Discrete Math (Honor) 2022-Fall Homework-4

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Problem 1. *(5 Points)*
Provide a predicate formula that is satisfiable for domain of discourse $D_1 = \{1,2,3\}$ but is not satisfiable for domain of discourse $D_2 = \{1,2\}$ or $D_3 = \{2,3\}$.

Answer: In D_2 or D_3 we can not find three different elements while we can do it in D_1. Let $R(x,y) : D \times D \to \{\mathbf{T}, \mathbf{F}\}$ satisfies that $R(x,y) = \mathbf{F}$ for $x = y$ and $R(x,y) = \mathbf{T}$ for $x \neq y$. Consider following predicate formula
$T(x,y,z) = R(x,y) \wedge R(x,z) \wedge R(y,z)$
We have $T(1,2,3) = \mathbf{T}$. But $T(x,y,z) = \mathbf{F}$ when we consider domain of discourse D_2 or D_3.

Problem 2. *(16 Points)*
Determine if each of the following statements is correct; justify your answer.

1. $(\forall x)P(x) = \mathbf{F}$, if and only if, for any $x_0 \in D$, we have $P(x_0) = \mathbf{F}$.
2. $(\forall x)(P(x) \wedge Q(x)) = \mathbf{F}$, if and only if, for any $x_0 \in D$, we have $P(x_0) = \mathbf{F}$ and $Q(x_0) = \mathbf{F}$.
3. $P(a) \to Q(b) \Rightarrow (\exists x)(P(x) \to Q(x))$
4. $P(a) \to Q(b) \Rightarrow (\exists x)(\exists y)(P(x) \to Q(y))$

Answer

1. wrong: $(\forall x)P(x) = \mathbf{F} \Leftrightarrow \neg(\forall x)P(x) = \mathbf{T} \Leftrightarrow (\exists x)\neg P(x) = \mathbf{T} \Leftrightarrow$ exists $x_0 \in D$ $P(x_0) = \mathbf{F}$.
2. wrong: Consider $D = \{1,2\}$, $P(1) = Q(2) = \mathbf{F}$, $P(2) = Q(1) = \mathbf{T}$.
3. correct: $P(a) \to Q(b) = \neg P(a) \vee Q(b) \Rightarrow (\neg P(a) \vee Q(a)) \vee (\neg P(b) \vee Q(b)) = (P(a) \to Q(a)) \vee (P(b) \to Q(b)) \Rightarrow (\exists x)(P(x) \to Q(x))$
4. correct: $P(a) \to Q(b) \Rightarrow (\exists y)(P(a) \to Q(y)) \Rightarrow (\exists x)(\exists y)(P(x) \to Q(y))$

Problem 3. *(10 Points)*
Prove each of the following equivalences.

1. $(\forall x)P(x) \to q = (\exists x)(P(x) \to q)$
2. $(\forall y)(\exists x)((P(x) \to q) \vee S(y)) = ((\forall x)P(x) \to q) \vee (\forall y)S(y)$

Answer

1. $(\forall x)P(x) \to q$
   $= \neg(\forall x)P(x) \vee q$
   $= (\exists x)\neg P(x) \vee q$
   $= (\exists x)(\neg P(x) \vee q)$
   $= (\exists x)(P(x) \to q)$

2. $(\forall y)(\exists x)((P(x) \to q) \vee S(y))$
   $= (\forall y)((\exists x)(P(x) \to q) \vee S(y))$
   $= (\exists x)(P(x) \to q) \vee (\forall y)S(y)$
   $= (\exists x)(\neg P(x) \vee q) \vee (\forall y)S(y)$
   $= ((\exists x)(\neg P(x)) \vee q) \vee (\forall y)S(y)$
   $= (\neg(\forall x)P(x) \vee q) \vee (\forall y)S(y)$
   $= ((\forall x)P(x) \to q) \vee (\forall y)S(y)$

Problem 4. *(8 Points)*
Formalize the following inference and prove it.

Students in Discrete Math class either like studying or like playing games. Bob does not like playing games. Therefore, if Bob is a student in Discrete Math class, then he likes studying.

Answer Formalize:
$S(x)$: x is a student
$DM(x)$: x is in Discrete Math class
$LS(x)$: x likes studying
$LPG(x)$: x likes playing games
b: Bob

Conclusion:
$(\forall x)((S(x) \wedge DM(x)) \to (LS(x) \oplus LPG(x))) \wedge \neg LPG(b) \wedge S(b) \wedge DM(b) \vdash LS(b)$

- $(\forall x)((S(x) \wedge DM(x)) \to (LS(x) \oplus LPG(x)))$
- $(S(x) \wedge DM(x)) \to (LS(x) \oplus LPG(x))$
- $S(b)$
- $DM(b)$
- $S(b) \wedge DM(b)$
- $LS(b) \oplus LPG(b)$
- $(LS(b) \wedge \neg LPG(b)) \vee (\neg LS(b) \wedge LPG(b))$
- $\neg LPG(b)$
- $(LS(b) \wedge \neg LPG(b))$
- $LS(b)$

Problem 5. *(15 Points)*
Put the correct relation in "(?)" and briefly explains why.

