The assertion follows from Theorem 0.3. □

Remark. The Weibull distribution can be symmetric. Its skewness is $\tilde{\mu}_3 = \frac{2\Gamma\left(1+\frac{1}{\alpha}\right)^3-3\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\Gamma\left(1+\frac{3}{\alpha}\right)}{\left(\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2\right)^{3/2}}$. Denote the solution of $\tilde{\mu}_3 = 0$ as $\alpha_0 \approx 3.602$. The above proof implies that when α is close to α_0, the bias function of SQA is no longer monotonic.

Then, the bias function of the trimmed mean for the Weibull distribution is

$$B_{\mathrm{TM}}(\epsilon,\alpha) = \frac{\frac{\Gamma\left(1+\frac{1}{\alpha},-\ln(1-\epsilon)\right)-\Gamma\left(1+\frac{1}{\alpha},-\ln(\epsilon)\right)}{1-2\epsilon}-\Gamma\left(1+\frac{1}{\alpha}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}},$$

Numerical solutions and the plot in Figure S1 indicate that, when $\alpha = 3.259, \epsilon = 0.1$, $\frac{\partial B_{\mathrm{TM}}}{\partial\epsilon} \approx -0.164$, when $\epsilon = 0.4$, $\frac{\partial B_{\mathrm{TM}}}{\partial\epsilon} \approx -0.001$, when $\epsilon = 0.499$, $\frac{\partial B_{\mathrm{TM}}}{\partial\epsilon} \approx 7.66\times10^{-8}$. So, the bias function of the trimmed mean can also be non-monotonic when $\alpha \geq \frac{1}{1-\ln(2)}$.

Additionally, the second derivative of the bias function of SQA for the Weibull distribution is,

$\frac{\partial^2 B_{\mathrm{SQA}}}{\partial^2\epsilon} = \frac{-\frac{(-\ln(\epsilon))^{\frac{1}{\alpha}-1}}{\alpha^2\epsilon^2\ln(\epsilon)}-\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon^2\ln^2(\epsilon)}-\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon^2\ln(\epsilon)}+\frac{\left(\frac{1}{\alpha}-1\right)(-\ln(1-\epsilon))^{\frac{1}{\alpha}-2}}{(1-\epsilon)(\alpha-\alpha\epsilon)}+\frac{\alpha(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1}}{(\alpha-\alpha\epsilon)^2}}{2\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$. The numerical solutions show that when $\alpha < 3$, $\frac{\partial^2 B_{\mathrm{SQA}}}{\partial^2\epsilon} > 0$, if $0 < \epsilon < \frac{1}{2}$. The flip of the signs also occurs when α is close to $\frac{1}{1-\ln(2)} \approx 3.259$. When

