Semiparametric robust mean estimations based on the orderliness of quantile averages

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Hodges–Lehmann inequality and γ-U-orderliness

The Hodges–Lehmann estimator stands out as a unique robust location estimator due to its definition being substantially dissimilar from conventional L-estimators, R-estimators, and M-estimators. In their landmark paper, *Estimates of location based on rank tests*, Hodges and Lehmann (1) proposed two methods for computing the H-L estimator: the Wilcoxon score R-estimator and the median of pairwise means. The Wilcoxon score R-estimator is a location estimator based on signed-rank test, or R-estimator, (1) and was later independently discovered by Sen (1963) (2, 3). However, the median of pairwise means is a generalized L-statistic and a trimmed U-statistic, as classified by Serfling in his novel conceptualized study in 1984 (4). Serfling further advanced the understanding by generalizing the H-L kernel as $hl_k\left(x_1,\ldots,x_k\right)=\frac{1}{k}\sum_{i=1}^{k}x_i$, where $k\in\mathbb{N}$ (4). Here, the weighted H-L kernel is defined as $whl_k\left(x_1,\ldots,x_k\right)=\frac{\sum_{i=1}^{k}x_i\mathbf{w}_i}{\sum_{i=1}^{k}\mathbf{w}_i}$, where $\mathbf{w}_i$s are the weights applied to each element.

By using the weighted H-L kernel and the L-estimator, it is now clear that the Hodges-Lehmann estimator is an LL-statistic, the definition of which is provided as follows:

$$LL_{k,\epsilon,\gamma,n}\coloneqq L_{\epsilon_0,\gamma,n}\left(\text{sort}\left(\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)\right),$$

where $L_{\epsilon_0,\gamma,n}\left(Y\right)$ represents the ϵ_0,γ-L-estimator that uses the sorted sequence, $\text{sort}\left(\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)$, as input. The upper asymptotic breakdown point of $LL_{k,\epsilon,\gamma}$ is $\epsilon=1-\left(1-\epsilon_0\right)^{\frac{1}{k}}$, as proven in DSSM II. There are two ways to adjust the breakdown point: either by setting k as a constant and adjusting ϵ_0, or by setting ϵ_0 as a constant and adjusting k. In the above definition, k is discrete, but the bootstrap method can be applied to ensure the continuity of k, also making the breakdown point continuous. Specifically, if $k\in\mathbb{R}$, let the bootstrap size be denoted by b, then first sampling the original sample $(1-k+\lfloor k\rfloor)b$ times with each sample size of $\lfloor k\rfloor$, and then subsequently sampling $(1-\lceil k\rceil+k)b$ times with each sample size of $\lceil k\rceil$, $(1-k+\lfloor k\rfloor)b\in\mathbb{N}$, $(1-\lceil k\rceil+k)b\in\mathbb{N}$. The corresponding kernels are computed separately, and the pooled sorted sequence is used as the input for the L-estimator. Let $\mathbf{S}_k$ represent the sorted sequence. Indeed, for any finite sample, X, when $k=n$, $\mathbf{S}_k$ becomes a single point, $whl_{k=n}\left(X_1,\ldots,X_n\right)$. When $\mathbf{w}_i=1$, the minimum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k}X_i$, due to the property of order statistics. The maximum of $\mathbf{S}_k$ is $\frac{1}{k}\sum_{i=1}^{k}X_{n-i+1}$. The monotonicity of the order statistics implies the monotonicity of the extrema with respect to k, i.e., the support of $\mathbf{S}_k$ shrinks monotonically. For unequal $\mathbf{w}_i$s, the shrinkage of the support of $\mathbf{S}_k$ might not be strictly monotonic, but the general trend remains, since all LL-statistics converge to the same point, as $k\to n$. Therefore, if $\frac{\sum_{i=1}^{n}X_i\mathbf{w}_i}{\sum_{i=1}^{n}\mathbf{w}_i}$ approaches the population mean when $n\to\infty$, all LL-statistics based on such consistent kernel function approach the population mean as $k\to\infty$. For example, if $whl_k=\mathrm{BM}_{\nu,\epsilon_k,n=k}$, $\nu\ll\epsilon_k^{-1}$, $\epsilon_k\to0$, such kernel function is consistent. These cases are termed the LL-mean ($\mathrm{LLM}_{k,\epsilon,\gamma,n}$). By substituting the $\mathrm{WA}_{\epsilon_0,\gamma,n}$ for the $L_{\epsilon_0,\gamma,n}$ in LL-statistic, the resulting statistic is referred to as the weighted L-statistic ($\mathrm{WL}_{k,\epsilon,\gamma,n}$). The case having a consistent kernel function is termed as the weighted L-mean ($\mathrm{WLM}_{k,\epsilon,\gamma,n}$). The $w_i=1$ case of $\mathrm{WLM}_{k,\epsilon,\gamma,n}$ is termed the weighted Hodges-Lehmann mean ($\mathrm{WHLM}_{k,\epsilon,\gamma,n}$). The $\mathrm{WHLM}_{k=1,\epsilon,\gamma,n}$ is the weighted average. If $k\geq2$ and the WA in WHLM is set as TM_{ϵ_0}, it is called the trimmed H-L mean (Figure **??**, $k=2$, $\epsilon_0=\frac{15}{64}$). The $\mathrm{THLM}_{k=2,\epsilon,\gamma=1,n}$ appears similar to the Wilcoxon's one-sample statistic investigated by Saleh in 1976 (5), which involves first censoring the sample, and then computing the mean of the number of events that the pairwise mean is greater than zero. The $\mathrm{THLM}_{k=2,\epsilon=1-\left(1-\frac{1}{2}\right)^{\frac{1}{2}},\gamma=1,n}$ is the Hodges-Lehmann estimator, or more generally, a special case of the median Hodges-Lehmann mean ($m\mathrm{HLM}_{k,n}$). $m\mathrm{HLM}_{k,n}$ is asymptotically equivalent to the $\mathrm{MoM}_{k,b=\frac{n}{k}}$ as discussed previously, Therefore, it is possible to define a series of location estimators, analogous to the WHLM, based on MoM. For example, the γ-median of means, $\gamma m o\mathrm{M}_{k,b=\frac{n}{k},n}$, is defined by replacing the median in $\mathrm{MoM}_{k,b=\frac{n}{k},n}$ with the γ-median.

The hl_k kernel distribution, denoted as F_{hl_k}, can be defined as the probability distribution of the sorted sequence $\text{sort}\left(\left(hl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)$. For any real value y, the cdf of the hl_k kernel distribution is given by: $F_{h_{\mathbf{k}}}(y)=\Pr(Y_i\leq y)$, where Y_i represents an individual element from the sorted sequence. The overall hl_k kernel distributions possess a two-dimensional structure, encompassing n kernel distributions with varying k values, from 1 to n. where one dimension is inherent to each individual kernel distribution, while the other is formed by the alignment of the same percentiles across all kernel distributions. As k increases, all percentiles converge to the same point, leading to the concept of γ-U-orderliness:

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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