Turn in problems 54, 55, 57, 58, 59, 60, 61, 62, 63, 64, 65.

Problem 54. If $f \in \mathbb{K}[X]$ (with $\mathbb{K}$ field) has degree n and $\mathbb{F}$ is a splitting field of f over $\mathbb{K}$, prove that $[\mathbb{F} : \mathbb{K}] \mid n!$.

Proof. f has at most n distinct zeros. If $n = 0$, then f is constant and splits in $\mathbb{K} = \mathbb{F}$ so $[\mathbb{F} : \mathbb{K}] = 1 \mid n!$ (arguably). Suppose $n \geq 1$. Let (a_i) be the k-tuple consisting of all $1 \leq k \leq n$ distinct zeros of f in (strictly) ascending order; $a_1 < \cdots < a_k$. Then $\mathbb{F} \cong \mathbb{K}(a_1, \ldots, a_k)$. Now consider $[\mathbb{F} : \mathbb{K}]$.

□

Problem 55. If $K \subseteq F$ is a field extension, F is algebraically closed, and E is the set of all elements of F that are algebraic over K, prove that E is an algebraic closure of K.

Problem 57. If $[F : K] = 2$, then $K \subseteq F$ is a normal extension.

Problem 58. If d is a nonnegative rational number, then $\mathrm{Aut}_{\mathbb{Q}}\big(\mathbb{Q}(\sqrt{d})\big)$ is the identity or is isomorphic to $\mathbb{Z}_2$.

Problem 59. What is the Galois group of $\mathbb{Q}\Big(\sqrt{2}, \sqrt{3}, \sqrt{5}\Big)$ over $\mathbb{Q}$?

Problem 60. Assume K is a field of characteristic 0. Let G be the subgroup of $\mathrm{Aut}_K\big(K(X)\big)$ generated by the K-automorphism induced by $X \mapsto X + 1$. Prove that G is an infinite cyclic group. What is the fixed field E of G? What is $[K(X) : E]$?

Problem 61. Let k be a finite field of characteristic $p > 0$.

(a) Prove that for every $n > 0$ there exists an irreducible polynomial $f \in k[X]$ of degree n.

(b) Prove that for every irreducible polynomial $P \in k[X]$ there exists $n \geq 0$ such that P divides $X^{p^n} - X$.

Problem 62. Let p be a prime and $\mathbb{F}_q$ (with $q = p^s$) be the finite field with q elements. Let $f \in \mathbb{F}_q[X]$ be an irreducible polynomial. Prove that f is irreducible in $\mathbb{F}_{q^m}[X]$ if and only if m and $\deg f$ are relatively prime.

Problem 63. Prove that $E = \mathbb{F}_2[X]/(X^4 + X^3 + 1)$ is a field with 16 elements. What are the roots of $X^4 + X^3 + 1$ in E?

Problem 64. Prove that an algebraic extension of a perfect field is a perfect field.

Problem 65. Show that the extension $\mathbb{Q} \subseteq \mathbb{Q}\left(\sqrt[4]{2}, i\right)$ is Galois. Find its Galois group.