Title of the Paper

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Abstract

This paper is a summary of the Thesis "Seven Edge Forest Designs" by Daniel Mauricio Banegas who was advised by Professor Bryan Freyberg of the University of Minnesota: Duluth. The main goal of this write up is to introduce G-decompositions, summarize various methods for constructing them, show some examples of these methods in action, and then touch on the role of modern computing in this area of research.

1 Background

To begin we introduce fundamentals of graph theory in detail. Then we introduce some algebraic machinery which is alluded to in this paper, but never properly introduced, as an alternative source of intuition behind the various methods used throughout this work. Lastly, we cover the most important objects, which are focused on in the main results of this research project.

1.1 Graph Theory

In Graph Theory, we study these things called *Graphs*, which consist of points called *vertices* and lines between them called *edges*. This paper focuses on *simple undirected* graphs which have at most one edge between any two vertices, no edges from vertices to themselves, no directionality to edges. Formally, a graph is an ordered pair $G = (V(G), E(G))$ where $V(G)$ and $E(G)$ are the set of all vertices and edges of G, respectively. We denote an edge between vertices u and v via $uv = vu$. Sometimes, we denote them as ordered pairs $(u, v) = (v, u)$ to make for easier constructions. Typically, Graph Theorists think of graphs through their visualizations.

For example, let $V(G) = \mathbb{Z}_5$ and $E(G) = \{uv \mid u, v \in \mathbb{Z}_5 \text{ and } u \neq v\}$. The figure below is a visual representation of G.

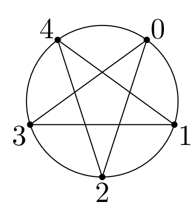

Figure 1: G

Definition 1 (Graph Union)**.** The *union* of two graphs G_1, G_2 is defined as follows.

$$G_1 \cup G_2 = (V(G_1) \cup V(G_2), E(G_1) \cup E(G_2)).$$

We use $\sqcup$ to denote a union between two graphs who share no vertices (or edges).

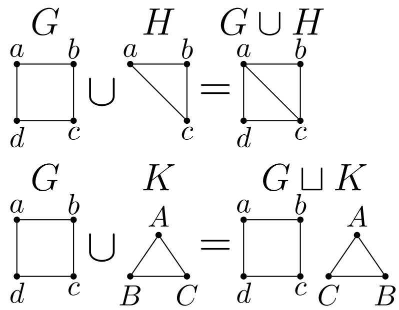

Figure 2: (above) $G \cup H$ and (below) $G \sqcup K$.

We say two graphs G_1, G_2 are isomorphic if there exists a bijection $f : V(G_1) \to V(G_2)$ between their vertices that preserves edge structure;

$$f(a) = \alpha,\ f(b) = \beta,\ \text{and } ab \in E(G_1) \implies \alpha\beta \in E(G_2), \forall a, b \in V(G_1).$$

Graph theorists typically view all graphs in the same isomorphism class as the same graph. Another way to think of isomorphism between two graphs is through their visual representations. If you can draw one graph, and then arrange the nodes (without deleting or adding edges) so that it is identical to the visual representation of another graph, then the two are isomorphic.

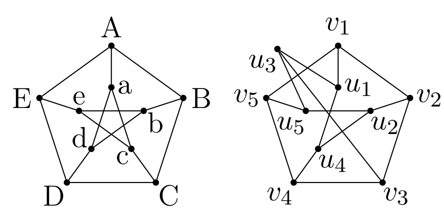

Figure 3: (left) $G \cong H$ (right).

We wrap up the basics with some important concepts and operations needed to understand the work done in this thesis. This is an excerpt from the paper itself.

Definition 2 (Subgraph)**.** A subgraph $G \subseteq K$ is a graph whose vertices and edges are subsets of the vertices and edges of K; $G \subseteq K$ if $V(G) \subseteq V(K)$ and $E(G) \subseteq E(K)$.

Definition 3 (Vertex-induced Subgraph)**.** A *vertex-induced* subgraph $G \subseteq K$ is one whose vertices are some subset S of $V(K)$ and whose edges are all edges between those vertices in K; $V(G) = S \subseteq V(K)$ and $E(G) = \{uv \in E(K) \mid u, v \in S\}$. If G is such a subgraph we say that G is induced by $S = V(G) \subseteq V(K)$.

