

Semiparametric robust mean estimations based on the orderliness of quantile averages

Tuban Lee

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As one of the most fundamental problems in statistics, the robust location estimation has many prominent solutions, such as the symmetric trimmed mean, symmetric Winsorized mean, Hodges–Lehmann estimator, Huber M -estimator, and median of means. Recent studies suggest that their maximum biases concerning the mean can be quite different in asymmetric distributions, but the underlying mechanisms and average performance remain largely unclear. This study establishes several forms of orderliness among quantile averages, similar to the mean-median-mode inequality, within a wide range of semiparametric distributions, particularly highlighting the unique role of γ -symmetric distributions. From this, a sequence of advanced robust mean estimators emerges, which also explains why the Winsorized mean and median of means typically have smaller biases compared to the trimmed mean. Building on the U -orderliness, the superiority of the median Hodges–Lehmann mean is discussed.

semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

In 1823, Gauss (1) proved that for any unimodal distribution, $|m - \mu| \leq \sqrt{\frac{3}{4}}\omega$ and $\sigma \leq \omega \leq 2\sigma$, where μ is the population mean, m is the population median, ω is the root mean square deviation from the mode, and σ is the population standard deviation. This pioneering work revealed that despite potential bias in robust mean estimates, the deviation remains bounded in units of a scale parameter under certain assumptions. Bernard, Kazzi, and Vanduffel (2020) (2) further derived asymptotic bias bounds of any quantile for unimodal distributions with finite second moments, by reducing this optimization problem to a parametric one, which can be solved analytically. They showed that m has the smallest maximum distance to μ among all symmetric quantile averages (SQA _{ϵ}). Daniell, in 1920, (3) analyzed a class of estimators, linear combinations of order statistics, and identified that ϵ -symmetric trimmed mean (STM _{ϵ}) belongs to this class. Another popular choice, the ϵ -symmetric Winsorized mean (SWM _{ϵ}), named after Winsor and introduced by Tukey (4) and Dixon (5) in 1960, is also an L -estimator. Bieniek (2016) derived exact bias upper bounds of the Winsorized mean based on Danielak and Rychlik's work (2003) on the trimmed mean for any distribution with a finite second moment and confirmed that the former is smaller than the latter (6, 7). In 1963, Hodges and Lehmann (8) proposed a class of nonparametric location estimators based on rank tests and, from the Wilcoxon signed-rank statistic (9), deduced the median of pairwise means as a robust location estimator for a symmetric population. Both L -statistics and R -statistics achieve robustness essentially by removing a certain proportion of extreme values. In 1964, Huber (10) generalized maximum likelihood estimation to the minimization of the sum of a specific loss function, which measures the residuals between the data points and the model's parameters. Some L -estimators are also M -estimators, e.g., the sample mean is an M -estimator with a squared error loss

function, the sample median is an M -estimator with an absolute error loss function (10). The Huber M -estimator is obtained by applying the Huber loss function that combines elements of both squared error and absolute error to achieve robustness against gross errors and high efficiency for contaminated Gaussian distributions (10). Sun, Zhou, and Fan (2020) examined the concentration bounds of Huber M -estimator (11). Mathieu (2022) (12) further derived the concentration bounds of M -estimators and demonstrated that, by selecting the tuning parameter which depends on the variance, Huber M -estimator can also be a sub-Gaussian estimator. The concept of the median of means (MoM _{$k, b = \frac{n}{k}, n$}), which was first introduced by Nemirovsky and Yudin (1983) in their work on stochastic optimization (13). Given its good performance even for distributions with infinite second moments, MoM has received increasing attention over the past decade (14–17). Devroye, Lerasle, Lugosi, and Oliveira (2016) showed that MoM _{$k, b = \frac{n}{k}, n$} nears the optimum of sub-Gaussian mean estimation with regards to concentration bounds when the distribution has a heavy tail (15). Laforge, Clemencon, and Bertail (2019) proposed the median of randomized means (MoRM _{k, b, n}) (16), wherein, rather than partitioning, an arbitrary number, b , of blocks are built independently from the sample, and showed that MoRM _{k, b, n} has a better non-asymptotic sub-Gaussian property compared to MoM _{$k, b = \frac{n}{k}, n$} . In fact, asymptotically, the Hodges–Lehmann (H-L) estimator is equivalent to MoM _{$k=2, b = \frac{n}{k}$} and MoRM _{$k=2, b$} , and they can be seen as the pairwise mean distribution is approximated by the sampling without replacement and bootstrap, respectively. When $k \ll n$, the difference between sampling with replacement and without replacement is negligible. For the asymptotic validity, readers are referred to the foundational works of Efron (1979) (18), Bickel and Freedman (1981, 1984) (19, 20), and Helmers, Janssen, and Veraverbeke (1990) (21).

Significance Statement

In 1964, van Zwet introduced the convex transformation order for comparing the skewness of two distributions. This paradigm shift played a fundamental role in defining robust measures of distributions, from spread to kurtosis. Here, instead of examining the stochastic ordering between two distributions, the orderliness of quantile averages within a distribution is investigated. By classifying distributions through the signs of derivatives, a series of sophisticated robust mean estimators is deduced. Nearly all common nonparametric robust location estimators are found to be special cases thereof.

