

Semiparametric robust mean estimations based on the orderliness of quantile averages

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This manuscript was compiled on June 13, 2023

semiparametric | mean-median-mode inequality | asymptotic | unimodal
| Hodges-Lehmann estimator

Remarkably, in 2018, Li, Shao, Wang, Yang (1) proved the bias bound of any quantile for arbitrary continuous distributions with finite second moments. Here, let $\mathcal{P}_{\mu,\sigma}$ denotes the set of continuous distributions whose mean is μ and standard deviation is σ . The bias upper bound of the quantile average for $P \in \mathcal{P}_{\mu=0,\sigma=1}$ is given in the following theorem.

Theorem .1. *The bias upper bound of the quantile average for any continuous distribution whose mean is zero and standard deviation is one is*

$$\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma) = \frac{1}{2} \left(\sqrt{\frac{\gamma\epsilon}{1-\gamma\epsilon}} + \sqrt{\frac{1-\epsilon}{\epsilon}} \right), \quad [1]$$

where $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.

Proof. Since $\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} \frac{1}{2}(Q(\gamma\epsilon) + Q(1-\epsilon)) \leq \frac{1}{2}(\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} Q(\gamma\epsilon) + \sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} Q(1-\epsilon))$, the assertion follows directly from the Lemma 2.6 in (1). \square

In 2020, Bernard et al. (2) further refined these bounds for unimodal distributions and derived the bias bound of the symmetric quantile average. Here, the bias upper bound of the quantile average, $0 \leq \gamma < 5$, for $P \in \mathcal{P}_U \cap \mathcal{P}_{\mu=0,\sigma=1}$ is given as

$$\sup_{P \in \mathcal{P}_U \cap \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma) = \begin{cases} \frac{1}{2} \left(\sqrt{\frac{4}{9\epsilon} - 1} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}} \right) & 0 \leq \epsilon \leq \frac{1}{6} \\ \frac{1}{2} \left(\sqrt{\frac{3(1-\epsilon)}{4-3(1-\epsilon)}} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}} \right) & \frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma} \end{cases}$$

The proof based on the bias bounds of any quantile (2) and the $\gamma \geq 5$ case are given in the SI Text. Subsequent theorems reveal the safeguarding role these bounds play in defining estimators based on ν th γ -orderliness. The proof of Theorem .2 is provided in the SI Text.

Theorem .2. *$\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma)$ is monotonic decreasing with respect to ϵ over the interval $[0, \frac{1}{1+\gamma}]$, when $0 \leq \gamma \leq 1$.*

Theorem .3. *$\sup_{P \in \mathcal{P}_U \cap \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma)$ is monotonic decreasing with respect to ϵ over the interval $[0, \frac{1}{1+\gamma}]$, when $0 \leq \gamma \leq 1$.*

Proof. When $0 \leq \epsilon \leq \frac{1}{6}$, $\frac{\partial \sup QA}{\partial \epsilon} = \frac{\gamma}{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}(4-3\epsilon\gamma)^2}} - \frac{1}{3\sqrt{\frac{4}{9\epsilon}-9\epsilon^2}}$. When $\gamma = 0$, $\frac{\partial \sup QA}{\partial \epsilon} = \frac{\gamma}{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}(4-3\epsilon\gamma)^2}} = 0$. $\frac{\partial \sup QA}{\partial \epsilon} = -\frac{1}{3\sqrt{\frac{4}{9\epsilon}-9\epsilon^2}} < 0$ if $0 < \epsilon \leq \frac{1}{6}$.

When $\epsilon \rightarrow 0^+$, $\lim_{\epsilon \rightarrow 0^+} \left(\frac{\gamma}{(4-3\gamma\epsilon)^2 \sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}} - \frac{1}{3\sqrt{\frac{4}{9\epsilon}-9\epsilon^2}} \right) = \lim_{\epsilon \rightarrow 0^+} \left(\frac{\gamma\sqrt{3}}{\sqrt{4^3\epsilon\gamma}} - \frac{1}{6\sqrt{\epsilon^3}} \right) \rightarrow -\infty$, so, $\frac{\partial \sup QA}{\partial \epsilon} < 0$. \square

Inequalities related to weighted averages

So far, it is quite natural to hypothesize that the value of ϵ, γ -trimmed mean should be monotonically related to the breakdown point in a semiparametric distribution, since it is a linear combination of quantile averages as shown in Section ???. Analogous to the γ -orderliness, the γ -trimming inequality for a right-skewed distribution is defined as $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}$, $TM_{\epsilon_1, \gamma} \geq TM_{\epsilon_2, \gamma}$. γ -orderliness is a sufficient condition for the γ -trimming inequality, as proven in the SI Text. The next theorem shows a relation between the ϵ, γ -quantile average and the ϵ, γ -trimmed mean under the γ -trimming inequality, suggesting the γ -orderliness is not a necessary condition for the γ -trimming inequality.

