

Semiparametric robust mean estimations based on the orderliness of quantile averages

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As the most fundamental problem in statistics, robust location estimation has many prominent solutions, such as the trimmed mean, Winsorized mean, Hodges–Lehmann estimator, Huber M -estimator, and median of means. Recent studies suggest that their maximum biases concerning the mean can be quite different, but the underlying mechanisms largely remain unclear. This study exploited a semiparametric method to classify distributions by the asymptotic orderliness of quantile combinations with varying breakdown points, showing their interrelations and connections to parametric distributions. Further deductions explain why the Winsorized mean typically has smaller biases compared to the trimmed mean; two sequences of semiparametric robust mean estimators emerge, particularly highlighting the superiority of the median Hodges–Lehmann mean.

semiparametric | mean-median-mode inequality | asymptotic | unimodal
| Hodges–Lehmann estimator

In 1823, Gauss (1) proved that for any unimodal distribution, $|m - \mu| \leq \sqrt{\frac{3}{4}}\omega$ and $\sigma \leq \omega \leq 2\sigma$, where μ is the population mean, m is the population median, ω is the root mean square deviation from the mode, and σ is the population standard deviation. This pioneering work revealed that, the potential bias of the median, the most fundamental robust location estimator, with respect to the mean is bounded in units of a scale parameter under certain assumptions. Bernard, Kazzi, and Vanduffel (2020) (2) further derived asymptotic bias bounds for any quantile in unimodal distributions with finite second moments. They showed that m has the smallest maximum distance to μ among all symmetric quantile averages (SQA _{ϵ}). Daniell, in 1920, (3) analyzed a class of estimators, linear combinations of order statistics, and identified that the ϵ -symmetric trimmed mean (STM _{ϵ}) belongs to this class. Another popular choice, the ϵ -symmetric Winsorized mean (SWM _{ϵ}), named after Winsor and introduced by Tukey (4) and Dixon (5) in 1960, is also an L -estimator. Bieniek (2016) derived exact bias upper bounds of the Winsorized mean based on Danielak and Rychlik's work (2003) on the trimmed mean for any distribution with a finite second moment and confirmed that the former is smaller than the latter (6, 7). In 1963, Hodges and Lehmann (8) proposed a class of nonparametric location estimators based on rank tests and, from the Wilcoxon signed-rank statistic (9), deduced the median of pairwise means as a robust location estimator for a symmetric population. Both L -statistics and R -statistics achieve robustness essentially by removing a certain proportion of extreme values. In 1964, Huber (10) generalized maximum likelihood estimation to the minimization of the sum of a specific loss function, which measures the residuals between the data points and the model's parameters. Some L -estimators are also M -estimators, e.g., the sample mean is an M -estimator with a squared error loss function, the sample median is an M -estimator with an ab-

solute error loss function (10). The Huber M -estimator is obtained by applying the Huber loss function that combines elements of both squared error and absolute error to achieve robustness against gross errors and high efficiency for contaminated Gaussian distributions (10). Sun, Zhou, and Fan (2020) examined the concentration bounds of the Huber M -estimator (11). The concept of the median of means (MoM _{$k, b = \frac{n}{k}, n$}) was first introduced by Nemirovsky and Yudin (1983) in their work on stochastic optimization (12). Given its good performance even for distributions with infinite second moments, the MoM has received increasing attention over the past decade (13–16). Devroye, Lerasle, Lugosi, and Oliveira (2016) showed that MoM _{$k, b = \frac{n}{k}, n$} nears the optimum of sub-Gaussian mean estimation with regards to concentration bounds when the distribution has a heavy tail (14). Laforge, Clemencon, and Bertail (2019) proposed the median of randomized means (MoRM _{k, b, n}) (15), wherein, rather than partitioning, an arbitrary number, b , of blocks are built independently from the sample, and showed that MoRM _{k, b, n} has a better non-asymptotic sub-Gaussian property compared to MoM _{$k, b = \frac{n}{k}, n$} . In fact, asymptotically, the Hodges–Lehmann (H–L) estimator is equivalent to MoM _{$k=2, b = \frac{n}{k}$} and MoRM _{$k=2, b$} , and they can be seen as the pairwise mean distribution is approximated by the sampling without replacement and bootstrap, respectively. When $k \ll n$, the difference between sampling with replacement and without replacement is negligible. For the asymptotic validity, readers are referred to the foundational works of Efron (1979) (17), Bickel and Freedman (1981, 1984) (18, 19), and Helmers, Janssen, and Veraverbeke (1990) (20).

Significance Statement

In 1964, van Zwet introduced the convex transformation order for comparing the skewness of two distributions. This paradigm shift played a fundamental role in defining robust measures of distributions, from spread to kurtosis. Here, instead of examining the stochastic ordering between two distributions, the orderliness of quantile combinations within a distribution is investigated. By classifying distributions through the signs of derivatives, a series of sophisticated robust mean estimators is deduced. Nearly all common nonparametric robust location estimators are found to be special cases thereof.

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Here, the ϵ, b -stratified mean is defined as

$$\text{SM}_{\epsilon, b, n} := \frac{b}{n} \left(\sum_{j=1}^{\frac{b-1}{2b\epsilon}} \sum_{i_j = \frac{(2bj-b-1)n\epsilon}{b-1}}^{\frac{(2bj-b+1)n\epsilon}{b-1}} X_{i_j} \right),$$

where $X_1 \leq \dots \leq X_n$ denote the order statistics of a sample of n independent and identically distributed random variables X_1, \dots, X_n . $b \in \mathbb{N}$, $b \geq 3$. The definition was further refined to guarantee the continuity of the breakdown point by incorporating an additional block in the center when $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 0$, or by adjusting the central block when $\lfloor \frac{b-1}{2b\epsilon} \rfloor \bmod 2 = 1$ (SI Text). If the subscript n is omitted, only the asymptotic behavior is considered. If b is omitted, $b = 3$ is assumed. $\text{SM}_{\epsilon, b=3}$ is equivalent to STM_{ϵ} , when $\epsilon > \frac{1}{6}$. When $\frac{b-1}{2\epsilon} \in \mathbb{N}$ and $b \bmod 2 = 1$, the basic idea of the stratified mean is to distribute the data into $\frac{b-1}{2\epsilon}$ equal-sized non-overlapping blocks according to their order. Then, further sequentially group these blocks into b equal-sized strata and compute the mean of the middle stratum, which is the median of means of each stratum. In situations where $i \bmod 1 \neq 0$, a potential solution is to generate multiple smaller samples that satisfy the equality by sampling without replacement, and subsequently calculate the mean of all estimations. The details of determining the smaller sample size and the number of sampling times are provided in the SI Text. Although the principle resembles that of the median of means, $\text{SM}_{\epsilon, b, n}$ is different from $\text{MoM}_{k=\frac{n}{b}, b, n}$ as it does not include the random shift. Additionally, the stratified mean differs from the mean of the sample obtained through stratified sampling methods, introduced by Neyman (1934) (21) or ranked set sampling (22), introduced by McIntyre in 1952, as these sampling methods aim to obtain more representative samples or improve the efficiency of sample estimates, but the sample means based on them are not robust. When $b \bmod 2 = 1$, the stratified mean can be regarded as replacing the other equal-sized strata with the middle stratum, which, in principle, is analogous to the Winsorized mean that replaces extreme values with less extreme percentiles. Furthermore, while the bounds confirm that the Winsorized mean and median of means outperform the trimmed mean (6, 7, 14) in worst-case performance, the complexity of bound analysis makes it difficult to achieve a complete and intuitive understanding of these results. Also, a clear explanation for the average performance of them remains elusive. The aim of this paper is to define a series of semiparametric models using the signs of derivatives, reveal their elegant interrelations and connections to parametric models, and show that by exploiting these models, a set of sophisticated mean estimators can be deduced, which exhibit strong robustness to departures from assumptions.

Quantile Average and Weighted Average

The symmetric trimmed mean, symmetric Winsorized mean, and stratified mean are all L -estimators. More specifically, they are symmetric weighted averages, which are defined as

$$\text{SWA}_{\epsilon, n} := \frac{\sum_{i=1}^{\lceil \frac{n}{2} \rceil} \frac{X_i + X_{n-i+1}}{2} w_i}{\sum_{i=1}^{\lceil \frac{n}{2} \rceil} w_i},$$

where w_i s are the weights applied to the symmetric quantile averages according to the definition of the corresponding L -estimators. For example, for the ϵ -symmetric trimmed mean,

$w_i = \begin{cases} 0, & i < n\epsilon \\ 1, & i \geq n\epsilon \end{cases}$, when $n\epsilon \in \mathbb{N}$. The mean and median are indeed two special cases of the symmetric trimmed mean.

