Semiparametric robust mean estimations based on the orderliness of quantile averages

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Weighted Inequalities and Binomial Mean

Analogous to the γ-orderliness, the γ-trimming inequality for a right-skewed distribution is defined as $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \mathrm{TM}_{\epsilon_1,\gamma} \geq \mathrm{TM}_{\epsilon_2,\gamma}$. γ-orderliness is a sufficient condition for the γ-trimming inequality, as proven in the SI Text. The next theorem shows a relation between the ϵ,γ-quantile average and the ϵ,γ-trimmed mean under the γ-trimming inequality, suggesting the γ-orderliness is not a necessary condition for the γ-trimming inequality.

Theorem .1. *For a distribution that is right-skewed and follows the γ-trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. According to the definition of the γ-trimming inequality: $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon+2\delta}\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right)du \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$, where δ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q\left(u\right)du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du \geq 0 \Leftrightarrow \int_{1-\epsilon}^{1-\epsilon+\delta} Q\left(u\right)du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q\left(u\right)du - \frac{2\delta}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du \geq 0$. Since $\delta \rightarrow 0^+$, $\frac{1}{2\delta}\left(\int_{1-\epsilon}^{1-\epsilon+\delta} Q\left(u\right)du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q\left(u\right)du\right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$, the proof is complete. □

An analogous result about the relation between the ϵ,γ-trimmed mean and the ϵ,γ-Winsorized mean can be obtained in the following theorem.

Theorem .2. *For a right-skewed distribution following the γ-trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 \leq \gamma \leq 1$. If assuming γ-orderliness, the inequality is valid for any non-negative γ.*

Proof. According to Theorem .1, $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du \Leftrightarrow \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \geq \left(\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$. Then, if $0 \leq \gamma \leq 1$, $\left(1-\frac{1}{1-\epsilon-\gamma\epsilon}\right)\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \geq 0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \gamma\epsilon Q\left(\gamma\epsilon\right) + \epsilon Q\left(1-\epsilon\right) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \gamma\epsilon\left(Q\left(\gamma\epsilon\right)+Q\left(1-\epsilon\right)\right) \geq \frac{1}{1-\epsilon-\gamma\epsilon}\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du$, the proof of the first assertion is complete. The second assertion is established in Theorem 0.3. in the SI Text. □

Replacing the TM in the γ-trimming inequality with WA forms the definition of the γ-weighted inequality. The γ-orderliness also implies the γ-Winsorization inequality when $0 \leq \gamma \leq 1$, as proven in the SI Text. The same rationale as presented in Theorem **??**, for a location-scale distribution characterized by a location parameter μ and a scale parameter λ, asymptotically, any $\mathrm{WA}(\epsilon,\gamma)$ can be expressed as $\lambda\mathrm{WA}_0(\epsilon,\gamma)+\mu$, where $\mathrm{WA}_0(\epsilon,\gamma)$ is an function of $Q_0(p)$ according to the definition of the weighted average. Adhering to the rationale present in Theorem **??**, for any probability distribution within a location-scale family, a necessary and sufficient condition for whether it follows the γ-weighted inequality is whether the family of probability distributions also adheres to the γ-weighted inequality.

To construct weighted averages based on the νth γ-orderliness and satisfying the corresponding weighted inequality, when $0 \leq \gamma \leq 1$, let $\mathcal{B}_i = \int_{i\epsilon}^{(i+1)\epsilon} \mathrm{QA}\left(u,\gamma\right)du$, $ka = k\epsilon + c$. From the γ-orderliness for a right-skewed distribution, it follows that, $-\frac{\partial \mathrm{QA}}{\partial \epsilon} \geq 0 \Leftrightarrow \forall 0 \leq a \leq 2a \leq \frac{1}{1+\gamma}, -\frac{(\mathrm{QA}(2a,\gamma)-\mathrm{QA}(a,\gamma))}{a} \geq 0 \Rightarrow \mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, if $0 \leq \gamma \leq 1$. Suppose that $\mathcal{B}_i = \mathcal{B}_0$. Then, the ϵ,γ-block Winsorized mean, is defined as

$$\mathrm{BWM}_{\epsilon,\gamma,n} := \frac{1}{n}\left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i\right),$$

which is double weighting the leftest and rightest blocks having sizes of $\gamma\epsilon n$ and ϵn, respectively. As a consequence of $\mathcal{B}_i - \mathcal{B}_{i+1} \geq 0$, the γ-block Winsorization inequality is valid, provided that $0 \leq \gamma \leq 1$. The block Winsorized mean uses two blocks to replace the trimmed parts, not two single quantiles. The subsequent theorem provides an explanation for this difference.

