Local Report

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2.1. Data Cleaning and Exploratory Analysis

In this study we first work with a dataset featuring 703 observed hurricanes in the North Atlantic since 1950, and 8 variables including the location (longitude and latitude) and maximum wind speed every 6 hours for each hurricanes. To verify how well our estimated Bayesian model fits the data, we partition this data set - 80% of the observations for each hurricane (by ID) will be used as a training data set while the remaining 20% will be used as testing observations.

The second data set contains the damages and deaths caused by 46 hurricanes in the United States. There are 14 variables and 43 observations.

2.2. Posterior Distributions

The following Bayesian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where $Y_i(t)$ the wind speed at time t (i.e. 6 hours earlier), $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ are the changes of latitude, longitude and wind speed between t and $t+6$, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{7,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim \mathcal{N}(\boldsymbol{\beta}, \boldsymbol{\Sigma}),$$

and we assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\beta}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)}\exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

Note from given Bayesian model:

$$\epsilon_i(t) = Y_i(t+6) - \left(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\right) \overset{i.i.d}{\sim} N(0,\sigma^2)$$

or

$$Y_i(t+6) \sim N(\boldsymbol{X}_i(t)\boldsymbol{\beta}_i^T, \sigma^2)$$

where $\boldsymbol{X}_i(t) = (1, Y_i(t), \Delta_{i,1}(t), \Delta_{i,2}(t), \Delta_{i,3}(t))$, and $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, \beta_{2,i}, \beta_{3,i}, \beta_{4,i})$. Therefore,

$$f_{Y_i(t+6)}(y_i(t+6) \mid \boldsymbol{X}_i(t), \boldsymbol{\beta}_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{ -\frac{1}{2\sigma^2}\left(y_i(t+6) - \boldsymbol{X}_i(t)\boldsymbol{\beta}_i^T\right)^2\right\}$$

for hurricane i at time t. To show the likelihood function for hurricane i across all time points, t, we can write the multivariate normal distribution

$$(\boldsymbol{Y}_i \mid \boldsymbol{X}_i, \boldsymbol{\beta}_i, \sigma^2) \sim \mathcal{N}(\boldsymbol{X}_i\boldsymbol{\beta}_i^T, \sigma^2 I)$$

where Y_i is an m_i-dimensional vector and $\boldsymbol{X}_i$ is a $m_i \times d$ matrix. Finally, the joint likelihood function of all hurricanes can be expressed as

$$L_Y(\mathbf{B}, \sigma^2 I) = \prod_{i=1}^{n}\left\{ \det(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^T)^T(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^T)\right)\right\},$$

where I is an identity matrix with dimension consistent with Y_i. We can find the posterior distribution for Θ by

$$\pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma^{-1} \mid Y) \propto L_Y(\mathbf{B}, \sigma^2 I) \times \pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma^{-1}) \times \pi(\boldsymbol{\beta}) \times \pi(\sigma^2) \times \pi(\Sigma^{-1}),$$

where $\pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma)$ is the joint multivariate normal density of β,

$$\pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma^{-1}) = \prod_{i=1}^{n}\left\{ \det(2\pi\Sigma)^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})\Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^T)\right\}.$$

So we have the following joint posterior distribution of Θ:

$$\pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma^{-1} \mid Y) \propto \prod_{i=1}^{n}\left\{ (2\pi\sigma^2)^{-m_i/2} \exp\left\{ -\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^T)^T(\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i\boldsymbol{\beta}_i^T)\right\}\right\}$$

$$\times \prod_{i=1}^{n}\left\{ \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left\{ -\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})\Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^T\right\}\right\} \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp\left\{ -\frac{1}{2}\Sigma^{-1}\right\}.$$

We can use the joint posterior distribution to derive conditional posterior distributions for each of our parameters.

