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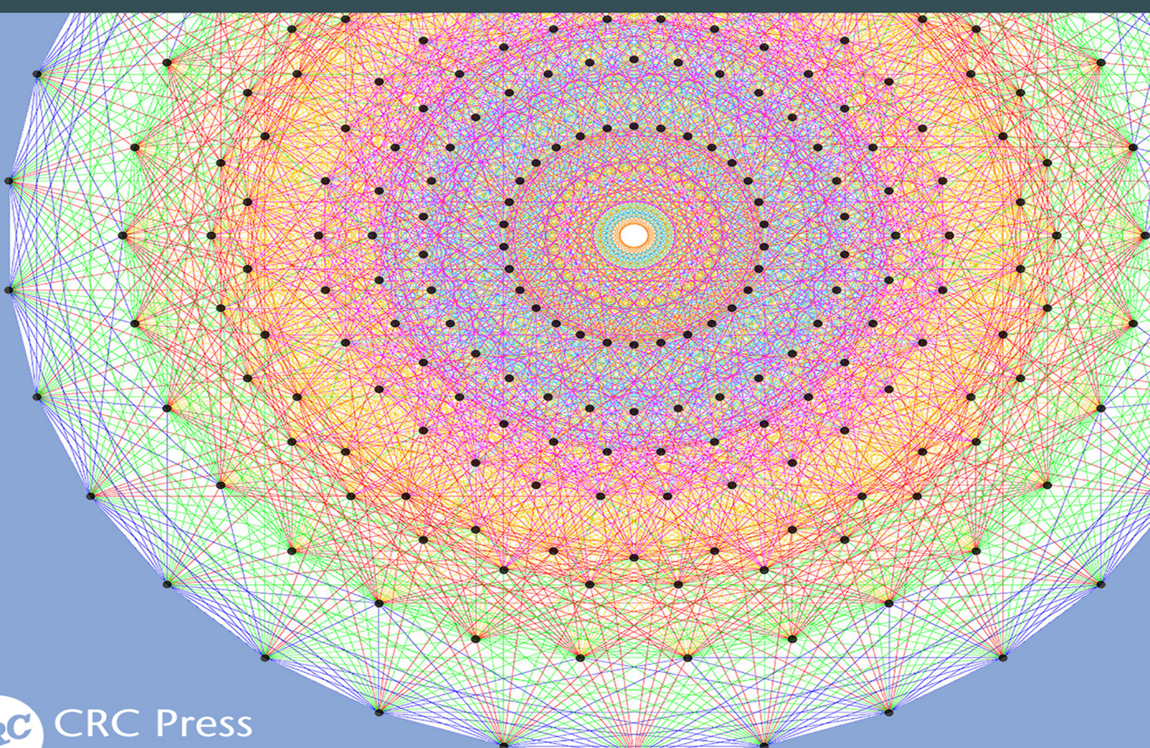
TENTH EDITION

Joseph A. Gallian



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Contemporary Abstract Algebra

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Contemporary Abstract Algebra

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To Char, Krissy, and Joey

Contents

Notations

SET THEORY	$\cap_{i\in I} S_i$	intersection of sets $S_i, i \in I$
	$\cup_{i\in I} S_i$	union of sets $S_i, i \in I$
	$[a]$	$\{x \in S \mid x \sim a\}$, equivalence class of S containing a, 16
	$\lvert s\rvert$	number of elements in the set of S
SPECIAL SETS	Z	integers, additive groups of integers, ring of integers
	Q	rational numbers, field of rational numbers
	Q^+	multiplicative group of positive rational numbers
	F^*	set of nonzero elements of F
	$\mathbf{R}$	real numbers, field of real numbers
	$\mathbf{R}^+$	multiplicative group of positive real numbers
	$\mathbf{C}$	complex numbers
FUNCTIONS AND ARITHMETIC	f^{-1}	inverse of the function f
	$t \mid s$	t divides s, 1
	$t \nmid s$	t does not divide s, 1
	$\gcd(a, b)$	greatest common divisor of the integers a and b, 3
	$\operatorname{lcm}(a, b)$	least common multiple of the integers a and b, 5
	$\lvert a+b\rvert$	$\sqrt{a^2+b^2}$, 10
	$\phi(a)$	image of a under ϕ, 19
	$\phi : A \to B$	mapping of A to B, 19
	$gf, \alpha\beta$	composite function, 19
ALGEBRAIC SYSTEMS	D_4	dihedral group of order, 29
	D_n	dihedral group of order $2n$, 30
	e	identity element, 40
	Z_n	group $\{0, 1, \dots, n-1\}$ under addition modulo n, 41
	$\det A$	the determinant of A, 42
	$U(n)$	group of units modulo n (that is, the set of integers less than n and relatively prime to n under multiplication modulo n), 43

$\mathbf{R}^n$	$\{(a_1, a_2, \ldots, a_n) \mid a_1, a_2, \ldots, a_n \in \mathbf{R}\}$, 45
$SL(2, F)$	group of 2×2 matrices over F with determinant 1, 45
$GL(2, F)$	2×2 matrices of nonzero determinants with coefficients from the field F (the general linear group), 46
g^{-1}	multiplicative inverse of g, 49
$-g$	additive inverse of g, 49
$\|G\|$	order of the group G, 59
$\|g\|$	order of the element g, 59
$H \leq G$	subgroup inclusion, 60
$H < G$	subgroup $H \neq G$, 61
$\langle a \rangle$	$\{a^n \mid n \in Z\}$, cyclic group generated by a, 64
$\langle S \rangle$	subgroup generated by the set S, 66
$Z(G)$	$\{a \in G \mid ax = xa$ for all x in $G\}$, the center of G, 67
$C(a)$	$\{g \in G \mid ga = ag\}$, the centralizer of a in G, 68
$C(H)$	$\{x \in G \mid xh = hx$ for all $h \in H\}$, the centralizer of H, 72
$\phi(n)$	Euler phi function of n, 86
S_n	group of one-to-one functions from $\{1, 2, \ldots, n\}$ to itself, 99
A_n	alternating group of degree n, 110
$G \approx \overline{G}$	G and $\overline{G}$ are isomorphic, 128
ϕ_a	mapping given by $\phi_a(x) = axa^{-1}$ for all x, 135
$\mathrm{Aut}(G)$	group of automorphisms of the group G, 135
$\mathrm{Inn}(G)$	group of inner automorphisms of G, 135
aH	$\{ah \mid h \in H\}$, 149
aHa^{-1}	$\{aha^{-1} \mid h \in H\}$, 149
$\|G : H\|$	the index of H in G, 154
HK	$\{hk \mid h \in H, k \in K\}$, 156
$\mathrm{stab}_G(i)$	$\{\phi \in G \mid \phi(i) = i\}$, the stabilizer of i under the permutation group G, 158
$\mathrm{orb}_G(i)$	$\{\phi(i) \mid \phi \in G\}$, the orbit of i under the permutation group G, 158
$G_1 \oplus G_2 \oplus \cdots \oplus G_n$	external direct product of groups $G_1, G_2, \ldots, G_n$, 171
$U_k(n)$	$\{x \in U(n) \mid x \bmod k = 1\}$, 176
$H \lhd G$	H is a normal subgroup of G, 193
G/H	factor group, 196
$H \times K$	internal direct product of H and K, 203
$H_1 \times H_2 \times \cdots \times H_n$	internal direct product of $H_1, \ldots, H_n$, 205
$\mathrm{Ker}\ \phi$	kernel of the homomorphism ϕ, 220
$\phi^{-1}(g')$	inverse image of g' under ϕ, 221
$\phi^{-1}(\overline{K})$	inverse image of $\overline{K}$ under ϕ, 222
$Z[x]$	ring of polynomials with integer coefficients, 256
$M_2(Z)$	ring of all 2×2 matrices with integer entries, 256
$R_1 \oplus R_2 \oplus \cdots \oplus R_n$	direct sum of rings, 257
nZ	ring of multiples of n, 259
$Z[i]$	ring of Gaussian integers, 260
$U(R)$	group of units of the ring R, 262

