

# Semiparametric robust mean estimations based on the orderliness of quantile averages

Tuban Lee

This manuscript was compiled on May 8, 2023

As one of the most fundamental problems in statistics, robust location estimation has many prominent solutions, such as the trimmed mean, Winsorized mean, Hodges–Lehmann estimator, Huber  $M$ -estimator, and median of means. Recent research findings suggest that their biases concerning the mean can be quite different in asymmetric distributions, but the underlying mechanisms remain largely unclear. In this article, similar to the mean-median-mode inequality, it is proven that in the context of nearly all common unimodal distributions, there exists an orderliness of symmetric quantile averages with different breakdown points. Further deductions explain why the Winsorized mean and median of means generally have smaller biases compared to the trimmed mean. Building on the  $U$ -orderliness, the superiority of the median Hodges–Lehmann mean is discussed.

semiparametric | mean-median-mode inequality | asymptotic | unimodal  
| Hodges–Lehmann estimator

In 1823, Gauss (1) proved that for any unimodal distribution with a finite second moment,  $|m - \mu| \leq \sqrt{\frac{3}{4}}\omega$ , where  $\mu$  is the population mean,  $m$  is the population median, and  $\omega$  is the root mean square deviation from the mode,  $M$ . This pioneering work revealed that despite potential bias with respect to the mean in robust estimates, the deviation remains bounded in unit of a scale parameter under certain assumptions. Bernard, Kazzi, and Vanduffel (2020) (2) further derived asymptotic bias bounds of any quantile for unimodal distributions by reducing this optimization problem to a parametric one, which can be solved analytically. They showed that the population median,  $m$ , has the smallest maximum distance to the population mean,  $\mu$ , among all symmetric quantile averages ( $SQA_\epsilon$ ). Daniell, in 1920, (3) analyzed a class of estimators, linear combinations of order statistics, and identified that  $\epsilon$ -symmetric trimmed mean ( $STM_\epsilon$ ) belongs to this class. Another popular choice, the  $\epsilon$ -symmetric Winsorized mean ( $SWM_\epsilon$ ), named after Winsor and introduced by Tukey (4) and Dixon (5) in 1960, is also an  $L$ -estimator. Bieniek (2016) derived exact bias upper bounds of the Winsorized mean based on Danielak and Rychlik's work (2003) on the trimmed mean for any distribution with a finite second moment and confirmed that the former is smaller than the latter (6, 7). In 1963, Hodges and Lehmann (8) proposed a class of nonparametric location estimators based on rank tests and, from the Wilcoxon signed-rank statistic (9), deduced the median of pairwise means as a robust location estimator for a symmetric population. Both  $L$ -statistics and  $R$ -statistics achieve robustness essentially by removing a certain proportion of extreme values. In 1964, Huber (10) generalized maximum likelihood estimation to the minimization of the sum of a specific loss function, which measures the residuals between the data points and the model's parameters. Some  $L$ -statistics are also  $M$ -statistics, e.g., the sample mean is an  $M$ -estimator with a squared error loss function, while the sample median is an  $M$ -estimator with an absolute error loss function (10). The

