

P8160 - Hurricane Project

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4/26/2022

Given Information

The following Bayesian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where $Y_i(t)$ the wind speed at time t (i.e. 6 hours earlier), $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ are the changes of latitude, longitude and wind speed between t and $t+6$, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2 , independent across t .

In the model, $\beta_i = (\beta_{0,i}, \beta_{1,i}, \dots, \beta_{7,i})$ are the random coefficients associated the i th hurricane, we assume that

$$\beta_i \sim N(\beta, \Sigma)$$

follows a multivariate normal distributions with mean β and covariance matrix Σ .

We assume the following non-informative or weak prior distributions for σ^2 , β and Σ .

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\beta) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right)$$

d is dimension of β .

Task 1:

Let $\mathbf{B} = (\beta_1^\top, \dots, \beta_n^\top)^\top$, derive the posterior distribution of the parameters $\Theta = (\mathbf{B}^\top, \beta^\top, \sigma^2, \Sigma)$.

Note from given Bayesian model:

$$\epsilon_i(t) = Y_i(t+6) - \left(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\right) \stackrel{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \stackrel{i.i.d}{\sim} N(\mu_i, \sigma^2)$$

where $\mu_i = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)$. Therefore,

$$f_{Y_i(t+6)}(Y_i | \beta_i, \sigma^2) = \frac{1}{\sqrt{2\pi\sigma}} \exp\left\{-\frac{1}{2\sigma^2}\left[Y_i(t+6) - \left(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\right)\right]^2\right\}$$

We can find the posterior distribution for Θ by

$$\pi(\mathbf{B}, \beta, \sigma^2, \Sigma \mid Y_i) \propto f_{Y_i(t+6)}(Y_i \mid \beta, \sigma^2) \times \pi(\mathbf{B}) \times \pi(\beta) \times \pi(\sigma^2) \times \pi(\Sigma),$$

where $\pi(\mathbf{B})$ is the joint multivariate normal density of β ,

$$\pi(\mathbf{B}) = \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta)\right).$$

So we have the following posterior distribution, from which we can derive conditional posteriors to perform Gibbs sampling.

$$\begin{aligned} \pi(\mathbf{B}, \beta, \sigma^2, \Sigma \mid Y_i) &\propto \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{1}{2\sigma^2}\left(Y_i(t+6) - \mu_i\right)^2\right\} \\ &\times \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta)\right) \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right) \end{aligned}$$

Old work (I think incorrect, above is better approach)

$$\begin{aligned} \pi(\beta \mid Y_i) &\propto f_{Y_i(t+6)}(Y_i \mid \beta, \sigma^2) \times \pi(\beta) \times \pi(\sigma^2) \\ \Rightarrow \pi(\beta \mid Y_i) &\propto \frac{1}{\sqrt{2\pi}\sigma^3} \exp\left\{-\frac{1}{2\sigma^2}\left(Y_i(t+6) - \left(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,1}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\right)\right)^2\right\} \end{aligned}$$

with joint likelihood

$$\pi(\beta \mid Y_i) \propto \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma^3} \exp\left\{-\frac{1}{2\sigma^2}\left(Y_i(t+6) - \left(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,1}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)\right)\right)^2\right\}$$

$$\log \pi(\beta \mid Y_i) \propto \sum_{i=1}^n -\frac{1}{2} \log(2\pi) - 3 \log \sigma - \frac{1}{2\sigma^2} \left(Y_i(t+6) - (\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,1}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t))\right)^2$$

Please, feel free to make edits or add work to this.

For \mathbf{B} , maybe we try something like this:

$$\pi(\mathbf{B} \mid \beta_i) \propto f_{\mathbf{B}}(\mathbf{B} \mid \beta_i, \Sigma^{-1}) \times \pi(\beta_i) \times \pi(\Sigma^{-1})$$

Since, $\beta_i \sim N(\beta, \Sigma)$,

$$f_{\mathbf{B}}(\mathbf{B} \mid \beta_i, \Sigma^{-1}) = \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta)\right)$$

The above is the pdf for a multivariate normal distribution, but I am not sure if we use this here. My thinking is that \mathbf{B} is joint multivariate normal for all independent (could be incorrect assumption) β_i . But this might actually be the distribution for $\pi(\beta_i)$.