Semiparametric robust mean estimations based on the orderliness of quantile averages

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As one of the most fundamental problems in statistics, robust location estimation has many prominent solutions, such as the symmetric trimmed mean, symmetric Winsorized mean, Hodges–Lehmann estimator, Huber M-estimator, and median of means. Recent studies suggest that their maximum biases concerning the mean can be quite different in asymmetric distributions, but the underlying mechanisms and average performance remain largely unclear. In this article, similar to the mean-median-mode inequality, it is proven that within the context of nearly all common unimodal distributions, there is an orderliness of symmetric quantile averages with varying breakdown points. Further deductions explain why the Winsorized mean and median of means typically have smaller biases compared to the trimmed mean. Building on the U-orderliness, the superiority of the median Hodges–Lehmann mean is discussed.

semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

In 1823, Gauss (1) proved that for any unimodal distribution with a finite second moment, $|m-\mu| \leq \sqrt{\frac{3}{4}}\omega$, where μ is the population mean, m is the population median, and ω is the root mean square deviation from the mode, M. This pioneering work revealed that despite potential bias with respect to the mean in robust estimates, the deviation remains bounded in unit of a scale parameter under certain assumptions. Bernard, Kazzi, and Vanduffel (2020) (2) further derived asymptotic bias bounds of any quantile for unimodal distributions with finite second moments by reducing this optimization problem to a parametric one, which can be solved analytically. They showed that the population median, m, has the smallest maximum distance to the population mean, μ, among all symmetric quantile averages (SQA_ϵ). Daniell, in 1920, (3) analyzed a class of estimators, linear combinations of order statistics, and identified that ϵ-symmetric trimmed mean (STM_ϵ) belongs to this class. Another popular choice, the ϵ-symmetric Winsorized mean (SWM_ϵ), named after Winsor and introduced by Tukey (4) and Dixon (5) in 1960, is also an L-estimator. Bieniek (2016) derived exact bias upper bounds of the Winsorized mean based on Danielak and Rychlik's work (2003) on the trimmed mean for any distribution with a finite second moment and confirmed that the former is smaller than the latter (6, 7). In 1963, Hodges and Lehmann (8) proposed a class of nonparametric location estimators based on rank tests and, from the Wilcoxon signed-rank statistic (9), deduced the median of pairwise means as a robust location estimator for a symmetric population. Both L-statistics and R-statistics achieve robustness essentially by removing a certain proportion of extreme values. In 1964, Huber (10) generalized maximum likelihood estimation to the minimization of the sum of a specific loss function, which measures the residuals between the data points and the model's parameters. Some L-estimators are also M-estimators, e.g., the sample mean is an M-estimator with a squared error loss function, the sample median is an M-estimator with an absolute error loss function (10). The Huber M-estimator is obtained by applying the Huber loss function that combines elements of both squared error and absolute error to achieve robustness against gross errors and high efficiency for contaminated Gaussian distributions (10). Sun, Zhou, and Fan (2020) examined the concentration bounds of Huber M-estimator (11). Mathieu (2022) (12) further derived the concentration bounds of M-estimators and demonstrated that, by selecting the tuning parameter which depends on the variance, Huber M-estimator can also be a sub-Gaussian estimator. The concept of median of means ($\mathrm{MoM}_{k,b=\frac{n}{k}}$, k is the number of size in each block, b is the number of blocks) was implicitly introduced several times in Nemirovsky and Yudin (1983) (13), Jerrum, Valiant, and Vazirani (1986), (14) and Alon, Matias and Szegedy (1996) (15)'s works. Given its good performance even for distributions with infinite second moments, MoM has received increasing attention over the past decade (16–18). Devroye, Lerasle, Lugosi, and Oliveira (2016) showed that MoM nears the optimum of sub-Gaussian mean estimation with regards to concentration bounds when the distribution has a heavy tail (17). For a comparison of concentration bounds of trimmed mean, Huber M-estimator, median of means and other relavent estimators, readers are directed to Gobet, Lerasle, and Métivier's paper (2022) (19). Laforgue, Clemencon, and Bertail (2019) proposed the median of randomized means ($\mathrm{MoRM}_{k,b}$) (18), wherein, rather than partitioning, an arbitrary number, b, of blocks are built independently from the sample, and showed that MoRM has a better non-asymptotic sub-Gaussian property compared to MoM. In fact, asymptotically, the Hodges-Lehmann (H-L) estimator is equivalent to $\mathrm{MoM}_{k=2,b=\frac{n}{k}}$ and $\mathrm{MoRM}_{k=2,b}$, and they can be seen as the pairwise mean distribution is approximated by the sampling without replacement and bootstrap, respectively. For the asymptotic validity, readers are referred to the foundational works of Efron (1979) (20), Bickel and

Significance Statement

In 1964, van Zwet introduced the convex transformation order for comparing the skewness of two distributions. This paradigm shift played a fundamental role in defining robust measures of distributions, from spread to kurtosis. Here, instead of examining the stochastic ordering between two distributions, the orderliness of quantile averages within a distribution is investigated. By classifying distributions through the signs of derivatives, a series of sophisticated robust mean estimators are deduced. Nearly all common nonparametric robust location estimators are found to be special cases thereof.

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Freedman (1981, 1984) (21, 22), and Helmers, Janssen, and Veraverbeke (1990) (23).

Here, the ϵ,b-stratified mean is defined as

$$\mathrm{SM}_{\epsilon,b,n} := \frac{b}{n}\left(\sum_{j=1}^{\frac{b-1}{2b\epsilon}} \sum_{i_j=\frac{(2bj-b-1)n\epsilon}{b-1}+1}^{\frac{(2bj-b+1)n\epsilon}{b-1}} X_{i_j}\right),$$