Huber  $M$ -estimator is obtained by applying the Huber loss function that combines elements of both squared error and absolute error to achieve robustness against gross errors and high efficiency for contaminated Gaussian distributions (10). Sun, Zhou, and Fan (2020) examined the concentration bounds of Huber  $M$ -estimator (11). Mathieu (2022) (12) further derived the concentration bounds of  $M$ -statistics and demonstrated that, by selecting the tuning parameter which depends on the variance, Huber  $M$ -estimator can also be a sub-Gaussian estimator. The concept of median of means ( $MoM_{k,b} = \frac{n}{k}$ ,  $k$  is the number of size in each block,  $b$  is the number of blocks) was implicitly introduced several times in Nemirovsky and Yudin (1983) (13), Jerrum, Valiant, and Vazirani (1986), (14) and Alon, Matias and Szegedy (1996) (15)'s works. Given its good performance even for distributions with infinite second moments, MoM has received increasing attention over the past decade (16–23). Devroye, Lerasle, Lugosi, and Oliveira (2016) showed that MoM nears the optimum of sub-Gaussian mean estimation with regards to concentration bounds when the distribution has a heavy tail (21). For a comparison of concentration bounds of trimmed mean, Huber  $M$ -estimator, median of means and other relevant estimators, readers are directed to Gobet, Lerasle, and Métivier's paper (2022) (24). Laforgue, Clemencon, and Bertail (2019) proposed the median of randomized means ( $MoRM_{k,b}$ ) (23), wherein, rather than partitioning, an arbitrary number,  $b$ , of blocks are built independently from the sample, and showed that MoRM has better non-asymptotic sub-Gaussian property compared to MoM. In fact, asymptotically, the Hodges–Lehmann (H-L) estimator is equivalent to  $MoM_{k=2,b=\frac{n}{k}}$  and  $MoRM_{k=2,b}$ , and they can be seen as the pairwise mean distribution is approximated by the sampling without replacement and bootstrap, respectively. For the asymptotic validity, readers are referred to the foundational works of Efron (1979) (25), Bickel and Freedman (1981, 1984) (26, 27), and Helmers, Janssen, and Veraverbeke (1990)

## Significance Statement

In 1964, van Zwet introduced the convex transformation order for comparing the skewness of two distributions. This paradigm shift played a fundamental role in defining robust measures of distributions, from spread to kurtosis. Here, rather than the stochastic ordering between two distributions, the orderliness of quantile averages within a distribution is investigated. By classifying distributions through the signs of derivatives, a series of sophisticated robust mean estimators are deduced. Nearly all common nonparametric robust location estimators are found to be special cases thereof.

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

<sup>1</sup>To whom correspondence should be addressed. E-mail: tl@biomathematics.org

72 (28).

Here, the  $\epsilon, b$ -stratified mean is defined as

$$SM_{\epsilon, b} := \frac{b}{n} \left( \sum_{j=1}^{\frac{b-1}{2b\epsilon}} \sum_{i_j=\frac{(2bj-b-1)n\epsilon}{b-1}}^{\frac{(2bj-b+1)n\epsilon}{b-1}} X_{i_j} \right),$$

73 where  $X_1 \leq \dots \leq X_n$  denote the order statistics of a sample  
 74 of  $n$  independent and identically distributed random variables  
 75  $X_1, \dots, X_n$ .  $b \in \mathbb{N}$ ,  $b \geq 3$ . The definition was further refined to  
 76 guarantee the continuity of the breakdown point by incorporat-  
 77 ing an additional block in the center when  $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 0$ ,  
 78 or by adjusting the central block when  $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 1$  (SI  
 79 Text). If the subscript  $n$  is omitted, only the asymptotic  
 80 behavior is considered. If  $b$  is omitted,  $b = 3$  is assumed.  
 81  $SM_{\epsilon, b=3}$  is equal to  $STM_{\epsilon}$ , when  $\epsilon > \frac{1}{6}$ . The basic idea of  
 82 the stratified mean, when  $\frac{b-1}{2\epsilon} \in \mathbb{N}$ ,  $b \bmod 2 = 1$  is to dis-  
 83 tribute the data into  $\frac{b-1}{2\epsilon}$  equal-sized non-overlapping blocks  
 84 according to their order, then further sequentially group these  
 85 blocks into  $b$  equal-sized strata and compute the mean of the  
 86 middle stratum, which is the median of means of each stratum.  
 87 In situations where  $i \bmod 1 \neq 0$ , a potential solution is to  
 88 generate multiple smaller samples that satisfy the equality by  
 89 sampling without replacement, and subsequently calculate the  
 90 mean of all estimations, the details of determining the sample  
 91 size and sampling times are included in the SI Text. Although  
 92 the principle is similar to that of the median of means, with-  
 93 out the random shift, the result is different from  $MoM_{k=\frac{n}{b}, b}$ .  
 94 Additionally, the stratified mean differs from the mean of the  
 95 sample obtained through stratified sampling methods, intro-  
 96 duced by Neymean (1934) (29) or ranked set sampling (30),  
 97 introduced by McIntyre in 1952, as these sampling methods  
 98 are designed to obtain more representative samples or improve  
 99 the efficiency of sample estimates, but the sample mean based  
 100 on them are not robust. When  $b \bmod 2 = 1$ , the stratified  
 101 mean can be regarded as replacing the other equal-sized strata  
 102 with the middle stratum, which, in principle, is analogous to  
 103 the Winsorized mean that replaces extreme values with less  
 104 extreme percentiles. Furthermore, while the bounds confirm  
 105 that the Winsorized mean and median of means outperform  
 106 the trimmed mean (6, 7, 21, 24) in worst-case performance,  
 107 the complexity of bound analysis makes it difficult to achieve a  
 108 complete and intuitive understanding of these results. Also, a  
 109 clear explanation for the average performance of them remains  
 110 elusive. The aim of this paper is to define a series of semi-  
 111 parametric models using the signs of derivatives, reveal their  
 112 elegant interrelations and connections to parametric models,  
 113 and show that by exploiting these models, a set of sophisti-  
 114 cated robust mean estimators can be deduced, which have  
 115 strong robustness to departures from assumptions.

