一、选择题

1.
$$\begin{aligned}|-3(A^TB^{-1})^2A^*| &= (-3)^3|A|^2|B^{-1}|^2|A^*| \\ &= -27\times 1\times\frac{1}{4}\times(-1)\times\frac{1}{-1} = -\frac{27}{4}\end{aligned}\tag{9}$$

选D.

2.
$$\begin{aligned}\begin{vmatrix}1 & \lambda & \lambda^2\\ 1 & -1 & 1\\ 2 & 4 & 8\end{vmatrix} &= 0\\ -8+4\lambda^2+2\lambda+2\lambda^2-8\lambda-4 &= 0\\ (\lambda-2)(\lambda+1) &= 0\end{aligned}\tag{10}$$

选C.

3. $|AB|=0 \implies |A|=0$或$|B|=0$ 选C.

4. $r(A)=r(B) \implies |A|\neq 0 \Leftrightarrow |B|\neq 0$ 选C.

5. C

6. $A(BC)=E \implies A^{-1}=BC \implies BCA=E$ 选D.

7. C

8.
$$\begin{aligned}PQ &= \begin{bmatrix}A & \boldsymbol{\alpha}\\ \mathbf{0} & -\boldsymbol{\alpha}^TA^*\boldsymbol{\alpha}+|A|\end{bmatrix}\\ &= |A|\begin{bmatrix}\frac{A}{|A|} & \frac{\boldsymbol{\alpha}}{|A|}\\ \mathbf{0} & -\boldsymbol{\alpha}^TA^{-1}\boldsymbol{\alpha}+1\end{bmatrix}\end{aligned}\tag{11}$$

由A可逆知选B

9. $r(AB)\geq r(A)+r(B)-n \implies r(A)+r(B)\leq n$ 选B.

10. $(A^*)^*=|A^*|(A^*)^{-1}=||A|A^{-1}||A^{-1}|A=|A|^{n-2}A$

11. $B^*=(E(1,2)A)^*=|E(1,2)A|(E(1,2)A)^{-1}=-|A|A^{-1}E^{-1}(1,2)=-A^*E(1,2)$

选C.

12. 设$A=[\boldsymbol{\alpha}_1\quad\boldsymbol{\alpha}_2\quad\boldsymbol{\alpha}_3]$，有$|A|A^{-1}=A^T \implies |A|E=A^TA$

$$\begin{gathered}\begin{bmatrix}|A| & & \\ & |A| & \\ & & |A|\end{bmatrix}=\begin{bmatrix}\boldsymbol{\alpha}_1^T\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_2^T\boldsymbol{\alpha}_1 & \boldsymbol{\alpha}_3^T\boldsymbol{\alpha}_1\\ \boldsymbol{\alpha}_1^T\boldsymbol{\alpha}_2 & \boldsymbol{\alpha}_2^T\boldsymbol{\alpha}_2 & \boldsymbol{\alpha}_3^T\boldsymbol{\alpha}_2\\ \boldsymbol{\alpha}_1^T\boldsymbol{\alpha}_3 & \boldsymbol{\alpha}_2^T\boldsymbol{\alpha}_3 & \boldsymbol{\alpha}_3^T\boldsymbol{\alpha}_3\end{bmatrix}\\ \begin{cases}\boldsymbol{\alpha}_1^T\boldsymbol{\alpha}_1=|A|\\ |A|^3=|A^TA|=|A|^2\end{cases}\end{gathered}\tag{12}$$

从而$3a_{11}^2=|A|=1$，$a_{11}=\frac{\sqrt{3}}{3}$

选A.

13. $|\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1,(\boldsymbol{\beta}_1+\boldsymbol{\beta}_2)|=|\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_1|+|\boldsymbol{\alpha}_3,\boldsymbol{\alpha}_2,\boldsymbol{\alpha}_1,\boldsymbol{\beta}_2|=(-1)^3m+(-1)^4n=-m+n$

选C.

14. $r\left(\begin{bmatrix}1 & a & 1 & 1\\ 1 & b & 1 & 0\\ 1 & c & 0 & 0\end{bmatrix}\right)=3$ 选C.

15.

$$[\boldsymbol{\alpha}_1 \quad \boldsymbol{\alpha}_2 \quad \boldsymbol{\alpha}_3 \quad \boldsymbol{\alpha}_4 \quad \boldsymbol{\alpha}_5] = \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ -1 & 3 & 0 & -2 & 1 \\ 2 & 1 & 7 & 2 & 5 \\ 4 & 2 & 14 & 0 & 10 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 3 & 3 & -1 & 3 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 2 & 2 & -4 & 2 \end{bmatrix} \to \begin{bmatrix} 1 & 0 & 3 & 1 & 2 \\ 0 & 1 & 1 & 0 & 1 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -4 & 0 \end{bmatrix} \tag{13}$$

选B.