To extend the symmetric quantile average to the asymmetric case, two definitions for the ϵ, γ -quantile average ($\text{QA}_{\epsilon, \gamma, n}$) are proposed. The first definition is:

$$\frac{1}{2}(\hat{Q}_n(\gamma\epsilon) + \hat{Q}_n(1 - \epsilon)), \quad [1]$$

and the second definition is:

$$\frac{1}{2}(\hat{Q}_n(\epsilon) + \hat{Q}_n(1 - \gamma\epsilon)), \quad [2]$$

where $\hat{Q}_n(p)$ is the empirical quantile function; γ is used to adjust the degree of asymmetry, $\gamma \geq 0$; and $0 \leq \epsilon \leq \frac{1}{1+\gamma}$. For trimming from both sides, [1] and [2] are essentially equivalent. The first definition along with $\gamma \geq 0$ and $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ are assumed in the rest of this article unless otherwise specified, since many common asymmetric distributions are right-skewed, and [1] allows trimming only from the right side by setting $\gamma = 0$.

Analogously, the weighted average can be defined as

$$\text{WA}_{\epsilon, \gamma, n} := \frac{\int_0^{\frac{1}{1+\gamma}} \text{QA}(\epsilon_0, \gamma, n) w(\epsilon_0) d\epsilon_0}{\int_0^{\frac{1}{1+\gamma}} w(\epsilon_0) d\epsilon_0}.$$

For any weighted average, if γ is omitted, it is assumed to be 1. The ϵ, γ -trimmed mean ($\text{TM}_{\epsilon, \gamma, n}$) is a weighted average with a left trim size of $n\gamma\epsilon$ and a right trim size of $n\epsilon$, where $w(\epsilon_0) = \begin{cases} 0, & \epsilon_0 < \epsilon \\ 1, & \epsilon_0 \geq \epsilon \end{cases}$. Using this definition, regardless of whether $n\gamma\epsilon \notin \mathbb{N}$ or $n\epsilon \notin \mathbb{N}$, the TM computation remains the same, since this definition is based on the empirical quantile function. However, in this article, considering the computational cost in practice, non-asymptotic definitions of various types of weighted averages are primarily based on order statistics. Unless stated otherwise, the solution to their decimal issue is the same as that in SM.

Furthermore, for weighted averages, separating the breakdown point into upper and lower parts is necessary.

Definition .1 (Upper/lower breakdown point). The upper breakdown point is the breakdown point generalized in Davies and Gather (2005)'s paper (?). The finite-sample upper breakdown point is the finite sample breakdown point defined by Donoho and Huber (1983) (23) and also detailed in (?). The (finite-sample) lower breakdown point is replacing the infinity symbol in these definitions with negative infinity.

Classifying Distributions by the Signs of Derivatives

Let $\mathcal{P}_{\mathbb{R}}$ denote the set of all continuous distributions over \mathbb{R} and $\mathcal{P}_{\mathbb{X}}$ denote the set of all discrete distributions over a countable set \mathbb{X} . The default of this article will be on the class of continuous distributions, $\mathcal{P}_{\mathbb{R}}$. However, it's worth noting that most discussions and results can be extended to encompass the discrete case, $\mathcal{P}_{\mathbb{X}}$, unless explicitly specified otherwise. Besides fully and smoothly parameterizing them by a Euclidean parameter or merely assuming regularity conditions, there exist additional methods for classifying distributions based on their characteristics, such as their skewness, peakedness, modality, and supported interval. In 1956, Stein initiated the

study of estimating parameters in the presence of an infinite-dimensional nuisance shape parameter (24) and proposed a necessary condition for this type of problem, a contribution later explicitly recognized as initiating the field of semiparametric statistics (25). In 1982, Bickel simplified Stein's general heuristic necessary condition (24), derived sufficient conditions, and used them in formulating adaptive estimates (25). A notable example discussed in these groundbreaking works was the adaptive estimation of the center of symmetry for an unknown symmetric distribution, which is a semiparametric model. In 1993, Bickel, Klaassen, Ritov, and Wellner published an influential semiparametrics textbook (26), which categorized most common statistical models as semiparametric models, considering parametric and nonparametric models as two special cases within this classification. Yet, there is another old and commonly encountered class of distributions that receives little attention in semiparametric literature: the unimodal distribution. It is a very unique semiparametric model because its definition is based on the signs of derivatives, i.e., $(f'(x) > 0 \text{ for } x \leq M) \wedge (f'(x) < 0 \text{ for } x \geq M)$, where $f(x)$ is the probability density function (pdf) of a random variable X , M is the mode. Let \mathcal{P}_U denote the set of all unimodal distributions. There was a widespread misbelief that the median of an arbitrary unimodal distribution always lies between its mean and mode until Runnenburg (1978) and van Zwet (1979) (27, 28) endeavored to determine sufficient conditions for the mean-median-mode inequality to hold, thereby implying the possibility of its violation. The class of unimodal distributions that satisfy the mean-median-mode inequality constitutes a subclass of \mathcal{P}_U , denoted by $\mathcal{P}_{MMM} \subsetneq \mathcal{P}_U$. To further investigate the relations of location estimates within a distribution, the γ -orderliness for a right-skewed distribution is defined as

$$\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}, \text{QA}(\epsilon_1, \gamma) \geq \text{QA}(\epsilon_2, \gamma).$$

The necessary and sufficient condition below hints at the relation between the mean-median-mode inequality and the γ -orderliness.

Theorem .1. *A distribution is γ -ordered if and only if its pdf satisfies the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$ or $f(Q(\gamma\epsilon)) \leq f(Q(1-\epsilon))$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. Without loss of generality, consider the case of right-skewed distribution. From the above definition of γ -orderliness, it is deduced that $\frac{Q(\gamma\epsilon-\delta)+Q(1-\epsilon+\delta)}{2} \geq \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \Leftrightarrow Q(\gamma\epsilon-\delta) - Q(\gamma\epsilon) \geq Q(1-\epsilon) - Q(1-\epsilon+\delta) \Leftrightarrow Q'(1-\epsilon) \geq Q'(\gamma\epsilon)$, where δ is an infinitesimal positive quantity. Observing that the quantile function is the inverse function of the cumulative distribution function (cdf), $Q'(1-\epsilon) \geq Q'(\gamma\epsilon) \Leftrightarrow F'(Q(\gamma\epsilon)) \geq F'(Q(1-\epsilon))$, thereby completing the proof, since the derivative of cdf is pdf. \square

According to Theorem .1, if a probability distribution is right-skewed and monotonic decreasing, it will always be γ -ordered. For a right-skewed unimodal distribution, if $Q(\gamma\epsilon) > M$, then the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ holds. The principle is extendable to unimodal-like distributions. Suppose there is a right-skewed unimodal-like distribution with the first mode, denoted as M_1 , having the greatest probability density, while there are several smaller modes located towards the higher values of the distribution. Furthermore, assume

that this distribution follows the mean- γ -median-first mode inequality, and the γ -median, $Q(\frac{\gamma}{1+\gamma})$, falling within the first dominant mode (i.e., if $x > Q(\frac{\gamma}{1+\gamma})$, $f(Q(\frac{\gamma}{1+\gamma})) \geq f(x)$). Then, if $Q(\gamma\epsilon) > M_1$, the inequality $f(Q(\gamma\epsilon)) \geq f(Q(1-\epsilon))$ also holds. In other words, even though a distribution following the mean- γ -median-mode inequality may not be strictly γ -ordered, the inequality defining the γ -orderliness remains valid for most quantile averages. The mean- γ -median-mode inequality can also indicate possible bounds for γ in practice, e.g., for any distributions, when $\gamma \rightarrow \infty$, the γ -median will be greater than the mean and the mode, when $\gamma \rightarrow 0$, the γ -median will be smaller than the mean and the mode, a reasonable γ should maintain the validity of the mean- γ -median-mode inequality.

The definition above of γ -orderliness for a right-skewed distribution implies a monotonic decreasing behavior of the quantile average function with respect to the breakdown point. Therefore, consider the sign of the partial derivative, it can also be expressed as:

$$\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}, \frac{\partial \text{QA}}{\partial \epsilon} \leq 0.$$

The left-skewed case can be obtained by reversing the inequality $\frac{\partial \text{QA}}{\partial \epsilon} \leq 0$ to $\frac{\partial \text{QA}}{\partial \epsilon} \geq 0$ and employing the second definition of QA, as given in [2]. For simplicity, the left-skewed case will be omitted in the following discussion. If $\gamma = 1$, the γ -ordered distribution is referred to as ordered distribution.

