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Normal Subgroups. Factor (Quotient) groups

G group, $N \leq G$ subgr.

We say that N is **normal** in G ($N \trianglelefteq G$) if

$$aNa^{-1} \subseteq N \quad \text{for all } a \in G.$$

(i.e. $\underbrace{ana^{-1}}_{\text{conjugate of } n} \in N$ for all $a \in G$ and all $n \in N$.)

Obs If $N \trianglelefteq G$, then

(a) $aNa^{-1} = N$ for all $a \in G$

(b) $aN = Na$ for all $a \in G$

(c) $\equiv_l$ and $\equiv_r$ coincide

Proof (a) '$\subseteq$' def

Also $N = a^{+1}\underbrace{(a^{-1}Na^{+1})}_{\subseteq N} \cdot a^{-1} \subseteq a^{+1}Na^{-1}$ $\boxtimes$

Ex G abelian $\Rightarrow$ every subgr. is normal

Ex $\left.\begin{array}{l} H \leq G \\ [G:H] = 2 \end{array}\right\} \Longrightarrow H \trianglelefteq G$ (exercise)

Ex $\left.\begin{array}{l} H \leq G \\ [G:H] = p \text{ prime} \\ \text{and } p \text{ is the smallest} \\ \text{prime that divides } |G| \end{array}\right\} \Longrightarrow H \trianglelefteq G$

__Theorem__ (Construction)

Assume $N \trianglelefteq G$

Then $G/N = (G/N)_{\equiv l} = (G/N)_{\equiv r}$

is a group with operation:

$$aN \cdot bN \overset{def}{=} (ab)N$$

well-defined. Why?

__Proof__ $\left.\begin{aligned} aN &= a'N \\ bN &= b'N \end{aligned}\right\} \overset{?}{\Longrightarrow} abN = a'b'N$

Write

$a = a' \cdot n_1$ with $n_1 \in N$

$b = b' \cdot n_2$ with $n_2 \in N$. Then $a \cdot b = a' \underbrace{n_1} b' n_2$

But $b'N = Nb'$.

Write $n_1 b' = b' \cdot n_3$ with $n_3 \in N$.

Then $a \cdot b = a' b' \underbrace{n_3 n_2}_{\in N} \in a'b'N$. So $abN = a'b'N$.

- Identity element $eN = N$ (Recall $aN = N \Longleftrightarrow a \in N$)
- $(aN)^{-1} = a^{-1}N$.

__Theorem__ $K, N \leq G$ with $N \trianglelefteq G$. Then

(i) $N \cap K \trianglelefteq K$

(ii) $N \trianglelefteq N \vee K \overset{\text{notation}}{=} \langle N \cup K \rangle$

(iii) $NK = N \vee K = KN$

In part, $NK = KN$ is a <u>subgroup</u> of G.

(iv) If $K \trianglelefteq G$ and $N \cap K = \{e\}$, then

$n \cdot k = k \cdot n$ for all $n \in N$, $k \in K$.

(v) If $K \trianglelefteq G$, then $KN \trianglelefteq G$ (normal subgr.)

__Proof__ (i) $\alpha \in N \cap K$, $k \in K$

Then $\underbrace{k \cdot \alpha \cdot k^{-1}}_{\in K} \in N$ (because $N \trianglelefteq G$).

Thus $k \alpha k^{-1} \in N \cap K$.

(ii) clear. More generally

$$\left.\begin{array}{l} N \trianglelefteq G \\ N \leq H \leq G \end{array}\right\} \Longrightarrow N \trianglelefteq H.$$

(iii) $N \cdot K \subseteq \langle N \cup K \rangle = N \vee K$.

<u>**Claim**</u> NK subgroup of G.

$$(n_1 k_1) \cdot (n_2 k_2) = n_1 \underbrace{k_1 n_2 k_1^{-1}}_{\in N} \underbrace{k_1 k_2}_{\in K} \in NK.$$

So $NK \leq G$ (subgr).

But $N \cup K \subseteq NK \Longrightarrow \langle N \cup K \rangle \subseteq NK$

Thus $NK = \langle N \cup K \rangle$.

Similarly, we start with $K \cdot N \subseteq \langle N \cup K \rangle$

<u>Claim</u> KN subgr of G.

$(k_1 n_1) \cdot (k_2 n_2) = \underbrace{k_1 k_2}_{\in K} \cdot \underbrace{k_2^{-1} \cdot n_1 k_2}_{\in N} n_2 \in KN.$

Then $K \cup N \subseteq \underbrace{KN}_{\text{subgr.}} \Rightarrow \langle K \cup N \rangle \subseteq KN$

Thus $\langle K \cup N \rangle = KN$

(iv) $n \in N$, $k \in K$

$nkn^{-1} \in K$ because $K \trianglelefteq G$.

Also $\cancel{nkn^{-1}k^{-1}} \in K$

$n \underbrace{kn^{-1}k^{-1}}_{\in N} \in N \cap K = \{e\}$

So $nkn^{-1}k^{-1} = e$, i.e. $nk = kn$ ☒

Exercise In this situation, $HK \cong H \times K$

(v) $g \in G$

$$g(kn) \cdot g^{-1} = \underbrace{gkg^{-1}}_{\in K} \underbrace{gng^{-1}}_{\in N} \in K \cdot N \qquad \blacksquare$$