

# P8160 - Hurricane Project

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## Given Information

The following Bayesian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where  $Y_i(t)$  the wind speed at time  $t$  (i.e. 6 hours earlier),  $\Delta_{i,1}(t)$ ,  $\Delta_{i,2}(t)$  and  $\Delta_{i,3}(t)$  are the changes of latitude, longitude and wind speed between  $t$  and  $t+6$ , and  $\epsilon_{i,t}$  follows a normal distributions with mean zero and variance  $\sigma^2$ , independent across  $t$ .

In the model,  $\beta_i = (\beta_{0,i}, \beta_{1,i}, \dots, \beta_{7,i})$  are the random coefficients associated the  $i$ th hurricane, we assume that

$$\beta_i \sim N(\beta, \Sigma)$$

follows a multivariate normal distributions with mean  $\beta$  and covariance matrix  $\Sigma$ .

We assume the following non-informative or weak prior distributions for  $\sigma^2$ ,  $\beta$  and  $\Sigma$ .

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\beta) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right)$$

$d$  is dimension of  $\beta$ .

## Task 1:

Let  $\mathbf{B} = (\beta_1^\top, \dots, \beta_n^\top)^\top$ , derive the posterior distribution of the parameters  $\Theta = (\mathbf{B}^\top, \beta^\top, \sigma^2, \Sigma)$ .

Note from given Bayesian model:

$$\epsilon_i(t) = Y_i(t+6) - \left( \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) \right) \stackrel{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \stackrel{i.i.d}{\sim} N(\mu_i, \sigma^2)$$

where  $\mu_i = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)$ . Therefore,

$$f_{Y_i(t+6)}(Y_i | \beta_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2} \left[ Y_i(t+6) - \left( \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) \right) \right]^2 \right\}$$

We can find the posterior distribution for  $\Theta$  by

$$\pi(\mathbf{B}, \beta, \sigma^2, \Sigma | Y_i) \propto f_{Y_i(t+6)}(Y_i | \beta, \sigma^2) \times \pi(\mathbf{B}) \times \pi(\beta) \times \pi(\sigma^2) \times \pi(\Sigma),$$

where  $\pi(\mathbf{B})$  is the joint multivariate normal density of  $\beta$ ,

$$\pi(\mathbf{B}) = \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta)\right).$$

So we have the following posterior distribution, from which we can derive conditional posteriors to perform Gibbs sampling.

$$\begin{aligned} \pi(\mathbf{B}, \beta, \sigma^2, \Sigma | Y_i) &\propto \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2} (Y_i(t+6) - \mu_i)^2 \right\} \\ &\times \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp \left( -\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta) \right) \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp(-\frac{1}{2}\Sigma^{-1}) \end{aligned}$$

Rearranged,

$$\pi(\mathbf{B}, \beta, \sigma^2, \Sigma | Y_i) \propto \frac{\det(2\pi\Sigma)^{-1/2}}{\sqrt{2\pi}\sigma^3} |\Sigma|^{-(d+1)} \exp \left( -\frac{1}{2}\Sigma^{-1} \right) \prod_{i=1}^n \exp \left\{ -\frac{1}{2\sigma^2} (Y_i(t+6) - \mu_i)^2 - \frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta) \right\}$$

To find the conditional posterior distributions for Gibbs sampling, we can consider:

$$\pi(\beta | \mathbf{B}, \sigma^2, \Sigma, Y_i) = \frac{\pi(\beta, \mathbf{B}, \sigma^2, \Sigma | Y_i)}{\pi(\mathbf{B}, \sigma^2, \Sigma)}.$$

In the equation above,  $\pi(\mathbf{B}, \sigma^2, \Sigma)$  only depends on  $\mathbf{B}, \sigma^2, \Sigma$ , so we can focus only on the part of the joint posterior that depends on all four variables  $\beta, \mathbf{B}, \sigma^2$ , and  $\Sigma$ .

$$\begin{aligned} \pi(\beta | \mathbf{B}, \sigma^2, \Sigma, Y_i) &\propto \frac{1}{\sigma^2} \exp \left\{ -\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta) \right\} \\ &\propto \frac{1}{\sigma^2} \exp \left\{ -\frac{1}{2} \left( \beta_i^\top \Sigma^{-1} \beta_i - \beta_i^\top \Sigma^{-1} \beta - \beta^\top \Sigma^{-1} \beta_i + \beta^\top \Sigma^{-1} \beta \right) \right\} \\ &\propto \frac{1}{\sigma^2} \exp \left\{ \beta^\top \Sigma^{-1} \beta_i - \frac{1}{2} \beta^\top \Sigma^{-1} \beta \right\} \end{aligned}$$

I believe we want to expand and simplify to find a recognizable distribution, but I'm not sure what to do from here. Seems like I've made an error somewhere.