Semiparametric robust mean estimations based on the orderliness of quantile averages

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1. Hodges–Lehmann inequality and γ-U-orderliness

The Hodges–Lehmann estimator stands out as a unique robust location estimator due to its definition being substantially dissimilar from conventional L-estimators, R-estimators, and M-estimators. In their landmark paper, *Estimates of location based on rank tests*, Hodges and Lehmann [1] proposed two methods for computing the H-L estimator: the Wilcoxon score R-estimator and the median of pairwise means. The Wilcoxon score R-estimator is a location estimator based on signed-rank test, or R-estimator, [1] and was later independently discovered by Sen (1963) [2, 3]. However, the median of pairwise means is a generalized L-statistic and a trimmed U-statistic, as classified by Serfling in his novel conceptualized study in 1984 [4]. Serfling further advanced the understanding by generalizing the H-L kernel as $hl_k(x_1, \ldots, x_k) = \frac{1}{k}\sum_{i=1}^{k} x_i$, where $k \in \mathbb{N}$ [4]. Here, the weighted H-L kernel is defined as $whl_k(x_1, \ldots, x_k) = \frac{\sum_{i=1}^{k} x_i \mathbf{w}_i}{\sum_{i=1}^{k} \mathbf{w}_i}$, where $\mathbf{w}_i$s are the weights applied to each element.

By using the weighted H-L kernel and the L-estimator, it is now clear that the Hodges-Lehmann estimator is an LL-statistic, the definition of which is provided as follows:

$$LL_{k,\epsilon,\gamma,n} := L_{\epsilon_0,\gamma,n}\left(\text{sort}\left((whl_k(X_{N_1}, \cdots, X_{N_k}))_{N=1}^{\binom{n}{k}}\right)\right),$$

where $L_{\epsilon_0,\gamma,n}(Y)$ represents the ϵ_0,γ-L-estimator that uses the sorted sequence, $\text{sort}(whl_k(X_{N_1}, \cdots, X_{N_k}))_{N=1}^{\binom{n}{k}}$, as input. The upper asymptotic breakdown point of $LL_{k,\epsilon,\gamma}$ is $\epsilon = 1 - (1-\epsilon_0)^{\frac{1}{k}}$, as proven in DSSM II. There are two ways to adjust the breakdown point: either by setting k as a constant and adjusting ϵ_0, or by setting ϵ_0 as a constant and adjusting k. In the above definition, k is discrete, but the bootstrap method can be applied to ensure the continuity of k, also making the breakdown point continuous. Specifically, if $k \in \mathbb{R}$, let the bootstrap size be denoted by b, then first sampling the original sample $(1 - k + \lfloor k \rfloor)b$ times with each sample size of $\lfloor k \rfloor$, and then subsequently sampling $(1 - \lceil k \rceil + k)b$ times with each sample size of $\lceil k \rceil$, $(1 - k + \lfloor k \rfloor)b \in \mathbb{N}$, $(1 - \lceil k \rceil + k)b \in \mathbb{N}$. The corresponding kernels are computed separately, and the pooled sorted sequence is used as the input for the L-estimator. Let $\mathbf{S}_k$ represent the sorted sequence. Indeed, for any finite sample, X, when $k = n$, $\mathbf{S}_k$ becomes a single point, $whl_{k=n}(X_1, \ldots, X_n)$. When $\mathbf{w}_i = 1$, the minimum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k} X_i$, due to the property of order statistics. The maximum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k} X_{n-i+1}$. The monotonicity of the order statistics implies the monotonicity of the extrema with respect to k, i.e., the support of $\mathbf{S}_k$ shrinks monotonically. For unequal $\mathbf{w}_i$s, the shrinkage of the support of $\mathbf{S}_k$ might not be strictly monotonic, but the general trend remains, since all LL-statistics converge to the same point, as $k \to n$. Therefore, if $\frac{\sum_{i=1}^{n} X_i \mathbf{w}_i}{\sum_{i=1}^{n} \mathbf{w}_i}$ approaches the population mean when $n \to \infty$, all LL-statistics based on such consistent kernel function approach the population mean as $k \to \infty$. For example, if $whl_k = \text{BM}_{\nu,\epsilon_k,n=k}$, $\nu \ll \epsilon_k^{-1}$, $\epsilon_k \to 0$, such kernel function is consistent. These cases are termed the LL-mean ($\text{LLM}_{k,\epsilon,\gamma,n}$). By substituting the $\text{WA}_{\epsilon_0,\gamma,n}$ for the L-estimator in LL, the resulting statistic is referred to as the weighted L-statistic ($\text{WL}_{k,\epsilon,\gamma,n}$).

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

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