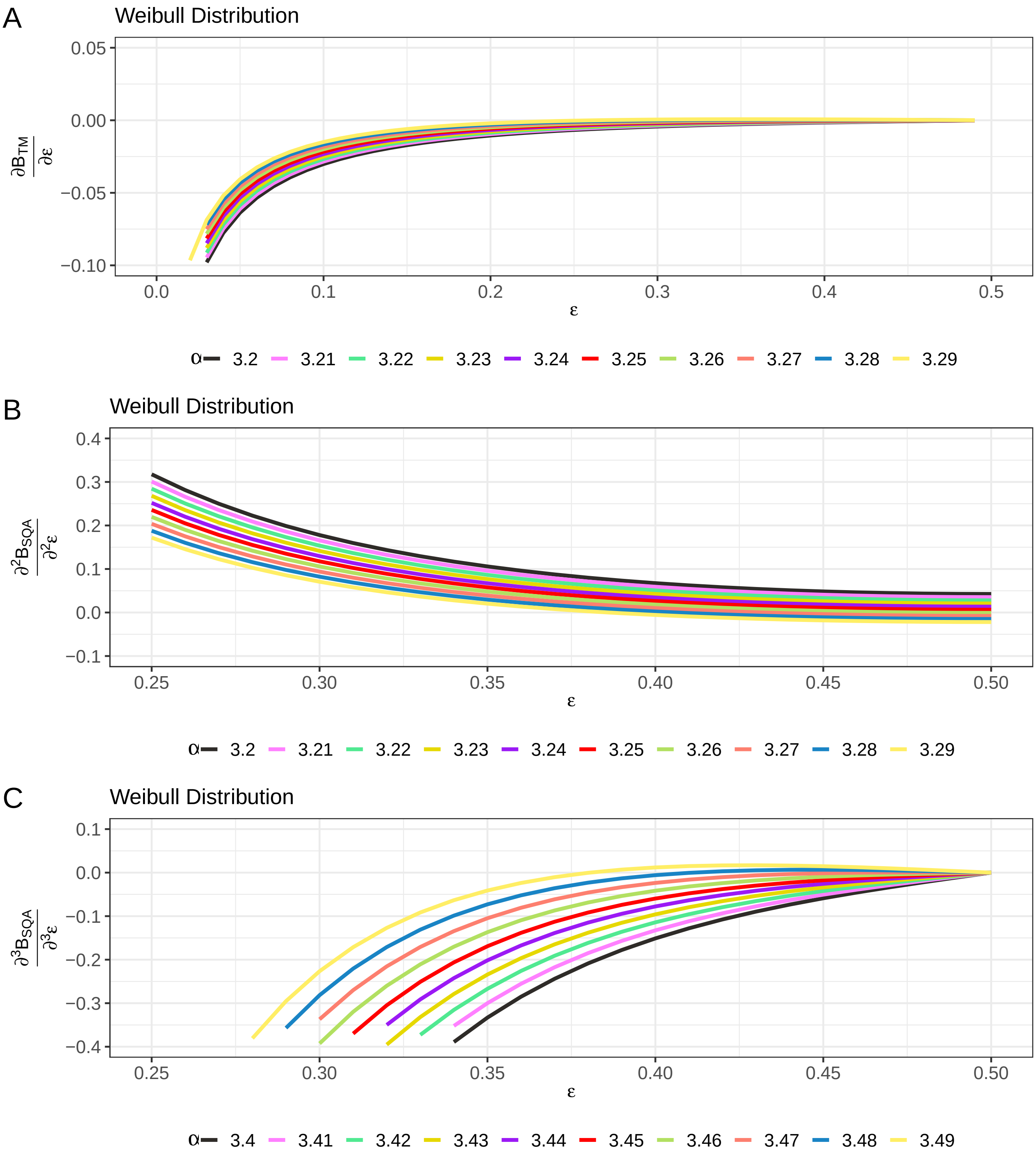

Fig. S1. A. The first derivative of the bias function of TM for the Weibull distribution as a function of the breakdown point ϵ. B. The second derivative of the bias function of SQA for the Weibull distribution as a function of the breakdown point ϵ. C. The third derivative of the bias function of SQA for the Weibull distribution as a function of the breakdown point ϵ.

$\alpha = \frac{1}{1-\ln(2)}, \epsilon = 0.1$, $\frac{\partial^2 B_{\text{SQA}}}{\partial^2 \epsilon} \approx 3.259$, when $\epsilon = 0.4$, $\frac{\partial^2 B_{\text{SQA}}}{\partial^2 \epsilon} \approx 0.020$, when $\epsilon = 0.5$, $\frac{\partial^2 B_{\text{SQA}}}{\partial^2 \epsilon} \approx -7.34 \times 10^{-16}$. A plot of $\frac{\partial^2 B_{\text{SQA}}}{\partial^2 \epsilon}$ for $0.25 < \epsilon < 0.5$ is given in Figure S1.

Then, the third derivative of the SQA for the Weibull distribution is,

$\frac{\partial^3 B_{\text{SQA}}}{\partial^3 \epsilon} = -\frac{(\alpha-1)(2\alpha-1)(\epsilon-1)^3(-\ln(\epsilon))^{\frac{1}{\alpha}-3} - 3(\alpha-1)\alpha(\epsilon-1)^3(-\ln(\epsilon))^{\frac{1}{\alpha}-2}}{2\alpha^3(\epsilon-1)^3\epsilon^3\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$

