Discrete Math (Honor) 2022-Fall Homework-5

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Problem 1. *(12 Points)*
Write the prenex normal form for each of the following formulas.

1. $(\forall x)(\forall y)((\exists z)P(x,y,z) \wedge ((\exists u)Q(x,u) \to (\exists v)Q(y,v)))$
2. $(\neg(\exists x)F(x) \vee (\forall y)G(y)) \wedge (F(a) \to (\forall z)H(z))$

Answer:

1. $(\forall x)(\forall y)((\exists z)P(x,y,z) \wedge (\neg(\exists u)Q(x,u) \vee (\exists v)Q(y,v)))$
   $(\forall x)(\forall y)(\exists z)(P(x,y,z) \wedge (\neg(\exists u)Q(x,u) \vee (\exists v)Q(y,v)))$
   $(\forall x)(\forall y)(\exists z)(P(x,y,z) \wedge ((\forall u)\neg Q(x,u) \vee (\exists v)Q(y,v)))$
   $(\forall x)(\forall y)(\exists z)(P(x,y,z) \wedge (\forall u)(\neg Q(x,u) \vee (\exists v)Q(y,v)))$
   $(\forall x)(\forall y)(\exists z)(P(x,y,z) \wedge (\forall u)(\exists v)(\neg Q(x,u) \vee Q(y,v)))$
   $(\forall x)(\forall y)(\exists z)(\forall u)(\exists v)(P(x,y,z) \wedge (\neg Q(x,u) \vee Q(y,v)))$

2. $(\neg(\exists x)F(x) \vee (\forall y)G(y)) \wedge (F(a) \to (\forall z)H(z))$
   $((\forall x)\neg F(x) \vee (\forall y)G(y)) \wedge (\neg F(a) \vee (\forall z)H(z))$
   $(\forall x)(\forall y)(\neg F(x) \vee G(y)) \wedge (\forall z)(\neg F(a) \vee H(z))$
   $(\forall x)(\forall y)((\neg F(x) \vee G(y)) \wedge (\forall z)(\neg F(a) \vee H(z)))$
   $(\forall x)(\forall y)(\forall z)((\neg F(x) \vee G(y)) \wedge (\neg F(a) \vee H(z)))$

Problem 2. *(12 Points)*
Write the Skolem normal form for each of the following formulae.

1. $(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$
2. $(\forall x)(P(x) \to (\exists y)Q(x,y)) \vee (\forall z)R(z)$

Answer

1. Step 1:
   $(\forall x)(P(x) \to Q(x)) \to ((\exists x)P(x) \to (\exists x)Q(x))$
   $(\forall x)(\neg P(x) \vee Q(x)) \to (\neg(\exists x)P(x) \vee (\exists x)Q(x))$
   $(\forall x)(\neg P(x) \vee Q(x)) \to ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
   $\neg(\forall x)(\neg P(x) \vee Q(x)) \vee ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
   $(\exists x)\neg(\neg P(x) \vee Q(x)) \vee ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
   $(\exists x)(P(x) \wedge \neg Q(x)) \vee ((\forall x)\neg P(x) \vee (\exists x)Q(x))$
   $(\exists x)(\forall y)(P(x) \wedge \neg Q(x)) \vee (\neg P(y) \vee (\exists x)Q(x))$
   $(\exists x)(\forall y)(\exists z)(P(x) \wedge \neg Q(x)) \vee (\neg P(y) \vee Q(z))$

   Step 2:
   $(\forall y)(\exists z)(P(u) \wedge \neg Q(u)) \vee (\neg P(y) \vee Q(z))$
   $(\forall y)(P(u) \wedge \neg Q(u)) \vee (\neg P(y) \vee Q(f(y)))$
   or $(\forall y)(P(a) \to Q(a)) \to (P(y) \to Q(f(y)))$