char R	characteristic of R, 271
$\langle a \rangle$	principal ideal generated by a, 280
$\langle a_1, a_2, \ldots, a_n \rangle$	ideal generated by $a_1, a_2, \ldots, a_n$, 280
R/A	factor ring, 280
$A + B$	sum of ideals A and B, 287
AB	product of ideals A and B, 287
$\mathrm{Ann}(A)$	annihilator of A, 291
$N(A)$	nil radical of A, 291
$F(x)$	field of quotients of $F[x]$, 302
$R[x]$	ring of polynomials over R, 311
$\deg f(x)$	degree of the polynomial, 313
$\Phi_p(x)$	pth cyclotomic polynomial, 330
$F(a_1, a_2, \ldots, a_n)$	extension of F by $a_1, a_2, \ldots, a_n$, 367
$f'(x)$	the derivative of $f(x)$, 374
$[E{:}F]$	degree of E over F, 387
$\mathrm{GF}(p^n)$	Galois field of order p^n, 402
$\mathrm{GF}(p^n)^*$	nonzero elements of $\mathrm{GF}(p^n)$, 403
$\mathrm{cl}(a)$	$\{xax^{-1} \mid x \in G\}$, the conjugacy class of a, 427
n_p	the number of Sylow p-subgroups of a group, 434
$W(S)$	set of all words from S, 468
$\langle a_1, a_2, \ldots, a_n \mid w_1 = w_2 = \cdots = w_t \rangle$	group with generators $a_1, a_2, \ldots, a_n$ and relations $w_1 = w_2 = \ldots = w_t$, 471
Q_4	quarternions, 476
Q_6	dicyclic group of order 12, 476
D_∞	infinite dihedral group, 476
$\mathrm{fix}(\phi)$	$\{i \in S \mid \phi(i) = i\}$, elements fixed by ϕ, 493
$\mathrm{Cay}(S : G)$	Cayley digraph of the group G with generating set S, 503
$k * (a, b, \ldots, c)$	concatenation of k copies of $(a, b, \ldots, c)$, 511
(n, k)	linear code, k-dimensional subspace of F^n, 530
F^n	$F \oplus F \oplus \cdots \oplus F$, direct product of n copies of the field F, 530
$d(u, v)$	Hamming distance between vectors u and v, 531
$\mathrm{wt}(u)$	the number of nonzero components of the vector u (the Hamming weight of u), 531
$\mathrm{Gal}(E/F)$	the automorphism group of E fixing F, 556
E_H	fixed field of H, 556
$\Phi_n(x)$	nth cyclotomic polynomial, 574

Preface

> Set your pace to a stroll. Stop whenever you want. Interrupt, jump back and forth, I won't mind. This book should be as easy as laughter. It is stuffed with small things to take away. Please help yourself.
>
> Willis Goth Regier, *In Praise of Flattery*, 2007

Although I wrote the first edition of this book more than thirty years ago, my goals for it remain the same. I want students to receive a solid introduction to the traditional topics. I want readers to come away with the view that abstract algebra is a contemporary subject–that its concepts and methodologies are being used by working mathematicians, computer scientists, physicists, and chemists. I want students to see the connections between abstract algebra and number theory and geometry. I want students to be able to do computations and to write proofs. I want students to enjoy reading the book. And I want to convey to the reader my enthusiasm for this beautiful subject.

Educational research has shown that an effective way of learning mathematics is to interweave worked-out examples and practice problems. Thus, I have made examples and exercises the heart of the book. The examples elucidate the definitions, theorems, and proof techniques. The exercises facilitate understanding, provide insight, and develop the ability of the students to do proofs. There is a large number of exercises ranging from straightforward to difficult and enough at each level so that instructors have plenty to choose from that are most appropriate for their students. The exercises often foreshadow definitions, concepts, and theorems to come. Many exercises focus on special cases and ask the reader to generalize. Generalizing is a skill that students should develop but rarely do. Even if an instructor chooses not to spend class time on the applications in the book, I feel that having them there demonstrates to students the utility of the theory.

Changes for the tenth edition include new exercises, new examples, new quotes, and a freshening of the discussion portions. These changes accentuate and enhance the hallmark features that have made previous editions of the book a comprehensive, lively, and engaging introduction to the subject:

- Extensive coverage of groups, rings, and fields, plus a variety of non-traditional special topics
- A good mixture of approximately 1900 computational and theoretical exercises appearing in each chapter that synthesize concepts from multiple chapters
- Back-of-the-book skeleton solutions and hints to the odd-numbered exercises
- Approximately 300 worked-out examples ranging from routine computations to quite challenging
- Computer exercises that utilize interactive software available on my website that stress guessing and making conjectures
- Many of applications from scientific and computing fields, as well as some from everyday life
- Numerous historical notes and biographies that spotlight the people and events behind the mathematics
- Motivational and humorous quotations
- Hundreds of figures, photographs, and tables

The Student Solutions Manual (ISBN: 9780367766801) is available from CRC Press. It has detailed solutions for all odd-numbered exercises and a large number of even-numbered exercises. Instructor resources are available from CRC Press for qualified adopters. Please go to https://www.routledge.com/Contemporary-Abstract-Algebra/Gallian/p/book/9780367651787. The website www.d.umn.edu/∼jgallian also offers a wealth of additional online resources supporting the book, including:

- True/false questions with comments
- Flash cards
- Essays on learning abstract algebra, doing proofs, and reasons why abstract algebra is a valuable subject to learn
- Links to abstract algebra-related websites and software packages and much, much more.

I wish to express my gratitude to Robert Ross for the opportunity to publish this edition with CRC Press. I am greatly indebted to Shahriyar Roshan Zamir, Shashi Kumar, and Scott Martin for creating the LaTex files for this edition. John Stembridge kindly granted permission to use his spectacular image of the simple Lie group E_8 on the cover.

0 Preliminaries

When we see it [modular arithmetic] for the first time, it looks so abstract that it seems impossible something like this could have any real-world applications.

Edward Frenkel, *Love and Math: The Heart of Hidden Reality*

The whole of science is nothing more than a refinement of everyday thinking.

Albert Einstein, *Physics and Reality*

Properties of Integers

It should not come as a surprise that much of abstract algebra involves properties of integers and sets. In this chapter we collect the properties we need for future reference. An important property of the integers, which we will often use, is the so-called Well Ordering Principle. Since this property cannot be proved from the usual properties of arithmetic, we will take it as an axiom.

Well Ordering Principle

Every nonempty set of positive integers contains a smallest member.

The concept of divisibility plays a fundamental role in the theory of numbers. We say a nonzero integer t is a *divisor* of an integer s if there is an integer u such that $s = tu$. In this case, we write $t \mid s$ (read "t divides s"). When t is not a divisor of s, we write $t \nmid s$. A *prime* is a positive integer greater than 1 whose only positive divisors are 1 and itself. We say an integer s is a *multiple* of an integer t if there is an integer u such that $s = tu$ or, equivalently, if t is a divisor of s.

As our first application of the Well Ordering Principle, we establish a fundamental property of integers that we will use often.

Theorem 0.1 Division Algorithm

Let a and b be integers with $b > 0$. Then there exist unique integers q and r with the property that $a = bq + r$, where $0 \leq r < b$.

