Semiparametric robust mean estimations based on the orderliness of quantile averages

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Inequalities related to weighted averages

So far, it is quite natural to hypothesize that the value of ϵ,γ-trimmed mean should be monotonically related to the breakdown point in a semiparametric distribution, since it is a linear combination of quantile averages as shown in Section **??**. Analogous to the γ-orderliness, the γ-trimming inequality for a right-skewed distribution is defined as $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \mathrm{TM}_{\epsilon_1,\gamma} \geq \mathrm{TM}_{\epsilon_2,\gamma}$. γ-orderliness is a sufficient condition for the γ-trimming inequality, as proven in the SI Text. The next theorem shows a relation between the ϵ,γ-quantile average and the ϵ,γ-trimmed mean under the γ-trimming inequality, suggesting the γ-orderliness is not a necessary condition for the γ-trimming inequality.

Theorem .1. *For a distribution that is right-skewed and follows the γ-trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. According to the definition of the γ-trimming inequality: $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$, where δ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u)\,du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du \geq 0 \Leftrightarrow \int_{1-\epsilon}^{1-\epsilon+\delta} Q(u)\,du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u)\,du - \frac{2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du \geq 0$. Since $\delta \to 0^+$, $\frac{1}{2\delta}\left(\int_{1-\epsilon}^{1-\epsilon+\delta} Q(u)\,du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u)\,du\right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$, the proof is complete. $\square$

An analogous result about the relation between the ϵ,γ-trimmed mean and the ϵ,γ-Winsorized mean can be obtained in the following theorem.

Theorem .2. *For a right-skewed distribution following the γ-trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, provided that $0 \leq \gamma \leq 1$. If assuming γ-orderliness, the inequality is valid for any non-negative γ.*

Proof. According to Theorem .1, $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du \Leftrightarrow \gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right) \geq \left(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$. Then, if $0 \leq \gamma \leq 1$, $\left(1-\frac{1}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right) \geq 0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon\left(Q(\gamma\epsilon)+Q(1-\epsilon)\right) \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du$, the proof of the first assertion is complete. The second assertion is established in Theorem 0.3. in the SI Text. $\square$

Replacing the TM in the γ-trimming inequality with WA forms the definition of the γ-weighted inequality. The γ-orderliness also implies the γ-Winsorization inequality when $0 \leq \gamma \leq 1$, as proven in the SI Text. To construct weighted averages based on the νth γ-orderliness and satisfying the corresponding weighted inequality, when $0 \leq \gamma \leq 1$, let $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} \mathrm{QA}(u,\gamma)\,du$, $ka = k\epsilon + c$. From the γ-orderliness, it follows that, $-\frac{\partial \mathrm{QA}}{\partial \epsilon} \geq 0 \Leftrightarrow \forall 0 \leq a \leq 2a \leq \frac{1}{1+\gamma}, -\frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a} \geq 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, if $0 \leq \gamma \leq 1$. Suppose that $\mathcal{B}_i = \mathcal{B}_0$. Then, the ϵ,γ-block Winsorized mean, is defined as

$$\mathrm{BWM}_{\epsilon,\gamma,n} := \frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i\right),$$

which is double weighting the leftest and rightest blocks having sizes of $\gamma\epsilon n$ and ϵn, respectively. As a consequence of $\mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, the γ-block Winsorization inequality is valid, provided that $0 \leq \gamma \leq 1$. The block Winsorized mean uses two blocks to replace the trimmed parts, not two single quantiles. The subsequent theorem provides an explanation for this difference.

Theorem .3. *Asymptotically, for a right-skewed γ-ordered distribution, the Winsorized mean is always greater than or equal to the corresponding block Winsorized mean with the same ϵ and γ, if $0 \leq \gamma \leq 1$.*

Proof. From the definitions of BWM and WM, the statement necessitates $\int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q(u)\,du + \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u)\,du + \int_{1-2\epsilon}^{1-\epsilon} Q(u)\,du \Leftrightarrow \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u)\,du + \int_{1-2\epsilon}^{1-\epsilon} Q(u)\,du$. Define $\mathrm{WM}l(x) = Q(\gamma\epsilon)$ and $\mathrm{BWM}l(x) = Q(x)$. In both functions, the interval for x is specified as $[\gamma\epsilon, 2\gamma\epsilon]$. Then, define $\mathrm{WM}u(y) = Q(1-\epsilon)$ and $\mathrm{BWM}u(y) = Q(y)$. In both functions, the interval for y is specified as $[1-2\epsilon, 1-\epsilon]$. The function $y : [\gamma\epsilon, 2\gamma\epsilon] \to [1-2\epsilon, 1-\epsilon]$ defined by $y(x) = 1 - \frac{x}{\gamma}$ is a bijection. $\mathrm{WM}l(x) + \mathrm{WM}u(y(x)) = Q(\gamma\epsilon) + Q(1-\epsilon) \geq \mathrm{BWM}l(x) + \mathrm{BWM}u(y(x)) = Q(x) + Q\left(1-\frac{x}{\gamma}\right)$ is valid for all $x \in [\gamma\epsilon, 2\gamma\epsilon]$, according to the definition of γ-orderliness. Integration of the left side yields, $\int_{\gamma\epsilon}^{2\gamma\epsilon}\left(\mathrm{WM}l(u) + \mathrm{WM}u(y(u))\right)du = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon)\,du + \int_{y(\gamma\epsilon)}^{y(2\gamma\epsilon)} Q(1-\epsilon)\,du = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon)\,du + \int_{1-2\epsilon}^{1-\epsilon} Q(1-\epsilon)\,du = \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon)$, while integration of the right side