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¹To whom correspondence should be addressed. E-mail: tl@biomathematics.org

Here, the ϵ, b -stratified mean is defined as

$$\text{SM}_{\epsilon, b, n} := \frac{b}{n} \left(\sum_{j=1}^{\lfloor \frac{b-1}{2b\epsilon} \rfloor} \sum_{i_j = \frac{(2bj-b-1)n\epsilon}{b-1}}^{\frac{(2bj-b+1)n\epsilon}{b-1}} X_{i_j} \right),$$

where $X_1 \leq \dots \leq X_n$ denote the order statistics of a sample of n independent and identically distributed random variables X_1, \dots, X_n . $b \in \mathbb{N}$, $b \geq 3$. The definition was further refined to guarantee the continuity of the breakdown point by incorporating an additional block in the center when $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 0$, or by adjusting the central block when $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 1$ (SI Text). If the subscript n is omitted, only the asymptotic behavior is considered. If b is omitted, $b = 3$ is assumed. $\text{SM}_{\epsilon, b=3}$ is equivalent to STM_{ϵ} , when $\epsilon > \frac{1}{6}$. When $\frac{b-1}{2\epsilon} \in \mathbb{N}$, $b \bmod 2 = 1$, the basic idea of the stratified mean is to distribute the data into $\frac{b-1}{2\epsilon}$ equal-sized non-overlapping blocks according to their order, then further sequentially group these blocks into b equal-sized strata and compute the mean of the middle stratum, which is the median of means of each stratum. In situations where $i \bmod 1 \neq 0$, a potential solution is to generate multiple smaller samples that satisfy the equality by sampling without replacement, and subsequently calculate the mean of all estimations. The details of determining the sample size and sampling times are provided in the SI Text. Although the principle resembles that of the median of means, $\text{SM}_{\epsilon, b, n}$ is different from $\text{MoM}_{k=\frac{n}{b}, b, n}$ as it does not include the random shift. Additionally, the stratified mean differs from the mean of the sample obtained through stratified sampling methods, introduced by Neyman (1934) (22) or ranked set sampling (23), introduced by McIntyre in 1952, as these sampling methods aim to obtain more representative samples or improve the efficiency of sample estimates, but the sample means based on them are not robust. When $b \bmod 2 = 1$, the stratified mean can be regarded as replacing the other equal-sized strata with the middle stratum, which, in principle, is analogous to the Winsorized mean that replaces extreme values with less extreme percentiles. Furthermore, while the bounds confirm that the Winsorized mean and median of means outperform the trimmed mean (6, 7, 15) in worst-case performance, the complexity of bound analysis makes it difficult to achieve a complete and intuitive understanding of these results. Also, a clear explanation for the average performance of them remains elusive. The aim of this paper is to define a series of semi-parametric models using the signs of derivatives, reveal their elegant interrelations and connections to parametric models, and show that by exploiting these models, a set of sophisticated mean estimators can be deduced, which exhibit strong robustness to departures from assumptions.

Quantile average and weighted average

The symmetric trimmed mean, symmetric Winsorized mean, and stratified mean are all L -estimators. More specifically, they are symmetric weighted averages, which are defined as

$$\text{SWA}_{\epsilon, n} := \frac{\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{X_i + X_{n-i+1}}{2} w_i}{\sum_{i=1}^{\lceil \frac{n}{2} \rceil} w_i},$$

where w_i s are the weights applied to the symmetric quantile averages according to the definition of the corresponding L -estimators. For example, for the ϵ -symmetric trimmed mean,

$w_i = \begin{cases} 0, & i < n\epsilon \\ 1, & i \geq n\epsilon \end{cases}$, provided that $n\epsilon \in \mathbb{N}$. The mean and median are indeed two special cases of the symmetric trimmed mean.

To extend the symmetric quantile average to the asymmetric case, two definitions for the ϵ, γ -quantile average ($\text{QA}(\epsilon, \gamma, n)$) are proposed. The first definition is:

$$\frac{1}{2}(\hat{Q}_n(\gamma\epsilon) + \hat{Q}_n(1-\epsilon)), \quad [1]$$

and the second definition is:

$$\frac{1}{2}(\hat{Q}_n(\epsilon) + \hat{Q}_n(1-\gamma\epsilon)), \quad [2]$$

where $\hat{Q}_n(p)$ is the empirical quantile function; γ is used to adjust the degree of asymmetry, $\gamma \geq 0$; and $0 \leq \epsilon \leq \frac{1}{1+\gamma}$. For trimming from both sides, [1] and [2] are essentially equivalent. The first definition along with $\gamma \geq 0$ and $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ are assumed in the remainder of this article unless otherwise specified, since many common asymmetric distributions are right-skewed, and [1] allows trimming only from the right side by setting $\gamma = 0$.

