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Supporting Information for

Semiparametric robust mean estimations based on the orderliness of quantile averages

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Supporting text
Figs. S1 to S2
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SI References

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Dataset S1

Supporting Information Text

Methods

The asymptotic biases of all robust location estimators proposed in this article for the Weibull, gamma, Pareto, and lognormal distributions were computed within specified kurtosis ranges and standardized by the standard deviations of the respective distributions. For Weibull, gamma, and lognormal distributions, the kurtosis range is from 3.1 to 15 in steps of 0.1 (there are two shape parameter solutions for the Weibull distribution, the lower one is used here). For Pareto, the range is from 9.1 to 21.

To approximate the asymptotic results, a quasi-Monte Carlo study (1, 2) was conducted by generating a large quasi-random sample with sample size 3.686 million for corresponding distributions and quasi-subsampling the sample 3.686 million times (3) to approximate the distributions of the kernels of the weighted Hodges-Lehmann mean. The accuracies of these results were checked by comparing the sample mean and mean H-L mean to the population mean. The errors were found to be smaller than 0.001σ.

For the generalized Gaussian distribution with a kurtosis range from 3.1 to 15, the standard errors of all estimators were computed by approximating the sampling distribution using 1000 pseudorandom samples of size $n = 4096$. Common random numbers were used for better comparison.

The complete definition of the stratified mean

When $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 1$, the ϵ,b-stratified mean is defined as

$$\mathrm{SM}_{\epsilon,b,n} := \frac{1}{n}\left(\sum_{i=\frac{\left(b\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor -1\right)n\epsilon}{b-1}+1}^{n\left(1-\frac{\left(b\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor -1\right)\epsilon}{b-1}\right)} X_i + \sum_{i=\frac{\left(b\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor -1\right)n\epsilon}{b-1}+1}^{\frac{b\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor n\epsilon}{b-1}} (X_i + X_{n-i+1})(b-1) + \sum_{j=1}^{\frac{1}{2}\left(\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor -1\right)} \sum_{i_j=\frac{(2bj-b-1)n\epsilon}{b-1}+1}^{\frac{(2bj-b+1)n\epsilon}{b-1}} (X_{i_j} + X_{n-i_j+1})b \right).$$

When $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 0$, the definition is

$$\mathrm{SM}_{\epsilon,b,n} := \frac{1}{n}\left(\sum_{i=\frac{bn\epsilon}{b-1}\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor+1}^{n\left(1-\frac{b\epsilon}{b-1}\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor\right)} X_i + \sum_{j=1}^{\frac{1}{2}\left\lfloor \frac{b-1}{2b\epsilon}\right\rfloor} \sum_{i_j=\frac{(2bj-b-1)n\epsilon}{b-1}+1}^{\frac{(2bj-b+1)n\epsilon}{b-1}} (X_{i_j} + X_{n-i_j+1})b \right).$$

In situations where $i \bmod 1 \neq 0$, a potential solution is to generate t smaller samples that satisfy the equality by sampling without replacement, the sample sizes are divided into two cases, 1, sampling the original sample $(\lfloor n\epsilon \rfloor \frac{1}{\epsilon} \bmod 1)t$ times with sample size $\lfloor \lfloor n\epsilon \rfloor \frac{1}{\epsilon} \rfloor$, 2, sampling the original sample $(1 - \lfloor n\epsilon \rfloor \frac{1}{\epsilon} \bmod 1)t$ times with sample size $\lfloor \lfloor n\epsilon \rfloor \frac{1}{\epsilon} \rfloor$. Since rational numbers are closed under multiplication and division, excluding division by zero, the continuity can always be ensured as long as t is large enough.

Orderliness and weighted average inequality

Unlike the mean-median-mode inequality, for which computing necessary and sufficient conditions is often challenging, the following result highlights another advantage of the trimming inequality.

Theorem 0.1. *A necessary and sufficient condition of the γ-trimming inequality for a right-skewed distributions is the monotonic decreasing of the bias of trimmed mean as a function of the breakdown point ϵ.*

Proof. From the definition of γ-trimming inequality, since $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}$, $\mathrm{TM}_{\epsilon_1,\gamma} \geq \mathrm{TM}_{\epsilon_2,\gamma}$, therefore

$$\frac{\mathrm{TM}_{\epsilon_1,\gamma} - \mu}{\sigma} \geq \frac{\mathrm{TM}_{\epsilon_2,\gamma} - \mu}{\sigma} \iff \mathrm{B}_{\mathrm{TM}_{\epsilon_1,\gamma}} \geq \mathrm{B}_{\mathrm{TM}_{\epsilon_2,\gamma}} \iff \mathrm{B}_{\mathrm{TM}}(\epsilon_1, \gamma) \geq \mathrm{B}_{\mathrm{TM}}(\epsilon_2, \gamma),$$

which implies the monotonic decreasing of $\mathrm{B}_{\mathrm{TM}}(\epsilon, \gamma)$ with respect to ϵ. □

The bias function is free of scale parameter, so the derivatives are much easier to compute. A useful sufficient condition for the γ-trimming inequality is γ-orderliness.

Theorem 0.2. *A sufficient condition of the γ-trimming inequality for a right-skewed distribution is γ-orderliness.*

Proof. The γ-trimming inequality is equivalent to, $\forall 0 \le \epsilon \le \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du \ge \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$, where δ is an infinitesimal quantity.