PROOF We begin with the existence portion of the theorem. Consider the set $S = \{a - bk \mid k \text{ is an integer and } a - bk \geq 0\}$. If $0 \in S$, then b divides a and we may obtain the desired result with $q = a/b$ and $r = 0$. Now assume $0 \notin S$. Since S is nonempty [if $a > 0$, $a - b \cdot 0 \in S$; if $a < 0, a - b(2a) = a(1 - 2b) \in S; a \neq 0$ since $0 \notin S$], we may apply the Well Ordering Principle to conclude that S has a smallest member, say $r = a - bq$. Then $a = bq + r$ and $r \geq 0$, so all that remains to be proved is that $r < b$.

If $r \geq b$, then $a - b(q + 1) = a - bq - b = r - b \geq 0$, so that $a - b(q + 1) \in S$. But $a - b(q + 1) < a - bq$, and $a - bq$ is the *smallest* member of S. So, $r < b$.

To establish the uniqueness of q and r, let us suppose that there are integers q, q', r, and r' such that

$$a = bq + r, 0 \leq r < b, \text{ and } \quad a = bq' + r', \quad 0 \leq r' < b.$$

For convenience, we may also suppose that $r' \geq r$. Then $bq + r = bq' + r'$ and $b(q - q') = r' - r$. So, b divides $r' - r$ and $0 \leq r' - r \leq r' < b$. It follows that $r' - r = 0$, and therefore $r' = r$ and $q = q'$. ∎

The integer q in the division algorithm is called the *quotient* upon dividing a by b; the integer r is called the *remainder* upon dividing a by b.

EXAMPLE 1 For $a = 17$ and $b = 5$, the division algorithm gives $17 = 5 \cdot 3 + 2$; for $a = -23$ and $b = 6$, the division algorithm gives $-23 = 6(-4) + 1$. ∎

There are many instances in this book where there are integers a and b and we will want to show that a is divisible by b. In such cases it is usually best to proceed by writing $a = bq + r$, where $0 \leq r < b$ and use properties of a and b to show that $r = 0$. The proof of Theorem 0.2 is one such instance.

Definitions Greatest Common Divisor, Relatively Prime Integers
The *greatest common divisor* of two nonzero integers a and b is the largest of all common divisors of a and b. We denote this integer by $\gcd(a, b)$. When $\gcd(a, b) = 1$, we say a and b are *relatively prime*.

The following property of the greatest common divisor of two integers plays a critical role in abstract algebra. The proof provides an application of the division algorithm and our second application of the Well Ordering Principle.

Theorem 0.2 GCD is a Linear Combination

For any nonzero integers a and b, there exist integers s and t such that $\gcd(a, b) = as + bt$. Moreover, $\gcd(a, b)$ is the smallest positive integer of the form $as + bt$.

PROOF Consider the set $S = \{am + bn \mid m, n$ are integers and $am + bn > 0\}$. Since S is obviously nonempty (if some choice of m and n makes $am + bn < 0$, then replace m and n by $-m$ and $-n$), the Well Ordering Principle asserts that S has a smallest member, say, $d = as + bt$. We claim that $d = \gcd(a, b)$. To verify this claim, use the division algorithm to write $a = dq + r$, where $0 \le r < d$. If $r > 0$, then $r = a - dq = a - (as + bt)q = a - asq - btq = a(1 - sq) + b(-tq) \in S$, contradicting the fact that d is the smallest member of S. So, $r = 0$ and d divides a. Analogously (or, better yet, by symmetry), d divides b as well. This proves that d is a common divisor of a and b. Now suppose d' is another common divisor of a and b and write $a = d'h$ and $b = d'k$. Then $d = as + bt = (d'h)s + (d'k)t = d'(hs + kt)$, so that d' is a divisor of d. Thus, among all common divisors of a and b, d is the greatest.
∎

The special case of Theorem 0.2 when a and b are relatively prime is so important in abstract algebra that we single it out as a corollary.

Corollary

Integers a and b are relatively prime if and only if there exist integers s and t such that $as + bt = 1$.

EXAMPLE 2 $\gcd(4, 15) = 1$; $\gcd(4, 10) = 2$; $\gcd(2^2 \cdot 3^2 \cdot 5, 2 \cdot 3^3 \cdot 7^2) = 2 \cdot 3^2$. Note that 4 and 15 are relatively prime, whereas 4 and 10 are not. Also, $4 \cdot 4 + 15(-1) = 1$ and $4(-2) + 10 \cdot 1 = 2$. ∎

The corollary of Theorem 0.2 provides a convenient method to show that two integers represented by polynomial expressions are relatively prime.

EXAMPLE 3 For any integer n the integers $n+1$ and n^2+n+1 are relatively prime. To verify this we observe that $n^2+n+1-n(n+1) = 1$. ∎

The next lemma is frequently used. It appeared in Euclid's *Elements*.

Lemma 0.1 Euclid's Lemma $p \mid ab$ Implies $p \mid a$ or $p \mid b$

If p is a prime that divides ab, then p divides a or p divides b.

PROOF Suppose p is a prime that divides ab but does not divide a. We must show that p divides b. Since p does not divide a, there are integers s and t such that $1 = as + pt$. Then $b = abs + ptb$, and since p divides the right-hand side of this equation, p also divides b. ∎

Note that Euclid's Lemma may fail when p is not a prime, since $6 \mid (4 \cdot 3)$ but $6 \nmid 4$ and $6 \nmid 3$.

Why are primes important? Our next property shows that the primes are the building blocks for all integers. We will often use this property without explicitly saying so.

Theorem 0.3 Fundamental Theorem of Arithmetic

Every integer greater than 1 is a prime or a product of primes. This product is unique, except for the order in which the factors appear. That is, if $n = p_1 p_2 \cdots p_r$ and $n = q_1 q_2 \cdots q_s$, where the p's and q's are primes, then $r = s$ and, after renumbering the q's, we have $p_i = q_i$ for all i.

We will prove the existence portion of Theorem 0.3 later in this chapter (Example 14). The uniqueness portion is a consequence of Euclid's Lemma (Exercise 31). Here is a fun application of Theorem 0.3.

EXAMPLE 4 Let n be any integer greater than 1, $\sqrt[n]{2}$ is irrational. For if $\sqrt[n]{2} = a/b$ where a and b are integers, and a/b is in lowest terms, then $a^n = 2b^n$. By Theorem 0.3 we know 2 divides a, say $a = 2c$. Then $2^n c^n = 2b^n$ and therefore $2^{n-1}c^n = b^n$. But this implies that 2 divides b. This contradicts our assumption that a/b was in lowest terms. ∎

Another concept that frequently arises is that of the least common multiple of two integers.

Definition Least Common Multiple
The *least common multiple* of two nonzero integers a and b is the smallest positive integer that is a multiple of both a and b. We will denote this integer by lcm(a, b).

We leave it as an exercise (Exercise 10) to prove that every common multiple of a and b is a multiple of lcm(a, b).

EXAMPLE 5 $\text{lcm}(4, 6) = 12$; $\text{lcm}(4, 8) = 8$; $\text{lcm}(10, 12) = 60$; $\text{lcm}(6, 5) = 30$; $\text{lcm}(2^2 \cdot 3^2 \cdot 5, 2 \cdot 3^3 \cdot 7^2) = 2^2 \cdot 3^3 \cdot 5 \cdot 7^2$. ∎

Modular Arithmetic

Another application of the division algorithm that will be important to us is modular arithmetic. Modular arithmetic is an abstraction of a method of counting that you often use. For example, if it is now September, what month will it be 25 months from now? Of course, the answer is October, but the interesting fact is that you didn't arrive at the answer by starting with September and counting off 25 months. Instead, without even thinking about it, you simply observed that $25 = 2 \cdot 12 + 1$, and you added 1 month to September. Similarly, if it is now Wednesday, you know that in 23 days it will be Friday. This time, you arrived at your answer by noting that $23 = 7 \cdot 3 + 2$, so you added 2 days to Wednesday instead of counting off 23 days. If your electricity is off for 26 hours, you must advance your clock 2 hours, since $26 = 2 \cdot 12 + 2$. Surprisingly, this simple idea has numerous important applications in mathematics and computer science. You will see a few of them in this section. We shall see many more in later chapters.