Theorem .4. *For a distribution that is right-skewed and follows the γ -trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ , for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. According to the definition of the γ -trimming inequality: $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta} \int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u) du \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$, where δ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u) du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \geq 0 \Leftrightarrow \int_{1-\epsilon}^{1-\epsilon+\delta} Q(u) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u) du - \frac{2\delta}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \geq 0$. Since $\delta \rightarrow 0^+$, $\frac{1}{2\delta} \left(\int_{1-\epsilon}^{1-\epsilon+\delta} Q(u) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u) du \right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$, the proof is complete. \square

An analogous result about the relation between the ϵ, γ -trimmed mean and the ϵ, γ -Winsorized mean can be obtained in the following theorem.

Theorem .5. *For a right-skewed distribution following the γ -trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ , for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 \leq \gamma \leq 1$. If assuming γ -orderliness, the inequality is valid for any non-negative γ .*

T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

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67 *Proof.* According to Theorem 4, $\frac{Q(\gamma\epsilon) + Q(1-\epsilon)}{2} \geq$
68 $\frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \Leftrightarrow \gamma\epsilon(Q(\gamma\epsilon) + Q(1-\epsilon)) \geq$
69 $(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}) \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du.$ Then, if $0 \leq \gamma \leq$
70 $1, (1 - \frac{1}{1-\epsilon-\gamma\epsilon}) \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon(Q(\gamma\epsilon) + Q(1-\epsilon)) \geq$
71 $0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du +$
72 $\gamma\epsilon(Q(\gamma\epsilon) + Q(1-\epsilon)) \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du,$ the proof
73 of the first assertion is complete. The second assertion is
74 established in Theorem 0.3. in the SI Text. \square

Replacing the TM in the γ -trimming inequality with WA forms the definition of the γ -weighted inequality. The γ -orderliness also implies the γ -Winsorization inequality when $0 \leq \gamma \leq 1$ for a right-skewed distribution, as proven in the SI Text. To construct weighted averages based on the ν th γ -orderliness and satisfying the corresponding weighted inequality, when $0 \leq \gamma \leq 1$, let $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} QA(u, \gamma) du$, $ka = k\epsilon + c$. From the γ -orderliness for a right-skewed distribution, it follows that, $-\frac{\partial QA}{\partial \epsilon} \geq 0 \Leftrightarrow \forall 0 \leq a \leq 2a \leq \frac{1}{1+\gamma}, -\frac{(QA(2a, \gamma) - QA(a, \gamma))}{a} \geq 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, if $0 \leq \gamma \leq 1$. Suppose that $\mathcal{B}_i = \mathcal{B}_0$. Then, the ϵ, γ -block Winsorized mean, is defined as

$$\text{BWM}_{\epsilon, \gamma, n} := \frac{1}{n} \left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i \right),$$

75 which is double weighting the leftest and rightest blocks hav-
76 ing sizes of $\gamma\epsilon n$ and ϵn , respectively. As a consequence of
77 $\mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, the γ -block Winsorization inequality is valid,
78 provided that $0 \leq \gamma \leq 1$. The block Winsorized mean uses
79 two blocks to replace the trimmed parts, not two single quan-
80 tiles. The subsequent theorem provides an explanation for
81 this difference.

82 **Theorem .6.** *Asymptotically, for a right-skewed γ -ordered*
83 *distribution, the Winsorized mean is always greater than or*
84 *equal to the corresponding block Winsorized mean with the*
85 *same ϵ and γ , for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, if $0 \leq \gamma \leq 1$.*