Then, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du = \lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon+2)}\sum_{i=n\gamma\epsilon}^{n(1-\epsilon)+1} X_i\right) \ge$
$\lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon+2)}\left(X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}+\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)\right) =$
$\lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon)}\left(\left(\frac{n-n\epsilon-n\gamma\epsilon}{n-n\epsilon-n\gamma\epsilon+2}\right)\left(X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}\right)+\left(\frac{n-n\epsilon-n\gamma\epsilon}{n-n\epsilon-n\gamma\epsilon+2}\right)\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)\right) =$
$\lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon)}\left(2\left(\frac{n-n\epsilon-n\gamma\epsilon}{n-n\epsilon-n\gamma\epsilon+2}\right)\left(X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}\right)+\left(\frac{n-n\epsilon-n\gamma\epsilon}{n-n\epsilon-n\gamma\epsilon+2}\right)\sum_{i=n\gamma\epsilon+2}^{n(1-\epsilon)-1} X_i\right)\right) =$
$\lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon)}\left(X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}+\left(\frac{n-n\epsilon-n\gamma\epsilon-4}{n-n\epsilon-n\gamma\epsilon+2}\right)\left(X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}\right)+\left(\frac{n-n\epsilon-n\gamma\epsilon}{n-n\epsilon-n\gamma\epsilon+2}\right)\sum_{i=n\gamma\epsilon+2}^{n(1-\epsilon)-1} X_i\right)\right) \ge$
$\lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon)}\left(X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}+\left(\frac{2}{n-n\epsilon-n\gamma\epsilon+2}\right)\sum_{i=n\gamma\epsilon+2}^{n(1-\epsilon)-1} X_i+\left(\frac{n-n\epsilon-n\gamma\epsilon}{n-n\epsilon-n\gamma\epsilon+2}\right)\sum_{i=n\gamma\epsilon+2}^{n(1-\epsilon)-1} X_i\right)\right) =$
$\lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon)}\left(X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}+\sum_{i=n\gamma\epsilon+2}^{n(1-\epsilon)-1} X_i\right)\right) =$
$\lim_{n\to\infty}\left(\frac{1}{(n-n\epsilon-n\gamma\epsilon)}\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right) = \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du.$ □

Theorem 0.3. *For a right-skewed continuous distribution following the γ-orderliness, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ.*

Proof. For a distribution following the γ-orderliness,
$\lim_{n\to\infty}\left((1+\gamma)\frac{1}{n-n\epsilon-n\gamma\epsilon}\left(\left(\frac{n\gamma}{1+\gamma}-n\gamma\epsilon\right)X_{n\gamma\epsilon+1}+\left(n(1-\epsilon)-\frac{n\gamma}{1+\gamma}\right)X_{n(1-\epsilon)}\right) \ge (1+\gamma)\frac{1}{n-n\epsilon-n\gamma\epsilon}\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$
$\iff \lim_{n\to\infty}\left(\left(\frac{1+\gamma}{1-\epsilon-\gamma\epsilon}\right)\left(\frac{\gamma-(1+\gamma)\gamma\epsilon}{1+\gamma}\right)X_{n\gamma\epsilon+1}+\left(\frac{1+\gamma}{1-\epsilon-\gamma\epsilon}\right)\left(\frac{(1-\epsilon)(1+\gamma)}{1+\gamma}-\frac{\gamma}{1+\gamma}\right)X_{n(1-\epsilon)} \ge (1+\gamma)\frac{1}{n-n\epsilon-n\gamma\epsilon}\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$
$\iff \lim_{n\to\infty}\left(\frac{\gamma-\gamma\epsilon-\gamma^2\epsilon}{1-\epsilon-\gamma\epsilon}X_{n\gamma\epsilon+1}+\left(\frac{1-\epsilon-\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right)X_{n(1-\epsilon)} \ge (1+\gamma)\frac{1}{n-n\epsilon-n\gamma\epsilon}\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$
$\iff \lim_{n\to\infty}\left(\left(\gamma X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}\right) \ge (1+\gamma)\frac{1}{n-n\epsilon-n\gamma\epsilon}\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$
$\iff \lim_{n\to\infty}\left((n-n\epsilon-n\gamma\epsilon)\left(\gamma X_{n\gamma\epsilon+1}+X_{n(1-\epsilon)}\right) \ge (1+\gamma)\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$
$\iff \lim_{n\to\infty}\left(\frac{n-n\epsilon-n\gamma\epsilon}{n}\left(n\epsilon\gamma X_{n\gamma\epsilon+1}+n\epsilon X_{n(1-\epsilon)}\right) \ge \frac{n\epsilon+n\gamma\epsilon}{n}\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$
$\iff \lim_{n\to\infty}\left(\frac{n-n\epsilon-n\gamma\epsilon}{n}\left(n\epsilon\gamma X_{n\gamma\epsilon+1}+n\epsilon X_{n(1-\epsilon)}+\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right) \ge \sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$
$\iff \lim_{n\to\infty}\left(\frac{1}{n}\left(n\epsilon\gamma X_{n\gamma\epsilon+1}+n\epsilon X_{n(1-\epsilon)}+\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right) \ge \frac{1}{n-n\epsilon-n\gamma\epsilon}\sum_{i=n\gamma\epsilon+1}^{n(1-\epsilon)} X_i\right)$, the proof is finished.
□

Theorem 0.4. *A sufficient condition of the γ-Winsorization inequality for a right-skewed distribution is the monotonic decreasing of the quantile average function with respect to the breakdown point ϵ, if $0 \le \gamma \le 1$.*

Proof. The γ-Winsorization inequality is equivalent to,
$\forall 0 \le \epsilon \le \frac{1}{1+\gamma}$, $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du + (\gamma\epsilon-\delta)Q(\gamma\epsilon-\delta)+(\epsilon-\delta)Q(1-\epsilon+\delta) \ge \int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon Q(\gamma\epsilon)+\epsilon Q(1-\epsilon)$, where δ is an infinitesimal quantity.

