MATH 266 - NDSU
Fall 2025
Exam 1 Solution
October 3rd, 2025

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1. (10 points) Find the explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{y-1}{x+3}, \quad y(-1) = 0.$$

This equation is separable, so we multiply both sides by $\frac{1}{y-1}dx$ to get

$$\frac{1}{y-1}dy = \frac{1}{x+3}dx.$$

Integrating both sides, we have

$$\int \frac{1}{y-1}\,dy = \int \frac{1}{x+3}\,dx.$$

This gives us $\ln|y-1| = \ln|x+3| + C$.

To get an explicit solution, we have

$$e^{\ln|y-1|} = e^{\ln|x+3|+C} = e^C e^{\ln|x+3|} = Ce^{\ln|x+3|}.$$

That is, $|y-1| = C|x+3|$, which implies that $y-1 = \pm C(x+3) = C(x+3)$ as C is arbitrary.

To find the value C, we plug in $x=-1$ and $y=0$ to get $-1 = 2C$, so $C = -\frac{1}{2}$. Therefore, the explicit solution is $y = 1 - \frac{1}{2}(x+3)$.

It is not necessary to simplify the right hand side, but the answer must be of the explicit form $y = \ldots$.

2. (10 points) Find the general solution of the equation

$$xy' + 2y = \frac{1}{x^3}, \quad x > 0.$$

Write your answer as an explicit solution.

This is a linear equation, so we solve it using integrating factors. Note that the coefficient of y' is not 1, so we first divide both sides of the equation by x to get

$$y' + \frac{2}{x}y = \frac{1}{x^4}.$$

The integrating factor

$$\mu(x) = e^{\int \frac{2}{x}\ dx} = e^{2\ln|x|} = e^{2\ln x} = x^2.$$

Multiplying both sides of the equation by $\mu(x) = x^2$ gives us

$$x^2y' + 2xy = \frac{1}{x^2}.$$

That is,

$$\frac{d}{dx}(x^2y) = \frac{1}{x^2}.$$

Integrating both sides, we get

$$\begin{aligned} x^2y &= \int \frac{1}{x^2}\ dx \\ &= -\frac{1}{x} + C. \end{aligned}$$

Finally, we divide both sides of the expression above by x^2, we get $y = -\frac{1}{x^3} + \frac{C}{x^2}$.

3. (10 points) Consider $\dfrac{dy}{dt} = y^2(4 - y^2)$.

   (a) (4 points) Find all equilibrium solutions.

   We factor the right-hand side see that $y' = y^2(2+y)(2-y)$. Setting $y' = 0$, we get $y = 0, 2, -2$ are equilibrium solutions.

   (b) (6 points) Sketch the phase diagram and classify all equilibrium solutions.

   The equilibrium solutions we found in (a) divide the real number line into 4 intervals. We test whether y' is positive or negative by plugging in appropriate numbers from each interval.

   Pick $y = -3$ $(y < -2)$, then $y' = (-3)^2 \cdot (2+(-3)) \cdot (2-(-3)) < 0$.

   Pick $y = -1$ $(-2 < y < 0)$, then $y' = (-1)^2 \cdot (2 + (-2)) \cdot (2 - (-2)) > 0$.

   Pick $y = 1$ $(0 < y < 2)$, then $y' = 1^2 \cdot (2+1) \cdot (2-1) > 0$.

   Pick $y = 3$ $(y > 2)$, then $y' = 3^2 \cdot (2+3) \cdot (2-3) < 0$.

   From the phase diagram, we see that $y = 2$ is stable, and $y = 0, -2$ are unstable.

   It is not necessary to compute the exact values of y'. Getting $y' > 0$ or $y' < 0$ is enough.

4. (10 points) Compute the Wronskian of two function $y_1 = e^x$ and $y_2 = xe^x$. Are y_1 and y_2 linearly independent? Justify your answer.

   We compute directly that $y_1' = e^x$. Using the product rule, we get $y_2' = xe^x + e^x$.

   Thus, the Wronskian is

$$\begin{aligned} W[y_1, y_2] &= e^x(xe^x + e^x) - (xe^x)(e^x) \\ &= xe^{2x} + e^{2x} - xe^{2x} \\ &= e^{2x}. \end{aligned}$$

   Since $W[y_1, y_2] \neq 0$ for all x, we conclude that y_1 and y_2 are linearly independent.

5. (10 points) Solve the equation

$$6x^2(1+\ln y)\ dx + \Big(\frac{2x^3}{y} - e^y\Big)\ dy = 0.$$

We first test for exactness: Note that $M(x,y) = 6x^2(1+\ln y)$ and $N(x,y) = \frac{2x^3}{y} - e^y$, so we compute $M_y = \frac{6x}{y}$ and $N_x = \frac{6x}{y}$. This tells us that the equation is exact.

Thus, we integrate $M(x,y)$ with respect to x to get

$$\begin{aligned}\psi(x,y) &= \int 6x^2(1+\ln y)\ dx \\ &= 2x^3(1+\ln y) + g(y).\end{aligned}$$

It then follows that $\psi_y = \frac{2x^3}{y} + g'(y)$. Since the equation is exact, we have $\psi_y = N(x.,y) = \frac{2x^3}{y} - e^y$. Therefore, we get $g'(y) = -e^y$, giving us $g(y) = -e^y$.

Hence, $\psi(x,y) = 2x^3(1+\ln y) - e^y$, which implies that the general solution is

$$2x^3(1+\ln y) - e^y = C.$$