86 *Proof.* From the definitions of BWM and WM, the state-
87 ment necessitates $\int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq$
88 $\int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du + \int_{1-2\epsilon}^{1-\epsilon} Q(u) du \Leftrightarrow \gamma\epsilon Q(\gamma\epsilon) +$
89 $\epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du + \int_{1-2\epsilon}^{1-\epsilon} Q(u) du.$ Define $\text{WML}(x) =$
90 $Q(\gamma\epsilon)$ and $\text{BWM}(x) = Q(x)$. In both functions, the
91 interval for x is specified as $[\gamma\epsilon, 2\gamma\epsilon]$. Then, define
92 $\text{WMu}(y) = Q(1-\epsilon)$ and $\text{BWMu}(y) = Q(y)$. In both
93 functions, the interval for y is specified as $[1-2\epsilon, 1-\epsilon]$.
94 The function $y : [\gamma\epsilon, 2\gamma\epsilon] \rightarrow [1-2\epsilon, 1-\epsilon]$ defined by
95 $y(x) = 1 - \frac{x}{\gamma}$ is a bijection. $\text{WML}(x) + \text{WMu}(y(x)) =$
96 $Q(\gamma\epsilon) + Q(1-\epsilon) \geq \text{BWM}(x) + \text{BWMu}(y(x)) = Q(x) +$
97 $Q(1 - \frac{x}{\gamma})$ is valid for all $x \in [\gamma\epsilon, 2\gamma\epsilon]$, according to the
98 definition of γ -orderliness. Integration of the left side
99 yields, $\int_{\gamma\epsilon}^{2\gamma\epsilon} (\text{WML}(u) + \text{WMu}(y(u))) du = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon) du +$
100 $\int_{y(\gamma\epsilon)}^{y(2\gamma\epsilon)} Q(1-\epsilon) dy = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon) du + \int_{1-2\epsilon}^{1-\epsilon} Q(1-\epsilon) dy =$
101 $\gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon)$, while integration of the right side
102 yields $\int_{\gamma\epsilon}^{2\gamma\epsilon} (\text{BWM}(x) + \text{BWMu}(y(x))) dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du +$
103 $\int_{\gamma\epsilon}^{2\gamma\epsilon} Q(1 - \frac{x}{\gamma}) dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du + \int_{1-2\epsilon}^{1-\epsilon} Q(u) du$, which are
104 the left and right sides of the desired inequality. Given that the
105 upper limits and lower limits of the integrations are different

for each term, the condition $0 \leq \gamma \leq 1$ is necessary for the
desired inequality to be valid. \square

From the second γ -orderliness for a right-skewed dis-
tribution, $\frac{\partial^2 QA}{\partial^2 \epsilon} \geq 0 \Rightarrow \forall 0 \leq a \leq 2a \leq 3a \leq$
 $\frac{1}{1+\gamma}, \frac{1}{a} \left(\frac{(QA(3a, \gamma) - QA(2a, \gamma))}{a} - \frac{(QA(2a, \gamma) - QA(a, \gamma))}{a} \right) \geq 0 \Rightarrow$ if
 $0 \leq \gamma \leq 1, \mathcal{B}_i - 2\mathcal{B}_{i+1} + \mathcal{B}_{i+2} \geq 0$. SM_ϵ can thus be interpreted
as assuming $\gamma = 1$ and replacing the two blocks, $\mathcal{B}_i + \mathcal{B}_{i+2}$
with one block $2\mathcal{B}_{i+1}$. From the ν th γ -orderliness for a right-
skewed distribution, the recurrence relation of the derivatives
naturally produces the alternating binomial coefficients,

$$(-1)^\nu \frac{\partial^\nu QA}{\partial \epsilon^\nu} \geq 0 \Rightarrow \forall 0 \leq a \leq \dots \leq (\nu+1)a \leq \frac{1}{1+\gamma},$$

$$\frac{(-1)^\nu}{a} \left(\frac{\frac{QA(\nu a + a, \gamma)}{a} - \dots - \frac{QA(2a, \gamma)}{a}}{a} - \frac{\frac{QA(\nu a, \gamma)}{a} - \dots - \frac{QA(a, \gamma)}{a}}{a} \right)$$

$$\geq 0 \Leftrightarrow \frac{(-1)^\nu}{a^\nu} \left(\sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} QA((\nu-j+1)a, \gamma) \right) \geq 0$$

$$\Rightarrow \text{if } 0 \leq \gamma \leq 1, \sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} \mathcal{B}_{i+j} \geq 0.$$

Based on the ν th orderliness, the ϵ, γ -binomial mean is intro-
duced as

$$\text{BM}_{\nu, \epsilon, \gamma, n} := \frac{1}{n} \left(\sum_{i=1}^{\frac{1}{2}\epsilon^{-1}(\nu+1)^{-1}} \sum_{j=0}^{\nu} \left(1 - (-1)^j \binom{\nu}{j} \right) \mathfrak{B}_{i,j} \right),$$

where $\mathfrak{B}_{i,j} = \sum_{l=n\gamma\epsilon(j+(i-1)(\nu+1)+1)}^{n\epsilon(j+(i-1)(\nu+1)+1)} (X_l + X_{n-l+1})$. If ν is
not indicated, it defaults to $\nu = 3$. Since the alternating sum
of binomial coefficients equals zero, when $\nu \ll \epsilon^{-1}$ and $\epsilon \rightarrow 0$,
 $\text{BM} \rightarrow \mu$. The solutions for the continuity of the breakdown
point is the same as that in SM and not repeated here. The
equalities $\text{BM}_{\nu=1, \epsilon} = \text{BWM}_\epsilon$ and $\text{BM}_{\nu=2, \epsilon} = \text{SM}_{\epsilon, b=3}$ hold,
when $\gamma = 1$ and their respective ϵ s are identical. Interestingly,
the biases of the $\text{SM}_{\epsilon=\frac{1}{3}, b=3}$ and the $\text{WM}_{\epsilon=\frac{1}{3}}$ are nearly indis-
tinguishable in common asymmetric unimodal distributions
such as Weibull, gamma, lognormal, and Pareto (SI Dataset
S1). This indicates that their robustness to departures from
the symmetry assumption is practically similar under uni-
modality, even though they are based on different orders of
orderliness. If single quantiles are used, based on the second
 γ -orderliness, the stratified quantile mean can be defined as

$$\text{SQM}_{\epsilon, \gamma, n} := 4\epsilon \sum_{i=1}^{\frac{1}{4\epsilon}} \frac{1}{2} (\hat{Q}_n((2i-1)\gamma\epsilon) + \hat{Q}_n(1 - (2i-1)\epsilon)),$$