Furthermore, many common right-skewed distributions, such as the Weibull, gamma, lognormal, and Pareto distributions, are partially bounded, indicating a convex behavior of the QA function with respect to ϵ as ϵ approaches 0. By further assuming convexity, the second γ -orderliness can be defined for a right-skewed distribution as follows,

$$\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}, \frac{\partial^2 \text{QA}}{\partial \epsilon^2} \geq 0 \wedge \frac{\partial \text{QA}}{\partial \epsilon} \leq 0.$$

Analogously, the ν th γ -orderliness of a right-skewed distribution can be defined as $(-1)^\nu \frac{\partial^\nu \text{QA}}{\partial \epsilon^\nu} \geq 0 \wedge \dots \wedge -\frac{\partial \text{QA}}{\partial \epsilon} \geq 0$. If $\gamma = 1$, the ν th γ -orderliness is referred as to ν th orderliness. Let \mathcal{P}_O denote the set of all distributions that are ordered and \mathcal{P}_{O_ν} and $\mathcal{P}_{\gamma O_\nu}$ represent the sets of all distributions that are ν th ordered and ν th γ -ordered, respectively. When the shape parameter of the Weibull distribution, α , is smaller than 3.258, it can be shown that the Weibull distribution belongs to $\mathcal{P}_U \cap \mathcal{P}_O \cap \mathcal{P}_{O_2} \cap \mathcal{P}_{O_3}$ (SI Text). At $\alpha \approx 3.602$, the Weibull distribution is symmetric, and as $\alpha \rightarrow \infty$, the skewness of the Weibull distribution approaches 1. Therefore, the parameters that prevent it from being included in the set correspond to cases when it is near-symmetric, as shown in the SI Text. Nevertheless, computing the derivatives of the QA function is often intricate and, at times, challenging. The following theorems establish the relationship between \mathcal{P}_O , \mathcal{P}_{O_ν} , and $\mathcal{P}_{\gamma O_\nu}$, and a wide range of other semi-parametric distributions. They can be used to quickly identify some parametric distributions in \mathcal{P}_O , \mathcal{P}_{O_ν} , and $\mathcal{P}_{\gamma O_\nu}$.

Theorem .2. *For any random variable X whose probability distribution belongs to a location-scale family, the distribution is ν th γ -ordered if and only if the family of probability distributions is ν th γ -ordered.*

Proof. Let Q_0 denote the quantile function of the standard distribution without any shifts or scaling. After a location-scale transformation, the quantile function becomes $Q(p) = \lambda Q_0(p) + \mu$, where λ is the scale parameter and μ is the location parameter. According to the definition of the ν th γ -orderliness, the signs of derivatives of the QA function are invariant after this transformation. As the location-scale transformation is reversible, the proof is complete. \square

Theorem .2 demonstrates that in the analytical proof of the ν th γ -orderliness of a parametric distribution, both the location and scale parameters can be regarded as constants. It is also instrumental in proving other theorems.

Theorem .3. Define a γ -symmetric distribution as one for which the quantile function satisfies $Q(\gamma\epsilon) = 2Q(\frac{\gamma}{1+\gamma}) - Q(1-\epsilon)$ for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$. Any γ -symmetric distribution is ν th γ -ordered.

Proof. The equality, $Q(\gamma\epsilon) = 2Q(\frac{\gamma}{1+\gamma}) - Q(1-\epsilon)$, implies that $\frac{\partial Q(\gamma\epsilon)}{\partial \epsilon} = \gamma Q'(\gamma\epsilon) = \frac{\partial(-Q(1-\epsilon))}{\partial \epsilon} = Q'(1-\epsilon)$. From the first definition of QA, the QA function of the γ -symmetric distribution is a horizontal line, since $\frac{\partial QA}{\partial \epsilon} = \gamma Q'(\gamma\epsilon) - Q'(1-\epsilon) = 0$. So, the ν th order derivative of QA is always zero. \square

Theorem .4. A symmetric distribution is a special case of the γ -symmetric distribution when $\gamma = 1$, provided that the cdf is monotonic.

Proof. A symmetric distribution is a probability distribution such that for all x , $f(x) = f(2m - x)$. Its cdf satisfies $F(x) = 1 - F(2m - x)$. Let $x = Q(p)$, then, $F(Q(p)) = p = 1 - F(2m - Q(p))$ and $F(Q(1-p)) = 1 - p \Leftrightarrow p = 1 - F(Q(1-p))$. Therefore, $F(2m - Q(p)) = F(Q(1-p))$. Since the cdf is monotonic, $2m - Q(p) = Q(1-p) \Leftrightarrow Q(p) = 2m - Q(1-p)$. Choosing $p = \epsilon$ yields the desired result. \square

Since the generalized Gaussian distribution is symmetric around the median, it is ν th ordered, as a consequence of Theorem .3.

Theorem .5. Any right-skewed distribution whose quantile function Q satisfies $Q^{(\nu)}(p) \geq 0 \wedge \dots \wedge Q^{(i)}(p) \geq 0 \wedge Q^{(2)}(p) \geq 0$, $i \bmod 2 = 0$, is ν th γ -ordered, provided that $0 \leq \gamma \leq 1$.

Proof. Since $(-1)^i \frac{\partial^i QA}{\partial \epsilon^i} = \frac{1}{2}((- \gamma)^i Q^{(i)}(\gamma\epsilon) + Q^{(i)}(1-\epsilon))$ and $1 \leq i \leq \nu$, when $i \bmod 2 = 0$, $(-1)^i \frac{\partial^i QA}{\partial \epsilon^i} \geq 0$ for all $\gamma \geq 0$. When $i \bmod 2 = 1$, if further assuming $0 \leq \gamma \leq 1$, $(-1)^i \frac{\partial^i QA}{\partial \epsilon^i} \geq 0$, since $Q^{(i+1)}(p) \geq 0$. \square

This result makes it straightforward to show that the Pareto distribution follows the ν th γ -orderliness, provided that $0 \leq \gamma \leq 1$, since the quantile function of the Pareto distribution is $Q(p) = x_m(1-p)^{-\frac{1}{\alpha}}$, where $x_m > 0$, $\alpha > 0$, and so $Q^{(\nu)}(p) \geq 0$ for all $\nu \in \mathbb{N}$ according to the chain rule.

Theorem .6. A right-skewed distribution with a monotonic decreasing pdf is second γ -ordered.

Proof. Given that a monotonic decreasing pdf implies $f'(x) = F^{(2)}(x) \leq 0$, let $x = Q(F(x))$, then by differentiating both sides of the equation twice, one can obtain $0 = Q^{(2)}(F(x))(F'(x))^2 + Q'(F(x))F^{(2)}(x) \Rightarrow Q^{(2)}(F(x)) = -\frac{Q'(F(x))F^{(2)}(x)}{(F'(x))^2} \geq 0$, since $Q'(p) \geq 0$. Theorem .1 already established the γ -orderliness for all $\gamma \geq 0$, which means $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{\partial QA}{\partial \epsilon} \leq 0$. The desired result is then derived from the proof of Theorem .5, since $(-1)^2 \frac{\partial^2 QA}{\partial \epsilon^2} \geq 0$ for all $\gamma \geq 0$. \square

Theorem .6 provides valuable insights into the relation between modality and second γ -orderliness. The conventional definition states that a distribution with a monotonic pdf is still considered unimodal. However, within its supported interval, the mode number is zero. Theorem .1 implies that the number of modes and their magnitudes within a distribution are closely related to the likelihood of γ -orderliness being valid. This is because, for a distribution satisfying the necessary and sufficient condition in Theorem .1, it is already implied that the probability density of the left-hand side of the γ -median is always greater than the corresponding probability density of the right-hand side of the γ -median, so although counterexamples can always be constructed for non-monotonic distributions, the general shape of a γ -ordered distribution should have a single dominant mode. It can be easily established that the gamma distribution is second γ -ordered when $\alpha \leq 1$, as the pdf of the gamma distribution is $f(x) = \frac{\lambda^{-\alpha} x^{\alpha-1} e^{-\frac{x}{\lambda}}}{\Gamma(\alpha)}$, where $x \geq 0$, $\lambda > 0$, $\alpha > 0$, and Γ represents the gamma function. This pdf is a product of two monotonic decreasing functions under constraints. For $\alpha > 1$, analytical analysis becomes challenging. Numerical results show that orderliness is valid until $\alpha > 00.000$, the second orderliness is valid until $\alpha > 00.000$, and the third orderliness is valid until $\alpha > 00.000$ (SI Text). It is instructive to consider that when $\alpha \rightarrow \infty$, the gamma distribution converges to a Gaussian distribution with mean $\mu = \alpha\lambda$ and variance $\sigma = \alpha\lambda^2$. The skewness of the gamma distribution, $\frac{\alpha+2}{\sqrt{\alpha(\alpha+1)}}$, is monotonic with respect to α , since $\frac{\partial \mu_3(\alpha)}{\partial \alpha} = \frac{-3\alpha-2}{2(\alpha(\alpha+1))^{3/2}} < 0$. When $\alpha = 00.000$, $\mu_3(\alpha) = 1.027$. Therefore, similar to the Weibull distribution, the parameters which make these distributions fail to be included in $\mathcal{P}_U \cap \mathcal{P}_O \cap \mathcal{P}_{O_2} \cap \mathcal{P}_{O_3}$ also correspond to cases when it is near-symmetric.

