Discrete Math (Honor) 2022-Fall Homework-1: Solution

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Problem 1. *(8 Points)*
Determine whether or not each of the following sentences is a proposition.

1. SJTU is better than Fudan.
2. $2 + 3 = 5$.
3. $5 + 6 = 12$.
4. Answer this question.
5. This sentence is wrong.
6. Is this sentence true?
7. I do not know whether or not what you said is true.
8. Given any Turning machine and input, one can determine if it sucks.

Answers: 1.Y 2.Y 3.Y 4.N 5. N 6.N 7.Y 8.Y

A proposition is a declarative sentence/assertion that is **either true or false** (but not both). Sentence in 5 uses self-reference which makes it neither true nor false. Sentence in 7 does not have this situation. Therefore, 5 is not a proposition while 7 is a proposition.

Problem 2. *(3 Points)*
We define the following propositions

P: I like grape juice Q: Grape juice is expensive R: I will buy grape juice.

Express the sentence "*If I don't like grape juice and it is expensive, I will not buy grape juice*" using the above atomic propositions.

Answers: $\neg P \wedge Q \to \neg R$ (answering $Q \wedge \neg P \to \neg R$, $\neg P \to Q \to \neg R$ or $Q \to \neg P \to \neg R$ is also OK)

Problem 3. *(8 Points)*
Explain intuitively in words under what situation each of the following propositions is true.

1. $(P \wedge Q) \vee (\neg P \wedge \neg Q)$
2. $P \to (Q \to R)$
3. $(P \vee \neg Q) \wedge (Q \vee \neg R) \wedge (R \vee \neg P)$
4. $(P \vee Q \vee R) \wedge (\neg P \vee \neg Q \vee \neg R)$

Answers: Any natural language sentences that is reasonable (or not unreasonable) is OK

1. P and Q are both True or both False
2. R is True, or P is False, or Q is False (P is False, or P is True and Q is False, or P is True and Q is True and R is True)
3. P, Q, R are all True or P, Q, R are all False (P, Q, R are the same)
4. P, Q, R are not all True or False at the same time

Problem 4. *(8 Points)*
Determine if each of the following propositions is tautology, contradiction or satisfiable, and justify your answer.

1. $\neg((P \vee Q) \to (Q \vee P))$
2. $(Q \to R) \to ((P \vee Q) \to (P \vee R))$
3. $(Q \to R) \to ((P \to Q) \to (P \to R))$
4. $(P \to Q) \to (\neg Q \to \neg P)$

Answer:

1. Contradiction.

   Since $P \vee Q = Q \vee P$ and $R \to R$ is tautology, $\neg((P \vee Q) \to (Q \vee P))$ is contradiction.
2. Tautology

   If $(P \vee Q) \to (P \vee R)$ is false, $P \vee Q$ is true and $P \vee R$ is false. Then Q is true, P and R are false. Then $Q \to R$ is false. However, if 2 is false, $(P \vee Q) \to (P \vee R)$ is false and $Q \to R$ is true. Therefore, 2 is tautology.
3. Tautology

   Since $(P \to Q) \to (P \to R) = P \to (Q \to R)$, we consider proposition $(Q \to R) \to (P \to (Q \to R))$. If $P \to (Q \to R)$ is false, $Q \to R$ is false. Then $(Q \to R) \to (P \to (Q \to R))$ is true. Thus $(Q \to R) \to (P \to (Q \to R))$ is tautology.
4. Tautology

   Since $P \to Q = \neg Q \to \neg P$, 4 is tautology.

Problem 5. *(8 Points)*
Draw the truth table for each of the following propositions. ($\oplus$ means "exclusive or")

1. $(P \leftrightarrow Q) \oplus (P \leftrightarrow \neg Q)$
2. $(P \vee Q) \to (P \oplus Q)$
3. $(\neg P \leftrightarrow \neg Q) \leftrightarrow (Q \leftrightarrow R)$
4. $((P \to Q) \to R) \to S$

Answer: (should discover the answer before instead of after drawing the truth table...think how you finish Problem 3) Free to using 0/1 or T/F

1. It's a tautology
2. The table shows $(P \vee Q) \to (P \oplus Q)$ is F only when P and Q are both T:

P	Q	$(P \vee Q) \to (P \oplus Q)$
F	F	T
F	T	T
T	F	T
T	T	F

3. You should find that the result is T iff P, Q are different and Q, R are also different.

P	Q	R	$(\neg P \leftrightarrow \neg Q) \leftrightarrow (Q \leftrightarrow R)$
0	0	0	1
0	0	1	0
0	1	0	1
0	1	1	0
1	0	0	0
1	0	1	1
1	1	0	0
1	1	1	1

4. You can think like this and draw the table:
   (a) When S is T, the proposition is always T (save you 50% time)
   (b) When S is F but R is T (then $(P \to Q) \to R$ is T), the proposition is always F (save you 25% time)
   (c) When S R are both F, then the proposition is F iff $P \to Q$ is F iff P is T but Q if F

P	Q	R	S	$((P \to Q) \to R) \to S$
0	0	0	0	1
0	0	0	1	1
0	0	1	0	0
0	0	1	1	1
0	1	0	0	1
0	1	0	1	1
0	1	1	0	0
0	1	1	1	1
1	0	0	0	0
1	0	0	1	1
1	0	1	0	0
1	0	1	1	1
1	1	0	0	1
1	1	0	1	1
1	1	1	0	0
1	1	1	1	1

Problem 6. *(8 Points)*

1. Show that conjunction and disjunction can be expressed using $\neg$ and $\to$.
2. We define "NAND $\uparrow$" by $P \uparrow Q = \neg(P \wedge Q)$. Show that negation, conjunction and disjunction can be expressed only using $\uparrow$.
3. We define "NOR $\downarrow$" by $P \downarrow Q = \neg(P \vee Q)$. Show that negation, conjunction and disjunction can be expressed only using $\downarrow$.

