一、矩阵的定义

$$\text{设}K\text{为数域}, a_{ij} \in K, 1 \leqslant i \leqslant m, 1 \leqslant j \leqslant n,$$

$$\text{令}\boldsymbol{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix},$$

$$\text{则称}\boldsymbol{A}\text{为数域}K\text{上的一个}m\times n\text{矩阵}, a_{ij}\text{叫做矩阵}\boldsymbol{A}\text{的}(i,j)-\text{分量或}(i,j)-\text{元素}.$$

$$\text{当}m=n\text{时}, \boldsymbol{A}\text{叫做}K\text{上的一个}n\text{阶方阵}.$$

***一些特殊的矩阵**

$$1.\ \text{单位矩阵：}\boldsymbol{A} = \begin{pmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{pmatrix}$$

$$2.\ \text{对角矩阵：}A = \begin{pmatrix} a_{11} & & & \\ & a_{22} & & \\ & & \ddots & \\ & & & a_{nn} \end{pmatrix}$$

$$3.\ \text{上三角矩阵：}\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ & a_{22} & \cdots & a_{2n} \\ & & \ddots & \vdots \\ & & & a_{nn} \end{pmatrix}, \text{（下三角矩阵同理）}$$

$$4.\ \text{转置矩阵：若}\boldsymbol{A} = (a_{ij})_{m\times n} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \boldsymbol{B} = (a_{ij})_{n\times m} = \begin{pmatrix} a_{11} & a_{21} & \cdots & a_{m1} \\ a_{12} & a_{22} & \cdots & a_{m2} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{mn} \end{pmatrix},$$

$$\text{则称}B\text{为}A\text{的转置矩阵}.$$

$$\text{转置矩阵的性质：}1)\text{对任意}\boldsymbol{A},\boldsymbol{B}\in K^{n\times n}, k\in K, \text{都有}(\boldsymbol{A}+\boldsymbol{B})^{\mathrm{T}} = \boldsymbol{A}^{\mathrm{T}} + \boldsymbol{B}^{\mathrm{T}}, \quad (k\boldsymbol{A})^{\mathrm{T}} = k\boldsymbol{A}^{\mathrm{T}}.$$

$$2)\ \text{对任意}\boldsymbol{A}\in K^{m\times l}, \boldsymbol{B}\in K^{l\times n}, \text{都有}(\boldsymbol{AB})^{\mathrm{T}} = \boldsymbol{B}^{\mathrm{T}}\boldsymbol{A}^{\mathrm{T}}.$$

$$5.\ \text{对称矩阵：}\boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{12} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{1n} & a_{2n} & \cdots & a_{nn} \end{pmatrix}$$

$$(*)\text{若}\boldsymbol{A}^2 = \boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}, \text{则}\boldsymbol{A}\text{为实对称矩阵}.$$

$$6.\ \text{反对称矩阵：}A = \begin{pmatrix} 0 & a_{12} & \cdots & a_{1n} \\ -a_{12} & 0 & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ -a_{1n} & -a_{2n} & \cdots & 0 \end{pmatrix}$$

$$7.\ n\text{维行向量（}m=1\text{）：}\ \alpha = (a_1, a_2, \cdots, a_n) \in K^{1\times n}$$

$$8.\ m\text{维列向量（}n=1\text{）：}\ \beta=\begin{pmatrix}b_1\\b_2\\\vdots\\b_m\end{pmatrix}\in K^{m\times 1}$$

二、矩阵运算

1.矩阵的线性运算

$$\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix}+\begin{pmatrix}b_{11}&b_{12}&\cdots&b_{1n}\\b_{21}&b_{22}&\cdots&b_{2n}\\\vdots&\vdots&&\vdots\\b_{m1}&b_{m2}&\cdots&b_{mn}\end{pmatrix}=\begin{pmatrix}a_{11}+b_{11}&a_{12}+b_{12}&\cdots&a_{1n}+b_{1n}\\a_{21}+b_{21}&a_{22}+b_{22}&\cdots&a_{2n}+b_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}+b_{m1}&a_{m2}+b_{m2}&\cdots&a_{mn}+b_{mn}\end{pmatrix}$$

$$K\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix}=\begin{pmatrix}ka_{11}&ka_{12}&\cdots&ka_{1n}\\ka_{21}&ka_{22}&\cdots&ka_{2n}\\\vdots&\vdots&&\vdots\\ka_{m1}&ka_{m2}&\cdots&ka_{mn}\end{pmatrix}$$

