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Last time

$H_1 \trianglelefteq G,\ H_2 \trianglelefteq G$

$G = H_1 H_2$

$H_1 \cap H_2 = \{e\}$

with these assumptions, we know that $h_1 h_2 = h_2 h_1$ for $h_1 \in H_1$, $h_2 \in H_2$

Then

$$\boxed{\begin{array}{c} H_1 \times H_2 \cong G \\ (h_1, h_2) \to h_1 h_2 \end{array}}$$

(We say that G is the <u>internal</u> <u>direct</u> <u>product</u> of H_1 and H_2

External Direct Product

G_1, G_2 groups

$$\underbrace{G_1 \times G_2}_{\text{group}} = \{(g_1, g_2) \mid g_1 \in G_1,\ g_2 \in G_2\}$$

$$(g_1, g_2)(g_1', g_2') = (g_1 g_1', g_2 g_2')$$

Observation G_1, G_2 groups. Then $\underbrace{G_1 \times \{e_{G_2}\}}_{\cong G_1} \trianglelefteq G_1 \times G_2$

$\underbrace{\{e_{G_1}\} \times G_2}_{\cong G_2} \trianglelefteq G_1 \times G_2$

and $G_1 \times G_2$ is the internal direct prod. of $G_1 \times \{e\}$ and $\{e\} \times G_2$

Assume

Proposition (i) $H_1 \trianglelefteq G, H_2 \trianglelefteq G, \ldots, H_k \trianglelefteq G$

(ii) $G = H_1 H_2 \cdots H_k$

(iii) $H_i \cap (H_1 H_2 \cdots H_{i-1} H_{i+1} \cdots H_k) = \{e\}$

for all $i = 1, \ldots, k$.

Then $H_1 \times H_2 \times \cdots \times H_k \cong G$

$(h_1, h_2, \ldots, h_k) \to h_1 h_2 \cdots h_k$

(G is the int. dir. product of $H_1, H_2, \ldots, H_k$)

Proof by induction (exercise)

Obs $H_1 \trianglelefteq G$, $H_2 \trianglelefteq G$, $H_1 \cap H_2 = \{e\}$, $G = H_1 H_2$

Then $G/H_1 \cong H_2/\{e\} = H_2$

and $G/H_2 \cong H_1$

Semidirect product

~~Def H ⊴ G, K ≤ G.~~

Def H, K groups

and $\tau : K \to \mathrm{Aut}(H)$

group homom.

$k \longrightarrow \tau(k) = \tau_k \in \mathrm{Aut}(H)$

(This means $\tau_{k_1 k_2} = \tau_{k_1} \circ \tau_{k_2}$ for all $k_1, k_2 \in K$)

$H \rtimes_\tau K$ (semidirect product of H and K)

is the set $H \times K$ with binary operation

$$(h, k) \cdot (h', k') = (h \tau_k(h'), k k')$$

Remark The semidirect product is the direct product

$\iff \tau_k(h') = h'$ for all k and all h'

Notation

$\underbrace{\mathrm{Aut}(H)}_{\text{this is a group}} = \{ f : H \to H \mid f \text{ bijective hom} \}$

$\Leftrightarrow Z_k = id_H$ for all k

$\Leftrightarrow Z$ is the trivial group hom. (i.e $Z(k) = Z_k = id_H$ for all $k \in K$)

Proposition $H \rtimes_Z K$ is a group.

Proof The operation is associative because $Z : K \to Aut(H)$ is a group homomorphism.

(Prove associativity)

Also, for all $(h, k) \in H \rtimes_Z K$ we have:

Check this! (exercise)

(1) $(h, k) \cdot (e_H, e_K) = (h, k)$

and $(e_H, e_K)(h, k) = (h, k)$

(2) $(h, k) \cdot (Z_{k^{-1}}(h^{-1}), k^{-1}) = (e_H, e_K)$ and

$(Z_{k^{-1}}(h^{-1}), k^{-1})(h, k) = (e_H, e_K)$

Proposition Let $G = H \rtimes_z K$

Let $H' = H \times \{e_K\}$ and $K' = \{e_H\} \times K$

Then $H' \trianglelefteq G$ (normal)

$K' \leqslant G$ (not nec. normal)

$H' \cap K' = \{(e_H, e_K)\}$

$G = H'K'$

Proof Define $f : G \to K$ by $f(h,k) = k$

Claim • f is a group hom. because

$$f\big(\underbrace{(h,k) * (h',k')}_{= (*,\, kk')}\big) \overset{?}{=} k \cdot k'$$

- f is surjective
- $\operatorname{Ker} f = H \times \{e\} = H'$

Thus $H' \trianglelefteq G$ (normal) and $G/H' \cong K$

Next : <u>prove</u> $K' \leq G$

$(e,k), (e,k') \in K' = \{e\} \times K$

$(e,k) \cdot (e,k') = (e\, \underbrace{\tau_k(e)}_{= e}, kk') = (e, kk') \in K'$

$(e,k) \in K' = \{e\} \times K$

$(e,k)^{-1} = (\tau_{k^{-1}}(e^{-1}), k^{-1}) = (e, k^{-1}) \in K'$

So $K' \leq G$.

To prove: $G = H'K'$

~~check this~~

$(h,k) \in G$. Then $(h,k) = \underbrace{(h,e) \cdot (e,k)}_{(h\tau_e(e),\, k)}$