Semiparametric robust mean estimations based on the orderliness of quantile averages

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Furthermore, for weighted averages, separating the breakdown point into upper and lower parts is necessary.

Definition .1 (Upper/lower breakdown point). The upper breakdown point is the breakdown point generalized in Davies and Gather (2005)'s paper (**?**). The finite-sample upper breakdown point is the finite sample breakdown point defined by Donoho and Huber (1983) (1) and also detailed in (**?**). The (finite-sample) lower breakdown point is replacing the infinity symbol in these definitions with negative infinity.

Classifying Distributions by the Signs of Derivatives

Let $\mathcal{P}_{\mathbb{R}}$ denote the set of all continuous distributions over $\mathbb{R}$ and $\mathcal{P}_{\mathbb{X}}$ denote the set of all discrete distributions over a countable set $\mathbb{X}$. The default of this article will be on the class of continuous distributions, $\mathcal{P}_{\mathbb{R}}$. However, it's worth noting that most discussions and results can be extended to encompass the discrete case, $\mathcal{P}_{\mathbb{X}}$, unless explicitly specified otherwise. Besides fully and smoothly parameterizing them by a Euclidean parameter or merely assuming regularity conditions, there exist additional methods for classifying distributions based on their characteristics, such as their skewness, peakedness, modality, and supported interval. In 1956, Stein initiated the study of estimating parameters in the presence of an infinite-dimensional nuisance shape parameter (2) and proposed a necessary condition for this type of problem, a contribution later explicitly recognized as initiating the field of semiparametric statistics (3). In 1982, Bickel simplified Stein's general heuristic necessary condition (2), derived sufficient conditions, and used them in formulating adaptive estimates (3). A notable example discussed in these groundbreaking works was the adaptive estimation of the center of symmetry for an unknown symmetric distribution, which is a semiparametric model. In 1993, Bickel, Klaassen, Ritov, and Wellner published an influential semiparametrics textbook (4), which categorized most common statistical models as semiparametric models, considering parametric and nonparametric models as two special cases within this classification. Yet, there is another old and commonly encountered class of distributions that receives little attention in semiparametric literature: the unimodal distribution. It is a very unique semiparametric model because its definition is based on the signs of derivatives, i.e., $(f'(x) > 0 \text{ for } x \leq M) \wedge (f'(x) < 0 \text{ for } x \geq M)$, where $f(x)$ is the probability density function (pdf) of a random variable X, M is the mode. Let $\mathcal{P}_U$ denote the set of all unimodal distributions. There was a widespread misbelief that the median of an arbitrary unimodal distribution always lies between its mean and mode until Runnenburg (1978) and van Zwet (1979) (5, 6) endeavored to determine sufficient conditions for the mean-median-mode inequality to hold, thereby implying the possibility of its violation. The class of unimodal distributions that satisfy the mean-median-mode inequality constitutes a subclass of $\mathcal{P}_U$, denoted by $\mathcal{P}_{MMM} \subsetneq \mathcal{P}_U$. To further investigate the relations of location estimates within a distribution, the γ-orderliness for a right-skewed distribution is defined as

$$\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \mathrm{QA}(\epsilon_1, \gamma) \geq \mathrm{QA}(\epsilon_2, \gamma).$$

The necessary and sufficient condition below hints at the relation between the mean-median-mode inequality and the γ-orderliness.