Answer:

1. $\neg P$: $\neg P$; $P \vee Q$: $\neg P \to Q$; $P \wedge Q$: $\neg(P \to \neg Q)$
2. $\neg P$: $P \uparrow P$; $P \vee Q$: $(P \uparrow P) \uparrow (Q \uparrow Q)$; $P \wedge Q$: $(P \uparrow Q) \uparrow (P \uparrow Q)$
3. $\neg P$: $P \downarrow P$; $P \vee Q$: $(P \downarrow Q) \downarrow (P \downarrow Q)$; $P \wedge Q$: $(P \downarrow P) \downarrow (Q \downarrow Q)$

Problem 7. *(8 Points)*

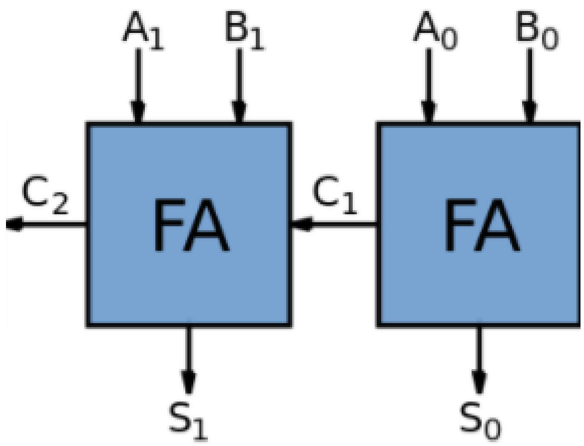
A *full adder* adds binary numbers and accounts for values carried in as well as out. A one-bit full-adder adds three one-bit numbers, often written as A, B, and C_{in}; A and B are the operands, and C_{in} is a bit carried in from the previous less-significant stage. Output carry and sum are represented by the signals C_{out} and S, respectively. The truth table of the full adder is shown as follows.

Inputs			Outputs	
A	B	C_{in}	C_{out}	S
0	0	0	0	0
0	0	1	0	1
0	1	0	0	1
0	1	1	1	0
1	0	0	0	1
1	0	1	1	0
1	1	0	1	0
1	1	1	1	1

1. Can you write C_{out} and S, respectively, using A, B and C_{in} and logical operators? (hint: it is more convenient to use $\oplus$.)
2. Can you combine a half adder and a full adder together to design a circuit that supports the addition of two 2-bits numbers? For example, it can compute $10 + 11 = 101$.

Answer:

1. $S = (A \oplus B) \oplus C_{in}$, $C_{out} = (A \wedge B) \vee (A \wedge C_{in}) \vee (B \wedge C_{in})$ (or $(A \wedge (B \vee C_{in})) \vee (B \wedge C_{in})$)
2. You should show the connection of two adders by drawing a figure or showing in logic expression. Specifically, you can draw like figure below or express like $S_1 = (A_1 \oplus B_1) \oplus C_1$, $C_1 = A_0 \oplus B_0$. If you write $S_1 = (A_1 \oplus B_1) \oplus (A_0 \wedge B_0)$, it just a more complicated logic expression compared with first question. You need to show how two adders work together. It is a thought of building complex system with simple components.



Problem 8. *(10 Points)*
Prove the following equivalences using any method you like.

1. $P \rightarrow (Q \wedge R) = (P \rightarrow Q) \wedge (P \rightarrow R)$
2. $(P \leftrightarrow Q) \leftrightarrow ((P \wedge \neg Q) \vee (Q \wedge \neg P)) = P \wedge \neg P$
3. $((P \rightarrow \neg Q) \rightarrow (Q \rightarrow \neg P)) \wedge R = R$
4. $P \rightarrow (Q \rightarrow R) = (P \wedge Q) \rightarrow R$
5. $\neg(P \leftrightarrow Q) = (P \wedge \neg Q) \vee (\neg P \wedge Q)$

Answer: You may also use truth table to prove all of them (but might be too trivial). The following proofs use conclusions from logical equivalence :

1. $P \rightarrow (Q \wedge R) = \neg P \vee (Q \wedge R) = (\neg P \vee Q) \wedge (\neg P \vee R) = (P \rightarrow Q) \wedge (P \rightarrow R)$
2. use truth table might be a more concise way
3. $left = ((\neg P \vee \neg Q) \rightarrow (\neg Q \vee \neg P)) \wedge R = ((\neg P \vee \neg Q) \rightarrow (\neg P \vee \neg Q)) \wedge R = \mathbf{T} \wedge R = R$
4. $left = \neg P \vee (\neg Q \vee R) = (\neg P \vee \neg Q) \vee R = \neg(P \wedge Q) \vee R = right$
5. use truth table might be a more concise way