3 Set Theory

Relationy

- In **naive set theory (朴素集合论)**, a set is described as a well-defined collection of objects. These objects are called **elements (元素)** or **members** of the set. Objects can be anything: numbers, people, other sets, etc, and the set can be finite or infinite.

  Definition: Set (Naive Definition)

  A set is a collection of objects treated as a single entity. We use notation $a \in A$ to denote that "*object a is an element of set A*" and use $a \notin A$ if not.

- There are two approaches to represent a set. The simplest way to describe a set is by its **extension (外延)**, i.e., to list all its elements between curly braces. For example, $A = \{a, \{b, \{c\}\}\}$ is a set having two elements: a and $\{b, \{c\}\}$, where the second element is a set which again has two elements. Also, we make the following remarks:
  - The order of elements is irrelevant, e.g., $\{1, 2\} = \{2, 1\}$.
  - Repetition of elements is irrelevant, e.g., $\{1, 2, 2\} = \{1, 1, 1, 2\} = \{1, 2\}$.
  - For any element a and set A, we have either $a \in A$ or $a \notin A$.
- The other way to describe a set is by its **intension (内涵)**. Specifically, let $P(x)$ be a predicate. We use notation $\{x : P(x)\}$ or $\{x \mid P(x)\}$ to denote the set containing all objects for which predicate $P(\cdot)$ holds, e.g., $\mathbb{N} = \{x : x \text{ is a natural number}\}$.

  Sometimes, we use notation $\{x \in A : P(x)\} = \{x : x \in A \wedge P(x)\}$ to denote the set of all x that is already a member of A such that the condition P holds for x. Also, we define
  - **empty set (空集)** as a set with no element, i.e., $\emptyset = \{x : x \neq x\} = \{\}$;
  - **universal set (全集)** as a set with all elements, i.e., $E = \{x : x = x\}$.

  Example

  Let us consider the following set described by its intension

  $$A = \{x : x = a \vee (\exists y)\,(y \in A \wedge x = \{y\})\}$$

  Then we have $a \in A$, and since $a \in A$, we have $\{a\} \in A$, and since $\{a\} \in A$, we have $\{\{a\}\} \in A, \cdots$. Therefore, we have $A = \{a, \{a\}, \{\{a\}\}, \cdots\}$.

- Given a set A, its **Cardinality (基数)** is the number of elements in it, and we usually use notations $|A|, \#(A)$ or $\mathrm{Card}(A)$ to denote the cardinality of A. For example, we have $|\{a, b, c\}| = 3, |\emptyset| = 0, |\{1, 2, 2, 3\}| = 3, |\{a, \{a, \{b\}\}\}| = 2$. We will discuss later what is the cardniality of an infinite set such as $\mathbb{N}$ or $\mathbb{R}$.

Set Relations

- According to the above definition, if two sets contain exactly the same elements, then we should consider them as the same set. Furthermore, if one set contains all elements in the other set, then the former should be considered "larger" than the latter.

Definition: Set Equivalence, Subset & Proper-Subset

Let A and B be two sets. We say

- A, B are **equivalent (相等)**, denoted by $A = B$, if $(\forall x)(x \in A \leftrightarrow x \in B)$;
- A is a **subset (子集)** of B, denoted by $A \subseteq B$, if $(\forall x)(x \in A \rightarrow x \in B)$;
- A is a **proper-subset (真子集)** of B, denoted by $A \subset B$, if $A \subseteq B$ and $A \neq B$.

We denote by $A \not\subseteq B$ if A is not a subset of B, i.e.,

$$\neg(\forall x)(x \in A \rightarrow x \in B) = (\exists x)(x \in A \wedge x \notin B)$$

Similarly, we denote by $A \not\subset B$ if A is not a proper-subset of B, i.e., either $A = B$ or A contains an element that is not in B. For example, we have

- for $A = \{a, \{b\}\}$, we have $\{b\} \in A$, $\{b\} \not\subseteq A$ and $\{\{b\}\} \subseteq A$.
- for $B = \{a, b, \{a\}\}$, we have $\{a, b\} \subseteq B$, $\{a, b\} \notin B$, $\{a\} \in B$, $\{a\} \not\subseteq B$ and $\{\{a\}\} \subseteq B$.

Note that if $A \subset B$, then we naturally have $A \subseteq B$.

- To prove that two sets are equivalent, it suffices to prove they are subsets of each other.