$-\frac{2\alpha^2\epsilon^3(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1} + 2\alpha^2(\epsilon-1)^3(-\ln(\epsilon))^{\frac{1}{\alpha}-1} + (1-\alpha)(1-2\alpha)\epsilon^3(-\ln(1-\epsilon))^{\frac{1}{\alpha}-3} + 3(1-\alpha)\alpha\epsilon^3(-\ln(1-\epsilon))^{\frac{1}{\alpha}-2}}{2\alpha^3(\epsilon-1)^3\epsilon^3\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$. The numerical solutions show that when $\alpha < 3$, $\frac{\partial^3 B_{\text{SQA}}}{\partial^3 \epsilon} < 0$, if $0 < \epsilon < \frac{1}{2}$. The flip of the signs occurs when $\alpha = 3.471$. When $\epsilon = 0.4$, $\frac{\partial^3 B_{\text{SQA}}}{\partial^3 \epsilon} \approx -0.0218$, when $\epsilon = 0.499$, $\frac{\partial^3 B_{\text{SQA}}}{\partial^3 \epsilon} \approx 4.928 \times 10^{-5}$. A plot of $\frac{\partial^3 B_{\text{SQA}}}{\partial^3 \epsilon}$ for $0.3 < \epsilon < 0.5$ is given in Figure S1.

Because the kurtosis range used here for the Weibull distribution starts from 3.1, the corresponding α is 2.133, so among the kurtosis range discussed here, the numerical results show that the Weibull distribution follows the first three orderlinesses.

The pdf of the gamma distribution is $f(x) = \frac{\lambda^{-\alpha}x^{\alpha-1}e^{-\frac{x}{\lambda}}}{\Gamma(\alpha)}$, $x \geq 0$, the quantile function is $Q(p) = \lambda P^{-1}(\alpha, p)$, $1 \geq p \geq 0$, $\alpha > 0, \lambda > 0$, P is the regularized incomplete gamma function. So, $E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = \alpha\lambda$. Similarly, the variance is $\alpha\lambda^2$. Then, the standardized bias of a symmetric quantile average with a breakdown point ϵ, is

$$B_{\text{SQA}}(\epsilon, \alpha) = \frac{\text{SQA}_\epsilon - \mu}{\sigma} = \frac{\frac{1}{2}(\lambda P^{-1}(\alpha, 1-\epsilon) + \lambda P^{-1}(\alpha, \epsilon)) - \alpha\lambda}{\sqrt{\alpha\lambda^2}}.$$

$\frac{\partial B_{\text{SQA}}}{\partial \epsilon} = \frac{\Gamma(a)}{2\sqrt{\alpha}}(e^{P^{-1}(\alpha,\epsilon)}P^{-1}(\alpha,\epsilon)^{1-\alpha} - e^{P^{-1}(\alpha,1-\epsilon)}P^{-1}(\alpha,1-\epsilon)^{1-\alpha})$. It is trivial to show that when $\alpha \leq 1$, $P^{-1}(\alpha,\epsilon)$ is monotonic increasing in ϵ, if $0 < \epsilon < \frac{1}{2}$. Then $\frac{\partial B_{\text{SQA}}}{\partial \epsilon} < 0$, $B_{\text{SQA}}(\epsilon, \alpha)$ is monotonic decreasing in ϵ over the interval $(0, \frac{1}{2})$. However, the analytical analysis of $\alpha > 1$ is hard. Numerical results shows that the flip of signs of $\frac{\partial B_{\text{SQA}}}{\partial \epsilon}$ occurs when $\alpha \approx 139.5$ (Figure S2). The second derivative of the bias function for the gamma distribution is $\frac{\partial^2 B_{\text{SQA}}}{\partial^2 \epsilon} = \frac{\Gamma(\alpha)^2}{2\sqrt{\alpha}}((1-\alpha)e^{2P^{-1}(\alpha,1-\epsilon)}P^{-1}(\alpha,1-\epsilon)^{1-2\alpha} + e^{2P^{-1}(\alpha,1-\epsilon)}P^{-1}(\alpha,1-\epsilon)^{2-2\alpha} + (1-\alpha)e^{2P^{-1}(\alpha,\epsilon)}P^{-1}(\alpha,\epsilon)^{1-2\alpha} + e^{2P^{-1}(\alpha,\epsilon)}P^{-1}(\alpha,\epsilon)^{2-2\alpha})$. The flip of signs of $\frac{\partial^2 B_{\text{SQA}}}{\partial^2 \epsilon}$ occurs when $\alpha \approx 78$ (Figure S2). The third derivative is much more cumbersome; numerical results show that the flip of signs of $\frac{\partial^3 B_{\text{SQA}}}{\partial^3 \epsilon}$ occurs when $\alpha \approx 55$ (Figure S2). Since the kurtosis range of the gamma distribution starts from 3.1, the corresponding α is 60. The second point is $\alpha = 30$. Besides the first point, the numerical results show that the gamma distribution follows the first three orderlinesses within the kurtosis setting here.

