MATH 266 – HW 10 Solutions (Problems 1, 3, 5)

Problem 1. Solve $\vec{x}' = A\vec{x}$, $\vec{x}(0) = \binom{1}{1}$, where

$$A = \begin{pmatrix} 1 & 5 \\ 1 & -3 \end{pmatrix}.$$

The characteristic polynomial is

$$\det(A - \lambda I) = \lambda^2 + 2\lambda - 8 = (\lambda+4)(\lambda-2),$$

so $\lambda_1 = -4$, $\lambda_2 = 2$.

For $\lambda_1 = -4$, $(A+4I)\vec{v}_1 = 0$ gives $x_1 + x_2 = 0$, so take

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -1 \end{pmatrix}.$$

For $\lambda_2 = 2$, $(A-2I)\vec{v}_2 = 0$ gives $-x_1 + 5x_2 = 0$, so take

$$\vec{v}_2 = \begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Thus the general solution is

$$\vec{x}(t) = C_1 e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2 e^{2t}\begin{pmatrix} 5 \\ 1 \end{pmatrix}.$$

Using $\vec{x}(0) = \binom{1}{1}$,

$$C_1\begin{pmatrix} 1 \\ -1 \end{pmatrix} + C_2\begin{pmatrix} 5 \\ 1 \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad \Rightarrow \quad C_1 = -\tfrac{2}{3},\ C_2 = \tfrac{1}{3}.$$

Hence

$$\boxed{\vec{x}(t) = -\frac{2}{3}e^{-4t}\begin{pmatrix} 1 \\ -1 \end{pmatrix} + \frac{1}{3}e^{2t}\begin{pmatrix} 5 \\ 1 \end{pmatrix}}$$

Problem 3. Solve $\vec{x}' = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 1 & 2 \\ 1 & 2 & 1 \\ 2 & 1 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda = 1, 4, -1$ with eigenvectors

$$\vec{v}_1 = \begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix}, \quad \vec{v}_2 = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad \vec{v}_3 = \begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}.$$

Thus

$$\boxed{\vec{x}(t) = C_1 e^{t}\begin{pmatrix} 1 \\ -2 \\ 1 \end{pmatrix} + C_2 e^{4t}\begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix} + C_3 e^{-t}\begin{pmatrix} -1 \\ 0 \\ 1 \end{pmatrix}}$$

Problem 5. Solve $\vec{x}' = A\vec{x}$, where

$$A = \begin{pmatrix} 1 & 2 & -1 \\ 0 & 1 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

The eigenvalues are $\lambda = 1,\ 1 \pm i$. An eigenvector for $\lambda = 1$ is $\binom{1}{0}0$. For $\lambda = 1+i$, take

$$\vec{v} = \begin{pmatrix} -1-2i \\ 1 \\ i \end{pmatrix} = \vec{a} + i\vec{b}, \quad \vec{a} = \begin{pmatrix} -1 \\ 1 \\ 0 \end{pmatrix}, \vec{b} = \begin{pmatrix} -2 \\ 0 \\ 1 \end{pmatrix}.$$

Real solutions are

$$\vec{x}_2(t) = e^{t}\begin{pmatrix} -\cos t + 2\sin t \\ \cos t \\ -\sin t \end{pmatrix}, \quad \vec{x}_3(t) = e^{t}\begin{pmatrix} -2\cos t - \sin t \\ \sin t \\ \cos t \end{pmatrix}.$$

Hence

$$\boxed{\vec{x}(t) = C_1 e^{t}\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} + C_2 e^{t}\begin{pmatrix} -\cos t + 2\sin t \\ \cos t \\ -\sin t \end{pmatrix} + C_3 e^{t}\begin{pmatrix} -2\cos t - \sin t \\ \sin t \\ \cos t \end{pmatrix}}$$