The following notation is convenient. When $a = qn + r$, where q is the quotient and r is the remainder upon dividing a by n, we write $a \bmod n = r$. Thus,

$$\begin{aligned}
3 \bmod 2 &= 1 \text{ since } 3 = 1 \cdot 2 + 1,\\
6 \bmod 2 &= 0 \text{ since } 6 = 3 \cdot 2 + 0,\\
11 \bmod 3 &= 2 \text{ since } 11 = 3 \cdot 3 + 2,\\
62 \bmod 85 &= 62 \text{ since } 62 = 0 \cdot 85 + 62,\\
-2 \bmod 15 &= 13 \text{ since } -2 = (-1)15 + 13.
\end{aligned}$$

In general, if a and b are integers and n is a positive integer, then $a \bmod n = b \bmod n$ if and only if n divides $a - b$ (Exercise 9).

In our applications, we will use addition and multiplication mod n. When you wish to compute $ab \bmod n$ or $(a + b) \bmod n$, and a or b is greater than n, it is easier to "mod first." For example, to compute $(27 \cdot 36) \bmod 11$, we note that $27 \bmod 11 = 5$ and $36 \bmod 11 = 3$, so $(27 \cdot 36) \bmod 11 = (5 \cdot 3) \bmod 11 = 4$. (See Exercise 11.)

Modular arithmetic is often used in assigning an extra digit to identification numbers for the purpose of detecting forgery or errors. We present two such applications.

EXAMPLE 6 The United States Postal Service money order shown in Figure 0.1 has an identification number consisting of 10 digits together with an extra digit called a *check*. The check digit is the 10-digit number modulo 9. Thus, the number 3953988164 has the check digit 2, since $3953988164 \bmod 9 = 2$.[1] If the number 39539881642 were incorrectly entered into a computer (programmed to calculate the check digit) as, say, 39559881642 (an error in the fourth position), the machine would calculate the check digit as 4, whereas the entered check digit would be 2. Thus, the error would be detected. ∎

The method used by the Postal Service does not detect all single-digit errors (see Exercise 41). However, detection of all

[1]The value of $N \bmod 9$ is easy to compute with a calculator. If $N = 9q + r$, where r is the remainder upon dividing N by 9, then on a calculator screen $N \div 9$ appears as $q.rrrrr\ldots$, so the first decimal digit is the check digit. For example, $3953988164 \div 9 = 439332018.222$, so 2 is the check digit. If N has too many digits for your calculator, replace N by the sum of its digits and divide that number by 9. Thus, $3953988164 \bmod 9 = 56 \bmod 9 = 2$. The value of $3953988164 \bmod 9$ can also be computed by searching Google for "3953988164 mod 9."

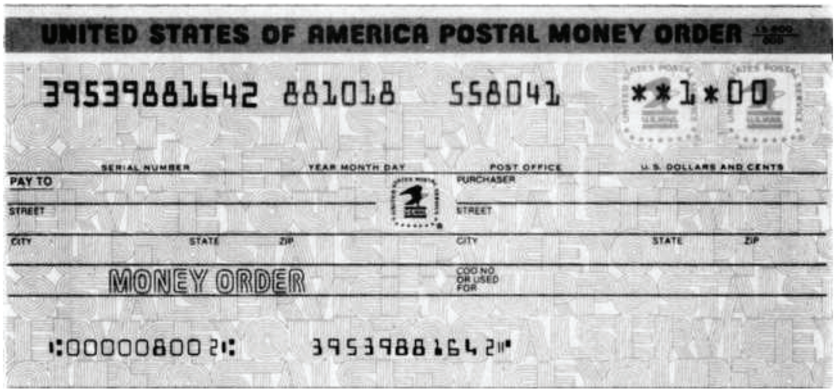


Figure 0.1 Postal money order.

single-digit errors, as well as nearly all errors involving the transposition of two adjacent digits, is easily achieved. One method that does this is the one used to assign the so-called Universal Product Code (UPC) to most retail items (see Figure 0.2). A UPC identification number has 12 digits. The first six digits identify the manufacturer, the next five identify the product, and the last is a check. (For many items, the 12th digit is not printed, but it is always bar-coded.) In Figure 0.2, the check digit is 8.

Figure 0.2 UPC bar code.

To explain how the check digit is calculated, it is convenient to introduce the dot product notation for two k-tuples:

$$(a_1, a_2, \ldots, a_k) \cdot (w_1, w_2, \ldots, w_k) = a_1w_1 + a_2w_2 + \cdots + a_kw_k.$$

An item with the UPC identification number $a_1a_2 \cdots a_{12}$ satisfies the condition

$$(a_1, a_2, \ldots, a_{12}) \cdot (3, 1, 3, 1, \ldots, 3, 1) \bmod 10 = 0.$$

To verify that the number in Figure 0.2 satisfies this condition, we calculate

$$(0 \cdot 3 + 2 \cdot 1 + 1 \cdot 3 + 0 \cdot 1 + 0 \cdot 3 + 0 \cdot 1 + 6 \cdot 3 + 5 \cdot 1$$

$$+ 8 \cdot 3 + 9 \cdot 1 + 7 \cdot 3 + 8 \cdot 1) \bmod 10 = 90 \bmod 10 = 0.$$

The fixed k-tuple used in the calculation of check digits is called the *weighting vector.*

Now suppose a single error is made in entering the number in Figure 0.2 into a computer. Say, for instance, that 021000958978 is entered (notice that the seventh digit is incorrect). Then the computer calculates

$$\begin{gathered} 0 \cdot 3 + 2 \cdot 1 + 1 \cdot 3 + 0 \cdot 1 + 0 \cdot 3 + 0 \cdot 1 + 9 \cdot 3 \\ + 5 \cdot 1 + 8 \cdot 3 + 9 \cdot 1 + 7 \cdot 3 + 8 \cdot 1 = 99. \end{gathered}$$

Since $99 \bmod 10 \neq 0$, the entered number cannot be correct.

In general, any single error will result in a sum that is not 0 modulo 10.

The advantage of the UPC scheme is that it will detect nearly all errors involving the transposition of two adjacent digits as well as all errors involving one digit. For doubters, let us say that the identification number given in Figure 0.2 is entered as 021000658798. Notice that the last two digits preceding the check digit have been transposed. But by calculating the dot product, we obtain $94 \bmod 10 \neq 0$, so we have detected an error. In fact, the only undetected transposition errors of adjacent digits a and b are those where $|a - b| = 5$. To verify this, we observe that a transposition error of the form

$$a_1 a_2 \cdots a_i a_{i+1} \cdots a_{12} \to a_1 a_2 \cdots a_{i+1} a_i \cdots a_{12}$$

is undetected if and only if

$$(a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_{12}) \cdot (3, 1, 3, 1, \ldots, 3, 1) \bmod 10 = 0.$$

That is, the error is undetected if and only if

$$\begin{gathered} (a_1, a_2, \ldots, a_{i+1}, a_i, \ldots, a_{12}) \cdot (3, 1, 3, 1, \ldots, 3, 1) \bmod 10 \\ = (a_1, a_2, \ldots, a_i, a_{i+1}, \ldots, a_{12}) \cdot (3, 1, 3, 1, \ldots, 3, 1) \bmod 10. \end{gathered}$$

This equality simplifies to either

$$(3a_{i+1} + a_i) \bmod 10 = (3a_i + a_{i+1}) \bmod 10$$

or

$$(a_{i+1} + 3a_i) \bmod 10 = (a_i + 3a_{i+1}) \bmod 10,$$

depending on whether i is even or odd. Both cases reduce to $2(a_{i+1} - a_i) \mod 10 = 0$. It follows that $|a_{i+1} - a_i| = 5$, if $a_{i+1} \neq a_i$.