Theorem .3. *Asymptotically, for a right-skewed distribution following the γ-orderliness, the Winsorized mean is always greater than or equal to the corresponding block Winsorized mean with the same ϵ and γ, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 \leq \gamma \leq 1$.*

Proof. From the definitions of BWM and WM, the statement necessitates $\int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \gamma\epsilon Q\left(\gamma\epsilon\right) + \epsilon Q\left(1-\epsilon\right) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q\left(u\right)du + \int_{\gamma\epsilon}^{2\gamma\epsilon} Q\left(u\right)du + \int_{1-2\epsilon}^{1-\epsilon} Q\left(u\right)du \Leftrightarrow \gamma\epsilon Q\left(\gamma\epsilon\right) + \epsilon Q\left(1-\epsilon\right) \geq \int_{\gamma\epsilon}^{2\gamma\epsilon} Q\left(u\right)du + \int_{1-2\epsilon}^{1-\epsilon} Q\left(u\right)du$. Define $\mathrm{WM}l(x) = Q\left(\gamma\epsilon\right)$ and $\mathrm{BWM}l(x) = Q\left(x\right)$. In both functions, the interval for x is specified as $[\gamma\epsilon, 2\gamma\epsilon]$. Then, define $\mathrm{WM}u(y) = Q\left(1-\epsilon\right)$ and $\mathrm{BWM}u(y) = Q\left(y\right)$. In both

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functions, the interval for y is specified as $[1-2\epsilon, 1-\epsilon]$. The function $y : [\gamma\epsilon, 2\gamma\epsilon] \to [1-2\epsilon, 1-\epsilon]$ defined by $y(x) = 1 - \frac{x}{\gamma}$ is a bijection. $\mathrm{WM}l(x) + \mathrm{WM}u(y(x)) = Q(\gamma\epsilon) + Q(1-\epsilon) \geq \mathrm{BWM}l(x) + \mathrm{BWM}u(y(x)) = Q(x) + Q\left(1 - \frac{x}{\gamma}\right)$ is valid for all $x \in [\gamma\epsilon, 2\gamma\epsilon]$, according to the definition of γ-orderliness. Integration of the left side yields, $\int_{\gamma\epsilon}^{2\gamma\epsilon} (\mathrm{WM}l(u) + \mathrm{WM}u(y(u)))\,du = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon)\,du + \int_{y(\gamma\epsilon)}^{y(2\gamma\epsilon)} Q(1-\epsilon)\,du = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon)\,du + \int_{1-2\epsilon}^{1-\epsilon} Q(1-\epsilon)\,du = \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon)$, while integration of the right side yields $\int_{\gamma\epsilon}^{2\gamma\epsilon} (\mathrm{BWM}l(x) + \mathrm{BWM}u(y(x)))\,dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u)\,du + \int_{\gamma\epsilon}^{2\gamma\epsilon} Q\left(1 - \frac{x}{\gamma}\right)dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u)\,du + \int_{1-2\epsilon}^{1-\epsilon} Q(u)\,du$, which are the left and right sides of the desired inequality. Given that the upper limits and lower limits of the integrations are different for each term, the condition $0 \leq \gamma \leq 1$ is necessary for the desired inequality to be valid.

□

From the second γ-orderliness for a right-skewed distribution, $\frac{\partial^2 \mathrm{QA}}{\partial^2 \epsilon} \geq 0 \Rightarrow \forall 0 \leq a \leq 2a \leq 3a \leq \frac{1}{1+\gamma}, \frac{1}{a}\left(\frac{(\mathrm{QA}(3a,\gamma) - \mathrm{QA}(2a,\gamma))}{a} - \frac{(\mathrm{QA}(2a,\gamma) - \mathrm{QA}(a,\gamma))}{a}\right) \geq 0 \Rightarrow$ if $0 \leq \gamma \leq 1$, $\mathcal{B}_i - 2\mathcal{B}_{i+1} + \mathcal{B}_{i+2} \geq 0$. SM_ϵ can thus be interpreted as assuming $\gamma = 1$ and replacing the two blocks, $\mathcal{B}_i + \mathcal{B}_{i+2}$ with one block $2\mathcal{B}_{i+1}$. From the νth γ-orderliness for a right-skewed distribution, the recurrence relation of the derivatives naturally produces the alternating binomial coefficients,

$$(-1)^\nu \frac{\partial^\nu \mathrm{QA}}{\partial \epsilon^\nu} \geq 0 \Rightarrow \forall 0 \leq a \leq \ldots \leq (\nu+1)a \leq \frac{1}{1+\gamma},$$

$$\frac{(-1)^\nu}{a}\left(\frac{\frac{\mathrm{QA}(\nu a + a,\gamma)^{\cdot^{\cdot^{\cdot}}}}{a} - \frac{\cdot^{\cdot^{\cdot}}\mathrm{QA}(2a,\gamma)}{a}}{a} - \frac{\frac{\mathrm{QA}(\nu a,\gamma)^{\cdot^{\cdot^{\cdot}}}}{a} - \frac{\cdot^{\cdot^{\cdot}}\mathrm{QA}(a,\gamma)}{a}}{a}\right)$$

$$\geq 0 \Leftrightarrow \frac{(-1)^\nu}{a^\nu}\left(\sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} \mathrm{QA}\left((\nu - j + 1)\,a, \gamma\right)\right) \geq 0$$

$$\Rightarrow \text{if } 0 \leq \gamma \leq 1, \sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} \mathcal{B}_{i+j} \geq 0.$$