For the lognormal distribution, the pdf of it is $f(x) = \frac{e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x}$, $x \geq 0$, the quantile function is $Q(p) = e^{\mu - \sqrt{2}\sigma\text{erfc}^{-1}(2p)}$, $1 \geq p \geq 0$, $\sigma > 0, \lambda > 0$. So, $E[X] = \int_{-\infty}^{\infty} xf(x)\,dx = e^{\mu + \frac{\sigma^2}{2}}$. Similarly, the variance is $\left(e^{\sigma^2} - 1\right)e^{2\mu+\sigma^2}$. Then, the standardized bias of a symmetric quantile average with a breakdown point ϵ, is

$$B_{\text{SQA}}(\epsilon, \sigma) = \frac{\frac{1}{2}\left(e^{\mu-\sqrt{2}\sigma\text{erfc}^{-1}(2\epsilon)} + e^{\mu-\sqrt{2}\sigma\text{erfc}^{-1}(2(1-\epsilon))}\right) - e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}}}.$$

The first two orderlinesses for the lognormal distribution were already discussed in the Main Text, the numerical results show that the third orderliness is also valid within the kurtosis setting here.

Bias bound

As stated in the Main Text, Bernard et al. (2020) (4) derived the bias bound of the symmetric quantile average for $\mathcal{P}_U$,

$$B_{\text{SQAB}}(\epsilon) = \begin{cases} \frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{3\epsilon}{4-3\epsilon}}\right) & \frac{1}{6} \geq \epsilon \geq 0 \\ \frac{1}{2}\left(\sqrt{\frac{1-\epsilon}{\epsilon+\frac{1}{3}}} + \sqrt{\frac{3\epsilon}{4-3\epsilon}}\right) & \frac{1}{2} \geq \epsilon > \frac{1}{6}. \end{cases}$$

Theorem 0.7. *The above bias bound function, $B_{SQAB}(\epsilon)$, is monotonic decreasing over the interval $(0, \frac{1}{2})$.*

Proof. When $\frac{1}{6} \geq \epsilon \geq 0$, $B'_{\text{SQAB}}(\epsilon) = \frac{1}{(4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}}} - \frac{1}{3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2}$. To prove $B'_{\text{SQAB}} < 0$, it is equivalent to proving $(4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}} > 3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2$. Let $L(\epsilon) = (4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}}$, $R(\epsilon) = 3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2$, then $\frac{L(\epsilon)}{\epsilon^2} = \frac{(4-3\epsilon)^2}{\epsilon^2}\sqrt{\frac{\epsilon}{12-9\epsilon}} = \left(\frac{4}{\epsilon}-3\right)^2\sqrt{\frac{1}{\frac{12}{\epsilon}-9}}$, $\frac{R(\epsilon)}{\epsilon^2} = 3\sqrt{\frac{4}{\epsilon}-9}$. Assuming, $\frac{1}{\epsilon} \in \left(\frac{9}{4}, \infty\right)$, $\frac{L(\epsilon)}{\epsilon^2} > \frac{R(\epsilon)}{\epsilon^2} \iff \left(\frac{4}{\epsilon}-3\right)^2\sqrt{\frac{1}{\frac{12}{\epsilon}-9}} > 3\sqrt{\frac{4}{\epsilon}-9} \iff \left(\frac{4}{\epsilon}-3\right)^2 > 3\sqrt{\frac{4}{\epsilon}-9}\sqrt{\frac{12}{\epsilon}-9}$. Let $LmR\left(\frac{1}{\epsilon}\right) = \left(\frac{4}{\epsilon}-3\right)^4 - 9\left(\frac{4}{\epsilon}-9\right)\left(\frac{12}{\epsilon}-9\right)$,