Then, similar to Theorem 0.2, by deducing
$\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du+(\gamma\epsilon-\delta)Q(\gamma\epsilon-\delta)+(\epsilon-\delta)Q(1-\epsilon+\delta) = 2\int_{\epsilon-\delta}^{\frac{1}{1+\gamma}} QA(u)\,du+(\gamma\epsilon-\delta)Q(\gamma\epsilon-\delta)+(\epsilon-\delta)Q(1-\epsilon+\delta)$
$= \lim_{n\to\infty}\left(\frac{1}{n}\left(\sum_{i=n\gamma\epsilon}^{\frac{\gamma}{1+\gamma}n+1} X_i+\sum_{i=n\epsilon-1}^{\frac{1}{1+\gamma}n} X_{n-i}+(n\gamma\epsilon-1)X_{n\gamma\epsilon}+(n\epsilon-1)X_{n-n\epsilon+1}\right)\right)$
$\ge \lim_{n\to\infty}\left(\frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{\frac{\gamma}{1+\gamma}n} X_i+\sum_{i=n\epsilon}^{\frac{1}{1+\gamma}n} X_{n-i}+X_{n\gamma\epsilon+1}+X_{n-n\epsilon}+(n\gamma\epsilon-1)X_{n\gamma\epsilon}+(n\epsilon-1)X_{n-n\epsilon+1}\right)\right)$
$\ge \lim_{n\to\infty}\left(\frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{\frac{\gamma}{1+\gamma}n} X_i+\sum_{i=n\epsilon}^{\frac{1}{1+\gamma}n} X_{n-i}+X_{n\gamma\epsilon+1}+X_{n-n\epsilon}+(n\gamma\epsilon-1)X_{n\gamma\epsilon+1}+(n\epsilon-1)X_{n-n\epsilon}\right)\right)$
$= \lim_{n\to\infty}\left(\frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{\frac{\gamma}{1+\gamma}n} X_i+\sum_{i=n\epsilon}^{\frac{1}{1+\gamma}n} X_{n-i}+(n\gamma\epsilon)X_{n\gamma\epsilon}+(n\epsilon)X_{n-n\epsilon}\right)\right) = \int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du+\gamma\epsilon Q(\gamma\epsilon)+\epsilon Q(1-\epsilon)$, the proof is complete. □

Theorem 0.5. *A necessary and sufficient condition of the νth γ-orderliness is the sign of the νth derivative of the bias function of the quantile average with respect to the breakdown point ϵ satisfy the same constraint.*

Proof. The proof is analogous to Theorem 0.1. □

Then, the orderliness for parametric distributions will be discussed. For simplicity, $0 < \epsilon < \frac{1}{2}$ is assumed in the following proofs unless otherwise specified.

Theorem 0.6. *The Weibull distribution is ordered if the shape parameter $\alpha \leqslant \frac{1}{1-\ln(2)} \approx 3.259$.*

Proof. The pdf of the Weibull distribution is $f(x)=\frac{\alpha e^{-\left(\frac{x}{\lambda}\right)^{\alpha}}\left(\frac{x}{\lambda}\right)^{\alpha-1}}{\lambda}$, $x\geq 0$, the quantile function is $F^{-1}(p)=\lambda(-\ln(1-p))^{1/\alpha}$, $1\geq p\geq 0$, $\alpha>0,\lambda>0$. Then, the standardized bias of a symmetric quantile average with a breakdown point ϵ, is

$$\mathrm{B}_{\mathrm{SQA}}(\epsilon,\alpha)=\frac{\mathrm{SQA}_{\epsilon}-\mu}{\sigma}=\frac{\frac{1}{2}\left(\lambda(-\ln(1-\epsilon))^{1/\alpha}+\lambda(-\ln(\epsilon))^{1/\alpha}\right)-\lambda\Gamma\left(1+\frac{1}{\alpha}\right)}{\sqrt{\lambda^2\left(\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2\right)}}.$$

$\frac{\partial \mathrm{B}_{\mathrm{SQA}}}{\partial\epsilon}=\frac{\frac{(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1}}{\alpha-\alpha\epsilon}+\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon\ln(\epsilon)}}{2\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$. Let $g(\epsilon,\alpha)=\frac{(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1}}{\alpha-\alpha\epsilon}+\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon\ln(\epsilon)}=-\left(-\ln(1-\epsilon)\right)^{\frac{1}{\alpha}}\left(\left(1-\epsilon\right)\left(\ln\left(1-\epsilon\right)\right)\right)^{-1}+\left(-\ln(\epsilon)\right)^{\frac{1}{\alpha}}\left(\epsilon\ln(\epsilon)\right)^{-1}$. Arranging the equation $g(\epsilon,\alpha)=0$, it can be shown that $\frac{\epsilon}{(1-\epsilon)}=\left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)^{\frac{1}{\alpha}-1}$. Let $L(\epsilon)=\frac{\epsilon}{(1-\epsilon)}$, $R(\epsilon,\alpha)=\left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)^{\frac{1}{\alpha}-1}$, $LmR(\epsilon,\alpha)=L(\epsilon,\alpha)-R(\epsilon,\alpha)$, then $\frac{\partial LmR}{\partial\alpha}=\frac{\ln\left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)\left(\frac{\ln(\epsilon)}{\ln(1-\epsilon)}\right)^{\frac{1}{\alpha}-1}}{\alpha^2}$. For $0<\epsilon<\frac{1}{2}$, $\frac{\partial LmR}{\partial\alpha}>0$, so $LmR(\epsilon,\alpha)$ is monotonic with respect to α. When $\alpha=\frac{1}{1-\ln(2)}$, $g(\epsilon)=-\frac{1}{\epsilon(-\ln(\epsilon))^{\ln(2)}}+\frac{1}{(1-\epsilon)(-\ln(1-\epsilon))^{\ln(2)}}$ Let $h(\epsilon)=\epsilon\left(-\ln(\epsilon)\right)^{\ln(2)}$), $h'(\epsilon)=\frac{(-\ln(\epsilon))^{\ln(2)}\ln(2\epsilon)}{\ln(\epsilon)}$, for $0<\epsilon<e^{-ln(2)}=\frac{1}{2}$, $h'(\epsilon)>0$. As a result, $h(\epsilon)$ is monotonic increasing, $-h(1-\epsilon)$ is monotonic increasing, $h(\epsilon)-h(1-\epsilon)$ is also monotonic increasing. So, if $0<\epsilon<\frac{1}{2}$, $h(\epsilon)-h(1-\epsilon)<h\left(\frac{1}{2}\right)-h\left(1-\frac{1}{2}\right)=0$, $g(\epsilon,\alpha)<0$. So, $\frac{\partial \mathrm{B}_{\mathrm{SQA}}}{\partial\epsilon}<0$, $\mathrm{B}_{\mathrm{SQA}}(\epsilon,\alpha)$ is monotonic decreasing in ϵ when $\alpha\leqslant\frac{1}{1-\ln(2)}$. The assertion follows from Theorem 0.5. □

Remark. The Weibull distribution can be symmetric. Its skewness is $\tilde{\mu}_3=\frac{2\Gamma\left(1+\frac{1}{\alpha}\right)^3-3\Gamma\left(1+\frac{2}{\alpha}\right)\Gamma\left(1+\frac{1}{\alpha}\right)+\Gamma\left(1+\frac{3}{\alpha}\right)}{\left(\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2\right)^{3/2}}$. Denote the solution of $\tilde{\mu}_3=0$ as $\alpha_0\approx 3.602$. The above proof implies that when α is close to α_0, the bias function of SQA is no longer monotonic.

