Name: ______________________________

Definition. Let $[n] := \{1, 2, \dots, n\}$.

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1. How many subsets of $[n]$ are there? How many ways can you prove it?

2. How many subsets of $[n]$ are there that have exactly k elements? (Define this to be $\binom{n}{k}$.) Prove it.

3. Prove that $\displaystyle\sum_{k=0}^{n} \binom{n}{k} = 2^n$ in as many ways as possible. Which methods are best? Why?

4. Verify for $a, b, n \in \mathbb{Z}_{\geq 0}$ that

$$\sum_{i=0}^{n} \binom{a}{i} \binom{b}{n-i} = \binom{a+b}{n}.$$

Give two proofs: one using the binomial theorem, $(1+x)^n = \sum_{k \geq 0} \binom{n}{k} x^k$, and a *combinatorial proof*.

5. Give combinatorial proofs of the following. (Most from Stanley, *Enumerative Combinatorics*, Exercise 1.3.)

   (a) $\displaystyle\sum_{k=0}^{n} k\binom{n}{k} = n2^{n-1}$

   (b) $\displaystyle\sum_{k=1}^{n} k = \binom{n+1}{2}.$

   (c) *At home problem* $\displaystyle\sum_{k=0}^{n} \binom{x+k}{k} = \binom{x+n+1}{n}$

   (d) $\displaystyle\sum_{k=0}^{n}(-1)^k\binom{n}{k} = 0$ for $n \geq 1$.

Combinatorial objects. In order to see the subtleties in different types of counting problems, it will help to have a bank of common combinatorial objects that we know how to count. Then, we'll often be able to map other counting problems back to some problem in this bank.

Definition. Let n be a natural number. A *composition of* n is an ordered sum of positive integers that total n. Each summand is called a *part*.

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6. What are the compositions of 2? 3? 4? 5?

7. How many compositions are there of n? Prove your answer.

8. How many compositions are there of n with k parts? Prove your answer.

9. How many solutions are there to the following equation, where x_1, x_2, and x_3 are positive integers?

$$x_1 + x_2 + x_3 = 25$$

10. How many solutions are there if x_1, x_2, and x_3 need only be non-negative integers? (These are called *weak compositions* of 25.)

11. How many integer solutions are there to the equation

$$x_1 + x_2 + x_3 \leq 30$$

with $x_1 \geq 0$, $x_2 > 4$, and $x_3 > 7$?

12. *At home problem* How many ways can I distribute 50 identical blank sheets of paper to distinct printers A, B, and C so that printer A gets at least 10 sheets of paper?

13. Let $\left(\!\binom{n}{k}\!\right)$ be the multisets of size k from $[n]$, meaning that elements in the multiset may be picked more than once and that order does not matter. Using stars and bars, find a formula for $\left(\!\binom{n}{k}\!\right)$.

14. Now, find a combinatorial bijection between $\left(\!\binom{n}{k}\!\right)$ and your answer from the previous problem (which can be interpreted as regular subsets).

Multinomials. The *multinomial theorem* tells us:

$$(x_1 + x_2 + \cdots + x_m)^n = \sum_{a_1+\cdots+a_m=n} \binom{n}{a_1, a_2, \ldots, a_m} x_1^{a_1} \cdots x_m^{a_m}$$

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15. Given the theorem above, how can we phrase $\binom{n}{a_1, a_2, \ldots, a_m}$ in terms of choosing objects?

16. Find a formula for $\binom{n}{a_1, a_2, \ldots, a_m}$.