where $X_1 \leq \ldots \leq X_n$ denote the order statistics of a sample of n independent and identically distributed random variables $X_1, \ldots, X_n$. $b \in \mathbb{N}$, $b \geq 3$. The definition was further refined to guarantee the continuity of the breakdown point by incorporating an additional block in the center when $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 0$, or by adjusting the central block when $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 1$ (SI Text). If the subscript n is omitted, only the asymptotic behavior is considered. If b is omitted, $b = 3$ is assumed. $\mathrm{SM}_{\epsilon,b=3}$ is equivalent to STM_{ϵ}, when $\epsilon > \frac{1}{6}$. The basic idea of the stratified mean, when $\frac{b-1}{2\epsilon} \in \mathbb{N}$, $b \bmod 2 = 1$, is to distribute the data into $\frac{b-1}{2\epsilon}$ equal-sized non-overlapping blocks according to their order, then further sequentially group these blocks into b equal-sized strata and compute the mean of the middle stratum, which is the median of means of each stratum. In situations where $i \bmod 1 \neq 0$, a potential solution is to generate multiple smaller samples that satisfy the equality by sampling without replacement, and subsequently calculate the mean of all estimations. The details of determining the sample size and sampling times are provided in the SI Text. Although the principle resembles that of the median of means, without the random shift, $\mathrm{SM}_{\epsilon,b,n}$ is different from $\mathrm{MoM}_{k=\frac{n}{b},b}$. Additionally, the stratified mean differs from the mean of the sample obtained through stratified sampling methods, introduced by Neyman (1934) (24) or ranked set sampling (25), introduced by McIntyre in 1952, as these sampling methods aim to obtain more representative samples or improve the efficiency of sample estimates, but the sample means based on them are not robust. When $b \bmod 2 = 1$, the stratified mean can be regarded as replacing the other equal-sized strata with the middle stratum, which, in principle, is analogous to the Winsorized mean that replaces extreme values with less extreme percentiles. Furthermore, while the bounds confirm that the Winsorized mean and median of means outperform the trimmed mean (6, 7, 17, 19) in worst-case performance, the complexity of bound analysis makes it difficult to achieve a complete and intuitive understanding of these results. Also, a clear explanation for the average performance of them remains elusive. The aim of this paper is to define a series of semi-parametric models using the signs of derivatives, reveal their elegant interrelations and connections to parametric models, and show that by exploiting these models, a set of sophisticated mean estimators can be deduced, which exhibit strong robustness to departures from assumptions.

Quantile average and weighted average

The symmetric trimmed mean, symmetric Winsorized mean, and stratified mean are all L-estimators. More specifically, they are symmetric weighted averages, which are defined as

$$\mathrm{SWA}_{\epsilon,n} := \frac{\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{X_i + X_{n-i+1}}{2} w_i}{\sum_{i=1}^{\lceil \frac{n}{2} \rceil} w_i},$$

where w_is are the weights applied to the symmetric quantile averages according to the definition of the corresponding L-estimators. For example, for the ϵ-symmetric trimmed mean, $w_i = \begin{cases} 0, & i < n\epsilon \\ 1, & i \geq n\epsilon \end{cases}$, provided that $n\epsilon \in \mathbb{N}$. The mean and median are indeed two special cases of the symmetric trimmed mean.

To extend the symmetric quantile average to the asymmetric case, there are two possible definitions for the ϵ,γ-quantile average ($\mathrm{QA}(\epsilon, \gamma, n)$), i.e.,

$$\frac{1}{2}(\hat{Q}_n(\gamma\epsilon) + \hat{Q}_n(1-\epsilon)), \qquad [1]$$

and

$$\frac{1}{2}(\hat{Q}_n(\epsilon) + \hat{Q}_n(1-\gamma\epsilon)), \qquad [2]$$

where $\gamma \geq 0$ and $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\hat{Q}_n(p)$ is the empirical quantile function. For trimming from both sides, [1] and [2] are essentially equivalent. [1] is assumed in this article unless otherwise specified, since many common asymmetric distributions are right-skewed, and [1] allows trimming only from the right side by setting $\gamma = 0$.

Analogously, the weighted average can be defined as

$$\mathrm{WA}_{\epsilon,\gamma,n} := \frac{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} \mathrm{QA}\left(\epsilon_0, \gamma, n\right) w_{\epsilon_0}}{\int_{\epsilon_0=0}^{\frac{1}{1+\gamma}} w_{\epsilon_0}}.$$

For instance, the ϵ,γ-trimmed mean ($\mathrm{TM}_{\epsilon,\gamma,n}$) is a weighted average with a left trim size of $\gamma\epsilon n$ and a right trim size of ϵn, where $w_{\epsilon_0} = \begin{cases} 0, & \epsilon_0 < \epsilon \\ 1, & \epsilon_0 \geq \epsilon \end{cases}$. Using this definition, even $\gamma\epsilon n \notin \mathbb{N}$ or $\epsilon n \notin \mathbb{N}$, the TM computation remains unaltered since this definition is based on the empirical quantile function. However, considering the computational cost in practice, here, the non-asymptotic definitions of various types of weighted averages, in most cases, are essentially based on order statistics. The solution to the decimal issue of them is the same as that in SM, unless stated otherwise.

Classifying distributions by the signs of derivatives

Let $\mathcal{P}_k$ denote the set of all distributions over $\mathbb{R}$ whose moments, from the first to the kth, are all finite. Without loss of generality, the discussion of all the classes outlined below is restricted to the intersection with the nonparametric class of distributions $\mathcal{P}_{\Upsilon}^{k} := \{\text{All continuous distribution } P \in \mathcal{P}_k\}$. Besides fully and smoothly parameterizing by a Euclidean parameter or just assuming regularity conditions, there are many ways to classify distributions. In 1956, Stein initiated the problem of estimating parameters in the presence of an infinite dimensional nuisance shape parameter (26). A notable example discussed in his groundbreaking work was the estimation of the center of symmetry for an unknown symmetric distribution. In 1993, Bickel, Klaassen, Ritov, and Wellner published an influential semiparametrics textbook (27) which systematically categorized many common models into three classes: parametric, nonparametric, and semiparametric. Yet, there is another old and commonly encountered class of distributions that receives little attention in semiparametric literature: the unimodal distribution. It is a very unique semiparametric model because its definition is based on the signs of derivatives, i.e., for a continuous distribution, $(f'(x) > 0 \text{ for } x \leq M) \wedge (f'(x) < 0 \text{ for } x \geq M)$. Let $\mathcal{P}_U$ denote the set of all unimodal distributions. There was a widespread misbelief that the median of an arbitrary unimodal distribution always lies between its mean and mode until Runnenburg (1978) and van Zwet (1979) (28, 29) endeavored to determine sufficient conditions for the inequality to hold, thereby implying the possibility of its violation. The class of distributions that satisfy the mean-median-mode inequality constitutes a subclass of $\mathcal{P}_U$. By analogy, a right-skewed distribution is called γ-ordered, if and only if

$$\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \mathrm{QA}_{\epsilon_1,\gamma} \geq \mathrm{QA}_{\epsilon_2,\gamma}.$$

The necessary and sufficient condition below hints at the relation between the mean-median-mode inequality and the γ-orderliness.

