Observation $N \trianglelefteq G$. Then

$$\begin{array}{rcl} G & \xrightarrow{\pi} & G/N \\ g & \rightsquigarrow & gN \end{array}$$ is a group hom and $\boxed{\operatorname{Ker}\pi = N}$

$(g \in \operatorname{Ker}\pi \Leftrightarrow gN = N \Leftrightarrow g \in N)$

Theorem (First Iso Theorem)

$f: G_1 \to G_2$ group hom. Then

$\operatorname{Ker} f \trianglelefteq G_1$ and

$$G_1/\operatorname{Ker} f \underset{\substack{\Uparrow \\ \text{iso}}}{\cong} \operatorname{Im} f$$ ↙ subgr. of G_2

Proof $\operatorname{Ker} f \leq G_1$ subgr (exercise)

normality: $\alpha \in \operatorname{Ker} f$, $g \in G$

Then $f(g\alpha g^{-1}) = f(g)\underbrace{f(\alpha)}_{\text{id.}} f(g^{-1})$

$= f(g) f(g^{-1}) = f(g \cdot g^{-1}) = f(e_G)$

$= e_{G_2}$ ✓

Define

$$G_1/\mathrm{Ker}\, f \xrightarrow{\bar{f}} \mathrm{Im}\, f$$

$$a(\mathrm{Ker} f) \rightsquigarrow f(a)$$

Well defined because

$a(\mathrm{Ker} f) = b(\mathrm{Ker} f) \Rightarrow a^{-1}b \in \mathrm{Ker} f \Rightarrow f(a^{-1}b) = e$

$\Rightarrow f(a)^{-1} \cdot f(b) = e \Rightarrow f(a) = f(b)$

Check that $\bar{f}$ is a group homom.

$\bar{f}$ surj. (clear).

$\bar{f}$ inj $\quad \bar{f}(a \mathrm{Ker} f) = \bar{f}(b \mathrm{Ker} f)$

$\Rightarrow f(a) = f(b) \Rightarrow a^{-1}b \in \mathrm{Ker} f$

$\Rightarrow a \mathrm{Ker} f = b \mathrm{Ker} f$. ☒

Obs $\quad f$ injective $\Leftrightarrow \mathrm{Ker} f = \{e_{G_1}\}$

Th (Second Iso Thm)

$K, N \leq G$, $N \trianglelefteq G$ Then

$$NK/_{N} \cong K/_{K \cap N}$$

Proof

$$\begin{tikzcd} K \arrow[r, hook, "i"] \arrow[rr, bend right, "f \text{ (group hom)}"'] & NK \arrow[r, "\pi"] & NK/_{N} \end{tikzcd}$$

$f(k) = kN$

$\operatorname{Ker} f = \{ k \mid k \in K, \ kN = N \} = K \cap N$

Claim $\operatorname{Im} f = NK/_{N}$

"$\supseteq$" $nkN = k\underbrace{n'N}$ for some $n' \in N$

$= kN = f(k)$.

By prev. theorem $K/_{K \cap N} \cong NK/_{N}$

$\square$

Th (Third iso thm) $H \trianglelefteq G$, $K \trianglelefteq G$, $K \subseteq H$

Then $H/K \trianglelefteq G/K$ and

$$\frac{G/K}{H/K} \cong G/H$$

Proof Define $G/K \xrightarrow{f} G/H$, $f(xK) = xH$.

Well-defined because
$xK = yK \Rightarrow y^{-1}x \in K \subseteq H \Rightarrow y^{-1}x \in H \Rightarrow$
$\Rightarrow xH = x\cancel{H}$.

- f is a group hom. ✓
- $\text{Ker } f = \{xK \mid xH = H\} = \{xK \mid x \in H\}$
$= H/K$

(in part $H/K \trianglelefteq G/K$)
normal

- $\operatorname{Im} f = G/H$.

By first iso thm: $\dfrac{G/K}{H/K} \cong G/H$ $\square$

Theorem $K \trianglelefteq G$. Then there exists a bijection

$$\{\text{subgr. of } G/K\} \longleftrightarrow \left\{\begin{matrix}\text{subgr. of } G \\ \text{that contain } K\end{matrix}\right\}$$

$$H/K \longleftarrow H$$

$$H' \longrightarrow \{x \in G \mid xK \in H'\}$$

Proof (exercise) check that the above maps are inv. of each other.

Obs By third iso theorem, normal subgroups correspond to normal subgroups.

Ex $G = \mathbb{Z}$, $K = n\mathbb{Z}$

The subgroups of $\mathbb{Z}/n\mathbb{Z}$ are of the form $m\mathbb{Z}/n\mathbb{Z}$ where $m \mid n$

The Symmetric Group

$$S_n = \{ f \mid f : \{1, \dots, n\} \to \{1, \dots, n\}$$
$$f \text{ bijective} \}$$

$(S_n, \circ)$ group, $|S_n| = n!$

$$\sigma \in S_n, \quad \sigma = \begin{pmatrix} 1 & 2 & 3 & \dots & n \\ \sigma(1) & \sigma(2) & \dots & & \sigma(n) \end{pmatrix}$$

Obs For an arbitrary set, one can define

$$S_A = \{ f \mid f : A \to A, f \text{ bij} \}$$

Similarly $(S_A, \circ)$ is a group.

k-cycle $\quad (i_1\, i_2 \dots i_k) = \begin{pmatrix} i_1 & i_2 & \dots & i_k \\ i_2 & i_3 & & i_1 \end{pmatrix}$

2-cycle $\quad (i\,j) = \begin{pmatrix} i & j \\ j & i \end{pmatrix}$, $i \neq j$ (transposition)

Theorem Every permutation in S_n can be written uniquely as a product of disjoint cycles.

Every perm is a product of transpositions.

Obs Disjoint cycles commute.

Ex $\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 4 & 6 & 1 & 3 & 2 & 5 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 3 \\ 4 & 3 & 1 \end{pmatrix} \cdot \begin{pmatrix} 2 & 6 & 5 \\ 6 & 5 & 2 \end{pmatrix}$

$= (1\,4\,3)(2\,6\,5)$

Note $(i_1\, i_2 \dots i_k) = (i_1\, i_k)(i_1\, i_{k-1}) \dots (i_1\, i_2)$

So $\sigma = (1\,3)(1\,4)(2\,5)(2\,6)$

Note $\sigma = (14)(34)(26)(56)$ (no uniqueness as product of transpositions)

Signature

$$\varepsilon : S_n \to \{\pm 1\}$$

$$\varepsilon(\sigma) = \prod_{\substack{i>j \\ i,j \in \{1,\dots,n\}}} \frac{\sigma(i) - \sigma(j)}{i - j}$$

$\binom{n}{2}$ factors

Note $\varepsilon(\sigma) = (-1)^{\alpha}$ where $\alpha = \left|\left\{(i,j) \,\middle|\, \begin{array}{l} i>j \\ \sigma(i)<\sigma(j) \end{array}\right\}\right|$

Theorem

ε is a group homomorphism.

Proof (next time).