Homework 1 - due Monday, January 26

1. Prove that if $0 = 1$ in a ring R, then R is the zero ring (that is, its only element is 0).

2. A category $\mathcal{C}'$ is a **subcategory** of a category $\mathcal{C}$ if
   1) $\mathrm{Obj}(\mathcal{C}') \subseteq \mathrm{Obj}(\mathcal{C})$ (i.e. the objects in $\mathcal{C}'$ are all objects in $\mathcal{C}$), and
   2) for all $A, B \in \mathrm{Obj}(\mathcal{C}')$, $\mathrm{Hom}_{\mathcal{C}'}(A, B) \subseteq \mathrm{Hom}_{\mathcal{C}}(A, B)$ (i.e. the morphisms in $\mathcal{C}'$ are all morphisms in $\mathcal{C}$)
   It is a **full subcategory** if in addition $\mathrm{Hom}_{\mathcal{C}'}(A, B) = \mathrm{Hom}_{\mathcal{C}}(A, B)$ for all $A, B \in \mathrm{Obj}(\mathcal{C}')$.
   Ring is a subcategory of Rng (you do not need to prove this). Show that it is not a full subcategory.

3. Let a and b be zero-divisors in a ring R. Either prove $a + b$ is always a zero-divisor OR provide a specific counterexample.

4. The **center** of a ring R is $\{z \in R \mid rz = zr \text{ for all } r \in R\}$. Prove that the center of a ring R is a subring of R.

5. An element x in a ring R is called **nilpotent** if $x^m = 0$ for some $m \in \mathbb{Z}^+$ (Here, x^m denotes $\underbrace{x \cdot x \cdots x}_{m \text{ times}}$)
   Prove that the nilpotent elements of a commutative ring R form an ideal (this is called the **nilradical** of R).

6. Let $n \in \mathbb{N}$.
   (a) Show that if $n = ab$ for some integers a, b, then ab is a nilpotent element of $\mathbb{Z}/n\mathbb{Z}$.
   (b) If $a \in \mathbb{Z}$, show that the element $a \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent if and only if every prime divisor of n divides a. In particular, determine the nilpotent elements of $\mathbb{Z}/72\mathbb{Z}$.

7. Let R and S be rings.
   (a) Prove that the direct product $R \times S = \{(r, s) \mid r \in R, s \in S\}$ forms a ring under componentwise addition and multiplication.
   (b) Prove that $R \times S$ is commutative if and only if both R and S are commutative.

8. Let R be a commutative ring. Define the ring $R[[x]]$ of formal power series by
   $$R[[x]] = \left\{ \sum_{n=0}^{\infty} a_n x^n \mid a_i \in R \right\}$$
   (a) Prove that $R[[x]]$ is a commutative ring, and be sure to explain how to add and multiply elements.
   (b) Show that $1 - x$ is a unit with inverse $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$.
   (c) (Optional Challenge): Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R.

9. Decide which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:
   (a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$
   (b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$
   (c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$
10. Let
$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \mid a, b, d \in \mathbb{Z} \right\}.$$
Prove that the map
$$\varphi : R \to \mathbb{Z} \times \mathbb{Z}, \qquad \varphi\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} = (a, d)$$
is a surjective ring homomorphism and describe its kernel.
11. Decide which of the following are ideals of $\mathbb{Z} \times \mathbb{Z}$:
   (a) $\{(a, a) \mid a \in \mathbb{Z}\}$
   (b) $\{(2a, 2b) \mid a, b \in \mathbb{Z}\}$
   (c) $\{(2a, 0) \mid a \in \mathbb{Z}\}$
   (d) $\{(a, -a) \mid a \in \mathbb{Z}\}$
12. The **characteristic** of a ring R (denoted char R) is the smallest $n \in \mathbb{N}$ such that $\underbrace{1_R + \cdots + 1_R}_{n \text{ times}} = 0$, and if there is no such n we say the characteristic of R is 0.
Prove that an integral domain has characteristic 0 or a prime.