Discrete Math (Honor) 2022-Fall Homework-3

Instructor: Xiang YIN

Problem 1. *(15 Points)*
Prove the following inferences.

1. $P \vee Q, P \to S, Q \to R \quad \vdash \quad S \vee R$
2. $\neg R \vee S, S \to Q, \neg Q \quad \vdash \quad Q \leftrightarrow R$
3. $\neg Q \vee S, (E \to \neg U) \to \neg S \quad \vdash \quad Q \to E$

Answer:
Note: proof of the following rules (named casually) are omitted in this solution sheet:

- Transition rule (Trans):$P \to Q, Q \to R \vdash P \to R$
- And-intros rule (AI): $P \to R, Q \to R \vdash P \vee Q \to R$

1. (a) $P \to S$
   (b) $\neg P \vee S$ (equivalence a)
   (c) $\neg P \vee S \vee R$ ($P \Rightarrow P \vee Q$ b)
   (d) $P \to (S \vee R)$ (equivalence c)
   (e) $Q \to R$
   (f) $\neg Q \vee R$ (equivalence e)
   (g) $\neg Q \vee S \vee R$ ($P \Rightarrow P \vee Q$ f)
   (h) $Q \to (S \vee R)$ (equivalence g)
   (i) $(P \vee Q) \to (S \vee R)$ (AI d h)
   (j) $P \vee Q$
   (k) $S \vee R$ (MP i j)
2. (a) $\neg R \vee S$
   (b) $R \to S$ (equivalence a)
   (c) $S \to Q$
   (d) $R \to Q$ (Trans b c)
   (e) $\neg Q$
   (f) $\neg Q \vee R$ ($P \Rightarrow P \vee Q$ e)
   (g) $Q \to R$ (equivalence f)
   (h) $(Q \to R) \wedge (R \to Q)$
   (i) $Q \leftrightarrow R$ (equivalence h)
3. (a) Q (conditional proof)
   (b) $\neg Q \vee S$
   (c) $Q \to S$ (equivalence b)

(d) S (MP a c)

(e) $(E \to \neg U) \to \neg S$

(f) $S \to \neg(E \to \neg U)$ (equivalence e)

(g) $\neg(E \to \neg U)$ (MP d f)

(h) $\neg(\neg E \vee \neg U)$ (equivalence g)

(i) $E \wedge U$ (equivalence h)

(j) E ($P \wedge Q \Rightarrow P$ i)

(k) $Q \to E$ (conditional proof, out)

Problem 2. *(12 Points)*
Prove the following inferences using resolution method.

1. $(P \vee Q) \wedge (P \to R) \wedge (Q \to R) \Rightarrow R$
2. $(S \to \neg Q) \wedge (P \to Q) \wedge (R \vee S) \wedge (R \to \neg Q) \Rightarrow \neg P$

Answer:
$S = \{...\}$

1. (a) $P \vee Q, \neg P \vee R, \neg Q \vee R, \neg R$
   (b) $Q \vee R, \neg Q \vee R, \neg R$
   (c) $R, \neg R$
   (d) $\square$

2. (a) $(\neg S \vee \neg Q), (\neg P \vee Q), (R \vee S), (\neg R \vee \neg Q), \neg\neg P$
   (b) $(\neg S \vee \neg Q), (\neg P \vee Q), (R \vee S), (\neg R \vee \neg Q), P$
   (c) $(\neg S \vee \neg Q), Q, (R \vee S), (\neg R \vee \neg Q)$
   (d) $(\neg S \vee \neg Q), Q, (R \vee S), (\neg R \vee \neg Q)$
   (e) $\neg S, Q, (R \vee S), (\neg R \vee \neg Q)$
   (f) $Q, R, (\neg R \vee \neg Q)$
   (g) $Q, \neg Q$
   (h) $\square$

Problem 3. *(21 Points)*
Formalize each of the following sentences using predicate logic formula. (You can define your own predicate needed)

1. Everyone loves everyone except himself.
2. Every student except Alice is a friend of Bob.
3. Every student who walks talks.
4. Every boy who loves Alice hates every boy who Alice loves.
5. Every boy who loves Alice hates some boy who Cauchy loves.
6. There exists an unique boy who loves Alice but hates Bob.
7. There exists an unique boy who hates everyone who loves Bob.

Answer:
Note: you can get all score if **the Quantifiers in 4 and 5 are right** and **you formalize "unique" in 6 and 7 correctly**.