Based on the νth orderliness, the ϵ,γ-binomial mean is introduced as

$$\mathrm{BM}_{\nu,\epsilon,\gamma,n} \coloneqq \frac{1}{n}\left(\sum_{i=1}^{\frac{1}{2}\epsilon^{-1}(\nu+1)^{-1}} \sum_{j=0}^{\nu} \left(1 - (-1)^j \binom{\nu}{j}\right) \mathfrak{B}_{i_j}\right),$$

where $\mathfrak{B}_{i_j} = \sum_{l=n\gamma\epsilon(j+(i-1)(\nu+1))+1}^{n\epsilon(j+(i-1)(\nu+1)+1)} (X_l + X_{n-l+1})$. If ν is not indicated, it defaults to $\nu = 3$. Since the alternating sum of binomial coefficients equals zero, when $\nu \ll \epsilon^{-1}$ and $\epsilon \to 0$, $\mathrm{BM} \to \mu$. The solutions for the continuity of the breakdown point is the same as that in SM and not repeated here. The equalities $\mathrm{BM}_{\nu=1,\epsilon} = \mathrm{BWM}_\epsilon$ and $\mathrm{BM}_{\nu=2,\epsilon} = \mathrm{SM}_{\epsilon,b=3}$ hold, when $\gamma = 1$ and their respective ϵs are identical. Interestingly, the biases of the $\mathrm{SM}_{\epsilon=\frac{1}{9},b=3}$ and the $\mathrm{WM}_{\epsilon=\frac{1}{9}}$ are nearly indistinguishable in common asymmetric unimodal distributions such as Weibull, gamma, lognormal, and Pareto (SI Dataset S1). This indicates that their robustness to departures from the symmetry assumption is practically similar under unimodality, even though they are based on different orders of orderliness. If single quantiles are used, based on the second γ-orderliness, the stratified quantile mean can be defined as

$$\mathrm{SQM}_{\epsilon,\gamma,n} \coloneqq 4\epsilon \sum_{i=1}^{\frac{1}{4\epsilon}} \frac{1}{2}\left(\hat{Q}_n\left((2i-1)\gamma\epsilon\right) + \hat{Q}_n\left(1 - (2i-1)\epsilon\right)\right),$$

$\mathrm{SQM}_{\epsilon=\frac{1}{4}}$ is the Tukey's midhinge (1). In fact, SQM is a subcase of SM when $\gamma = 1$ and $b \to \infty$, so the solution for the continuity of the breakdown point, $\frac{1}{\epsilon} \bmod 4 \neq 0$, is identical. However, since the definition is based on the empirical quantile function, no decimal issues related to order statistics will arise. The next theorem explains another advantage.

Theorem .4. *For a right-skewed second γ-ordered distribution, asymptotically, $\mathrm{SQM}_{\epsilon,\gamma}$ is always greater or equal to the corresponding $\mathrm{BM}_{\nu=2,\epsilon,\gamma}$ with the same ϵ and γ, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, if $0 \leq \gamma \leq 1$.*

Proof. For simplicity, suppose the order statistics of the sample are distributed into $\epsilon^{-1} \in \mathbb{N}$ blocks in the computation of both $\mathrm{SQM}_{\epsilon,\gamma}$ and $\mathrm{BM}_{\nu=2,\epsilon,\gamma}$. The computation of $\mathrm{BM}_{\nu=2,\epsilon,\gamma}$ alternates between weighting and non-weighting, let '0' denote the block assigned with a weight of zero and '1' denote the block assigned with a weighted of one, the sequence indicating the weighted or non-weighted status of each block is: $0, 1, 0, 0, 1, 0, \ldots$. Let this sequence be denoted by $a_{\mathrm{BM}_{\nu=2,\epsilon,\gamma}}(j)$, its formula is $a_{\mathrm{BM}_{\nu=2,\epsilon,\gamma}}(j) = \left\lfloor \frac{j \bmod 3}{2} \right\rfloor$. Similarly, the computation of $\mathrm{SQM}_{\epsilon,\gamma}$ can be seen as positioning quantiles (p) at the beginning of the blocks if $0 < p < \frac{1}{1+\gamma}$, and at the end of the blocks if $p > \frac{1}{1+\gamma}$. The sequence of denoting whether each block's quantile is weighted or not weighted is: $0, 1, 0, 1, 0, 1, \ldots$. Let the sequence be denoted by $a_{\mathrm{SQM}_{\epsilon,\gamma}}(j)$, the formula of the sequence is $a_{\mathrm{SQM}_{\epsilon,\gamma}}(j) = j \bmod 2$. If pairing all blocks in $\mathrm{BM}_{\nu=2,\epsilon,\gamma}$ and all quantiles in $\mathrm{SQM}_{\epsilon,\gamma}$, there are two possible pairings of $a_{\mathrm{BM}_{\nu=2}}(j)$ and $a_{\mathrm{SQM}_{\epsilon,\gamma}}(j)$. One pairing occurs when $a_{\mathrm{BM}_{\nu=2,\epsilon,\gamma}}(j) = a_{\mathrm{SQM}_{\epsilon,\gamma}}(j) = 1$, while the other involves the sequence $0, 1, 0$ from $a_{\mathrm{BM}_{\nu=2,\epsilon,\gamma}}(j)$ paired with $1, 0, 1$ from $a_{\mathrm{SQM}_{\epsilon,\gamma}}(j)$. By leveraging the same principle as Theorem .3 and the second γ-orderliness (replacing the two quantile averages with one quantile average between them), the desired result follows. □

