MATH 266 - NDSU
Fall 2024
Exam 1 Solution
October 4th, 2024

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1. (10 points) Find the explicit solution of the initial value problem

$$\frac{dy}{dx} = \frac{y+1}{x-3}, \quad y(1) = 0.$$

This equation is separable, so we multiply both sides by $\frac{1}{y+1}dx$ to get

$$\frac{1}{y+1}dy = \frac{1}{x-3}dx.$$

Integrating both sides, we have

$$\int \frac{1}{y+1}\,dy = \int \frac{1}{x-3}\,dx.$$

This gives us $\ln|y+1| = \ln|x-3| + C$.

To get an explicit solution, we have

$$e^{\ln|y+1|} = e^{\ln|x-3|+C} = e^C e^{\ln|x-3|} = Ce^{\ln|x-3|}.$$

That is, $|y+1| = C|x-3|$, which implies that $y+1 = \pm C(x-3) = C(x-3)$ as C is arbitrary.

To find the value C, we plug in $x = 1$ and $y = 0$ to get $1 = -2C$, so $C = -\frac{1}{2}$. Therefore, the explicit solution is $y = -1 - \frac{1}{2}(x-3)$.

It is not necessary to simplify the right hand side, but the answer must be of the explicit form $y = \dots$.

2. (10 points) Find the general solution of the equation

$$xy' - 3y = x^{-2}, \quad x > 0.$$

Write your answer as an explicit solution.

This is a linear equation, so we solve it using integrating factors. Note that the coefficient of y' is not 1, so we first divide both sides of the equation by x to get

$$y' - \frac{3}{x}y = \frac{1}{x^3}.$$

The integrating factor

$$\mu(x) = e^{\int -\frac{3}{x}\ dx} = e^{-3\ln|x|} = e^{-3\ln x} = x^{-3}.$$

Multiplying both sides of the equation by $\mu(x) = x^{-3}$ gives us

$$x^{-3}y' - 3x^{-4}y = \frac{1}{x^6}.$$

That is,

$$\frac{d}{dx}(x^{-3}y) = \frac{1}{x^6}.$$

Integrating both sides, we get

$$\begin{aligned} x^{-3}y &= \int \frac{1}{x^6}\ dx \\ &= -\frac{1}{5x^5} + C. \end{aligned}$$

Finally, we divide both sides of the expression above by x^3, we get $y = -\frac{1}{5x^2} + \frac{C}{x^3}$.

3. (10 points) Consider $\dfrac{dy}{dt} = y^2(4 - y^2)$.

   (a) (4 points) Find all equilibrium solutions.

   We factor the right-hand side see that $y' = y^2(2+y)(2-y)$. Setting $y' = 0$, we get $y = 0, 2, -2$ are equilibrium solutions.

   (b) (6 points) Sketch the phase diagram and classify all equilibrium solutions.

   The equilibrium solutions we found in (a) divide the real number line into 4 intervals. We test whether y' is positive or negative by plugging in appropriate numbers from each interval.

   Pick $y = -3$ $(y < -2)$, then $y' = (-3)^2 \cdot (2+(-3)) \cdot (2-(-3)) < 0$.

   Pick $y = -1$ $(-2 < y < 0)$, then $y' = (-1)^2 \cdot (2 + (-2)) \cdot (2 - (-2)) > 0$.

   Pick $y = 1$ $(0 < y < 2)$, then $y' = 1^2 \cdot (2+1) \cdot (2-1) > 0$.

   Pick $y = 3$ $(y > 2)$, then $y' = 3^2 \cdot (2+3) \cdot (2-3) < 0$.

   From the phase diagram, we see that $y = 2$ is stable, and $y = 0, -2$ are unstable.

   It is not necessary to compute the exact values of y'. Getting $y' > 0$ or $y' < 0$ is enough.

4. (10 points) Compute the Wronskian of two functions $y_1 = e^{3x}$ and $y_2 = x^2e^{3x}$, $x \neq 0$. Are y_1 and y_2 linearly independent? Justify your answer.

   We compute directly that $y_1' = 3e^{3x}$. Using the product rule, we get $y_2' = 3x^2e^{3x} + 2xe^{3x}$.

   Thus, the Wronskian is

$$\begin{aligned} W[y_1, y_2] &= e^{3x}(3x^2e^{3x} + 2xe^{3x}) - (x^2e^{3x})(3e^{3x}) \\ &= 3x^2e^{6x} + 2xe^{6x} - 3x^2e^{6x} \\ &= 2xe^{6x}. \end{aligned}$$

   Since $W[y_1, y_2] \neq 0$ for all x, we conclude that y_1 and y_2 are linearly independent.

5. (10 points) Solve the equation

$$3x^2(1+\ln y)\ dx + \big(\frac{x^3}{y} + e^y\big)\ dy = 0, \quad y > 0.$$

We first test for exactness: Note that $M(x,y) = 3x^2(1+\ln y)$ and $N(x,y) = \frac{x^3}{y} - e^y$, so we compute $M_y = \frac{3x^2}{y}$ and $N_x = \frac{3x^2}{y}$. This tells us that the equation is exact.

Thus, we integrate $M(x,y)$ with respect to x to get

$$\begin{aligned}\psi(x,y) &= \int 3x^2(1+\ln y)\ dx \\ &= x^3(1+\ln y) + g(y).\end{aligned}$$

It then follows that $\psi_y = \frac{x^3}{y} + g'(y)$. Since the equation is exact, we have $\psi_y = N(x,y) = \frac{x^3}{y} - e^y$. Therefore, we get $g'(y) = e^y$, giving us $g(y) = e^y$.

Hence, $\psi(x,y) = x^3(1+\ln y) + e^y$, which implies that the general solution is

$$x^3(1+\ln y) + e^y = C.$$