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Risk622 Coursework:
Reliability Analysis of Systems with Advanced Monte Carlo Methods

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Introduction

In performance critical systems, such as advanced engineering projects in nuclear or chemical industries, reliability analysis is critical to understanding how the system might fail. One method to analyze reliability is to construct a performance function $g(\boldsymbol{X})$ such that the system is considered to have failed when g enters a certain region D_f. $\boldsymbol{X}$ is a vector of random variables describing the state of the system. A common design of performance function takes the form 'capacity' minus 'demand'. In this form $g(\boldsymbol{X}) = 0$ is the limit state function, and when $g(\boldsymbol{X}) \leq 0$ the system has failed. From this $P(g(\boldsymbol{X}) \leq 0) = p_f$, the probability of failure, can be found.

$$p_f = \int_{D_f} g(\boldsymbol{X})\, dx$$

Equation 1

Computing p_f analytically can become infeasible when g becomes complex or depends on many random variables as it requires solving the integral equation given in equation 1. Therefore numerical methods are usually used.

The Monte Carlo Method randomly samples from g and tests the result with an indicator function I such that $I = 0$ if $g > 0$ or $I = 1$ if $g \leq 0$. The ratio of the sum of all I over the sample size N is an estimator for p_f.

$$\widehat{p_f} = \frac{1}{N}\sum_{i=1}^{N} I\big(g(\boldsymbol{X}_i)\big)$$

Equation 2

This report will analyse a performance function using a Monte Carlo Simulation, and compare it with two advanced methods, Subset Simulation, and Line Sampling.

The performance function is

$$g(\boldsymbol{X}) = 2 - X_2 + (4\,X_1)^4$$

Equation 3

where $X_{1,2} \sim N(0,1)$.

Advanced Monte Carlo Methods - Review

History

The idea of using random sampling to estimate parameters has been around since at least the 18th century with Buffon's Needle experiment (Metropolis, 1987) but it wasn't until the invention of modern computers during world war two that the method became doable, first being done by John Von Neumann during the Manhattan Project (Kalos & Whitlock, 2008). There is no formal definition for the method, but it is usually applied to describe a process of using a *large number* of random samples to simulate a probabilistic model.

Monte Carlo Simulations (MCSs) require a number of samples of an order inverse to the order of the probability in question. This makes computing small probabilities very computationally intensive. In recent years advanced methods have been found that can find small probabilities with a smaller number of samples. Subset Simulation (SS) and Line sampling (LS) are two such methods. (E. Zio, 2009).

Subset Simulation

Subset simulation considers the fact that a rare event is a subset of less rare events and repeats over multiple *conditional levels* using conditional probabilities to reduce sample sizes by reduced the probability of failure at each level.

$$P(F) = P(F_1)\prod_{i=1}^{m-1} P(F_{i+1}|F_i)$$

Equation 4

If F is a failure event, we can safely assume F is within an intermediate event F_m which itself can be in F_{m-1} and so on until F_1. This means if P_f is small enough as to be difficult to compute, we can chose a large enough m such that the conditional probabilities $P(F_{i+1}\,|F_i)$ are computable. (S. K. Au, 2001). The conditional pdfs needed to find these probabilities are derived from a Markov Chain. If F corresponds to $g \leq 0$ for instance, where $g(\boldsymbol{X})$ if a performance function, then F_i corresponds to $g \leq y_i$ where $y_1 > y_2 > \cdots > y_m$. m is often chosen such that each intermediate probability ≈ 0.1 as this has been shown to be efficient (S. K. Au, 2001).

Line Sampling

To reduce the number of samples needed to compute an unlikely probability, line sampling replaces a random sample in the input space to a sample along line. This also reduced the number of dimensions in the problem to one.

First the input space needs to be transformed so that each input variable becomes a standard normal variable. This can be done with Rosenblatt's or Nataf's transformations, but is a linear transformation if the original input variables are normal (B Huang, 2006). This transforms input vector $\boldsymbol{X}$ into $\boldsymbol{\theta}$. Computing the performance function $g(\boldsymbol{\theta})$ requires mapping back into the input space and applying g. A line in then found pointing towards the failure region, specifically a point α. α can be the nearest point on the limit state function ($g(\boldsymbol{X}) = 0$), it could be the centre of mass of the failure regions, or it could point in the direction of steepest negative gradient of g. The point is then normalised to a unit directional vector defining the line used for the sample. Samples are taken a long this one-dimensional line and used to estimate P_f. (E. Zio, 2009).