The biases of $\mathrm{SQM}_{\epsilon=\frac{1}{8}}$, which is based on the second orderliness with a quantile approach, are notably similar to those of $\mathrm{BM}_{\nu=3,\epsilon=\frac{1}{8}}$, which is based on the third orderliness with a block approach, in common asymmetric unimodal distributions (Figure **??**).

Hodges–Lehmann inequality and γ-U-orderliness

The Hodges–Lehmann estimator stands out as a unique robust location estimator due to its definition being substantially dissimilar from conventional L-estimators, R-estimators, and M-estimators. In their landmark paper, *Estimates of location based on rank tests*, Hodges and Lehmann (2) proposed two methods for computing the H-L estimator: the Wilcoxon score R-estimator and the median of pairwise means. The Wilcoxon score R-estimator is a location estimator based on signed-rank test, or R-estimator, (2) and was later independently discovered by Sen (1963) (3, 4). However, the median of pairwise means is a generalized L-statistic and a trimmed U-statistic, as classified by Serfling in his novel conceptualized study in 1984 (5). Serfling further advanced the understanding by generalizing the H-L kernel as $hl_k(x_1,\ldots,x_k)=\frac{1}{k}\sum_{i=1}^{k}x_i$, where $k\in\mathbb{N}$ (5). Here, the weighted H-L kernel is defined as $whl_k(x_1,\ldots,x_k)=\frac{\sum_{i=1}^{k}x_i\mathbf{w}_i}{\sum_{i=1}^{k}\mathbf{w}_i}$, where $\mathbf{w}_i$s are the weights applied to each element.

By using the weighted H-L kernel and the L-estimator, it is now clear that the Hodges-Lehmann estimator is an LL-statistic, the definition of which is provided as follows:

$$LL_{k,\epsilon,\gamma,n} \coloneqq L_{\epsilon_0,\gamma,n}\left(\mathrm{sort}\left(\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)\right),$$