Theorem: Set Equivalence

$A = B$ iff $A \subseteq B$ and $B \subseteq A$.

Proof:

$$\begin{aligned} A = B &\text{ iff } (\forall x)(x \in A \leftrightarrow x \in B) \\ &\text{ iff } (\forall x)((x \in A \rightarrow x \in B) \wedge (x \in B \rightarrow x \in A)) \\ &\text{ iff } (\forall x)(x \in A \rightarrow x \in B) \wedge (\forall x)(x \in B \rightarrow x \in A) \\ &\text{ iff } A \subseteq B \wedge B \subseteq A \end{aligned}$$

- Also, we can prove that the empty set is unique and it is a subset of any set.

Theorem: Properties of Empty Set

① For any set A, we have $\emptyset \subseteq A$.
② Empty set $\emptyset$ is unique.

Proof: To see ①, suppose that $\emptyset \not\subseteq A$ for some set A. Then $(\exists x)(x \in \emptyset \wedge x \notin A)$. However, this is a contradiction since $\emptyset$ has no element.
To see ②, suppose that there are two empty sets $\emptyset$ and $\emptyset'$ satisfying the definition. Then based on ①, we know that $\emptyset \subseteq \emptyset'$ and $\emptyset' \subseteq \emptyset$, which means that $\emptyset = \emptyset'$.

Basic Set Operations

- Let A and B be two sets and E be the universal set. we define the following sets
  - **union (并集)**: $A \cup B = \{x : x \in A \vee x \in B\}$
  - **intersection (交集)**: $A \cap B = \{x : x \in A \wedge x \in B\}$
  - **difference (差集)**: $A - B = \{x : x \in A \wedge x \notin B\}$, also as $A \setminus B$ (B 对 A 的补)
  - **complement (余集)**: $A^c = \{x : x \notin A\} = E \setminus A$ (A 的绝对补)
  - **symmetric difference (对称差)**: $A \oplus B = \{x : x \in A \oplus x \in B\}$
- The above operators have the following properties:

Basic Properties of Set Operations

①	$A \cup B = B \cup A$	$A \cap B = B \cap A$
②	$(A \cup B) \cup C = A \cup (B \cup C)$	$(A \cap B) \cap C = A \cap (B \cap C)$
③	$A \cup (B \cap C) = (A \cup B) \cap (A \cup C)$	$A \cap (B \cup C) = (A \cap B) \cup (A \cap C)$
④	$A \cup A = A$	$A \cap A = A$
⑤	$A \cup (A \cap B) = A$	$A \cap (A \cup B) = A$
⑥	$A - (B \cup C) = (A - B) \cap (A - C)$	$A - (B \cap C) = (A - B) \cup (A - C)$
⑦	$(A \cup B)^c = A^c \cap B^c$	$(A \cap B)^c = A^c \cup B^c$

Proof of ⑥: $x \in A - (B \cup C)$ iff $x \in A \wedge x \notin (B \cup C)$ iff $x \in A \wedge x \notin B \wedge x \notin C$ iff $(x \in A \wedge x \notin B) \wedge (x \in A \wedge x \notin C)$ iff $x \in A - B \wedge x \in A - C$ iff $x \in (A - B) \cap (A - C)$.

- The above defined operators are all binary operators by manipulating two sets. Later in axiomatic set theory, we need to use the following unitary operators for a **family of sets (集合族)**. Formally, A is called a family of sets over S if each element in A is a subset of S. That is, assuming S is the set of all elements, then A is a set of sets.

Definition: Generalized Union & Intersection

Let A be a family of sets. Then we define its

- **Generalized Union (广义并)** as the union of all elements in A

$$\bigcup A = \{x : (\exists z)(z \in A \wedge x \in z)\}$$

- **Generalized Intersection (广义交)** as the intersection of all elements in A

$$\bigcap A = \{x : (\forall z)(z \in A \rightarrow x \in z)\}$$

For example, for $A = \{\{a, b, c\}, \{a, b\}, \{b, c, d\}\}$, we have $\bigcup A = \{a, b, c, d\}$ and $\bigcap A = \{b\}$.

- **Remark:** For technical reason, we define $\bigcup \emptyset = \emptyset$, but $\bigcap \emptyset$ is undefined.

Power Set and Its Properties

- The **power set (幂集)** is a very important concept in set theory as it always gives us a new set with "higher order" than the original one. Its definition is as follows:

Definition: Power Set

Let A be a set. The power set of A is the collection of all subsets of A, i.e., $2^A = \{x : x \subseteq A\}$. We also use notations $P(A)$ or $Pwr(A)$ to denote the power set.

