# Semiparametric robust mean estimations based on the orderliness of quantile averages

**Tuban Lee**

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semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

*Proof.* □

## Inequalities related to weighted averages

So far, it is quite natural to hypothesize that the value of $\epsilon,\gamma$-trimmed mean should be monotonically related to the breakdown point in a semiparametric distribution, since it is a linear combination of quantile averages as shown in Section **??**. Analogous to the $\gamma$-orderliness, the $\gamma$-trimming inequality for a right-skewed distribution is defined as $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \mathrm{TM}_{\epsilon_1,\gamma} \geq \mathrm{TM}_{\epsilon_2,\gamma}$. $\gamma$-orderliness is a sufficient condition for the $\gamma$-trimming inequality, as proven in the SI Text. The next theorem shows a relation between the $\epsilon,\gamma$-quantile average and the $\epsilon,\gamma$-trimmed mean under the $\gamma$-trimming inequality, suggesting the $\gamma$-orderliness is not a necessary condition for the $\gamma$-trimming inequality.

**Theorem .1.** *For a distribution that is right-skewed and follows the $\gamma$-trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same $\epsilon$ and $\gamma$, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

*Proof.* According to the definition of the $\gamma$-trimming inequality: $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$, where $\delta$ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du \geq 0 \Leftrightarrow \int_{1-\epsilon}^{1-\epsilon+\delta} Q(u)\,du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u)\,du - \frac{2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du \geq 0$. Since $\delta \to 0^+$, $\frac{1}{2\delta}\left(\int_{1-\epsilon}^{1-\epsilon+\delta} Q(u)\,du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u)\,du\right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$, the proof is complete. □

An analogous result about the relation between the $\epsilon,\gamma$-trimmed mean and the $\epsilon,\gamma$-Winsorized mean can be obtained in the following theorem.

**Theorem .2.** *For a right-skewed distribution following the $\gamma$-trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same $\epsilon$ and $\gamma$, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 \leq \gamma \leq 1$. If assuming $\gamma$-orderliness, the inequality is valid for any non-negative $\gamma$.*

*Proof.* According to Theorem .1, $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du \Leftrightarrow \gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right) \geq \left(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$. Then, if $0 \leq \gamma \leq 1$, $\left(1-\frac{1}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right) \geq 0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right) \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$, the proof of the first assertion is complete. The second assertion is established in Theorem 0.3. in the SI Text. □

Replacing the TM in the $\gamma$-trimming inequality with WA forms the definition of the $\gamma$-weighted inequality. The $\gamma$-orderliness also implies the $\gamma$-Winsorization inequality when $0 \leq \gamma \leq 1$ for a right-skewed distribution, as proven in the SI Text. To construct weighted averages based on the $\nu$th $\gamma$-orderliness and satisfying the corresponding weighted inequality, when $0 \leq \gamma \leq 1$, let $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} \mathrm{QA}(u,\gamma)\,du$, $ka = k\epsilon + c$. From the $\gamma$-orderliness for a right-skewed distribution, it follows that, $-\frac{\partial \mathrm{QA}}{\partial \epsilon} \geq 0 \Leftrightarrow \forall 0 \leq a \leq 2a \leq \frac{1}{1+\gamma}, -\frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a} \geq 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, if $0 \leq \gamma \leq 1$. Suppose that $\mathcal{B}_i = \mathcal{B}_0$. Then, the $\epsilon,\gamma$-block Winsorized mean, is defined as

$$\mathrm{BWM}_{\epsilon,\gamma,n} \coloneqq \frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i\right),$$

which is double weighting the leftest and rightest blocks having sizes of $\gamma\epsilon n$ and $\epsilon n$, respectively. As a consequence of $\mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, the $\gamma$-block Winsorization inequality is valid, provided that $0 \leq \gamma \leq 1$. The block Winsorized mean uses two blocks to replace the trimmed parts, not two single quantiles. The subsequent theorem provides an explanation for this difference.

**Theorem .3.** *Asymptotically, for a right-skewed $\gamma$-ordered distribution, the Winsorized mean is always greater than or equal to the corresponding block Winsorized mean with the same $\epsilon$ and $\gamma$, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, if $0 \leq \gamma \leq 1$.*

**Data Availability.** Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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[1] To whom correspondence should be addressed. E-mail: tl@biomathematics.org