where $L_{\epsilon_0,\gamma,n}(Y)$ represents the ϵ_0,γ-L-estimator that uses the sorted sequence, $\mathrm{sort}\left(\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)$, as input. The upper asymptotic breakdown point of $LL_{k,\epsilon,\gamma}$ is $\epsilon=1-(1-\epsilon_0)^{\frac{1}{k}}$, as proven in DSSM II. There are two ways to adjust the breakdown point: either by setting k as a constant and adjusting ϵ_0, or by setting ϵ_0 as a constant and adjusting k. In the above definition, k is discrete, but the bootstrap method can be applied to ensure the continuity of k, also making the breakdown point continuous. Specifically, if $k\in\mathbb{R}$, let the bootstrap size be denoted by b, then first sampling the original sample $(1-k+\lfloor k\rfloor)b$ times with each sample size of $\lfloor k\rfloor$, and then subsequently sampling $(1-\lceil k\rceil+k)b$ times with each sample size of $\lceil k\rceil$, $(1-k+\lfloor k\rfloor)b\in\mathbb{N}$, $(1-\lceil k\rceil+k)b\in\mathbb{N}$. The corresponding kernels are computed separately, and the pooled sorted sequence is used as the input for the L-estimator. Let $\mathbf{S}_k$ represent the sorted sequence. Indeed, for any finite sample, X, when $k=n$, $\mathbf{S}_k$ becomes a single point, $whl_{k=n}(X_1,\ldots,X_n)$. When $\mathbf{w}_i=1$, the minimum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k}X_i$, due to the property of order statistics. The maximum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k}X_{n-i+1}$. The monotonicity of the order statistics implies the monotonicity of the extrema with respect to k, i.e., the support of $\mathbf{S}_k$ shrinks monotonically. For unequal $\mathbf{w}_i$s, the shrinkage of the support of $\mathbf{S}_k$ might not be strictly monotonic, but the general trend remains, since all LL-statistics converge to the same point, as $k\to n$. Therefore, if $\frac{\sum_{i=1}^{n}X_i\mathbf{w}_i}{\sum_{i=1}^{n}\mathbf{w}_i}$ approaches the population mean when $n\to\infty$, all LL-statistics based on such consistent kernel function approach the population mean as $k\to\infty$. For example, if $whl_k=\mathrm{BM}_{\nu,\epsilon_k,n=k}$, $\nu\ll\epsilon_k^{-1}$, $\epsilon_k\to 0$, such kernel function is consistent. These cases are termed the LL-mean ($\mathrm{LLM}_{k,\epsilon,\gamma,n}$). By substituting the $\mathrm{WA}_{\epsilon_0,\gamma,n}$ for the $L_{\epsilon_0,\gamma,n}$ in LL-statistic, the resulting statistic is referred to as the weighted L-statistic ($\mathrm{WL}_{k,\epsilon,\gamma,n}$). The case having a consistent kernel function is termed as the weighted L-mean ($\mathrm{WLM}_{k,\epsilon,\gamma,n}$). The $w_i=1$ case of $\mathrm{WLM}_{k,\epsilon,\gamma,n}$ is termed the weighted Hodges-Lehmann mean ($\mathrm{WHLM}_{k,\epsilon,\gamma,n}$). The $\mathrm{WHLM}_{k=1,\epsilon,\gamma,n}$ is the weighted average. If $k\ge 2$ and the WA in WHLM is set as TM_{ϵ_0}, it is called the trimmed H-L mean (Figure ??, $k=2$, $\epsilon_0=\frac{15}{64}$). The $\mathrm{THLM}_{k=2,\epsilon,\gamma=1,n}$ appears similar to the Wilcoxon's one-sample statistic investigated by Saleh in 1976 (6), which involves first censoring the sample, and then computing the mean of the number of events that the pairwise mean is greater than zero. The $\mathrm{THLM}_{k=2,\epsilon=1-\left(1-\frac{1}{2}\right)^{\frac{1}{2}},\gamma=1,n}$ is the Hodges-Lehmann estimator, or more generally, a special case of the median Hodges-Lehmann mean ($m\mathrm{HLM}_{k,n}$). $m\mathrm{HLM}_{k,n}$ is asymptotically equivalent to the $\mathrm{MoM}_{k,b=\frac{n}{k}}$ as discussed previously, Therefore, it is possible to define a series of location estimators, analogous to the WHLM, based on MoM. For example, the γ-median of means, $\gamma m\mathrm{oM}_{k,b=\frac{n}{k},n}$, is defined by replacing the median in $\mathrm{MoM}_{k,b=\frac{n}{k},n}$ with the γ-median.

The hl_k kernel distribution, denoted as F_{hl_k}, can be defined as the probability distribution of the sorted sequence $\mathrm{sort}\left(\left(hl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)$. For any real value y, the cdf of the hl_k kernel distribution is given by: $F_{h_{\mathbf{k}}}(y)=\Pr(Y_i\le y)$, where Y_i represents an individual element from the sorted sequence. The overall hl_k kernel distributions possess a two-dimensional structure, encompassing n kernel distributions with varying k values, from 1 to n, where one dimension is inherent to each individual kernel distribution, while the other is formed by the alignment of the same percentiles across all kernel distributions. As k increases, all percentiles converge to $\bar{X}$, leading to the concept of γ-U-orderliness:

$$\left(\forall k_2\ge k_1\ge 1, \gamma m\mathrm{HLM}_{k_2,\epsilon=1-\left(\frac{\gamma}{1+\gamma}\right)^{\frac{1}{k_2}},\gamma}\ge\gamma m\mathrm{HLM}_{k_1,\epsilon=1-\left(\frac{\gamma}{1+\gamma}\right)^{\frac{1}{k_1}},\gamma}\right)\vee$$
$$\left(\forall k_2\ge k_1\ge 1, \gamma m\mathrm{HLM}_{k_2,\epsilon=1-\left(\frac{\gamma}{1+\gamma}\right)^{\frac{1}{k_2}},\gamma}\le\gamma m\mathrm{HLM}_{k_1,\epsilon=1-\left(\frac{\gamma}{1+\gamma}\right)^{\frac{1}{k_1}},\gamma}\right),$$