## 116 Quantile average and weighted average

The symmetric trimmed mean, symmetric Winsorized mean,  
 and stratified mean are all  $L$ -statistics. More specifically, they  
 are symmetric weighted averages, which are defined as

$$SWA_{\epsilon, n} := \frac{\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{X_i + X_{n-i+1}}{2} w_i}{\sum_{i=1}^{\frac{n}{2}} w_i},$$

117 where  $w_i$ s are the weights applied to the symmetric quantile  
 118 averages according to the definition of the corresponding  $L$ -  
 119 statistic. For example, for the  $\epsilon$ -symmetric trimmed mean,

$w_i = \begin{cases} 0, & i < n\epsilon \\ 1, & i \geq n\epsilon \end{cases}$ , provided that  $n\epsilon \in \mathbb{N}$ . The mean and  
 median are indeed two special cases of the symmetric trimmed  
 mean.

To extend the symmetric quantile average to the asymmet-  
 ric case, there are two possible definitions for the  $\epsilon, \gamma$ -quantile  
 average (QA( $\epsilon, \gamma, n$ )), i.e.,

$$\frac{1}{2}(\hat{Q}_n(\gamma\epsilon) + \hat{Q}_n(1-\epsilon)), \quad [1]$$

and

$$\frac{1}{2}(\hat{Q}_n(\epsilon) + \hat{Q}_n(1-\gamma\epsilon)), \quad [2]$$

where  $\gamma \geq 0$  and  $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ ,  $\hat{Q}_n(p)$  is the empirical quantile  
 function. For trimming from both sides, [1] and [2] are equiva-  
 lent. [1] is assumed in this article unless otherwise specified,  
 since many common asymmetric distributions are right skewed,  
 and [1] allows trimming only from the right side by setting  
 $\gamma = 0$ .

Analogously, the weighted average can be defined as

$$WA_{\epsilon, \gamma} := \frac{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} QA(\epsilon_0, \gamma) w_{\epsilon_0}}{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} w_{\epsilon_0}}.$$

For instance, the  $\epsilon, \gamma$ -trimmed mean ( $TM_{\epsilon, \gamma}$ ) is a weighted  
 average with a left trim size of  $\gamma\epsilon n$  and a right trim size of  $\epsilon n$ ,

$$w_{\epsilon_0} = \begin{cases} 0, & \epsilon_0 < \epsilon \\ 1, & \epsilon_0 \geq \epsilon \end{cases}.$$