2.矩阵乘法

1）定义

$$\text{设}\boldsymbol{A}=(a_{ij})_{m\times n}\in K^{m\times n},\quad \boldsymbol{B}=(b_{ij})_{n\times s}\in K^{n\times s},$$

$$\text{令}\boldsymbol{C}=(c_{ij})_{m\times s},\quad \text{其中}c_{ij}=\sum_{k=1}^{n}a_{ik}b_{kj},\quad 1\leqslant i\leqslant m,\quad 1\leqslant j\leqslant s,$$

$$\text{则称}m\times s\text{矩阵}\boldsymbol{C}\text{为矩阵}\boldsymbol{A}\text{与}\boldsymbol{B}\text{的乘积, 记作}\boldsymbol{C}=\boldsymbol{AB}.$$

2）性质

$$(1)\text{设}k\in K,\boldsymbol{A}\in K^{m\times n},\boldsymbol{B}\in K^{n\times s},\text{则 }k(\boldsymbol{AB})=(k\boldsymbol{A}),\ \boldsymbol{B}=\boldsymbol{A}(k\boldsymbol{B}).$$

$$(2)\text{设}\boldsymbol{A}\in K^{m\times n},\text{则 }\boldsymbol{O}_{l\times m}\boldsymbol{A}=\boldsymbol{O}_{l\times n},\ \boldsymbol{A}\boldsymbol{O}_{n\times s}=\boldsymbol{O}_{m\times s};\ \boldsymbol{E}_m\boldsymbol{A}=\boldsymbol{A}\boldsymbol{E}_n=\boldsymbol{A}.$$

$$(3)\text{乘法结合律：设}\boldsymbol{A}\in K^{m\times n},\boldsymbol{B}\in K^{n\times s},\boldsymbol{C}\in K^{s\times t},\text{则}(\boldsymbol{AB})\boldsymbol{C}=\boldsymbol{A}(\boldsymbol{BC}).$$

$$(4)\text{乘法对于加法的分配律：设}\boldsymbol{A}\in K^{m\times n},\boldsymbol{B}_1,\boldsymbol{C}_1\in K^{n\times s},\boldsymbol{B}_2,\boldsymbol{C}_2\in K^{l\times m},$$
$$\text{则}A\left(B_1+C_1\right)=AB_1+AC_1,\quad (B_2+C_2)A=B_2A+C_2A.$$

3）注意!(与数的乘法的不同)

$$1.\text{存在矩阵}A\neq O,B\neq O,\text{然而}AB=O;$$
$$2.\text{矩阵乘法一般不满足交换律, 即存在矩阵}A\text{与}B\text{使}AB\neq BA;$$
$$3.\text{消去律一般不成立, 即由}AB=AC\text{且}A\neq O\text{不能推出}B=C.$$

3.矩阵的幂

1）定义

$$\text{设}\boldsymbol{A}\in K^{n\times n},k\text{为非负整数, 令}\boldsymbol{A}^k=\begin{cases}\boldsymbol{E}_n,&\text{若 }k=0,\\\boldsymbol{A},&\text{若 }k=1,\\\boldsymbol{A}\boldsymbol{A}^{k-1},&\text{若 }k\geqslant 1,\end{cases}\quad\text{称}\boldsymbol{A}^k\text{为}\boldsymbol{A}\text{的}k\text{次幂.}$$