where $\gamma m\mathrm{HLM}_k$ sets the WA in WHLM as γ-median, with γ being constant. The direction of the inequality depends on the relative magnitudes of $\gamma m\mathrm{HLM}_{k=1,\epsilon,\gamma}=\gamma m$ and $\gamma m\mathrm{HLM}_{k=\infty,\epsilon,\gamma}=\mu$. The Hodges-Lehmann inequality can be defined as a special case of the γ-U-orderliness when $\gamma=1$. When $\gamma\in\{0,\infty\}$, the γ-U-orderliness is valid for any distribution as previously shown. If $\gamma\notin\{0,\infty\}$, analytically proving the validity of the γ-U-orderliness for a parametric distribution is pretty challenging. As an example, the hl_2 kernel distribution has a probability density function $f_{hl_2}(x)=\int_0^{2x}2f(t)\,f(2x-t)\,dt$ (a result after the transformation of variables); the support of the original distribution is assumed to be $[0,\infty)$ for simplicity. The expected value of the H-L estimator is the positive solution of $\int_0^{\text{H-L}}\left(f_{hl_2}(s)\right)ds=\frac{1}{2}$. For the exponential distribution, $f_{hl_2,exp}(x)=4\lambda^{-2}xe^{-2\lambda^{-1}x}$, λ is a scale parameter, $E[\text{H-L}]=\frac{-W_{-1}\left(-\frac{1}{2e}\right)-1}{2}\lambda\approx 0.839\lambda$, where W_{-1} is a branch of the Lambert W function which cannot be expressed in terms of elementary functions. However, the violation of the γ-U-orderliness is bounded under certain assumptions, as shown below.

Theorem .5. *For any distribution with a finite second central moment, σ^2, the following concentration bound can be established for the γ-median of means,*

$$\mathbb{P}\left(\gamma moM_{k,b=\frac{n}{k},n}-\mu>\frac{t\sigma}{\sqrt{k}}\right)\le e^{-\frac{2n}{k}\left(\frac{1}{1+\gamma}-\frac{1}{k+t^2}\right)^2}.$$

Proof. Denote the mean of each block as $\widehat{\mu_i}$, $1\le i\le b$. Observe that the event $\left\{\gamma m\mathrm{oM}_{k,b=\frac{n}{k},n}-\mu>\frac{t\sigma}{\sqrt{k}}\right\}$ necessitates the condition that there are at least $b(1-\frac{\gamma}{1+\gamma})$ of $\widehat{\mu_i}$s larger than μ by more than $\frac{t\sigma}{\sqrt{k}}$, i.e., $\left\{\gamma m\mathrm{oM}_{k,b=\frac{n}{k},n}-\mu>\frac{t\sigma}{\sqrt{k}}\right\}\subset\left\{\sum_{i=1}^{b}\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}\ge b\left(1-\frac{\gamma}{1+\gamma}\right)\right\}$, where $\mathbf{1}_A$ is the indicator of event A. Assuming a finite second central moment, σ^2, it follows from one-sided Chebeshev's inequality that $\mathbb{E}\left(\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}\right)=\mathbb{P}\left(\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}\right)\le\frac{\sigma^2}{k\sigma^2+t^2\sigma^2}$.

Given that $\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}\in[0,1]$ are independent and identically distributed random variables, according to the aforementioned inclusion relation, the one-sided Chebeshev's inequality and the one-sided Hoeffding's inequality, $\mathbb{P}\left(\gamma m o\mathrm{M}_{k,b=\frac{n}{k},n}-\mu>\frac{t\sigma}{\sqrt{k}}\right) \leq \mathbb{P}\left(\sum_{i=1}^{b}\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}\geq b\left(1-\frac{\gamma}{1+\gamma}\right)\right) = \mathbb{P}\left(\frac{1}{b}\sum_{i=1}^{b}\left(\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}-\mathbb{E}\left(\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}\right)\right)\geq \left(1-\frac{\gamma}{1+\gamma}\right)-\mathbb{E}\left(\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}\right)\right) \leq e^{-2b\left(\left(1-\frac{\gamma}{1+\gamma}\right)-\mathbb{E}\left(\mathbf{1}_{\left(\widehat{\mu_i}-\mu\right)>\frac{t\sigma}{\sqrt{k}}}\right)\right)^2} \leq e^{-2b\left(1-\frac{\gamma}{1+\gamma}-\frac{\sigma^2}{k\sigma^2+t^2\sigma^2}\right)^2} = e^{-2b\left(\frac{1}{1+\gamma}-\frac{1}{k+t^2}\right)^2}$. □

Theorem .6. *Let* $B(k,\gamma,t,n) = e^{-\frac{2n}{k}\left(\frac{1}{1+\gamma}-\frac{1}{k+t^2}\right)^2}$. *If* $n\in\mathbb{N}$, $\gamma\geq 0$, $0\leq t^2<\gamma+1$, *and* $\gamma-t^2+1\leq k\leq \frac{1}{2}\sqrt{9\gamma^2+18\gamma-8\gamma t^2-8t^2+9}+\frac{1}{2}\left(3\gamma-2t^2+3\right)$, *$B$ is monotonic decreasing with respect to k.*