Theorem .1. *A distribution is γ-ordered if and only if its pdf satisfies the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ or $f(Q(\gamma\epsilon)) \leq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. Without loss of generality, consider the case of right-skewed distribution. From the above definition of γ-orderliness, it is deduced that $\frac{Q(\gamma\epsilon-\delta)+Q(1-\epsilon+\delta)}{2} \geq \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \Leftrightarrow Q(\gamma\epsilon-\delta) - Q(\gamma\epsilon) \geq Q(1-\epsilon) - Q(1-\epsilon+\delta) \Leftrightarrow Q'(1-\epsilon) \geq Q'(\gamma\epsilon)$, where δ is an infinitesimal positive quantity. Observing that the quantile function is the inverse function of the cumulative distribution function (cdf), $Q'(1-\epsilon) \geq Q'(\gamma\epsilon) \Leftrightarrow F'(Q(\gamma\epsilon)) \geq F'(Q(1-\epsilon))$, thereby completing the proof, since the derivative of cdf is pdf. $\square$

According to Theorem .1, if a probability distribution is right-skewed and monotonic decreasing, it will always be γ-ordered. For a right-skewed unimodal distribution, if $Q(\gamma\epsilon) > M$, then the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ holds. The principle is extendable to unimodal-like distributions. Suppose there is a right-skewed unimodal-like distribution with the first mode, denoted as M_1, having the greatest probability density, while there are several smaller modes located towards the higher values of the distribution. Furthermore, assume that this distribution follows the mean-γ-median-first mode inequality, amd the γ-median, $Q(\frac{\gamma}{1+\gamma})$, falling within the first dominant mode (i.e., if $x > Q(\frac{\gamma}{1+\gamma})$, $f(Q(\frac{\gamma}{1+\gamma})) \geq f(x)$). Then, if $Q(\gamma\epsilon) > M_1$, the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ also holds. In other words, even though a distribution following the mean-γ-median-mode inequality may not be strictly γ-ordered, the inequality defining the γ-orderliness remains valid for most quantile averages. The mean-γ-median-mode inequality can also indicate possible bounds for γ in practice, e.g., for any distributions, when $\gamma \to \infty$, the γ-median will be greater than the mean and the mode, when $\gamma \to 0$, the γ-median will be smaller than the mean and

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the mode, a reasonable γ should maintain the validity of the mean-γ-median-mode inequality.

The definition above of γ-orderliness for a right-skewed distribution implies a monotonic decreasing behavior of the quantile average function with respect to the breakdown point. Therefore, consider the sign of the partial derivative, it can also be expressed as:

$$\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}, \frac{\partial \mathrm{QA}}{\partial \epsilon} \leq 0.$$

The left-skewed case can be obtained by reversing the inequality $\frac{\partial \mathrm{QA}}{\partial \epsilon} \leq 0$ to $\frac{\partial \mathrm{QA}}{\partial \epsilon} \geq 0$ and employing the second definition of QA, as given in [**??**]. For simplicity, the left-skewed case will be omitted in the following discussion. If $\gamma = 1$, the γ-ordered distribution is referred to as ordered distribution.

Furthermore, many common right-skewed distributions, such as the Weibull, gamma, lognormal, and Pareto distributions, are partially bounded, indicating a convex behavior of the QA function with respect to ϵ as ϵ approaches 0. By further assuming convexity, the second γ-orderliness can be defined for a right-skewed distribution as follows,

$$\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}, \frac{\partial^2 \mathrm{QA}}{\partial \epsilon^2} \geq 0 \wedge \frac{\partial \mathrm{QA}}{\partial \epsilon} \leq 0.$$

Analogously, the νth γ-orderliness of a right-skewed distribution can be defined as $(-1)^{\nu} \frac{\partial^{\nu} \mathrm{QA}}{\partial \epsilon^{\nu}} \geq 0 \wedge \ldots \wedge -\frac{\partial \mathrm{QA}}{\partial \epsilon} \geq 0$. If $\gamma = 1$, the νth γ-orderliness is referred as to νth orderliness. Let $\mathcal{P}_O$ denote the set of all distributions that are ordered and $\mathcal{P}_{O_\nu}$ and $\mathcal{P}_{\gamma O_\nu}$ represent the sets of all distributions that are νth ordered and νth γ-ordered, respectively.

Data Availability. Data for Figure **??** are given in SI Dataset S1. All codes have been deposited in GitHub.

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