2）性质

$$
\begin{gathered}
\text{设}\boldsymbol{A} \in K^{n \times n}, l\text{与}k\text{为非负整数, 则} \\
(1) A^l A^k = A^k A^l = A^{k+l}. \\
(2) \left(A^l\right)^k = A^{lk}. \\
(3)\text{设}\boldsymbol{A}\text{和}\boldsymbol{B}\text{为}n\text{阶方阵, 若}\boldsymbol{A}\text{与}\boldsymbol{B}\text{可交换, 则}(\boldsymbol{A}+\boldsymbol{B})^m = \sum_{i=0}^{m} \mathrm{C}_m^i \boldsymbol{A}^i \boldsymbol{B}^{m-i}. \\
(4)\text{设}\boldsymbol{A}\text{和}\boldsymbol{B}\text{为}n\text{阶方阵, 若}\boldsymbol{A}\text{与}\boldsymbol{B}\text{可交换, 则}\boldsymbol{A}^m - \boldsymbol{B}^m = (\boldsymbol{A}-\boldsymbol{B})\left(\boldsymbol{A}^{m-1} + \boldsymbol{A}^{m-2}\boldsymbol{B} + \cdots + \boldsymbol{A}\boldsymbol{B}^{m-2} + \boldsymbol{B}^{m-1}\right)
\end{gathered}
$$

4.矩阵的迹

1）定义

$$
\text{设}\boldsymbol{A} = (a_{ij})_{n \times n}\text{，则称其主对角线上的元素之和为}\boldsymbol{A}\text{的迹，记作}\operatorname{tr}(\boldsymbol{A})\text{，即 }\operatorname{tr}(\boldsymbol{A}) = \sum_{i=1}^{n} a_{ii}.
$$

2）性质

$$
\begin{gathered}
\text{设}\boldsymbol{A}\text{和}\boldsymbol{B}\text{为}n\text{阶方阵, }k\text{为数, 则：} \\
(1) \operatorname{tr}\left(\boldsymbol{A}^{\mathrm{T}}\right) = \operatorname{tr}(\boldsymbol{A}). \\
(2) \operatorname{tr}(\boldsymbol{A}+\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{A}) + \operatorname{tr}(\boldsymbol{B}). \\
(3) \operatorname{tr}(k\boldsymbol{A}) = k \operatorname{tr}(\boldsymbol{A}) \\
(4) \operatorname{tr}(\boldsymbol{A}\boldsymbol{B}) = \operatorname{tr}(\boldsymbol{B}\boldsymbol{A}). \\
(5)\text{若}\operatorname{tr}\left(\boldsymbol{A}\boldsymbol{A}^{\mathrm{T}}\right) = 0, \text{则}\boldsymbol{A} = \boldsymbol{O};
\end{gathered}
$$

三、行列式

1.行列式的定义

$$
D = \left|a_{ij}\right|_n = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \sum_{j_1 j_2 \cdots j_n} (-1)^{\tau(j_1 j_2 \cdot \cdots \cdot j_n)} a_{1j_1} \cdot a_{2j_2} \cdot \cdots \cdot a_{nj_n},
$$

$$
\text{其中}\tau\left(j_1 j_2 \cdots j_n\right)\text{是}n\text{阶排列}j_1 j_2 \cdots j_n\text{的逆序数.}
$$

2.行列式的性质

$$
1\text{）} D^{\mathrm{T}} = D
$$

$$
2\text{）} \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1}+b_{i1} & a_{i2}+b_{i2} & \cdots & a_{in}+b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} = \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ a_{i1} & a_{i2} & \cdots & a_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix} + \begin{vmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ \vdots & \vdots & & \vdots \\ b_{i1} & b_{i2} & \cdots & b_{in} \\ \vdots & \vdots & & \vdots \\ a_{n1} & a_{n2} & \cdots & a_{nn} \end{vmatrix}.
$$

$$
\begin{gathered}
3\text{） 设}\boldsymbol{A}\text{与}\boldsymbol{B}\text{为数域}K\text{上的两个}n\text{阶方阵, 则}|\boldsymbol{A}\boldsymbol{B}| = |\boldsymbol{A}||\boldsymbol{B}|, \\
\text{即两个同阶方阵乘积的行列式等于这两个方阵的行列式的乘积.}
\end{gathered}
$$

