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Groups

Def $(G, \cdot)$ set $\quad G \times G \longrightarrow G$

$\qquad (a, b) \longrightarrow a \cdot b = ab$

(a) assoc. $(ab)c = a(bc)$

(b) there exists $e \in G$ s.t. $ae = ea = a$ for all $a \in G$.

(c) for every $a \in G$ there exists a^{-1} st. $a \cdot a^{-1} = a^{-1} \cdot a = e$.

If $a \cdot b = b \cdot a$ for all $a, b \in G$, we say that G is commutative (abelian).

Examples $(\mathbb{Z}, +)$, $(\mathbb{Z}_n, +)$, $(\mathbb{Q}^*, \cdot)$, $(\mathbb{Q}, +)$

$\mathbb{Q}^* = \mathbb{Q} \setminus \{0\}$

Ex $GL_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \,\middle|\, ad - bc \neq 0 \right\}$

$(GL_2(\mathbb{R}), \cdot)$ group (not abelian).

Ex G_1, G_2 groups. Define $G_1 \times G_2 = \{ (a, b) \mid a \in G_1, b \in G_2 \}$

$(a,b)\cdot(c,d) = (ac, bd)$

Then $(G_1 \times G_2, \cdot)$ is a group.

Group homomorphisms

G_1, G_2 groups $\quad f: G_1 \longrightarrow G_2$ st.

$$f(xy) = f(x)\cdot f(y) \text{ for all } x, y \in G_2$$

- monomorphism : injective homom.
- epimorphism : surjective homom.
- isomorphism : bijective homom.

$\operatorname{Ker} f = \{x \in G_1 \mid f(x) = e_{G_2}\}$

$\operatorname{Im} f = \{f(x) \mid x \in G_1\}$.

Prop $f: G_1 \longrightarrow G_2$ homom. Then

(1) f mono $\Longleftrightarrow \operatorname{Ker} f = \{e_{G_1}\}$

(2) f isom $\Longrightarrow f^{-1}$ isom.

Sketch of Proof (1)

"$\Longrightarrow$" Let $x \in \operatorname{Ker} f$. Then $f(x) = e_{G_2} = f(e_{G_1})$

But f mono, so $x = e_{G_1}$ $\boxtimes$

"$\Longleftarrow$" $f(x) = f(y)$ for $x, y \in G_1$.

Then $f(x \cdot y^{-1}) = f(x) \cdot \underbrace{f(y^{-1})}_{\text{prove it.}} = f(x) \cdot f(y)^{-1}$

$\phantom{Then f(x \cdot y^{-1}) = f(x) \cdot f(y^{-1})} = e_{G_2}$.

So $xy^{-1} \in \operatorname{Ker} f = \{e_{G_1}\}$. Then $xy^{-1} = e_{G_1}$,

i.e. $x = y$. $\boxtimes$

Subgroups

G group, $H \subseteq G$, $H \neq \emptyset$ st. H is a group w.r.t. the same operation. Equivalently, H satisfies:

(a) $x, y \in H \Longrightarrow xy \in H$

(b) $x \in H \Longrightarrow x^{-1} \in H$. We write $\boxed{H \leq G}$

__Examples__ (1) $(\mathbb{Z}, +)$ For $n \in \mathbb{Z}$

$n\mathbb{Z} \leq \mathbb{Z}$

(2) $A_n \leq S_n$

even permutations

__Exercise__ G group, $\{H_i\}_{i \in \Lambda}$ $H_i \leq G$ for all $i \in \Lambda$. subgr.

Then $\bigcap_{i \in \Lambda} H_i$ subgr. of G

__Def__ $X \subseteq G$ subset. What is the smallest subgr. of G that contains X?

group.

Let $\langle X \rangle = \bigcap_{\substack{H \leq G \\ X \subseteq H}} H$

Then $\langle X \rangle$ is the smallest subgr. of G that contains X (the subgr. gen. by X).

Proof By prev. exercise, $\langle X \rangle$ is a subgr. of G.

Need to prove:

$$X \subseteq L \overset{\text{subgroup}}{\leq} G \implies L \supseteq \langle X \rangle \text{ (clear)}$$

$\boxtimes$

Ex $X = \{2\} \subseteq \mathbb{Z}$. Then $\langle \{2\} \rangle = 2\mathbb{Z}$.

Def If $H = \langle \{a_1, \dots, a_n\} \rangle$, we say that H is finitely generated.

Theorem $X \subseteq \underset{\text{group}}{G}$ subset

$$\text{Then } \langle X \rangle = \overbrace{\left\{ a_1^{n_1} a_2^{n_2} \cdots a_k^{n_k} \;\middle|\; \begin{array}{l} a_i \in X,\ k \in \mathbb{N} \\ n_i \in \mathbb{Z} \end{array} \right\}}^{H}$$

Sketch of proof First, check that H is a subgroup.

Second $X \subseteq H$ ✓

Let $X \subseteq L \leq G$. ~~Then~~ Need to prove
$\underset{\substack{\uparrow \\ \text{subgr.}}}{}$

$H \subseteq L$. $\blacksquare$