

MATH 266 - NDSU  
Fall 2025  
Final Exam  
December 17th, 2025  
Time Limit: 120 Minutes

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This exam contains 10 pages (including this cover page) and 8 questions. Total of points is 100. Read all of the following information before starting the exam:

- Present your work clearly and in order, and justify your conclusions. We reserve the right to take off points if we cannot see how you arrived your answers or cannot read your handwriting clearly.
- No calculators are allowed. You may use a double-sided full page note sheet.
- Write your answers on the space provided. Feel free to use the other side of each page if more space is needed, but please indicate you have done so, otherwise the back side of each page will be regarded as scratch work and **will not be graded**.
- A Laplace transform table is provided separately.
- Good luck!

Grade Table

Question	Points	Score
1	10	3
2	15	8
3	20	12
4	10	1
5	10	4
6	10	0
7	10	10
8	15	9
Total:	100	47

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1. (10 points) A tank contains 10 gal of brine made by dissolving 6 lb of salt in water. Salt water containing 1 lb of salt per gallon runs in at the rate of 2 gal/min and the well-stirred mixture runs out at the rate of 3 gal/min. Find the amount of salt in the tank after  $t$  minutes.

$r$ : rate  
 $C$ : concentration

$$A'(t) = r_{in} C_{in} - r_{out} C_{out}$$

$$C_{out} = \frac{A}{V}$$

$$A(0) = 6$$

$$V = 10$$

$$A(t) =$$

$$r_{in} = 2 \text{ gal/min}$$

$$r_{out} = 3 \text{ gal/min}$$

$$C_{in} = 1 \text{ lb/gal}$$

$$C_{out} =$$

$$\int \frac{3}{10x} dx = 3 \int \frac{1}{10x} dx = 3 \ln(x)$$

$$A'(t) = (2)(1) - 3\left(\frac{A}{10}\right)$$

$$A'(t) = 2 - \frac{3}{10}A$$

$$A'(t) + \frac{3}{10}A = 2$$

$$A = e^{\int \frac{3}{10} dx} = 3 + e$$

$$y' + p(x) = q(x)$$

3/10

2. (15 points) Solve the initial value problem

Laplace?

$$y'' + 2y' + y = u(x-3), \quad y(0) = 0, \quad y'(0) = 1.$$

$$\mathcal{L}\{y''\} + 2\mathcal{L}\{y'\} + \mathcal{L}\{y\} = \mathcal{L}\{u(x-3)\}$$

$$[s^2 F(s) - s f(0) - f'(0)] + 2[s F(s) - f(0)] + F(s) = \frac{e^{-3s}}{s}$$

$$s^2 F - s(0) - (1) \quad \left[ \begin{array}{l} 2[s F(s) - (0)] \\ 2s F(s) \end{array} \right] + F(s)$$

$$s^2 F(s) - 1$$

$$F(s)(s^2 + 2s + 1) - 1 = \frac{e^{-3s}}{s}$$

$$F(s)(s+1)(s+1) = \frac{e^{-3s}}{s} + 1$$

$$F(s) = \frac{e^{-3s}}{s(s+1)(s+1)} + \frac{1}{(s+1)(s+1)}$$

$$F(s) = \frac{e^{-3s}}{s} \cdot \frac{1}{(s+1)^2} + \frac{1}{(s+1)^2}$$

$$\mathcal{L}^{-1}\left\{\frac{e^{-3s}}{s}\right\} = u(x-3)$$

$$\mathcal{L}^{-1}\left\{\frac{1}{(s+1)^2}\right\} = e^{-x}x$$

$$y(x) = u(x-3)e^x + e^x x$$

3. (20 points) This problem consists of two parts.

(a) (10 points) Find the general solution of the system of equations

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x}.$$

$$x' = Ax$$

$$A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 1 \\ 1 & -2-\lambda \end{vmatrix}$$

$$(-2-\lambda)(-2-\lambda) - (1)(1)$$

$$4 + 2\lambda + 2\lambda + \lambda^2 - 1$$

$$\lambda^2 + 4\lambda + 3$$

$$(\lambda + 3)(\lambda + 1)$$

$$\lambda_1 = -3 \quad \lambda_2 = -1$$

$$(A - (-3)I) = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$x + y = 0$$

$$x = -y$$

$$\begin{matrix} x=1 \\ y=-1 \end{matrix} \quad v_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$(A - (-1)I) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 8 \\ 8 \end{bmatrix}$$

$$-x + y = 0$$

$$x = y$$

$$x = y$$

$$\begin{matrix} x=1 \\ y=1 \end{matrix} \quad v_2 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$x_1(t) = Ce^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + Ce^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

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<sup>1</sup>This problem continues on next page.

