**Homework 2 - due Monday, February 2**

1. In the following problems, we investigate the relationship between integral domains and fields.
   (a) Prove that every field is an integral domain.
   (b) Prove that if $R$ is a *finite*[1] integral domain, then $R$ is a field. Hint: Consider the set function $a : R \to R$ given by multiplication by an element $a$ (it will not be a ring homomorphism). If $a$ is a non-zero-divisor, prove this is an injective function, and so must also be surjective since $R$ has finitely many elements.
   (c) Find an example of an integral domain which is not a field.

2. Let $I, J$ be ideals in a ring $R$ such that $I + J = (1)$. Prove that $IJ = I \cap J$.

3. Let $\varphi : R \to S$ be a ring homomorphism, and let $J$ be an ideal of $S$. Prove that $I = \varphi^{-1}(J) = \{i \in R \mid \varphi(i) \in J\}$ is an ideal of $R$.

4. Let $\varphi : R \to S$ be a ring homomorphism, and let $J$ be an ideal of $S$.
   (a) Prove that $\varphi(J) = \{\varphi(j) \mid j \in J\}$ need not be an ideal of $S$
   (b) Prove that if $\varphi$ is surjective, then $\varphi(J)$ is an ideal of $S$.
   (c) Prove that if $\varphi$ is surjective, and $I = \ker \varphi$, then $S \cong R/I$ and if we let $\bar{J} \subseteq R/I$ be the image of $\varphi(J)$ under this isomorphism, then
   $$\frac{R/I}{\bar{J}} \cong \frac{R}{I + J}$$

5. Let $R$ be a commutative ring, $a \in R$, and let $f_1(x), \ldots, f_r(x) \in R[x]$.
   (a) Prove that $R[x]/(x - a) \cong R$.
   (b) Prove the equality of ideals $(f_1(x), \ldots, f_r(x), x - a) = (f_1(a), \ldots, f_r(a), x - a)$.
   (c) Prove the useful substitution trick
   $$\frac{R[x]}{(f_1(x), \ldots, f_r(x), x - a)} \cong \frac{R}{(f_1(a), \ldots, f_r(a))}$$
   Hint: Use part (c) of the previous problem.

6. If $k$ is an algebraic closed field, then the only maximal ideals of $k[x]$ are of the form $(x - a)$ where $a \in k$. In this problem, we'll see that this is not true when $k$ is not algebraically closed.
   (a) Use the first isomorphism theorem to show that $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.
   (b) Prove that $(x^2 + 1)$ is a maximal ideal of $\mathbb{R}[x]$.

7. Let $R$ be a commutative ring with $0 \neq 1$. In this problem, we will prove that every proper ideal of $R$ is contained in some maximal ideal.
   (a) Look up Zorn's Lemma and record it here.
   (b) Define $S = \{J \mid J$ is a proper ideal of $R$ and $J \supseteq I\}$. Explain why $S$ is a partially ordered set (what is the ordering?).

[1] meaning $|R| < \infty$

(c) Given a chain $C$ in $S$, prove that $\bigcap_{J \in C} J$ is an ideal of $R$ (You will use that $C$ is totally ordered), and further that this ideal is in the set $S$.

(d) Conclude using Zorn's Lemma that $S$ has a maximal element.

8. Let $k$ be a field. In this problem, we will prove that the only maximal ideal of $k[[x]]$ is $(x)$, which makes $k[[x]]$ a **local ring**.

   (a) Explain why $(x) = \{f \in k[[x]] \mid f \text{ has no constant term}\}$.

   (b) Compute $k[[x]]/(x)$, and then explain why $(x)$ is a maximal ideal.

   (c) You may freely use the following result from the optional hint last week: $f \in k[[x]]$ is a unit if and only $f$ has a nonzero constant term. Use Proposition 1.41 to show that the only maximal ideal of $k[[x]]$ is $(x)$.

9. Let $d$ be an integer which is not the square of an integer, and consider

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{B}\}.$$

   (a) Prove that $\mathbb{Q}(\sqrt{d})$ is a subring of $\mathbb{C}$

   (b) Define a function $N : \mathbb{Q}(\sqrt{d}) \to \mathbb{Q}$ by $N(a+b\sqrt{d}) : a^2+b^2 d$. Prove that $N(zw) = N(z)N(w)$ and that $N(z) \neq 0$ if $z \neq 0$. This function is often called the **norm**.

   (c) Prove that $\mathbb{Q}(\sqrt{d})$ is a field and is the smallest subfield of $\mathbb{C}$ containing both $\mathbb{Q}$ and $\sqrt{d}$ (Use $N$).

   (d) Prove that $\mathbb{Q}(\sqrt{d}) \cong \mathbb{Q}[t]/(t^2 - d)$