Theorem .7. Consider a γ -symmetric random variable X . Let it be transformed using a function $\phi(x)$ such that $\phi^{(2)}(x) \geq 0$ over the interval supported, the resulting convex transformed distribution is γ -ordered. Moreover, if the quantile function of X satisfies $Q^{(2)}(p) \leq 0$, the convex transformed distribution is second γ -ordered.

Proof. Let $\phi QA(\epsilon, \gamma) = \frac{1}{2}(\phi(Q(\gamma\epsilon)) + \phi(Q(1-\epsilon)))$. Then, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{\partial \phi QA}{\partial \epsilon} = \frac{1}{2}(\gamma \phi'(Q(\gamma\epsilon)) Q'(\gamma\epsilon) - \phi'(Q(1-\epsilon)) Q'(1-\epsilon)) = \frac{1}{2} \gamma Q'(\gamma\epsilon) (\phi'(Q(\gamma\epsilon)) - \phi'(Q(1-\epsilon))) \leq 0$, since for a γ -symmetric distribution, $Q(\frac{1}{1+\gamma}) - Q(\gamma\epsilon) = Q(1-\epsilon) - Q(\frac{1}{1+\gamma})$, differentiating both sides, $-\gamma Q'(\gamma\epsilon) = -Q'(1-\epsilon)$, where $Q'(p) \geq 0$, $\phi^{(2)}(x) \geq 0$. If further differentiating the equality, $\gamma^2 Q^{(2)}(\gamma\epsilon) = -Q^{(2)}(1-\epsilon)$. Since $\frac{\partial^2 \phi QA}{\partial \epsilon^2} = \frac{1}{2}(\gamma^2 \phi^2(Q(\gamma\epsilon)) (Q'(\gamma\epsilon))^2 + \phi^2(Q(1-\epsilon)) (Q'(1-\epsilon))^2) + \frac{1}{2}(\gamma^2 \phi'(Q(\gamma\epsilon)) (Q^2(\gamma\epsilon)) + \phi'(Q(1-\epsilon)) (Q^2(1-\epsilon))) =$

321 $\frac{1}{2} \left((\phi^{(2)}(Q(\gamma\epsilon)) + \phi^{(2)}(Q(1-\epsilon))) (\gamma^2 Q'(\gamma\epsilon))^2 \right) +$
 322 $\frac{1}{2} \left((\phi'(Q(\gamma\epsilon)) - \phi'(Q(1-\epsilon))) \gamma^2 Q^{(2)}(\gamma\epsilon) \right)$. If $Q^{(2)}(p) \leq 0$,
 323 for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{\partial^{(2)} \phi QA}{\partial \epsilon^{(2)}} \geq 0$. \square

324 An application of Theorem 7 is that the lognormal
 325 distribution is ordered as it is exponentially transformed
 326 from the Gaussian distribution. The quantile function of
 327 the Gaussian distribution meets the condition $Q^{(2)}(p) =$
 328 $-2\sqrt{2\pi}\sigma e^{2\text{erfc}^{-1}(2p)^2} \text{erfc}^{-1}(2p) \leq 0$, where σ is the standard
 329 deviation of the Gaussian distribution and erfc denotes the
 330 complementary error function. Thus, the lognormal distribu-
 331 tion is second ordered. Numerical results suggest that it is
 332 also third ordered, although analytically proving this result is
 333 challenging.

334 Theorem 7 also reveals a relation between convex transfor-
 335 mation and orderliness, since ϕ is the non-decreasing convex
 336 function in van Zwet's trailblazing work *Convex transforma-*
 337 *tions of random variables* (29) if adding an additional con-
 338 straint that $\phi'(x) \geq 0$. Consider a near-symmetric distribution
 339 S , such that the SQA(ϵ) as a function of ϵ fluctuates from 0
 340 to $\frac{1}{2}$. By definition, S is not ordered. Let s be the pdf of S .
 341 Applying the transformation $\phi(x)$ to S decreases $s(Q_S(\epsilon))$,
 342 and the decrease rate, due to the order, is much smaller for
 343 $s(Q_S(1-\epsilon))$. As a consequence, as $\phi^{(2)}(x)$ increases, even-
 344 tually, after a point, for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $s(Q_S(\epsilon))$ becomes
 345 greater than $s(Q_S(1-\epsilon))$ even if it was not previously. Thus,
 346 the SQA(ϵ) function becomes monotonically decreasing, and S
 347 becomes ordered. Accordingly, in a family of distributions that
 348 differ by a skewness-increasing transformation in van Zwet's
 349 sense, violations of orderliness typically occur only when the
 350 distribution is near-symmetric.

351 Pearson proposed using the 3 times standardized mean-
 352 median difference, $\frac{3(\mu-m)}{\sigma}$, as a measure of skewness in 1895
 353 (30). Bowley (1926) proposed a measure of skewness based on
 354 the SQA $_{\epsilon=\frac{1}{4}}$ -median difference SQA $_{\epsilon=\frac{1}{4}} - m$ (31). Groeneveld
 355 and Meeden (1984) (32) generalized these measures of skewness
 356 based on van Zwet's convex transformation (29) while explor-
 357 ing their properties. A distribution is called monotonically
 358 right-skewed if and only if $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{2}$, SQA $_{\epsilon_1} - m \geq$
 359 SQA $_{\epsilon_2} - m$. Since m is a constant, the monotonic skewness
 360 is equivalent to the orderliness. For a nonordered distribu-
 361 tion, the signs of SQA $_{\epsilon} - m$ with different breakdown points
 362 might be different, implying that some skewness measures
 363 indicate left-skewed distribution, while others suggest right-
 364 skewed distribution. Although it seems reasonable that such a
 365 distribution is likely to be generally near-symmetric, counterex-
 366 amples can be constructed. For example, first consider the
 367 Weibull distribution, when $\alpha > \frac{1}{1-\ln(2)}$, it is near-symmetric
 368 and nonordered, the non-monotonicity of the SQA function
 369 arises when ϵ is close to $\frac{1}{2}$, but if then replacing the third quar-
 370 tile with one from a right-skewed heavy-tailed distribution
 371 leads to a right-skewed, heavy-tailed, and nonordered distri-
 372 bution. Therefore, the validity of robust measures of skewness
 373 based on the SQA-median difference is closely related to the
 374 orderliness of the distribution.

375 Remarkably, in 2018, Li, Shao, Wang, Yang (33) proved the
 376 bias bound of any quantile for arbitrary continuous distribu-
 377 tions with finite second moments. Here, let $\mathcal{P}_{\mu,\sigma}$ denotes the
 378 set of continuous distributions whose mean is μ and standard
 379 deviation is σ . The bias upper bound of the quantile average

for $P \in \mathcal{P}_{\mu=0,\sigma=1}$ is given in the following theorem. 380

Theorem .8. *The bias upper bound of the quantile average for any continuous distribution whose mean is zero and standard deviation is one is*

$$\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma) = \frac{1}{2} \left(\sqrt{\frac{\gamma\epsilon}{1-\gamma\epsilon}} + \sqrt{\frac{1-\epsilon}{\epsilon}} \right),$$

where $0 \leq \epsilon \leq \frac{1}{1+\gamma}$. 381

Proof. Since $\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} \frac{1}{2}(Q(\gamma\epsilon) + Q(1-\epsilon)) \leq$ 382
 $\frac{1}{2}(\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} Q(\gamma\epsilon) + \sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} Q(1-\epsilon))$, the 383
 assertion follows directly from the Lemma 2.6 in (33). \square 384