Examples

- for $A = \{a, b\}$, we have $2^A = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$
- for $A = \{1, 2, 3\}$, we have $2^A = \{\emptyset, \{1\}, \{2\}, \{3\}, \{1, 2\}, \{1, 3\}, \{2, 3\}, \{1, 2, 3\}\}$
- for $\emptyset$, we have $2^\emptyset = \{\emptyset\} \neq \emptyset$. However, for $\{\emptyset\}$, we have $2^{\{\emptyset\}} = \{\emptyset, \{\emptyset\}\}$

- Based on the above examples, we see the following properties for power set
  1. Since we always have $\emptyset \in 2^A$, <u>the power set is always non-empty</u>.
  2. If $|A| = n$, then we have $|2^A| = C_n^0 + C_n^1 + \cdots C_n^n = 2^n$ elements. Therefore, <u>2^A always has "more" elements than that of A</u>. Of course, we need to discuss later what do we mean "more" when both A and 2^A have infinite number of elements.
- The power set also has the following properties:

Theorem: Properties of Power Sets

① $A \subseteq B \Leftrightarrow 2^A \subseteq 2^B$ ② $2^A \in 2^B \Rightarrow A \in B$
③ $2^A \cap 2^B = 2^{A \cap B}$ ④ $2^A \cup 2^B \subseteq 2^{A \cup B}$

The proofs of the above results are as follows:

① "$\Rightarrow$" Suppose that $A \subseteq B$. Then for any $x \in 2^A$, we have $x \subseteq A \subseteq B$, which means that $x \in 2^B$. Therefore, we have $2^A \subseteq 2^B$.

"$\Leftarrow$" Suppose $2^A \subseteq 2^B$. Then for any $x \in A$, we have $\{x\} \subseteq A$, which means that $\{x\} \in 2^A \subseteq 2^B$. This means that $\{x\} \subseteq B$ or $x \in B$ and we have $A \subseteq B$.

② $2^A \in 2^B \Rightarrow 2^A \subseteq B \Rightarrow (A \in 2^A) \wedge (2^A \subseteq B) \Rightarrow A \in B$.

③ $x \in 2^A \cap 2^B$ iff $x \in 2^A \wedge x \in 2^B$ iff $x \subseteq A \wedge x \subseteq B$ iff $x \subseteq A \cap B$ iff $x \in 2^{A \cap B}$.

④ $x \in 2^A \cup 2^B$ iff $x \subseteq A \vee x \subseteq B \boxed{\Rightarrow} x \subseteq A \cup B$ iff $x \in 2^{A \cup B}$.

- Note that "$\Leftarrow$" of ② is NOT correct. For example, we consider $A = \{\emptyset\}$ and $B = \{\{\emptyset\}\}$, we have $A \in B$, but $2^A = \{\emptyset, \{\emptyset\}\}, 2^B = \{\emptyset, \{\{\emptyset\}\}\}, 2^A \notin 2^B$.

- Also, we note that $2^{A \cup B} \subseteq 2^A \cup 2^B$ is NOT correct. For example, we consider $A = \{a\}$ and $B = \{b\}$, then we have $2^{\{a,b\}} = \{\emptyset, \{a\}, \{b\}, \{a, b\}\}$ but $2^{\{a\}} \cup 2^{\{b\}} = \{\emptyset, \{a\}, \{b\}\}$.

Transitive Sets

- **Transitive Sets (传递集)** are important class of sets for defining ordinal numbers. Its definition is given follows:

Definition: Transitive Sets

Let A be a *family of sets*. Then A is said to be a **transitive set** if any element of an element in A is still an element in A, i.e.,

$$(\forall x)(\forall y)((y \in A \wedge x \in y) \rightarrow x \in A)$$

Examples

- $A = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is a transitive set
- $A = \{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$ is NOT a transitive set because $\emptyset \in \{\emptyset\} \in A$ but $\emptyset \notin A$
- $A = \{x : x = \{a\} \vee (\exists y)(y \in A \wedge x = \{y\})\} = \{\{a\}, \{\{a\}\}, \{\{\{a\}\}\}, \cdots\}$ is NOT a transitive set because $a \in \{a\} \in A$ but $a \notin A$

- We note that, for any transitive set A, <u>if $x \in A$, then $x \subseteq A$</u>. This is due to the transitive property, i.e., for any $x \in A$, we have $(\forall z \in x)(z \in A)$, which means that $x \subseteq A$.

