P8160 - Hurricane Project

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4/26/2022

Given Information

The following Bayesian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where $Y_i(t)$ the wind speed at time t (i.e. 6 hours earlier), $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ are the changes of latitude, longitude and wind speed between t and $t+6$, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2, independent across t.

In the model, $\boldsymbol{\beta}_i = (\beta_{0,i}, \beta_{1,i}, ..., \beta_{7,i})$ are the random coefficients associated the ith hurricane, we assume that

$$\boldsymbol{\beta}_i \sim N(\boldsymbol{\beta}, \boldsymbol{\Sigma})$$

follows a multivariate normal distributions with mean $\boldsymbol{\beta}$ and covariance matrix Σ.

We assume the following non-informative or weak prior distributions for σ^2, $\boldsymbol{\beta}$ and Σ.

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\boldsymbol{\beta}) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1})$$

d is dimension of β.

Task 1:

Let $\mathbf{B} = (\boldsymbol{\beta}_1^\top, ..., \boldsymbol{\beta}_n^\top)^\top$, derive the posterior distribution of the parameters $\Theta = (\mathbf{B}^\top, \boldsymbol{\beta}^\top, \sigma^2, \Sigma)$.

Note from given Bayesian model:

$$\epsilon_i(t) = Y_i(t+6) - \Big(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\Big) \overset{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \overset{i.i.d}{\sim} N(\boldsymbol{X}_i(t)^\top \boldsymbol{\beta}_i, \sigma^2)$$

where $\boldsymbol{X}_i(t)^\top \boldsymbol{\beta}_i = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,1}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)$. Therefore,

$$f_{Y_i(t+6)}(Y_i \mid \boldsymbol{\beta}_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\Big\{-\frac{1}{2\sigma^2}\Big[Y_i(t+6) - \Big(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\Big)\Big]^2\Big\}$$

for hurricane i at time t. To show the likelihood function for hurricane i across all time points, t, we can write the multivariate normal distribution

$$f_{Y_i}(Y_i \mid \boldsymbol{\beta}_i, \sigma^2) \sim N(\boldsymbol{X}_i^\top \boldsymbol{\beta}_i, \sigma^2 I).$$

Finally, the joint likelihood function of all hurricanes can be expresses as

$$L_{Y_i}(\boldsymbol{\beta}_i, \sigma^2 I) = \prod_{i=1}^n \det(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp\Big(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i)^\top (\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i)\Big).$$

We can find the posterior distribution for Θ by

$$\pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto L_{Y_i}(\boldsymbol{\beta}_i, \sigma^2 I) \times \pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma) \times \pi(\boldsymbol{\beta}) \times \pi(\sigma^2) \times \pi(\Sigma),$$

where $\pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma)$ is the joint multivariate normal density of β,

$$\pi(\mathbf{B} \mid \boldsymbol{\beta}, \Sigma) = \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})).$$

So we have the following posterior distribution, from which we can derive conditional posteriors to perform Gibbs sampling.

$$\pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto \prod_{i=1}^n \det(2\pi\sigma^2 I)^{-\frac{1}{2}} \exp\Big\{-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i)^\top (\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i)\Big\}$$
$$\times \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\Big\{-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})\Big\} \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp\Big\{-\frac{1}{2}\Sigma^{-1}\Big\}$$

Rearranged,

$$\pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto \det(2\pi\Sigma)^{-1/2} \det(2\pi\sigma^2 I)^{-1/2} \frac{1}{\sigma^2} |\Sigma|^{-(d+1)} \prod_{i=1}^n \exp\Big(-\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i)^\top (\sigma^2 I)^{-1}(\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i) - \frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-$$

And therefore,

$$\log \pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto -\frac{1}{2}\log(\det(2\pi\Sigma)) - \frac{1}{2}\log(\det(2\pi\sigma^2 I)) - 2\log\sigma - (d+1)\log|\Sigma| +$$

$$\sum_{i=1}^{n} -\frac{1}{2}(\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i)^\top (\sigma^2 I)^{-1} (\boldsymbol{Y}_i - \boldsymbol{X}_i^\top \boldsymbol{\beta}_i) - \frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1} (\boldsymbol{\beta}_i - \boldsymbol{\beta}).$$

To find the conditional posterior distributions for Gibbs sampling, we can consider:

$$\pi(\boldsymbol{\beta} \mid \mathbf{B}, \sigma^2, \Sigma, Y_i) = \frac{\pi(\beta, \mathbf{B}, \sigma^2, \Sigma \mid Y_i)}{\pi(\mathbf{B}, \sigma^2, \Sigma)} \propto ??$$

In the equation above, $\pi(\mathbf{B}, \sigma^2, \Sigma)$ only depends on $\mathbf{B}, \sigma^2, \Sigma$, so we can focus only on the part of the joint posterior that depends on all four variables β, $\mathbf{B}$, σ^2, and Σ.

$$\begin{aligned}
\pi(\boldsymbol{\beta} \mid \mathbf{B}, \sigma^2, \Sigma, Y_i) &\propto \frac{1}{\sigma^2} \exp\Big\{ -\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1} (\boldsymbol{\beta}_i - \boldsymbol{\beta}) \Big\} \\
\propto \frac{1}{\sigma^2} \exp\Big\{ -\frac{1}{2}\Big(\boldsymbol{\beta}_i^\top \Sigma^{-1} \boldsymbol{\beta}_i - \boldsymbol{\beta_i}^\top \Sigma^{-1} \boldsymbol{\beta} - \boldsymbol{\beta}^\top \Sigma^{-1} \boldsymbol{\beta}_i - \boldsymbol{\beta}^\top \Sigma^{-1} \boldsymbol{\beta}\Big) \Big\} \\
&\propto \frac{1}{\sigma^2} \exp\Big\{ \boldsymbol{\beta}^\top \Sigma^{-1} \boldsymbol{\beta}_i - \frac{1}{2} \boldsymbol{\beta}^\top \Sigma^{-1} \boldsymbol{\beta} \Big\}
\end{aligned}$$

I believe we want to expand and simplify to find a recognizable distribution, but I'm not sure what to do from here. Seems like I've made an error somewhere? Would it be easier to do Metropolis-Hastings?

The log of the posterior can be written as:

$$\log \pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto \log f_{Y_i(t+6)}(Y_i \mid \boldsymbol{\beta}_i, \sigma^2) + \log \pi(\Theta),$$

where

$$\pi(\Theta) \propto \prod_{i=1}^{n} \det(2\pi\Sigma)^{-\frac{1}{2}} \exp(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})) \times 1 \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1}).$$

Therefore,

$$\log \pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto \sum_{i=1}^{n} -\frac{1}{2}\log 2\pi - \log\sigma - \frac{1}{2\sigma^2}\Big(Y_i(t+6) - \mu_i\Big)^2 +$$

$$\sum_{i=1}^{n} -\frac{1}{2}\log(\det(2\pi\Sigma)) - \frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta}) - 2\log\sigma - (d+1)\log|\Sigma| - \frac{1}{2}\Sigma^{-1}.$$

Using this equation, I think we could implement a similar component-wise M-H algorithm to the Example: Hierarchical Poisson Model from lecture note 9, page 45.