Analysis

The performance function in equation 4 is

$$g(\boldsymbol{X}) = 2 - X_2 + (4\,X_1)^4$$

where $X_{1,2} \sim N(0,1)$.

Comparison

To perform advanced Monte Carlo analysis the OpenCossan MATLAB toolbox is used. The number of samples is fixed to 1,00,000. The number of subsets for the SS method is set at 4. The number of lines for the LS methods is set at 20. All simulations are ran 10 times. The results are show in table 1 and plotted in figure 1.

Method	P(fail)										Runtime
MCS	3.18E-03	3.23E-03	3.18E-03	3.24E-03	3.21E-03	3.22E-03	3.25E-03	3.22E-03	3.31E-03	3.19E-03	~12s
SS	5.49E-03	3.98E-03	2.24E-03	2.47E-03	1.94E-03	3.46E-03	4.51E-03	3.04E-03	2.36E-03	1.87E-03	~0.6s
ls	2.05E-03	5.53E-07	3.63E-03	1.82E-03	1.50E-03	2.58E-03	2.39E-03	3.42E-03	3.74E-03	1.27E-03	~0.5s

Table 1

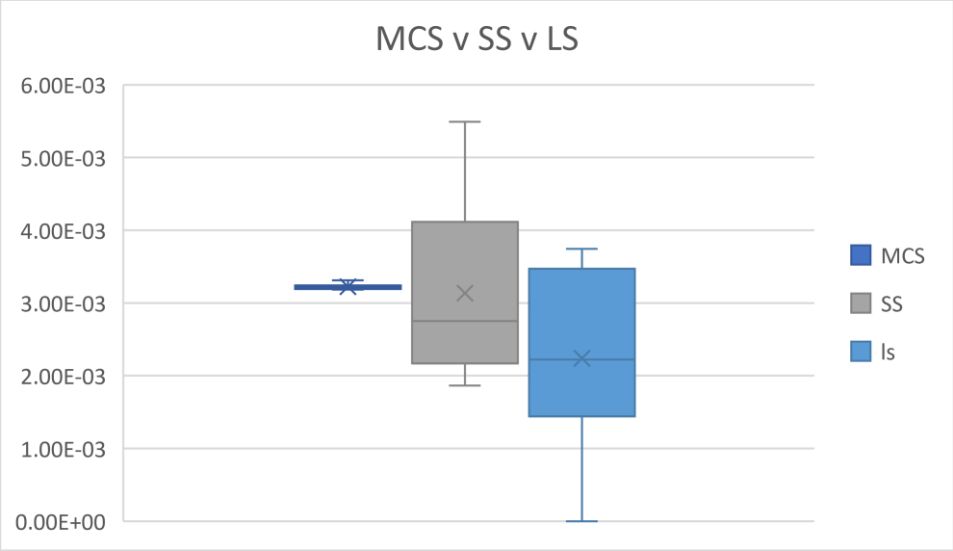

Figure 1

At 1,000,000 samples the traditional MCS is able to consistently find a failure probability about 3.2×10^{-3} with a smaller margin of error. This will be used as the reference solution.

The SS method approximately agrees with this on average however the margins of error are larger. The method generally varies greatly but with the much shorter run time it could be repeated and averaged to get a more accurate solution.

The LS method on the other hand consistently under-estimates the failure probability and has a greater range. In this example LS cannot be used for a reliable estimation.

Effect of the number of intermediate levels on the SS method

The sample size is kept at 1,000,000 and the number of intermediate levels is varied from 1 to 6 on the SS method. The results are shown in table 2 and plotted in figure 2. Each level is repeated 10 times.

Number of levels	P(fail)									
1	4.00E-04	2.00E-03	1.20E-02	2.00E-03	0.00E+00	0.00E+00	0.00E+00	0.00E+00	6.00E-03	6.00E-03
2	1.80E-03	3.40E-03	3.40E-04	4.00E-04	5.40E-03	2.80E-03	2.20E-03	5.80E-03	1.00E-03	8.00E-04
3	6.63E-03	1.98E-03	3.31E-03	2.51E-03	7.71E-03	3.39E-03	6.96E-03	5.60E-03	5.61E-03	2.40E-03
4	5.49E-03	3.98E-03	2.24E-03	2.47E-03	1.94E-03	3.46E-03	4.51E-03	3.04E-03	2.36E-03	1.87E-03
5	2.12E-03	3.77E-03	3.37E-03	2.74E-03	3.16E-03	3.10E-03	1.56E-03	3.84E-03	2.78E-03	6.38E-03
6	2.44E-03	4.68E-03	2.99E-03	3.04E-03	3.08E-03	4.67E-03	2.06E-03	3.06E-03	7.48E-03	4.28E-03

