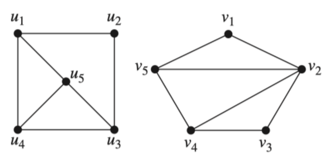
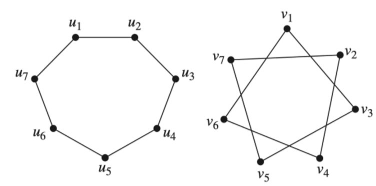
Discrete Math (Honor) 2022-Fall Homework-10

Instructor: Xiang YIN

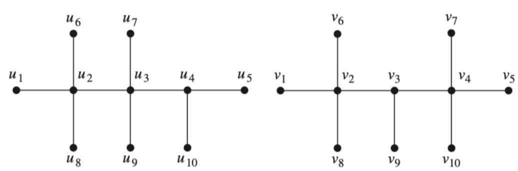
Problem 1. *(12 Points)*
Determine whether each of the following pairs of graphs is isomorphic. If so, find a bijection mapping; otherwise, explain why none exists.



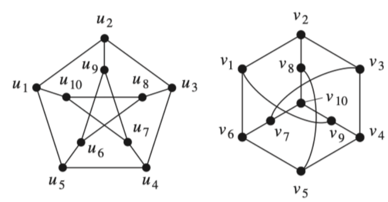
(a) Pair 1



(b) Pair 2



(c) Pair 3



(d) Pair 4

Answer:

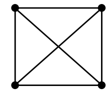
1 No. $d(v_2) = 4$, we cannot find a vertex u in the left graph such that $d(u) = 4$.

2 Yes. $f(u_1) = v_1, f(u_2) = v_3, f(u_3) = v_5, f(u_4) = v_7, f(u_5) = v_2, f(u_6) = v_4, f(u_7) = v_6$.

3 No. We can find u_2, u_3 in left graph such that $d(u_2) = d(u_3) = 4$ and (u_2, u_3) is an edge. But we can find such pair of vertex in right graph.

4 Yes. $f(u_1) = v_1, f(u_2) = v_2, f(u_3) = v_8, f(u_4) = v_{10}, f(u_5) = v_9, f(u_6) = v_4, f(u_7) = v_7, f(u_8) = v_5, f(u_9) = v_3, f(u_{10}) = v_6$.

Problem 2. *(8 Points)*
Prove that, there is a unique (in the sense of isomorphic) simple graph such that it has $3n - 6$ edges and all vertices have the same degree 3, where n is the number of vertices in G.



Answer:$|E| = 3n/2 = 3n - 6$, so $n = 4$, the number of vertices is 4 and the number of edges is 6. Since this graph is a complete graph, the graph is unique.

Problem 3. *(8 Points)*
Prove that, for simple graph $G = \langle V, E \rangle$, if $|E| > \frac{1}{2}(|V|-1)(|V|-2)$, then G has no isolated vertex.

Answer: Suppose that G has an isolated vertex u, then

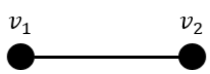
$$|E| \leq \frac{1}{2}(|V| - \{u\})(|V| - \{u\} - 1) = \frac{1}{2}(|V|-1)(|V|-2)$$

It contradicts with the condition, so G has no isolated vertex.

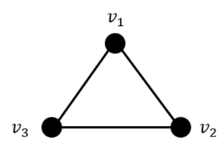
Problem 4. *(15 Points)*
A simple graph is said to be k-*regular* if all vertices have the same degree $k \geq 1$. For a k-regular graph G, its *line graph*, denoted by $L(G)$, is a new simple graph whose vertices are the edges of G, and two vertices in $L(G)$ are adjacent if and only if the corresponding edges in G share a vertex.

1. Provide examples for 1-regular graph, 2-regular graph and 3-regular graph.
2. What are the line graphs for the k-regular graphs you provided in the previous problem.
3. Prove that, if a k-regular graph is connected, then its line graph has an Euler circuit.

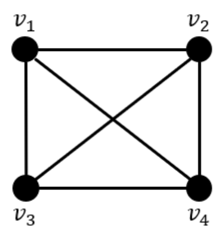
Answer:
1.



(e) 1-1

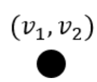


(f) 1-2

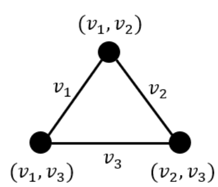


(g) 1-3

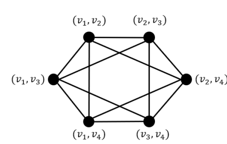
2.



(h) 1-1



(i) 1-2



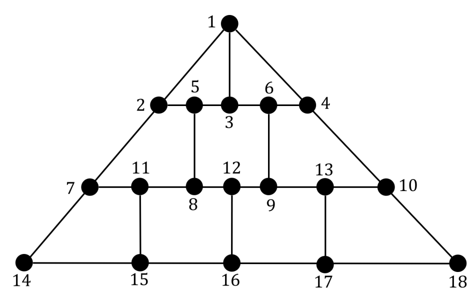
(j) 1-3

3. Assume k-regular graph $G = \langle V, E \rangle$, its line graph $G' = \langle V', E' \rangle$. Consider any $v' \in V', v'$ represents $(u, v) \in E$, for $u \in V$, we have $d(u) = k$, so there are $k-1$ edges sharing u with (u, v), and similarly there are $k-1$ edges sharing v with (u, v). In total there are $2(k-1)$ edges sharing a vertex with (u, v), so there are $2(k-1)$ vertices adjacent with v', which means $d(v') = 2(k-1)$. Thus we have $\forall v' \in V' : d(v')$ is even, namely G' has an Euler circuit.

Problem 5. *(10 Points)*
1. Determine whether the following graph contains an Euler circuit. If so, provide one; otherwise, justify why none exists.
2. Determine whether the following graph contains a Hamilton circuit. If so, provide one; otherwise, justify why none exists.

Answer:

- Euler circuit: No; $d(v_3) = 3$, so Euler circuit does not exist.
- Hamilton circuit: Yes. For example, $P = (1, 2, 5, 8, 11, 7, 14, 15, 16, 12, 9, 13, 17, 18, 10, 4, 6, 3, 1)$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