(b) (10 points) Find a particular solution to the nonhomogeneous system of equations

$$\vec{x}' = \begin{pmatrix} -2 & 1 \\ 1 & -2 \end{pmatrix} \vec{x} + \begin{pmatrix} 2e^{-t} \\ -2e^{-t} \end{pmatrix}.$$

Hint: You solved the homogeneous system in part (a).

homogeneous:  $x_h(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$

$$x_p(t) = \begin{bmatrix} a \\ b \end{bmatrix} \Rightarrow x_p' = 0$$

$$0 = Ax_p + b \Rightarrow \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} + \begin{bmatrix} 2e^{-t} \\ -2e^{-t} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$-2a + b + 2e^{-t} = 0$$

$$a - 2b - 2e^{-t} = 0$$

$$(a - 2b - 2e^{-t}) + (-2a + b + 2e^{-t}) = 0$$

$$-a - b = 0$$

$$-a = b \quad a = 1$$

$$-b = a \quad b = -1$$

$$x_p(t) = C_1 e^{-3t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + C_2 e^{-t} \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

4. (10 points) Use the method of **undetermined coefficients** to find the general solution of the equation

 $y_h$ 

$$y'' - 2y' - 3y = -3e^{-x}.$$

$$y'' - 2y' - 3y = 0$$

$$r^2 - 2r - 3 = 0$$

$$(r-3)(r+1)$$

$$r=3, r=-1$$

$$y_h = c_1 e^{3x} + c_2 e^{-x}$$

$$y = y_p + y_h$$

$$y_p =$$

5. (10 points) Solve the system of equations

$$\vec{x}' = \begin{pmatrix} 2 & -4 \\ 2 & -2 \end{pmatrix} \vec{x} \quad C_1 e^{\alpha t} \begin{bmatrix} \cos \beta t \\ \sin \beta t \end{bmatrix} + C_2 e^{\alpha t} \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$$

$$\vec{x}' = A\vec{x}$$

$$A = \begin{bmatrix} 2 & -4 \\ 2 & -2 \end{bmatrix}$$

$$\det(A - \lambda I)$$

$$\begin{bmatrix} 2-\lambda & -4 \\ 2 & -2-\lambda \end{bmatrix}$$

$$(2-\lambda)(-2-\lambda) - (-4)(2)$$

$$-4 - 2\lambda + 2\lambda + \lambda^2 + 8$$

$$\lambda^2 + 2\lambda - 2\lambda + 8 - 4$$

$$\lambda^2 + 4$$

$$\lambda = 1 \pm 2i$$

$$= \alpha + \beta i$$

$$(A - 2iI)$$

$$\begin{bmatrix} 2i & -4 \\ 2 & -4i \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$2xi - 4y = 0 \quad y = \frac{2x}{4}i$$

$$2x - 4yi = 0$$

$$2x - 4\left(\frac{2x}{4}i\right)i = 0$$

$$2x - (4xi)i = 0$$

4/10

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$$\vec{x}(t) = C_1 e^t \begin{bmatrix} \cos(2t) \\ \sin(2t) \end{bmatrix} + C_2 e^t \begin{bmatrix} \sin(2t) \\ \cos(2t) \end{bmatrix}$$

6. (10 points) Find the general solution of the equation

$$\int \frac{1}{1+y^2} dy = \arctan(y)$$

$$\left(2\sqrt{y} + \frac{1}{x^2}\right) dx + \left(\frac{x}{\sqrt{y}} + \frac{1}{1+y^2}\right) dy = 0.$$

