Semiparametric robust mean estimations based on the orderliness of quantile averages

Tuban Lee

This manuscript was compiled on June 6, 2023

As one of the most fundamental problems in statistics, robust location estimation has many prominent solutions, such as the symmetric trimmed mean, symmetric Winsorized mean, Hodges–Lehmann estimator, Huber M-estimator, and median of means. Recent studies suggest that their biases concerning the mean can be quite different in asymmetric distributions, but the underlying mechanisms remain largely unclear. This study establishes two forms of orderliness within a wide range of semiparametric distributions. From this, two sequences of semiparametric robust mean estimators emerge. Further deductions explain why the Winsorized mean typically have smaller biases compared to the trimmed mean. Building on the γ-U-orderliness, the superiority of the median Hodges–Lehmann mean is discussed.

semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

In 1823, Gauss (1) proved that for any unimodal distribution, $|m-\mu| \leq \sqrt{\frac{3}{4}}\omega$ and $\sigma \leq \omega \leq 2\sigma$, where μ is the population mean, m is the population median, ω is the root mean square deviation from the mode, and σ is the population standard deviation. This pioneering work revealed that despite potential bias in robust mean estimates, the deviation remains bounded in units of a scale parameter under certain assumptions. Bernard, Kazzi, and Vanduffel (2020) (2) further derived asymptotic bias bounds of any quantile for unimodal distributions with finite second moments, by reducing this optimization problem to a parametric one, which can be solved analytically. They showed that m has the smallest maximum distance to μ among all symmetric quantile averages (SQA_ϵ). Daniell, in 1920, (3) analyzed a class of estimators, linear combinations of order statistics, and identified that the ϵ-symmetric trimmed mean (STM_ϵ) belongs to this class. Another popular choice, the ϵ-symmetric Winsorized mean (SWM_ϵ), named after Winsor and introduced by Tukey (4) and Dixon (5) in 1960, is also an L-estimator. Bieniek (2016) derived exact bias upper bounds of the Winsorized mean based on Danielak and Rychlik's work (2003) on the trimmed mean for any distribution with a finite second moment and confirmed that the former is smaller than the latter (6, 7). In 1963, Hodges and Lehmann (8) proposed a class of nonparametric location estimators based on rank tests and, from the Wilcoxon signed-rank statistic (9), deduced the median of pairwise means as a robust location estimator for a symmetric population. Both L-statistics and R-statistics achieve robustness essentially by removing a certain proportion of extreme values. In 1964, Huber (10) generalized maximum likelihood estimation to the minimization of the sum of a specific loss function, which measures the residuals between the data points and the model's parameters. Some L-estimators are also M-estimators, e.g., the sample mean is an M-estimator with a squared error loss function, the sample median is an M-estimator with an absolute error loss function (10). The Huber M-estimator is obtained by applying the Huber loss function that combines elements of both squared error and absolute error to achieve robustness against gross errors and high efficiency for contaminated Gaussian distributions (10). Sun, Zhou, and Fan (2020) examined the concentration bounds of the Huber M-estimator (11). Mathieu (2022) (12) further derived the concentration bounds of M-estimators and demonstrated that, by selecting the tuning parameter which depends on the variance, the Huber M-estimator can also be a sub-Gaussian estimator. The concept of the median of means ($\mathrm{MoM}_{k,b=\frac{n}{k},n}$) was first introduced by Nemirovsky and Yudin (1983) in their work on stochastic optimization (13). Given its good performance even for distributions with infinite second moments, the MoM has received increasing attention over the past decade (14–17). Devroye, Lerasle, Lugosi, and Oliveira (2016) showed that $\mathrm{MoM}_{k,b=\frac{n}{k},n}$ nears the optimum of sub-Gaussian mean estimation with regards to concentration bounds when the distribution has a heavy tail (15). Laforgue, Clemencon, and Bertail (2019) proposed the median of randomized means ($\mathrm{MoRM}_{k,b,n}$) (16), wherein, rather than partitioning, an arbitrary number, b, of blocks are built independently from the sample, and showed that $\mathrm{MoRM}_{k,b,n}$ has a better non-asymptotic sub-Gaussian property compared to $\mathrm{MoM}_{k,b=\frac{n}{k},n}$. In fact, asymptotically, the Hodges-Lehmann (H-L) estimator is equivalent to $\mathrm{MoM}_{k=2,b=\frac{n}{k}}$ and $\mathrm{MoRM}_{k=2,b}$, and they can be seen as the pairwise mean distribution is approximated by the sampling without replacement and bootstrap, respectively. When $k \ll n$, the difference between sampling with replacement and without replacement is negligible. For the asymptotic validity, readers are referred to the foundational works of Efron (1979) (18), Bickel and Freedman (1981, 1984) (19, 20), and Helmers, Janssen, and Veraverbeke (1990) (21).

