Conditional posterior distributions

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Conditional posterior distributions

For σ^2

Let $\iota = \frac{1}{\sigma^2}$, then

$$(\iota|\beta, B, \Sigma^{-1}, Y) \propto \iota^{1+\frac{\sum_{i=1}^{n} m_i}{2}} exp(-\iota \times \frac{1}{2}\sum_{i=1}^{n}(Y_i - X_i\beta_i^T)^T I^{-1}(Y_i - X_i\beta_i^T)$$

Thus, ι is from gamma distribution

$$\iota \sim -(2 + \frac{\sum_{i=1}^{n} m_i}{2}, \frac{1}{2}\sum_{i=1}^{n}(Y_i - X_i\beta_i^T)(Y_i - X_i\beta_i^T)$$

For B

$$\begin{aligned}
\pi(B|\beta, \sigma^2, \Sigma^{-1}, Y) &\propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I)^{-1}(Y_i - X_i\beta_i^T) + (\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T\right]) \tag{1} \\
&\propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[\beta_i(X_i^T(\sigma^2 I)^{-1}X_i + \Sigma^{-1})\beta_i^T - 2\beta_i(X_i(\sigma^2 I)^{-1})Y_i + \Sigma^{-1}\beta^T\right]) \tag{2} \\
& \tag{3}
\end{aligned}$$

Let $V_i = X_i^T(\sigma^2 I)^{-1})X_i + \Sigma^{-1}$, and $U_i = X_i(\sigma^2 I)^{-1})Y_i + \Sigma^{-1}\beta^T$.
Then

$$(\beta_i|\beta, \sigma^2, \Sigma^{-1}, Y) \sim \mathcal{M}VN(V_i^{-1}U_i, V_i^{-1})$$

For β

$$\begin{aligned}
\pi(\beta|B, \sigma^2, \Sigma^{-1}, Y) &\propto exp(-\frac{1}{2}\sum_{i=1}^{n}(\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T) \tag{4} \\
&\propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[\beta\Sigma^{-1}\beta^T - 2\beta\Sigma^{-1}\beta_i^T\right]) \tag{5} \\
& \tag{6}
\end{aligned}$$

Let $V = n\Sigma^{-1}, U = \sum_{i=1}^{n}\beta_i^T$.
Then

$$(\beta|B, \sigma^2, \Sigma^{-1}, Y) \sim \mathcal{M}VN(V^{-1}U, V^{-1})$$

For Σ^{-1}

$$\pi(\Sigma^{-1}|\beta, B, \sigma^2, Y) \propto |\Sigma|^{-(d+1)} exp(-\frac{1}{2}tr(\Sigma^{-1})|\Sigma|^{-\frac{n}{2}} exp(-\frac{1}{2}\sum_{i=1}^{n}(\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T) \tag{7}$$

$$\propto |\Sigma|^{-(d+1+\frac{n}{2})} exp(-\frac{1}{2}\left[tr(\Sigma^{-1}) + tr((\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T)\right] \tag{8}$$

$$\propto |\Sigma|^{\frac{-(d+1+n+d+1)}{2}} exp(-\frac{1}{2}tr(\left[I + \sum_{i=1}^{n}(\beta_i - \beta)^T(\beta_i - \beta)\right]\Sigma^{-1})) \tag{9}$$

$$\tag{10}$$

Thus,

$$\Sigma \sim \mathcal{W}_d^{-1}(\psi, v)$$

where $v = d + 1 + n$, and $\psi = I + \sum_{i=1}^{n}(\beta_i - \beta)^T(\beta_i - \beta)$.
Accordingly,

$$\Sigma^{-1} \sim \mathcal{W}_d(V, v)$$

where $V = \psi^{-1}$.