Discrete Math (Honor) 2022-Fall Homework-6

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Problem 1. *(12 Points)*
Determine whether each of the following propositions is true or false.

1. $\emptyset \subseteq \emptyset$
2. $\emptyset \in \emptyset$
3. $\emptyset \subseteq \{\emptyset\}$
4. $\emptyset \in \{\emptyset\}$
5. $\{\emptyset\} \subseteq \{\emptyset\}$
6. $\{\emptyset\} \in \{\emptyset\}$
7. $\{\emptyset\} \subseteq \{\{\emptyset\}\}$
8. $\{\emptyset\} \in \{\{\emptyset\}\}$
9. $\{a, b\} \in \{a, b, \{a, b\}\}$
10. $\{a, b\} \subseteq \{a, b, \{a, b\}\}$
11. $\{a, b\} \in \{a, b, \{\{a, b\}\}\}$
12. $\{a, b\} \subseteq \{a, b, c, \{\{a, b\}\}\}$

Answer:
1. T
2. F
3. T
4. T
5. T
6. F
7. F
8. T
9. T
10. T
11. F
12.T

Problem 2. *(6 Points)*
Let $A = 2^{2^{2^{\emptyset}}}$. Determine whether each of the following propositions is true or false.

1. $\emptyset \in A$
2. $\emptyset \subseteq A$
3. $\{\emptyset\} \in A$
4. $\{\emptyset\} \subseteq A$
5. $\{\{\emptyset\}\} \in A$
6. $\{\{\emptyset\}\} \subseteq A$

Answer:
$2^{\emptyset} = \{\emptyset\}$
$2^{\{\emptyset\}} = \{\emptyset, \{\emptyset\}\}$
$2^{\{\emptyset,\{\emptyset\}\}} = \{\emptyset, \{\emptyset\}, \{\{\emptyset\}\}, \{\emptyset, \{\emptyset\}\}\}$

1. T 2. T 3. T 4. T 5. T 6. T

Problem 3. *(6 Points)*
Write down the following sets by listing their elements.

1. $2^{\{\emptyset,\{1,\{2\}\}\}}$
2. $\bigcup\{\{a,b\},\{\{a\},\{b\}\},\{a,\{b\}\},\{\{a\},b\}\}$
3. $\bigcap\{2^{\emptyset}, 2^{2^{\emptyset}}, 2^{2^{2^{\emptyset}}}\}$

Answer:

1. $\{\emptyset, \{\emptyset\}, \{\{1,\{2\}\}\}, \{\emptyset,\{1,\{2\}\}\}\}$
2. $\{a, b, \{a\}, \{b\}\}$
3. $\{\emptyset\}$

Problem 4. *(8 Points)*
Find two sets A and B such that $(\bigcap A) \cap (\bigcap B) \neq \bigcap(A \cap B)$.
Find two sets C and D such that $(\bigcap C) \cap (\bigcap D) = \bigcap(C \cap D)$.

Answer:
Example:
$A = \{\emptyset, \{\emptyset\}\}, B = \{\{\emptyset\}\}$
any $C = D$ is an example

Problem 5. *(5 Points)*
Write $\langle a, \langle b, c\rangle, d\rangle$ by set representation, e.g., $\langle a, b\rangle = \{\{a\}, \{a, b\}\}$.

Answer: $\langle a, \langle b, c\rangle, d\rangle$
$= \{\{\langle a, \langle b, c\rangle\rangle\}, \{\langle a, \langle b, c\rangle\rangle, d\}\}$
$= \{\{\{\{a\}, \{a, \langle b, c\rangle\}\}\}, \{\{\{a\}, \{a, \langle b, c\rangle\}\}, d\}\}$
$= \{\{\{\{a\}, \{a, \{\{b\}, \{b, c\}\}\}\}\}, \{\{\{a\}, \{a, \{\{b\}, \{b, c\}\}\}\}, d\}\}$

Problem 6. *(16 Points, 4+4+8)*
Let A be a set of sets. Then we say A is a *transitive set* if

$$(\forall x)(\forall y)\left((y \in A \wedge x \in y) \rightarrow x \in A\right)$$

1. Is $\{\ \{\emptyset\}\ ,\ \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\ \}$ a transitive set?
2. The *transitive closure* of a set A is the smallest transitive set that contains A. What is the transitive closure of $\{\ \{\emptyset\}\ ,\ \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\ \}$? ($A$ is smaller than B if $A \subset B$)
3. Prove that A is a transitive set *if and only if* $\bigcup A \subseteq A$.

Answer:

1. No, otherwise $\emptyset$ should be one element
2. $\{\emptyset, \{\emptyset\}, \{\emptyset, \{\emptyset\}\}, \{\emptyset, \{\emptyset, \{\emptyset\}\}\}, \{\ \{\emptyset\}\ ,\ \{\emptyset, \{\emptyset, \{\emptyset\}\}\}\ \}\}$

3. (a) $\Rightarrow$: suppose A is a transitive set.
   For any $x \in \cup A$, exists some y such that $x \in y$ and $y \in A$. Since A is transitive, we have $x \in A$, therefore, $\bigcup A \subseteq A$.
   (b) $\Leftarrow$: suppose $\bigcup A \subseteq A$.
   For any x, y such that $x \in y$ and $y \in A$, then we have $x \in \cup A$. Therefore, A is transitive.

Note: We require that A is contained in the transitive closure rather than a subset set of transitive closure.