$\frac{dLmR(1/\epsilon)}{d(1/\epsilon)} = 32\left(32\left(\frac{1}{\epsilon}\right)^3 - 72\left(\frac{1}{\epsilon}\right)^2 + 27\frac{1}{\epsilon} + 27\right)$, $\frac{d^2LmR(1/\epsilon)}{d^2(1/\epsilon)} = 32\left(96\left(\frac{1}{\epsilon}\right)^2 - 144\left(\frac{1}{\epsilon}\right) + 27\right) > 0$, let $\frac{1}{\epsilon} = \frac{9}{4}$, $\frac{dLmR(1/\epsilon)}{d(1/\epsilon)} > 0$, therefore, $\frac{dLmR(1/\epsilon)}{d(1/\epsilon)} > 0$, for $\frac{1}{\epsilon} \in \left(\frac{9}{4}, \infty\right)$. Also, $LmR\left(\frac{9}{4}\right) > 0$, so, $LmR\left(\frac{1}{\epsilon}\right) > 0$ for $\epsilon \in \left(0, \frac{4}{9}\right)$. The first part is finished.

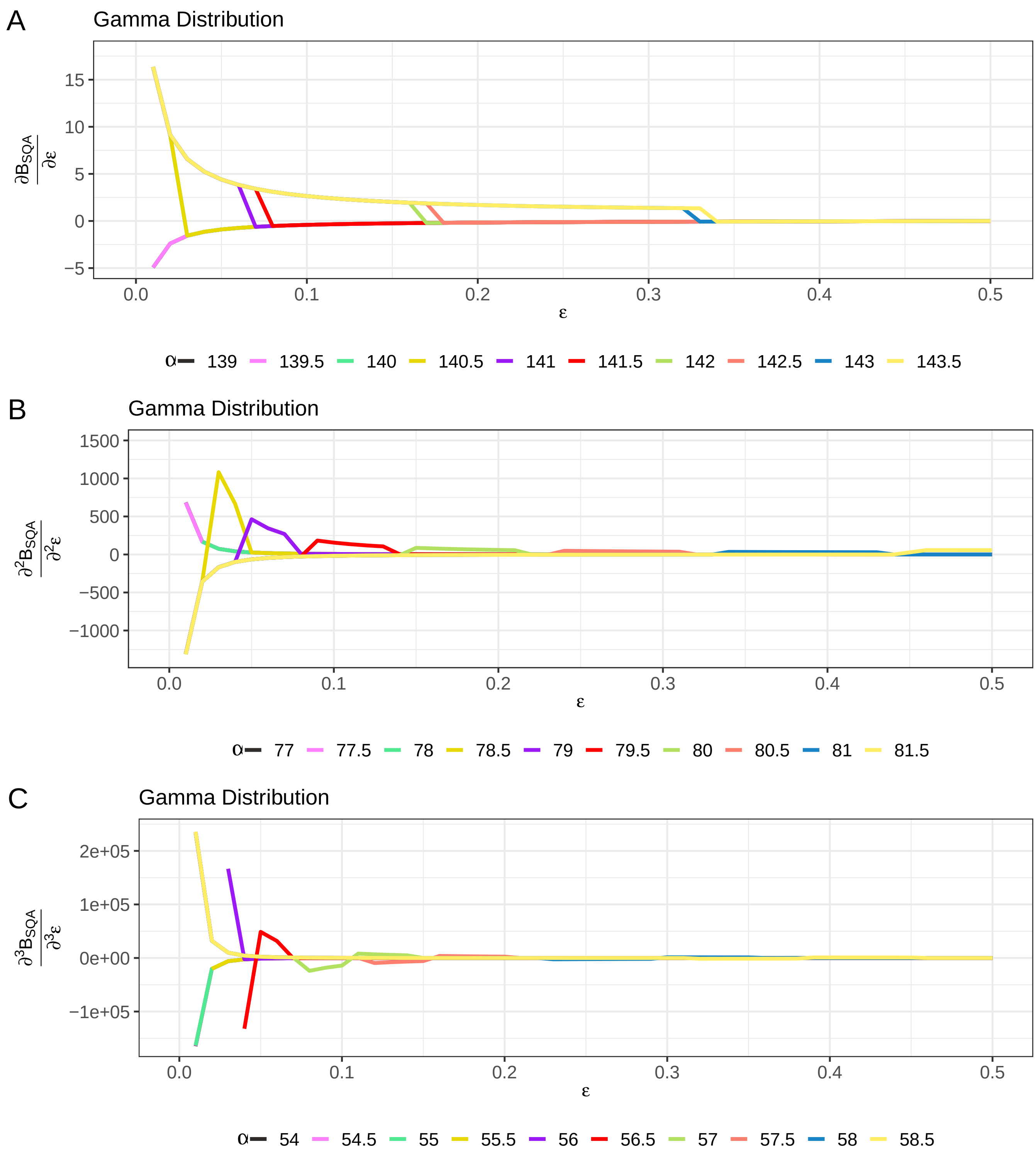

Fig. S2. A. The first derivative of the bias function of SQA for the gamma distribution as a function of the breakdown point ϵ. B. The second derivative of the bias function of SQA for the gamma distribution as a function of the breakdown point ϵ. C. The third derivative of the bias function of SQA for the gamma distribution as a function of the breakdown point ϵ.

When $\frac{1}{2} \geq \epsilon > \frac{1}{6}$, $B'_{\text{SQAB}}(\epsilon) = \frac{1}{(4-3\epsilon)^2\sqrt{\frac{\epsilon}{12-9\epsilon}}} - \frac{1}{(3\epsilon+1)^2\sqrt{\frac{1-\epsilon}{9\epsilon+3}}}$. To check whether $B'_{\text{SQAB}}(\epsilon) < 0$, first using the two identities $\sqrt{\frac{1}{12-9\epsilon}} = \sqrt{\frac{1}{3(4-3\epsilon)}}$ and $\sqrt{\frac{1}{3+9\epsilon}} = \sqrt{\frac{1}{3(1+3\epsilon)}}$ to simplify the expression, and then the inequality becomes, $(4-3\epsilon)^{\frac{3}{2}}\sqrt{\epsilon} > (3\epsilon+1)^{\frac{3}{2}}\sqrt{1-\epsilon}\sqrt{\frac{1}{3}} \iff (4-3\epsilon)^{\frac{3}{2}}\sqrt{\epsilon} > (3\epsilon+1)^{\frac{3}{2}}\sqrt{1-\epsilon}\sqrt{\frac{1}{3}} \iff 3(4-3\epsilon)^3\epsilon > (3\epsilon+1)^3(1-\epsilon) \iff -54\epsilon^4 + 324\epsilon^3 - 450\epsilon^2 + 184\epsilon - 1 > 0$. Since when $\epsilon < 1$, $-54\epsilon^4 + 54\epsilon^3 > 0$, just consider the condition that $270\epsilon^3 - 450\epsilon^2 + 184\epsilon - 1 > 0 \iff \epsilon\left(270\epsilon^2 - 450\epsilon + 174\right) + 10\epsilon - 1 > 0$. Since $270\epsilon^2 - 450\epsilon + 174 > 0$ is valid for $\epsilon < \frac{1}{30}\left(25 - 3\sqrt{5}\right)$, so just need $10\epsilon - 1 > 0$, $10\epsilon > 1$, $\epsilon > \frac{1}{10}$. So, the inequality is valid for $\frac{1}{30}\left(25 - 3\sqrt{5}\right) \approx 0.610 > \epsilon > \frac{1}{10}$, within the range of $\frac{1}{2} \geq \epsilon > \frac{1}{6}$, therefore, $B'_{\text{SQAB}} < 0$ for $\frac{1}{2} \geq \epsilon > \frac{1}{6}$. The first and second formula, when $\epsilon = \frac{1}{6}$, are all equal to $\frac{1}{2}\left(\sqrt{\frac{5}{3}} + \frac{1}{\sqrt{7}}\right)$. It follows that $B_{\text{SQAB}}(\epsilon)$ is continuous over $\left(0, \frac{1}{2}\right)$. Hence, $B'_{\text{SQAB}}(\epsilon) < 0$ is valid for $0 < \epsilon < \frac{1}{2}$, which leads to the assertion of this theorem. □

