Discrete Math (Honor) 2022-Fall Homework-9

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Problem 1. *(15 Points)*
Read the following definitions carefully and then answer the questions.

Let $\langle A, \leq \rangle$ be a poset and $B \subseteq A$ be a subset. We say

- B is a **Chain** if any two elements in B are comparable. The number of elements in B is called the *length* of the chain;
- B is a **Anti-Chain** if any two elements in B are incomparable. The number of elements in B is called the *length* of the chain.

where "x and y are incomparable" means that "neither $x \leq y$ nor $y \leq x$".

Now, let $\langle 2^{\{a,b,c\}}, \subseteq \rangle$ be a poset. Then

1. Write down two different chains with length 4.
2. Write down an anti-chains with length 3.
3. The *lower set* of an anti-chain B is defined as $L_B = \{x : x \in A \wedge (\exists y \in B)(x \leq y)\}$. What is the lower set of the anti-chain you provided in the above problem.

Answer:

1. $\{\} \subseteq \{a\} \subseteq \{a,b\} \subseteq \{a,b,c\}$, $\{\} \subseteq \{b\} \subseteq \{a,b\} \subseteq \{a,b,c\}$
2. $\{a,b\}, \{b,c\}, \{a,c\}$
3. $\{\}, \{a\}, \{b\}, \{c\}, \{a,b\}, \{b,c\}, \{a,c\}$

Problem 2. *(10 Points)*
Let A be a set and R be a relation on $2^A \times 2^A$ defined by

$$R = \{\langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in (2^A \times 2^A) \times (2^A \times 2^A) : x_1 \oplus x_2 \subseteq y_1 \oplus y_2\}$$

where "$\oplus$" is the symmetric difference of sets, i.e., $X \oplus Y = (X - Y) \cup (Y - X)$. Determine if R is a partial order. If so, prove it; otherwise, provide a counter-example and explain why.

Answer: Not a partial order. You can give some counter-examples s.t.
$\langle x_1, x_2 \rangle \neq \langle y_1, y_2 \rangle \wedge x_1 \oplus x_2 = y_1 \oplus y_2 \wedge \langle \langle x_1, x_2 \rangle, \langle y_1, y_2 \rangle \rangle \in R \wedge \langle \langle y_1, y_2 \rangle, \langle x_1, x_2 \rangle \rangle \in R$.
For example, let $A = \{a,b,c\}$ and $x_1 = \{a\}, x_2 = \{a,c\}, y_1 = \{b\}, y_2 = \{b,c\}$.

Problem 3. *(10 Points)*

1. Prove that set $[0,1] \subseteq \mathbb{R}$ and set $(0,1) \subseteq \mathbb{R}$ are equinumerous.
2. Prove that set $[0,1] \times [0,1] \subseteq \mathbb{R} \times \mathbb{R}$ and set $[0,1] \subseteq \mathbb{R}$ are equinumerous.

Answer:

1
- $(0,1)$ to $[0,1]$, $f(x) = x$.
- $[0,1]$ to $(0,1)$, Table 1

x	$f(x)$	x	$f(x)$	x	$f(x)$
1	$\frac{1}{2}$	0	$\frac{1}{3}$	other x	x
$\frac{1}{2}$	$\frac{1}{2^2}$	$\frac{1}{3}$	$\frac{1}{3^2}$	$\vdots$	$\vdots$
$\frac{1}{2^2}$	$\frac{1}{2^3}$	$\frac{1}{3^2}$	$\frac{1}{3^3}$	$\vdots$	$\vdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$

Table 1: The function from $[0, 1]$ to $(0, 1)$.

2 Suppose $a_1 = 0.x_1x_2 \cdots x_k \cdots$, $a_1 = 0.y_1y_2 \cdots y_k \cdots$ and $b = 0x_1y_1x_2y_2 \cdots x_ky_k \cdots$, we define a map $f : [0, 1] \times [0, 1] \to [0, 1]$ such that $f(a_1, a_2) = b$. For finite decimal, we use the infinite decimal expression. For example, for $a = 0.2$, we use expression $a = 0.19999\ldots$. Moreover, for $a = 1$ and $b = 0$, we use expression $a = 0.9999\ldots$ and $b = 0$. Then the function f is an injection from $[0, 1] \times [0, 1]$ to $[0, 1]$. Moreover, it is easy to define a injection from $[0, 1]$ to $[0, 1] \times [0, 1]$. From S-B Theorem, we know that $[0, 1] \times [0, 1]$ and $[0, 1]$ are equinumerous.

Problem 4. *(10 Points)*
Given function $f : A \to B$, define a new function $g : B \to 2^A$ by $b \mapsto \{a : a \in A \wedge f(a) = b\}$.

1. Prove that, if f is a surjection, then g is an injection.
2. Provide an example that g is an injection but f is not a surjection.

Answer:

1. Suppose g is not an injection,i.e.

$$(\exists x \in B)(\exists y \in B)(x \neq y \wedge g(x) = g(y))$$

$$\Rightarrow \{a : a \in A \wedge f(a) = x\} = \{a : a \in A \wedge f(a) = y\} \neq \emptyset \text{ (Because } f \text{ is surjection.)}$$

$$\Rightarrow f(a) = x = y$$

$$\textit{Contradiction.}$$

2. $A = B = \{1, 2\}$,
$f = \{\langle 1, 1\rangle, \langle 2, 1\rangle\}$,
$g = \{\langle 1, \{1, 2\}\rangle, \langle 2, \emptyset\rangle\}$,