Problem List[1]

Sagan

Problem 5.1. (**Sagan, Chapter 5, Exercise 1**)
This exercise refers to the list of examples just after the definition of a poset.

(a) Verify that they satisfy the definition of a poset.

(b) Show that the partial order in Π_n is equivalent to defining $\rho \leq \pi$ if every block of π is a union of blocks of ρ.

(c) Describe the cover relations in the list. For example, in C_n the covers are of the form $i \prec i+1$ for $0 \leq i < n$.

Proof. **(a) Two Posets** We prove that the set of all positive divisors of $n \in \mathbb{Z}$ with $|$:$(D_n, |)$ and the set of all subspaces of a vector space $\mathcal{V}$ over $\mathbb{F}_q$ with subspace containment $\leq$: $(L(\mathcal{V}), \leq)$ are posets. For any $a, b, c \in D_n$ and any $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \in L(\mathcal{V})$,

(reflexivity)	$a \mid a$	$\mathcal{U}_1 \leq \mathcal{U}_1$
(antisymmetry)	$a \neq b$ and $a \mid b \implies a < b \implies b \nmid a$	$\mathcal{U}_1 < \mathcal{U}_2 \implies \mathcal{U}_2 \not\leq \mathcal{U}_1$
(transitivity)	$a \mid b \mid c \implies \exists k, q \in \mathbb{Z}$ s.t. $\underset{\implies akq=c}{\overset{ak=b}{bq=c}} \implies a \mid c$	$\mathcal{U}_1 \leq \mathcal{U}_2 \leq \mathcal{U}_3 \implies \mathcal{U}_1 \leq \mathcal{U}_3$

(c) In D_n, if $|n| = \prod_i p_i^{a_i}$ is a prime decomposition, then each positive divisor is of the form $d = \prod_i p_i^{b_i}$ where $0 \leq b_i \leq a_i$ is some 'subdecomposition'. So then each cover relation just looks like

$$\prod_i p_i^{b_i} \mid \prod_i p_i^{\beta_i} \text{ where each } \beta_i \geq b_i \text{ and } \sum_i (\beta_i - b_i) = 1$$

In $L(\mathcal{V})$, cover relations look like

$$\mathcal{W} \leq \mathcal{U} \text{ and } \mathcal{B}_{\mathcal{U}} = \mathcal{B}_{\mathcal{W}} \sqcup \{b\} \text{ is a basis for } \mathcal{U} \text{ and } b \in \mathcal{U}.$$

That is, $\mathcal{W} \leq \mathcal{U}$ and $\dim \mathcal{U} = \dim \mathcal{W} + 1$.

□

[1]
- **Sagan—Chapter 5:** 1(ac), 4, 5abc, 6a, 7bc, 8c, 9, 10a, 15 (1 poset not C_n), 17ab, 18ab, 19a (oneway), 20ab
- **Stanley—Chapter 3:** 7, 14a, 25, 30a, 34, 38, 33, 52, 62abc
- **Fulton—Chapter 1:** Exercises 1, 2 (pp. 15–16), compute product both ways
- **Fulton—Chapter 2:** Exercises 1, 2 (pp. 24–26)

Problem 5.4. **(Sagan, Chapter 5, Exercise 4)**
Complete the proof of Proposition 5.1.3. To show that $K_n \cong B_{n-1}$ it may be simpler to show that $K_n \cong B_{n-1}^*$ using the map ϕ from Section 1.7.

Proof. Recall that $B_{n-1} = (2^{[n-1]}, \subseteq)$ and that K_n is the set of all compositions of n, partially ordered by refinement.

Define $\phi : 2^{[n-1]} \to K_n$ as follows. If $S = \{s_1 < \cdots < s_k\} \subseteq [n-1]$, set $s_0 = 0$ and $s_{k+1} = n$, and define

$$\phi(S) = [s_1 - s_0,\ s_2 - s_1,\ \ldots,\ s_{k+1} - s_k].$$

This is the bijection from Section 1.7.