3.行列式的初等变换

1）定义

$$设\boldsymbol{A}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{n1}&a_{n2}&\cdots&a_{nn}\end{pmatrix}\in K^{n\times n},\ 又设D=|\boldsymbol{A}|,\ 1\leqslant i,j\leqslant n,i\neq j.则:$$

$$第一类行变换：交换D的第i行和第j行,记作r_i\leftrightarrow r_j;$$

$$第二类行变换：对k\in K,k\neq 0,用k去乘\boldsymbol{A}的第i行,记作k\cdot r_i;$$

$$第三类行变换：设k\in K,将D的第j行的k倍加到第i行,记作r_i+k\cdot r_j.$$

2)影响

$$(1)经过一次第一类初等变换,行列式值反号;$$

$$(2)经过一次第二类初等变换,行列式值为原行列式的非零常数倍;$$

$$(3)经过一次第三类初等变换,行列式值不变.$$

3）推论

$$(1)若行列式中有两行相同,则行列式的值为零.$$

$$(2)若行列式D中有两行的对应元素成比例,则D=0.$$

4.行列式的按行（列）展开

1）子式、余子式、代数余子式

$$设D为n阶行列式,1\leqslant k\leqslant n,\ 1\leqslant i_1<i_2<\cdots<i_k\leqslant n,\ 1\leqslant j_1<j_2<\cdots<j_k\leqslant n.$$

$$在D中取定第i_1,i_2,\cdots,i_k行与第j_1,j_2,\cdots,j_k列,$$

$$将位于这k行k列交点上的k^2个元素依原来的顺序组成一个k阶行列式M,称为D的一个k阶子式.$$

$$当k<n时,在D中划去这k行k列后余下的元素按原来顺序组成的n-k阶行列式M'称为M的余子式.$$

$$再令A=(-1)^{(i_1+i_2+\cdots+i_k)+(j_1+j_2+\cdots+j_k)}M',则A叫做M的代数余子式.$$

2)按行（列）展开

$$D=\left|a_{ij}\right|_n=a_{i1}A_{i1}+a_{i2}A_{i2}+\cdots+a_{in}A_{in}=\sum_{k=1}^{n}a_{ik}A_{ik},$$

$$从而\ \sum_{k=1}^{11}a_{ik}A_{jk}=a_{i1}A_{j1}+a_{i2}A_{j2}+\cdots+a_{in}A_{jn}=\begin{cases}D&(i=j),\\0&(i\neq j)\end{cases}$$

5. Laplace定理

$$设D为n阶行列式,1\leqslant k\leqslant n-1.在D中任意取定k行,$$

$$由这k行元素所组成的全体k阶子式记作M_1,M_2,\cdots,M_t,\ 此处t=\mathrm{C}_n^k.$$

$$对1\leqslant i\leqslant t,\ 令M_i的代数余子式为A_i,\ 则D=\sum_{i=1}^{t}M_iA_i=M_1A_1+M_2A_2+\cdots+M_tA_t.$$

6.一些行列式的小结论

1）爪形行列式

$$形如D_n=\begin{vmatrix} a_1 & a_2 & a_3 & \cdots & a_n \\ c_2 & b_2 & & & \\ c_3 & & b_3 & & \\ \vdots & & & \ddots & \\ c_n & & & & b_n \end{vmatrix}的行列式叫做n阶爪型行列式.$$

$$若b_2b_3\cdots b_n\neq 0,对2\leqslant j\leqslant n,将第j列的-\frac{c_j}{b_j}倍加到第1列,则得$$

$$D_n=\begin{vmatrix} a_1-\sum_{j=2}^{n}\frac{a_jc_j}{b_j} & a_2 & \cdots & a_n \\ 0 & b_2 & & \\ \vdots & & \ddots & \\ 0 & & & b_n \end{vmatrix}=\left(a_1-\sum_{j=2}^{n}\frac{a_jc_j}{b_j}\right)b_2b_3\cdots b_n.$$

$$若b_2b_3\cdots b_n=0,即存在某个k,2\leqslant k\leqslant n,使b_k=0.$$

$$此时，若c_k=0,则D_n的第k行元素都是零,因此D_n=0.$$

$$若b_k=0而c_k\neq 0,则对k<i\leqslant n,将第k行的-\frac{c_i}{c_k}倍加到第i行,然后再将第1列与第k列交换,即得$$

