Th ε is a group hom.

Proof $\quad \varepsilon(\sigma \circ \tau) = \displaystyle\prod_{i>j} \frac{\sigma(\tau(i)) - \sigma(\tau(j))}{\tau(i) - \tau(j)} \cdot \frac{\tau(i) - \tau(j)}{i - j}$

$$= \varepsilon(\sigma) \cdot \varepsilon(\tau) \qquad \blacksquare$$

Corollary $\quad A_n \underset{\substack{\uparrow \\ \text{notation}}}{:=} \operatorname{Ker} \varepsilon$ is a normal subgroup of S_n

$\varepsilon : S_n \to \{\pm 1\}$

and $\quad S_n / A_n \cong$ cyclic group of order 2.

$(n \geq 2)$.

even perm = perm of sign. equal to 1

Cayley's Thm $\quad$ Every group is isomorphic to a subgroup of a symmetric group S_A.

Proof $\quad G \xrightarrow{\phi} S_G$

$x \longrightarrow f_x : G \to G \quad$ where $f_x(y) = xy$

Prove that ϕ is an injective group hom. (exercise)
By 1st Iso Thm

$$G \cong \operatorname{Im}\phi \leq S_G$$

Obs If G finite with $|G| = n$, then $S_G = S_n$ □

Recall A_n = the subgr. of even perm in S_n

Theorem A_n is simple for $n \neq 4$ $(n \geq 2)$.

Def G is called simple group if $\{e\}$ and G are the only normal subgroups.

Obs A_2 has 1 element, $\Longrightarrow A_2$ simple

A_3 has $\frac{3!}{2} = 3 \Longrightarrow A_3$ simple

A_4 is not simple because

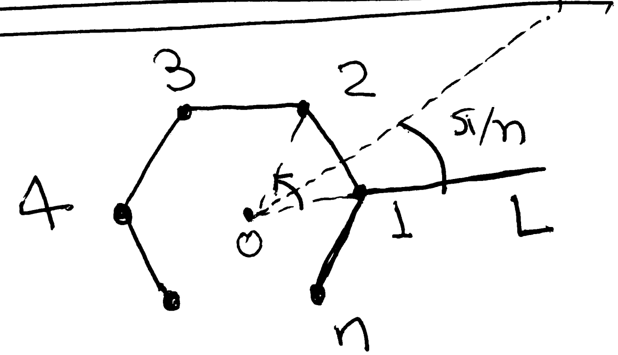
$\cancel{\{(1), (12)(3}$ $\{(1), (12)(34), (13)(24), (14)(23)\} \trianglelefteq A_4$ exercise

A_n simple for $n \geq 5$ (proof later).

Going back to signature:

$\varepsilon(\sigma) = (-1)^{\alpha}$ where α is the number of transp. in a decomp of σ as a product of transpositions.

The Dihedral Group



α = rotation $\frac{2\pi}{n}$

$\alpha^2, \alpha^3, \dots, \alpha^{n-1}, \alpha^n = \mathrm{id}$

β = reflection (flip) L, $\beta^2 = \mathrm{id}$

$$D_n = \{e, \alpha, \alpha^2, \dots, \alpha^{n-1}, \underbrace{\beta}_{\text{refl.}}, \underbrace{\alpha\beta}_{\text{refl.}}, \alpha^2\beta, \dots, \underbrace{\alpha^{n-1}\beta}_{\text{refl.}}\}$$

where $\begin{cases} \alpha^n = e \\ \beta^2 = e \\ \beta\alpha = \alpha^{n-1}\beta \end{cases}$

$$\alpha\beta = \begin{pmatrix} 1 & 2 & 3 & \cdot & n \\ 2 & 1 & n & \cdots & 3 \end{pmatrix}$$

This is the dihedral group with $2n$ elements (denoted D_n or D_{2n} in different sources)

<u>Obs</u> D_n (dihedral group with $2n$ elts) is a subgroup of S_n

<u>Direct Products</u>

G_1, G_2 groups

$G_1 \times G_2 = \{ (h_1, h_2) \mid h_1 \in G_1, h_2 \in G_2 \}$

$\downarrow$

group with op. $(h_1, h_2) \cdot (l_1, l_2) = (h_1 l_1, h_2 l_2)$

<u>Prop</u> Consider $H_1 \trianglelefteq G$, $H_2 \trianglelefteq G$

Assume that

(1) $G = H_1 H_2$

(2) $H_1 \cap H_2 = \{e\}$

Then $H_1 \times H_2 \cong G$ isomorphism

$(h_1, h_2) \xrightarrow{\varphi} h_1 h_2$.

<u>Proof</u> • φ is a group homomorphism (because every elt. of H_1 commutes with every elt. of H_2) (check this).

• φ is surjective $(G = H_1 H_2)$

• φ is injective. Why.

$(h,k) \in \ker\varphi \Rightarrow h \cdot k = e \Rightarrow h = k^{-1} \in K \cap H = \{e\}$

so $h = k = e$.

<u>Obs</u> In other words, every elt of G can be written <u>uniquely</u> as a prod. of an elt in H_1 and an elt. in H_2

$\blacksquare$