Theorem: Properties of Transitive Sets

① A is a transitive set, if and only if, 2^A is a transitive set.
② If A is a transitive set, then $\bigcup A$ is a transitive set.
③ If any element in A is a transitive set, then $\bigcup A$ is a transitive set.

The proofs of the above results are as follows:

① "$\Rightarrow$" Let us consider arbitrary $x \in y \in 2^A$. Note that $y \in 2^A$ implies that $y \subseteq A$, which means that $x \in A$. However, since A is transitive, we know that $x \subseteq A$, i.e., $x \in 2^A$, which proves that 2^A is transitive.

"$\Leftarrow$" Let us consider arbitrary $x \in y \in A$. Note that $y \in A$ implies that $\{y\} \subseteq A$, i.e., $\{y\} \in 2^A$. Since 2^A is transitive, we also have $y \in 2^A$, i.e., $y \subseteq A$. This means that $x \in y \subseteq A$, which proves that A is transitive.

② $[x \in y \in \bigcup A]$ iff $[x \in y \wedge (\exists z)(y \in z \wedge z \in A)] \overset{A \text{ is trans.}}{\Longrightarrow} [x \in y \in A] \Rightarrow [x \in \bigcup A]$

③ $[x \in y \in \bigcup A]$ iff $[x \in y \wedge (\exists z)(y \in z \wedge z \in A)] \overset{z \text{ is trans.}}{\Longrightarrow} [x \in y \wedge (\exists z)(y \subseteq z \wedge z \in A)]$
$\Rightarrow [(\exists z)(x \in z \wedge z \in A)] \Rightarrow [x \in \bigcup A]$

- The converse of ② is NOT correct. For example, for $A = \{\{\emptyset\}, \{\{\emptyset\}\}\}$, which is not transitive, we have $\bigcup A = \{\emptyset, \{\emptyset\}\}$, which is transitive.

- The converse of ③ is NOT correct. For example, for $A = \{\{\emptyset\}, \{\{\emptyset\}\}\}$, $\bigcup A = \{\emptyset, \{\emptyset\}\}$ is transitive, but $\{\{\emptyset\}\} \in A$ is not.

Ordered Pair and Cartisian Product

- We note that elements in a set are unordered in the sense that $\{a, b\} = \{b, a\}$. In case that we need to emphasize the order of elements a and b, e.g., a point in a coordinate system, we need to introduce the notion of **Ordered Pair (有序对)**.

- Formally, an order pair, also called a two-tuple (二元组), consists of two elements in order, denoted by $\langle x, y\rangle$. Then we have $\langle x_1, y_1\rangle = \langle x_2, y_2\rangle$ iff $x_1 = x_2$ and $y_1 = y_2$. Therefore, if $x \neq y$, then $\langle x, y\rangle \neq \langle y, x\rangle$.

- Note that $\langle x, y\rangle$ is just a notation for the sake of simplicity. Essentially, it should be define as the following set
$$\langle x, y\rangle := \{\{x\}, \{x, y\}\}$$
Note that $\{x, y\}$ is not sufficient for the purpose of ordering. Therefore, we also need $\{x\}$ to denote which element is the first one.

- By understanding that $\langle x, y\rangle$ is essentially $\{\{x\}, \{x, y\}\}$, we have the following result:

  Let A be a set and $x, y \in A$. Then we have $\langle x, y\rangle \in 2^{2^A}$.

  Proof: First, we note that $x \in A$ iff $\{x\} \subseteq A$ iff $\{x\} \in 2^A$. Also, we know that $x \in A \wedge y \in A$ iff $\{x, y\} \subseteq A$ iff $\{x, y\} \in 2^A$. Therefore, $\langle x, y\rangle = \{\{x\}, \{x, y\}\} \subseteq 2^A$, which means that $\langle x, y\rangle \in 2^{2^A}$.

- The idea of 2-tuple can be extended to order n elements. Specifically, let $x_1, x_2, \cdots, x_n$ be n elements, where $n \geq 2$. Then n-tuple $\langle x_1, x_2, \cdots, x_n\rangle$ is defined recursively by
  - for $n = 2$, 2-tuple $\langle x_1, x_2\rangle$ is an ordered pair;
  - for $n > 2$, we have $\langle x_1, x_2, \cdots, x_n\rangle = \langle\langle x_1, \cdots, x_{n-1}\rangle, x_n\rangle$

  For example, we have $\langle a, b, c, d\rangle = \langle\langle a, b, c\rangle, d\rangle = \langle\langle\langle a, b\rangle, c\rangle, d\rangle$.