Theorem .1. *Let P_Υ^k represent an arbitrary distribution in the set $\mathcal{P}_\Upsilon^k$. P_Υ^k is γ-ordered if and only if the probability density function (pdf) satisfies the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ or $f(Q(\gamma\epsilon)) \leq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, where $\gamma \geq 0$.*

Proof. Without loss of generality, consider the case of right-skewed continuous distribution. From the definition of γ-orderliness, it is deduced that $\frac{Q(\gamma\epsilon-\delta)+Q(1-\epsilon+\delta)}{2} \geq \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \Leftrightarrow Q(\gamma\epsilon-\delta)-Q(\gamma\epsilon) \geq Q(1-\epsilon)-Q(1-\epsilon+\delta) \Leftrightarrow Q'(1-\epsilon) \geq Q'(\gamma\epsilon)$, where δ is an infinitesimal positive quantity. Observing that the quantile function is the inverse function of the cumulative distribution function (cdf), $Q'(1-\epsilon) \geq Q'(\gamma\epsilon) \Leftrightarrow F'(Q(\gamma\epsilon)) \geq F'(Q(1-\epsilon))$, thereby completing the proof, given that the derivative of cdf is pdf. □

According to Theorem .1, if a probability distribution is right-skewed and monotonic, it will always be γ-ordered, provided $\gamma \geq 0$. For a right-skewed continuous unimodal distribution, if $Q(\gamma\epsilon) > M$, the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ holds. The principle is extendable to unimodal-like distributions. Suppose there is a right-skewed continuous multimodal distribution following the mean-γ-median-first mode inequality with several smaller modes on the right side, with the first mode, M, having the greatest probability density, and the γ-median, $Q(\frac{\gamma}{1+\gamma})$, falling within the first dominant mode (i.e., if $x > Q(\frac{\gamma}{1+\gamma})$, $f(Q(\frac{\gamma}{1+\gamma})) \geq f(x)$), then if $Q(\gamma\epsilon) > M$, the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ also holds. In other words, while a distribution following the mean-γ-median-mode inequality may not be strictly γ-ordered, the inequality that defines γ-orderliness remains valid for most quantile averages. The mean-γ-median-mode inequality can also indicate possible bounds for γ in practice, e.g., for any distributions, when $\gamma \to \infty$, the γ-median will be greater than the mean and the mode, when $\gamma \to 0$, the γ-median will be smaller than the mean and the mode.

Consider the sign of the derivative of the quantile average with respect to the breakdown point, the above definition of γ-orderliness can also be expressed as

$$\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}, \frac{\partial \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon} \leq 0.$$

The left-skewed case can be obtained by reversing the inequality $\frac{\partial \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon} \leq 0$ to $\frac{\partial \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon} \geq 0$ and employing the second definition of QA, as given in [2]; for simplicity, it will be omitted in the following discussion. If $\gamma = 1$, the γ-ordered distribution is referred to as ordered.

Furthermore, many common right-skewed distributions are partial bounded, indicating a convex behavior of the QA function when $\epsilon \to 0$. By further assuming convexity, the second γ-orderliness can be defined as follows for a right-skewed distribution,

$$\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}, \frac{\partial^2 \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon^2} \geq 0 \wedge \frac{\partial \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon} \leq 0.$$

Analogously, the νth γ-orderliness of a right-skewed distribution can be defined as $(-1)^\nu \frac{\partial^\nu \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon^\nu} \geq 0 \wedge \ldots \wedge -\frac{\partial \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon} \geq 0$. If $\gamma = 1$, the νth γ-orderliness is referred as νth orderliness. Let $\mathcal{P}_O$ denote the set of all distributions that are ordered and $\mathcal{P}_{O_\nu}$ and $\mathcal{P}_{\gamma O_\nu}$ represent the sets of all distributions that are νth ordered and νth γ-ordered, respectively. When the shape parameter of the Weibull distribution, α, is smaller than 3.258, it can be shown that the Weibull distribution belong to $\mathcal{P}_U \cap \mathcal{P}_O \cap \mathcal{P}_{O_2} \cap \mathcal{P}_{O_3}$ (SI Text). At $\alpha \approx 3.602$, the Weibull distribution is symmetric, and as $\alpha \to \infty$, the skewness of the Weibull distribution reaches 1. Therefore, the parameters that let it not be included in the set correspond to cases when it is near-symmetric, as shown in the SI Text. Nevertheless, computing the derivatives of the QA function is often intricate and, at times, challenging. The following theorems establish the relationship between $\mathcal{P}_O$, $\mathcal{P}_{O_\nu}$, and $\mathcal{P}_{\gamma O_\nu}$, and a wide range of other semi-parametric distributions. They can be used to quickly identify some parametric distributions in $\mathcal{P}_O$, $\mathcal{P}_{O_\nu}$, and $\mathcal{P}_{\gamma O_\nu}$.

Theorem .2. *For any random variable X whose probability distribution function belongs to a location-scale family, the distribution is νth γ-ordered if and only if the family of probability distributions is νth γ-ordered.*

Proof. A continuous distribution belonging to a location–scale family, parametrized by a location parameter μ and a scale parameter λ, takes the form $F(x) = F_0\left(\frac{x-\mu}{\lambda}\right)$, where F_0 is a standard distribution without any shifts or scaling. Consequently, after a location-scale transformation, $F(x) = Q^{-1}(x) \to x = Q(p) = \lambda Q_0(p) + \mu$. According to the definition of the νth γ-orderliness, the signs of derivatives of the QA function are invariant after this transformation. As the location-scale transformation is reversible, the proof is complete. □

Theorem .2 demonstrates that in the analytical proof of the νth γ-orderliness of a parametric distribution, both the location and scale parameters can be regarded as constants. It is also instrumental in proving other theorems, as illustrated below.

Theorem .3. *Any symmetric distribution with a finite second moment is νth ordered.*

Proof. Without loss of generality, assuming continuity and $m = 0$. A symmetric distribution is a probability distribution such that for all x, $f(x) = f(-x)$. Its cdf satisfies $F(x) = 1 - F(-x)$. Let $x = Q(p)$, then, $F(Q(p)) = p = 1 - F(-Q(p))$ and $F(Q(1-p)) = 1 - p \Leftrightarrow p = 1 - F(Q(1-p))$. Therefore, $F(-Q(p)) = F(Q(1-p))$. Since the cdf is monotonic, $-Q(p) =$

$Q(1-p) \Leftrightarrow Q(p)+Q(1-p)=0$. As a result, all symmetric quantile averages coincide; the νth order derivative is zero. The case of $m \neq 0$ follows directly from Theorem .2. $\square$

As a consequence of Theorem .3 and the fact that generalized Gaussian distribution is symmetric around the median, it is νth ordered.