Table 2

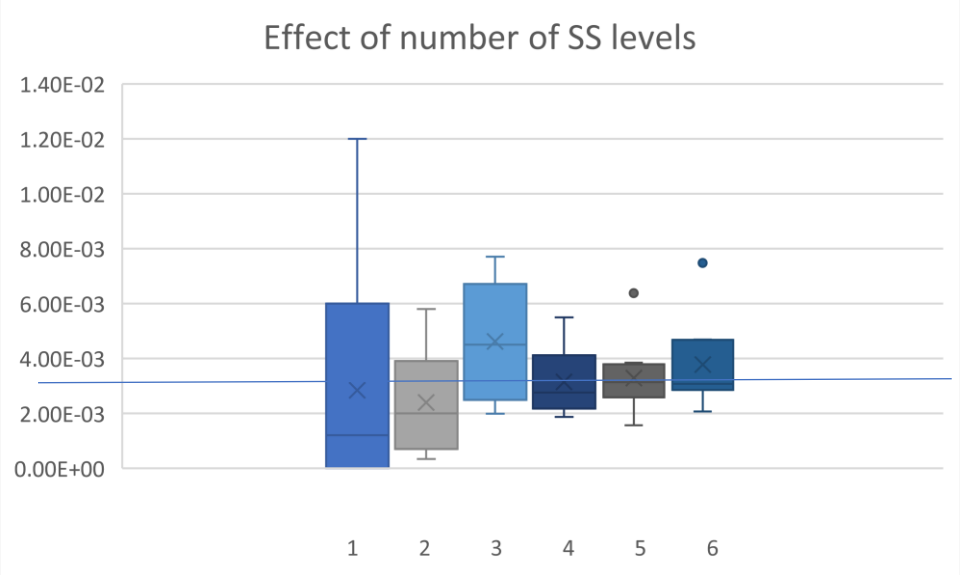

Figure 2

Accuracy improves with the number of levels and is most accurate with 4 and 5 levels in this data. The reference value is marked with a horizontal blue line. As discussed before the probability at each level should be about 0.1, this would imply level 3 would be the most accurate but this data suits an extra level or two.

Effect of the number of lines on the LS method

The number of samples is fixed at 1,000,000 and the number of lines is varies through 1, 5, 10, 15, 20, and 25 lines. Each number is repeated 10 times. The results are shown in table 3 and plotted in figure 3.

Number of lines	P(fail)									
1	0	0	5.68E-03	0	0	0	0	5.21E-03	0	0
5	9.08E-03	5.90E-03	4.57E-03	9.07E-04	1.25E-02	7.05E-03	1.09E-04	8.58E-03	2.50E-05	3.65E-03
10	2.68E-03	5.68E-03	2.09E-03	6.81E-03	2.46E-03	7.51E-03	2.06E-03	2.34E-03	4.16E-03	2.53E-03
15	2.61E-07	7.22E-04	1.50E-03	6.46E-03	2.86E-03	2.66E-03	2.07E-03	3.01E-03	5.93E-03	2.56E-03
20	2.05E-03	5.53E-07	3.63E-03	1.82E-03	1.50E-03	2.58E-03	2.39E-03	3.42E-03	3.74E-03	1.27E-03
25	9.38E-04	3.77E-03	3.39E-03	4.99E-03	3.06E-03	5.32E-03	4.11E-03	2.64E-03	3.61E-03	1.61E-03