- With the concept of ordered pairs, we define the **Cartisian Product (笛卡尔积)**:

  Definition: Cartisian Product

  Let A and B be two sets. The **Cartisian product** of A and B is
$$A \times B := \{\langle x, y\rangle : x \in A \wedge y \in B\}$$
  Also, for n sets $A_1, \ldots, A_n$, we define their **n-Cartisian product** as
$$A_1 \times A_2 \times \cdots \times A_n = \{\langle x_1, x_2, \cdots, x_n\rangle : x_1 \in A_1 \wedge \cdots \wedge x_n \in A_n\}$$

- For example, for $A = \{a, b\}$ and $B = \{1, 2, 3\}$, we have
$$A \times B = \{\langle a, 1\rangle, \langle a, 2\rangle, \langle a, 3\rangle, \langle b, 1\rangle, \langle b, 2\rangle, \langle b, 3\rangle\} \text{ and } A \times A = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle\}$$

Properties of Cartisian Product

- The Cartisian product has the following basic properties.

Basic Properties of Set Operations

① $A \times \emptyset = \emptyset \times A = \emptyset$
② If $A \neq B$, and A, B are non-empty, then $A \times B \neq B \times A$
③ $(A \times B) \times C \neq A \times (B \times C)$
④ $A \times (B \cup C) = (A \times B) \cup (A \times C)$
⑤ $A \times (B \cap C) = (A \times B) \cap (A \times C)$
⑥ $(B \cup C) \times A = (B \times A) \cup (C \times A)$
⑦ $(B \cap C) \times A = (B \times A) \cap (C \times A)$

③ follows from the definition of n-tuple that $\langle a, b, c\rangle = \langle\langle a, b\rangle, c\rangle \neq \langle a, \langle b, c\rangle\rangle$.
To see ④, we have

$$\langle x, y\rangle \in A \times (B \cup C) \Leftrightarrow x \in A \wedge y \in B \cup C \Leftrightarrow x \in A \wedge (y \in B \vee y \in C)$$
$$\Leftrightarrow (x \in A \wedge y \in B) \vee (x \in A \wedge y \in C) \Leftrightarrow (\langle x, y\rangle \in A \times B) \vee (\langle x, y\rangle \in A \times C)$$
$$\Leftrightarrow \langle x, y\rangle \in (A \times B) \cup (A \times C)$$

- For any two sets A, B and non-empty set C, we claim that the followings are equivalent:
① $A \subseteq B$ ② $A \times C \subseteq B \times C$ ③ $C \times A \subseteq C \times B$

Proof: ①⇒② Suppose $A \subseteq B$. Then for $\langle x, y\rangle \in A \times C$, we have $x \in A \wedge y \in C$, which means that $x \in B \wedge y \in C$. Therefore, we have $\langle x, y\rangle \in B \times C$.

②⇒① Suppose $A \times C \subseteq B \times C$. Then for $y \in C, x \in A$, we have $\langle x, y\rangle \in A \times C$, which means that $\langle x, y\rangle \in B \times C$. Therefore, we have $x \in B$.

The proof of ①⇔③ is the same as the above.

- Let A, B, C, D be non-empty sets. We have

$$A \times B \subseteq C \times D \text{ iff } A \subseteq C \text{ and } B \subseteq D$$

Proof: "⇒" Suppose $A \times B \subseteq C \times D$. For any $x \in A$, consider $y \in B$. We have

$$[x \in A \wedge y \in B] \text{ iff } [\langle x, y\rangle \in A \times B] \Rightarrow [\langle x, y\rangle \in C \times D] \Rightarrow [x \in C \wedge y \in D] \Rightarrow [x \in C]$$

Therefore, $A \subseteq C$, and $B \subseteq D$ for the same reason.

"⇐" For any x and y, we have the following

$$[\langle x, y\rangle \in A \times B] \text{ iff } [x \in A \wedge y \in B] \Rightarrow [x \in C \wedge y \in D] \text{ iff } [\langle x, y\rangle \in C \times D]$$

Axiomatic Set Theory

- So far, we have studied the naive set theory by considering a set as a collection of objects. However, this naive setting may lead to the following **Russel's Paradox (罗素悖论)**.