Theorem .4. *Any continuous right-skewed distribution whose quantile function Q satisfies $Q^{(\nu)}(p) \geq 0 \wedge \ldots Q^{(i)}(p) \geq 0 \ldots \wedge Q^{(2)}(p) \geq 0$, i mod $2=0$, is νth γ-ordered, provided that $0 \leq \gamma \leq 1$.*

Proof. Since $(-1)^i \frac{\partial^i \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon^i} = \frac{1}{2}((-\gamma)^i Q^i(\gamma\epsilon) + Q^i(1-\epsilon))$, for $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ and $1 \leq i \leq \nu$, when $i \bmod 2 = 0$, $(-1)^i \frac{\partial^i \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon^i} \geq 0$ for all $\gamma \geq 0$. When $i \bmod 2 = 1$, if further assuming $0 \leq \gamma \leq 1$, $(-1)^i \frac{\partial^i \mathrm{QA}_{\epsilon,\gamma}}{\partial \epsilon^i} \geq 0$, since $Q^{(i+1)}(p) \geq 0$. $\square$

It is now straightforward to prove that the Pareto distribution follows the νth γ-orderliness, provided that $0 \leq \gamma \leq 1$, since the quantile function of the Pareto distribution is $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, where $x_m > 0$, $\alpha > 0$, and so $Q^{(\nu)}(p) \geq 0$ for all $\nu \in \mathbb{N}$ according to the chain rule.

Theorem .5. *A right-skewed continuous distribution with a monotonic decreasing pdf is second γ-ordered.*

Proof. A monotonic decreasing pdf implies $f'(x) = F^{(2)}(x) \leq 0$. Since $Q'(p) \geq 0$, let $x = Q(F(x))$, then by differentiating both sides of the equation twice, one can obtain $0 = Q^{(2)}(F(x))(F'(x))^2 + Q'(F(x))F^{(2)}(x) \Leftrightarrow Q^{(2)}(F(x)) = -\frac{Q'(F(x))F^{(2)}(x)}{(F'(x))^2} \geq 0$. The desired result is derived from Theorem .1 and .4. $\square$

Theorem .5 provides valuable insights into the relation between modality and orderliness. The conventional definition states that a distribution with a monotonic pdf is still considered unimodal. However, within its supported interval, the mode number is zero. The number of modes and their magnitudes within a distribution are closely related to the possibility of orderliness being valid, although counterexamples can always be constructed for non-monotonic distributions. A proof of the second γ-orderliness, if $\gamma > 0$, can be easily established for the gamma distributions when $\alpha \leq 1$ as the pdf of the gamma distribution is $f(x) = \frac{\lambda^{-\alpha} x^{\alpha-1} e^{-\frac{x}{\lambda}}}{\Gamma(\alpha)}$, where $x \geq 0$, $\lambda > 0$, $\alpha > 0$, Γ is the gamma function, it is a product of two monotonic decreasing functions under constraints. For $\alpha > 1$, the proof becomes challenging. Numerical results show that the orderliness is valid until $\alpha > 140$, the second orderliness is valid until $\alpha > 78$, and the third orderliness is valid until $\alpha > 55$ (SI Text). It is instructive to consider that when $\alpha \to \infty$ the gamma distribution converges to a Gaussian distribution with mean $\mu = \alpha\lambda$ and variance $\sigma = \alpha\lambda^2$. The skewness of the gamma distribution, $\frac{\alpha+2}{\sqrt{\alpha(\alpha+1)}}$, is monotonic with respect to α, since $\frac{\partial \tilde{\mu}_3(\alpha)}{\partial \alpha} = \frac{-3\alpha-2}{2(\alpha(\alpha+1))^{3/2}} < 0$. When $\alpha = 55$, $\tilde{\mu}_3(\alpha) = 1.027$. Theorefore, similar to the Weibull distribution, the parameters that let the distribution not be included in $\mathcal{P}_U \cap \mathcal{P}_O \cap \mathcal{P}_{O_2} \cap \mathcal{P}_{O_3}$ also correspond to cases when it is near-symmetric.

Theorem .6. *Consider a symmetric random variable X. Let it be transformed using a function $\phi(x)$ such that $\phi^{(2)}(x) \geq 0 \wedge \phi'(x) \geq 0$ over the interval supported, the resulting convex transformed distribution is ordered. Moreover, if the quantile function of X satifies $Q^{(2)}(\epsilon) \leq 0$, the convex transformed distribution is second ordered.*

Proof. Let $\phi\mathrm{SQA}(\epsilon) = \frac{1}{2}(\phi(Q(\epsilon)) + \phi(Q(1-\epsilon)))$, $0 \leq \epsilon \leq \frac{1}{2}$. Then, $\frac{d\phi\mathrm{SQA}}{d\epsilon} = \frac{1}{2}(\phi'(Q(\epsilon))Q'(\epsilon) - \phi'(Q(1-\epsilon))Q'(1-\epsilon)) = \frac{1}{2}Q'(\epsilon)(\phi'(Q(\epsilon)) - \phi'(Q(1-\epsilon))) \leq 0$, since for a symmetric distribution, $m - Q(\epsilon) = Q(1-\epsilon) - m$, differentiating both sides, $-Q'(\epsilon) = -Q'(1-\epsilon)$, where $Q'(\epsilon) \geq 0, \phi^{(2)}(x) \geq 0$. If further differentiating the equality, $Q^{(2)}(\epsilon) = -Q^{(2)}(1-\epsilon)$. Since $\frac{d^{(2)}\phi\mathrm{SQA}}{d\epsilon^{(2)}} = \frac{1}{2}\left(\phi^2(Q(\epsilon))(Q'(\epsilon))^2 + \phi^2(Q(1-\epsilon))(Q'(1-\epsilon))^2\right) + \frac{1}{2}\left(\phi'(Q(\epsilon))\left(Q^2(\epsilon)\right) + \phi'(Q(1-\epsilon))\left(Q^2(1-\epsilon)\right)\right) = \frac{1}{2}\left(\left(\phi^{(2)}(Q(\epsilon)) + \phi^{(2)}(Q(1-\epsilon))\right)(Q'(\epsilon))^2\right) + \frac{1}{2}\left((\phi'(Q(\epsilon)) - \phi'(Q(1-\epsilon)))Q^{(2)}(\epsilon)\right)$. If $Q^{(2)}(\epsilon) \leq 0$, $\frac{d^{(2)}\phi\mathrm{SQA}}{d\epsilon^{(2)}} \geq 0$. $\square$

The mean-median-mode inequality for distributions of the powers and roots of the variates of a given distribution was investigated by Henry Rietz in 1927 (30), but the most straightforward solution is the exponential transformation since the derivatives are invariably positive. An application of Theorem .6 is that the lognormal distribution is ordered as it is exponentially transformed from the Gaussian distribution. The quantile function of the Gaussian distribution meets the condition $Q^{(2)}(\epsilon) = -2\sqrt{2}\pi\sigma e^{2\mathrm{erfc}^{-1}(2\epsilon)^2}\mathrm{erfc}^{-1}(2\epsilon) \leq 0$, where σ is the standard deviation, erfc denotes the complementary error function. Thus, the lognormal distribution is second ordered. Numerical results suggest that it is also third ordered, although an analytical proof is challenging.