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yields $\int_{\gamma\epsilon}^{2\gamma\epsilon} \left(\mathrm{BWM}l\left(x\right) + \mathrm{BWM}u\left(y\left(x\right)\right)\right) dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q\left(u\right) du + \int_{\gamma\epsilon}^{2\gamma\epsilon} Q\left(1-\frac{x}{\gamma}\right) dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q\left(u\right) du + \int_{1-2\epsilon}^{1-\epsilon} Q\left(u\right) du$, which are the left and right sides of the desired inequality. Given that the upper limits and lower limits of the integrations are different for each term, the condition $0 \leq \gamma \leq 1$ is necessary for the desired inequality to be valid.

$\square$

From the second γ-orderliness, $\frac{\partial^2 \mathrm{QA}}{\partial^2 \epsilon} \geq 0 \Rightarrow \forall 0 \leq a \leq 2a \leq 3a \leq \frac{1}{1+\gamma}, \frac{1}{a}\left(\frac{(\mathrm{QA}(3a,\gamma)-\mathrm{QA}(2a,\gamma))}{a} - \frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a}\right) \geq 0 \Rightarrow$ if $0 \leq \gamma \leq 1$, $\mathcal{B}_i - 2\mathcal{B}_{i+1} + \mathcal{B}_{i+2} \geq 0$. SM_ϵ can thus be interpreted as assuming $\gamma = 1$ and replacing the two blocks, $\mathcal{B}_i + \mathcal{B}_{i+2}$ with one block $2\mathcal{B}_{i+1}$. From the νth γ-orderliness, the recurrence relation of the derivatives naturally produces the alternating binomial coefficients,

$$(-1)^\nu \frac{\partial^\nu \mathrm{QA}}{\partial \epsilon^\nu} \geq 0 \Rightarrow \forall 0 \leq a \leq \ldots \leq (\nu+1)a \leq \frac{1}{1+\gamma},$$

$$\frac{(-1)^\nu}{a}\left(\frac{\frac{\mathrm{QA}(\nu a+a,\gamma)^{\cdot^{\cdot^{\cdot}}}}{a} - \frac{^{\cdot^{\cdot^{\cdot}}}\mathrm{QA}(2a,\gamma)}{a}}{a} - \frac{\frac{\mathrm{QA}(\nu a,\gamma)^{\cdot^{\cdot^{\cdot}}}}{a} - \frac{^{\cdot^{\cdot^{\cdot}}}\mathrm{QA}(a,\gamma)}{a}}{a}\right)$$

$$\geq 0 \Leftrightarrow \frac{(-1)^\nu}{a^\nu}\left(\sum_{j=0}^{\nu}(-1)^j \binom{\nu}{j} \mathrm{QA}\left(\left(\nu - j + 1\right)a, \gamma\right)\right) \geq 0$$

$$\Rightarrow \text{if } 0 \leq \gamma \leq 1, \sum_{j=0}^{\nu}(-1)^j \binom{\nu}{j} \mathcal{B}_{i+j} \geq 0.$$

Based on the νth orderliness, the ϵ,γ-binomial mean is introduced as

$$\mathrm{BM}_{\nu,\epsilon,\gamma,n} := \frac{1}{n}\left(\sum_{i=1}^{\frac{1}{2}\epsilon^{-1}(\nu+1)^{-1}} \sum_{j=0}^{\nu}\left(1-(-1)^j\binom{\nu}{j}\right)\mathfrak{B}_{i_j}\right),$$

where $\mathfrak{B}_{i_j} = \sum_{l=n\gamma\epsilon(j+(i-1)(\nu+1))+1}^{n\epsilon(j+(i-1)(\nu+1)+1)}\left(X_l + X_{n-l+1}\right)$. If ν is not indicated, it defaults to $\nu = 3$. Since the alternating sum of binomial coefficients equals zero, when $\nu \ll \epsilon^{-1}$ and $\epsilon \to 0$, $\mathrm{BM} \to \mu$. The solutions for the continuity of the breakdown point is the same as that in SM and not repeated here. The equality $\mathrm{BM}_{\nu=1,\epsilon} = \mathrm{BWM}_\epsilon$ holds. Similarly, $\mathrm{BM}_{\nu=2,\epsilon} = \mathrm{SM}_{\epsilon,b=3}$, when $\gamma = 1$ and their respective ϵs are identical.

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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