Russel's Paradox

Since we consider any $\{x : P(x)\}$ as a set, we can construct the following set

$$A = \{x : x \notin x\}$$

Of course, you can argue that there is no x such that $x \in x$; if so, A should be the empty set, but we do not care. Let us ask whether or not we have $A \in A$:

- if $A \in A$, then A does not satisfy the predicate, meaning that $A \notin A$;
- if $A \notin A$, then A does satisfy the predicate, meaning that $A \in A$.

This is actually a paradox making the entire naive set theory problematic.

- The problem of the Russel's paradox is that we cannot just describe a set of things satisfying a predicate as a set *without any restriction*. The key idea for resolving this issue is that *sets are constructed from axioms*, which leads to the **Axiomatic Set Theory (公理集合论)**. Hereafter, we introduce the axioms in the ZF-set theory developed by (Zermelo-Fraenkel).

- The first axiom is the **Axiom of Extensionality (外延公理)** specifying when two sets are the same.

$$(\forall x)(\forall y)(x = y \leftrightarrow (\forall z)(z \in x \leftrightarrow z \in y))$$

- Now we need the **Axiom of Empty Set (空集存在公理)** in order to generate the first set in axiomatic set theory, the empty set.

$$(\exists x)(\forall y)(y \notin x)$$

The above set x that exists is actually the empty $\emptyset$. In fact, we can use the first two axioms to prove that $\emptyset$, a state such that nothing belongs to it, is unique.

- Suppose that we have two sets. We can put they together in a bigger set based on the **Axiom of Pairing (配对公理)**.

$$(\forall x)(\forall y)(\exists z)(\forall u)(u \in z \leftrightarrow (u = x \vee u = y))$$

The above z is actually $\{x, y\}$. Recall that the axiom of empty set gives us $\emptyset$. Then using the axiom of pairing, we can construct $\{\emptyset\}$ taking $x = y = \emptyset$. Similarly, we can construct $\{\{\emptyset\}\}, \{\{\{\emptyset\}\}\}, \ldots$, and also, for example, $\{\emptyset, \{\emptyset\}\}$ and $\{\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\} = \langle \emptyset, \{\emptyset\} \rangle$.

Axiomatic Set Theory

- Now, using $\emptyset$ and the axiom of pairing, we are able to construct infinite number of finite sets. However, this only allows us to put two sets as elements in a new set. What if we need to put all their elements together to obtain their union. This is done by the **Axiom of Union (并集公理)** which allows us to unpack a set of sets and create a flatter set.

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\exists u)(z \in u \wedge u \in x))$$

The above y is actually is *generalized union* $\bigcup x$. This axiom together with the axiom of pairing imply that for any two sets, there is a set (called their union) that contains exactly the elements of these two sets. For example, we can define $x \cup y$ by first introducing $\{x, y\}$, and then introducing $\bigcup\{x, y\}$, which is $x \cup y$.

- Recall that, in naive set theory, we consider $\{x : P(x)\}$ as a set. However, in axiomatic set theory, because sets are "constructed" from axioms, we need to first show a family of sets exists and then separate some of them with particular property as a new set. This is called the **Axiom Schema of Separation (分离公理模式)**, we call it "schema" because it is parameterized by predicate $P(\cdot)$.

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow z \in x \wedge P(z))$$

The above y is actually set $\{a \in x : P(a)\} = \{a : a \in x \wedge P(a)\}$, i.e., the collection of all elements in x satisfying predicate $P(\cdot)$.

  - Using this axiom schema, we can define set intersection and set complement by $A \cap B = \{a \in A : a \in B\}$ and $A - B = \{a \in A : a \notin B\}$.
  - Based on this axiom, we argue that <u>there is no "universal set" in axiomatic set theory</u>. Suppose that there exists such E containing all sets as its elements. Then this actually means that we can use the axiom schema of separation freely without any restriction. Therefore, if we choose $P(a)$ as $a \notin a$, then $A = \{x \in E : x \notin x\}$ is a well-defined set. However, this leads to a contradiction because $A \in A \Leftrightarrow A \notin A$.