Theorem .6 also reveals a relation between convex transformation and orderliness, since ϕ is the non-decreasing convex function in van Zwet's trailblazing work *Convex transformations of random variables* (31). Consider a near-symmetric distribution S, such that SQA_ϵ as a function of ϵ fluctuates from 0 to $\frac{1}{2}$, with $\mu = m$. By definition, S is not ordered. Let s be the pdf of S. Applying the transformation $\phi(x)$ to S decreases $s(Q_S(\epsilon))$, and the decrease rate, due to the order, is much smaller for $s(Q_S(1-\epsilon))$. As a consequence, as the second derivative of $\phi(x)$ increases, eventually, after a point, $s(Q_S(\epsilon))$ becomes greater than $s(Q_S(1-\epsilon))$ even if it was not previously. Thus, the SQA_ϵ function becomes monotonically decreasing, and S becomes ordered. Accordingly, in a family of distributions that differ by a skewness-increasing transformation in van Zwet's sense, violations of orderliness typically occur only when the distribution is near-symmetric.

Pearson proposed using the mean-median difference $\mu - m$ as a measure of skewness after standardization in 1895 (32). Bowley (1926) proposed a measure of skewness based on the SQA-median difference $\mathrm{SQA}_\epsilon - m$ (33). Groeneveld and Meeden (1984) (34) generalized these measures of skewness based on van Zwet's convex transformation (31) while exploring their properties. A distribution is called monotonically right-skewed if and only if $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{2}, \mathrm{SQA}_{\epsilon_1} - m \geq \mathrm{SQA}_{\epsilon_2} - m$. Since m is a constant, the monotonic skewness is equivalent to the orderliness. For a nonordered distribution, the signs of

$\mathrm{SQA}_\epsilon - m$ with different breakdown points might be different, implying that some skewness measures indicate left-skewed distribution, while others suggest right-skewed distribution. Although it seems reasonable that such a distribution is likely be generally near-symmetric, however, counterexamples can be constructed. For example, consider the Weibull distribution, when $\alpha > \frac{1}{1-\ln(2)}$, it is near-symmetric and nonordered, the non-monotonicity of the SQA function arises when ϵ is close to $\frac{1}{2}$. Replacing the third quartile with one from a right-skewed heavy-tailed distribution leads to a right-skewed, heavy-tailed, and nonordered distribution. Therefore, the validity of robust measures of skewness based on the SQA-median difference is closely related to the orderliness of the distribution.

Remarkably, in 2020, Bernard et al. (2) proved the bias bounds of any quantile for $P \in \mathcal{P}_U \cap \mathcal{P}^2_\Upsilon$. They further derived the bias bound of the symmetric quantile average. Here, let $\mathcal{P}_{\mu,\sigma}$ denotes the set of continuous distributions whose mean is μ and standard deviation is σ, the bias upper bound of the quantile average, $0 \leq \gamma < 5$, is given as

$$\sup_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} \mathrm{QA}(\epsilon,\gamma) = \begin{cases} \frac{1}{2}\left(\sqrt{\frac{4}{9\epsilon}-1}+\sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & 0 \leq \epsilon \leq \frac{1}{6} \\ \frac{1}{2}\left(\sqrt{\frac{3(1-\epsilon)}{4-3(1-\epsilon)}}+\sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}}\right) & \frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma} \end{cases}$$

The proof based on the bias bounds of any quantile (2) and the $\gamma \geq 5$ case are given in the SI Text. The next theorem highlights its safeguarding role in defining estimators based on νth γ-orderliness.

Theorem .7. *The above bias upper bound function, $\sup_{P\in\mathcal{P}_U\cap\mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon,\gamma)$, is monotonic decreasing with respect to ϵ over the interval $[0, \frac{1}{1+\gamma}]$, when $0 \leq \gamma \leq 1$.*

