

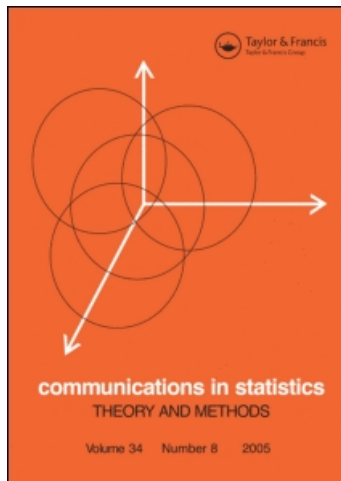
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Patrick Fitzsimmons <sup>a</sup>;Tucker McElroy <sup>b</sup>

<sup>a</sup> Department of Mathematics, University of California, San Diego, California, USA <sup>b</sup> Statistical Research Division, U.S. Bureau of the Census, Washington, DC, USA

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# On Joint Fourier–Laplace Transforms

PATRICK FITZSIMMONS<sup>1</sup> AND TUCKER McELROY<sup>2</sup>

<sup>1</sup>Department of Mathematics, University of California, San Diego,  
California, USA

<sup>2</sup>Statistical Research Division, U.S. Bureau of the Census,  
Washington, DC, USA

*Weak convergence of random variables is characterized by pointwise convergence of the Fourier transform of the respective distributions, and in some cases can also be characterized through the Laplace transform. For some distributions, the Laplace transform is easier to compute and provides an alternative approach to the method of characteristic functions that facilitates proving weak convergence. We show that, for a bivariate distribution, a joint Fourier–Laplace transform always characterizes the distribution when the second variate is positive almost surely.*

**Keywords** Characteristic functions; Joint asymptotics; Weak convergence.

**Mathematics Subject Classification** 62E20; 60F05.

## Disclaimer

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## 1. Introduction

The joint Fourier–Laplace transform of bivariate random variables is, for some statistical problems, a convenient characterization of weak convergence. Given the existence in a suitable interval of the Laplace transform of the second variate's distribution, which is up to sign the moment generating function of the variable, such a function characterizes the distribution and can be used to prove weak convergence of a sequence of variables. We focus on the joint Fourier–Laplace transform, which is essentially a characteristic function in the first variable and a moment generating function in the second. Below, we present the main

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Address correspondence to Tucker McElroy, Statistical Research Division, U.S. Bureau of the Census, 4600 Silver Hill Road, Washington, DC 20233-9100, USA; E-mail: tucker.s.mcelroy@census.gov

analytical result that justifies the use of such transforms. We briefly mention some applications.

It sometimes occurs in statistical problems that the joint convergence of sample mean and sample variance must be determined. In this case, we have a pair of random variables for which the second variate is positive with probability one. Therefore, the Laplace transform is typically well defined for the second variate, and a joint Fourier–Laplace transform may be easier to calculate than a straight Fourier transform. See McElroy and Politis (2007a,b) for some examples.

We now consider a pair of random variables  $(X, Y)$  such that  $Y \geq 0$  almost surely. The joint Fourier–Laplace (FL) transform of  $X$  and  $Y$  is defined for  $t \in \mathbb{R}$  and  $u \in U = [0, \infty)$  by

$$\phi(t, u) := \mathbf{E}[\exp(itX - uY)].$$

In some cases, the moment generating function  $u \mapsto \phi(0, u)$  is defined for  $U = (-s_0, s_0)$  or  $U = [0, s_0)$  for some  $s_0 > 0$ , but we will focus on the case  $U = [0, \infty)$ . The following theorem summarizes the important properties of the FL transform as concerns weak convergence.

**Theorem 1.1.**

- (i) *The joint distribution of  $(X, Y)$  is uniquely determined by the FL transform  $\phi$ .*
- (ii) *If  $(X_n, Y_n)$  is a sequence of random pairs with  $Y_n \geq 0$  almost surely for each  $n$ , then  $(X_n, Y_n)$  converges weakly (to some pair  $(X, Y)$ ) provided the sequence  $(\phi_n)$  of FL transforms converges pointwise on  $\mathbb{R} \times [0, \infty)$  to a function  $\phi$  on  $\mathbb{R} \times [0, \infty)$  that is continuous at  $(0, 0)$ . In this case,  $Y \geq 0$  almost surely, and the FL transform of  $(X, Y)$  is  $\phi$ .*

**Remark 1.1.** By an example in Mukherjea et al. (2006), the converse of the second statement of Theorem 1.1 need not be true, i.e., there exist sequences of distributions whose characteristic functions converge, but whose moment generating functions do not.