- Now, given a set, we want to always be able to build a "larger" set (we will understand the need later when we discuss the difference between countable and uncountable set). This is done by the **Axiom of Power Set (幂集公理)**,

$$(\forall x)(\exists y)(\forall z)(z \in y \leftrightarrow (\forall u)(u \in z \rightarrow u \in x))$$

The above y is actually 2^x. With this axiom, we can construct $A \times B$ as follows. First, we define $2^{2^{A \cup B}}$ as the space $A \times B$ "lives". Then we separate $A \times B$ precisely by

$$A \times B := \{z : z \in 2^{2^{A \cup B}} \wedge z = \{\{x\}, \{x, y\}\} \wedge x \in A \wedge y \in B\}$$

Axiomatic Set Theory

- In order to introduce the natural numbers $\mathbb{N}$, we need an infinite set. However, all the above axioms cannot guarantee the existence of an infinite set. Therefore, we have to introduce the **Axiom of Infinity (无穷公理)** that guarantees the existence of at least one infinite set, namely a set containing the natural numbers.

$$(\exists x)(\emptyset \in x \wedge (\forall y)(y \in x \to y \cup \{y\} \in x))$$

With this axiom, we can actually define natural numbers as follows. For any set A, we define $A^+ = A \cup \{A\}$ as the *successor* (后继数) of A. Then we define

  - $0 := \emptyset$;
  - $1 := 0^+ = 0 \cup \{0\} = \{0\} = \{\emptyset\}$;
  - $2 := 1^+ = 1 \cup \{1\} = \{0, 1\} = \{\emptyset, \{\emptyset\}\}$;
  - $3 := 2^+ = 2 \cup \{2\} = \{0, 1, 2\} = \{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}\}$;
  - $\cdots$ and $n + 1 := n^+$.

The axiom of infinity guarantees that putting them together is a set. For any m, n, by defining $m < n \Leftrightarrow m \subset n$ and $m \leq n \Leftrightarrow m \subseteq n$, we get the structural of natural numbers.

- To avoid some contradiction, we also need the **Axiom of Regularity (正则公理)** saying that that every non-empty set A contains an element that is disjoint from A.

$$(\forall x)(x \neq \emptyset \to (\exists y)(y \in x \wedge (x \cap y = \emptyset)))$$

This axiom together with the axiom of pairing imply that no set is an element of itself, and that no infinite sequence $A_1, A_2, \ldots$ such that $A_{i+1} \in A_i$. To see this, suppose that there exists A such that $A \in A$. Then we can construct $\{A\}$. By the axiom of regularity, we have $(\exists y)(y \in \{A\} \wedge \{A\} \cap y = \emptyset)$. But y can only be A itself, which means that $\{A\} \cap A = \emptyset$. However, since $A \in A$, we know $\{A\} \cap A \neq \emptyset$, which is a contradiction.

- In some cases, we also need the **Axiom Schema of Replacement (替换公理模式)**. The need is non-trivial and we will not discuss it here rather than just listing it.

$$(\forall x)(\exists! y)P(x, y) \to (\forall t)(\exists s)(\forall u)(u \in s \leftrightarrow (\exists z)(z \in t \wedge P(z, u)))$$

where $\exists!$ means "uniquely exists"

$$(\forall x)(\exists! y)P(x, y) = (\forall y)(\exists y)(P(x, y) \wedge (\forall z)(P(x, z) \to z = y))$$

With this axiom, in fact, the axioms of pairing and separation can be its consequences.

Sometimes we also need the **Axiom of Choice (选择公理式)**. It is not initial included in the ZF-set theory. But when we talk about ZFC, "C" standards for the "axiom of choice". We will mention this after we discuss what is a function.

Why the Axiom of Separation Resolves the Russel's Paradox

- Based on the axiom of separation, we are allowed to construct the following set

$$S = \{x \in A : x \notin x\} = \{x : x \in A \wedge x \notin x\}$$

only when A **is already given as set**.

Now, let us see why for this case there is no contradiction.

  - If $S \notin S$, then it means that $S \notin A$ **or** $S \in S$. The second case has been ruled out by $S \notin S$. Therefore, we can conclude that $S \notin A$, for which there is no contradiction.
  - If $S \in S$, then it means that $S \in A$ **and** $S \notin S$. For this case, there is a contradiction. But we do not care because the case of $S \notin S$ already provides us a possible explanation. Actually, if we further introduce the axiom of regularity, then we will exclude the case of $S \in S$, but this has nothing to do with how we resolve the Russel's paradox.

Therefore, we can concluded that $S \notin S$ and $S \notin A$.