Problem List[1]

Sagan

Problem 5.1. (**Sagan, Chapter 5, Exercise 1**)
This exercise refers to the list of examples just after the definition of a poset.

(a) Verify that they satisfy the definition of a poset.

(b) Show that the partial order in Π_n is equivalent to defining $\rho \leq \pi$ if every block of π is a union of blocks of ρ.

(c) Describe the cover relations in the list. For example, in C_n the covers are of the form $i \prec i+1$ for $0 \leq i < n$.

Proof. **(a) Two Posets** We prove that the set of all positive divisors of $n \in \mathbb{Z}$ with $|:(D_n, |)$ and the set of all subspaces of a vector space $\mathcal{V}$ over $\mathbb{F}_q$ with subspace containment $\leq$: $(L(\mathcal{V}), \leq)$ are posets. For any $a,b,c \in D_n$ and any $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \in L(\mathcal{V})$,

(reflexivity)	$a \mid a$	$\mathcal{U}_1 \leq \mathcal{U}_1$
(antisymmetry)	$a \neq b$ and $a \mid b \implies a < b \implies b \nmid a$	$\mathcal{U}_1 < \mathcal{U}_2 \implies \mathcal{U}_2 \not\leq \mathcal{U}_1$
(transitivity)	$a \mid b \mid c \implies \exists k, q \in \mathbb{Z}$ s.t. $\underset{\implies akq=c}{\overset{ak=b}{bq=c}} \implies a \mid c$	$\mathcal{U}_1 \leq \mathcal{U}_2 \leq \mathcal{U}_3 \implies \mathcal{U}_1 \leq \mathcal{U}_3$

(c) In D_n, if $|n| = \prod_i p_i^{a_i}$ is a prime decomposition, then each positive divisor is of the form $d = \prod_i p_i^{b_i}$ where $0 \leq b_i \leq a_i$ is some 'subdecomposition'. So then each cover relation just looks like

$$\prod_i p_i^{b_i} \mid \prod_i p_i^{\beta_i} \text{ where each } \beta_i \geq b_i \text{ and } \sum_i (\beta_i - b_i) = 1$$

In $L(\mathcal{V})$, cover relations look like

$$\mathcal{W} \leq \mathcal{U} \text{ and } \mathcal{B}_{\mathcal{U}} = \mathcal{B}_{\mathcal{W}} \sqcup \{b\} \text{ is a basis for } \mathcal{U} \text{ and } b \in \mathcal{U}.$$

That is, $\mathcal{W} \leq \mathcal{U}$ and $\dim \mathcal{U} = \dim \mathcal{W} + 1$.

□

[1]
- **Sagan—Chapter 5:** 1(ac), 4, 5abc, 6a, 7bc, 8c, 9 (2 parts), 10a, 15 (1 poset not C_n), 17ab, 18ab, 19a (oneway), 20ab
- **Stanley—Chapter 3:** 7, 14a, 25, 30a, 34, 38, 33, 52, 62abc
- **Fulton—Chapter 1:** Exercises 1, 2 (pp. 15–16), compute product both ways
- **Fulton—Chapter 2:** Exercises 1, 2 (pp. 24–26)

Problem 5.4. (Sagan, Chapter 5, Exercise 4)
Complete the proof of Proposition 5.1.3. To show that $K_n \cong B_{n-1}$ it may be simpler to show that $K_n \cong B^*_{n-1}$ using the map ϕ from Section 1.7.

Proof. $\square$

Problem 5.5. (Sagan, Chapter 5, Exercise 5)
Let $f : P \to Q$ be an isomorphism of posets.

(a) Show that f is also an isomorphism of P^* with Q^*.

(b) Show that if P has a $\hat{0}$, then so does Q.

(c) Show in two ways that if P has a $\hat{1}$, then so does Q: by mimicking the proof of part (b) and by using the result of (b) together with part (a).

Problem 5.6. (Sagan, Chapter 5, Exercise 6)

(a) Show that the axioms for a partially ordered set are satisfied by $P \sqcup Q$, $P+Q$, and $P \times Q$.

Problem 5.7. (Sagan, Chapter 5, Exercise 7)
Complete the proof of Proposition 5.2.1.

Problem 5.8. (Sagan, Chapter 5, Exercise 8)

(a) Show that if P is a ranked poset, then for any k we have $R_k(P)$ is an antichain.

(b) Let P be a ranked poset and assume $f : P \to Q$ is an isomorphism. Show that Q is also ranked and for all $x \in P$ we have $\mathrm{rk}_P x = \mathrm{rk}_Q f(x)$.

(c) Show that if P, Q are ranked posets, then so is $P \times Q$ with rank function

$$\mathrm{rk}_{P\times Q}(x,y) = \mathrm{rk}_P x + \mathrm{rk}_Q y.$$

Problem 5.9. (Sagan, Chapter 5, Exercise 9)
Prove Proposition 5.2.2.

Problem 5.10. (Sagan, Chapter 5, Exercise 10)

(a) Prove Proposition 5.3.1.

Problem 5.11. (Sagan, Chapter 5, Exercise 15)
Prove Proposition 5.3.4.

Problem 5.12. (Sagan, Chapter 5, Exercise 17)
Let P be a finite poset and let $L = J(P)$ be the corresponding distributive lattice. If $X \subseteq P$ is a lower-order ideal, then use the corresponding lowercase letter x to denote the associated element of L.

(a) Show that x covers y in L if and only if $Y = X - \{m\}$ where m is a maximal element of X.

(b) Show that x is join irreducible in L if and only if X is a principal ideal of P.

Problem 5.13. (Sagan, Chapter 5, Exercise 18)
Given a poset P, let $\mathcal{A}(P)$ be the set of antichains of P. Show that the map

$$f : \mathcal{A}(P) \to J(P)$$

given by $f(A) = I(A)$ (where $I(A)$ is the order ideal generated by A) is a bijection.

Problem 5.14. (Sagan, Chapter 5, Exercise 19)

(a) Rederive the formula for μ in B_n, equation (5.6), in two ways: by mimicking the proof of (5.7) and by constructing an $m \in P$ such that $D_m = B_n$ and then applying (5.7).

Problem 5.15. (Sagan, Chapter 5, Exercise 20)

(a) Let P be a locally finite poset with a $\hat{0}$. Show that if x covers exactly one element of P, then

$$\mu(x) = \begin{cases} -1 & \text{if } x \text{ covers } \hat{0}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Given any $n \in \mathbb{Z}$, construct a poset containing an element x with $\mu(x) = n$.