$$D_n=-b_2b_3\cdots b_{k-1}\left(a_kc_k\right)b_{k+1}\cdots b_n$$

2）Vandermonde 行列式

$$D_n=\begin{vmatrix} 1 & 1 & \cdots & 1 \\ x_1 & x_2 & \cdots & x_n \\ x_1^2 & x_2^2 & \cdots & x_n^2 \\ \vdots & \vdots & & \vdots \\ x_1^{n-1} & x_2^{n-1} & \cdots & x_n^{n-1} \end{vmatrix}=\prod_{1\leqslant i<j\leqslant n}\left(x_j-x_i\right)$$

四、可逆矩阵

1.可逆矩阵的定义

$$设\boldsymbol{A}\in K^{n\times n},若存在\boldsymbol{B}\in K^{n\times n},使AB=BA=E,则称\boldsymbol{A}为n阶可逆矩阵,\boldsymbol{B}为\boldsymbol{A}的逆矩阵,记作\boldsymbol{A}^{-1}.$$
$$显然,若B是A的逆矩阵,则\boldsymbol{A}也是\boldsymbol{B}的逆矩阵,即\boldsymbol{A}与\boldsymbol{B}互为逆矩阵.$$

2.伴随矩阵的定义

$$A_{ij}表示a_{ij}的代数余子式,则\boldsymbol{A}^*=\begin{pmatrix} A_{11} & A_{21} & \cdots & A_{n1} \\ A_{12} & A_{22} & \cdots & A_{n2} \\ \vdots & \vdots & & \vdots \\ A_{1n} & A_{2n} & \cdots & A_{nn} \end{pmatrix}.$$

则

$$\boldsymbol{A}=\left(a_{ij}\right)_{n\times n}为可逆矩阵的充要条件为|\boldsymbol{A}|\neq 0,且\boldsymbol{A}^{-1}=\frac{1}{|\boldsymbol{A}|}\boldsymbol{A}^*.$$

3.可逆矩阵的性质

$$\text{设}\boldsymbol{A}, B \in K^{n\times n}, \text{则}$$
$$(1)\text{若}\boldsymbol{A}\text{可逆，则 }\boldsymbol{A}^{-1}\text{也可逆 且 }\left(A^{-1}\right)^{-1} = A.$$
$$(2)\text{若}\boldsymbol{A}\text{可逆，则 }\boldsymbol{A}\text{的转置矩阵}\boldsymbol{A}^{\mathrm{T}}\text{也可逆 且 }\left(\boldsymbol{A}^{\mathrm{T}}\right)^{-1} = \left(\boldsymbol{A}^{-1}\right)^{\mathrm{T}}.$$
$$(3)\text{若}\boldsymbol{A}\text{与}\boldsymbol{B}\text{都可逆，则}\boldsymbol{AB}\text{也可逆 且 }(AB)^{-1} = B^{-1}A^{-1}.$$
$$(4)\text{若}\boldsymbol{A}\text{可逆, 则}k\boldsymbol{A}\text{可逆的充要条件为}k \neq 0, \text{且}(k\boldsymbol{A})^{-1} = \frac{1}{k}\boldsymbol{A}^{-1}.$$
$$(5)\text{若}\boldsymbol{A}, \boldsymbol{B}, \boldsymbol{A}+\boldsymbol{B}\text{都可逆, 则}(\boldsymbol{A}+\boldsymbol{B})^{-1} = \boldsymbol{A}^{-1}\left(\boldsymbol{A}^{-1}+\boldsymbol{B}^{-1}\right)^{-1}\boldsymbol{B}^{-1}.$$
$$(6)\boldsymbol{AA}^* = \boldsymbol{A}^*\boldsymbol{A} = |\boldsymbol{A}|\boldsymbol{E}.$$
$$(7)\text{若}\boldsymbol{A}\text{为}n\text{阶可逆矩阵, 则 }\left|\boldsymbol{A}^{-1}\right| = |\boldsymbol{A}|^{-1}, \quad |\boldsymbol{A}^*| = |\boldsymbol{A}|^{n-1}.$$

