

P8160 - Hurricane Project

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Given Information

The following Bayesian model was suggested.

$$Y_i(t+6) = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) + \epsilon_i(t)$$

where $Y_i(t)$ the wind speed at time t (i.e. 6 hours earlier), $\Delta_{i,1}(t)$, $\Delta_{i,2}(t)$ and $\Delta_{i,3}(t)$ are the changes of latitude, longitude and wind speed between t and $t+6$, and $\epsilon_{i,t}$ follows a normal distributions with mean zero and variance σ^2 , independent across t .

In the model, $\beta_i = (\beta_{0,i}, \beta_{1,i}, \dots, \beta_{7,i})$ are the random coefficients associated the i th hurricane, we assume that

$$\beta_i \sim N(\beta, \Sigma)$$

follows a multivariate normal distributions with mean β and covariance matrix Σ .

We assume the following non-informative or weak prior distributions for σ^2 , β and Σ .

$$P(\sigma^2) \propto \frac{1}{\sigma^2}; \quad P(\beta) \propto 1; \quad P(\Sigma^{-1}) \propto |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right)$$

d is dimension of β .

Task 1:

Let $\mathbf{B} = (\beta_1^\top, \dots, \beta_n^\top)^\top$, derive the posterior distribution of the parameters $\Theta = (\mathbf{B}^\top, \beta^\top, \sigma^2, \Sigma)$.

Note from given Bayesian model:

$$\epsilon_i(t) = Y_i(t+6) - \left(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) \right) \stackrel{i.i.d}{\sim} N(0, \sigma^2)$$

or

$$Y_i(t+6) \stackrel{i.i.d}{\sim} N(\mathbf{X}_i^\top \beta_i, \sigma^2)$$

where $\mathbf{X}_i^\top \beta_i = \beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t)$. Therefore,

$$f_{Y_i(t+6)}(Y_i | \beta_i, \sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2} \left[Y_i(t+6) - \left(\beta_{0,i} + \beta_{1,i}Y_i(t) + \beta_{2,i}\Delta_{i,1}(t) + \beta_{3,i}\Delta_{i,2}(t) + \beta_{4,i}\Delta_{i,3}(t) \right) \right]^2 \right\}$$

for hurricane i at time t . To show the likelihood function for hurricane i across all time points, t , we can write

$$f_{Y_i}(Y_i | \beta_i, \sigma^2) \sim MVN(\sigma^2 I)$$

(Need to find mean here by matrix multiplication of covariate matrix \mathbf{X} and beta coefficients and update derivations below)

We can find the posterior distribution for Θ by

$$\pi(\mathbf{B}, \beta, \sigma^2, \Sigma | Y_i) \propto f_{Y_i(t+6)}(Y_i | \beta, \sigma^2) \times \pi(\mathbf{B} | \beta, \Sigma) \times \pi(\beta) \times \pi(\sigma^2) \times \pi(\Sigma),$$

where $\pi(\mathbf{B} | \beta, \Sigma)$ is the joint multivariate normal density of β ,

$$\pi(\mathbf{B} | \beta, \Sigma) = \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta)\right).$$

So we have the following posterior distribution, from which we can derive conditional posteriors to perform Gibbs sampling.

$$\begin{aligned} \pi(\mathbf{B}, \beta, \sigma^2, \Sigma | Y_i) &\propto \prod_{i=1}^n \frac{1}{\sqrt{2\pi}\sigma} \exp \left\{ -\frac{1}{2\sigma^2} \left(Y_i(t+6) - \mathbf{X}_i^\top \beta_i \right)^2 \right\} \\ &\times \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp \left(-\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta) \right) \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right) \end{aligned}$$

Rearranged,

$$\pi(\mathbf{B}, \beta, \sigma^2, \Sigma | Y_i) \propto \frac{\det(2\pi\Sigma)^{-1/2}}{\sqrt{2\pi}\sigma^3} |\Sigma|^{-(d+1)} \exp \left(-\frac{1}{2}\Sigma^{-1} \right) \prod_{i=1}^n \exp \left\{ -\frac{1}{2\sigma^2} (Y_i(t+6) - \mathbf{X}_i^\top \beta_i)^2 - \frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta) \right\}$$

