MATH 266 Homework 1 Solution

Problem 1 (**20 points**) Consider the equation $y' = y^2$.

(a) (**15 points**) Find the equilibrium point(s), and use the phase diagram to mark each one stable or unstable.

We first find the equilibrium points by setting $y^2 = 0$. This gives us $y = 0$.

Let us next sketch the phase diagram. We see that $y = 0$ splits the y-axis into two intervals: $(0, \infty)$, $(-\infty, 0)$. For the interval $(0, \infty)$, we pick $y = 1$. This yields $y' = 1^2 > 0$. Turning attention to the interval $(-\infty, 0)$, by setting $y = -1$, we have $y' = (-1)^2 > 0$. Hence, the phase diagram is as follows.

Therefore, $y = 0$ is unstable.

(b) (**5 points**) Find $\lim\limits_{x\to\infty} y(x)$ for the solution y if $y(0) = 1$. Justify your answer.

We see from the phase diagram above that y is increasing on the interval $(0, \infty)$. Thus, if $y(0) = 1$, y will increase. As there are no equilibrium points above, y will increase without bound. Therefore, $\lim\limits_{x\to\infty} y(x) = \infty$.

Problem 2 (**20 points**) Consider the equation $y' = (y-1)(y-2)y^2$.

(a) (**15 points**) Find and classify the equilibrium points.

We first need to find equilibrium points. To this end, by setting $y' = 0$, we have $(y-1)(y-2)y^2 = 0$. We solve this equation to see that $y = 0, 1, 2$ are the equilibrium points.

We next sketch the y-axis and see that the three equilibrium points above split the y-axis into four intervals: $(-\infty, 0)$, $(0, 1)$, $(1, 2)$, and $(2, \infty)$.

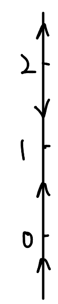
Let us now check whether y is increasing or decreasing on each interval. To this end, we first choose $y = -1$ on $(-\infty, 0)$ to get $y' = (-1-1)(-1-2)(-1)^2 > 0$.

We next pick $y = 0.5$ on $(0, 1)$, then we have $y' = (0.5-1)(0.5-2)(0.5)^2 > 0$.

Similarly, setting $y = 1.5$ on $(1, 2)$, we obtain $y' = (1.5-1)(1.5-2)(1.5)^2 < 0$.

Finally, we let $y = 3$ on $(2, \infty)$ to deduce $y' = (3-1)(3-2)3^2 > 0$.

The phase diagram is the following:



From the diagram, we see that $y = 1$ is stable, and $y = 0, 2$ are unstable.

(b) (**5 points**) Find $\lim_{x\to\infty} y(x)$ for the solution y if $y(0) = 0.5$. Justify your answer.

Since y is increasing on the interval $(0, 1)$, we see that y decreases as x increases if $y(0) = 0.5$. However, y will stop changing when it reaches the equilibrium point $y = 1$. Therefore, we see that $\lim_{x\to\infty} y(x) = 1$.

Problem 3 (**10 points**) Determine whether $y = e^x$ is a solution to the equation $y''' - 12y'' + 48y' - 64y = 0$.

We need to take the derivative of $y = e^x$ up to the third order. Clearly, we have $y' = y'' = y''' = e^x$. Plugging them into the given differential equation, we get

$$\begin{aligned} y''' - 12y'' + 48y' - 64y &= e^x - 12e^x + 48e^x - 64e^x \\ &= -27e^x \neq 0. \end{aligned}$$

Therefore, $y = e^x$ is not a solution to the given equation.

Problem 4 (**20 points**) Consider the equation $y'' - y' = 0$.

(a) (**10 points**) Show that $y = C_1 e^x + C_2$ is a solution to the equation for any constants C_1, C_2.

We compute y' and y''. To this end, we see that $y' = C_1 e^x$ and $y'' = C_1 e^x$. Thus, it follows that

$$y'' - y' = C_1 e^x - C_1 e^x = 0.$$

Therefore, $y = C_1 e^x + C_2$ is a solution to the equation.

(b) (**10 points**) Find C_1 and C_2 that satisfies $y(0) = 10$ and $y'(0) = 100$.

By substituting $x = 0$ into $y = C_1 e^x + C_2$, we get

$$y(0) = C_1 e^0 + C_2 = C_1 + C_2.$$

Given that $y(0) = 10$, we have $y(0) = 10$.

Similarly, we substitute $x = 0$ into $y' = C_1 e^x$, we have

$$y'(0) = C_1 e^0 = C_1.$$

Since it is given that $y'(0) = 100$, we see that $C_1 = 100$.

Finally, substituting $C_1 = 100$ into the equation $C_1 + C_2 = 10$, we obtain $C_2 = -90$.

Problem 5 (**10 points**) Find all the values of r such that $y = e^{rx}$ is a solution to the equation $y'' + 9y' - 10y = 0$.

[Hint: $e^{rx} \neq 0$ for any r and all real number x.]

By the chain rule, we compute that $y' = re^{rx}$ and $y'' = r^2 e^{rx}$. Then we get

$$\begin{aligned} y'' + 9y' - 10y &= r^2 e^{rx} + 9re^{rx} - 10e^{rx} \\ &= (r^2 + 9r - 10)e^{rx}. \end{aligned}$$

By hypothesis, we see that $(r^2+9r-10)e^{rx} = 0$. Since $e^{rx} \neq 0$, we must have $r^2+9r-10 = 0$. To solve this quadratic equation, we factor it to get $(r+10)(r-1) = 0$. This yields $r = -10$ and $r = 1$.