Then, the bias function of the trimmed mean for the Weibull distribution can be expressed as

$$\mathrm{B}_{\mathrm{TM}}(\epsilon,\alpha)=\frac{\frac{\Gamma\left(1+\frac{1}{\alpha},-\ln(1-\epsilon)\right)-\Gamma\left(1+\frac{1}{\alpha},-\ln(\epsilon)\right)}{1-2\epsilon}-\Gamma\left(1+\frac{1}{\alpha}\right)}{\sqrt{\Gamma\left(1+\frac{2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}.$$

Numerical solutions and the plot in Figure S1 indicate that, when $\alpha=3.259,\epsilon=0.1$, $\frac{\partial \mathrm{B}_{\mathrm{TM}}}{\partial\epsilon}\approx -0.164$; when $\epsilon=0.4$, $\frac{\partial \mathrm{B}_{\mathrm{TM}}}{\partial\epsilon}\approx -0.001$; and when $\epsilon=0.499$, $\frac{\partial \mathrm{B}_{\mathrm{TM}}}{\partial\epsilon}\approx 7.66\times 10^{-8}$. Thus, the bias function of the trimmed mean can also be non-monotonic when $\alpha\geq\frac{1}{1-\ln(2)}$.

Additionally, the second derivative of the bias function of SQA for the Weibull distribution is, $\frac{\partial^2 \mathrm{B}_{\mathrm{SQA}}}{\partial^2\epsilon}=\frac{-\frac{(-\ln(\epsilon))^{\frac{1}{\alpha}-1}}{\alpha^2\epsilon^2\ln(\epsilon)}-\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon^2\ln^2(\epsilon)}-\frac{(-\ln(\epsilon))^{1/\alpha}}{\alpha\epsilon^2\ln(\epsilon)}+\frac{\left(\frac{1}{\alpha}-1\right)(-\ln(1-\epsilon))^{\frac{1}{\alpha}-2}}{(1-\epsilon)(\alpha-\alpha\epsilon)}+\frac{\alpha(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1}}{(\alpha-\alpha\epsilon)^2}}{2\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$. Numerical solutions show that when $\alpha<3$, $0<\epsilon<\frac{1}{2}$, $\frac{\partial^2 \mathrm{B}_{\mathrm{SQA}}}{\partial^2\epsilon}>0$. The flip of signs occurs when α is close to $\frac{1}{1-\ln(2)}\approx 3.259$. When $\alpha=\frac{1}{1-\ln(2)},\epsilon=0.1$, $\frac{\partial^2 \mathrm{B}_{\mathrm{SQA}}}{\partial^2\epsilon}\approx 3.259$; when $\epsilon=0.4$, $\frac{\partial^2 \mathrm{B}_{\mathrm{SQA}}}{\partial^2\epsilon}\approx 0.020$; when $\epsilon=0.5$, $\frac{\partial^2 \mathrm{B}_{\mathrm{SQA}}}{\partial^2\epsilon}\approx -7.34\times 10^{-16}$. A plot of $\frac{\partial^2 \mathrm{B}_{\mathrm{SQA}}}{\partial^2\epsilon}$ for $0.25<\epsilon<0.5$ is given in Figure S1.

Then, the third derivative of the SQA for the Weibull distribution is, $\frac{\partial^3 \mathrm{B}_{\mathrm{SQA}}}{\partial^3\epsilon}=-\frac{(\alpha-1)(2\alpha-1)(\epsilon-1)^3(-\ln(\epsilon))^{\frac{1}{\alpha}-3}-3(\alpha-1)\alpha(\epsilon-1)^3(-\ln(\epsilon))^{\frac{1}{\alpha}-2}}{2\alpha^3(\epsilon-1)^3\epsilon^3\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$ $-\frac{2\alpha^2\epsilon^3(-\ln(1-\epsilon))^{\frac{1}{\alpha}-1}+2\alpha^2(\epsilon-1)^3(-\ln(\epsilon))^{\frac{1}{\alpha}-1}+(1-\alpha)(1-2\alpha)\epsilon^3(-\ln(1-\epsilon))^{\frac{1}{\alpha}-3}+3(1-\alpha)\alpha\epsilon^3(-\ln(1-\epsilon))^{\frac{1}{\alpha}-2}}{2\alpha^3(\epsilon-1)^3\epsilon^3\sqrt{\Gamma\left(\frac{\alpha+2}{\alpha}\right)-\Gamma\left(1+\frac{1}{\alpha}\right)^2}}$. Numerical solutions show that when $\alpha<3$, $0<\epsilon<\frac{1}{2}$, $\frac{\partial^3 \mathrm{B}_{\mathrm{SQA}}}{\partial^3\epsilon}<0$. The flip of signs occurs when $\alpha=3.471$.. When $\epsilon=0.4$, $\frac{\partial^3 \mathrm{B}_{\mathrm{SQA}}}{\partial^3\epsilon}\approx -0.0218$; when $\epsilon=0.499$, $\frac{\partial^3 \mathrm{B}_{\mathrm{SQA}}}{\partial^3\epsilon}\approx 4.928\times 10^{-5}$. A plot of $\frac{\partial^3 \mathrm{B}_{\mathrm{SQA}}}{\partial^3\epsilon}$ for $0.3<\epsilon<0.5$ is given in Figure S1.

Because the kurtosis range used here for the Weibull distribution is discrete and starts from 3.1, the corresponding α is 2.133, so among the kurtosis range discussed here, the numerical results show that the Weibull distribution follows the first three orderlinesses.