In 2020, Bernard et al. (2) further refined these bounds for unimodal distributions and derived the bias bound of the symmetric quantile average. Here, the bias upper bound of the quantile average, $0 \leq \gamma < 5$, for $P \in \mathcal{P}_U \cap \mathcal{P}_{\mu=0,\sigma=1}$ is given as

$$\sup_{P \in \mathcal{P}_U \cap \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma) = \begin{cases} \frac{1}{2} \left(\sqrt{\frac{4}{9\epsilon} - 1} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}} \right) & 0 \leq \epsilon \leq \frac{1}{6} \\ \frac{1}{2} \left(\sqrt{\frac{3(1-\epsilon)}{4-3(1-\epsilon)}} + \sqrt{\frac{3\gamma\epsilon}{4-3\gamma\epsilon}} \right) & \frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma}. \end{cases}$$

The proof based on the bias bounds of any quantile (2) and the $\gamma \geq 5$ case are given in the SI Text. Subsequent theorems reveal the safeguarding role these bounds play in defining estimators based on ν th γ -orderliness. The proof of Theorem 9 is provided in the SI Text. 385 386 387 388 389

Theorem .9. $\sup_{P \in \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma)$ is monotonic decreasing with respect to ϵ over $[0, \frac{1}{1+\gamma}]$, provided that $0 \leq \gamma \leq 1$. 390 391

Theorem .10. $\sup_{P \in \mathcal{P}_U \cap \mathcal{P}_{\mu=0,\sigma=1}} QA(\epsilon, \gamma)$ is a nonincreasing function with respect to ϵ on the interval $[0, \frac{1}{1+\gamma}]$, provided that $0 \leq \gamma \leq 1$. 392 393 394

Proof. When $0 \leq \epsilon \leq \frac{1}{6}$, $\frac{\partial \sup QA}{\partial \epsilon} = \frac{\gamma}{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2} -$ 395

$\frac{1}{3\sqrt{\frac{4}{9\epsilon} - 9\epsilon^2}} = \frac{\sqrt{\gamma}}{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2} - \frac{1}{3\sqrt{\frac{4}{9\epsilon} - 9\epsilon^2}}$. If $\gamma = 0$ 396
 and $\epsilon \rightarrow 0^+$, $\frac{\partial \sup QA}{\partial \epsilon} = -\frac{1}{3\sqrt{\frac{4}{9\epsilon} - 9\epsilon^2}} < 0$. If 397

$\epsilon \rightarrow 0^+$, $\lim_{\epsilon \rightarrow 0^+} \left(\frac{\gamma}{(4-3\epsilon\gamma)^2 \sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}} - \frac{1}{3\sqrt{\frac{4}{9\epsilon} - 9\epsilon^2}} \right) =$ 398

$\lim_{\epsilon \rightarrow 0^+} \left(\frac{\sqrt{3\gamma}}{\sqrt{43\epsilon}} - \frac{1}{6\sqrt{\epsilon^3}} \right) \rightarrow -\infty$, for all $0 \leq \gamma \leq 1$, 399
 so, $\frac{\partial \sup QA}{\partial \epsilon} < 0$. When $0 < \epsilon \leq \frac{1}{6}$ and 400

$0 < \gamma \leq 1$, to prove $\frac{\partial \sup QA}{\partial \epsilon} \leq 0$, it is equivalent 401

to showing $\frac{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2}{\gamma} \geq 3\sqrt{\frac{4}{9\epsilon} - 9\epsilon^2}$. Define 402

$$L(\epsilon, \gamma) = \frac{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2}{\gamma}, \quad R(\epsilon, \gamma) = 3\sqrt{\frac{4}{9\epsilon} - 9\epsilon^2}. \quad 403$$

$$\frac{L(\epsilon, \gamma)}{\epsilon^2} = \frac{\sqrt{\frac{\epsilon\gamma}{12-9\epsilon\gamma}}(4-3\epsilon\gamma)^2}{\gamma\epsilon^2} = \frac{1}{\gamma} \left(\frac{4}{\epsilon} - 3\gamma \right)^2 \sqrt{\frac{1}{\frac{12}{\epsilon\gamma} - 9}}, \quad 404$$

$$\frac{R(\epsilon, \gamma)}{\epsilon^2} = 3\sqrt{\frac{4}{\epsilon} - 9}. \quad \text{Then, the above inequality is} \quad 405$$

$$\text{equivalent to } \frac{L(\epsilon, \gamma)}{\epsilon^2} \geq \frac{R(\epsilon, \gamma)}{\epsilon^2} \Leftrightarrow \frac{1}{\gamma} \sqrt{\frac{1}{\frac{12}{\epsilon\gamma} - 9}} \left(\frac{4}{\epsilon} - 3\gamma \right)^2 \geq \quad 406$$

$$3\sqrt{\frac{4}{\epsilon} - 9} \Leftrightarrow \frac{1}{\gamma} \left(\frac{4}{\epsilon} - 3\gamma \right)^2 \geq 3\sqrt{\frac{12}{\epsilon\gamma} - 9} \sqrt{\frac{4}{\epsilon} - 9} \Leftrightarrow \quad 407$$

$$\frac{1}{\gamma^2} \left(\frac{4}{\epsilon} - 3\gamma \right)^4 \geq 9 \left(\frac{12}{\epsilon\gamma} - 9 \right) \left(\frac{4}{\epsilon} - 9 \right). \quad \text{Let } LmR\left(\frac{1}{\epsilon}\right) = \quad 408$$

$$\frac{1}{\gamma^2} \left(\frac{4}{\epsilon} - 3\gamma \right)^4 - 9 \left(\frac{12}{\epsilon\gamma} - 9 \right) \left(\frac{4}{\epsilon} - 9 \right). \quad \frac{\partial LmR(1/\epsilon)}{\partial (1/\epsilon)} = \frac{16(\frac{4}{\epsilon} - 3\gamma)^3}{\gamma^2} - \quad 409$$

$$36 \left(\frac{12}{\epsilon\gamma} - 9 \right) - \frac{108(4\frac{4}{\epsilon} - 9)}{\gamma} = \frac{4(4(\frac{4}{\epsilon} - 3\gamma)^3 - 27\gamma(\frac{4}{\epsilon} - 3\gamma) + 27(9 - \frac{4}{\epsilon})\gamma)}{\gamma^2} = \frac{4(256\frac{1}{\epsilon^3} - 576\frac{1}{\epsilon^2}\gamma + 432\frac{1}{\epsilon}\gamma^2 - 216\frac{1}{\epsilon}\gamma - 108\gamma^3 + 81\gamma^2 + 243\gamma)}{\gamma^2}. \quad \text{Since}$$

$256\frac{1}{\epsilon^3} - 576\frac{1}{\epsilon^2}\gamma + 432\frac{1}{\epsilon}\gamma^2 - 216\frac{1}{\epsilon}\gamma - 108\gamma^3 + 81\gamma^2 + 243\gamma \geq 1536\frac{1}{\epsilon^2} - 576\frac{1}{\epsilon^2} + 432\frac{1}{\epsilon}\gamma^2 - 216\frac{1}{\epsilon}\gamma - 108\gamma^3 + 81\gamma^2 + 243\gamma \geq 924\frac{1}{\epsilon^2} + 36\frac{1}{\epsilon^2} - 216\frac{1}{\epsilon} + 432\frac{1}{\epsilon}\gamma^2 - 108\gamma^3 + 81\gamma^2 + 243\gamma \geq 924\frac{1}{\epsilon^2} + 36\frac{1}{\epsilon^2} - 216\frac{1}{\epsilon} + 513\gamma^2 - 108\gamma^3 + 243\gamma > 0$,
 $\frac{\partial LmR(1/\epsilon)}{\partial(1/\epsilon)} > 0$. Also, $LmR(6) = \frac{81(\gamma-8)((\gamma-8)^3+15\gamma)}{\gamma^2} > 0 \iff \gamma^4 - 32\gamma^3 + 399\gamma^2 - 2168\gamma + 4096 > 0$. If $0 < \gamma \leq 1$, then $32\gamma^3 < 256$. Also, $\gamma^4 > 0$. So, it suffices to prove that $399\gamma^2 - 2168\gamma + 4096 > 256$. Applying the quadratic formula demonstrates the validity of $LmR(6) > 0$, if $0 < \gamma \leq 1$. Hence, $LmR(\frac{1}{\epsilon}) \geq 0$ for $\epsilon \in (0, \frac{1}{6}]$, if $0 < \gamma \leq 1$. The first part is finished.

