MATH 266 - NDSU
Fall 2025
Final Exam
December 17th, 2025
Time Limit: 120 Minutes

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This exam contains 10 pages (including this cover page) and 8 questions. Total of points is 100. Read all of the following information before starting the exam:

- Present your work clearly and in order, and justify your conclusions. We reserve the right to take off points if we cannot see how you arrived your answers or cannot read your handwriting clearly.
- **No calculators are allowed. You may use a double-sided full page note sheet.**
- Write your answers on the space provided. Feel free to use the other side of each page if more space is needed, but please indicate you have done so, otherwise the back side of each page will be regarded as scratch work and **will not be graded**.
- A Laplace transform table is provided separately.
- Good luck!

Grade Table

Question	Points	Score
1	10	10
2	15	13
3	20	20
4	10	10
5	10	9
6	10	10
7	10	10
8	15	15
Total:	100	97

$V_i = 10$ $A(0) = 6$

1. (10 points) A tank contains 10 gal of brine made by dissolving 6 lb of salt in water. Salt water containing 1 lb of salt per gallon runs in at the rate of 2 gal/min and the well-stirred mixture runs out at the rate of 3 gal/min. Find the amount of salt in the tank after t minutes.

$A'(t) = C_{in} V_{in} - C_{out} V_{out}$

$A'(t) = (1)(2) - \left(\frac{A(t)}{V(t)}\right) \cdot 3$

$A'(t) = 2 - \frac{3A(t)}{10-t}$

$A'(t) + \frac{3A(t)}{10-t} = 2$

integrating factor: $e^{\int \frac{3}{10-t} dt}$

$\int \frac{3}{10-t} dt = \int \frac{-3}{u} du = -3 \ln|u| = -3 \ln|10-t| = \ln|(10-t)^{-3}|$

$u = 10-t$

$du = -dt$

$e^{\ln|(10-t)^{-3}|} = (10-t)^{-3}$

$(10-t)^{-3} A'(t) + 3(10-t)^{-4} A(t) = 2(10-t)^{-3}$

$[(10-t)^{-3} A(t)]' = 2(10-t)^{-3}$

$(10-t)^{-3} A(t) = \int 2(10-t)^{-3} dt$ $u = 10-t$, $du = -dt$

$= -2 \int u^{-3} du$

$= \cancel{-2}\left(\frac{u^{-2}}{\cancel{-2}}\right) + C$

$= u^{-2} + C$

$= (10-t)^{-2} + C$

$A(t) = 10 - t + C(10-t)^3$

$A(0) = 10 + 1000C$

$6 = 10 + 1000C$

$-4 = 1000C$

$C = -\frac{1}{250}$

$A(t) = 10 - t - \frac{1}{250}(10-t)^3$

Check work

$-1 - 3C(10-t)^2 = 2 - (3 + 3C(10-t)^2)$

$-1 - 3C(10-t)^2 = -1 - 3C(10-t)^2$

$0 = 0$ ✓

10

2. (15 points) Solve the initial value problem

$$y'' + 2y' + y = u(x-3),\ y(0) = 0,\ y'(0) = 1.$$

$\mathcal{L}(y'') + 2\mathcal{L}(y') + \mathcal{L}(y) = \mathcal{L}(u(x-3))$

$s^2Y - sy(0) - y'(0) + 2sY - 2y(0) + Y = \frac{e^{-3s}}{s}$

$s^2Y - 1 + 2sY + Y = \frac{e^{-3s}}{s}$

$(s^2+2s+1)Y = \frac{e^{-3s}}{s} + 1$

$Y = \frac{e^{-3s}}{s(s+1)^2} + \frac{1}{(s+1)^2}$

$\frac{A}{s} + \frac{B}{s+1} + \frac{C}{(s+1)^2} = \frac{1}{s(s+1)^2}$

$A(s+1)^2 + Bs(s+1) + Cs = 1$

(s=0): $A = 1$

(s=-1): $-C = 1$ $\quad C = -1$

(s=1): $4 + 2B - 1 = 1$

$2B + 3 = 1$

$2B = -2$

$B = -1$

$\frac{e^{-3s}}{s+1} = e^{-as}F(s)$

$F(s) = \left(\frac{1}{s+1}\right)$

$f(x) = e^{-x}$

$f(x-3) = e^{-x+3}$

$\frac{e^{-3s}}{(s+1)^2}$

$F(s) = \frac{1}{(s+1)^2}$

$f(x) = e^{-x} \cdot x^1$

$f(x-3) = e^{-x+3}(x-3)$

$= xe^{-x+3} - 3e^{-x+3}$

$Y = \frac{e^{-3s}}{s} - \frac{e^{-3s}}{s+1} - \frac{e^{-3s}}{(s+1)^2} + \frac{1}{(s+1)^2}$

$y(x) = u(x-3) - u(x-3)e^{-x+3} - u(x-3)(xe^{-x+3} - 3e^{-x+3}) + xe^{-x}$

$= u(x-3)(1 - e^{-x+3} - xe^{-x+3} + 3e^{-x+3}) + xe^{-x}$

$= u(x-3)(1 + 2e^{-x+3} - xe^{-x+3}) + xe^{-x}$

$y(x) = u(x-3)(1 + \cancel{2}e^{-x+3} - \cancel{x}e^{-x+3}) + xe^{-x}$

x-3

13/15

3. (20 points) This problem consists of two parts.

