Submit the following from the problem list: 40, 41, 42, 45, 46, 47, 48, 49, 50, 51, 52.

Problem 40. Prove that an abelian group has a composition series if and only if it is finite.

Problem 41. Prove that a solvable simple group is abelian.

Problem 42. Prove that a solvable group that has a composition series is finite.

Problem 45. If $\mathbb{K} \subseteq \mathbb{F}$ is a field extension, $u, v \in \mathbb{F}$, v is algebraic over $\mathbb{K}(u)$, and v is transcendental over $\mathbb{K}$, then u is algebraic over $\mathbb{K}(v)$.

Problem 46. If $\mathbb{K} \subseteq \mathbb{F}$ is a field extension and $u \in \mathbb{F}$ is algebraic of odd degree over $\mathbb{K}$, then so is u^2 and $\mathbb{K}(u) = \mathbb{K}(u^2)$.

Problem 47. Let $\mathbb{K} \subseteq \mathbb{F}$ be a field extension. If $X^n - a \in \mathbb{K}[X]$ is irreducible and $u \in \mathbb{F}$ is a root of $X^n - a$ and m divides n, then the degree of u^m over $\mathbb{K}$ is n/m. What is the irreducible polynomial of u^m over $\mathbb{K}$?.

Problem 48. Let $\mathbb{K} \subseteq R \subseteq \mathbb{F}$ be an extension of rings with $\mathbb{K}, \mathbb{F}$ fields. If $\mathbb{K} \subseteq \mathbb{F}$ is algebraic, prove that R is a field.

Problem 49. Let $f = X^3 - 6X^2 + 9X + 3 \in \mathbb{Q}[X]$.

(a) Prove that f is irreducible in $\mathbb{Q}[X]$.

(b) Let u be a real root of f. Consider the extension $\mathbb{Q} \subseteq \mathbb{Q}(u)$. Express each of the following elements in terms of the basis $\{1, u, u^2\}$ of the $\mathbb{Q}$-vector space $\mathbb{Q}(u)$:

$$u^4, \quad u^5, \quad 3u^5 - u^4 + 2, \quad (u+1)^{-1}, \quad (u^2 - 6u + 8)^{-1}.$$

Problem 50. Let $F = \mathbb{Q}(\sqrt{2}, \sqrt{3})$. Find $[F : \mathbb{Q}]$ and a basis of $\mathbb{F}$ over $\mathbb{Q}$.

Problem 51. Let $\mathbb{K}$ be a field. In the field $\mathbb{K}(X)$, let $u = X^3/(X+1)$. What is $[\mathbb{K}(X) : \mathbb{K}(u)]$?

Problem 52. Let $\mathbb{K} \subseteq \mathbb{F}$ be a field extension. If $u, v \in \mathbb{F}$ are algebraic over $\mathbb{K}$ of degrees m and n, respectively, then $[\mathbb{K}(u,v) : \mathbb{K}] \leq mn$. If m and n are relatively prime, then $[\mathbb{K}(u,v) : \mathbb{K}] = mn$.