Proof. When $0 \leq \epsilon \leq \frac{1}{6}$, $\frac{\partial \sup \mathrm{QA}(\epsilon,\gamma)}{\partial\epsilon} = \frac{\gamma}{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2} - \frac{1}{3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2}$. When $\gamma = 0$, $\frac{\partial \sup \mathrm{QA}(\epsilon,\gamma)}{\partial\epsilon} = -\frac{1}{3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2} \leq 0$. When $\epsilon \to 0^+$, $\lim_{\epsilon\to 0^+}\left(\frac{\gamma}{(4-3\gamma\epsilon)^2\sqrt{\frac{\epsilon\gamma}{12-9\gamma\epsilon}}} - \frac{1}{3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2}\right) = \lim_{\epsilon\to 0^+}\left(\frac{\gamma\sqrt{3}}{\sqrt{4^3\epsilon\gamma}} - \frac{1}{6\sqrt{\epsilon^3}}\right) \to -\infty$. Assuming $\epsilon > 0$, when $0 < \gamma \leq 1$, to prove $\frac{\partial \sup \mathrm{QA}(\epsilon,\gamma)}{\partial\epsilon} \leq 0$, it is equivalent to showing $\frac{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2}{\gamma} \geq 3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2$. Define $L(\epsilon,\gamma) = \frac{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2}{\gamma}$, $R(\epsilon,\gamma) = 3\sqrt{\frac{4}{\epsilon}-9}\epsilon^2$. $\frac{L(\epsilon,\gamma)}{\epsilon^2} = \frac{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2}{\gamma\epsilon^2} = \frac{1}{\gamma}\left(\frac{4}{\epsilon}-3\gamma\right)^2\sqrt{\frac{1}{\frac{12}{\epsilon\gamma}-9}}$, $\frac{R(\epsilon,\gamma)}{\epsilon^2} = 3\sqrt{\frac{4}{\epsilon}-9}$. Then, $\frac{L(\epsilon,\gamma)}{\epsilon^2} \geq \frac{R(\epsilon,\gamma)}{\epsilon^2} \Leftrightarrow \frac{1}{\gamma}\sqrt{\frac{1}{\frac{12}{\epsilon\gamma}-9}}\left(\frac{4}{\epsilon}-3\gamma\right)^2 \geq 3\sqrt{\frac{4}{\epsilon}-9} \Leftrightarrow \frac{1}{\gamma}\left(\frac{4}{\epsilon}-3\gamma\right)^2 \geq 3\sqrt{\frac{12}{\epsilon\gamma}-9}\sqrt{\frac{4}{\epsilon}-9}$. Let $LmR\left(\frac{1}{\epsilon}\right) = \frac{1}{\gamma^2}\left(\frac{4}{\epsilon}-3\gamma\right)^4 - 9\left(\frac{12}{\epsilon\gamma}-9\right)\left(\frac{4}{\epsilon}-9\right)$. $\frac{\partial LmR(1/\epsilon)}{\partial(1/\epsilon)} = \frac{16\left(\frac{4}{\epsilon}-3\gamma\right)^3}{\gamma^2} - 36\left(\frac{12}{\epsilon\gamma}-9\right) - \frac{108\left(4\frac{4}{\epsilon}-9\right)}{\gamma} = \frac{4\left(4(\frac{4}{\epsilon}-3\gamma)^3-27\gamma(\frac{4}{\epsilon}-3\gamma)+27(9-\frac{4}{\epsilon})\gamma\right)}{\gamma^2} = \frac{4\left(256\frac{1}{\epsilon}^3-576\frac{1}{\epsilon}^2\gamma+432\frac{1}{\epsilon}\gamma^2-216\frac{1}{\epsilon}\gamma-108\gamma^3+81\gamma^2+243\gamma\right)}{\gamma^2}$. Since $256\frac{1}{\epsilon}^3 - 576\frac{1}{\epsilon}^2\gamma + 432\frac{1}{\epsilon}\gamma^2 - 216\frac{1}{\epsilon}\gamma - 108\gamma^3 + 81\gamma^2 + 243\gamma \geq 1536\frac{1}{\epsilon}^2 - 576\frac{1}{\epsilon}^2 + 432\frac{1}{\epsilon}\gamma^2 - 216\frac{1}{\epsilon}\gamma - 108\gamma^3 + 81\gamma^2 + 243\gamma \geq 924\frac{1}{\epsilon}^2 + 36\frac{1}{\epsilon}^2 - 216\frac{1}{\epsilon} + 432\frac{1}{\epsilon}\gamma^2 - 108\gamma^3 + 81\gamma^2 + 243\gamma \geq 924\frac{1}{\epsilon}^2 + 36\frac{1}{\epsilon}^2 - 216\frac{1}{\epsilon} + 513\gamma^2 - 108\gamma^3 + 243\gamma > 0$, $\frac{\partial LmR(1/\epsilon)}{\partial(1/\epsilon)} > 0$. Also, $LmR(6) = \frac{81(\gamma-8)\left((\gamma-8)^3+15\gamma\right)}{\gamma^2} > 0 \Longleftrightarrow \gamma^4 - 32\gamma^3 + 399\gamma^2 - 2168\gamma + 4096 > 0$. Since $\gamma^4 > 0$, if $0 < \gamma \leq 1$, then $32\gamma^3 < 256$, it suffices to prove that $399\gamma^2 - 2168\gamma + 4096 > 256$. Applying the quadratic formula demonstrates the validity of this inequality. Hence, $LmR\left(\frac{1}{\epsilon}\right) \geq 0$ for $\epsilon \in (0, \frac{1}{6}]$, provided that $0 < \gamma \leq 1$. The first part is finished.

When $\frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma}$, $\frac{\partial \sup \mathrm{QA}(\epsilon,\gamma)}{\partial\epsilon} = \sqrt{3}\left(\frac{\gamma}{\sqrt{\gamma\epsilon}(4-3\gamma\epsilon)^{\frac{3}{2}}} - \frac{1}{\sqrt{1-\epsilon}(3\epsilon+1)^{\frac{3}{2}}}\right)$. When $\gamma = 0$, $\frac{\gamma}{\sqrt{\gamma\epsilon}(4-3\gamma\epsilon)^{\frac{3}{2}}} = \frac{\sqrt{\gamma}}{\sqrt{\epsilon}(4-3\gamma\epsilon)^{\frac{3}{2}}} = 0$, $\frac{\partial \sup \mathrm{QA}(\epsilon,\gamma)}{\partial\epsilon} < 0$. For other cases, to determine whether $\frac{\partial \sup_{\mathrm{QA}}(\epsilon,\gamma)}{\partial\epsilon} \leq 0$, since $\sqrt{1-\epsilon}(3\epsilon+1)^{\frac{3}{2}} > 0$ and $\sqrt{\gamma\epsilon}(4-3\gamma\epsilon)^{\frac{3}{2}} > 0$, showing $\frac{\sqrt{\gamma\epsilon}(4-3\gamma\epsilon)^{\frac{3}{2}}}{\gamma} \geq \sqrt{1-\epsilon}(3\epsilon+1)^{\frac{3}{2}} \Leftrightarrow \frac{\gamma\epsilon(4-3\gamma\epsilon)^3}{\gamma^2} \geq (1-\epsilon)(3\epsilon+1)^3 \Leftrightarrow -27\gamma^2\epsilon^4 + 108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} + 27\epsilon^4 - 162\epsilon^2 - 8\epsilon - 1 \geq 0$ is sufficient. When $0 < \gamma \leq 1$, the inequality can be further simplified to $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 \geq 0$. Since $\epsilon \leq \frac{1}{1+\gamma}$, $\gamma \leq \frac{1}{\epsilon} - 1$. Also, as $0 < \gamma \leq 1$, $0 < \gamma \leq \min(1, \frac{1}{\epsilon} - 1)$. When $\frac{1}{6} < \epsilon \leq \frac{1}{2}$, $\frac{1}{\epsilon} - 1 > 1$, so $0 < \gamma \leq 1$. When $\frac{1}{2} \leq \epsilon < 1$, $0 < \gamma \leq \frac{1}{\epsilon} - 1$. Let $h(\gamma) = 108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma}$, $\frac{\partial h(\gamma)}{\partial\gamma} = 108\epsilon^3 - \frac{64\epsilon}{\gamma^2}$. When $\gamma \leq \sqrt{\frac{64\epsilon}{18\epsilon^3}}$, $\frac{\partial h(\gamma)}{\partial\gamma} \geq 0$, when $\gamma \geq \sqrt{\frac{64\epsilon}{18\epsilon^3}}$, $\frac{\partial h(\gamma)}{\partial\gamma} \leq 0$, therefore, the minimum of $h(\gamma)$ must be when γ is equal to the boundary point of the domain. When $\frac{1}{6} < \epsilon \leq \frac{1}{2}$, $0 < \gamma \leq 1$, since $h(0) \to \infty$, $h(1) = 108\epsilon^3 + 64\epsilon$, the minimum occurs at the boundary point $\gamma = 1$, $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 > 108\epsilon^3 + 56\epsilon - 162\epsilon^2 - 1$. Let $g(\epsilon) = 108\epsilon^3 + 56\epsilon - 162\epsilon^2 - 1$. $g'(\epsilon) = 324\epsilon^2 - 324\epsilon + 56$, when $\epsilon \leq \frac{2}{9}$, $g'(\epsilon) \geq 0$, when $\frac{2}{9} \leq \epsilon \leq \frac{1}{2}$, $g'(\epsilon) \leq 0$, since $g(\frac{1}{6}) = \frac{13}{3}$, $g(\frac{1}{2}) = 0$, so $g(\epsilon) \geq 0$, the simplified inequality is satisfied. When $\frac{1}{2} \leq \epsilon < 1$, $0 < \gamma \leq \frac{1}{\epsilon} - 1$. Since $h(\frac{1}{\epsilon} - 1) = 108(\frac{1}{\epsilon} - 1)\epsilon^3 + \frac{64\epsilon}{\frac{1}{\epsilon}-1}$, $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 > 108\left(\frac{1}{\epsilon} - 1\right)\epsilon^3 + \frac{64\epsilon}{\frac{1}{\epsilon}-1} - 162\epsilon^2 - 8\epsilon - 1 = \frac{-108\epsilon^4+54\epsilon^3-18\epsilon^2+7\epsilon+1}{\epsilon-1}$. Let $nu(\epsilon) = -108\epsilon^4 + 54\epsilon^3 - 18\epsilon^2 + 7\epsilon + 1$, then $nu'(\epsilon) = -432\epsilon^3 + 162\epsilon^2 - 36\epsilon + 7$, $nu''(\epsilon) = -1296\epsilon^2 + 324\epsilon - 36 < 0$. Since $nu'(\epsilon = \frac{1}{2}) = -\frac{49}{2} < 0$, $nu'(\epsilon) < 0$. Also, $nu(\epsilon = \frac{1}{2}) = 0$, so $nu(\epsilon) \leq 0$, the simplified inequality is also satisfied. As a result, the simplified inequality is also valid within the range of $\frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 < \gamma \leq 1$. Then, it validates $\frac{\partial \sup \mathrm{QA}(\epsilon,\gamma)}{\partial\epsilon} \leq 0$ for the same range of ϵ and γ.