   (a) (10 points) Find the general solution of the system of equations

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}.$$

$\begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix} = 0$

$4 + 4\lambda + \lambda^2 - 1 = 0$

$\lambda^2 + 4\lambda + 3 = 0$

$(\lambda+1)(\lambda+3) = 0$

$\lambda = -1, -3$

$-2x + y = -x$

$x - 2y = -y$

$-x + y = 0$

$x - y = 0$

$x = y$

$\lambda = -1, \vec{v} = \langle 1, 1 \rangle$

$\lambda = -3, \vec{v} = \langle 1, -1 \rangle$

$-2x + y = -3x$

$x - 2y = -3y$

$x + y = 0$

$x = -y$

$\vec{x}(t) = c_1 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + c_2 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

10

[1]This problem continues on next page.

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix} = \begin{bmatrix} ad-bc & 0 \\ 0 & ad-bc \end{bmatrix}$$

(b) (10 points) Find a particular solution to the nonhomogeneous system of equations

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix}\vec{x} + \begin{pmatrix} 2e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

$$X = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} \qquad \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix}\begin{bmatrix} 0 \\ e^{2t} \end{bmatrix} = \begin{bmatrix} e^{-t} \\ -e^{-t} \end{bmatrix}$$

Hint: You solved the homogeneous system in part (a).

$$\begin{vmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{vmatrix} = -e^{-4t} - e^{-4t} = -2e^{-4t} \qquad \hat{X} = \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{t} & -e^{-3t} \end{bmatrix} g(t)$$

$$\frac{1}{-2}e^{4t}\begin{bmatrix} -e^{-3t} & -e^{-3t} \\ -e^{-t} & e^{-t} \end{bmatrix} = \begin{bmatrix} \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \\ \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \end{bmatrix} = X^{-1}$$

$$\cancel{\begin{bmatrix} -e^{-t} & -3e^{-3t} \\ -e^{-t} & 3e^{-3t} \end{bmatrix}\vec{g}(t)} + \begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} g'(t) = \cancel{\begin{bmatrix} -e^{-t} & -3e^{-3t} \\ -e^{-t} & 3e^{-3t} \end{bmatrix}} g(t) + \begin{bmatrix} 2e^{-t} \\ -2e^{-t} \end{bmatrix}$$

$$\begin{bmatrix} e^{-t} & e^{-3t} \\ e^{-t} & -e^{-3t} \end{bmatrix} g'(t) = \begin{bmatrix} 2e^{-t} \\ -2e^{-t} \end{bmatrix}$$

$$\frac{1}{\cancel{t}} \quad e^{2t} + e^{2t}$$

$$g'(t) = \begin{bmatrix} \frac{1}{2}e^{t} & \frac{1}{2}e^{t} \\ \frac{1}{2}e^{3t} & -\frac{1}{2}e^{3t} \end{bmatrix}\begin{bmatrix} 2e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ 2e^{2t} \end{bmatrix}$$

$$g(t) = \begin{bmatrix} 0 \\ e^{2t} \end{bmatrix}$$

$$\vec{X}(t) = e^{-t}\begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

10

4. (10 points) Use the method of **undetermined coefficients** to find the general solution of the equation

$$y'' - 2y' - 3y = -3e^{-x}.$$

$r^2 - 2r - 3 = 0$

$(r+1)(r-3)$

$r = -1 \quad r = 3$

$y(x) = axe^{-x}$

$y'(x) = ae^{-x} - axe^{-x}$

$y''(x) = -ae^{-x} + axe^{-x} - ae^{-x}$

$y''(x) = -2ae^{-x} + axe^{-x}$

$-2y'(x) = -2ae^{-x} + 2axe^{-x}$

$-3y(x) = -3axe^{-x}$

$-3e^{-x} = -4ae^{-x}$

$a = \frac{3}{4}$

$y(x) = \frac{3}{4}xe^{-x}$

$Y(x) = C_1e^{-x} + C_2e^{3x} + \frac{3}{4}xe^{-x}$

Check w/ E:

$y'(x) = -C_1e^{-x} + 3C_2e^{3x} + \frac{3}{4}e^{-x} - \frac{3}{4}xe^{-x}$

$y''(x) = C_1e^{-x} + 9C_2e^{3x} - \frac{3}{4}e^{-x} - \frac{3}{4}e^{-x} + \frac{3}{4}xe^{-x}$

$y''(x) = C_1e^{-x} + 9C_2e^{3x} + \frac{3}{4}xe^{-x} - \frac{3}{2}e^{-x}$

$-2y'(x) = 2C_1e^{-x} - 6C_2e^{3x} + \frac{3}{2}xe^{-x} - \frac{3}{2}e^{-x}$

$-3y(x) = -3C_1e^{-x} - 3C_2e^{3x} - \frac{9}{4}xe^{-x}$

$-3e^{-x} = -3e^{-x}$ ✓

10

5. (10 points) Solve the system of equations

$$\vec{x}' = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix} \vec{x}.$$

$\begin{vmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{vmatrix} = 0$

$\lambda^2 - 4 + 8 = 0$

$\lambda^2 + 4 = 0$

$\lambda = \pm 2i$

$2x - 4y = 2ix$

$(2-2i)x = 4y$

$\frac{x}{4} = \frac{y}{2-2i}$

$\frac{x}{2} = \frac{y}{1-i}$

$\lambda = 2i, \vec{v} = \langle 2, 1-i \rangle$

$\lambda = 2i, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\lambda = -2i, \vec{v} = \begin{bmatrix} 2 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$

$\vec{x}_1 = (\cos(2t) + i\sin(2t))\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$

$= \begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos(2t) + \begin{bmatrix} 2 \\ 1 \end{bmatrix} i \sin(2t) + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(2t)$

$\vec{x}_2 = (\cos(2t) - i\sin(2t))\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix} - i \begin{bmatrix} 0 \\ -1 \end{bmatrix}\right)$

$= \left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos(2t) - i \begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin(2t) - i \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(2t)\right.$

$x_3 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(2t)$

$x_4 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(2t)$

$\vec{x}(t) = c_1\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(2t)\right) + c_2\left(\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin(2t) + \begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(2t)\right)$

Check work

$c_1 = 1$ $c_2 = 0$

$\vec{x}(t) = \begin{bmatrix} 2 \\ 1 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -1 \end{bmatrix}\sin(2t)$ ✓

$\vec{x}'(t) = -2\begin{bmatrix} 2 \\ 1 \end{bmatrix}\sin(2t) - 2\begin{bmatrix} 0 \\ -1 \end{bmatrix}\cos(2t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos(2t) - \begin{bmatrix} 4 \\ 2 \end{bmatrix}\sin(2t)$

$A\vec{x}(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\cos(t) - \begin{bmatrix} 4 \\ 2 \end{bmatrix}\sin(2t)$

$c_1 = 0$ $c_2 = 1$

$\vec{x}'(t) = \begin{bmatrix} 4 \\ 2 \end{bmatrix}\cos(2t) - \begin{bmatrix} 0 \\ -2 \end{bmatrix}\sin(2t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\sin(2t) + \begin{bmatrix} 4 \\ 2 \end{bmatrix}\cos(2t)$ ✓

$A\vec{x}(t) = \begin{bmatrix} 0 \\ 2 \end{bmatrix}\sin(2t) + \begin{bmatrix} 4 \\ 2 \end{bmatrix}\cos(2t)$

9/10

$2y^{\frac{1}{2}}$ $y^{-\frac{1}{2}}$

6. (10 points) Find the general solution of the equation

$$\left(2\sqrt{y} + \frac{1}{x^2}\right)\,dx + \left(\frac{x}{\sqrt{y}} + \frac{1}{1+y^2}\right)\,dy = 0.$$

Check for exactness

$N_x = \frac{1}{\sqrt{y}}$ $M_y = y^{-\frac{1}{2}} = \frac{1}{\sqrt{y}}$ $N_x = M_y$

$\int (2y^{\frac{1}{2}} + x^{-2})\,dx = 2xy^{\frac{1}{2}} - x^{-1} + C(y)$

$\frac{d}{dy}[2xy^{\frac{1}{2}} - x^{-1} + C(y)] = xy^{-\frac{1}{2}} + C'(y)$

$\frac{x}{\sqrt{y}} + C'(y) = \frac{x}{\sqrt{y}} + \frac{1}{1+y^2}$

$C'(y) = \frac{1}{1+y^2}$

$\int \frac{dy}{1+y^2}$

$y = \tan(\theta)$

$dy = \sec^2(\theta)\,d\theta$

$\theta = \tan^{-1}(y)$

$1 + \tan^2(\theta) = \sec^2(\theta)$

$\left(\frac{\sin}{\cos}\right)' = \frac{\cos^2 + \sin^2}{\cos^2} = \frac{1}{\cos^2} = \sec^2$