They also investigated the bias bounds of Range Value at Risk (5), which is

$$RVaR_{\alpha,\beta} = \frac{1}{\beta - \alpha}\int_{\alpha}^{\beta} VaR(u)\,du, 0 < \alpha < \beta < 1,$$

where $VaR(u) = \inf\{x \in \mathbb{R} : F_U(x) \geq u\}$. They pointed out that $VaR(u)$ is the quantile function, and it is obvious here that $RVaR_{\alpha,\beta}$ is the trimmed mean. Let $\alpha = \epsilon$, $\beta = 1 - \gamma\epsilon$, then because of asymmetry, if setting the $\mu = 0$, $\sigma = 1$, the upper bound $\sup_{P\in\mathcal{P}_U}(RVaR_{\alpha=\epsilon,\beta=1-\gamma\epsilon})$ and lower bound $\inf_{P\in\mathcal{P}_U}(RVaR_{\alpha=\epsilon,\beta=1-\gamma\epsilon})$ are not exactly opposite values. Also, $\sup_{P\in\mathcal{P}_U}(RVaR_{\alpha=\epsilon,\beta=1-\gamma\epsilon})$ and $\inf_{P\in\mathcal{P}_U}(RVaR_{\alpha=\epsilon,\beta=1-\gamma\epsilon})$ are very complex in form. If setting $\gamma = 1$, they are opposite values, i.e., the bias bound of symmetric trimmed mean is

$$B_{\text{STMB}}(\epsilon) = \frac{|\text{STM}_\epsilon - \mu|}{\sigma} = \frac{\epsilon\left(9\epsilon^2 + \left(4 - 3\sqrt{9\epsilon^2+4}\right)\epsilon - \sqrt{9\epsilon^2+4} + 2\right)}{(2\epsilon-1)\sqrt{-\frac{81\epsilon^4}{2} + 3\left(4\sqrt{9\epsilon^2+4} - 9\right)\epsilon^2 + 6\left(\sqrt{9\epsilon^2+4} - 2\right)\epsilon + \frac{4}{3}\left(\sqrt{9\epsilon^2+4} - 2\right) + \frac{9}{2}\left(3\sqrt{9\epsilon^2+4} - 8\right)\epsilon^3}}.$$

Theorem 0.8. *The above bias bound function, $B_{STMB}(\epsilon)$, is monotonic increasing with respect to ϵ over the interval $(0, \frac{1}{2})$.*

Proof. $\frac{dB_{\text{STMB}}(\epsilon)}{d\epsilon} = \frac{2\sqrt{6}\left(-5832\epsilon^7 + 232\left(\sqrt{9\epsilon^2+4}-2\right)\epsilon + 32\left(\sqrt{9\epsilon^2+4}-2\right) + 324\left(6\sqrt{9\epsilon^2+4}-25\right)\epsilon^6\right)}{(1-2\epsilon)^2\sqrt{9\epsilon^2+4}\left(-243\epsilon^4 + 18\left(4\sqrt{9\epsilon^2+4}-9\right)\epsilon^2 + 36\left(\sqrt{9\epsilon^2+4}-2\right)\epsilon + 8\left(\sqrt{9\epsilon^2+4}-2\right) + 27\left(3\sqrt{9\epsilon^2+4}-8\right)\epsilon^3\right)^{3/2}} + \frac{2\sqrt{6}\left(2\left(397\sqrt{9\epsilon^2+4}-830\right)\epsilon^2 + 54\left(50\sqrt{9\epsilon^2+4}-171\right)\epsilon^5 + 9\left(294\sqrt{9\epsilon^2+4}-779\right)\epsilon^4 + 9\left(193\sqrt{9\epsilon^2+4}-444\right)\epsilon^3\right)}{(1-2\epsilon)^2\sqrt{9\epsilon^2+4}\left(-243\epsilon^4 + 18\left(4\sqrt{9\epsilon^2+4}-9\right)\epsilon^2 + 36\left(\sqrt{9\epsilon^2+4}-2\right)\epsilon + 8\left(\sqrt{9\epsilon^2+4}-2\right) + 27\left(3\sqrt{9\epsilon^2+4}-8\right)\epsilon^3\right)^{3/2}}$.