$\text{SQM}_{\epsilon=\frac{1}{4}}$ is the Tukey's midhinge (3). In fact, SQM is a
subcase of SM when $\gamma = 1$ and $b \rightarrow \infty$, so the solution for the
continuity of the breakdown point, $\frac{1}{\epsilon} \bmod 4 \neq 0$, is identical.
However, since the definition is based on the empirical quantile
function, no decimal issues related to order statistics will arise.
The next theorem explains another advantage.

Theorem .7. *For a right-skewed second γ -ordered distribu-*
tion, asymptotically, $\text{SQM}_{\epsilon, \gamma}$ is always greater or equal to
the corresponding $\text{BM}_{\nu=2, \epsilon, \gamma}$ with the same ϵ and γ , for all
 $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, if $0 \leq \gamma \leq 1$.

119 *Proof.* For simplicity, suppose the order statistics of the sam-
 120 ple are distributed into $\epsilon^{-1} \in \mathbb{N}$ blocks in the computa-
 121 tion of both $\text{SQM}_{\epsilon, \gamma}$ and $\text{BM}_{\nu=2, \epsilon, \gamma}$. The computation of
 122 $\text{BM}_{\nu=2, \epsilon, \gamma}$ alternates between weighting and non-weighting,
 123 let ‘0’ denote the block assigned with a weight of zero and
 124 ‘1’ denote the block assigned with a weighted status of one, the se-
 125 quence indicating the weighted or non-weighted status of each
 126 block is: 0, 1, 0, 0, 1, 0, \dots . Let this sequence be denoted by
 127 $a_{\text{BM}_{\nu=2, \epsilon, \gamma}}(j)$, its formula is $a_{\text{BM}_{\nu=2, \epsilon, \gamma}}(j) = \lfloor \frac{j \bmod 3}{2} \rfloor$. Simi-
 128 larly, the computation of $\text{SQM}_{\epsilon, \gamma}$ can be seen as positioning
 129 quantiles (p) at the beginning of the blocks if $0 < p < \frac{1}{1+\gamma}$, and
 130 at the end of the blocks if $p > \frac{1}{1+\gamma}$. The sequence of denoting
 131 whether each block’s quantile is weighted or not weighted is:
 132 0, 1, 0, 1, 0, 1, \dots . Let the sequence be denoted by $a_{\text{SQM}_{\epsilon, \gamma}}(j)$,
 133 the formula of the sequence is $a_{\text{SQM}_{\epsilon, \gamma}}(j) = j \bmod 2$. If pair-
 134 ing all blocks in $\text{BM}_{\nu=2, \epsilon, \gamma}$ and all quantiles in $\text{SQM}_{\epsilon, \gamma}$, there
 135 are two possible pairings of $a_{\text{BM}_{\nu=2}}(j)$ and $a_{\text{SQM}_{\epsilon, \gamma}}(j)$. One
 136 pairing occurs when $a_{\text{BM}_{\nu=2, \epsilon, \gamma}}(j) = a_{\text{SQM}_{\epsilon, \gamma}}(j) = 1$, while the
 137 other involves the sequence 0, 1, 0 from $a_{\text{BM}_{\nu=2, \epsilon, \gamma}}(j)$ paired
 138 with 1, 0, 1 from $a_{\text{SQM}_{\epsilon, \gamma}}(j)$. By leveraging the same principle
 139 as Theorem 6 and the second γ -orderliness (replacing the two
 140 quantile averages with one quantile average between them),
 141 the desired result follows. \square

142 The biases of $\text{SQM}_{\epsilon=\frac{1}{8}}$, which is based on the second order-
 143 liness with a quantile approach, are notably similar to those
 144 of $\text{BM}_{\nu=3, \epsilon=\frac{1}{8}}$, which is based on the third orderliness with a
 145 block approach, in common asymmetric unimodal distributions
 146 (Figure ??).

147 **Data Availability.** Data for Figure ?? are given in SI Dataset
 148 S1. All codes have been deposited in [GitHub](#).

149 **ACKNOWLEDGMENTS.** I sincerely acknowledge the insightful
 150 comments from the editor which considerably elevated the lucidity
 151 and merit of this paper.

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