Semiparametric robust mean estimations based on the orderliness of quantile averages

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As one of the most fundamental problem in statistics, robust location estimation has many prominent solutions, such as the symmetric trimmed mean, symmetric Winsorized mean, Hodges–Lehmann estimator, Huber M-estimator, and median of means. Recent studies suggest that their maximum biases concerning the mean can be quite different in asymmetric distributions, but the underlying mechanisms and average performance remain largely unclear. This study establishes several forms of orderliness among quantile averages, similar to the mean-median-mode inequality, within a wide range of semiparametric distributions. From this, a sequence of advanced robust mean estimators emerges, which also explains why the Winsorized mean and median of means typically have smaller biases compared to the trimmed mean. Building on the U-orderliness, the superiority of the median Hodges–Lehmann mean is discussed.

semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

In 1823, Gauss (1) proved that for any unimodal distribution with a finite second moment, $|m - \mu| \leq \sqrt{\frac{3}{4}}\omega$, where μ is the population mean, m is the population median, and ω is the root mean square deviation from the mode, M. This pioneering work revealed that despite potential bias with respect to the mean in robust estimates, the deviation remains bounded in units of a scale parameter under a semi-parametric assumption. Bernard, Kazzi, and Vanduffel (2020) (2) further derived asymptotic bias bounds of any quantile for unimodal distributions with finite second moments, building on the works of Li, Shao, Wang, and Yang (2018) (3), by reducing this optimization problem to a parametric one, which can be solved analytically.

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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