2. Step 1:
$(\forall x)(P(x) \to (\exists y)Q(x,y)) \vee (\forall z)R(z)$
$(\forall x)(\neg P(x) \vee (\exists y)Q(x,y)) \vee (\forall z)R(z)$
$(\forall x)(\exists y)(\neg P(x) \vee Q(x,y)) \vee (\forall z)R(z)$
$(\forall x)(\exists y)(\neg P(x) \vee Q(x,y) \vee (\forall z)R(z))$
$(\forall x)(\exists y)(\forall z)(\neg P(x) \vee Q(x,y) \vee R(z))$

Step 2:
$(\forall x)(\forall z)(\neg P(x) \vee Q(x, f(x)) \vee R(z))$

Problem 3. *(8 Points)* Prove the following inference **by resolution**

$$(\forall x)(P(x) \vee Q(x)) \wedge (\forall x)(Q(x) \to \neg R(x)) \wedge (\forall x)R(x) \Rightarrow (\forall x)P(x)$$

Answer: Write the Skolem normal form of $\alpha \wedge \neg\beta$

$$\begin{aligned}
&(\forall x)(P(x) \vee Q(x)) \wedge (\forall x)(Q(x) \to \neg R(x)) \wedge (\forall x)R(x) \wedge \neg(\forall x)P(x) \\
=&(\forall x)(P(x) \vee Q(x)) \wedge (\forall x)(\neg Q(x) \vee \neg R(x)) \wedge (\forall x)R(x) \wedge (\exists x)\neg P(x) \\
=&(\forall x)(\forall y)(\forall z)(\exists v)((P(x) \vee Q(x)) \wedge (\neg Q(y) \vee \neg R(y)) \wedge R(z) \wedge \neg P(v))
\end{aligned}$$

The clause set is $S = \{P(x) \vee Q(x), \neg Q(y) \vee \neg R(y), R(z), \neg P(a)\}$

$$\left.\begin{array}{r} P(x) \vee Q(x) \\ \neg P(a) \end{array}\right\} \xrightarrow{\sigma=\{x/a\}} Q(a), \qquad \left.\begin{array}{r} \neg Q(y) \vee \neg R(y) \\ Q(a) \end{array}\right\} \xrightarrow{\sigma=\{y/a\}} \neg R(a)$$

$$R(z) \xrightarrow{\sigma=\{z/a\}} R(a)$$

This gives us a contradiction $R(a) \wedge \neg R(a)$

Problem 4. *(8 Points)*
Formalize the following inference and prove it **by resolution**.

All SJTU students are smart. Bob is both an SJTU student and an NBA player. Therefore, some NBA player is smart.

Answer: We define following predicates a : “Bob” $P(x)$:“ x is an SJTU student.” $Q(x)$: “ x is smart.” $R(x)$: “x is an NBA player.” We need to prove

$$(\forall x)(P(x) \to Q(x)) \wedge P(a) \wedge R(a) \Rightarrow (\exists y)(R(y) \wedge Q(y))$$

Write the Skolem normal form of $\alpha \wedge \neg\beta$

$$\begin{aligned}
&(\forall x)(P(x) \to Q(x)) \wedge P(a) \wedge R(a) \wedge \neg(\exists y)(R(y) \wedge Q(y)) \\
=&(\forall x)(\neg P(x) \vee Q(x)) \wedge P(a) \wedge R(a) \wedge (\forall y)(\neg R(y) \vee \neg Q(y)) \\
=&(\forall x)(\forall y)((\neg P(x) \vee Q(x)) \wedge (\neg R(y) \vee \neg Q(y)) \wedge P(a) \wedge R(a))
\end{aligned}$$

The clause set is $S = \{\neg P(x) \vee Q(x), \neg R(y) \vee \neg Q(y), P(a), R(a)\}$

$$\left.\begin{array}{r} \neg P(x) \vee Q(x) \\ P(a) \end{array}\right\} \xrightarrow{\sigma=\{x/a\}} \neg Q(a), \qquad \left.\begin{array}{r} Q(a) \\ \neg R(y) \vee \neg Q(y) \end{array}\right\} \xrightarrow{\sigma=\{y/a\}} \neg R(a)$$

This gives us a contradiction $R(a) \wedge \neg R(a)$