## Classifying distributions by the signs of derivatives

Let  $\mathcal{P}_k$  denote the set of all distributions over  $\mathbb{R}$  whose  
 moments, from the first to the  $k$ th, are all finite. With-  
 out loss of generality, all classes discussed in the following  
 are subclasses of the nonparametric class of distributions  
 $\mathcal{P}_\infty^k := \{\text{All continuous distribution } P \in \mathcal{P}_k\}$ . Besides fully  
 and smoothly parameterizing by a Euclidean parameter or  
 just assuming regularity conditions, there are many ways to  
 classify distributions. In 1956, Stein initiated the problem of  
 estimating parameters in the presence of an infinite dimen-  
 sional nuisance shape parameter (31). A notable example  
 discussed in his groundbreaking work was the estimation of  
 the center of symmetry for an unknown symmetric distribution.  
 In 1993, Bickel, Klaassen, Ritov, and Wellner published an  
 influential semiparametrics textbook (32) and systematically  
 classified many common models into three classes: paramet-  
 ric, nonparametric, and semiparametric. However, there is  
 another old and commonly encountered class of distributions  
 that receives little attention in semiparametric literature: the  
 unimodal distribution. It is a very unique semiparametric  
 model because its definition is based on the signs of deriva-  
 tives, i.e., assuming  $P$  is continuous,  $(f'(x) > 0 \text{ for } x \leq M) \wedge$   
 $(f'(x) < 0 \text{ for } x \geq M)$ . Let  $\mathcal{P}_U$  denote the set of all unimodal  
 distributions. Five parametric distributions in  $\mathcal{P}_U$  are detailed  
 as examples here: Weibull, gamma, Pareto, lognormal and  
 generalized Gaussian.

Consider the sign of the derivative of the quantile average  
 with respect to the breakdown point, a right-skewed distribu-  
 tion is called  $\gamma$ -ordered, if and only if

$$\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}, \frac{\partial QA_{\epsilon, \gamma}}{\partial \epsilon} \leq 0.$$

164 **Data Availability.** Data for Figure ?? are given in SI Dataset  
 165 S1. All codes have been deposited in [GitHub](#).

166 **ACKNOWLEDGMENTS.** I gratefully acknowledge the valuable  
 167 comments by the editor which substantially improved the clarity  
 168 and quality of this paper.