The pdf of the gamma distribution is $f(x)=\frac{\lambda^{-\alpha}x^{\alpha-1}e^{-\frac{x}{\lambda}}}{\Gamma(\alpha)}$, $x\geq 0$, the quantile function is $Q(p)=\lambda P^{-1}(\alpha,p)$, $1\geq p\geq 0$, $\alpha>0,\lambda>0$, P is the regularized incomplete gamma function. So, $E[X]=\int_{-\infty}^{\infty}xf(x)\,dx=\alpha\lambda$. Similarly, the variance is $\alpha\lambda^2$. Then, the standardized bias of a symmetric quantile average with a breakdown point ϵ, is

$$\mathrm{B}_{\mathrm{SQA}}(\epsilon,\alpha)=\frac{\mathrm{SQA}_{\epsilon}-\mu}{\sigma}=\frac{\frac{1}{2}(\lambda P^{-1}(\alpha,1-\epsilon)+\lambda P^{-1}(\alpha,\epsilon))-\alpha\lambda}{\sqrt{\alpha\lambda^2}}.$$

$\frac{\partial \mathrm{B}_{\mathrm{SQA}}}{\partial\epsilon}=\frac{\Gamma(a)}{2\sqrt{\alpha}}(e^{P^{-1}(\alpha,\epsilon)}P^{-1}(\alpha,\epsilon)^{1-\alpha}-e^{P^{-1}(\alpha,1-\epsilon)}P^{-1}(\alpha,1-\epsilon)^{1-\alpha})$. It is trivial to show that when $\alpha\leq 1$, $P^{-1}(\alpha,\epsilon)$ is monotonic increasing in ϵ, if $0<\epsilon<\frac{1}{2}$. Then $\frac{\partial \mathrm{B}_{\mathrm{SQA}}}{\partial\epsilon}<0$, $\mathrm{B}_{\mathrm{SQA}}(\epsilon,\alpha)$ is monotonic decreasing in ϵ over the interval $(0,\frac{1}{2})$.

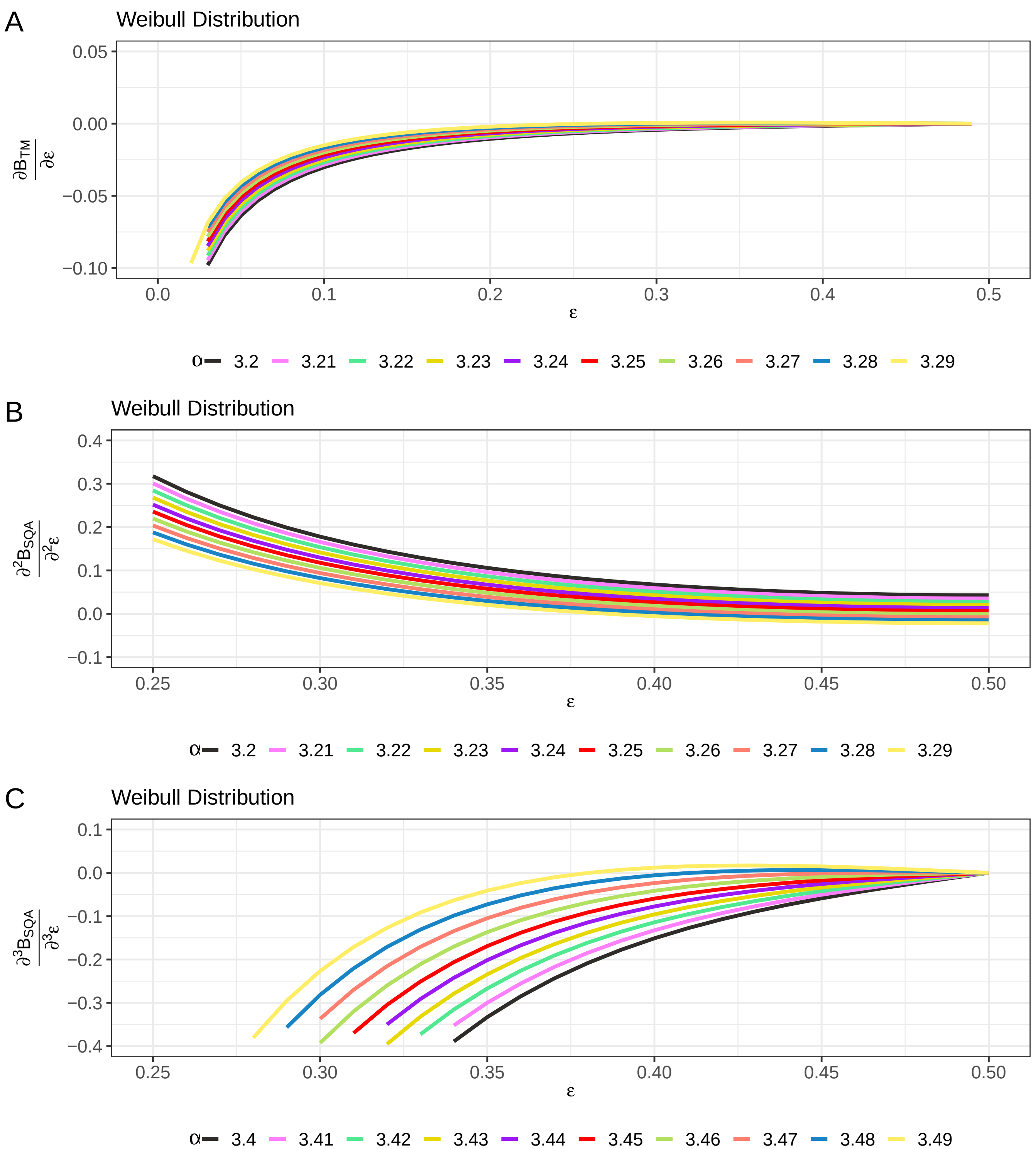

Fig. S1. A. The first derivative of the bias function of TM for the Weibull distribution with respect to the breakdown point ϵ. B. The second derivative of the bias function of SQA for the Weibull distribution with respect to the breakdown point ϵ. C. The third derivative of the bias function of SQA for the Weibull distribution with respect to the breakdown point ϵ.

