22. 设：V为n维线性空间，
$\alpha_1, \alpha_2 \cdots \alpha_n$ 一组基，
σ为V上线性变换，
$\sigma(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n) A$
证明：$\dim(Im(\sigma)) = r(A)$，$\dim(Ker(\sigma)) = n - r(A)$

设A为σ在基$\alpha_1 \cdots \alpha_n$下的矩阵，即
$\sigma(\alpha_1, \alpha_2 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n) A$
任意向量$\alpha \in V$，设α在基下坐标为x，即
$$\alpha = (\alpha_1 \cdots \alpha_n) x$$
$$\therefore Im(\sigma) = \{\sigma(\alpha) \mid \alpha \in V\}$$
$$= \{\sigma(\alpha_1 \cdots \alpha_n) x \mid x \in R^n\}$$
$$= \{(\alpha_1 \cdots \alpha_n) A x \mid x \in R^n\}$$

$$\{Ax \mid x \in R^n\} = L(\theta_1, \theta_2 \cdots \theta_n)$$
其中 $A = (\theta_1, \theta_2 \cdots \theta_n)$

20 × 20

$$\therefore \dim(Im(\sigma)) = r(A)$$

求 $\mathrm{Im}(\sigma)$ 的步骤:

① 取一组基 $\alpha_1 \cdots \alpha_n$

写出 σ 在 $\alpha_1 \cdots \alpha_n$ 下矩阵 A

即 $\sigma(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n) A$

② 求 $r(A)$, 则 $\dim(\mathrm{Im}(\sigma)) = r(A)$

③ 求 A 的列向量的一个极大线性无关组, 设为 $\theta_1, \theta_2 \cdots \theta_r$

④ $\mathrm{Im}(\sigma)$ 的基为

$$(\alpha_1, \alpha_2 \cdots \alpha_n)\theta_1$$
$$(\alpha_1, \alpha_2 \cdots \alpha_n)\theta_2$$
$$\vdots$$
$$(\alpha_1, \alpha_2 \cdots \alpha_n)\theta_r$$

$$= (\alpha_1 \cdots \alpha_n) A X$$
$$\sigma(\alpha_1 \cdots \alpha_n) X = 0$$

$$\mathrm{Ker}(\sigma) = \{\alpha \mid \alpha \in V, \sigma(\alpha) = 0\}$$
$$= \{(\alpha_1 \cdots \alpha_n)x \mid x \in R^n, Ax = 0\}$$
$$\Updownarrow$$
$$\{x \mid x \in R^n, Ax = 0\}$$

$\dim(\mathrm{Ker}(\sigma)) = n - r(A)$

求 $\mathrm{Ker}(\sigma)$ 的步骤

① $\sigma(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n) A$

② 求 $AX=0$ 的基础解系、设为 $\xi_1\ \xi_2 \cdots \xi_{n-r(A)}$

③ $\mathrm{Ker}(\sigma)$ 的基为

$(\alpha_1 \cdots \alpha_n)\xi_1$

$(\alpha_1 \cdots \alpha_n)\xi_2$

$\cdots$

$(\alpha_1 \cdots \alpha_n)\xi_{n-r(A)}$

$\mathrm{End}_V = \{\sigma \mid \sigma \text{是} V \text{到} V \text{的线性变换}\}$

取 $\sigma, \tau \in \mathrm{End}_V$.

定义: $\sigma+\tau$

任何 $\alpha \in V$, $(\sigma+\tau)(\alpha) = \sigma(\alpha)+\tau(\alpha)$

20 × 20

② 验证保持加法：$\alpha, \beta \in V$

$$
\begin{aligned}
&(\sigma+\tau)(\alpha+\beta) \\
&\triangleq \sigma(\alpha+\beta)+\tau(\alpha+\beta) \\
&= \sigma(\alpha)+\sigma(\beta)+\tau(\alpha)+\tau(\beta) \\
&\triangleq (\sigma+\tau)(\alpha)+(\sigma+\tau)(\beta)
\end{aligned}
$$

$$
\begin{aligned}
(\sigma+\tau)(k\alpha) &\triangleq \sigma(k\alpha)+\tau(k\alpha) \\
&= k\sigma(\alpha)+k\tau(\alpha) \\
&= k(\sigma(\alpha)+\tau(\alpha)) \\
&\triangleq k(\sigma+\tau)(\alpha)
\end{aligned}
$$

定义：$(k\sigma)(\alpha) = k\cdot\sigma(\alpha)$

$$
\begin{aligned}
(k\sigma)(\alpha+\beta) &\triangleq k\cdot\sigma(\alpha+\beta) \\
&= k(\sigma(\alpha)+\sigma(\beta)) \\
&= k\sigma(\alpha)+k\sigma(\beta) \\
&\triangleq (k\sigma)(\alpha)+(k\sigma)(\beta)
\end{aligned}
$$

End_V

关于加法、数乘构成线性空间

1-1映射 f. 同构.

$R^{n\times n}$

$$(\sigma\tau)(\alpha)=\sigma(\tau(\alpha))$$

$$(\sigma\tau)(\alpha+\beta)\triangleq\sigma(\tau(\alpha+\beta))$$
$$=\sigma(\tau(\alpha)+\tau(\beta))$$
$$=\sigma(\tau(\alpha))+\sigma(\tau(\beta))$$
$$\triangleq(\sigma\tau)(\alpha)+(\sigma\tau)(\beta)$$

已知 $\sigma(\alpha_1\cdots\alpha_n)=(\alpha_1\cdots\alpha_n)A$

$\tau(\alpha_1\cdots\alpha_n)=(\alpha_1\cdots\alpha_n)B$.

$$(\sigma+\tau)(\alpha_1\cdots\alpha_n)=\sigma(\alpha_1\cdots\alpha_n)+\tau(\alpha_1\cdots\alpha_n)$$
$$=(\alpha_1\cdots\alpha_n)A+(\alpha_1\cdots\alpha_n)B$$

20 × 20

$$= (\alpha_1 \cdots \alpha_n)(A+B)$$

$$(k\sigma)(\alpha_1 \cdots \alpha_n) = (\alpha_1 \cdots \alpha_n)\, kA$$

$$(\sigma\tau)(\alpha_1 \cdots \alpha_n) = \sigma\left(\tau(\alpha_1 \cdots \alpha_n)\right)$$

$$= \sigma\left((\alpha_1 \cdots \alpha_n)B\right)$$

$$= \left(\sigma(\alpha_1 \cdots \alpha_n)\right)B$$

$$= (\alpha_1 \cdots \alpha_n)AB$$

20 × 20