The first and second formulae, when $\epsilon = \frac{1}{6}$, are all equal to $\frac{1}{2}\left(\frac{\sqrt{\frac{\gamma}{4-\frac{\gamma}{2}}}}{\sqrt{2}} + \sqrt{\frac{5}{3}}\right)$. It follows that $\sup \mathrm{QA}(\epsilon,\gamma)$ is continuous over $[0, \frac{1}{1+\gamma}]$. Hence, $\frac{\partial \sup \mathrm{QA}(\epsilon,\gamma)}{\partial\epsilon} \leq 0$ holds for the entire range $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, when $0 \leq \gamma \leq 1$, which leads to the assertion of this theorem. □

For a right-skewed distribution, considering the upper bound is enough. The monotonicity of $\sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon}$ QA implies that the extent of any violations of the γ-orderliness, if $0 \leq \gamma \leq 1$, is bounded for a unimodal distribution with a finite second moment, e.g., for a right-skewed unimodal distribution in $\mathcal{P}^2_\Upsilon$, if $\exists 0 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \frac{1}{1+\gamma}$, $\mathrm{QA}_{\epsilon_2} \geq \mathrm{QA}_{\epsilon_3} \geq \mathrm{QA}_{\epsilon_1}$, QA_{ϵ_2} will not be too far away from QA_{ϵ_1}, since $\sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon} \mathrm{QA}_{\epsilon_1} > \sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon} \mathrm{QA}_{\epsilon_2} > \sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon} \mathrm{QA}_{\epsilon_3}$. The νth γ-orderliness, when $\nu \geq 2$, corresponds to the higher order derivatives of the QA function, so its violation is also bounded by $\sup_{P\in\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon}\mathrm{QA}$ for $\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon$.

Inequalities related to weighted averages

The bias bound of the ϵ-symmetric trimmed mean also exhibits monotonicity for $\mathcal{P}_U\cap\mathcal{P}^2_\Upsilon$, as proven in the SI Text using the formulae provided in Bernard et al.'s paper (2). So far, it appears clear that the bias of an estimator is closely related to its degree of robustness. In a right-skewed unimodal distribution, it is often observed that $\mu\geq\mathrm{STM}_\epsilon\geq m$. Analogous to the γ-orderliness, the γ-trimming inequality is defined as $\forall 0\leq\epsilon_1\leq\epsilon_2\leq\frac{1}{1+\gamma},\mathrm{TM}_{\epsilon_1,\gamma}\geq\mathrm{TM}_{\epsilon_2,\gamma}$. Since for a location-scale distribution, any $\mathrm{WA}(\epsilon,\gamma)$ can be expressed as $\lambda\mathrm{WA}_0(\epsilon,\gamma)+\mu$, where $\mathrm{WA}_0(\epsilon,\gamma)$ is an integral of $Q_0(p)$ according to the definition of the weighted average. According to Theorem .2, for a probability distribution belongs to a location-scale family, a necessary and sufficient condition for its γ-trimming inequality is whether the family of probability distributions follows the γ-trimming inequality. This condition is the same for other weighted average inequalities. While γ-orderliness is a sufficient condition for the γ-trimming inequality, as proven in the SI Text, it is not necessary.

Theorem .8. *For a distribution that is right-skewed and follows the γ-trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, provided that $0\leq\epsilon\leq\frac{1}{1+\gamma}$ and $\gamma\geq0$.*

Proof. Assume, without loss of generality, that the distribution is continuous. According to the definition of the γ-trimming inequality: $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta}Q(u)\,du\geq\frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du$, if $0\leq\epsilon\leq\frac{1}{1+\gamma}$ and $\gamma\geq0$, where δ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta}Q(u)\,du-\frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du\geq0\Leftrightarrow\int_{1-\epsilon}^{1-\epsilon+\delta}Q(u)\,du+\int_{\gamma\epsilon-\delta}^{\gamma\epsilon}Q(u)\,du-\frac{2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du\geq0$. Since $\delta\to0^+$, $\frac{1}{2\delta}\left(\int_{1-\epsilon}^{1-\epsilon+\delta}Q(u)\,du+\int_{\gamma\epsilon-\delta}^{\gamma\epsilon}Q(u)\,du\right)=\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2}\geq\frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du$, the proof is complete. □

An analogous result can be obtained in the following theorem.