When $\frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma}$, $\frac{\partial \sup QA}{\partial \epsilon} = \sqrt{3} \left(\frac{\gamma}{\sqrt{\gamma\epsilon(4-3\gamma\epsilon)}^{\frac{3}{2}}} - \frac{1}{\sqrt{1-\epsilon(3\epsilon+1)}^{\frac{3}{2}}} \right)$. If $\gamma = 0$, $\frac{\gamma}{\sqrt{\gamma\epsilon(4-3\gamma\epsilon)}^{\frac{3}{2}}} = 0$,

$\frac{\sqrt{\gamma}}{\sqrt{\epsilon(4-3\gamma\epsilon)}^{\frac{3}{2}}} = 0$, so $\frac{\partial \sup QA}{\partial \epsilon} = \sqrt{3} \left(-\frac{1}{\sqrt{1-\epsilon(3\epsilon+1)}^{\frac{3}{2}}} \right) < 0$,

for all $\frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma}$. If $\gamma > 0$, to determine whether $\frac{\partial \sup QA}{\partial \epsilon} \leq 0$, when $\frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma}$, since $\sqrt{1-\epsilon(3\epsilon+1)}^{\frac{3}{2}} > 0$

and $\sqrt{\gamma\epsilon(4-3\gamma\epsilon)}^{\frac{3}{2}} > 0$, showing $\frac{\sqrt{\gamma\epsilon(4-3\gamma\epsilon)}^{\frac{3}{2}}}{\gamma} \geq \sqrt{1-\epsilon(3\epsilon+1)}^{\frac{3}{2}} \iff \frac{\gamma\epsilon(4-3\gamma\epsilon)^3}{\gamma^2} \geq (1-\epsilon)(3\epsilon+1)^3 \iff$

$-27\gamma^2\epsilon^4 + 108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} + 27\epsilon^4 - 162\epsilon^2 - 8\epsilon - 1 \geq 0$ is sufficient. When $0 < \gamma \leq 1$, the inequality can be further simplified to $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 \geq 0$. Since $\epsilon \leq \frac{1}{1+\gamma}$,

$\gamma \leq \frac{1}{\epsilon} - 1$. Also, as $0 < \gamma \leq 1$ is assumed, the range of γ can be expressed as $0 < \gamma \leq \min(1, \frac{1}{\epsilon} - 1)$. When $\frac{1}{6} < \epsilon \leq \frac{1}{2}$,

$1 < \frac{1}{\epsilon} - 1$, so in this case, $0 < \gamma \leq 1$. When $\frac{1}{2} \leq \epsilon < 1$, so in this case, $0 < \gamma \leq \frac{1}{\epsilon} - 1$. Let $h(\gamma) = 108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma}$,

$\frac{\partial h(\gamma)}{\partial \gamma} = 108\epsilon^3 - \frac{64\epsilon}{\gamma^2}$. When $\gamma \leq \sqrt{\frac{64\epsilon}{18\epsilon^3}}$, $\frac{\partial h(\gamma)}{\partial \gamma} \geq 0$, when $\gamma \geq \sqrt{\frac{64\epsilon}{18\epsilon^3}}$, $\frac{\partial h(\gamma)}{\partial \gamma} \leq 0$, therefore, the minimum of $h(\gamma)$

must be when γ is equal to the boundary point of the domain. When $\frac{1}{6} < \epsilon \leq \frac{1}{2}$, $0 < \gamma \leq 1$, since $h(0) \rightarrow \infty$, $h(1) = 108\epsilon^3 + 64\epsilon$, the minimum occurs at the boundary point

$\gamma = 1$, $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 > 108\epsilon^3 + 56\epsilon - 162\epsilon^2 - 1$. Let $g(\epsilon) = 108\epsilon^3 + 56\epsilon - 162\epsilon^2 - 1$. $g'(\epsilon) = 324\epsilon^2 - 324\epsilon + 56$, when

$\epsilon \leq \frac{2}{9}$, $g'(\epsilon) \geq 0$, when $\frac{2}{9} \leq \epsilon \leq \frac{1}{2}$, $g'(\epsilon) \leq 0$, since $g(\frac{1}{6}) = \frac{13}{3}$, $g(\frac{1}{2}) = 0$, so $g(\epsilon) \geq 0$, $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 \geq 0$.

When $\frac{1}{2} \leq \epsilon < 1$, $0 < \gamma \leq \frac{1}{\epsilon} - 1$. Since $h(\frac{1}{\epsilon} - 1) = 108(\frac{1}{\epsilon} - 1)\epsilon^3 + \frac{64\epsilon}{\frac{1}{\epsilon} - 1}$, $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 >$

$108(\frac{1}{\epsilon} - 1)\epsilon^3 + \frac{64\epsilon}{\frac{1}{\epsilon} - 1} - 162\epsilon^2 - 8\epsilon - 1 = \frac{-108\epsilon^4 + 54\epsilon^3 - 18\epsilon^2 + 7\epsilon + 1}{\epsilon - 1}$.

Let $nu(\epsilon) = -108\epsilon^4 + 54\epsilon^3 - 18\epsilon^2 + 7\epsilon + 1$, then $nu'(\epsilon) = -432\epsilon^3 + 162\epsilon^2 - 36\epsilon + 7$, $nu''(\epsilon) = -1296\epsilon^2 + 324\epsilon - 36 < 0$.

Since $nu'(\epsilon = \frac{1}{2}) = -\frac{49}{2} < 0$, $nu'(\epsilon) < 0$. Also, $nu(\epsilon = \frac{1}{2}) = 0$, so $nu(\epsilon) \geq 0$, $108\gamma\epsilon^3 + \frac{64\epsilon}{\gamma} - 162\epsilon^2 - 8\epsilon - 1 \geq 0$ is also valid.

As a result, this simplified inequality is valid within the range of $\frac{1}{6} < \epsilon \leq \frac{1}{1+\gamma}$, when $0 < \gamma \leq 1$. Then, it validates

$\frac{\partial \sup QA}{\partial \epsilon} \leq 0$ for the same range of ϵ and γ .

The first and second formulae, when $\epsilon = \frac{1}{6}$, are all equal

to $\frac{1}{2} \left(\sqrt{\frac{\gamma}{4-\frac{\gamma}{2}}} + \sqrt{\frac{3}{5}} \right)$. It follows that $\sup QA(\epsilon, \gamma)$ is continuous over $[0, \frac{1}{1+\gamma}]$. Hence, $\frac{\partial \sup QA}{\partial \epsilon} \leq 0$ holds for the entire

range $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, when $0 \leq \gamma \leq 1$, which leads to the

assertion of this theorem. \square

Let \mathcal{P}_T^k denote the set of all continuous distributions whose moments, from the first to the k th, are all finite. For a right-skewed distribution, it suffices to consider the upper bound. The monotonicity of $\sup_{P \in \mathcal{P}_T^2} QA$ with respect to ϵ implies that the extent of any violations of the γ -orderliness, if $0 \leq \gamma \leq 1$, is bounded for any distribution with a finite second moment, e.g., for a right-skewed distribution in \mathcal{P}_T^2 , if $0 \leq \epsilon_1 \leq \epsilon_2 \leq \epsilon_3 \leq \frac{1}{1+\gamma}$, $QA_{\epsilon_2, \gamma} \geq QA_{\epsilon_3, \gamma} \geq QA_{\epsilon_1, \gamma}$, then $QA_{\epsilon_2, \gamma}$ will not be too far away from $QA_{\epsilon_1, \gamma}$, since $\sup_{P \in \mathcal{P}_T^2} QA_{\epsilon_1, \gamma} > \sup_{P \in \mathcal{P}_T^2} QA_{\epsilon_2, \gamma} > \sup_{P \in \mathcal{P}_T^2} QA_{\epsilon_3, \gamma}$. Moreover, a stricter bound can be established for unimodal distributions. The violation of ν th γ -orderliness, when $\nu \geq 2$, is also bounded, since the QA function is bounded, the ν th γ -orderliness corresponds to the higher-order derivatives of the QA function with respect to ϵ .

The Interplay of γ -Orderliness and Weighted Inequalities

Analogous to the γ -orderliness, the γ -trimming inequality for a right-skewed distribution is defined as $\forall 0 \leq \epsilon_1 \leq \epsilon_2 \leq \frac{1}{1+\gamma}$, $TM_{\epsilon_1, \gamma} \geq TM_{\epsilon_2, \gamma}$. γ -orderliness is a sufficient condition for the γ -trimming inequality, as proven in the SI Text. The next theorem shows a relation between the ϵ, γ -quantile average and the ϵ, γ -trimmed mean under the γ -trimming inequality, suggesting the γ -orderliness is not a necessary condition for the γ -trimming inequality.

