Name: ______________________

Review:

1. What is a combinatorial proof? Write down some explicit structures for combinatorial proofs we have seen.

2. Why are bijections useful?

3. Give an example of a formula we haven't yet seen that you can prove is true by using a combinatorial proof.

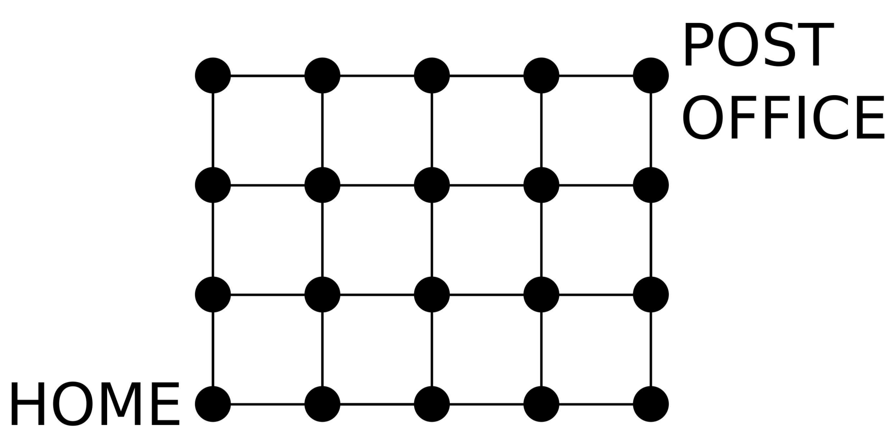
4. What tool do we use to count compositions?

5. What is a multinomial coefficient?

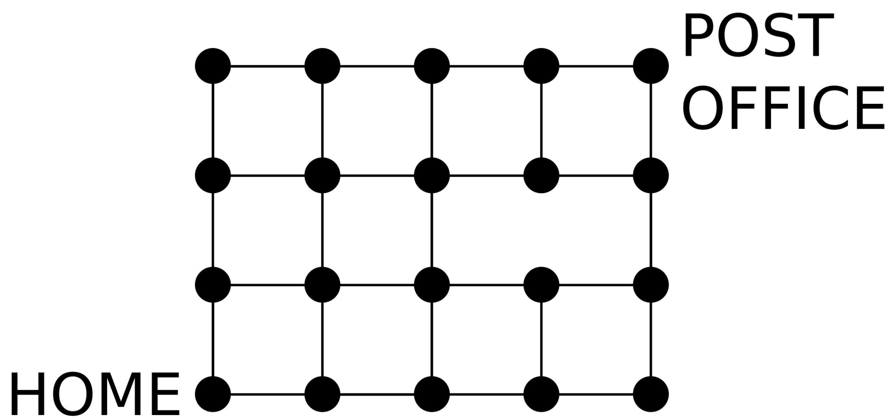
6. How many total (multiset) subsets of a multiset A are there if for each $x \in A$, A has $v(x)$ copies of x?

7. How many ways can we arrange the letters of "BANANAS" to form arrangements that look different?

8. A town is formed with roads on a grid, as shown below. How many different shortest paths are there between home and the post office? Explain.



9. *At home problem* A bridge was flooded in the town, removing one of the possible routes. How many shortest paths are there now?



10. How many shortest paths are there from the origin to the point $(a_1, a_2, \ldots, a_d)$ following edges in $\mathbb{Z}^d$?

11. (The following few questions are from Miklós Bóna's *A Walk Through Combinatorics.*) *At home problem* A gardener has five red flowers, three yellow flowers and two white flowers to plant in a row. In how many different ways can she do that?

12. How many k-digit positive integers are there?

13. A city has recently built ten intersections. Some of these will get traffic lights, and some of those that get traffic lights will also get a gas station. In how many different ways can this happen? Come up with the simplest answer possible.

14. A medical student has to work in a hospital for five days in January. However, he is not allowed to work two consecutive days in the hospital. In how many different ways can he choose the five days he will work in the hospital?

15. *At home problem* A hotel has 20 rooms in a line, and 8 guests want separate rooms. How many ways can the hotel fill the rooms so that no two guests are in adjacent rooms?

16. Assume that we play a lottery game where five numbers are drawn out of [90], but the numbers drawn are put back into the basket right after being selected. To win the jackpot, one must have played the same multiset of numbers as the one drawn (regardless of the order in which the numbers were drawn). How many lottery tickets do we have to buy to make sure that we win the jackpot?

17. A track and field championship has participants from 49 countries. The flag of each participating country consists of three horizontal stripes of different colors. However, no flag contains colors other than red, white, blue, and green. Is it true that there are three participating countries with identical flags?

18. A restaurant offers five different soups, ten main courses, and six desserts. Joe decided to order at most one soup, at most one main course, and at most one dessert. In how many ways can he do this?

19. How many five-digit positive integers are there with middle digit 6 that are divisible by three?

20. How many subsets $S \subseteq [2n]$ are there (of any size) so that there are no two elements x and y in S satisfying $x + y = 2n + 1$?

Definition. A *permutation* is an arrangement of the elements in a set or multiset.

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21. How many permutations are there of $[n]$? Explain.

22. In this exercise, the word "precede" does not mean "immediately precede."

    (a) In how many ways can the elements of $[n]$ be permuted if 1 is to precede 2 and 3 is to precede 4?

    (b) *At home problem* In how many ways can the elements of $[n]$ be permuted if 1 is to precede both 2 and 3?

Encoding permutations. There are many ways to represent permutations. Some common methods include:

- As a word, called *one-line notation*. A permutation of the multiset $S = \{A, A, A, B, N, N\}$ is BANANA.
- As a function. $f : [n] \to S$, where $f(i)$ is the ith digit of the permutation. In the previous example, $f(1) = B, f(2) = A, f(3) = N$, and so on.

Let the permutation ω (a function) be on a finite set with distinct elements, which we label $[n]$ without loss of generality. Then, we can consider the sequence, $x, \omega(x), \omega(\omega(x)), \ldots$. In this way, a permutation can be decomposed into *cycles* where the image of each element is whatever appears next in the cycle.

Example. (275)(1436) means $2 \to 7 \to 5 \to 2$ and $1 \to 4 \to 3 \to 6 \to 1$.

- (275)(1536) is *cycle notation* for the permutation.

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23. Write (275)(1436) as a function, and then in one-line notation.

24. If 41285736 is a permutation written in one-line notation, write it in cycle notation.

25. Consider an individual cycle within a permutation of $[n]$. What are the possible lengths of the cycle? Prove your answer.

26. The *standard representation* of cycle notation is where each cycle is written with its largest element first, and the cycles are written in increasing order of largest element. What is the standard representation of $(275)(1436)$?

27. Justify that the number of permutations with k cycles is the same as the number of permutations with k left-to-right maxima (meaning that in one-line notation, there are k times when you find a number larger than any number read so far).

28. Justify that the number of permutations on $[n]$ with c_1 cycles of length 1, c_2 cycles of length 2, and in general, c_i cycles of length i, is

$$\frac{n!}{1^{c_1}c_1! \cdot 2^{c_2}c_2! \cdots n^{c_n}c_n!}$$

29. *At home problem* How many permutations on [8] have no cycle of length 8?