Conditional posterior distributions

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For σ^2

Let $\tau = 1/\sigma^2$, then

$$(\tau|\boldsymbol{\beta}, \mathbf{B}, \Sigma^{-1}, Y) \propto \tau^{1+\frac{\sum_{i=1}^{n} m_i}{2}} exp(-\tau \times \frac{1}{2}\sum_{i=1}^{n}(Y_i - X_i\beta_i^T)^T(Y_i - X_i\beta_i^T)$$

Thus, σ^2 is from inverse-gamma distribution

$$(\sigma^2 \mid \boldsymbol{\beta}, \mathbf{B}, \Sigma^{-1}, Y) \sim \text{Inv-Gamma}(\frac{\sum_{i=1}^{n} m_i}{2}, \frac{1}{2}\sum_{i=1}^{n}(Y_i - X_i\beta_i^T)(Y_i - X_i\beta_i^T).$$

For $\mathbf{B}$

$$\pi(\mathbf{B}|\boldsymbol{\beta}, \sigma^2, \Sigma^{-1}, Y) \propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[(Y_i - X_i\beta_i^T)^T(\sigma^2 I)^{-1}(Y_i - X_i\beta_i^T) + (\beta_i - \beta)\Sigma^{-1}(\beta_1 - \beta)^T\right]) \quad (1)$$

$$\propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[\beta_i(X_i^T(\sigma^2 I)^{-1}X_i + \Sigma^{-1})\beta_i^T - 2\beta_i(X_i(\sigma^2 I)^{-1})Y_i + \Sigma^{-1}\beta^T\right]) \quad (2)$$

Let $V_i = X_i^T(\sigma^2 I)^{-1})X_i + \Sigma^{-1}$, and $U_i = X_i(\sigma^2 I)^{-1}Y_i + \Sigma^{-1}\beta^T$, then

$$(\boldsymbol{\beta}_i \mid \boldsymbol{\beta}, \Sigma^{-1}, \sigma^2, Y) \sim \mathcal{M}VN(V_i^{-1}U_i, V_i^{-1}).$$

For β

$$\pi(\boldsymbol{\beta}|\mathbf{B}, \sigma^2, \Sigma^{-1}, Y) \propto exp(-\frac{1}{2}\sum_{i=1}^{n}(\beta_i - \beta)\Sigma^{-1}(\beta_i - \beta)^T) \quad (3)$$

$$\propto exp(-\frac{1}{2}\sum_{i=1}^{n}\left[\beta\Sigma^{-1}\beta^T - 2\beta\Sigma^{-1}\beta_i^T\right]) \quad (4)$$

Let $V = n\Sigma^{-1}, U = \sum_{i=1}^{n}\Sigma^{-1}\beta_i^T$, then

$$(\boldsymbol{\beta}|\mathbf{B}, \sigma^2, \Sigma^{-1}, Y) \sim \mathcal{M}VN(V^{-1}U, V^{-1}).$$

For Σ^{-1}

$$\pi(\Sigma^{-1}|\boldsymbol{\beta},\mathbf{B},\sigma^2,Y) \propto |\Sigma|^{-(d+1)}exp(-\frac{1}{2}tr(\Sigma^{-1})|\Sigma|^{-\frac{n}{2}}exp(-\frac{1}{2}\sum_{i=1}^{n}(\beta_i-\beta)\Sigma^{-1}(\beta_i-\beta)^T) \tag{5}$$

$$\propto |\Sigma^{-1}|^{d+1+\frac{n}{2}}exp(-\frac{1}{2}\left[tr(\Sigma^{-1})+tr(\sum_{i=1}^{n}(\beta_i-\beta)\Sigma^{-1}(\beta_i-\beta)^T)\right] \tag{6}$$

$$\propto |\Sigma^{-1}|^{3d+3+n-d-1}exp(-\frac{1}{2}tr(\left[I+\sum_{i=1}^{n}(\beta_i-\beta)^T(\beta_i-\beta)\right]\Sigma^{-1})) \tag{7}$$

Thus,

$$\Sigma^{-1} \sim \mathcal{W}_d(\Psi, v),$$

where $v = 3d+3+n$, and $\Psi = I + \sum_{i=1}^{n}(\beta_i-\beta)^T(\beta_i-\beta)$.