In 2005, United States companies began to phase in the use of a 13th digit to be in conformance with the 13-digit product identification numbers used in Europe. The weighting vector for 13-digit numbers is $(1, 3, 1, 3, \ldots, 3, 1)$.

Identification numbers printed on bank checks (on the bottom left between the two colons) consist of an eight-digit number $a_1 a_2 \cdots a_8$ and a check digit a_9, so that

$$(a_1, a_2, \ldots, a_9) \cdot (7, 3, 9, 7, 3, 9, 7, 3, 9) \bmod 10 = 0.$$

As is the case for the UPC scheme, this method detects all single-digit errors and all errors involving the transposition of adjacent digits a and b except when $|a - b| = 5$. But it also detects most errors of the form $\cdots abc \cdots \to \cdots cba \cdots$, whereas the UPC method detects no errors of this form.

In Chapter 5, we will examine more sophisticated means of assigning check digits to numbers.

Modular arithmetic is often used to verify the validity of statements about divisibility regarding all positive integers by checking only finitely many cases.

EXAMPLE 7 Consider the statement, "The sum of the cubes of any three consecutive integers is divisible by 9." This statement is equivalent to checking that the equation $(n^3 + (n+1)^3 + (n+2)^3) \bmod 9 = 0$ is true for all integers n. Because of properties of modular arithmetic, to prove this, all we need to do is check the validity of the equation for $n = 0, 1, \ldots, 8$. You do the math. ∎

Modular arithmetic is occasionally used to show that certain equations have no rational number solutions.

EXAMPLE 8 We use mod 4 arithmetic to show that there are no integers x and y such that $x^2 - y^2 = 1002$. To see this, suppose that there are such integers. Then, taking both sides modulo 4, there is an integer solution to $x^2 - y^2 \bmod 4 = 2$. Note that for any integer n, if $n \bmod 4 = 0$ or 2, then $n^2 \bmod 4 = 0$ and if $n \bmod 4 = 1$ or 3 mod 4, then $n^2 \bmod 4 = 1$. But then the only differences of squares of integers modulo 4 are 0, 1, and $-1 = 3$, which gives a contradiction. ∎

At the dawn of the 20th century no one would have thought that strings of 0s and 1s added modulo 2 would provide the underpinning for a revolution in business, industry, technology, and science. The next example is an application of mod 2 arithmetic to circuit design. More applications of mod 2 arithmetic are given in later chapters.

EXAMPLE 9 Logic Gates In electronics a *logic gate* is a device that accepts as inputs two possible states (on or off) and produces one output (on or off). This can be conveniently modeled using 0 and 1 and modulo 2 arithmetic. The AND gate outputs 1 if and only if both inputs are 1; the OR (inclusive or) gate outputs 1 if at least one input is 1; the XOR (exclusive or) outputs 1 if and only if exactly one input is 1; MAJ (majority) takes three inputs and outputs 1 if and only if at least two inputs are 1. These and others can be conveniently modeled as functions using 0 and 1 and modulo 2 arithmetic as follows:

x AND y	xy
x OR y	$x+y+xy$
x XOR y	$x+y$
$\text{MAJ}(x,y,z)$	$xz+xy+yz.$

∎

Complex Numbers

Recall that complex numbers **C** are expressions of the form $a + b\sqrt{-1}$, where a and b are real numbers. The number $\sqrt{-1}$ is defined to have the property $\sqrt{-1}^2 = -1$. It is customary to use i to denote $\sqrt{-1}$. Then, $i^2 = -1$. Complex numbers written in the form $a+bi$ are said to be in *standard* form. In some instances it is convenient to write a complex number $a+bi$ in another form. To do this we represent $a+bi$ as the point $(a,\ b)$ in a plane coordinatized by a horizontal axis called the *real* axis and a vertical i axis called the *imaginary* axis . The distance from the point $a+bi$ to the origin is $r = \sqrt{a^2+b^2}$ and is often denoted by $|a+bi|$ and called the *norm* of $a+bi$. If we draw the line segment from the origin to $a+bi$ and denote the angle formed by the line segment and the positive real axis by θ, we can write $a+bi$ as $r(\cos\theta + i\sin\theta)$.

This form of $a + bi$ is called the *polar* form. An advantage of the polar form is demonstrated in parts 5 and 6 of Theorem 0.4.

Theorem 0.4 Properties of Complex Numbers

1. ***Closure under addition:*** $(a + bi) + (c + di) = (a + c) + (b + d)i$
2. ***Closure under multiplication:*** $(a + bi)(c + di) = (ac) + (ad)i + (bc)i + (bd)i^2 = (ac - bd) + (ad + bc)i$
3. ***Closure under division:*** $(c + di \neq 0)$: $\dfrac{(a + bi)}{(c + di)} = \dfrac{(a + bi)}{(c + di)}\dfrac{(c - di)}{(c - di)} = \dfrac{(ac + bd) + (bc - ad)i}{c^2 + d^2} = \dfrac{(ac + bd)}{c^2 + d^2} + \dfrac{(bc - ad)}{c^2 + d^2}i$
4. ***Complex conjugation:*** $(a + bi)(a - bi) = a^2 + b^2$
5. ***Inverses: For every nonzero complex number*** $a + bi$ ***there is a complex number*** $c + di$ ***such that*** $(a + bi)(c + di) = 1$ ***(i.e.,*** $(a + bi)^{-1}$ ***exists in*** C***).***
6. ***Powers: For every complex number*** $a + bi = r(\cos\theta + i\sin\theta)$ ***and every positive integer*** n***, we have*** $(a+bi)^n = (r(\cos\theta + i\sin\theta))^n = r^n(\cos n\,\theta + i\sin n\,\theta)$.
7. n^{th}***-roots of*** $a + bi$***: For any positive integer*** n ***the*** n ***distinct*** n^{th} ***roots of*** $a + bi = r(\cos\theta + i\sin\theta)$ ***are*** $\sqrt[n]{r}(\cos\frac{\theta+2\pi k}{n} + i\sin\frac{\theta+2\pi k}{n})$ ***for*** $k = 0, 1, \ldots n - 1$.

PROOF Parts 1 and 2 are definitions. Part 4 follows from part 2. Part 6 is proved in Example 15 in the next section of this chapter. Part 7 follows from part 6. ∎

The next two examples illustrate properties of complex numbers.