Theorem .9. *For a right-skewed continuous distribution following the γ-trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, provided that $0\leq\epsilon\leq\frac{1}{1+\gamma}$ and $0\leq\gamma\leq1$.*

Proof. According to Theorem .8, $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2}\geq\frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du\Leftrightarrow\gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right)\geq\left(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du$. Then, if $1\geq\gamma\geq0$, $\left(1-\frac{1}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du+\gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right)\geq0\Rightarrow\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du+\gamma\epsilon Q(\gamma\epsilon)+\epsilon Q(1-\epsilon)\geq\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du+\gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right)\geq\frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon}Q(u)\,du$, the proof is complete. □

Assuming γ-orderliness, the result in Theorem .9 can be extended to the $\gamma>1$ case, as proven in the SI Text. Replacing the trimmed mean in the γ-trimming inequality with Winsorized mean forms the definition of the γ-Winsorization inequality. γ-orderliness also implies the γ-Winsorization inequality when $0\leq\gamma\leq1$, as proven in the SI Text.

To construct weighted averages based on the γ-orderliness, let $\mathcal{B}_i=\int_{i\epsilon}^{(i+1)\epsilon}\mathrm{QA}(u,\gamma)\,du$, $ka=k\epsilon+c$. It follows from the γ-orderliness that, $-\frac{\partial\mathrm{QA}_{\epsilon,\gamma}}{\partial\epsilon}\geq0\Rightarrow\forall0\leq a\leq2a\leq\frac{1}{1+\gamma},-\frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a}\geq0\Rightarrow\mathcal{B}_i-\mathcal{B}_{i+1}\geq0$. Suppose $\mathcal{B}_i=\mathcal{B}_0$, then, based on the γ-orderliness, the ϵ,γ-block Winsorized mean, is defined here for comparison in the SI Dataset S1 as

$$\mathrm{BWM}_{\epsilon,\gamma,n}:=\frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n}X_i+\sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1}X_i+\sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n}X_i\right),$$

which is double weighting the leftest and rightest blocks having the size $\gamma\epsilon n$ and ϵn. Since their sizes are different, the $0\leq\gamma\leq1$ is still necessary for the γ-block Winsorization inequality. If γ is omitted, $\gamma=1$ is assumed. This terminology is the same for other weighted averages. The solutions for the $i \bmod 1\neq0$ case are the same as that in the SM. From the second γ-orderliness, $\frac{\partial^2\mathrm{QA}_{\epsilon,\gamma}}{\partial^2\epsilon}\geq0\Rightarrow\forall0\leq a\leq2a\leq3a\leq\frac{1}{1+\gamma},\frac{1}{a}\left(\frac{(\mathrm{QA}(3a,\gamma)-\mathrm{QA}(2a,\gamma))}{a}-\frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a}\right)\geq0\Rightarrow\mathcal{B}_i-2\mathcal{B}_{i+1}+\mathcal{B}_{i+2}\geq0$. So, based on the second orderliness, SM_ϵ can be seen as assuming $\gamma=1$, replacing the two blocks, $\mathcal{B}_i+\mathcal{B}_{i+2}$ with one block $2\mathcal{B}_{i+1}$. From the νth γ-orderliness, the recurrence relation of the derivatives naturally produces the alternating binomial coefficients,

$$(-1)^\nu\frac{\partial^\nu\mathrm{QA}_{\epsilon,\gamma}}{\partial\epsilon^\nu}\geq0\Rightarrow\forall0\leq a\leq\ldots\leq(\nu+1)a\leq\frac{1}{1+\gamma},$$

$$\frac{(-1)^\nu}{a}\left(\frac{\frac{\mathrm{QA}(\nu a+a,\gamma)^{\cdot^{\cdot^{\cdot}}}}{a}-\frac{^{\cdot^{\cdot^{\cdot}}}\mathrm{QA}(2a,\gamma)}{a}}{a}-\frac{\frac{\mathrm{QA}(\nu a,\gamma)^{\cdot^{\cdot^{\cdot}}}}{a}-\frac{^{\cdot^{\cdot^{\cdot}}}\mathrm{QA}(a,\gamma)}{a}}{a}\right)$$

$$\geq0\Leftrightarrow\frac{(-1)^\nu}{a^\nu}\left(\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\mathrm{QA}\left((\nu-j+1)\,a,\gamma\right)\right)\geq0$$

$$\Rightarrow\sum_{j=0}^{\nu}(-1)^j\binom{\nu}{j}\mathcal{B}_{i+j}\geq0.$$

Based on the νth orderliness, the ϵ-binomial mean is introduced as

$$\mathrm{BM}_{\nu,\epsilon,n}:=\frac{1}{n}\left(\sum_{i=1}^{\frac{1}{2}\epsilon^{-1}(\nu+1)^{-1}}\sum_{j=0}^{\nu}\left(1-(-1)^j\binom{\nu}{j}\right)\mathfrak{B}_{i_j}\right),$$

where $\mathfrak{B}_{i_j}=\sum_{l=n\epsilon(j+(i-1)(\nu+1))+1}^{n\epsilon(j+(i-1)(\nu+1)+1)}(X_l+X_{n-l+1})$. If ν is not indicated, it is default as $\nu=3$. Since the alternating sum of binomial coefficients equals zero, when $\nu\ll\epsilon^{-1}$, $\epsilon\to0$, $\mathrm{BM}\to\mu$. If $\frac{1}{2}\epsilon^{-1}(\nu+1)^{-1}\in\mathbb{N}$, the asymmetry case is dividing the sample into ϵ^{-1} blocks in the same way as SM and then further weighting each block using binomial coefficients ($0\leq\gamma\leq1$ is needed). The solutions for the continuity of the breakdown point and $l \bmod 1\neq0$ are the same as that in SM and not repeated here. The equality $\mathrm{BM}_{\nu=1,\epsilon}=\mathrm{BWM}_\epsilon$ holds, and similarly, $\mathrm{BM}_{\nu=2,\epsilon}=\mathrm{SM}_{\epsilon,b=3}$, when $\gamma=1$ and their respective ϵs are identical. Interestingly, the biases of the $\mathrm{SM}_{\epsilon=\frac{1}{9},b=3}$ and the $\mathrm{WM}_{\epsilon=\frac{1}{9}}$ are nearly indistinguishable in common asymmetric unimodal distributions such as Weibull, gamma, lognormal, and Pareto (SI Text), indicating that their robustness to departures from the symmetry assumption is practically similar. The reason is that the Winsorized mean is using two single quantiles to replace the trimmed parts, not two blocks. The subsequent theorem provides an explanation for this difference.

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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