Discrete Math (Honor) 2022-Fall Homework-8

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Problem 1. *(8 Points)*
For any relations R and S. Prove that $R \circ S \subseteq S$ if and only if $t(R) \circ S \subseteq S$.

Answer:

- $\Rightarrow$, if $R \circ S \subseteq S$:
  For any $\langle x, z\rangle \in t(R) \circ S$, exists some y s.t. $\langle y, z\rangle \in t(R) \wedge \langle x, y\rangle \in S$
  From $\langle y, z\rangle \in t(R)$, we have two cases:
  - $\langle y, z\rangle \in R$
  - Exists finite $t_1, t_2, ..., t_k (k \geq 1)$ s.t. $\langle y, t_1\rangle \in R \wedge \langle t_1, t_2\rangle \in R \wedge ... \wedge ... \langle t_k, z\rangle \in R$. Then we have $\langle x, t_1\rangle \in R \circ S$, and from $R \circ S \subseteq S$ we immediately get $\langle x, t_2\rangle \in S$
    Similarly, $\langle x, t_3\rangle \in S, ...$ Finally we have $\langle x, t_k\rangle \in S$ and therefore $\langle x, z\rangle \in S$.
- $\Leftarrow$, if $t(R) \circ S \subseteq S$:
  For any $\langle x, z\rangle \in R \circ S$, exists some y s.t. $\langle y, z\rangle \in R \wedge \langle x, y\rangle \in S$
  Then $\langle y, z\rangle \in t(R)$, then $\langle x, z\rangle \in t(R) \circ S$. Therefore, $\langle x, z\rangle \in S$

Problem 2. *(5 Points)*
Let $A = \{a, b, c, d\}$ and $R = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle c, d\rangle, \langle d, c\rangle, \langle d, d\rangle\} \subseteq A \times A$ be a relation on A. Determine whether or not R is an equivalence relation; if so, determine the quotient set of A.

Answer:
Yes.
The quotient set of A:
$\{\{a, b\}, \{c, d\}\}$

Problem 3. *(15 Points)*
Let $R_1 \subseteq A \times A$ and $R_2 \subseteq A \times A$ be two non-empty equivalence relations on A. Determine whether or not each of the following relations is an equivalence relation. If so, prove it; otherwise, provide an counter example.

1. $(A \times A) - R_1$
2. $(R_1)^2$
3. $R_1 - R_2$
4. $r(R_1 - R_2)$
5. $t(R_1 \cup R_2)$

Answer:

1. No. Suppose $A = \{a\}, R = \{\langle a, a\rangle\}$
2. Yes. For $(x, y) \in R_1^2$, there exists $(x, z), (z, y) \in R_1$. Thus $(z, x), (y, z) \in R_1$. Then $(y, x) \in R_1^2$. Therefore, R_1^2 is symmetric. Since $(a, a) \in R_1$, we have $(a, a) \in R_1^2$. Thus R_1^2 is reflexive. Moreover, for $(x, y), (y, z) \in R_1^2$, there exists $(x, u), (u, y), (y, v), (v, z) \in R_1$. Since R_1 is transitive, we know that $(x, y), (y, z) \in R_1$. Thus $(x, z) \in R_1^2$. Therefore, R_1^2 is transitive.

3. No. Suppose any $R_1 = R_2$

4. No. Suppose $A = \{a, b, c\}, R_1 = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\langle a, b\rangle, \langle b, a\rangle, \langle a, c\rangle, \langle c, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$, $R_2 = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle, \langle a, b\rangle, \langle b, a\rangle\}$ then $R_1 - R_2 = \{\langle a, c\rangle, \langle c, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$
   $r(R_1 - R_2) = \{\langle a, a\rangle, \langle b, b\rangle, \langle c, c\rangle\langle a, c\rangle, \langle c, a\rangle, \langle b, c\rangle, \langle c, b\rangle\}$ is not transitive ($\langle b, c\rangle, \langle c, a\rangle \in r(R_1 - R_2)$)

5. Yes. Since R_1 and R_2 are symmetric and reflexive, we know that $R_1 \cup R_2$ is symmetric and reflexive. Moreover, from Theorem in page 10 of Note-4, $t(R_1 \cup R_2)$ are transitive, symmetric and reflexive.

Problem 4. *(8 Points)*
Let $\langle S, \leq_1\rangle$ be $\langle T, \leq_2\rangle$ be two posets. Prove that $\langle S \times T, \leq\rangle$ is also a poset, where $\leq$ is defined by: $\langle s, t\rangle \leq \langle u, v\rangle$ iff $s \leq_1 u$ and $t \leq_2 v$.

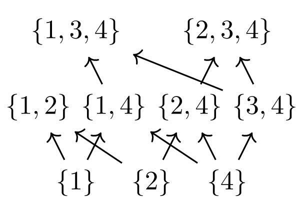
Answer:

- Reflexive: ... (obvious)
- Transitive: ... (easy)
- Anti-symmetric: For any s, t, u, v s.t. $\langle s, t\rangle \leq \langle u, v\rangle$ and $\langle u, v\rangle \leq \langle s, t\rangle$. we have $s \leq_1 u \wedge u \leq_1 s$ and also $t \leq_2 v \wedge v \leq_2 t$, so $s = u$ and $t = v$, and $\langle u, v\rangle = \langle s, t\rangle$.

Problem 5. *(15 Points)*
Let $A = \{\{1\}, \{2\}, \{4\}, \{1, 2\}, \{1, 4\}, \{2, 4\}, \{3, 4\}, \{1, 3, 4\}, \{2, 3, 4\}\}$ and $\langle A, \subseteq\rangle$ be a poset, where "$\subseteq$" is the "subset relation". Then answer the following questions.

1. Draw the Hasse diagram of the poset.
2. Find the minimal element and the maximal element of A.
3. Do the greatest element and the least element exist for A?
4. Find all upper bounds and the least upper bound of $B = \{\{2\}, \{4\}\}$.
5. Find all lower bounds and the greatest lower bound of $B = \{\{1, 3, 4\}, \{2, 3, 4\}\}$.

Answer:

1 

2 minimal element: $\{1\}, \{2\}, \{4\}$, maximal element: $\{1, 3, 4\}, \{2, 3, 4\}, \{1, 2\}$.

3 No, No.

4 upper bounds: $\{2, 4\}, \{2, 3, 4\}$, least upper bound: $\{2, 4\}$.

5 lower bounds: $\{3, 4\}, \{4\}$, greatest lower bound: $\{3, 4\}$.