4.奇异矩阵的定义

$$\text{设}\boldsymbol{A} \in K^{n\times n}, \text{若}|\boldsymbol{A}| = 0, \text{则称}\boldsymbol{A}\text{为}n\text{阶奇异矩阵,}$$
$$\text{若}|A| \neq 0, \text{则称}A\text{为非奇异矩阵.}$$

5. Cramer法则

$$\text{若线性方组}(1.6.16)\text{的系数矩阵}A\text{的行列式不等于零, 则}$$
$$\text{方程组}\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = b_n \end{cases} \text{有唯一解且其解为}x_j = \frac{D_j}{D}, \quad 1 \leqslant j \leqslant n.$$
$$\text{其中}D = |\boldsymbol{A}|, \quad D_j\text{是将}\boldsymbol{A}\text{的第}\boldsymbol{j}\text{列用常数列}\boldsymbol{\beta} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_n \end{pmatrix}\text{代替所得到的}\boldsymbol{n}\text{阶行列式.}$$

推论

$$\text{如果上述方程组中}, b_i = 0(i = 1, 2, \cdots, n), \text{即}$$
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = 0 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = 0 \\ \cdots\cdots\cdots \\ a_{n1}x_1 + a_{n2}x_2 + \cdots + a_{nn}x_n = 0 \end{cases}, \quad \text{则称为齐次线性方程组.}$$
$$\text{相应地}, b_i\text{不全为零时, 称为非齐次线性方程组.}$$

$$\text{若齐次线性方程组的系数行列式}D \neq 0, \text{则它只有零解.}$$
$$\text{若齐次线性方程组有非零解, 则其系数行列式}D = 0.$$

五、分块矩阵

1.定义

$$\text{设}\boldsymbol{A}\text{为数域}K\text{上的}m\times n\text{矩阵}$$
$$\boldsymbol{A}=\begin{pmatrix}a_{11}&a_{12}&\cdots&a_{1n}\\a_{21}&a_{22}&\cdots&a_{2n}\\\vdots&\vdots&&\vdots\\a_{m1}&a_{m2}&\cdots&a_{mn}\end{pmatrix}$$
$$\text{我们考虑对矩阵}\boldsymbol{A}\text{作适当的分块, 设}$$
$$m=m_1+m_2+\cdots+m_s,\quad m_i\geqslant 1,\quad 1\leqslant i\leqslant s,$$
$$n=n_1+n_2+\cdots+n_t,\quad n_j\geqslant 1,\quad 1\leqslant j\leqslant t.$$
$$\text{令}\boldsymbol{A}_{ij}\text{表示由}\boldsymbol{A}\text{的第}m_1+\cdots+m_{i-1}+1,m_1+\cdots+m_{i-1}+2,\cdots,m_1+\cdots+m_{i-1}+m_i\text{各行与}$$
$$\text{第}n_1+\cdots+n_{j-1}+1,n_1+\cdots+n_{j-1}+2,\cdots,n_1+\cdots+n_{j-1}+n_j\text{各列组成的}m_i\times n_j\text{子矩阵,}$$
$$\text{则}\boldsymbol{A}\text{可表示成以下分块矩阵形式：}\boldsymbol{A}=\begin{pmatrix}\boldsymbol{A}_{11}&\boldsymbol{A}_{12}&\cdots&\boldsymbol{A}_{1t}\\\boldsymbol{A}_{21}&\boldsymbol{A}_{22}&\cdots&\boldsymbol{A}_{2t}\\\vdots&\vdots&&\vdots\\\boldsymbol{A}_{s1}&\boldsymbol{A}_{s2}&\cdots&\boldsymbol{A}_{st}\end{pmatrix}$$