Analogously, the weighted average can be defined as

$$\text{WA}_{\epsilon, \gamma, n} := \frac{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} \text{QA}(\epsilon_0, \gamma, n) w_{\epsilon_0}}{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} w_{\epsilon_0}}.$$

For any weighted average, if γ is omitted, $\gamma = 1$ is assumed. The ϵ, γ -trimmed mean ($\text{TM}_{\epsilon, \gamma, n}$) is a weighted average with a left trim size of $\gamma\epsilon n$ and a right trim size of ϵn , where $w_{\epsilon_0} = \begin{cases} 0, & \epsilon_0 < \epsilon \\ 1, & \epsilon_0 \geq \epsilon \end{cases}$. Using this definition, the TM computation remains the same, regardless of whether $\gamma\epsilon n \notin \mathbb{N}$ or $\epsilon n \notin \mathbb{N}$, since this definition is based on the empirical quantile function. However, in this article, considering the computational cost in practice, the non-asymptotic definitions of various types of weighted averages are essentially based on order statistics, in most cases. Unless stated otherwise, the solution to their decimal issue is the same as that in SM.

Classifying distributions by the signs of derivatives

Let $\mathcal{P}_{\mathcal{R}}$ denote the set of all continuous distribution over \mathbb{R} . Without loss of generality, the discussion of all the classes outlined below is restricted to the intersection with this non-parametric class of distributions. Besides fully and smoothly parameterizing by a Euclidean parameter or just assuming regularity conditions, there are many ways to classify distributions. In 1956, Stein initiated the problem of estimating parameters in the presence of an infinite dimensional nuisance shape parameter (24). A notable example discussed in his groundbreaking work was the estimation of the center of symmetry for an unknown symmetric distribution. In 1993, Bickel, Klaassen, Ritov, and Wellner published an influential semi-parametrics textbook (25) which systematically categorized many common models into three classes: parametric, non-parametric, and semiparametric. Yet, there is another old and commonly encountered class of distributions that receives little attention in semiparametric literature: the unimodal distribution. It is a very unique semiparametric model because its definition is based on the signs of derivatives, i.e.,

($f'(x) > 0$ for $x \leq M$) \wedge ($f'(x) < 0$ for $x \geq M$), where $f(x)$ is the probability density function (pdf) of a random variable X , M is the mode. Let \mathcal{P}_U denote the set of all unimodal distributions. There was a widespread misbelief that the median of an arbitrary unimodal distribution always lies between its mean and mode until Runnenburg (1978) and van Zwet (1979) (26, 27) endeavored to determine sufficient conditions for the inequality to hold, thereby implying the possibility of its violation. The class of distributions that satisfy the mean-median-mode inequality constitutes a subclass of \mathcal{P}_U . By analogy, a right-skewed distribution is called γ -ordered, if and only if

$$\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, Q_{A_{\epsilon_1, \gamma}} \geq Q_{A_{\epsilon_2, \gamma}}.$$

The necessary and sufficient condition below hints at the relation between the mean-median-mode inequality and the γ -orderliness.

Theorem .1. *A distribution is γ -ordered if and only if its pdf satisfies the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ or $f(Q(\gamma\epsilon)) \leq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. Without loss of generality, consider the case of right-skewed distribution. From the definition of γ -orderliness, it is deduced that $\frac{Q(\gamma\epsilon-\delta)+Q(1-\epsilon+\delta)}{2} \geq \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \Leftrightarrow Q(\gamma\epsilon-\delta) - Q(\gamma\epsilon) \geq Q(1-\epsilon) - Q(1-\epsilon+\delta) \Leftrightarrow Q'(1-\epsilon) \geq Q'(\gamma\epsilon)$, where δ is an infinitesimal positive quantity. Observing that the quantile function is the inverse function of the cumulative distribution function (cdf), $Q'(1-\epsilon) \geq Q'(\gamma\epsilon) \Leftrightarrow F'(Q(\gamma\epsilon)) \geq F'(Q(1-\epsilon))$, thereby completing the proof, given that the derivative of cdf is pdf. \square

According to Theorem .1, if a probability distribution is right-skewed and monotonic, it will always be γ -ordered. For a right-skewed unimodal distribution, if $Q(\gamma\epsilon) > M$, the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ holds. The principle is extendable to unimodal-like distributions. Suppose there is a right-skewed multimodal distribution following the mean- γ -median-first mode inequality with several smaller modes on the right side, with the first mode, M_1 , having the greatest probability density, and the γ -median, $Q(\frac{\gamma}{1+\gamma})$, falling within the first dominant mode (i.e., if $x > Q(\frac{\gamma}{1+\gamma})$, $f(Q(\frac{\gamma}{1+\gamma})) \geq f(x)$), then if $Q(\gamma\epsilon) > M_1$, the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ also holds. In other words, while a distribution following the mean- γ -median-mode inequality may not be strictly γ -ordered, the inequality that defines γ -orderliness remains valid for most quantile averages. The mean- γ -median-mode inequality can also indicate possible bounds for γ in practice, e.g., for any distributions, when $\gamma \rightarrow \infty$, the γ -median will be greater than the mean and the mode, when $\gamma \rightarrow 0$, the γ -median will be smaller than the mean and the mode.

Data Availability. Data for Figure ?? are given in SI Dataset S1. All codes have been deposited in [GitHub](#).

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