EXAMPLE 10 $(3 + 5i) + (-5 + 2i) = -2 + 7i$;
$(3 + 5i)(-5 + 2i) = -25 + (-19)i = -25 - 19i$;
$\dfrac{3 + 5i}{-2 + 7i} = \dfrac{3 + 5i}{-2 + 7i}\dfrac{-2 - 7i}{-2 - 7i} = \dfrac{29 - 31i}{53} = \dfrac{29}{53} + \dfrac{-31}{53}i$;
$(3 + 5i)(3 - 5i) = 9 + 25 = 34$;
$(3 + 5i)^{-1} = \dfrac{3}{34} - \dfrac{5}{34}i$. ∎

EXAMPLE 11 $(-1+i)^4 = \left(\sqrt{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\right)^4 =$
$\sqrt{2^4}\left(\cos\frac{4\cdot 3\pi}{4} + i\sin\frac{4\cdot 3\pi}{4}\right) = 4(\cos 3\pi + i\sin 3\pi) = -4.$

The three cube roots of $i = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2}$ are

$$\cos\frac{\pi}{6} + i\sin\frac{\pi}{6} = \frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$\cos\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{2\pi}{3}\right) = -\frac{\sqrt{3}}{2} + \frac{1}{2}i$$
$$\cos\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) + i\sin\left(\frac{\pi}{6} + \frac{4\pi}{3}\right) = -i.$$

Mathematical Induction

There are two forms of proof by mathematical induction that we will use. Both are equivalent to the Well Ordering Principle. The explicit formulation of the method of mathematical induction came in the 16th century. Francisco Maurolico (1494–1575), a teacher of Galileo, used it in 1575 to prove that $1 + 3 + 5 + \cdots + (2n-1) = n^2$, and Blaise Pascal (1623–1662) used it when he presented what we now call Pascal's triangle for the coefficients of the binomial expansion. The term *mathematical induction* was coined by Augustus De Morgan.

Theorem 0.5 First Principle of Mathematical Induction

Let S be a set of integers containing a. Suppose S has the property that whenever some integer $n \geq a$ belongs to S, then the integer $n+1$ also belongs to S. Then, S contains every integer greater than or equal to a.

PROOF The proof is left as an exercise (Exercise 33).

So, to use induction to prove that a statement involving positive integers is true for every positive integer, we must first verify that the statement is true for the integer 1. We then *assume* the statement is true for the integer n and use this assumption to prove that the statement is true for the integer $n+1$.

Our next example uses some facts about plane geometry. Recall that given a straightedge and compass, we can construct a right angle.

EXAMPLE 12 We use induction to prove that given a straightedge, a compass, and a unit length, we can construct a line segment of length $\sqrt{n}$ for every positive integer n. The case when $n = 1$ is given. Now we assume that we can construct a line segment of length $\sqrt{n}$. Then use the straightedge and compass to construct a right triangle with height 1 and base $\sqrt{n}$. The hypotenuse of the triangle has length $\sqrt{n+1}$. So, by induction, we can construct a line segment of length $\sqrt{n}$ for every positive integer n. ∎

EXAMPLE 13 **DeMoivre's Theorem** We use induction to prove that for every positive integer n and every real number $\theta, (\cos\theta + i\sin\theta)^n = \cos n\theta + i\sin n\theta$, where i is the complex number $\sqrt{-1}$. Obviously, the statement is true for $n = 1$. Now assume it is true for n. We must prove that $(\cos\theta + i\sin\theta)^{n+1} = \cos(n+1)\theta + i\sin(n+1)\theta$. Observe that

$$\begin{aligned}(\cos\theta + i\sin\theta)^{n+1} &= (\cos\theta + i\sin\theta)^n(\cos\theta + i\sin\theta)\\ &= (\cos n\theta + i\sin n\theta)(\cos\theta + i\sin\theta)\\ &= \cos n\theta\cos\theta + i(\sin n\theta\cos\theta\\ &\quad + \sin\theta\cos n\theta) - \sin n\theta\sin\theta.\end{aligned}$$

Now, using trigonometric identities for $\cos(\alpha+\beta)$ and $\sin(\alpha+\beta)$, we see that this last term is $\cos(n+1)\theta + i\sin(n+1)\theta$. So, by induction, the statement is true for all positive integers. ∎

In many instances, the assumption that a statement is true for an integer n does not readily lend itself to a proof that the statement is true for the integer $n+1$. In such cases, the following equivalent form of induction may be more convenient. Some authors call this formulation the *strong form* of induction.

Theorem 0.6 Second Principle of Mathematical Induction

Let S be a set of integers containing a. Suppose S has the property that n belongs to S whenever every integer less than n and greater than or equal to a belongs to S. Then, S contains every integer greater than or equal to a.

PROOF The proof is left to the reader. ∎

To use this form of induction, we first show that the statement is true for the integer a. We then *assume* that the statement is true for *all* integers that are greater than or equal to a and less than n, and use this assumption to prove that the statement is true for n.

▮ **EXAMPLE 14** We will use the Second Principle of Mathematical Induction with $a = 2$ to prove the existence portion of the Fundamental Theorem of Arithmetic. Let S be the set of integers greater than 1 that are primes or products of primes. Clearly, $2 \in S$. Now we assume that for some integer n, S contains all integers k with $2 \leq k < n$. We must show that $n \in S$. If n is a prime, then $n \in S$ by definition. If n is not a prime, then n can be written in the form ab, where $1 < a < n$ and $1 < b < n$.
Since we are assuming that both a and b belong to S, we know that each of them is a prime or a product of primes. Thus, n is also a product of primes. This completes the proof. ▮

Notice that it is more natural to prove the Fundamental Theorem of Arithmetic with the Second Principle of Mathematical Induction than with the First Principle. Knowing that a particular integer factors as a product of primes does not tell you anything about factoring the next larger integer. (Does knowing that 5280 is a product of primes help you to factor 5281 as a product of primes?)

The following problem appeared in the "Brain Boggler" section of the January 1988 issue of the science magazine *Discovery.*[2] Problems like this one are often called chicken McNugget problems, postage stamp problems, or Frobenius coin problems. Originally, McDonald's sold its chicken nuggets in packs of 9 and 20. The largest number of nuggets that could not have been bought with these packs is 151.

▮ **EXAMPLE 15** The Quakertown Poker Club plays with blue chips worth \$5.00 and red chips worth \$8.00. What is the largest bet that cannot be made?

To gain insight into this problem, we try various combinations of blue and red chips and obtain 5, 8, 10, 13, 15, 16, 18, 20, 21, 23, 24, 25, 26, 28, 29, 30, 31, 32, 33, 34, 35, 36, 37, 38, 39, 40. It appears that the answer is 27. But how can we be sure? Well, we need only prove that every integer greater than 27 can be written

[2]"Brain Boggler" by Maxwell Carver. Copyright © 1988 by *Discover Magazine.* Used by permission.

in the form $a \cdot 5 + b \cdot 8$, where a and b are nonnegative integers. This will solve the problem, since a represents the number of blue chips and b the number of red chips needed to make a bet of $a \cdot 5 + b \cdot 8$. For the purpose of contrast, we will give two proofs—one using the First Principle of Mathematical Induction and one using the Second Principle.

Let S be the set of all integers greater than or equal to 28 of the form $a \cdot 5 + b \cdot 8$, where a and b are nonnegative. Obviously, $28 \in S$. Now assume that some integer $n \in S$, say, $n = a \cdot 5 + b \cdot 8$. We must show that $n + 1 \in S$. First, note that since $n \geq 28$, we cannot have both a and b less than 3. If $a \geq 3$, then

$$\begin{aligned} n + 1 &= (a \cdot 5 + b \cdot 8) + (-3 \cdot 5 + 2 \cdot 8) \\ &= (a - 3) \cdot 5 + (b + 2) \cdot 8. \end{aligned}$$

(Regarding chips, this last equation says that we may increase a bet from n to $n + 1$ by removing three blue chips from the pot and adding two red chips.) If $b \geq 3$, then

$$\begin{aligned} n + 1 &= (a \cdot 5 + b \cdot 8) + (5 \cdot 5 - 3 \cdot 8) \\ &= (a + 5) \cdot 5 + (b - 3) \cdot 8. \end{aligned}$$

(The bet can be increased by 1 by removing three red chips and adding five blue chips.) This completes the proof.

