

# Semiparametric robust mean estimations based on the orderliness of quantile averages

Tuban Lee

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semiparametric | mean-median-mode inequality | asymptotic | unimodal  
| Hodges-Lehmann estimator

*Proof.*

## Inequalities related to weighted averages

So far, it is quite natural to hypothesize that the value of  $\epsilon, \gamma$ -trimmed mean should be monotonically related to the breakdown point in a semiparametric distribution, since it is a linear combination of quantile averages as shown in Section ???. Analogous to the  $\gamma$ -orderliness, the  $\gamma$ -trimming inequality for a right-skewed distribution is defined as  $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}$ ,  $\text{TM}_{\epsilon_1, \gamma} \geq \text{TM}_{\epsilon_2, \gamma}$ .  $\gamma$ -orderliness is a sufficient condition for the  $\gamma$ -trimming inequality, as proven in the SI Text. The next theorem shows a relation between the  $\epsilon, \gamma$ -quantile average and the  $\epsilon, \gamma$ -trimmed mean under the  $\gamma$ -trimming inequality, suggesting the  $\gamma$ -orderliness is not a necessary condition for the  $\gamma$ -trimming inequality.

**Theorem .1.** *For a distribution that is right-skewed and follows the  $\gamma$ -trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same  $\epsilon$  and  $\gamma$ , for all  $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ .*

*Proof.* According to the definition of the  $\gamma$ -trimming inequality:  $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$ ,  $\frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u) du \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$ , where  $\delta$  is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives  $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u) du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \geq 0 \Leftrightarrow \int_{1-\epsilon}^{1-\epsilon+\delta} Q(u) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u) du - \frac{2\delta}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \geq 0$ . Since  $\delta \rightarrow 0^+$ ,  $\frac{1}{2\delta} \left( \int_{1-\epsilon}^{1-\epsilon+\delta} Q(u) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u) du \right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$ , the proof is complete.  $\square$

An analogous result about the relation between the  $\epsilon, \gamma$ -trimmed mean and the  $\epsilon, \gamma$ -Winsorized mean can be obtained in the following theorem.

**Theorem .2.** *For a right-skewed distribution following the  $\gamma$ -trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same  $\epsilon$  and  $\gamma$ , for all  $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ , provided that  $0 \leq \gamma \leq 1$ . If assuming  $\gamma$ -orderliness, the inequality is valid for any non-negative  $\gamma$ .*

*Proof.* According to Theorem .1,  $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \Leftrightarrow \gamma\epsilon(Q(\gamma\epsilon)+Q(1-\epsilon)) \geq \left(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right) \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$ . Then, if  $0 \leq \gamma \leq 1$ ,  $1, \left(1 - \frac{1}{1-\epsilon-\gamma\epsilon}\right) \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon(Q(\gamma\epsilon)+Q(1-\epsilon)) \geq$

$0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon(Q(\gamma\epsilon)+Q(1-\epsilon)) \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$ , the proof of the first assertion is complete. The second assertion is established in Theorem 0.3. in the SI Text.  $\square$

Replacing the TM in the  $\gamma$ -trimming inequality with WA forms the definition of the  $\gamma$ -weighted inequality. The  $\gamma$ -orderliness also implies the  $\gamma$ -Winsorization inequality when  $0 \leq \gamma \leq 1$  for a right-skewed distribution, as proven in the SI Text. To construct weighted averages based on the  $\nu$ th  $\gamma$ -orderliness and satisfying the corresponding weighted inequality, when  $0 \leq \gamma \leq 1$ , let  $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} \text{QA}(u, \gamma) du$ ,  $ka = k\epsilon + c$ . From the  $\gamma$ -orderliness for a right-skewed distribution, it follows that,  $-\frac{\partial \text{QA}}{\partial \epsilon} \geq 0 \Leftrightarrow \forall 0 \leq a \leq 2a \leq \frac{1}{1+\gamma}$ ,  $-\frac{(\text{QA}(2a, \gamma) - \text{QA}(a, \gamma))}{a} \geq 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$ , if  $0 \leq \gamma \leq 1$ . Suppose that  $\mathcal{B}_i = \mathcal{B}_0$ . Then, the  $\epsilon, \gamma$ -block Winsorized mean, is defined as

$$\text{BWM}_{\epsilon, \gamma, n} := \frac{1}{n} \left( \sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i \right),$$

which is double weighting the leftest and rightest blocks having sizes of  $\gamma\epsilon n$  and  $\epsilon n$ , respectively. As a consequence of  $\mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$ , the  $\gamma$ -block Winsorization inequality is valid, provided that  $0 \leq \gamma \leq 1$ . The block Winsorized mean uses two blocks to replace the trimmed parts, not two single quantiles. The subsequent theorem provides an explanation for this difference.

**Data Availability.** Data for Figure ?? are given in SI Dataset S1. All codes have been deposited in [GitHub](#).

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<sup>1</sup>To whom correspondence should be addressed. E-mail: [tl@biomathematics.org](mailto:tl@biomathematics.org)