2.运算

1）线性运算

$$\text{设}\boldsymbol{A}\text{与}\boldsymbol{B}\text{为数域}K\text{上的两个}m\times n\text{矩阵.}$$
$$\text{设}m=m_1+m_2+\cdots+m_s,\quad m_i\geqslant 1,\quad 1\leqslant i\leqslant s,$$
$$n=n_1+n_2+\cdots+n_t,\quad n_j\geqslant 1,\quad 1\leqslant j\leqslant t.$$
$$\text{将}A\text{与}B\text{作如下分块：}\boldsymbol{A}=\begin{pmatrix}\boldsymbol{A}_{11}&\boldsymbol{A}_{12}&\cdots&\boldsymbol{A}_{1t}\\\boldsymbol{A}_{21}&\boldsymbol{A}_{22}&\cdots&\boldsymbol{A}_{2t}\\\vdots&\vdots&&\vdots\\\boldsymbol{A}_{s1}&\boldsymbol{A}_{s2}&\cdots&\boldsymbol{A}_{st}\end{pmatrix},\quad\boldsymbol{B}=\begin{pmatrix}\boldsymbol{B}_{11}&\boldsymbol{B}_{12}&\cdots&\boldsymbol{B}_{1t}\\\boldsymbol{B}_{21}&\boldsymbol{B}_{22}&\cdots&\boldsymbol{B}_{2t}\\\vdots&\vdots&&\vdots\\\boldsymbol{B}_{s1}&\boldsymbol{B}_{s2}&\cdots&\boldsymbol{B}_{st}\end{pmatrix}$$
$$\text{则有}\boldsymbol{A}+\boldsymbol{B}=\begin{pmatrix}\boldsymbol{A}_{11}+\boldsymbol{B}_{11}&\boldsymbol{A}_{12}+\boldsymbol{B}_{12}&\cdots&\boldsymbol{A}_{1t}+\boldsymbol{B}_{1t}\\\boldsymbol{A}_{21}+\boldsymbol{B}_{21}&\boldsymbol{A}_{22}+\boldsymbol{B}_{22}&\cdots&\boldsymbol{A}_{2t}+\boldsymbol{B}_{2t}\\\vdots&\vdots&&\vdots\\\boldsymbol{A}_{s1}+\boldsymbol{B}_{s1}&\boldsymbol{A}_{s2}+\boldsymbol{B}_{s2}&\cdots&\boldsymbol{A}_{st}+\boldsymbol{B}_{st}\end{pmatrix}$$

$$k\boldsymbol{A}=\begin{pmatrix}k\boldsymbol{A}_{11}&k\boldsymbol{A}_{12}&\cdots&k\boldsymbol{A}_{1t}\\k\boldsymbol{A}_{21}&k\boldsymbol{A}_{22}&\cdots&k\boldsymbol{A}_{2t}\\\vdots&\vdots&&\vdots\\k\boldsymbol{A}_{s1}&k\boldsymbol{A}_{s2}&\cdots&k\boldsymbol{A}_{st}\end{pmatrix}$$

2）乘法

$$
\begin{gathered}
\text{设}\boldsymbol{A}\text{与}\boldsymbol{B}\text{分别为数域}K\text{上}m\times n\text{矩阵与}n\times p\text{矩阵},\\
\text{设}m=m_1+m_2+\cdots+m_s,\quad m_i\geqslant 1,\quad 1\leqslant i\leqslant s,\\
n=n_1+n_2+\cdots+n_t,\quad n_j\geqslant 1,\quad 1\leqslant j\leqslant t,\\
p=p_1+n_2+\cdots+p_u,\quad p_k\geqslant 1,\quad 1\leqslant k\leqslant u.\\
\boldsymbol{A}=\begin{pmatrix}\boldsymbol{A}_{11}&\boldsymbol{A}_{12}&\cdots&\boldsymbol{A}_{1t}\\\boldsymbol{A}_{21}&\boldsymbol{A}_{22}&\cdots&\boldsymbol{A}_{2t}\\\vdots&\vdots&&\vdots\\\boldsymbol{A}_{s1}&\boldsymbol{A}_{s2}&\cdots&\boldsymbol{A}_{st}\end{pmatrix},\quad \boldsymbol{B}=\begin{pmatrix}\boldsymbol{B}_{11}&\boldsymbol{B}_{12}&\cdots&\boldsymbol{B}_{1u}\\\boldsymbol{B}_{21}&\boldsymbol{B}_{22}&\cdots&\boldsymbol{B}_{2u}\\\vdots&\vdots&&\vdots\\\boldsymbol{B}_{t1}&\boldsymbol{B}_{t2}&\cdots&\boldsymbol{B}_{tu}\end{pmatrix}\\
\text{其中}\boldsymbol{A}_{ij}\text{为}m_i\times n_j\text{矩阵},\boldsymbol{B}_{jk}\text{为}n_j\times p_k\text{矩阵. 对}1\leqslant i\leqslant s,1\leqslant k\leqslant u,\text{令}\\
\boldsymbol{C}_{ik}=\sum_{j=1}^{t}\boldsymbol{A}_{ij}\boldsymbol{B}_{jk}=\boldsymbol{A}_{i1}\boldsymbol{B}_{1k}+\boldsymbol{A}_{i2}\boldsymbol{B}_{2k}+\cdots+\boldsymbol{A}_{it}\boldsymbol{B}_{tk}C=\begin{pmatrix}C_{11}&C_{12}&\cdots&C_{1u}\\C_{21}&C_{22}&\cdots&C_{2u}\\\vdots&\vdots&&\vdots\\C_{s1}&C_{s2}&\cdots&C_{su}\end{pmatrix}\\
\text{则}C=AB.
\end{gathered}
$$

