Math 721 Problem List (last updated on 2025-09-12)

1. If p is a prime number, prove that the nonzero elements of $\mathbb{Z}_p$ form a multiplicative group of order $p-1$. Show that this statement is false if p is not a prime.

2. (a) Prove that the relation given by $a \sim b \iff a - b \in \mathbb{Z}$ is an equivalence relation on the additive group $\mathbb{Q}$.

   (b) Prove that $\mathbb{Q}/\mathbb{Z}$ is an infinite abelian group.

3. Let p be a prime number and let $\mathbb{Z}(p^\infty)$ be the following subset of the group $\mathbb{Q}/\mathbb{Z}$:

$$\mathbb{Z}(p^\infty) = \{\overline{(a/b)} \in \mathbb{Q}/\mathbb{Z} \mid, a, b \in \mathbb{Z} \text{ and } b = p^i \text{ for some } i \geq 0\}$$

   Prove that $\mathbb{Z}(p^\infty)$ is an infinite subgroup of $\mathbb{Q}/\mathbb{Z}$.

4. If G is a finite group of even order, prove that G has an element of order two.

5. Let Q_8 be the multiplicative group generated by the complex matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

   Observe that $A^4 = B^4 = I_2$ and $BA = AB^3$. Prove that Q_8 is a group of order 8.

6. Let G be a group and let $\mathrm{Aut}(G)$ denote the set of all automorphisms of G.

   (a) Prove that $\mathrm{Aut}(G)$ is a group with composition of functions as binary operation.

   (b) Prove that $\mathrm{Aut}(\mathbb{Z}) \cong \mathbb{Z}_2$, $\mathrm{Aut}(\mathbb{Z}_6) \cong \mathbb{Z}_2$, $\mathrm{Aut}(\mathbb{Z}_8) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, $\mathrm{Aut}(\mathbb{Z}_p) \cong \mathbb{Z}_{p-1}$ (p prime)

7. Let G be an infinite group that is isomorphic to each of its proper subgroups. Prove that $G \cong \mathbb{Z}$.

8. Let G be the multiplicative group of 2×2 invertible matrices with rational entries. Show that

$$A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \text{ and } B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$$

   have finite orders but AB has infinite order.

9. Let G be an abelian group containing elements a and b of orders m and n, respectively. Prove that G contains an element of order $\mathrm{lcm}(m, n)$.

10. Let H, K be subgroups of a group G. Prove that HK is a subgroup of G if and only if $HK = KH$.

11. Let H, K be subgroups of finite index of a group G such that $[G : H]$ and $[G : K]$ are relatively prime. Prove that $G = HK$.

12. Let H, K, N be subgroups of G such that $H \subseteq N$. Prove that $HK \cap N = H(K \cap N)$.

13. Let H, K, N be subgroups of G such that $H \subseteq K, H \cap N = K \cap N$ and $HN = KN$. Prove that $H = K$.

14. Let H be a subgroup of G. For $a \in G$, prove that aHa^{-1} is a subgroup of G that is isomorphic to H.

15. Let G be a finite group and H a subgroup of G of order n. If H is the only subgroup of G of order n, prove that H is normal in G.

16. If H is a cyclic normal subgroup of a group G, then every subgroup of H is normal in G.

17. What is $Z(S_n)$ for $n \geq 2$?

18. If H is a normal subgroup of G such that H and G/H are finitely generated, then G is finitely generated.

19. If N is a normal subgroup of G, $[G : N]$ is finite, H is a subgroup of G, $|H|$ is finite, and $[G : N]$ and $|H|$ are relatively prime, then H is a subgroup of N.

20. If N is a normal subgroup of G, $|N|$ is finite, H is a subgroup of G, $[G : H]$ is finite, and and $[G : H]$ and $|N|$ are relatively prime, then N is a subgroup of H.

21. If G is a finite group and H, K are subgroups of G, then

$$[G : H \cap K] \leq [G : H][G : H]$$

22. If H, K, L are subgroups of a finite group G such that $H \subseteq K$, then

$$[K : H] \geq [L \cap K : L \cap H]$$

23. Let H, K be subgroups of a group G. Assume that $H \cup K$ is a subgroup of G. Prove that either $H \subseteq K$ or $K \subseteq K$.

24. Let G be an abelian group, H a subgroup of G such that G/H is an infinit cyclic group. Prove that $G \cong H \times G/H$.