Discrete Math (Honor) 2022-Fall Homework-7

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Problem 1. *(8 Points)*
Let $R = \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\}$ be a relation. Determine each of the followings.

1. $R \circ R$ 2. R^{-1} 3. $R\restriction\emptyset$ 4. $R\restriction\{\emptyset\}$ 5. $R\restriction\{\emptyset, \{\emptyset\}\}$ 6. $R[\emptyset]$ 7. $R[\{\emptyset\}]$ 8. $R[\{\emptyset, \{\emptyset\}\}]$

Answer:
1. $= \{\langle\{\emptyset\}, \{\emptyset, \{\emptyset\}\}\rangle\}$
2. $= \{\langle\{\emptyset, \{\emptyset\}\}, \emptyset\rangle, \langle\emptyset, \{\emptyset\}\rangle\}$
3. $= \emptyset$
4. $= \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle\}$
5. $= R, \{\langle\emptyset, \{\emptyset, \{\emptyset\}\}\rangle, \langle\{\emptyset\}, \emptyset\rangle\}$
6. $= \emptyset$
7. $= \{\{\emptyset, \{\emptyset\}\}\}$
8. $= \{\{\emptyset, \{\emptyset\}\}, \emptyset\}$

Problem 2. *(8 Points)*
Let $A = \{a, b, c, d\}$, $R_1 = \{\langle a, a\rangle, \langle a, b\rangle, \langle b, d\rangle\} \subseteq A \times A$ and $R_2 = \{\langle a, d\rangle, \langle b, c\rangle, \langle b, d\rangle, \langle c, b\rangle\} \subseteq A \times A$. Draw the relation graph for each of the following relations.

1.$R_1 \circ R_2$ 2.$R_2 \circ R_1$ 3.$R_1 \circ R_1$ 4.$R_2 \circ R_2$

Answer:
1. $= \{\langle c, d\rangle\}$
2. $= \{\langle a, d\rangle, \langle a, c\rangle\}$
3. $= \{\langle a, a\rangle, \langle a, b\rangle, \langle a, d\rangle\}$
4. $= \{\langle b, b\rangle, \langle c, c\rangle, \langle c, d\rangle\}$

Problem 3. *(8 Points)*
Let $A = \{a, b, c\}$ be a set.

1. Provide a relation $R \subseteq A \times A$ that is symmetric, anti-symmetric and transitive.
2. Provide a relation $R \subseteq A \times A$ that is not symmetric, not anti-symmetric and transitive.

Answer:
1. R $= \{\langle a, a\rangle\}$
2. R $= \{\langle a, a\rangle, \langle a, b\rangle, \langle b, a\rangle, \langle b, b\rangle, \langle b, c\rangle, \langle a, c\rangle\}$

Problem 4. *(12 Points)*
Let $A = \{1, 2, \ldots, 10\}$ be a set. Determine if each of the following relations is reflexive, inreflexive, symmetric, anti-symmetric or transitive. (You do not need to write down each relation explicitly)

1. $R_1 = \{\langle x, y\rangle : x + y = 10\}$
2. $R_2 = \{\langle x, y\rangle : x + y \text{ is odd}\}$
3. $R_3 = \{\langle x, y\rangle : x + y \text{ is even}\}$

Answer:
1. not reflexive, not in-reflexive, symmetric, not anti-symmetric, not transitive
2. not reflexive, in-reflexive, symmetric, not anti-symmetric, not transitive
3. reflexive, not in-reflexive, symmetric, not anti-symmetric, transitive

Problem 5. *(15 Points)*
For any relation $R \subseteq A \times A$, $R^n \subseteq A \times A$ is a new relation on A defined inductively by

$$R^0 := I_A, \text{ and for any } k \geq 0, \text{ we have } R^{k+1} := R^k \circ R$$

Then answer the following questions:

1. Let $A = \{a, b, c, d\}$ and $R = \{\langle a, b\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle c, d\rangle\} \subseteq A \times A$. Compute R^2, R^3 and R^4. (You can show your result by relation graph)
2. Prove that $R^m \circ R^n = R^{m+n}$ for any n and m.
3. Prove that, if R is symmetric, then R^n is also symmetric for any n.

Answer:
1.
$R^2 = \{\langle a, c\rangle, \langle b, b\rangle, \langle b, d\rangle, \langle c, c\rangle\}$
$R^3 = \{\langle a, b\rangle, \langle a, d\rangle, \langle b, c\rangle, \langle c, b\rangle, \langle c, d\rangle\}$
$R^4 = \{\langle a, c\rangle, \langle b, b\rangle, \langle b, d\rangle, \langle c, c\rangle\}$

2.
You should use mathematical induction Induction on n:
i. Base Step(n=0):
$R^m \circ R^0 = R^m$ (details omitted)
ii. Induction Step (suppose it was found when n = k, consider n = k+ 1):
$R^m \circ R^{k+1} = R^m \circ (R^k \circ R^1) = (R^m \circ R^k) \circ R^1 = (R^{m+k}) \circ R^1 = R^{m+k+1}$

3.
You should use mathematical induction Induction on n:
i. Base Step (n=0):
R^0 is symmetric (obviously) ii. Induction Step (n=k+1):
For any $\langle x, y\rangle \in R^{k+1} = R^k \circ R$,
$\exists z, \langle x, z\rangle \in R^k \wedge \langle z, y\rangle \in R$
Then $\exists z, \langle z, x\rangle \in R^k \wedge \langle y, z\rangle \in R$
Then $\langle y, x\rangle \in R \circ R^k = R^{1+k} = R^{k+1}$

Problem 6. *(15 Points)*
Let $R = A \times A$ be a relation on A. Prove the followings formally:

1. R is reflexive if and only if $I_A \subseteq R$.
2. R is inreflexive if and only if $I_A \cap R = \emptyset$.
3. R is transitive if and only if $R \circ R \subseteq R$.

Answer:
1. if R is reflexive, then for any $\langle x, x\rangle \in I_A, \langle x, x\rangle \in R$;
if $I_A \subseteq R$, then for any $x \in A, \langle x, x\rangle \in I_A$, so $\langle x, x\rangle \in R$

2. if R is in-reflexive, then suppose any $\langle x, x\rangle \in I_A \cap R$, then $\langle x, x\rangle \in R$, which is impossible;
if $I_A \cap R = \emptyset$, then suppose $\langle x, x\rangle \in R$, since $\langle x, x\rangle \in I_A$, $\langle x, x\rangle \in I_A \cap R$, which is impossible;

3. if R is transitive, then for any $\langle x, y\rangle \in R \circ R$, $\exists z, \langle x, z\rangle \in R \wedge \langle z, y\rangle \in R$
Then $\langle x, y\rangle \in R$;
if $R \circ R \subseteq R$, then for any $\langle x, y\rangle, \langle y, z\rangle \in R$, $\langle x, z\rangle \in R \circ R$. Therefore, $\langle x, z\rangle \in R$