2.性质

1)

$$
\begin{gathered}
\text{若}\boldsymbol{A}_{m\times l},\boldsymbol{B}_{l\times n},\\
\text{记}\boldsymbol{B}=(\beta_1,\beta_2,\cdots,\beta_n),\text{其中}\beta_j\text{为}\boldsymbol{B}\text{的第}\boldsymbol{j}\text{列},\\
\text{记}\boldsymbol{A}=\left(\alpha_1^{\mathrm{T}},\alpha_2^{\mathrm{T}},\cdots,\alpha_m^{\mathrm{T}}\right)^{\mathrm{T}},\text{其中}\alpha_i\text{是}\boldsymbol{A}\text{的第}\boldsymbol{i}\text{行, 则}\\
\boldsymbol{AB}=\boldsymbol{A}(\beta_1,\beta_2,\cdots,\beta_n)=(\boldsymbol{A}\beta_1,\boldsymbol{A}\beta_2,\cdots,\boldsymbol{A}\beta_n),\\
\boldsymbol{AB}=\begin{bmatrix}\alpha_1\\\alpha_2\\\vdots\\\alpha_m\end{bmatrix}\boldsymbol{B}=\begin{bmatrix}\alpha_1\boldsymbol{B}\\\alpha_2\boldsymbol{B}\\\vdots\\\alpha_m\boldsymbol{B}\end{bmatrix}
\end{gathered}
$$

2)

$$
\begin{gathered}
2)\text{若方阵}\boldsymbol{A}=\begin{bmatrix}\boldsymbol{A}_1&0\\\mathbf{0}&\boldsymbol{A}_2\end{bmatrix},\text{其中}\boldsymbol{A}_1,\boldsymbol{A}_2\text{也是方阵, 则}\boldsymbol{A}\text{为准对角阵, 有}\\
(1)\ |\boldsymbol{A}|=|\boldsymbol{A}_1|\,|\boldsymbol{A}_2|.\\
(2)\ \text{若}\boldsymbol{A}\text{可逆},\boldsymbol{A}^{-1}=\begin{bmatrix}\boldsymbol{A}_1&\mathbf{0}\\\mathbf{0}&\boldsymbol{A}_2\end{bmatrix}^{-1}=\begin{bmatrix}\boldsymbol{A}_1^{-1}&0\\\mathbf{0}&\boldsymbol{A}_2^{-1}\end{bmatrix}.\\
(3)\ \boldsymbol{A}^*=\begin{bmatrix}\boldsymbol{A}_1&\mathbf{0}\\\mathbf{0}&\boldsymbol{A}_2\end{bmatrix}^*=\begin{bmatrix}|\boldsymbol{A}_2|\boldsymbol{A}_1^*&0\\0&|\boldsymbol{A}_1|\boldsymbol{A}_2^*\end{bmatrix}.
\end{gathered}
$$