We claim that ϕ is order-reversing:

$$S \subseteq T \quad \iff \quad \phi(T) \leq_{K_n} \phi(S).$$

Indeed, S records the cut positions of $\phi(S)$. If $S \subseteq T$, then T has all the cuts of S (and possibly more), so $\phi(T)$ is obtained from $\phi(S)$ by inserting additional cuts, i.e. by refining parts. Hence $\phi(T) \leq_{K_n} \phi(S)$.

Conversely, if $\phi(T) \leq_{K_n} \phi(S)$, then $\phi(T)$ is a refinement of $\phi(S)$, so every cut used to form $\phi(S)$ also appears among the cuts used to form $\phi(T)$. Thus $S \subseteq T$.

Therefore ϕ is an isomorphism of posets $B_{n-1}^* \to K_n$, i.e. $K_n \cong B_{n-1}^*$. Since $B_{n-1} \cong (B_{n-1}^*)^*$, this also yields $K_n \cong B_{n-1}$ up to duality as claimed in Proposition 5.1.3. □

Problem 5.5. **(Sagan, Chapter 5, Exercise 5)**
Let $f : P \to Q$ be an isomorphism of posets.

(a) Show that f is also an isomorphism of P^* with Q^*.

(b) Show that if P has a $\hat{0}$, then so does Q.

(c) Show in two ways that if P has a $\hat{1}$, then so does Q: by mimicking the proof of part (b) and by using the result of (b) together with part (a).

Problem 5.6. **(Sagan, Chapter 5, Exercise 6)**

(a) Show that the axioms for a partially ordered set are satisfied by $P \sqcup Q$, $P + Q$, and $P \times Q$.

Problem 5.7. **(Sagan, Chapter 5, Exercise 7)**
Complete the proof of Proposition 5.2.1.

Problem 5.8. **(Sagan, Chapter 5, Exercise 8)**

(a) Show that if P is a ranked poset, then for any k we have $R_k(P)$ is an antichain.

(b) Let P be a ranked poset and assume $f : P \to Q$ is an isomorphism. Show that Q is also ranked and for all $x \in P$ we have $\mathrm{rk}_P x = \mathrm{rk}_Q f(x)$.

(c) Show that if P, Q are ranked posets, then so is $P \times Q$ with rank function

$$\mathrm{rk}_{P\times Q}(x,y) = \mathrm{rk}_P x + \mathrm{rk}_Q y.$$

Problem 5.9. (**Sagan, Chapter 5, Exercise 9**)
Prove Proposition 5.2.2.

Problem 5.10. (**Sagan, Chapter 5, Exercise 10**)

(a) Prove Proposition 5.3.1.

Problem 5.11. (**Sagan, Chapter 5, Exercise 15**)
Prove Proposition 5.3.4.

Problem 5.12. (**Sagan, Chapter 5, Exercise 17**)
Let P be a finite poset and let $L = J(P)$ be the corresponding distributive lattice. If $X \subseteq P$ is a lower-order ideal, then use the corresponding lowercase letter x to denote the associated element of L.

(a) Show that x covers y in L if and only if $Y = X - \{m\}$ where m is a maximal element of X.

(b) Show that x is join irreducible in L if and only if X is a principal ideal of P.

Problem 5.13. (**Sagan, Chapter 5, Exercise 18**)
Given a poset P, let $\mathcal{A}(P)$ be the set of antichains of P. Show that the map

$$f : \mathcal{A}(P) \to J(P)$$

given by $f(A) = I(A)$ (where $I(A)$ is the order ideal generated by A) is a bijection.

Problem 5.14. (**Sagan, Chapter 5, Exercise 19**)

(a) Rederive the formula for μ in B_n, equation (5.6), in two ways: by mimicking the proof of (5.7) and by constructing an $m \in P$ such that $D_m = B_n$ and then applying (5.7).

Problem 5.15. (**Sagan, Chapter 5, Exercise 20**)

(a) Let P be a locally finite poset with a $\hat{0}$. Show that if x covers exactly one element of P, then

$$\mu(x) = \begin{cases} -1 & \text{if } x \text{ covers } \hat{0}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Given any $n \in \mathbb{Z}$, construct a poset containing an element x with $\mu(x) = n$.