- 169 1. CF Gauss, *Theoria combinationis observationum erroribus minimis obnoxiae*. (Henricus  
 170 Dieterich), (1823).
- 171 2. C Bernard, R Kazzi, S Vanduffel, Range value-at-risk bounds for unimodal distributions under  
 172 partial information. *Insur. Math. Econ.* **94**, 9–24 (2020).
- 173 3. P Daniell, Observations weighted according to order. *Am. J. Math.* **42**, 222–236 (1920).
- 174 4. JW Tukey, A survey of sampling from contaminated distributions in *Contributions to probability*  
 175 *and statistics*. (Stanford University Press), pp. 448–485 (1960).
- 176 5. WJ Dixon, Simplified Estimation from Censored Normal Samples. *The Annals Math. Stat.* **31**,  
 177 385–391 (1960).
- 178 6. M Bieniek, Comparison of the bias of trimmed and winsorized means. *Commun. Stat. Methods*  
 179 **45**, 6641–6650 (2016).
- 180 7. K Danielak, T Rychlik, Theory & methods: Exact bounds for the bias of trimmed means. *Aust.*  
 181 *& New Zealand J. Stat.* **45**, 83–96 (2003).
- 182 8. J Hodges Jr, E Lehmann, Estimates of location based on rank tests. *The Annals Math. Stat.*  
 183 **34**, 598–611 (1963).
- 184 9. F Wilcoxon, Individual comparisons by ranking methods. *Biom. Bull.* **1**, 80–83 (1945).
- 185 10. PJ Huber, Robust estimation of a location parameter. *Ann. Math. Stat.* **35**, 73–101 (1964).
- 186 11. Q Sun, WX Zhou, J Fan, Adaptive huber regression. *J. Am. Stat. Assoc.* **115**, 254–265 (2020).
- 187 12. T Mathieu, Concentration study of m-estimators using the influence function. *Electron. J. Stat.*  
 188 **16**, 3695–3750 (2022).
- 189 13. AS Nemirovskij, DB Yudin, *Problem complexity and method efficiency in optimization*. (Wiley-  
 190 Interscience), (1983).
- 191 14. MR Jerrum, LG Valiant, VV Vazirani, Random generation of combinatorial structures from a  
 192 uniform distribution. *Theor. computer science* **43**, 169–188 (1986).
- 193 15. N Alon, Y Matias, M Szegedy, The space complexity of approximating the frequency moments  
 194 in *Proceedings of the twenty-eighth annual ACM symposium on Theory of computing*. pp.  
 195 20–29 (1996).
- 196 16. PL Bühlmann, Bagging, subbagging and bragging for improving some prediction algorithms  
 197 in *Research report/Seminar für Statistik, Eidgenössische Technische Hochschule (ETH)*.  
 198 (Seminar für Statistik, Eidgenössische Technische Hochschule (ETH), Zürich), Vol. 113,  
 199 (2003).
- 200 17. JY Audibert, O Catoni, Robust linear least squares regression. *The Annals Stat.* **39**, 2766–2794  
 201 (2011).
- 202 18. D Hsu, S Sabato, Heavy-tailed regression with a generalized median-of-means in *International*  
 203 *Conference on Machine Learning*. (PMLR), pp. 37–45 (2014).
- 204 19. S Minsker, Geometric median and robust estimation in banach spaces. *Bernoulli* **21**, 2308–  
 205 2335 (2015).
- 206 20. C Brownlees, E Joly, G Lugosi, Empirical risk minimization for heavy-tailed losses. *The Annals*  
 207 *Stat.* **43**, 2507–2536 (2015).
- 208 21. L Devroye, M Lerasle, G Lugosi, RI Oliveira, Sub-gaussian mean estimators. *The Annals Stat.*  
 209 **44**, 2695–2725 (2016).
- 210 22. E Joly, G Lugosi, Robust estimation of u-statistics. *Stoch. Process. their Appl.* **126**, 3760–3773  
 211 (2016).
- 212 23. P Laforgue, S Cléménçon, P Bertail, On medians of (randomized) pairwise means in *Interna-*  
 213 *tional Conference on Machine Learning*. (PMLR), pp. 1272–1281 (2019).
- 214 24. E Gobet, M Lerasle, D Métivier, Mean estimation for Randomized Quasi Monte Carlo method.  
 215 working paper or preprint (2022).
- 216 25. B Efron, Bootstrap methods: Another look at the jackknife. *The Annals Stat.* **7**, 1–26 (1979).
- 217 26. PJ Bickel, DA Freedman, Some asymptotic theory for the bootstrap. *The annals statistics* **9**,  
 218 1196–1217 (1981).
- 219 27. PJ Bickel, DA Freedman, Asymptotic normality and the bootstrap in stratified sampling. *The*  
 220 *annals statistics* **12**, 470–482 (1984).
- 221 28. R Helmers, P Janssen, N Veraverbeke, *Bootstrapping U-quantiles*. (CWI. Department of  
 222 Operations Research, Statistics, and System Theory [BS]), (1990).
- 223 29. J Neyman, On the two different aspects of the representative method: The method of stratified  
 224 sampling and the method of purposive selection. *J. Royal Stat. Soc.* **97**, 558–606 (1934).
- 225 30. G McIntyre, A method for unbiased selective sampling, using ranked sets. *Aust. journal*  
 226 *agricultural research* **3**, 385–390 (1952).
- 227 31. C Stein, , et al., Efficient nonparametric testing and estimation in *Proceedings of the third*  
 228 *Berkeley symposium on mathematical statistics and probability*. Vol. 1, pp. 187–195 (1956).
- 229 32. P Bickel, CA Klaassen, Y Ritov, JA Wellner, *Efficient and adaptive estimation for semiparamet-*  
 230 *ric models*. (Springer) Vol. 4, (1993).