Theorem .11. *For a distribution that is right-skewed and follows the γ -trimming inequality, it is asymptotically true that the quantile average is always greater or equal to the corresponding trimmed mean with the same ϵ and γ , for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$.*

Proof. According to the definition of the γ -trimming inequality: $\forall 0 \leq \epsilon \leq \frac{1}{1+\gamma}$, $\frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u) du \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$, where δ is an infinitesimal positive quantity. Subsequently, rewriting the inequality gives $\int_{\gamma\epsilon-\delta}^{1-\epsilon+\delta} Q(u) du - \frac{1-\epsilon-\gamma\epsilon+2\delta}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \geq 0 \iff \int_{1-\epsilon}^{1-\epsilon+\delta} Q(u) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u) du - \frac{2\delta}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \geq 0$. Since $\delta \rightarrow 0^+$, $\frac{1}{2\delta} \left(\int_{1-\epsilon}^{1-\epsilon+\delta} Q(u) du + \int_{\gamma\epsilon-\delta}^{\gamma\epsilon} Q(u) du \right) = \frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$, the proof is complete. \square

An analogous result about the relation between the ϵ, γ -trimmed mean and the ϵ, γ -Winsorized mean can be obtained in the following theorem.

Theorem .12. *For a right-skewed distribution following the γ -trimming inequality, asymptotically, the Winsorized mean is always greater or equal to the corresponding trimmed mean with the same ϵ and γ , for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided that $0 \leq \gamma \leq 1$. If assuming γ -orderliness, the inequality is valid for any non-negative γ .*

Proof. According to Theorem .11, $\frac{Q(\gamma\epsilon)+Q(1-\epsilon)}{2} \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du \iff \gamma\epsilon(Q(\gamma\epsilon) + Q(1-\epsilon)) \geq (\frac{2\gamma\epsilon}{1-\epsilon-\gamma\epsilon}) \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$. Then, if $0 \leq \gamma \leq 1$, $1, (1 - \frac{1}{1-\epsilon-\gamma\epsilon}) \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon(Q(\gamma\epsilon) + Q(1-\epsilon)) \geq$

512 $0 \Rightarrow \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du +$
513 $\gamma\epsilon(Q(\gamma\epsilon) + Q(1-\epsilon)) \geq \frac{1}{1-\epsilon-\gamma\epsilon} \int_{\gamma\epsilon}^{1-\epsilon} Q(u) du$, the proof
514 of the first assertion is complete. The second assertion is
515 established in Theorem 0.3. in the SI Text. \square

516 Replacing the TM in the γ -trimming inequality with WA
517 forms the definition of the γ -weighted inequality. The γ -
518 orderliness also implies the γ -Winsorization inequality when
519 $0 \leq \gamma \leq 1$, as proven in the SI Text. The same rationale
520 as presented in Theorem .2, for a location-scale distribu-
521 tion characterized by a location parameter μ and a scale
522 parameter λ , asymptotically, any $WA(\epsilon, \gamma)$ can be expressed
523 as $\lambda WA_0(\epsilon, \gamma) + \mu$, where $WA_0(\epsilon, \gamma)$ is an function of $Q_0(p)$
524 according to the definition of the weighted average. Adhering
525 to the rationale present in Theorem .2, for any probability
526 distribution within a location-scale family, a necessary and
527 sufficient condition for whether it follows the γ -weighted in-
528 equality is whether the family of probability distributions also
529 adheres to the γ -weighted inequality.

To construct weighted averages based on the ν th γ -
orderliness and satisfying the corresponding weighted in-
equality, when $0 \leq \gamma \leq 1$, let $B_i = \int_{i\epsilon}^{(i+1)\epsilon} QA(u, \gamma) du$,
 $ka = k\epsilon + c$. From the γ -orderliness for a right-skewed dis-
tribution, it follows that, $-\frac{\partial QA}{\partial \epsilon} \geq 0 \Leftrightarrow \forall 0 \leq a \leq 2a \leq$
 $\frac{1}{1+\gamma}, -\frac{(QA(2a, \gamma) - QA(a, \gamma))}{a} \geq 0 \Rightarrow B_i - B_{i+1} \geq 0$, if $0 \leq \gamma \leq 1$.
Suppose that $B_i = B_0$. Then, the ϵ, γ -block Winsorized mean,
is defined as

$$BWM_{\epsilon, \gamma, n} := \frac{1}{n} \left(\sum_{i=n\gamma\epsilon+1}^{(1-\epsilon)n} X_i + \sum_{i=n\gamma\epsilon+1}^{2n\gamma\epsilon+1} X_i + \sum_{i=(1-2\epsilon)n}^{(1-\epsilon)n} X_i \right),$$

530 which is double weighting the leftest and rightest blocks hav-
531 ing sizes of $\gamma\epsilon n$ and ϵn , respectively. As a consequence of
532 $B_i - B_{i+1} \geq 0$, the γ -block Winsorization inequality is valid,
533 provided that $0 \leq \gamma \leq 1$. The block Winsorized mean uses
534 two blocks to replace the trimmed parts, not two single quan-
535 tities. The subsequent theorem provides an explanation for
536 this difference.

537 **Theorem .13.** *Asymptotically, for a right-skewed distribution*
538 *following the γ -orderliness, the Winsorized mean is always*
539 *greater than or equal to the corresponding block Winsorized*
540 *mean with the same ϵ and γ , for all $0 \leq \epsilon \leq \frac{1}{1+\gamma}$, provided*
541 *that $0 \leq \gamma \leq 1$.*

542 *Proof.* From the definitions of BWM and WM, the state-
543 ment necessitates $\int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon) \geq$
544 $\int_{\gamma\epsilon}^{1-\epsilon} Q(u) du + \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du + \int_{1-2\epsilon}^{1-\epsilon} Q(u) du \Leftrightarrow \gamma\epsilon Q(\gamma\epsilon) +$
545 $\epsilon Q(1-\epsilon) \geq \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du + \int_{1-2\epsilon}^{1-\epsilon} Q(u) du$. Define $WML(x) =$
546 $Q(\gamma\epsilon)$ and $BWML(x) = Q(x)$. In both functions, the
547 interval for x is specified as $[\gamma\epsilon, 2\gamma\epsilon]$. Then, define
548 $WMu(y) = Q(1-\epsilon)$ and $BWMu(y) = Q(y)$. In both
549 functions, the interval for y is specified as $[1-2\epsilon, 1-\epsilon]$.
550 The function $y : [\gamma\epsilon, 2\gamma\epsilon] \rightarrow [1-2\epsilon, 1-\epsilon]$ defined by
551 $y(x) = 1 - \frac{x}{\gamma}$ is a bijection. $WML(x) + WMu(y(x)) =$
552 $Q(\gamma\epsilon) + Q(1-\epsilon) \geq BWML(x) + BWMu(y(x)) = Q(x) +$
553 $Q(1 - \frac{x}{\gamma})$ is valid for all $x \in [\gamma\epsilon, 2\gamma\epsilon]$, according to the
554 definition of γ -orderliness. Integration of the left side
555 yields, $\int_{\gamma\epsilon}^{2\gamma\epsilon} (WML(u) + WMu(y(u))) du = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon) du +$
556 $\int_{y(\gamma\epsilon)}^{y(2\gamma\epsilon)} Q(1-\epsilon) du = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(\gamma\epsilon) du + \int_{1-2\epsilon}^{1-\epsilon} Q(1-\epsilon) du =$

$\gamma\epsilon Q(\gamma\epsilon) + \epsilon Q(1-\epsilon)$, while integration of the right side
yields $\int_{\gamma\epsilon}^{2\gamma\epsilon} (BWML(x) + BWMu(y(x))) dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du +$
 $\int_{\gamma\epsilon}^{2\gamma\epsilon} Q(1 - \frac{x}{\gamma}) dx = \int_{\gamma\epsilon}^{2\gamma\epsilon} Q(u) du + \int_{1-2\epsilon}^{1-\epsilon} Q(u) du$, which are
the left and right sides of the desired inequality. Given that the
upper limits and lower limits of the integrations are different
for each term, the condition $0 \leq \gamma \leq 1$ is necessary for the
desired inequality to be valid. \square

