Name: ____________________

1. How many lattice paths taking only steps $\rightarrow$ and $\uparrow$ start at $(0,0)$ and end at (n,n)?

2. How many such lattice paths start above the line $y = x$ but cross over the line at some point? To count these paths, use the **reflection principle**: if a path does cross over the line $y = x$, then consider the steps it takes *strictly after* it has already crossed over. Swap all of these steps between $\rightarrow$ and $\uparrow$. Draw a picture. What sort of path are we left with? Find a bijection, and carefully show it is correct.

3. How many paths C_n are there from $(0,0)$ to (n,n) that always stay (weakly) above the line $y = x$? Simplify your expression. C_n is called the nth *Catalan number*, and the sequence $\{C_n\}$ is ubiquitous.

4. A valid parenthesization is a list of parentheses that can correctly be parsed, like (()) or ()(), but not)((). How many valid parenthesizations are there with n sets of parentheses? Prove your answer.

5. Justify each of the following statements with a combinatorial proof.

   (a) $\binom{2n}{n} = \sum_{k=0}^{n} \binom{n}{k}^2$

   (b) $2\binom{2n-1}{n} = \binom{2n}{n}$

   (c) $\sum_{k=0}^{n} k^2 = \binom{n+1}{2} + 2\binom{n+1}{3}$. Hint: show that both sides correspond to picking an ordered triple (r, s, t) where $0 \leq r, s < t$.

6. Let $\sigma \in \mathcal{S}_n$ be a permutation on $[n]$, written as a function $\sigma : [n] \to [n]$. Characterize the permutations where $\sigma^6(i) = i$ for all i in terms of the cycle structure of σ.

7. Let $c(n, k)$ be the number of permutations on $[n]$ with exactly k cycles.

   (a) Show that the number of permutations σ on $[n]$ with k cycles and $\sigma(n) = n$ is $c(n-1, k-1)$.

(b) Show that the number of permutations on σ on $[n]$ where $\sigma(n) \neq n$ is $(n-1)c(n-1,k)$.

(c) Explain why $c(n,k) = (n-1)c(n-1,k) + c(n-1,k-1)$.

Note: The recurrence above could be used to find a *bivariate generating function* encoding the number of permutations on $[n]$ with k cycles. We'll save our generating function perspective until later in the semester.

Permutation matrices. The permutation matrix P_σ of a permutation $\sigma : [n] \to [n]$ is defined to have entries

$$(P_\sigma)_{i,j} = \begin{cases} 1, & \sigma(i) = j, \\ 0, & \text{otherwise.} \end{cases}$$

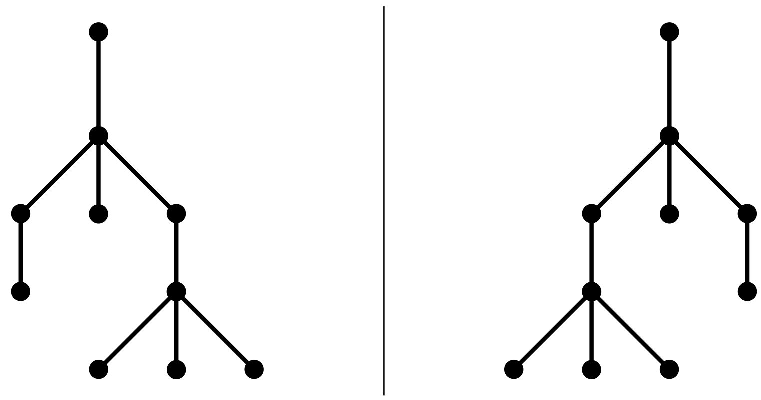
8. How many ones could be in P_σ if $\sigma : [n] \to [n]$? Where can the ones appear?

9. Write the permutation matrix for $\sigma = (92)(5784)(316)$.

10. Let $\sigma, \tau \in \mathcal{S}_n$. What can we say about $P_{\sigma \circ \tau}$? Prove your answer.

At-home problems:

11. A *tree* is a collection of nodes connected by edges that has no cycles. A *rooted plane tree* is a tree with a special node (called the root) where all other nodes are drawn downwards from this node, and where the order of the nodes in the picture makes a difference. Below are two distinct rooted plane trees on 10 vertices. Count the number of rooted plane trees on n nodes, and prove your answer is correct.



12. An election between candidates A and B receives 200 votes total: exactly 100 for candidate A and 100 for candidate B. What is the probability that as the votes are tallied, candidate A is never behind candidate B?

13. Give a bijection between weak compositions of n with $k+1$ parts and lattice paths from $(0,0)$ to (n,k).

14. Give a combinatorial proof of the identity, $\binom{2n}{2} = 2\binom{n}{2} + n^2$

15. Express the permutation 5342671 in cycle notation and as a permutation matrix.

16. How many permutations on [8] have σ^6 equal to the identity function?