*Proof of Theorem 1.1.* (i) For fixed real  $t$  and  $x$ , define

$$f(y, z) = e^{itx} e^{-zy}, \quad y \geq 0, z \in W,$$

where  $W = \{z \in \mathbb{C} \mid \operatorname{Re}(z) > 0\}$ . Then  $f(y, \cdot)$  is analytic in the open set  $W$ , with derivative  $\frac{\partial}{\partial z} f(y, z) = -y f(y, z)$ . Notice that  $|\frac{\partial}{\partial z} f(y, z)| = y e^{-\operatorname{Re}(z)y} \leq [e \operatorname{Re}(z)]^{-1}$ , since  $\operatorname{Re}(z) > 0$ . Also,  $f(\cdot, z)$  is measurable and the magnitude of  $f$  is bounded by 1. It follows from these observations and exercise 16.6 of Billingsley (1995) that  $z \mapsto \mathbf{E}[\exp(itX - zY)]$  is analytic on  $W$ . Moreover, by the Dominated Convergence Theorem, the extension  $\Phi(t, z) := \mathbf{E}[\exp(itX - zY)]$  of  $\phi$  is continuous on  $\mathbb{R} \times \overline{W}$ , where  $\overline{W} = \{z \in \mathbb{C} \mid \operatorname{Re}(z) \geq 0\}$  is the closure of  $W$ .

Now suppose that  $(X^*, Y^*)$  is a second pair of random variables (such that  $Y^* \geq 0$  almost surely) with FL transform equal to  $\phi$  on all of  $\mathbb{R} \times [0, \infty)$ . Then for each real  $t$ , the two expectations  $\mathbf{E}[\exp(itX - zY)]$  and  $\mathbf{E}[\exp(itX^* - zY^*)]$  are analytic functions of  $z \in W$  and coincide (with  $\phi(t, \cdot)$ ) on the axis  $\{z \in W \mid \operatorname{Im}(z) = 0\}$ . It follows that

$$\mathbf{E}[\exp(itX - zY)] = \mathbf{E}[\exp(itX^* - zY^*)], \quad (t, z) \in \mathbb{R} \times W.$$

By the continuity noted earlier, this equality extends to  $\mathbb{R} \times \overline{W}$ . Thus, writing  $z = -iv$  for complex  $z$  with  $\operatorname{Re}(z) = 0$ ,

$$\mathbf{E}[\exp(itX + ivY)] = \mathbf{E}[\exp(itX^* + ivY^*)], \quad \forall (t, v) \in \mathbb{R}^2.$$

Thus,  $(X, Y)$  and  $(X^*, Y^*)$  have the same bivariate characteristic function, hence the same distribution.

(ii) The assumptions imply that the limits

$$\lim_n \mathbf{E}[\exp(itX_n)] = \lim_n \phi_n(t, 0) = \phi(t, 0) \quad t \in \mathbb{R},$$

and

$$\lim_n \mathbf{E}[\exp(-uY_n)] = \lim_n \phi_n(0, u) = \phi(0, u) \quad u \geq 0,$$

exist and are continuous at the origin. It follows that both  $(X_n)$  and  $(Y_n)$  converge in distribution. (See Corollary 1, p. 350 and Exercise 26.7, p. 354 of Billingsley, 1995) This, in turn, implies that both  $(X_n)$  and  $(Y_n)$  are tight. Consequently, the bivariate sequence  $(X_n, Y_n)$  is tight. Since  $\phi(t, u) = \lim_n \phi_n(t, u)$  for all  $(t, u) \in \mathbb{R} \times [0, \infty)$ , the weak limits of all subsequences of  $(X_n, Y_n)$  have the same FL transform (namely  $\phi$ ), hence the same distribution, by assertion (i). It follows from the Corollary on p. 381 of Billingsley (1995) that  $(X_n, Y_n)$  converges in distribution to a limit  $(X, Y)$  with FL transform  $\phi$ , and clearly  $Y \geq 0$  almost surely.  $\square$

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