Proof. Since $\frac{\partial B}{\partial k} = \left(\frac{2n\left(\frac{1}{\gamma+1}-\frac{1}{k+t^2}\right)^2}{k^2}-\frac{4n\left(\frac{1}{\gamma+1}-\frac{1}{k+t^2}\right)}{k\left(k+t^2\right)^2}\right) e^{-\frac{2n\left(\frac{1}{\gamma+1}-\frac{1}{k+t^2}\right)^2}{k}}$ and $n\in\mathbb{N}$, $\frac{\partial B}{\partial k}\leq 0 \Leftrightarrow \frac{2n\left(\frac{1}{\gamma+1}-\frac{1}{k+t^2}\right)^2}{k^2}-\frac{4n\left(\frac{1}{\gamma+1}-\frac{1}{k+t^2}\right)}{k\left(k+t^2\right)^2}\leq 0 \Leftrightarrow \frac{2n\left(-\gamma+k+t^2-1\right)\left(k^2-3(\gamma+1)k+2kt^2+t^2\left(-\gamma+t^2-1\right)\right)}{(\gamma+1)^2k^2\left(k+t^2\right)^3}\leq 0 \Leftrightarrow \left(-\gamma+k+t^2-1\right)\left(k^2-3(\gamma+1)k+2kt^2+t^2\left(-\gamma+t^2-1\right)\right)\leq 0$. When the factors are expanded, it yields a cubic inequality in terms of k: $k^3+k^2\left(3t^2-4(\gamma+1)\right)+3k\left(\gamma-t^2+1\right)^2+t^2\left(\gamma-t^2+1\right)^2\leq 0$. Assuming $0\leq t^2<\gamma+1$ and $\gamma\geq 0$, using the factored form and subsequently applying the quadratic formula, the inequality is valid if $\gamma-t^2+1\leq k\leq \frac{1}{2}\sqrt{9\gamma^2+18\gamma-8\gamma t^2-8t^2+9}+\frac{1}{2}\left(3\gamma-2t^2+3\right)$. □

Let X be a random variable and $\bar{Y}=\frac{1}{k}(Y_1+\cdots+Y_k)$ be the average of k independent, identically distributed copies of X. Applying the variance operation gives: $\mathrm{Var}(\bar{Y}) = \mathrm{Var}\left(\frac{1}{k}(Y_1+\cdots+Y_k)\right) = \frac{1}{k^2}(\mathrm{Var}(Y_1)+\cdots+\mathrm{Var}(Y_k)) = \frac{1}{k^2}(k\sigma^2) = \frac{\sigma^2}{k}$, since the variance operation is a linear operator for independent variables, and the variance of a scaled random variable is the square of the scale times the variance of the variable, i.e., $\mathrm{Var}(cX)=E[(cX-E[cX])^2]=E[(cX-cE[X])^2]=E[c^2(X-E[X])^2]=c^2E[((X)-E[X])^2]=c^2\mathrm{Var}(X)$. Thus, the standard deviation of the hl_k kernel distribution, asymptotically, is $\frac{\sigma}{\sqrt{k}}$. By utilizing the asymptotic bias bound of any quantile for any continuous distribution with a finite second central moment, σ^2,(7), a conservative asymptotic bias bound of $\gamma m o\mathrm{M}_{k,b=\frac{n}{k}}$ can be established as $\gamma m o\mathrm{M}_{k,b=\frac{n}{k}}-\mu\leq\sqrt{\frac{\frac{\gamma}{1+\gamma}}{1-\frac{\gamma}{1+\gamma}}}\sigma_{hl_k}=\sqrt{\frac{\gamma}{k}}\sigma$. That implies in Theorem .5, $t<\sqrt{\gamma}$, so when $\gamma=1$, the upper bound of k, subject to the monotonic decreasing constraint, is $2+\sqrt{5}<\frac{1}{2}\sqrt{9+18-8t^2-8t^2+9}+\frac{1}{2}\left(3-2t^2+3\right)\leq 6$, the lower bound is $1<2-t^2\leq 2$. These analyses elucidate a surprising result: although the conservative asymptotic bound of $\mathrm{MoM}_{k,b=\frac{n}{k}}$ is monotonic with respect to k, its concentration bound is optimal when $k\in(2+\sqrt{5},6]$.

Then consider the structure within each individual hl_k kernel distribution. The sorted sequence $\mathbf{S}_k$, when $k=n-1$, has n elements and the corresponding hl_k kernel distribution can be seen as a location-scale transformation of the original distribution, so the corresponding hl_k kernel distribution is νth γ-ordered if and only if the original distribution is νth γ-ordered according to Theorem **??**. Analytically proving other cases is challenging. For example, $f_{hl_2}'(x)=4f(2x)f(0)+\int_0^{2x}4f(t)f'(2x-t)\,dt$, the strict negative of $f_{hl_2}'(x)$ is not guaranteed if just assuming $f'(x)<0$, so, even if the original distribution is monotonic decreasing, the hl_2 kernel distribution might be non-monotonic. Also, unlike the pairwise difference distribution, if the original distribution is unimodal, the pairwise mean distribution might be non-unimodal, as demonstrated by a counterexample given by Chung in 1953 and mentioned by Hodges and Lehmann in 1954 (8, 9). Theorem **??** implies that the violation of νth γ-orderliness within the hl_k kernel distribution is also bounded, and the bound monotonically shrinks as k increases because the bound is in unit of the standard deviation of the hl_k kernel distribution. If all hl_k kernel distributions are νth γ-ordered and the distribution itself is νth γ-ordered and γ-U-ordered, then the distribution is called νth γ-U-ordered. The following theorems highlight the significance of γ-symmetric distributions.