However, the analytical analysis of $\alpha > 1$ is hard. Numerical results shows that the flip of signs of $\frac{\partial \mathrm{B_{SQA}}}{\partial \epsilon}$ occurs when $\alpha \approx 139.5$ (Figure S2). The second derivative of the bias function for the gamma distribution is $\frac{\partial^2 \mathrm{B_{SQA}}}{\partial^2 \epsilon} = \frac{\Gamma(\alpha)^2}{2\sqrt{\alpha}}((1-\alpha)e^{2P^{-1}(\alpha,1-\epsilon)}P^{-1}(\alpha,1-\epsilon)^{1-2\alpha} + e^{2P^{-1}(\alpha,1-\epsilon)}P^{-1}(\alpha,1-\epsilon)^{2-2\alpha} + (1-\alpha)e^{2P^{-1}(\alpha,\epsilon)}P^{-1}(\alpha,\epsilon)^{1-2\alpha} + e^{2P^{-1}(\alpha,\epsilon)}P^{-1}(\alpha,\epsilon)^{2-2\alpha})$. The flip of signs of $\frac{\partial^2 \mathrm{B_{SQA}}}{\partial^2 \epsilon}$ occurs when $\alpha \approx 78$ (Figure S2). The third derivative is much more cumbersome; numerical results show that the flip of signs of $\frac{\partial^3 \mathrm{B_{SQA}}}{\partial^3 \epsilon}$ occurs when $\alpha \approx 55$ (Figure S2). Since the kurtosis range of the gamma distribution here is discrete and starts from 3.1, the corresponding α is 60. The second point is $\alpha = 30$. Besides the first point, the numerical results show that the gamma distribution follows the first three orderlinesses within the kurtosis setting here.

For the lognormal distribution, the pdf of it is $f(x) = \frac{e^{-\frac{(\ln(x)-\mu)^2}{2\sigma^2}}}{\sqrt{2\pi}\sigma x}$, $x \geq 0$, the quantile function is $Q(p) = e^{\mu - \sqrt{2}\sigma \mathrm{erfc}^{-1}(2p)}$, $1 \geq p \geq 0$, $\sigma > 0, \lambda > 0$. So, $E[X] = \int_{-\infty}^{\infty} x f(x)\,dx = e^{\mu + \frac{\sigma^2}{2}}$. Similarly, the variance is $\left(e^{\sigma^2} - 1\right)e^{2\mu+\sigma^2}$. Then, the standardized bias of a symmetric quantile average with a breakdown point ϵ, is

$$\mathrm{B_{SQA}}(\epsilon,\sigma) = \frac{\frac{1}{2}\left(e^{\mu-\sqrt{2}\sigma\mathrm{erfc}^{-1}(2\epsilon)} + e^{\mu-\sqrt{2}\sigma\mathrm{erfc}^{-1}(2(1-\epsilon))}\right) - e^{\mu+\frac{\sigma^2}{2}}}{\sqrt{\left(e^{\sigma^2}-1\right)e^{2\mu+\sigma^2}}}.$$

The first two orderlinesses for the lognormal distribution were already discussed in the Main Text, the numerical results show that the third orderliness is also valid within the kurtosis setting here.

Bias bound

As stated in the Main Text, Bernard et al. (2020) (4) derived the bias bound of the symmetric quantile average for $\mathcal{P}_U$,

$$B_{\mathrm{SQA}}(\epsilon) = \frac{|\mathrm{SQA}_\epsilon - \mu|}{\sigma} = \begin{cases} \frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{3\epsilon}{4-3\epsilon}}\right) & \frac{1}{6} \geq \epsilon \geq 0 \\ \frac{1}{2}\left(\sqrt{\frac{1-\epsilon}{\epsilon+\frac{1}{3}}} + \sqrt{\frac{3\epsilon}{4-3\epsilon}}\right) & \frac{1}{2} \geq \epsilon > \frac{1}{6}. \end{cases}$$

Theorem 0.7. *Extending the bound to the quantile average, there are two main cases for the upper bound:*

Main Case 1: *If* $5 > \gamma \geq 0$,

$$\sup_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon,\gamma) = \begin{cases} \frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & \frac{1}{6} \geq \epsilon \geq 0 \\ \frac{1}{2}\left(\sqrt{\frac{3(1-\epsilon)}{4-3(1-\epsilon)}} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & \frac{1}{1+\gamma} \geq \epsilon > \frac{1}{6}. \end{cases} \tag{1}$$

Main Case 2: *If* $\gamma \geq 5$,

$$\sup_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon,\gamma) = \begin{cases} \frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & \frac{5}{6\gamma} \geq \epsilon \geq 0 \\ \frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{4}{9(1-\gamma\epsilon)}-1}\right) & \frac{1}{1+\gamma} \geq \epsilon > \frac{5}{6\gamma}. \end{cases} \tag{2}$$

Proof. Since $\sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon} \frac{1}{2}(Q(\gamma\epsilon) + Q(1-\epsilon)) \leq \frac{1}{2}(\sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon} Q(\gamma\epsilon) + \sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon} Q(1-\epsilon))$, the assertion follows directly from the Theorem 1 in (4). ☐

Since the lower bound is for the left skewed distribution, so using the second definition of the quantile average is better. From the Corollary 1 in (4), there are also two main cases, which are identical to the opposite of the above upper bound if using the second definition of the quantile average:

Main Case 1: If $5 > \gamma \geq 0$,

$$\inf_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} \mathrm{QA}(\epsilon,\gamma) = \begin{cases} -\frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & \frac{1}{6} \geq \epsilon \geq 0 \\ -\frac{1}{2}\left(\sqrt{\frac{3(1-\epsilon)}{4-3(1-\epsilon)}} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & \frac{1}{1+\gamma} \geq \epsilon > \frac{1}{6}. \end{cases} \tag{3}$$

Main Case 2: If $\gamma \geq 5$,

$$\inf_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} \mathrm{QA}(\epsilon,\gamma) = \begin{cases} -\frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & \frac{5}{6\gamma} \geq \epsilon \geq 0 \\ -\frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1} + \sqrt{\frac{4}{9(1-\gamma\epsilon)}-1}\right) & \frac{1}{1+\gamma} \geq \epsilon > \frac{5}{6\gamma}. \end{cases} \tag{4}$$

Bernard et al. (2020) (4) also investigated the bias bounds of Range Value at Risk (5), which is

$$RVaR_{\alpha,\beta} = \frac{1}{\beta-\alpha}\int_\alpha^\beta VaR(u)\,du, 0 < \alpha < \beta < 1,$$

