Semiparametric robust mean estimations based on the orderliness of quantile averages

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This manuscript was compiled on June 17, 2023

semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

Proof. □

Inequalities related to weighted averages

So far, it is quite natural to hypothesize that the value of ϵ,γ-trimmed mean should be monotonically related to the breakdown point in a semiparametric distribution, since it is a linear combination of quantile averages as shown in Section **??**. Analogous to the γ-orderliness, the γ-trimming inequality for a right-skewed distribution is defined as $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \mathrm{TM}_{\epsilon_1,\gamma} \geq \mathrm{TM}_{\epsilon_2,\gamma}$. γ-orderliness is a sufficient condition for the γ-trimming inequality, as proven in the SI Text. The next theorem shows a relation between the ϵ,γ-quantile average and the ϵ,γ-trimmed mean under the γ-trimming inequality, suggesting the γ-orderliness is not a necessary condition for the γ-trimming inequality.

Theorem .1. *For a distribution that is right-skewed and follows the γ-trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. According to the definition of the γ-trimming inequality: $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right)du \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$, where δ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right)du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du \geq 0 \Leftrightarrow \int_{1-\epsilon}^{1-\epsilon+\delta} Q\left(u\right)du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q\left(u\right)du - \frac{2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du \geq 0$. Since $\delta \rightarrow 0^+$, $\frac{1}{2\delta}\left(\int_{1-\epsilon}^{1-\epsilon+\delta} Q\left(u\right)du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q\left(u\right)du\right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$, the proof is complete. □

An analogous result about the relation between the ϵ,γ-trimmed mean and the ϵ,γ-Winsorized mean can be obtained in the following theorem.

Theorem .2. *For a right-skewed distribution following the γ-trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 \leq \gamma \leq 1$. If assuming γ-orderliness, the inequality is valid for any non-negative γ.*

Proof. According to Theorem .1, $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du \Leftrightarrow \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \geq \left(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$. Then, if $0 \leq \gamma \leq 1$, $\left(1-\frac{1}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \geq 0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \gamma\epsilon Q\left(\gamma\epsilon\right) + \epsilon Q\left(1-\epsilon\right) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$, the proof of the first assertion is complete. The second assertion is established in Theorem 0.3. in the SI Text. □

Replacing the TM in the γ-trimming inequality with WA forms the definition of the γ-weighted inequality. The γ-orderliness also implies the γ-Winsorization inequality when $0 \leq \gamma \leq 1$ for a right-skewed distribution, as proven in the SI Text. To construct weighted averages based on the νth γ-orderliness and satisfying the corresponding weighted inequality, when $0 \leq \gamma \leq 1$, let $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} \mathrm{QA}\left(u,\gamma\right)du$, $ka = k\epsilon + c$. From the γ-orderliness for a right-skewed distribution, it follows that, $-\frac{\partial \mathrm{QA}}{\partial \epsilon} \geq 0 \Leftrightarrow \forall 0 \leq a \leq 2a \leq \frac{1}{1+\gamma}, -\frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a} \geq 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, if $0 \leq \gamma \leq 1$. Suppose that $\mathcal{B}_i = \mathcal{B}_0$. Then, the ϵ,γ-block Winsorized mean, is defined as

$$\mathrm{BWM}_{\epsilon,\gamma,n} := \frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i\right),$$

which is double weighting the leftest and rightest blocks having sizes of $\gamma\epsilon n$ and ϵn, respectively. As a consequence of $\mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, the γ-block Winsorization inequality is valid, provided that $0 \leq \gamma \leq 1$. The block Winsorized mean uses two blocks to replace the trimmed parts, not two single quantiles. The subsequent theorem provides an explanation for this difference.

Theorem .3. *Asymptotically, for a right-skewed γ-ordered distribution, the Winsorized mean is always greater than or equal to the corresponding block Winsorized mean with the same ϵ and γ, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 \leq \gamma \leq 1$.*

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

ACKNOWLEDGMENTS. I sincerely acknowledge the insightful comments from the editor which considerably elevated the lucidity and merit of this paper.

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

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