To prove the same statement by the Second Principle, we note that each of the integers 28, 29, 30, 31, and 32 is in S. Now assume that for some integer $n > 32$, S contains all integers k with $28 \leq k < n$. We must show that $n \in S$. Since $n - 5 \in S$, there are nonnegative integers a and b such that $n - 5 = a \cdot 5 + b \cdot 8$. But then $n = (a + 1) \cdot 5 + b \cdot 8$. Thus n is in S. ∎

Equivalence Relations

In mathematics, things that are considered different in one context may be viewed as equivalent in another context. We have already seen one such example. Indeed, the sums $2 + 1$ and $4 + 4$ are certainly different in ordinary arithmetic, but are the same under modulo 5 arithmetic. Congruent triangles that are situated differently in the plane are not the same, but they are often considered to be the same in plane geometry. In physics, vectors of the same magnitude and direction can produce different effects—a

10-pound weight placed 2 feet from a fulcrum produces a different effect than a 10-pound weight placed 1 foot from a fulcrum. But in linear algebra, vectors of the same magnitude and direction are considered to be the same. What is needed to make these distinctions precise is an appropriate generalization of the notion of equality; that is, we need a formal mechanism for specifying whether or not two quantities are the same in a given setting. This mechanism is an equivalence relation.

Definition Equivalence Relation

An *equivalence relation* on a set S is a set R of ordered pairs of elements of S such that

1. **$(a, a) \in R$ for all $a \in S$ (reflexive property).**
2. **$(a, b) \in R$ implies $(b, a) \in R$ (symmetric property).**
3. **$(a, b) \in R$ and $(b, c) \in R$ imply $(a, c) \in R$ (transitive property).**

When R is an equivalence relation on a set S, it is customary to write aRb instead of $(a, b) \in R$. Also, since an equivalence relation is just a generalization of equality, a suggestive symbol such as $\approx, \equiv$, or $\sim$ is usually used to denote the relation. Using this notation, the three conditions for an equivalence relation become $a \sim a; a \sim b$ implies $b \sim a$; and $a \sim b$ and $b \sim c$ imply $a \sim c$. If $\sim$ is an equivalence relation on a set S and $a \in S$, then the set $[a] = \{x \in S \mid x \sim a\}$ is called the *equivalence class of S containing a.*

EXAMPLE 16 Let S be the set of all triangles in a plane. If $a, b \in S$, define $a \sim b$ if a and b are similar—that is, if a and b have corresponding angles that are the same. Then $\sim$ is an equivalence relation on S. ∎

EXAMPLE 17 Let S be the set of all polynomials with real coefficients. If $f, g \in S$, define $f \sim g$ if $f' = g'$, where f' is the derivative of f. Then $\sim$ is an equivalence relation on S. Since two polynomials with equal derivatives differ by a constant, we see that for any f in S, $[f] = \{f + c \mid c \text{ is real}\}$. ∎

EXAMPLE 18 Let S be the set of integers and let n be a positive integer. If $a, b \in S$, define $a \equiv b$ if $a \bmod n = b \bmod n$ (i.e., if $a - b$ is divisible by n). Then $\equiv$ is an equivalence relation on S and

$[a] = \{a+kn \mid k \in S\}$. Since this particular relation is important in abstract algebra, we will take the trouble to verify that it is indeed an equivalence relation. Certainly, $a - a$ is divisible by n, so that $a \equiv a$ for all a in S. Next, assume that $a \equiv b$, say, $a - b = rn$. Then, $b - a = (-r)n$, and therefore $b \equiv a$. Finally, assume that $a \equiv b$ and $b \equiv c$, say, $a - b = rn$ and $b - c = sn$. Then, we have $a - c = (a - b) + (b - c) = rn + sn = (r + s)n$, so that $a \equiv c$. ∎

EXAMPLE 19 Let $\equiv$ be as in Example 18 and let $n = 7$. Then we have $16 \equiv 2; 9 \equiv -5$; and $24 \equiv 3$. Also, $[1] = \{\ldots, -20, -13, -6, 1, 8, 15, \ldots\}$ and $[4] = \{\ldots, -17, -10, -3, 4, 11, 18, \ldots\}$. ∎

EXAMPLE 20 Let $S = \{(a, b) \mid a, b \text{ are integers}, b \neq 0\}$. If $(a, b), (c, d) \in S$, define $(a, b) \approx (c, d)$ if $ad = bc$. Then $\approx$ is an equivalence relation on S. [The motivation for this example comes from fractions. In fact, the pairs (a, b) and (c, d) are equivalent if the fractions a/b and c/d are equal.]
To verify that $\approx$ is an equivalence relation on S, note that $(a, b) \approx (a, b)$ requires that $ab = ba$, which is true. Next, we assume that $(a, b) \approx (c, d)$, so that $ad = bc$. We have $(c, d) \approx (a, b)$ provided that $cb = da$, which is true from commutativity of multiplication. Finally, we assume that $(a, b) \approx (c, d)$ and $(c, d) \approx (e, f)$ and prove that $(a, b) \approx (e, f)$. This amounts to using $ad = bc$ and $cf = de$ to show that $af = be$. Multiplying both sides of $ad = bc$ by f and replacing cf by de, we obtain $adf = bcf = bde$. Since $d \neq 0$, we can cancel d from the first and last terms. ∎

Definition Partition
A *partition* of a set S is a collection of nonempty disjoint subsets of S whose union is S.

Figure 0.3 illustrates a partition of a set into four subsets.

EXAMPLE 21 The sets $\{0\}, \{1, 2, 3, \ldots\}$, and $\{\ldots, -3, -2, -1\}$ constitute a partition of the set of integers. ∎

EXAMPLE 22 The set of nonnegative integers and the set of nonpositive integers do not partition the integers, since both contain 0. ∎

The next theorem reveals that equivalence relations and partitions are intimately intertwined.

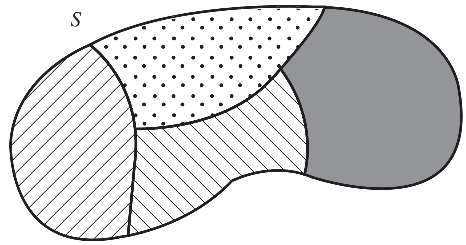


Figure 0.3 Partition of S into four subsets.

Theorem 0.7 Equivalence Classes Partition

The equivalence classes of an equivalence relation on a set S constitute a partition of S. Conversely, for any partition P of S, there is an equivalence relation on S whose equivalence classes are the elements of P.

PROOF Let $\sim$ be an equivalence relation on a set S. For any $a \in S$, the reflexive property shows that $a \in [a]$. So, $[a]$ is nonempty and the union of all equivalence classes is S. Now, suppose that $[a]$ and $[b]$ are distinct equivalence classes. We must show that $[a] \cap [b] = \emptyset$. On the contrary, assume $c \in [a] \cap [b]$. We will show that $[a] \subseteq [b]$. To this end, let $x \in [a]$. We then have $c \sim a, c \sim b$, and $x \sim a$. By the symmetric property, we also have $a \sim c$. Thus, by transitivity, $x \sim c$, and by transitivity again, $x \sim b$. This proves $[a] \subseteq [b]$. Analogously, $[b] \subseteq [a]$. Thus, $[a] = [b]$, in contradiction to our assumption that $[a]$ and $[b]$ are distinct equivalence classes.

To prove the converse, let P be a collection of nonempty disjoint subsets of S whose union is S. Define $a \sim b$ if a and b belong to the same subset in the collection. We leave it to the reader to show that $\sim$ is an equivalence relation on S (Exercise 53). ∎

Functions (Mappings)

Although the concept of a function plays a central role in nearly every branch of mathematics, the terminology and notation associated with functions vary quite a bit. In this section, we establish ours.

Definitions Function (Mapping)

A *function* (or *mapping*) ϕ *from a set* A *to a set* B is a rule that assigns to each element a of A exactly one element b of B. The set A is called the *domain of* ϕ, and B is called the *range of* ϕ. If ϕ assigns b to a, then b is called the *image of* a *under* ϕ. The subset of B comprising all the images of elements of A is called the *image of* A *under* ϕ.