And therefore,

$$\begin{aligned} \log \pi(\mathbf{B}, \beta, \sigma^2, \Sigma | Y_i) &\propto \sum_{i=1}^n -\frac{1}{2} \log 2\pi - \log \sigma - \frac{1}{2\sigma^2} \left(Y_i(t+6) - \mathbf{X}_i^\top \beta_i \right)^2 + \\ &\sum_{i=1}^n -\frac{1}{2} \log(\det(2\pi\Sigma)) - \frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta) - 2 \log \sigma - (d+1) \log |\Sigma| - \frac{1}{2}\Sigma^{-1}. \end{aligned}$$

To find the conditional posterior distributions for Gibbs sampling, we can consider:

$$\pi(\beta \mid \mathbf{B}, \sigma^2, \Sigma, Y_i) = \frac{\pi(\beta, \mathbf{B}, \sigma^2, \Sigma \mid Y_i)}{\pi(\mathbf{B}, \sigma^2, \Sigma)} \propto ??$$

In the equation above, $\pi(\mathbf{B}, \sigma^2, \Sigma)$ only depends on $\mathbf{B}, \sigma^2, \Sigma$, so we can focus only on the part of the joint posterior that depends on all four variables $\beta, \mathbf{B}, \sigma^2$, and Σ .

$$\begin{aligned} \pi(\beta \mid \mathbf{B}, \sigma^2, \Sigma, Y_i) &\propto \frac{1}{\sigma^2} \exp \left\{ -\frac{1}{2}(\beta_i - \beta)^\top \Sigma^{-1}(\beta_i - \beta) \right\} \\ &\propto \frac{1}{\sigma^2} \exp \left\{ -\frac{1}{2} \left(\beta_i^\top \Sigma^{-1} \beta_i - \beta_i^\top \Sigma^{-1} \beta - \beta^\top \Sigma^{-1} \beta_i + \beta^\top \Sigma^{-1} \beta \right) \right\} \\ &\propto \frac{1}{\sigma^2} \exp \left\{ \beta^\top \Sigma^{-1} \beta_i - \frac{1}{2} \beta^\top \Sigma^{-1} \beta \right\} \end{aligned}$$

I believe we want to expand and simplify to find a recognizable distribution, but I'm not sure what to do from here. Seems like I've made an error somewhere? Would it be easier to do Metropolis-Hastings?

The log of the posterior can be written as:

$$\log \pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) \propto \log f_{Y_i(t+6)}(Y_i \mid \boldsymbol{\beta}_i, \sigma^2) + \log \pi(\Theta),$$

where

$$\pi(\Theta) \propto \prod_{i=1}^n \det(2\pi\Sigma)^{-\frac{1}{2}} \exp\left(-\frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta})\right) \times 1 \times \frac{1}{\sigma^2} \times |\Sigma|^{-(d+1)} \exp\left(-\frac{1}{2}\Sigma^{-1}\right).$$

Therefore,

$$\begin{aligned} \log \pi(\mathbf{B}, \boldsymbol{\beta}, \sigma^2, \Sigma \mid Y_i) &\propto \sum_{i=1}^n -\frac{1}{2} \log 2\pi - \log \sigma - \frac{1}{2\sigma^2} \left(Y_i(t+6) - \mu_i\right)^2 + \\ &\sum_{i=1}^n -\frac{1}{2} \log(\det(2\pi\Sigma)) - \frac{1}{2}(\boldsymbol{\beta}_i - \boldsymbol{\beta})^\top \Sigma^{-1}(\boldsymbol{\beta}_i - \boldsymbol{\beta}) - 2 \log \sigma - (d+1) \log |\Sigma| - \frac{1}{2}\Sigma^{-1}. \end{aligned}$$

Using this equation, I think we could implement a similar component-wise M-H algorithm to the Example: Hierarchical Poisson Model from lecture note 9, page 45.