From the second γ -orderliness for a right-skewed dis-
tribution, $\frac{\partial^2 QA}{\partial \epsilon^2} \geq 0 \Rightarrow \forall 0 \leq a \leq 2a \leq 3a \leq$
 $\frac{1}{1+\gamma}, \frac{1}{a} \left(\frac{QA(3a, \gamma) - QA(2a, \gamma)}{a} - \frac{QA(2a, \gamma) - QA(a, \gamma)}{a} \right) \geq 0 \Rightarrow$ if
 $0 \leq \gamma \leq 1, B_i - 2B_{i+1} + B_{i+2} \geq 0$. SM_ϵ can thus be interpreted
as assuming $\gamma = 1$ and replacing the two blocks, $B_i + B_{i+2}$
with one block $2B_{i+1}$. From the ν th γ -orderliness for a right-
skewed distribution, the recurrence relation of the derivatives
naturally produces the alternating binomial coefficients,

$$(-1)^\nu \frac{\partial^\nu QA}{\partial \epsilon^\nu} \geq 0 \Rightarrow \forall 0 \leq a \leq \dots \leq (\nu+1)a \leq \frac{1}{1+\gamma},$$

$$\frac{(-1)^\nu}{a} \left(\frac{QA(\nu a + a, \gamma)}{a} - \frac{\dots QA(2a, \gamma)}{a} - \frac{QA(\nu a, \gamma)}{a} - \dots \frac{QA(a, \gamma)}{a} \right)$$

$$\geq 0 \Leftrightarrow \frac{(-1)^\nu}{a^\nu} \left(\sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} QA((\nu-j+1)a, \gamma) \right) \geq 0$$

$$\Rightarrow \text{if } 0 \leq \gamma \leq 1, \sum_{j=0}^{\nu} (-1)^j \binom{\nu}{j} B_{i+j} \geq 0.$$

Based on the ν th orderliness, the ϵ, γ -binomial mean is intro-
duced as

$$BM_{\nu, \epsilon, \gamma, n} := \frac{1}{n} \left(\sum_{i=1}^{\frac{1}{2}\epsilon^{-1}(\nu+1)-1} \sum_{j=0}^{\nu} \left(1 - (-1)^j \binom{\nu}{j} \right) \mathfrak{B}_{ij} \right),$$

where $\mathfrak{B}_{ij} = \sum_{l=n\gamma\epsilon(j+(i-1)(\nu+1)+1)}^{n\epsilon(j+(i-1)(\nu+1)+1)} (X_l + X_{n-l+1})$. If ν is
not indicated, it defaults to $\nu = 3$. Since the alternating sum
of binomial coefficients equals zero, when $\nu \ll \epsilon^{-1}$ and $\epsilon \rightarrow 0$,
 $BM \rightarrow \mu$. The solutions for the continuity of the breakdown
point is the same as that in SM and not repeated here. The
equalities $BM_{\nu=1, \epsilon} = BWM_\epsilon$ and $BM_{\nu=2, \epsilon} = SM_{\epsilon, b=3}$ hold,
when $\gamma = 1$ and their respective ϵ s are identical. Interestingly,
the biases of the $SM_{\epsilon=\frac{1}{9}, b=3}$ and the $WM_{\epsilon=\frac{1}{9}}$ are nearly indis-
tinguishable in common asymmetric unimodal distributions
such as Weibull, gamma, lognormal, and Pareto (SI Dataset
S1). This indicates that their robustness to departures from
the symmetry assumption is practically similar under uni-
modality, even though they are based on different orders of
orderliness. If single quantiles are used, based on the second
 γ -orderliness, the stratified quantile mean can be defined as

$$SQM_{\epsilon, \gamma, n} := 4\epsilon \sum_{i=1}^{\frac{1}{4\epsilon}} \frac{1}{2} (\hat{Q}_n((2i-1)\gamma\epsilon) + \hat{Q}_n(1 - (2i-1)\epsilon)),$$

$SQM_{\epsilon=\frac{1}{4}}$ is the Tukey's midhinge (34). In fact, SQM is a
subcase of SM when $\gamma = 1$ and $b \rightarrow \infty$, so the solution for the
continuity of the breakdown point, $\frac{1}{\epsilon} \bmod 4 \neq 0$, is identical.
However, since the definition is based on the empirical quantile
function, no decimal issues related to order statistics will arise.
The next theorem explains another advantage.

Data Availability. Data for Figure ?? are given in SI Dataset S1. All codes have been deposited in [GitHub](#).

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1. CF Gauss, *Theoria combinationis observationum erroribus minimis obnoxiae*. (Henricus Dieterich), (1823).
2. C Bernard, R Kazzi, S Vanduffel, Range value-at-risk bounds for unimodal distributions under partial information. *Insur. Math. Econ.* **94**, 9–24 (2020).
3. P Daniell, Observations weighted according to order. *Am. J. Math.* **42**, 222–236 (1920).
4. JW Tukey, A survey of sampling from contaminated distributions in *Contributions to probability and statistics*. (Stanford University Press), pp. 448–485 (1960).
5. WJ Dixon, Simplified Estimation from Censored Normal Samples. *The Annals Math. Stat.* **31**, 385–391 (1960).
6. K Danielak, T Rychlik, Theory & methods: Exact bounds for the bias of trimmed means. *Aust. & New Zealand J. Stat.* **45**, 83–96 (2003).
7. M Bieniek, Comparison of the bias of trimmed and winsorized means. *Commun. Stat. Methods* **45**, 6641–6650 (2016).
8. J Hodges Jr, E Lehmann, Estimates of location based on rank tests. *The Annals Math. Stat.* **34**, 598–611 (1963).
9. F Wilcoxon, Individual comparisons by ranking methods. *Biom. Bull.* **1**, 80–83 (1945).
10. PJ Huber, Robust estimation of a location parameter. *Ann. Math. Stat.* **35**, 73–101 (1964).
11. Q Sun, WX Zhou, J Fan, Adaptive huber regression. *J. Am. Stat. Assoc.* **115**, 254–265 (2020).
12. AS Nemirovskij, DB Yudin, *Problem complexity and method efficiency in optimization*. (Wiley-Interscience), (1983).
13. D Hsu, S Sabato, Heavy-tailed regression with a generalized median-of-means in *International Conference on Machine Learning*. (PMLR), pp. 37–45 (2014).
14. L Devroye, M Lerasle, G Lugosi, RI Oliveira, Sub-gaussian mean estimators. *The Annals Stat.* **44**, 2695–2725 (2016).
15. P Laforgue, S Cléménçon, P Bertail, On medians of (randomized) pairwise means in *International Conference on Machine Learning*. (PMLR), pp. 1272–1281 (2019).
16. G LECUÉ, M LERASLE, Robust machine learning by median-of-means: Theory and practice. *The Annals Stat.* **48**, 906–931 (2020).
17. B Efron, Bootstrap methods: Another look at the jackknife. *The Annals Stat.* **7**, 1–26 (1979).
18. PJ Bickel, DA Freedman, Some asymptotic theory for the bootstrap. *The annals statistics* **9**, 1196–1217 (1981).
19. PJ Bickel, DA Freedman, Asymptotic normality and the bootstrap in stratified sampling. *The annals statistics* **12**, 470–482 (1984).
20. R Helmers, P Janssen, N Veraverbeke, *Bootstrapping U-quantiles*. (CWI. Department of Operations Research, Statistics, and System Theory [BS]), (1990).
21. J Neyman, On the two different aspects of the representative method: The method of stratified sampling and the method of purposive selection. *J. Royal Stat. Soc.* **97**, 558–606 (1934).
22. G McIntyre, A method for unbiased selective sampling, using ranked sets. *Aust. journal agricultural research* **3**, 385–390 (1952).
23. DL Donoho, PJ Huber, The notion of breakdown point. *A festschrift for Erich L. Lehmann* **157184** (1983).
24. CM Stein, Efficient nonparametric testing and estimation in *Proceedings of the third Berkeley symposium on mathematical statistics and probability*. Vol. 1, pp. 187–195 (1956).
25. PJ Bickel, On adaptive estimation. *The Annals Stat.* **10**, 647–671 (1982).
26. P Bickel, CA Klaassen, Y Ritov, JA Wellner, *Efficient and adaptive estimation for semiparametric models*. (Springer) Vol. 4, (1993).
27. JT Runnenburg, Mean, median, mode. *Stat. Neerlandica* **32**, 73–79 (1978).
28. Wv Zwet, Mean, median, mode ii. *Stat. Neerlandica* **33**, 1–5 (1979).
29. WR van Zwet, *Convex Transformations of Random Variables: Nebst Stellingen*. (1964).
30. K Pearson, X. contributions to the mathematical theory of evolution.—ii. skew variation in homogeneous material. *Philos. Transactions Royal Soc. London.(A.)* **186**, 343–414 (1895).
31. AL Bowley, *Elements of statistics*. (King) No. 8, (1926).
32. RA Groeneveld, G Meeden, Measuring skewness and kurtosis. *J. Royal Stat. Soc. Ser. D (The Stat.)* **33**, 391–399 (1984).
33. L Li, H Shao, R Wang, J Yang, Worst-case range value-at-risk with partial information. *SIAM J. on Financial Math.* **9**, 190–218 (2018).
34. JW Tukey, *Exploratory data analysis*. (Reading, MA) Vol. 2, (1977).