Semiparametric robust mean estimations based on the orderliness of quantile averages

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Inequalities related to weighted averages

So far, it is quite natural to hypothesize that the value of ϵ,γ-trimmed mean should be monotonically related to the breakdown point in a semiparametric distribution, since it is a linear combination of quantile averages as shown in Section **??**. Analogous to the γ-orderliness, the γ-trimming inequality for a right-skewed distribution is defined as $\forall 0 \le \epsilon_1 \le \epsilon_2 \le \frac{1}{1+\gamma}, \mathrm{TM}_{\epsilon_1,\gamma} \ge \mathrm{TM}_{\epsilon_2,\gamma}$. γ-orderliness is a sufficient condition for the γ-trimming inequality, as proven in the SI Text. The next theorem shows a relation between the ϵ,γ-quantile average and the ϵ,γ-trimmed mean under the γ-trimming inequality, suggesting the γ-orderliness is not a necessary condition for the γ-trimming inequality.

Theorem .1. *For a distribution that is right-skewed and follows the γ-trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, $0 \le \epsilon \le \frac{1}{1+\gamma}$.*

Proof. According to the definition of the γ-trimming inequality: $\forall 0 \le \epsilon \le \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right) du \ge \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du$, where δ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right) du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du \ge 0 \Leftrightarrow \int_{1-\epsilon}^{1-\epsilon+\delta} Q\left(u\right) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q\left(u\right) du - \frac{2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du \ge 0$. Since $\delta \to 0^+$, $\frac{1}{2\delta}\left(\int_{1-\epsilon}^{1-\epsilon+\delta} Q\left(u\right) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q\left(u\right) du\right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \ge \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du$, the proof is complete. □

An analogous result about the relation between the ϵ,γ-trimmed mean and the ϵ,γ-Winsorized mean can be obtained in the following theorem.

Theorem .2. *For a right-skewed distribution following the γ-trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, provided that $0 \le \gamma \le 1$. If assuming γ-orderliness, the inequality is valid for any non-negative γ.*

Proof. According to Theorem .1, $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \ge \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du \Leftrightarrow \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \ge \left(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du$. Then, if $0 \le \gamma \le 1$, $\left(1-\frac{1}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du + \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \ge 0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du + \gamma\epsilon Q\left(\gamma\epsilon\right) + \epsilon Q\left(1-\epsilon\right) \ge \int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du + \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \ge \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right) du$, the proof of the first assertion is complete. The second assertion is established in Theorem 0.3. in the SI Text. □

Replacing the TM in the γ-trimming inequality with WA forms the definition of the γ-weighted inequality. The γ-orderliness also implies the γ-Winsorization inequality when $0 \le \gamma \le 1$, as proven in the SI Text. To construct weighted averages based on the νth γ-orderliness and satisfying the corresponding weighted inequality, when $0 \le \gamma \le 1$, let $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} \mathrm{QA}\left(u,\gamma\right) du$, $ka = k\epsilon + c$. From the γ-orderliness, it follows that, $-\frac{\partial \mathrm{QA}}{\partial \epsilon} \ge 0 \Leftrightarrow \forall 0 \le a \le 2a \le \frac{1}{1+\gamma}, -\frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a} \ge 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \ge 0$, if $0 \le \gamma \le 1$. Suppose that $\mathcal{B}_i = \mathcal{B}_0$. Then, the ϵ,γ-block Winsorized mean, is defined as

$$\mathrm{BWM}_{\epsilon,\gamma,n} := \frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i\right),$$

which is double weighting the leftest and rightest blocks having sizes of $\gamma\epsilon n$ and ϵn, respectively. As a consequence of $\mathcal{B}_i - \mathcal{B}_{i+1} \ge 0$, the γ-block Winsorization inequality is valid, provided that $0 \le \gamma \le 1$. The block Winsorized mean uses two blocks to replace the trimmed parts, not two single quantiles. The subsequent theorem provides an explanation for this difference.

Theorem .3. *Asymptotically, for a right-skewed γ-ordered distribution, the Winsorized mean is always greater than or equal to the corresponding block Winsorized mean with the same ϵ and γ, if $0 \le \gamma \le 1$.*

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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