Let $g(\epsilon) = -5832\epsilon^7 + 2\left(397\sqrt{9\epsilon^2+4} - 830\right)\epsilon^2 + 232\left(\sqrt{9\epsilon^2+4} - 2\right)\epsilon + 32\left(\sqrt{9\epsilon^2+4} - 2\right) + 324\left(6\sqrt{9\epsilon^2+4} - 25\right)\epsilon^6 + 54\left(50\sqrt{9\epsilon^2+4} - 171\right)\epsilon^5 + 9\left(294\sqrt{9\epsilon^2+4} - 779\right)\epsilon^4 + 9\left(193\sqrt{9\epsilon^2+4} - 444\right)\epsilon^3$ and $h(\epsilon)$ denotes the common denominator of $\frac{dB_{\text{STMB}}(\epsilon)}{d\epsilon}$. Then, for $0 < \epsilon < \frac{1}{2}$, $h(\epsilon) > 0$. To have $g(\epsilon) > 0$, it is equivalent to $2 \times 397\sqrt{9\epsilon^2+4}\epsilon^2 + 232\sqrt{9\epsilon^2+4}\epsilon + 32\sqrt{9\epsilon^2+4} + 324\epsilon^6 \times 6\sqrt{9\epsilon^2+4} + 54\epsilon^5 \times 50\sqrt{9\epsilon^2+4} + 9\epsilon^4 \times 294\sqrt{9\epsilon^2+4} + 9\epsilon^3 \times 193\sqrt{9\epsilon^2+4} > 5832\epsilon^7 + 2 \times 830\epsilon^2 + 2 \times 232\epsilon + 32 \times 2 + 25 \times 324\epsilon^6 + 54\epsilon^5 \times 171 + 9\epsilon^4 \times 779 + 444 \times 9\epsilon^3$. Squaring the left and right sides and then expanding, it is equivalent to $34012224\epsilon^{14} + 94478400\epsilon^{13} + 173315376\epsilon^{12} + 231367104\epsilon^{11} + 245524284\epsilon^{10} + 213603804\epsilon^9 + 155238849\epsilon^8 + 94957380\epsilon^7 + 48836664\epsilon^6 + 20951856\epsilon^5 + 7364752\epsilon^4 + 2051968\epsilon^3 + 427776\epsilon^2 + 59392\epsilon + 4096 > 34012224\epsilon^{14} + 94478400\epsilon^{13} + 173315376\epsilon^{12} + 231367104\epsilon^{11} + 245454300\epsilon^{10} + 213576588\epsilon^9 + 155256345\epsilon^8 + 94952088\epsilon^7 + 48850488\epsilon^6 + 20954880\epsilon^5 + 7361296\epsilon^4 + 2051968\epsilon^3 + 427776\epsilon^2 + 59392\epsilon + 4096 \iff 69984\epsilon^{10} + 27216\epsilon^9 - 17496\epsilon^8 + 5292\epsilon^7 - 13824\epsilon^6 - 3024\epsilon^5 + 3456\epsilon^4 > 0 \iff 108(1-2\epsilon)^2\epsilon^4\left(162\epsilon^4 + 225\epsilon^3 + 144\epsilon^2 + 100\epsilon + 32\right) > 0$. Then just need $162\epsilon^4 + 225\epsilon^3 + 144\epsilon^2 + 100\epsilon + 32 > 0$. Since $144\epsilon^2 + 100\epsilon + 32 > 0$ is valid for any $\epsilon \in \mathbb{R}$, $g(\epsilon) > 0$ is valid for any $\epsilon > 0$. So, $\frac{dB_{\text{STMB}}(\epsilon)}{d\epsilon} > 0$, which leads to the assertion of the theorem. □

Another interesting case is when $\gamma = 2$, setting the $\mu = 0$, $\sigma = 1$, $\sup_{P\in\mathcal{P}_U}(RVaR_{\alpha=\epsilon,\beta=1-2\epsilon}) = \frac{\sqrt{\epsilon(3\epsilon+8)}}{3}$, obviously monotonic, and $\inf_{P\in\mathcal{P}_U}(RVaR_{\alpha=\epsilon,\beta=1-2\epsilon}) = -\frac{\sqrt{(2\epsilon)(8+6\epsilon)}}{3}$, also obviously monotonic.

SI Dataset S1 (dataset_one.xlsx)

Raw data of asymptotic bias of all estimators shown in Figure 1 in the main text.

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