Definition 4 (Edge-induced Subgraph)**.** An *edge-induced* subgraph $G \subseteq K$ is one whose edges are some subset of $E(K)$ and whose vertices are all those who appear as an endpoint in that subset of edges; $E(G) \subseteq E(K)$ and $V(G) = \{u \in V(K) \mid uv \in E(G) \text{ for some } v \in V(K)\}$. If G is such a subgraph we say that G is induced by $S = E(G) \subseteq E(K)$

Here is a visual example of these types of graphs: Let K be the Petersen graph from Figure

Subgraph: $G \subseteq K$ where $V(G) = \{E, e, b\}$, $E(G) = \{Ee\}$.
Vertex-induced Subgraph: $H \subseteq K$ is induced by $\{a, A, B\} \subseteq V(K)$
Edge-induced Subgraph: $M \subseteq K$ is induced by $\{Dd, DC, Cc\} \subseteq E(K)$

The figure below shows K and it's color-coded subgraphs.

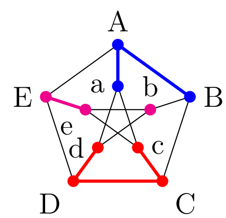

Figure 4: K and subgraphs $G, H, M \subseteq K$.

Definition 5 (Graph Complement)**.** The complement of a graph G denoted $\overline{G}$ is the graph obtained by removing all edges of G and then adding in all edges not originally present in G. Formally,

$$\overline{G} = (V(G), \{uv \mid u, v \in V(G) \text{ and } uv \notin G\})$$

Here is an example of a graph complement. Let $G = (\{a, b, c, d\}, \{ab, bc, cd, da\})$. Then $\overline{G} = (\{a, b, c, d\}, \{ac, bd\})$. These graphs are depicted in the figure below.

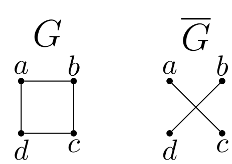

Figure 5: (left) G and $\overline{G}$ (right).

Definition 6 (Join)**.** Let G and H be vertex disjoint graphs. Their *join*, denoted $G \vee H$, is the graph obtained by taking the disjoint union of G and H and adding all possible edges between every vertex in G and every vertex in H. Formally:

$$G \vee H \;=\; \big(V(G) \cup V(H), E(G) \;\cup\; E(H) \;\cup\; \{\, xy \mid x \in V(G),\, y \in V(H)\}\big).$$

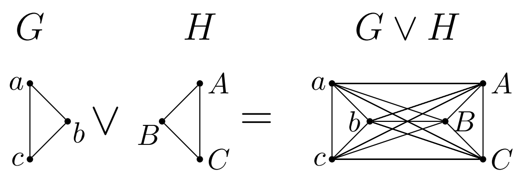

Figure 6: (left) G, H (middle), and $G \vee H$ (right).

1.2 Algebra

Some basic algebraic machinery can simplify many of the constructions in this thesis. This area of research lies in an intersection of Graph Theory, Combinatorics, and Design Theory. Depending on the context it can be helpful to think of the objects studied in this work from many different perspectives offered by these disciplines, but the thread connecting them all, constructively anyways, is Group Actions.

It is assumed that the reader understands knows what a group is. The only thing to cover on that front is that the group $\mathbb{Z}_n$ is presented as $\mathbb{Z}_n = \{0, 1, \ldots, n-1\}$ rather than $\{[0], [1], \ldots, [n-1]\}$. This is done for the sake of hygeine.

We now briefly define Group Actions and Orbits.

Definition 7 (Group Action)**.** We say a Group G acts on a set X, and denote that statement $G \curvearrowright S$ if there exists a mapping $\varphi : G \times X \to X$ such that

$$\textbf{[Identity]:}\ \varphi(e, x) = x,\ \forall x \in X$$
$$\textbf{[Compatibility]:}\ \varphi(gh, x) = \varphi(g, (\varphi(h, x)))$$

Notations for the actual mappings themselves can vary depending on the context. Often we use $\varphi_g(x)$ or $g(x)$ to denote $\varphi(g, x)$. Sometimes, we may even just denote an action using a binary operation such as $\cdot$, for example when a group is acting on one of it's subgroups.

Really, all that is necessary to take away from this is that a group action permutes the elements of X such that composition preserves multiplication between group elements. This allows us to essentially do some algebra with elements of sets. We only need one other concept for this research.

Definition 8 (Orbit)**.** If a group G acts on a set X, we call the set of all elements that x is permuted to via G, the orbit of x:

$$\mathrm{Orb}_G(x) = \{g(x) \mid g \in G\}$$

The next section will describe certain mappings called *labelings* which prove the existence of certain types of *graph decompositions*. The way they prove existence is constructively, through group actions. We could just define these labelings and call it a day since theorems say they prove existence (as many do), but that would be lame. That is like ordering a tasty intricate burger or sandwich and then asking the server to just throw it in your mouth and hammer it down your throat with a rubber mallet. We want to appreciate the sandwich, not just consume it for nutritional value.

2 Decompositions

Think back to our sandwich analogy in the previous section.