$\int \frac{\sec^2(\theta)\,d\theta}{1+\tan^2(\theta)} = \int \frac{\sec^2\theta}{\sec^2\theta}\,d\theta = \int d\theta = \theta$

$C(y) = \tan^{-1}(y)$

$2x\sqrt{y} - \frac{1}{x} + \tan^{-1}(y) = C$

10

7. (10 points) Use **any method learned in this class** to solve the initial value problem

$$y'' - 8y' + 16y = 0,\ y(0) = 5,\ y'(0) = 3.$$

$r^2 - 8r + 16 = 0$

$(r-4)^2 = 0$

$r = 4$

$y(x) = C_1 e^{4x} + C_2 x e^{4x}$

$y'(x) = 4C_1 e^{4x} + C_2 e^{4x} + 4C_2 x e^{4x}$

$y(0) = C_1 = 5$ $\qquad C_1 = 5$

$y'(0) = 4C_1 + C_2 = 3$

$4 \cdot 5 + C_2 = 3$

$20 + C_2 = 3$

$C_2 = -17$

$y(x) = 5e^{4x} - 17xe^{4x}$

10

8. (15 points) Solve the system of equations

note: triangle matrix has eigenvalues on diagonals

$$\vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} \vec{x}.$$

$\lambda_1 = \lambda_2 = 1$

$\lambda_3 = 3$

$x = x$ $\quad 0 = 0$

$x + 3y = y$ $\quad x + 2y = 0$

$y + z = z$ $\quad y = 0$

$x = 0$

$y = 0$

$\lambda = 1: \vec{v} = \langle 0, 0, 1 \rangle$

$\lambda = 3: \vec{v} = \langle 0, 2, 1 \rangle$

$x = 3x$

$x + 3y = 3y$

$y + z = 3z$

$0 = 2x \quad x = 0$

$0 + 3y = 3y$

$0 = 0$

$y = 2z \quad \frac{y}{2} = z$

$w = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$$

$$\begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} x_1 \\ x_1 + 3x_2 \\ x_2 + x_3 \end{bmatrix}$$

$x_1 = x_1$

$x_2 = x_1 + 3x_2$ $\quad x_1 + 2x_2 = 0$

$x_3 + 1 = x_2 + x_3$ $\quad x_2 = 1 \quad x_1 + 2 = 0$

$x_1 = -2$

$$\boxed{\vec{x}(t) = c_1 e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + c_2 \left(t e^t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + e^t \begin{bmatrix} -2 \\ 1 \\ 0 \end{bmatrix} \right) + c_3 e^{3t} \begin{bmatrix} 0 \\ 2 \\ 1 \end{bmatrix}}$$

$w = \langle -2, 1, t \rangle$

$t = 0$

15

Laplace Transform Table:

$f(x)$	$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-st} f(t)dt$
C	$\frac{C}{s}$
$x^n, n = 1, 2, 3, \ldots$	$\frac{n!}{s^{n+1}}$
e^{ax}	$\frac{1}{s-a}$
$\sin(ax)$	$\frac{a}{s^2+a^2}$
$\cos(ax)$	$\frac{s}{s^2+a^2}$
$e^{ax}x^n$	$\frac{n!}{(s-a)^{n+1}}$
$e^{ax}\sin(bx)$	$\frac{b}{(s-a)^2+b^2}$
$e^{ax}\cos(bx)$	$\frac{s-a}{(s-a)^2+b^2}$
$u(x-a)$	$\frac{e^{-as}}{s}$
$f'(x)$	$sF(s) - f(0)$
$f''(x)$	$s^2F(s) - sf(0) - f'(0)$
$f(x-a)u(x-a)$	$e^{-as}F(s)$
$e^{ax}f(x)$	$F(s-a)$
$-xf(x)$	$F'(s)$

$\vec{X}_2 = te^{\lambda t}\vec{V} + e^{\lambda t}\vec{W}$

$A\vec{X}_2 = \lambda te^{\lambda t}\vec{V} + e^{\lambda t}A\vec{W}$

$\vec{X}_2' = \lambda te^{\lambda t}\vec{V} + e^{\lambda t}\vec{V} + \lambda e^{\lambda t}\vec{W}$

$\lambda te^{\lambda t}\vec{V} + e^{\lambda t}\vec{V} + \lambda e^{\lambda t}\vec{W} = \lambda te^{\lambda t}\vec{V} + e^{\lambda t}A\vec{W}$

$\vec{V} = (A - \lambda I)\vec{W}$

$\lambda\vec{W} + \vec{V} = A\vec{W}$