16.选B.

17.由题知$\alpha_1 = \xi_1 - \xi_2, \alpha_2 = \xi_1 - \xi_3, \alpha_3 = \xi_1 - \xi_4$ 是 $AX = \mathbf{0}$的三个互不相同的解，从而 $r(A) < n$ 又由$A^* \neq O$ 知 $r(A) = n - 1$ 从而基础解系只有一个向量，选B.

18.$AX = \mathbf{0} \implies A^T AX = \mathbf{0}$ 显然成立，若$A^T AX = \mathbf{0}$则 $X^T A^T AX = \mathbf{0} \implies (AX)^T AX = \mathbf{0} \implies AX = \mathbf{0}$ 选A.

19.选B.

20.令$\alpha_1 = [1, 0, 0, 0], \alpha_2 = [0, 1, 0, 0], \beta_1 = [0, 0, 1, 0], \beta_2 = [0, 0, 0, 1]$，可知选D.

21.选D.

22.选C.

23.$a + 2 = 5$得$a = 3$ 选D.

24.$|A^*| = |A|^{n-1} = (-8)^3 = -512$ 选A

25.由$A\xi = -\xi$得$\lambda = -1$ 选A.

二、填空题

1. $(-1)^{\frac{n(n+3)}{2}}(n-1)!(1 - \sum_{i=2}^{n} \frac{1}{i})$

2. 0 28

3. 与选择题12题类似的方法得 -1

4. $(-1)^{mn}ab$

5. $|A| = -1$，A的特征值只能为± 1, 从而$|A + E| = 0$

6. $a = 0$

7. 令$B = \begin{bmatrix} 1 & 0 \\ 0 & -2 \end{bmatrix}, C = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$,

$$A^3 = \begin{bmatrix} B^3 & O \\ O & C^3 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & -8 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \tag{14}$$

8.

由6知A的特征值为$2,2,0$，对角化后得$A = \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2 & & \\ & 2 & \\ & & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T$

从而

$$\begin{aligned} A^n - 2A^{n-1} &= A^{n-1}(A-2E) \\ &= \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{bmatrix} 2^{n-1} & & \\ & 2^{n-1} & \\ & & 0 \end{bmatrix} \begin{bmatrix} 0 & \frac{\sqrt{2}}{2} & -\frac{\sqrt{2}}{2} \\ 1 & 0 & 0 \\ 0 & \frac{\sqrt{2}}{2} & \frac{\sqrt{2}}{2} \end{bmatrix}^T \begin{bmatrix} -1 & 0 & 1 \\ 0 & 0 & 0 \\ 1 & 0 & -1 \end{bmatrix} \\ &= O \end{aligned} \tag{15}$$

9. $A^{-1}(A^{-1}+B^{-1})B^{-1}$

10. 设$(E-A)^{-1} = A^2 + \lambda A + \mu E$，解得$\lambda = -1, \mu = 1$，从而$(E-A)^{-1} = A^2 - A + E$

11. $\frac{1}{10}A$

12. $\begin{bmatrix} |B|A^* & O \\ O & |A|B^* \end{bmatrix}$

13. 2

14.

$$r(A^*) = \begin{cases} n, & r(A) = n \\ 1, & r(A) = n-1 \\ 0, & otherwise \end{cases} \tag{16}$$

因此$r((A^*)^*) = 0$

15. 3

16.

α_1, α_2对应的特征值都为1，从而$Q^{-1}AQ = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$

17. 令$A = E_3$，解得$\boldsymbol{x} = b$

18. $k \notin \{0, -3, 3\}$

19. $a_1 + a_2 + a_3 + a_4 = 0$

20. 知A可逆，从而 $X = (A^T)^{-1}b = (\frac{1}{|A|}A^*)^T b = \frac{1}{|A|}(A^*)^T b$。利用代数余子式的性质不难得到 $(A^*)^T b = [1 \quad 0 \quad 0 \quad \cdots \quad 0]^T$

21. 线性无关

22. 知基础解系只有一个向量，由各行元素之和为零知$[1 \quad 1 \quad 1 \quad \cdots \quad 1]^T$为一个解，从而通解为 $k[1 \quad 1 \quad 1 \quad \cdots \quad 1]^T$其中$k$为任意常数.

23. B的特征值为-2，1，0，$|B+E|$的特征值为-1，2，1，从而$|B| = 0$，$|B+E| = -2$

24. 利用$A\alpha = \lambda\alpha$解得$k = 1$或-2

25. $A \begin{bmatrix} 1 & 0 & -2 \\ 0 & 3 & -1 \\ -1 & 2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -3 & 0 \\ -1 & -2 & 0 \end{bmatrix}$ 解得$A = \begin{bmatrix} -5 & 4 & -6 \\ 3 & -3 & 3 \\ 7 & -6 & 8 \end{bmatrix}$