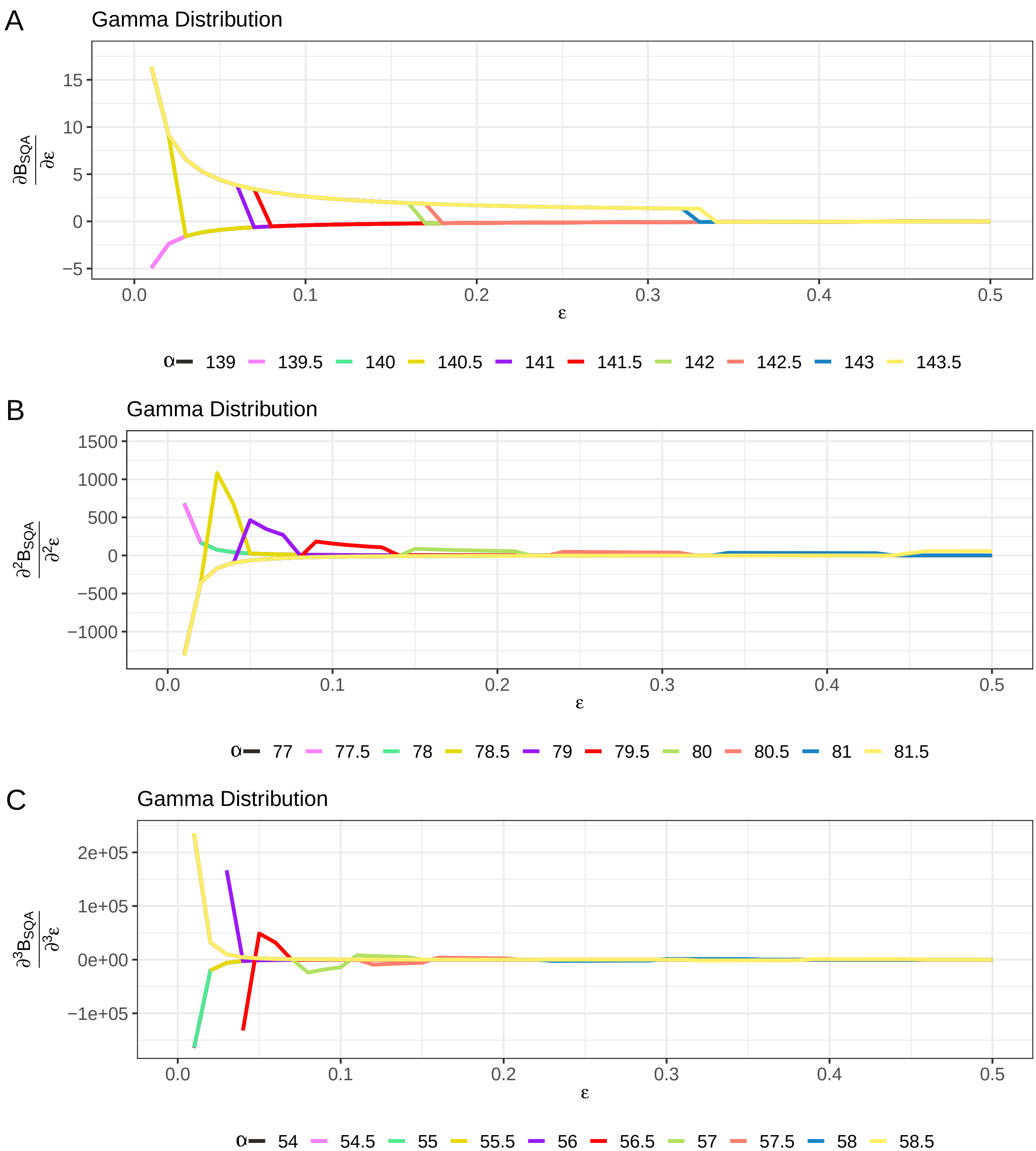

Fig. S2. A. The first derivative of the bias function of SQA for the gamma distribution with respect to the breakdown point ϵ. B. The second derivative of the bias function of SQA for the gamma distribution with respect to the breakdown point ϵ. C. The third derivative of the bias function of SQA for the gamma distribution with respect to the breakdown point ϵ.

where $VaR(u) = \inf\{x \in \mathbb{R} : F_U(x) \geq u\}$. They pointed out that $VaR(u)$ is the quantile function, and it is obvious here that $RVaR_{\alpha,\beta}$ is the trimmed mean. Let $\alpha = \gamma\epsilon$, $\beta = 1-\epsilon$, then because of asymmetry, the upper bound $\sup\limits_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} RVaR_{\alpha=\gamma\epsilon,\beta=1-\epsilon}$ and lower bound $\inf\limits_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} RVaR_{\alpha=\gamma\epsilon,\beta=1-\epsilon}$ are not exactly opposite values. Also, they are very complex in form. If setting $\gamma = 1$, they are opposite values, i.e., the bias bound of symmetric trimmed mean is

$$B_{\mathrm{STM}}(\epsilon) = \frac{|\mathrm{STM}_\epsilon - \mu|}{\sigma} = \frac{\epsilon\left(9\epsilon^2+\left(4-3\sqrt{9\epsilon^2+4}\right)\epsilon-\sqrt{9\epsilon^2+4}+2\right)}{(2\epsilon-1)\sqrt{-\frac{81\epsilon^4}{2}+3\left(4\sqrt{9\epsilon^2+4}-9\right)\epsilon^2+6\left(\sqrt{9\epsilon^2+4}-2\right)\epsilon+\frac{4}{3}\left(\sqrt{9\epsilon^2+4}-2\right)+\frac{9}{2}\left(3\sqrt{9\epsilon^2+4}-8\right)\epsilon^3}}.$$

Theorem 0.8. *The above bias bound function, $B_{STM}(\epsilon)$, is monotonic increasing with respect to ϵ over the interval $(0, \frac{1}{2})$.*