1. CF Gauss, *Theoria combinationis observationum erroribus minimis obnoxiae.* (Henricus Dieterich), (1823).
2. C Bernard, R Kazzi, S Vanduffel, Range value-at-risk bounds for unimodal distributions under partial information. *Insur. Math. Econ.* **94**, 9–24 (2020).
3. P Daniell, Observations weighted according to order. *Am. J. Math.* **42**, 222–236 (1920).

Significance Statement

In 1964, van Zwet introduced the convex transformation order for comparing the skewness of two distributions. This paradigm shift played a fundamental role in defining robust measures of distributions, from spread to kurtosis. Here, instead of examining the stochastic ordering between two distributions, the orderliness of quantile averages within a distribution is investigated. By classifying distributions through the signs of derivatives, two series of sophisticated robust mean estimators are deduced. Nearly all common nonparametric robust location estimators are found to be special cases thereof.

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

[1]To whom correspondence should be addressed. E-mail: tl@biomathematics.org

4. JW Tukey, A survey of sampling from contaminated distributions in *Contributions to probability and statistics*. (Stanford University Press), pp. 448–485 (1960).
5. WJ Dixon, Simplified Estimation from Censored Normal Samples. *The Annals Math. Stat.* **31**, 385 – 391 (1960).
6. K Danielak, T Rychlik, Theory & methods: Exact bounds for the bias of trimmed means. *Aust. & New Zealand J. Stat.* **45**, 83–96 (2003).
7. M Bieniek, Comparison of the bias of trimmed and winsorized means. *Commun. Stat. Methods* **45**, 6641–6650 (2016).
8. J Hodges Jr, E Lehmann, Estimates of location based on rank tests. *The Annals Math. Stat.* **34**, 598–611 (1963).
9. F Wilcoxon, Individual comparisons by ranking methods. *Biom. Bull.* **1**, 80–83 (1945).
10. PJ Huber, Robust estimation of a location parameter. *Ann. Math. Stat.* **35**, 73–101 (1964).
11. Q Sun, WX Zhou, J Fan, Adaptive huber regression. *J. Am. Stat. Assoc.* **115**, 254–265 (2020).
12. T Mathieu, Concentration study of m-estimators using the influence function. *Electron. J. Stat.* **16**, 3695–3750 (2022).
13. AS Nemirovskij, DB Yudin, *Problem complexity and method efficiency in optimization*. (Wiley-Interscience), (1983).
14. D Hsu, S Sabato, Heavy-tailed regression with a generalized median-of-means in *International Conference on Machine Learning*. (PMLR), pp. 37–45 (2014).
15. L Devroye, M Lerasle, G Lugosi, RI Oliveira, Sub-gaussian mean estimators. *The Annals Stat.* **44**, 2695–2725 (2016).
16. P Laforgue, S Clémençon, P Bertail, On medians of (randomized) pairwise means in *International Conference on Machine Learning*. (PMLR), pp. 1272–1281 (2019).
17. G LECUÉ, M LERASLE, Robust machine learning by median-of-means: Theory and practice. *The Annals Stat.* **48**, 906–931 (2020).
18. B Efron, Bootstrap methods: Another look at the jackknife. *The Annals Stat.* **7**, 1–26 (1979).
19. PJ Bickel, DA Freedman, Some asymptotic theory for the bootstrap. *The annals statistics* **9**, 1196–1217 (1981).
20. PJ Bickel, DA Freedman, Asymptotic normality and the bootstrap in stratified sampling. *The annals statistics* **12**, 470–482 (1984).
21. R Helmers, P Janssen, N Veraverbeke, *Bootstrapping U-quantiles*. (CWI. Department of Operations Research, Statistics, and System Theory [BS]), (1990).