0

7. (10 points) Use any method learned in this class to solve the initial value problem

$$y'' - 8y' + 16y = 0, y(0) = 5, y'(0) = 3. \quad \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

$$y'' + 8y' + 16y = 0$$

$$r^2 - 8r + 16 = 0$$

$$(r-4)^2 = 0$$

$$r_1 = r_2 = 4$$

$$y = C_1 e^{4x} + C_2 x e^{4x}$$

$$y' = 4C_1 e^{4x} + (4C_2 x e^{4x} + C_2 e^{4x})$$

$$y=5, x=0$$

$$5 = C_1$$

$$y'=3, x=0$$

$$3 = 4C_1 + C_2$$

$$3 = 4(5) + C_2$$

$$3 = 20 + C_2$$

$$-17 = C_2$$

$$y = C_1 e^{4x} + C_2 x e^{4x} \Rightarrow y = e^{4x}(C_1 + C_2 x)$$

$\Downarrow$

$$y = e^{4x}(5 - 17x) \Rightarrow y = 5e^{4x} - 17xe^{4x}$$

8. (15 points) Solve the system of equations

$$\vec{x}' = \begin{pmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{pmatrix} \vec{x}.$$

$$\lambda_1 = 1 \quad \lambda_2 = 3$$

$$x_1 = C_1 e^{t\sqrt{v}} \quad x_2 = C_2 e^{3t} w$$

$$A = \begin{bmatrix} 1 & 0 & 0 \\ 1 & 3 & 0 \\ 0 & 1 & 1 \end{bmatrix} \rightarrow \det(A - \lambda I) = \begin{vmatrix} 1-\lambda & 0 & 0 \\ 1 & 3-\lambda & 0 \\ 0 & 1 & 1-\lambda \end{vmatrix} = (1-\lambda) \det \begin{bmatrix} 3-\lambda & 0 \\ 1 & 1-\lambda \end{bmatrix}$$

$$(A - 3I)$$

$$\begin{bmatrix} -2 & 0 & 0 \\ 1 & 1 & 0 \\ 0 & 1 & -2 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$-2x = 0$$

$$x + y = 0$$

$$y - 2z = 0$$

$$x = 0$$

$$y = 0$$

$$z = 0$$

$$v_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$(A - I)$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$x + 2y = 0$$

$$y = 0$$

$$z = 1 \quad \text{free variable}$$

$$x = 0$$

$$y = 0$$

$$v_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$(3-\lambda)(1-\lambda) - (0)(1)$$

$$3 - 3\lambda - \lambda + \lambda^2$$

$$\lambda^2 - 4\lambda + 3$$

$$(\lambda - 3)(\lambda - 1)$$

$$x(t) = C_1 e^{t \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right)}$$

$$x(t) = C_2 e^{3t} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

$$x(t) = C_1 e^{-3t} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 \\ 1 & 2 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad w = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$$

$$x + 2y = 0 \quad z = 1$$

$$y = 1$$

$$x = -2$$

$$x(t) = C_1 e^{t \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}} + C_2 e^{t \left( \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix} \right)} + C_3 e^{-3t} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$$

### Laplace Transform Table:

$f(x)$	$\mathcal{L}\{f(x)\} = F(s) = \int_0^\infty e^{-st} f(t) dt$
$C$	$\frac{C}{s}$
$x^n, n = 1, 2, 3, \dots$	$\frac{n!}{s^{n+1}}$
$e^{ax}$	$\frac{1}{s - a}$
$\sin(ax)$	$\frac{a}{s^2 + a^2}$
$\cos(ax)$	$\frac{s}{s^2 + a^2}$
$e^{ax} x^n$	$\frac{n!}{(s - a)^{n+1}}$
$e^{ax} \sin(bx)$	$\frac{b}{(s - a)^2 + b^2}$
$e^{ax} \cos(bx)$	$\frac{s - a}{(s - a)^2 + b^2}$
$u(x - a)$	$\frac{e^{-as}}{s}$
$f'(x)$	$sF(s) - f(0)$
$f''(x)$	$s^2 F(s) - sf(0) - f'(0)$
$f(x - a)u(x - a)$	$e^{-as} F(s)$
$e^{ax} f(x)$	$F(s - a)$
$-xf(x)$	$F'(s)$

$$(\lambda+3)(\lambda+1)$$

$$\lambda^2 + \lambda + 3\lambda + 3$$

$$\lambda^2 + 4\lambda + 3$$

$$(\lambda-3)(\lambda-1)$$

$$\lambda^2 - \lambda - 3\lambda + 3$$

$$\lambda^2 - 4\lambda + 3$$

$$(r-4)(r-4)$$

$$r^2 - 4r - 4r + 16$$

$$r^2 - 8r + 16$$

$$(r-3)(r-1)$$

$$r^2 - 3r - r + 3$$

$$r^2 - 4r + 3$$