1. Answer 1:
   Define:
   $H(x)$: x is a human,
   $E(x,y)$: x is y,
   $Love(x,y)$: x loves y,
   Formalization:
   $(\forall x)(H(x) \rightarrow (\forall y)(\neg E(x,y) \rightarrow Love(x,y)),$

   Answer 2:
   Define:
   $H(x)$: x is a human,
   $P(x,y)$: $x = y$,
   $Q(x,y)$: x doesn't love y,
   Formalization:
   $(\forall x)(\forall y)(H(x) \wedge H(y) \rightarrow (P(x,y) \leftrightarrow Q(x,y))).$

2. Define:
   $P(x)$: x is a student
   $F(x)$: x is a friend of Bob,
   $E(x)$: x is $Alice$,
   Formalization:
   $(\forall x)(P(x) \wedge \neg E(x) \rightarrow F(x)).$

3. Define:
   $P(x)$: x is a student
   $T(x)$: x talks
   $W(x)$: x walks
   Formalization:
   $(\forall x)(P(x) \wedge W(x) \rightarrow T(x))$

4. Define:
   $B(x)$: x is a boy,
   $P(x)$: x loves Alice,
   $F(y)$: Alice loves y,
   $Q(x,y)$: x hates y,
   Formalization:
   $(\forall x)(\forall y)(B(x) \wedge P(x) \wedge B(y) \wedge F(y) \rightarrow Q(x,y)).$

5. Define:
   $B(x)$: x is a boy,
   $P(x)$: x loves Alice,
   $F(y)$: Cauchy loves y,
   $Q(x,y)$: x hates y,
   Formalization:
   $(\forall x)(B(x) \wedge P(x) \rightarrow (\exists y)(B(y) \wedge F(y) \wedge Q(x,y))).$

6. Define:
   $B(x)$: x is a boy,
   $P(x)$: x loves Alice,
   $Q(x)$: x hates Bob,
   $E(x,y)$: $x = y$.
   Formalization:
   $(\exists x)(B(x) \wedge P(x) \wedge Q(x) \wedge (\forall y)(B(y) \wedge P(y) \wedge Q(y) \rightarrow E(x,y)))$

7. Define:
   $H(x)$: x is a human,
   $P(x)$: x is a boy,
   $A(x, y)$: x hates y,
   $B(x)$: x loves Bob,
   $E(x, y)$: $x = y$.
   Formalization:
   $(\exists x)(P(x) \wedge (\forall y)(H(y) \wedge B(y) \to A(x, y)) \wedge (\forall z)(P(z) \wedge (\forall m)(H(m) \wedge B(m) \to A(z, m)) \to E(z, x)))$

Problem 4. *(10 Points)*
For each of the following formulae, determine free variables and bound variables in it and determine the scope of each quantifier.

1. $(\forall x)(P(x) \to Q(x, y))$
2. $(\forall x)P(x, y) \to (\exists y)Q(x, y)$
3. $(\forall x)(\exists y)(P(x, y) \wedge Q(y, z) \vee (\exists x)R(x, y, z))$
4. $(\exists x)(P(x) \to Q(x)) \to (\exists y)R(y) \to S(z)$
5. $(\forall x)(P(x) \wedge (\exists y)Q(y)) \vee ((\forall x)P(x) \to Q(z))$

Answer

1. Free: y Bound: x
   Scope: $\forall x \mid P(x) \to Q(x, y)$
2. Free: y in $P(x, y)$ and x in $Q(x, y)$ Bound: x in $P(x, y)$ and y in $Q(x, y)$
   Scope:
   $\forall x \mid P(x, y)$
   $\exists y \mid Q(x, y)$
3. Free: z Bound: x y
   Scope:
   $\forall x \mid (\exists y)(P(x, y) \wedge Q(y, z) \vee (\exists x)R(x, y, z))$
   $\exists y \mid (P(x, y) \wedge Q(y, z) \vee (\exists x)R(x, y, z))$
   $\exists x \mid R(x, y, z)$
4. Free: z Bound: x y
   Scope:
   $\exists x \mid P(x) \to Q(x)$
   $\exists y \mid R(y)$
5. Free: z Bound: x y
   Scope:
   $\forall x \mid P(x) \wedge (\exists y)Q(y)$
   $\exists y \mid Q(y)$
   $\forall x \mid P(x)$