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Direct Product

H, K groups $\boxed{H \times K}$

$H' = H \times \{e\} \trianglelefteq H \times K$

$K' = \{e\} \times K \trianglelefteq H \times K$

$H' \cong H$, $K' \cong K$

$H \times K = H'K'$

$H' \cap K' = \{e_{H \times K}\} = \{(e_H, e_K)\}$

Semidirect Product

H, K groups $\tau : K \to \mathrm{Aut}(H)$

$\boxed{H \rtimes_\tau K}$

$(h, k)(h', k') = (h\tau_k(h'), kk')$

$H' = H \times \{e\} \trianglelefteq H \rtimes_\tau K$

$K' = \{e\} \times K \leqslant H \rtimes_\tau K$

$H' \cong H$, $K' \cong K$

$H \rtimes_\tau K = H'K'$

$H' \cap K' = \{e_{H \rtimes_\tau K}\} = \{(e_H, e_K)\}$

Proposition G group, $H \trianglelefteq G$ (normal), $K \leq G$ such that $H \cap K = \{e\}$ and $G = HK$.

Then G is isomorphic to a semidirect product of H and K.

More precisely, let $\tau : K \longrightarrow \mathrm{Aut}(H)$ where

$$\tau_k(h) = k h k^{-1} \quad \text{(inner automorphism)}$$

$$\tau_k : H \longrightarrow H$$

(Check that τ is a group hom.)

Then $G \cong H \rtimes_\tau K$

Proof $f : H \rtimes_\tau K \longrightarrow G$

$$(h, k) \longrightarrow hk$$

- f is group hom.

$$f((h,k)(h',k')) = f(h\, \underbrace{\tau_k(h')}_{k h' k^{-1}},\ k k')$$

$$= hk h' \cancel{k^{-1}} \cancel{k} k'$$

$$= f(h,k) \cdot f(h',k') \quad \checkmark$$

- f is surj. because $G = HK$
- f is injective
  $(h,k) \in \operatorname{Ker} f \Rightarrow hk = e \Rightarrow h = k^{-1} \in H \cap K = \{e\}$
  & $h = e = k$

Remark If $K \trianglelefteq G$, then z is the trivial group hom. and we get the usual direct product.

Example $C_n = \{e, x, x^2, \ldots, x^{n-1}\}$, $x^n = e$
$C_2 = \{e, y\}$, $y^2 = e$

Let $G = C_n \rtimes_z C_2$ where
$$z : C_2 \to \operatorname{Aut}(C_n)$$
$$z_e = id_{C_n}$$
$$z_y(x) = x^{-1} \quad (\text{thus } z_y(x^k) = x^{-k})$$

$$C_n \cong C_n \times \{e\} = \{(e,e), \underbrace{(x,e)}_{a}, \ldots, \underbrace{(x^{n-1},e)}_{a^{n-1}}\}$$

Let $a = (x, e)$

$$C_2 \cong \{e\} \times C_2 = \{(e,e), \underbrace{(e,y)}_{b}\}$$

Then $C_n \rtimes_z C_2 = \{(x^k, y^t) \mid k \in \mathbb{Z},\ t \in \mathbb{Z}\}$

$$= \{e, a, a^2, \ldots, a^{n-1}, b, ab, \ldots, a^{n-1}b\}$$

where $a^n = e$, $b^2 = e$ (check this)

$$b \cdot a = (e, y) \cdot (x, e) = (e\, z_y(x), y \cdot e)$$

$$= (x^{-1}, y) = a^{-1} \cdot b = a^{n-1} \cdot b$$

So $C_n \rtimes_z C_2 \cong D_n$ $\boxtimes$

G group, $T \subseteq G$ subset, $x \in G$

Notation $C(T) = \{ a \mid a \in G \text{ and } ay = ya \text{ for all } y \in T \}$

centralizer of T

Claim $C(T)$ is a subgroup of G (exercise)

Notation $C(x) = C(\{x\})$

$N(T) = \{ a \mid a \in G \text{ and } aTa^{-1} = T \}$ normalizer of T

$N(x) = N(\{x\})$

Claim $N(T)$ is a subgroup of G (exercise)

Obs $N(x) = C(x)$

$$\begin{array}{ccc} N(x) & = & C(x) \\ \| & & \| \\ \{a \in G \mid axa^{-1} = x\} & & \{a \in G \mid ax = xa\} \end{array}$$

Remarks (1) $C(T) \leq N(T) \leqslant G$

(2) $\underset{\substack{\| \\ Z(G)}}{C(G)} \trianglelefteq G$ and $G = Z(G) \Longleftrightarrow$ G is abelian.

(3) If $H \leqslant G$ subgroup, then $N(H)$ is the largest subgr of G in which H is normal.

Proof (3) Claim $H \trianglelefteq N(H)$

$\alpha \in N(H) \overset{?}{\Longrightarrow} \alpha H \alpha^{-1} \subseteq H$

To prove : $H \trianglelefteq K \leq G \Longrightarrow K \subseteq N(H)$

$k \in K$. Then $k \cdot h \cdot k^{-1} \in H$ for all $h \in H$

$$\left.\begin{array}{l} \text{So } kHk^{-1} \subseteq H \\ \left(H \subseteq kHk^{-1} \text{ also holds} \right) \end{array}\right\} \Longrightarrow k \in N(H)$$

Corollary $H \trianglelefteq G \Longleftrightarrow N(H) = G$

<u>Notation</u> For $a \in G$

$c_a : G \to G$

$c_a(x) = axa^{-1}$, Note that $c_a \in \mathrm{Aut}(G)$

$\underbrace{\mathrm{Inn}\, G}_{\text{inner automorphisms}} \overset{\text{def}}{:=} \{ c_a \mid a \in G \}$

<u>Remark</u> $F : G \longrightarrow \mathrm{Aut}(G)$

$a \longrightarrow c_a$

Then F is a group homomorphism. (Why?)

$F(a \cdot b) = c_{ab}$ $\quad c_{ab}(x) = abx(ab)^{-1} = abxb^{-1}a^{-1}$

$F(a)F(b) = c_a c_b$ $\quad c_a c_b(x) = a(bxb^{-1})a^{-1}$

$\mathrm{Ker}\, F = \{ a \in G \mid c_a = \mathrm{id}_G \} = \{ a \in G \mid c_a(x) = x \text{ for all } x \in G \}$

$= \{ a \in G \mid axa^{-1} = x \text{ for all } x \in G \} = Z(G)$

$\operatorname{Im} F = \operatorname{Inn}(G)$

Thus $\boxed{G/Z(G) \cong \operatorname{Inn}(G)}$

$$\bar{a} \longleftrightarrow c_a$$