Proof. $$\frac{dB_{\mathrm{STM}}(\epsilon)}{d\epsilon} = \frac{2\sqrt{6}\left(-5832\epsilon^7+232\left(\sqrt{9\epsilon^2+4}-2\right)\epsilon+32\left(\sqrt{9\epsilon^2+4}-2\right)+324\left(6\sqrt{9\epsilon^2+4}-25\right)\epsilon^6\right)}{(1-2\epsilon)^2\sqrt{9\epsilon^2+4}\left(-243\epsilon^4+18\left(4\sqrt{9\epsilon^2+4}-9\right)\epsilon^2+36\left(\sqrt{9\epsilon^2+4}-2\right)\epsilon+8\left(\sqrt{9\epsilon^2+4}-2\right)+27\left(3\sqrt{9\epsilon^2+4}-8\right)\epsilon^3\right)^{3/2}} +$$
$$\frac{2\sqrt{6}\left(2\left(397\sqrt{9\epsilon^2+4}-830\right)\epsilon^2+54\left(50\sqrt{9\epsilon^2+4}-171\right)\epsilon^5+9\left(294\sqrt{9\epsilon^2+4}-779\right)\epsilon^4+9\left(193\sqrt{9\epsilon^2+4}-444\right)\epsilon^3\right)}{(1-2\epsilon)^2\sqrt{9\epsilon^2+4}\left(-243\epsilon^4+18\left(4\sqrt{9\epsilon^2+4}-9\right)\epsilon^2+36\left(\sqrt{9\epsilon^2+4}-2\right)\epsilon+8\left(\sqrt{9\epsilon^2+4}-2\right)+27\left(3\sqrt{9\epsilon^2+4}-8\right)\epsilon^3\right)^{3/2}}.$$
Let $g(\epsilon) = -5832\epsilon^7 + 2\left(397\sqrt{9\epsilon^2+4} - 830\right)\epsilon^2 + 232\left(\sqrt{9\epsilon^2+4} - 2\right)\epsilon + 32\left(\sqrt{9\epsilon^2+4} - 2\right) + 324\left(6\sqrt{9\epsilon^2+4} - 25\right)\epsilon^6 + 54\left(50\sqrt{9\epsilon^2+4} - 171\right)\epsilon^5 + 9\left(294\sqrt{9\epsilon^2+4} - 779\right)\epsilon^4 + 9\left(193\sqrt{9\epsilon^2+4} - 444\right)\epsilon^3$ and $h(\epsilon)$ denotes the common denominator of $\frac{dB_{\mathrm{STM}}(\epsilon)}{d\epsilon}$. Then, for $0 < \epsilon < \frac{1}{2}$, $h(\epsilon) > 0$. To have $g(\epsilon) > 0$, it is equivalent to $2 \times 397\sqrt{9\epsilon^2+4}\epsilon^2 + 232\sqrt{9\epsilon^2+4}\epsilon + 32\sqrt{9\epsilon^2+4} + 324\epsilon^6 \times 6\sqrt{9\epsilon^2+4} + 54\epsilon^5 \times 50\sqrt{9\epsilon^2+4} + 9\epsilon^4 \times 294\sqrt{9\epsilon^2+4} + 9\epsilon^3 \times 193\sqrt{9\epsilon^2+4} > 5832\epsilon^7 + 2 \times 830\epsilon^2 + 2 \times 232\epsilon + 32 \times 2 + 25 \times 324\epsilon^6 + 54\epsilon^5 \times 171 + 9\epsilon^4 \times 779 + 444 \times 9\epsilon^3$. Squaring the left and right sides and then expanding, it is equivalent to $34012224\epsilon^{14} + 94478400\epsilon^{13} + 173315376\epsilon^{12} + 231367104\epsilon^{11} + 245524284\epsilon^{10} + 213603804\epsilon^9 + 155238849\epsilon^8 + 94957380\epsilon^7 + 48836664\epsilon^6 + 20951856\epsilon^5 + 7364752\epsilon^4 + 2051968\epsilon^3 + 427776\epsilon^2 + 59392\epsilon + 4096 > 34012224\epsilon^{14} + 94478400\epsilon^{13} + 173315376\epsilon^{12} + 231367104\epsilon^{11} + 245454300\epsilon^{10} + 213576588\epsilon^9 + 155256345\epsilon^8 + 94952088\epsilon^7 + 48850488\epsilon^6 + 20954880\epsilon^5 + 7361296\epsilon^4 + 2051968\epsilon^3 + 427776\epsilon^2 + 59392\epsilon + 4096 \Longleftrightarrow 69984\epsilon^{10} + 27216\epsilon^9 - 17496\epsilon^8 + 5292\epsilon^7 - 13824\epsilon^6 - 3024\epsilon^5 + 3456\epsilon^4 > 0 \Longleftrightarrow 108(1-2\epsilon)^2\epsilon^4\left(162\epsilon^4 + 225\epsilon^3 + 144\epsilon^2 + 100\epsilon + 32\right) > 0$. Then just need $162\epsilon^4 + 225\epsilon^3 + 144\epsilon^2 + 100\epsilon + 32 > 0$. Since $144\epsilon^2 + 100\epsilon + 32 > 0$ is valid for any $\epsilon \in \mathbb{R}$, $g(\epsilon) > 0$ is valid for any $\epsilon > 0$. So, $\frac{dB_{\mathrm{STM}}(\epsilon)}{d\epsilon} > 0$, which leads to the assertion of the theorem. □

Another interesting case is when $\gamma = \frac{1}{2}$, setting the $\mu = 0$, $\sigma = 1$, $\sup\limits_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} RVaR_{\alpha=\frac{1}{2}\epsilon,\beta=1-\epsilon} = \dfrac{1}{3}\sqrt{\dfrac{1}{2}\epsilon\left(\dfrac{3}{2}\epsilon+8\right)}$, obviously monotonic, and $\inf\limits_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} RVaR_{\alpha=\frac{1}{2}\epsilon,\beta=1-\epsilon} = -\dfrac{1}{3}\sqrt{\epsilon(8+3\epsilon)}$, also obviously monotonic.

SI Dataset S1 (dataset_one.xlsx)

Raw data of asymptotic biases of all estimators shown in Figure 1 in the Main Text and the standard errors of these estimators for the generalized Gaussian distribution.

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