Problem 1. Prove that if $0 = 1$ in a ring R, then R is the zero ring (that is, its only element is 0).

Proof. $\forall r \in R$, $r = 1 \cdot r = 0 \cdot r = 0$. Thus, $R = \{0\}$.

□

Problem 2. A category C' is a subcategory of a category C if

1) $\text{Obj}(C') \subseteq \text{Obj}(C)$, and
2) for all $A, B \in \text{Obj}(C')$, $\text{Hom}_{C'}(A,B) \subseteq \text{Hom}_C(A,B)$.

It is a full subcategory if in addition $\text{Hom}_{C'}(A,B) = \text{Hom}_C(A,B)$ for all $A, B \in \text{Obj}(C')$.

Ring is a subcategory of Rng (you do not need to prove this). Show that it is not a full subcategory.

Proof. $\varphi : \mathbb{Z}_6 \mapsto \mathbb{Z}_6$ defined via $\varphi(a) = 2a$ belongs to $\text{Hom}_{\text{Rng}}(\mathbb{Z}_6, \mathbb{Z}_6)$ but not $\text{Hom}_{\text{Ring}}(\mathbb{Z}_6, \mathbb{Z}_6)$ since $\varphi(1) = 2 \neq 1$, and a ring homomorphism in $\text{Hom}_{\text{Ring}}(\mathbb{Z}_6, \mathbb{Z}_6)$ must preserve 1. Thus, $\text{Hom}_{\text{Ring}}(\mathbb{Z}_6, \mathbb{Z}_6) \subset \text{Hom}_{\text{Rng}}(\mathbb{Z}_6, \mathbb{Z}_6) \implies$ Ring is not a full subcategory of Rng.

□

Problem 3. Let a and b be zero-divisors in a ring R. Either prove $a + b$ is always a zero-divisor or provide a specific counterexample.

Proof. Consider $[2], [3] \in \mathbb{Z}_6$. $[2] \cdot [3] = [0]$ and both are non-zero, so they are both zero-divisors. Yet $[2] + [3] = [5]$, which isn't a zero divisor since $([5][m])_{m=0}^{5} = ([0], [5], [4], [3], [2], [1])$. Thus, the sum of two zero-divisors is not always a zero-divisor.

□

Problem 4. The center of a ring R is $Z(R) = \{z \in R \mid rz = zr \text{ for all } r \in R\}$. Prove that the center of a ring R is a subring of R.

Proof. $\forall a, b \in Z(R)$, and $\forall r \in R$,

$$[\mathbf{1}] : 1r = r1 = r \implies 1 \in Z(R)$$

$$[\leq_+] : (a-b)r = ar - br = (ra) - (rb) = r(a-b) \implies a - b \in Z(R)$$

$$[\cdot] : (ab)r = a(br) = a(rb) = (ar)b = (ra)b = r(ab) \implies ab \in Z(R).$$

Thus, $Z(R)$ is a subring of R.

□

Problem 5. An element x in a ring R is called nilpotent if $x^m = 0$ for some $m \in \mathbb{Z}^+$ (here x^m denotes $x \cdot x \cdots x$ (m times)).

Prove that the nilpotent elements of a commutative ring R form an ideal (this is called the nilradical of R).

Proof. Let R be a commutative ring with unity, and let $\text{nil}(R)$ be the set of all nilpotent elements of R; $\text{nil}(R) = \{x \in R \mid x^m = 0 \text{ for some } m \in \mathbb{Z}^+\}$. Let $x, y \in \text{nil}(R)$. Then $x^m = 0, y^n = 0$ for some $m, n \in \mathbb{Z}^+$. Observe.

$$(x-y)^{mn} = \sum_{i=0}^{m} \binom{m}{i} x^i (-y)^{mn-i} =$$

□

Problem 6. Let $n \in \mathbb{N}$.

(a) Show that if $n = ab$ for some integers a, b, then ab is a nilpotent element of $\mathbb{Z}/n\mathbb{Z}$.

(b) If $a \in \mathbb{Z}$, show that the element $a \in \mathbb{Z}/n\mathbb{Z}$ is nilpotent if and only if every prime divisor of n divides a. In particular, determine the nilpotent elements of $\mathbb{Z}/72\mathbb{Z}$.

Proof. □

Problem 7. Let R and S be rings.

(a) Prove that the direct product $R \times S = \{(r,s) \mid r \in R,\ s \in S\}$ forms a ring under componentwise addition and multiplication.

(b) Prove that $R \times S$ is commutative if and only if both R and S are commutative.

Proof. □

Problem 8. Let R be a commutative ring. Define the ring $R[[x]]$ of formal power series by

$$R[[x]] = \left\{ \sum_{n=0}^{\infty} a_n x^n \;\middle|\; a_i \in R \right\}.$$

(a) Prove that $R[[x]]$ is a commutative ring, and be sure to explain how to add and multiply elements.

(b) Show that $1 - x$ is a unit with inverse $1 + x + x^2 + \cdots = \sum_{n=0}^{\infty} x^n$.

(c) (Optional Challenge) Prove that $\sum_{n=0}^{\infty} a_n x^n$ is a unit in $R[[x]]$ if and only if a_0 is a unit in R.

Proof. □

Problem 9. Decide which of the following are ring homomorphisms from $M_2(\mathbb{Z})$ to $\mathbb{Z}$:

(a) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a$

(b) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto a + d$

(c) $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto ad - bc$

Proof. □

Problem 10. Let

$$R = \left\{ \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \;\middle|\; a, b, d \in \mathbb{Z} \right\}.$$

Prove that the map

$$\varphi : R \to \mathbb{Z} \times \mathbb{Z}, \qquad \varphi\left(\begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \right) = (a, d)$$

is a surjective ring homomorphism and describe its kernel.

Proof. □

Problem 11. Decide which of the following are ideals of $\mathbb{Z} \times \mathbb{Z}$:

(a) $\{(a,a) \mid a \in \mathbb{Z}\}$

(b) $\{(2a,2b) \mid a,b \in \mathbb{Z}\}$

(c) $\{(2a,0) \mid a \in \mathbb{Z}\}$

(d) $\{(a,-a) \mid a \in \mathbb{Z}\}$

Proof. □

Problem 12. The characteristic of a ring R (denoted $\operatorname{char} R$) is the smallest $n \in \mathbb{N}$ such that

$$\underbrace{1_R + \cdots + 1_R}_{n \text{ times}} = 0,$$

and if there is no such n we say the characteristic of R is 0.

Prove that an integral domain has characteristic 0 or a prime.

Proof. □