Table 3

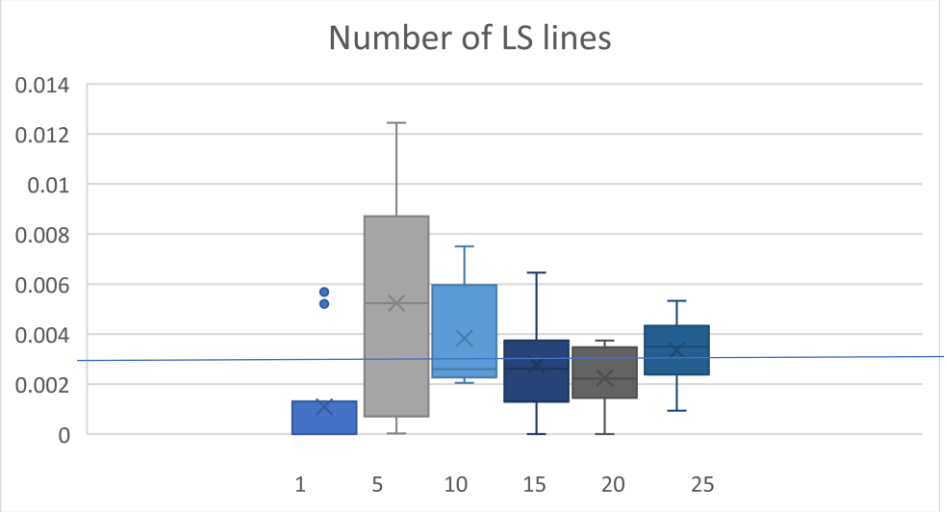

Figure 3

The LS method continues to show inconsistency in its predicted values even over different numbers of lines. The range does decrease as the number of lines increases but never enough for an accurate prediction, at least with this data.

Effect of Sample Size on MCS

On a MCS, the sample size is varied. Each size is repeated 10 times. The results are shown in table 4 and plotted in figure 4.

Sample Size	P(fail)									
1000	0.006	0	0.006	0.007	0.002	0.004	0.002	0.004	0.004	0.005
5000	0.0036	0.0022	0.0026	0.0034	0.0042	0.0026	0.0042	0.0026	0.0044	0.004
10000	0.003	0.0038	0.0031	0.0025	0.0036	0.0028	0.0034	0.003	0.0034	0.0025
15000	0.00293333	0.003533	0.0032	0.003067	0.003	0.002867	0.003	0.003333	0.003067	0.0032
20000	0.0039	0.00325	0.00375	0.0039	0.00265	0.003	0.00325	0.0032	0.003	0.0035
100000	0.00373	0.0034	0.00314	0.00313	0.00308	0.00316	0.00338	0.00351	0.0032	0.00282
500000	0.003168	0.003182	0.003156	0.003114	0.00326	0.003176	0.00325	0.003346	0.003192	0.00332
1000000	0.003177	0.003184	0.003255	0.00323	0.003303	0.003209	0.003193	0.00329	0.003316	0.003243

Table 4

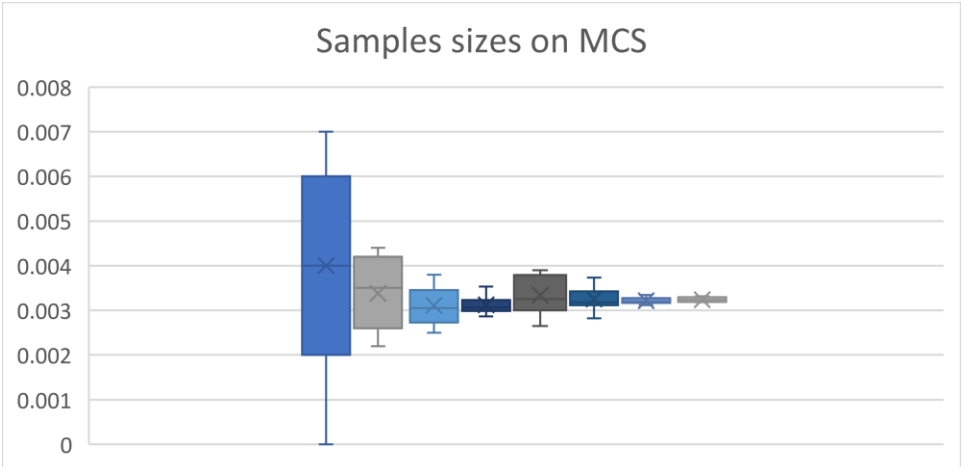

Figure 4

Its is clear and makes intuitive sense that as the sample size increase so does the accuracy of the predicted value, converging towards 3.2×10^{-3}.

Conclusions

The Monte Carlo method has been around as long as modern computers have. It is a simple but reliable method for simulating complex probabilistic models. From these comparisons it is difficult to recommend Subset Simulation or Line Sampling unless reducing the sample size is a priority. Even then it would make sense to repeat the simulation multiple times and take the mean. Subset simulation does achieve decent accuracy especially if the number of levels is chosen to fit the expected order of magnitude of the probability, but that might not be known. The tradition approach to the Monte Carlo method should always be preferred.

References

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