Semiparametric robust mean estimations based on the orderliness of quantile averages

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Hodges–Lehmann inequality and γ-U-orderliness

The Hodges–Lehmann estimator stands out as a unique robust location estimator due to its definition being substantially dissimilar from conventional L-estimators, R-estimators, and M-estimators. In their landmark paper, *Estimates of location based on rank tests*, Hodges and Lehmann (1) proposed two methods for computing the H-L estimator: the Wilcoxon score R-estimator and the median of pairwise means. The Wilcoxon score R-estimator is a location estimator based on signed-rank test, or R-estimator, (1) and was later independently discovered by Sen (1963) (2, 3). However, the median of pairwise means is a generalized L-statistic and a trimmed U-statistic, as classified by Serfling in his novel conceptualized study in 1984 (4). Serfling further advanced the understanding by generalizing the H-L kernel as $hl_k\left(x_1,\ldots,x_k\right)=\frac{1}{k}\sum_{i=1}^{k}x_i$, where $k\in\mathbb{N}$ (4). Here, the weighted H-L kernel is defined as $whl_k\left(x_1,\ldots,x_k\right)=\frac{\sum_{i=1}^{k}x_i\mathbf{w}_i}{\sum_{i=1}^{k}\mathbf{w}_i}$, where $\mathbf{w}_i$s are the weights applied to each element.

By using the whl_k kernel and the L-estimator, it is now clear that the Hodges-Lehmann estimator is an LL-statistic, the definition of which is provided as follows:

$$LL_{k,\epsilon,\gamma,n} := L_{\epsilon_0,\gamma,n}\left(\mathrm{sort}\left(\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}\right)\right),$$

where $L_{\epsilon_0,\gamma,n}\left(Y\right)$ represents the L-estimator that uses the sorted sequence, $\mathrm{sort}\left(whl_k\left(X_{N_1},\cdots,X_{N_k}\right)\right)_{N=1}^{\binom{n}{k}}$, as input, the upper asymptotic breakdown point of the L-estimator is ϵ_0, the lower asymptotic breakdown point is $\gamma\epsilon_0$. The upper asymptotic breakdown point of $LL_{k,\epsilon,\gamma}$ is $\epsilon=1-\left(1-\epsilon_0\right)^{\frac{1}{k}}$, as proven in another relevant paper. There are two ways to adjust the breakdown point: either by setting k as a constant and adjusting ϵ_0, or by setting ϵ_0 as a constant and adjusting k. In the above definition, k is discrete, but the bootstrap method can be applied to ensure the continuity of k, also making the breakdown point continuous. Specifically, if $k\in\mathbb{R}$, let the bootstrap size be denoted by b, then first sampling the original sample $(1-k+\lfloor k\rfloor)b$ times with each sample size of $\lfloor k\rfloor$, and then subsequently sampling $(1-\lceil k\rceil+k)b$ times with each sample size of $\lceil k\rceil$, $(1-k+\lfloor k\rfloor)b\in\mathbb{N}$, $(1-\lceil k\rceil+k)b\in\mathbb{N}$. The corresponding kernels are computed separately, and the pooled sorted sequence is used as the input for the L-estimator. Indeed, for any finite sample, X, when $k=n$, the corresponding whl_k kernel distribution becomes a single point, $\frac{\sum_{i=1}^{n}X_i\mathbf{w}_i}{\sum_{i=1}^{n}\mathbf{w}_i}$.

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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The author declares no competing interest.

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