Let $\tau = \sigma^2$, then

$$(\tau|\beta, B, \Sigma^{-1}, Y) \propto \tau^{1 + \frac{\sum_{i=1}^{n} m_i}{2}} exp(-\tau \times \frac{1}{2}\sum_{i=1}^{n}(Y_i - X_i\beta_i^T)^T I^{-1}(Y_i - X_i\beta_i^T)$$

Thus, σ^2 is from inverse-gamma distribution

$$\sigma^2 \sim \text{Inv-Gamma}(\frac{\sum_{i=1}^{n} m_i}{2}, \frac{1}{2}\sum_{i=1}^{n}(Y_i - X_i\beta_i^T)(Y_i - X_i\beta_i^T)).$$

Parameter $\mathbf{B}$ has the following conditional posterior:

$$\pi(B|\beta, \sigma^2, \Sigma^{-1}, Y) \propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I)^{-1}(Y_i - X_i\beta_i^T) + (\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T\right]) \quad (1)$$

$$\propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[\beta_i(X_i^T(\sigma^2 I)^{-1}X_i + \Sigma^{-1})\beta_i^T - 2\beta_i(X_i(\sigma^2 I)^{-1})Y_i + \Sigma^{-1}\beta^T\right]) \quad (2)$$

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Let $V_i = X_i^T(\sigma^2 I)^{-1})X_i + \Sigma^{-1}$, and $U_i = X_i(\sigma^2 I)^{-1}Y_i + \Sigma^{-1}\beta^T$, then

$$\boldsymbol{\beta}_i \sim \mathcal{MVN}(V_i^{-1}U_i, V_i^{-1}).$$

Similarly, parameter $\boldsymbol{\beta}$ has a conditional posterior of:

$$\pi(\beta|B, \sigma^2, \Sigma^{-1}, Y) \propto exp(-\frac{1}{2}\sum_{i=1}^{n}(\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T) \tag{3}$$

$$\propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[\beta\Sigma^{-1}\beta^T - 2\beta\Sigma^{-1}\beta_i^T\right]) \tag{4}$$

Let $V = n\Sigma^{-1}, U = \sum_{i=1}^{n}\Sigma^{-1}\beta_i^T$, then

$$(\boldsymbol{\beta}|B, \sigma^2, \Sigma^{-1}, Y) \sim \mathcal{MVN}(V^{-1}U, V^{-1}).$$

Finally, parameter Σ has the conditional posterior:

$$\pi(\Sigma^{-1}|\beta, B, \sigma^2, Y) \propto |\Sigma|^{-(d+1)}exp(-\frac{1}{2}tr(\Sigma^{-1})|\Sigma|^{-\frac{n}{2}}exp(-\frac{1}{2}\sum_{i=1}^{n}(\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T) \tag{5}$$

$$\propto |\Sigma|^{-(d+1+\frac{n}{2})}exp(-\frac{1}{2}\left[tr(\Sigma^{-1}) + tr((\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T)\right] \tag{6}$$

$$\propto |\Sigma|^{\frac{-(d+1+n+d+1)}{2}}exp(-\frac{1}{2}tr(\left[I + \sum_{i=1}^{n}(\beta_i - \beta)^T(\beta_i - \beta)\right]\Sigma^{-1})) \tag{7}$$

Thus,

$$\Sigma \sim \mathcal{W}_d^{-1}(\Psi, v),$$

where $v = d + 1 + n$, and $\Psi = I + \sum_{i=1}^{n}(\beta_i - \beta)^T(\beta_i - \beta)$.

2.3. Gibbs Sampling Algorithm

Now that we have conditional posterior distributions for each of our parameters, we can utilize the Gibbs Sampling MCMC algorithm to estimate model parameters. In Gibbs sampling, we use starting values $(\boldsymbol{\beta}_0, \Sigma_0, \sigma_0^2, \mathbf{B}_0)$ and for each $j = 1, 2, ..., n$:

1. Generate $\boldsymbol{\beta}_j$ from $\pi(\boldsymbol{\beta} \mid \Sigma = \Sigma_{j-1}, \sigma^2 = \sigma_{j-1}^2, \mathbf{B} = \mathbf{B}_{j-1})$;

2. Generate Σ_j from $\pi(\Sigma \mid \boldsymbol{\beta} = \boldsymbol{\beta}_j, \sigma^2 = \sigma_{j-1}^2, \mathbf{B} = \mathbf{B}_{j-1})$;

3. Generate σ^2 from $\pi(\sigma^2 \mid \boldsymbol{\beta} = \boldsymbol{\beta}_j, \Sigma = \Sigma_j, \mathbf{B} = \mathbf{B}_{j-1})$;

4. Generate $\mathbf{B}$ from $\pi(\mathbf{B} \mid \boldsymbol{\beta} = \boldsymbol{\beta}_j, \Sigma = \Sigma_j \sigma^2 = \sigma_j^2)$

to yield Θ_j. As j increases and the Markov Chain continues, estimates stabilize to yield our results.