Semiparametric robust mean estimations based on the orderliness of quantile averages

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This manuscript was compiled on June 11, 2023

semiparametric | mean-median-mode inequality | asymptotic | unimodal | Hodges–Lehmann estimator

Hodges–Lehmann inequality and γ-U-orderliness

The Hodges–Lehmann estimator stands out as a unique robust location estimator due to its definition being substantially dissimilar from conventional L-estimators, R-estimators, and M-estimators. In their landmark paper, *Estimates of location based on rank tests*, Hodges and Lehmann (1) proposed two methods for computing the H-L estimator: the Wilcoxon score R-estimator and the median of pairwise means. The Wilcoxon score R-estimator is a location estimator based on signed-rank test, or R-estimator, (1) and was later independently discovered by Sen (1963) (2, 3). However, the median of pairwise means is a generalized L-statistic and a trimmed U-statistic, as classified by Serfling in his novel conceptualized study in 1984 (4). Serfling further advanced the understanding by generalizing the H-L kernel as $hl_k\left(x_1,\ldots,x_k\right)=\frac{1}{k}\sum_{i=1}^{k}x_i$, where $k\in\mathbb{N}$ (4). Here, the weighted H-L kernel is defined as $whl_k\left(x_1,\ldots,x_k\right)=\frac{\sum_{i=1}^{k}x_i\mathbf{w}_i}{\sum_{i=1}^{k}\mathbf{w}_i}$, where $\mathbf{w}_i$s are the weights applied to each element.

By using the weighted H-L kernel and the L-estimator, it is now clear that the Hodges-Lehmann estimator is an LL-statistic, the definition of which is provided as follows:

$$LL_{k,\epsilon,\gamma,n}\coloneqq L_{\epsilon_0,\gamma,n}\left(\operatorname{sort}\left(\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)\right),$$

where $L_{\epsilon_0,\gamma,n}\left(Y\right)$ represents the ϵ_0,γ-L-estimator that uses the sorted sequence, $\operatorname{sort}\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}$, as input. The upper asymptotic breakdown point of $LL_{k,\epsilon,\gamma}$ is $\epsilon=1-\left(1-\epsilon_0\right)^{\frac{1}{k}}$, as proven in DSSM II. There are two ways to adjust the breakdown point: either by setting k as a constant and adjusting ϵ_0, or by setting ϵ_0 as a constant and adjusting k. In the above definition, k is discrete, but the bootstrap method can be applied to ensure the continuity of k, also making the breakdown point continuous. Specifically, if $k\in\mathbb{R}$, let the bootstrap size be denoted by b, then first sampling the original sample $(1-k+\lfloor k\rfloor)b$ times with each sample size of $\lfloor k\rfloor$, and then subsequently sampling $(1-\lceil k\rceil+k)b$ times with each sample size of $\lceil k\rceil$, $(1-k+\lfloor k\rfloor)b\in\mathbb{N}$, $(1-\lceil k\rceil+k)b\in\mathbb{N}$. The corresponding kernels are computed separately, and the pooled sorted sequence is used as the input for the L-estimator. Let $\mathbf{S}_k$ represent the sorted sequence. Indeed, for any finite sample, X, when $k=n$, $\mathbf{S}_k$ becomes a single point, $whl_{k=n}\left(X_1,\ldots,X_n\right)$. When $\mathbf{w}_i=1$, the minimum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k}X_i$, due to the property of order statistics. The maximum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k}X_{n-i+1}$. The monotonicity of the order statistics implies the monotonicity of the extrema with respect to k, i.e., the support of $\mathbf{S}_k$ shrinks monotonically. For unequal $\mathbf{w}_i$s, the shrinkage of the support of $\mathbf{S}_k$ might not be strictly monotonic, but the general trend remains, since all LL-statistics converge to the same point, as $k\to n$. Therefore, if $\frac{\sum_{i=1}^{n}X_i\mathbf{w}_i}{\sum_{i=1}^{n}\mathbf{w}_i}$ approaches the population mean when $n\to\infty$, all LL-statistics based on such consistent kernel function approach the population mean as $k\to\infty$. For example, if $whl_k=\mathrm{BM}_{\nu,\epsilon_k,n=k}$, $\nu\ll\epsilon_k^{-1}$, $\epsilon_k\to 0$, such kernel function is consistent. These cases are termed the LL-mean ($\mathrm{LLM}_{k,\epsilon,\gamma,n}$). By substituting the $\mathrm{WA}_{\epsilon_0,\gamma,n}$ for the $L_{\epsilon_0,\gamma,n}$ in LL-statistic, the resulting statistic is referred to as the weighted L-statistic ($\mathrm{WL}_{k,\epsilon,\gamma,n}$). The case having a consistent kernel function is termed as the weighted L-mean ($\mathrm{WLM}_{k,\epsilon,\gamma,n}$). The $w_i=1$ case of $\mathrm{WLM}_{k,\epsilon,\gamma,n}$ is termed the weighted Hodges-Lehmann mean ($\mathrm{WHLM}_{k,\epsilon,\gamma,n}$). The $\mathrm{WHLM}_{k=1,\epsilon,\gamma,n}$ is the weighted average. If $k\geq 2$ and the WA in WHLM is set as TM_{ϵ_0}, it is called the trimmed H-L mean (Figure **??**, $k=2$, $\epsilon_0=\frac{15}{64}$).

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

ACKNOWLEDGMENTS. I sincerely acknowledge the insightful comments from the editor which considerably elevated the lucidity and merit of this paper.

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T.L. designed research, performed research, analyzed data, and wrote the paper.

The author declares no competing interest.

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