Theorem .7. *Any γ-symmetric distribution is νth γ-U-ordered, provided that the γ is the same.*

The succeeding theorem shows that the whl_k kernel distribution is invariably a location-scale distribution if the original distribution belongs to a location-scale family with the same location and scale parameters.

Theorem .8. $whl_k\left(x_1=\lambda x_1+\mu,\ldots,x_k=\lambda x_k+\mu\right) = \lambda whl_k\left(x_1,\ldots,x_k\right)+\mu$.

Proof. $whl_k\left(x_1=\lambda x_1+\mu,\cdots,x_k=\lambda x_k+\mu\right) = \frac{\sum_{i=1}^{k}(\lambda x_i+\mu)w_i}{\sum_{i=1}^{k}w_i} = \frac{\sum_{i=1}^{k}\lambda x_iw_i+\sum_{i=1}^{k}\mu w_i}{\sum_{i=1}^{k}w_i} = \lambda\frac{\sum_{i=1}^{k}x_iw_i}{\sum_{i=1}^{k}w_i}+\frac{\sum_{i=1}^{k}\mu w_i}{\sum_{i=1}^{k}w_i} = \lambda\frac{\sum_{i=1}^{k}x_iw_i}{\sum_{i=1}^{k}w_i}+\mu = \lambda whl_k\left(x_1,\cdots,x_k\right)+\mu$. □

According to Theorem .8, the γ-weighted inequality for a right-skewed distribution can be modified as $\forall 0\leq\epsilon_{0_1}\leq\epsilon_{0_2}\leq\frac{1}{1+\gamma}$, $\mathrm{WLM}_{k,\epsilon=1-\left(1-\epsilon_{0_1}\right)^{\frac{1}{k}},\gamma}\geq\mathrm{WLM}_{k,\epsilon=1-\left(1-\epsilon_{0_2}\right)^{\frac{1}{k}},\gamma}$, which holds the same rationale as the γ-weighted inequality defined in the last section. If the νth γ-orderliness is valid for the whl_k kernel distribution, then all results in the last section can be directly implemented. From that, the binomial H-L mean (set the WA as BM) can be constructed (Figure **??**), while its maximum breakdown point is ≈ 0.065 if $\nu=3$. A comparison of the biases of $\mathrm{BM}_{\nu=3,\epsilon=\frac{1}{8}}$, $\mathrm{SQM}_{\epsilon=\frac{1}{8}}$, $\mathrm{THLM}_{k=2,\epsilon=\frac{1}{8}}$, $\mathrm{WHLM}_{k=2,\epsilon=\frac{1}{8}}$, $\mathrm{MHHLM}_{k=\frac{2\ln(2)-\ln(3)}{3\ln(2)-\ln(7)},\epsilon=\frac{1}{8}}$ (midhinge H-L mean), $m\mathrm{HLM}_{k=\frac{\ln(2)}{3\ln(2)-\ln(7)},\epsilon=\frac{1}{8}}$, $\mathrm{THLM}_{k=5,\epsilon=\frac{1}{8}}$, and $\mathrm{WHLM}_{k=5,\epsilon=\frac{1}{8}}$ is appropriate (Figure **??**, SI Dataset S1), given their same breakdown points, with $m\mathrm{HLM}_{k=\frac{\ln(2)}{3\ln(2)-\ln(7)},\epsilon=\frac{1}{8}}$ exhibiting the smallest biases. Another comparison among the H-L estimator, the trimmed mean, and the Winsorized mean, all with the same breakdown point, yields the same result that the H-L estimator has the smallest biases (SI Dataset S1). This aligns with Devroye et al.(2016)'s seminal work that MoM is nearly optimal with regards to concentration bounds for heavy-tailed distributions (10).

In 1958, Richtmyer introduced the concept of quasi-Monte Carlo simulation that utilizes low-discrepancy sequences, resulting in a significant reduction in computational expenses for large sample simulation (11). Among various low-discrepancy sequences, Sobol sequences are often favored in quasi-Monte Carlo methods (12). Building upon this principle, in 1991, Do and Hall extended it to bootstrap and found that the quasi-random approach resulted in lower variance compared to other bootstrap Monte Carlo procedures (13). By using a deterministic approach, the variance of $m\mathrm{HLM}_{k,n}$ is much lower than that of $\mathrm{MoM}_{k,b=\frac{n}{k}}$ (SI Dataset S1), when k is small. This highlights the superiority of the median Hodges-Lehmann mean over the median of means, as it not only can provide an accurate estimate for moderate sample sizes, but also allows the use of quasi-bootstrap, where the bootstrap size can be adjusted as needed.

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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