1. $(\exists x)(\forall y)(\forall z)P(x, y, z)$ (?) $(\forall y)(\exists x)(\exists z)P(x, y, z)$ ($\Rightarrow$ / $\Leftarrow$ / $\Leftrightarrow$ /None)
2. $(\exists x)(\forall y)(\forall z)P(x, y, z)$ (?) $(\exists y)(\forall x)(\exists z)P(x, y, z)$ ($\Rightarrow$ / $\Leftarrow$ / $\Leftrightarrow$ /None)
3. $(\exists x)(\forall y)(\forall z)P(x, y, z)$ (?) $(\forall z)(\exists y)(\forall x)P(x, y, z)$ ($\Rightarrow$ / $\Leftarrow$ / $\Leftrightarrow$ /None)
4. $(\forall x)(\exists y)(\forall z)P(x, y, z)$ (?) $(\forall z)(\exists y)(\forall x)P(x, y, z)$ ($\Rightarrow$ / $\Leftarrow$ / $\Leftrightarrow$ /None)
5. $(\exists y)(\forall x)(\exists z)P(x, y, z)$ (?) $(\exists z)(\exists y)(\forall x)P(x, y, z)$ ($\Rightarrow$ / $\Leftarrow$ / $\Leftrightarrow$ /None)

Answer

1. $(\exists x)(\forall y)(\forall z)P(x, y, z) \Rightarrow (\forall y)(\exists x)(\exists z)P(x, y, z)$.

2. $(\exists x)(\forall y)(\forall z)P(x,y,z)$ (None) $(\exists y)(\forall x)(\exists z)P(x,y,z)$
3. $(\exists x)(\forall y)(\forall z)P(x,y,z)$ (None) $(\forall z)(\exists y)(\forall x)P(x,y,z)$
4. $(\forall x)(\exists y)(\forall z)P(x,y,z)$ (None) $(\forall z)(\exists y)(\forall x)P(x,y,z)$
5. $(\exists y)(\forall x)(\exists z)P(x,y,z) \Leftarrow (\exists z)(\exists y)(\forall x)P(x,y,z)$

Explain

1. $(\exists x)(\forall y)(\forall z)P(x,y,z) \Rightarrow (\forall y)(\forall z)P(a,y,z) \Rightarrow (\forall y)P(a,y,b) \Rightarrow (\forall y)(\exists x)(\exists z)P(x,y,z)$
2. Let $P(x,y,z) = \mathbf{T}$ only when $x = a$, then $(\exists y)(\forall x)(\exists z)P(x,y,z) = \mathbf{F}$. Conversely, Let $P(x,y,z) = \mathbf{T}$ only when $y = b, z = c$, then $(\exists x)(\forall y)(\forall z)P(x,y,z) = \mathbf{F}$
3. Like prove in (2), we know that $(\exists x)(\forall y)(\forall z)P(x,y,z) \Rightarrow (\forall z)(\exists y)(\forall x)P(x,y,z)$ is wrong. Conversely, for $D = \{a,b\}$, let $P(x,a,b) = P(x,b,a) = \mathbf{T}$ and $P(x,y,z) = \mathbf{F}$ otherwise. Then $(\forall z)(\exists y)(\forall x)P(x,y,z) = \mathbf{T}$ and $(\exists x)(\forall y)(\forall z)P(x,y,z) = \mathbf{F}$
4. For $D = \{a,b\}$, let $P(a,a,z) = P(b,b,z) = \mathbf{T}$ and $P(x,y,z) = \mathbf{F}$ otherwise. Then $(\forall x)(\exists y)(\forall z)P(x,y,z) = \mathbf{T}$ and $(\forall z)(\exists y)(\forall x)P(x,y,z) = \mathbf{F}$. Two formulas are equal if we change the position of x and z. Therefore, we put None in (?)
5. $(\exists z)(\exists y)(\forall x)P(x,y,z) = (\exists y)(\exists z)(\forall x)P(x,y,z)$. Since $(\exists z)(\forall x)P(x,y,z) \Rightarrow (\forall x)(\exists z)P(x,y,z)$, the correct relation is $\Leftarrow$