We use the shorthand $\phi : A \to B$ to mean that ϕ is a mapping from A to B. We will write $\phi(a) = b$ or $\phi : a \to b$ to indicate that ϕ carries a to b.

There are often different ways to denote the same element of a set. In defining a function in such cases one must verify that the function values assigned to the elements depend not on the way the elements are expressed but on only the elements themselves. For example, the correspondence ϕ from the rational numbers to the integers given by $\phi(a/b) = a+b$ does not define a function since $1/2 = 2/4$, but $\phi(1/2) \neq \phi(2/4)$. To verify that a correspondence is a function, you assume that $x_1 = x_2$ and prove that $\phi(x_1) = \phi(x_2)$.

Definition Composition of Functions

Let $\phi : A \to B$ and $\psi : B \to C$. The *composition* $\psi\phi$ is the mapping from A to C defined by $(\psi\phi)(a) = \psi(\phi(a))$ for all a in A.

In calculus courses, the composition of f with g is written $(f \circ g)(x)$ and is defined by $(f \circ g)(x) = f(g(x))$. When we compose functions, we omit the "circle."

▮ **EXAMPLE 23** Let $f(x) = 2x + 3$ and $g(x) = x^2 + 1$. Then $(fg)(5) = f(g(5)) = f(26) = 55$; $(gf)(5) = g(f(5)) = g(13) = 170$. More generally, $(fg)(x) = f(g(x)) = f(x^2 + 1) = 2(x^2 + 1) + 3 = 2x^2 + 5$ and $(gf)(x) = g(f(x)) = g(2x + 3) = (2x + 3)^2 + 1 = 4x^2 + 12x + 9 + 1 = 4x^2 + 12x + 10$. Note that the function fg is not the same as the function gf. ▮

There are several kinds of functions that occur often enough to be given names.

Definition One-to-One Function

A function ϕ from a set A is called *one-to-one* if for every $a_1, a_2 \in A$, $\phi(a_1) = \phi(a_2)$ implies $a_1 = a_2$.

The term *one-to-one* is suggestive, since the definition ensures that one element of B can be the image of only one element of

A. Alternatively, ϕ is one-to-one if $a_1 \neq a_2$ implies $\phi(a_1) \neq \phi(a_2)$. That is, different elements of A map to different elements of B. See Figure 0.4.

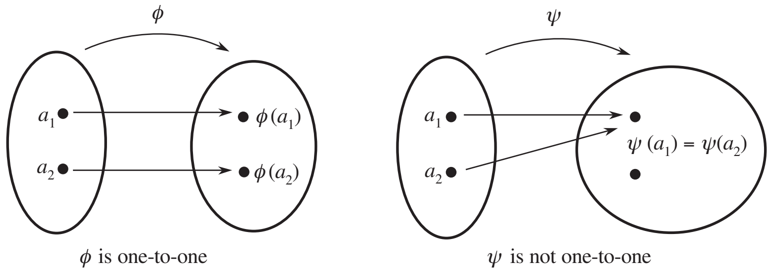


Figure 0.4 Mappings.

Definition Function from A onto B

A function ϕ from a set A to a set B is said to be *onto* B if each element of B is the image of at least one element of A. In symbols, $\phi : A \to B$ is onto if for each b in B there is at least one a in A such that $\phi(a) = b$. See Figure 0.5.

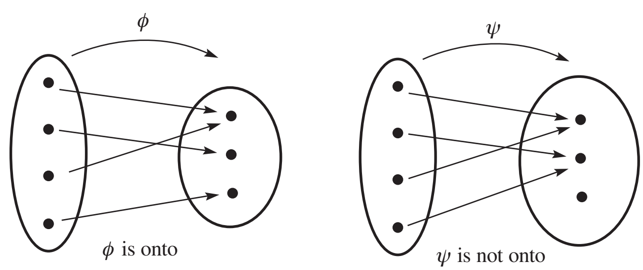


Figure 0.5 Functions.

The next theorem summarizes the facts about functions we will need.

Theorem 0.8 Properties of Functions

Given functions $\alpha : A \to B, \beta : B \to C$, and $\gamma : C \to D$, then

1. $\gamma(\beta\alpha) = (\gamma\beta)\alpha$ *(associativity).*

2. ***If α and β are one-to-one, then $\beta\alpha$ is one-to-one.***

3. ***If α and β are onto, then $\beta\alpha$ is onto.***

4. ***If α is one-to-one and onto, then there is a function α^{-1} from B onto A such that $(\alpha^{-1}\alpha)(a) = a$ for all a in A and $(\alpha\alpha^{-1})(b) = b$ for all b in B.***

PROOF We prove only part 1. The remaining parts are left as exercises (Exercise 55). Let $a \in A$. Then $(\gamma(\beta\alpha))(a) = \gamma((\beta\alpha)(a)) = \gamma(\beta(\alpha(a)))$. On the other hand, $((\gamma\beta)\alpha)(a) = (\gamma\beta)(\alpha(a)) = \gamma(\beta(\alpha(a)))$. So, $\gamma(\beta\alpha) = (\gamma\beta)\alpha$. ∎

It is useful to note that if α is one-to-one and onto, the function α^{-1} described in part 4 of Theorem 0.8 has the property that if $\alpha(s) = t$, then $\alpha^{-1}(t) = s$. That is, the image of t under α^{-1} is the unique element s that maps to t under α. In effect, α^{-1} "undoes" what α does.

EXAMPLE 24 Let $\mathbf{Z}$ denote the set of integers, $\mathbf{R}$ the set of real numbers, and $\mathbf{N}$ the set of nonnegative integers. The following table illustrates the properties of one-to-one and onto.

Domain	Range	Rule	One-to-One	Onto
$\mathbf{Z}$	$\mathbf{Z}$	$x \to x^3$	Yes	No
$\mathbf{R}$	$\mathbf{R}$	$x \to x^3$	Yes	Yes
$\mathbf{Z}$	$\mathbf{N}$	$x \to \lvert x \rvert$	No	Yes
$\mathbf{Z}$	$\mathbf{Z}$	$x \to x^2$	No	No

To verify that $x \to x^3$ is one-to-one in the first two cases, notice that if $x^3 = y^3$, we may take the cube roots of both sides of the equation to obtain $x = y$. Clearly, the mapping from $\mathbf{Z}$ to $\mathbf{Z}$ given by $x \to x^3$ is not onto, since 2 is the cube of no integer. However, $x \to x^3$ defines an onto function from $\mathbf{R}$ to $\mathbf{R}$, since every real number is the cube of its cube root (i.e., $\sqrt[3]{b} \to b$). The remaining verifications are left to the reader. ∎

Exercises

I was interviewed in the Israeli Radio for five minutes and I said that more than 2000 years ago, Euclid proved that there are infinitely many primes. Immediately the host interrupted me and asked: "Are there still infinitely many primes?"

Noga Alon

1. For $n = 5, 8, 12, 20$, and 25, find all positive integers less than n and relatively prime to n.

2. Determine

a. gcd(2,10) lcm(2,10)
b. gcd(20,8) lcm(20,8)
c. gcd(12,40) lcm(12,40)
d. gcd(21,50) lcm(21,50)
e. $\gcd(p^2q^2, pq^3)$ $\operatorname{lcm}(p^2q^2, pq^3)$ where p and q are distinct primes

3. Determine 51 mod 13, 342 mod 85, 62 mod 15, 10 mod 15, $(82 \cdot 73)$ mod 7, $(51+68)$ mod 7, $(35 \cdot 24)$ mod 11, and $(47+68)$ mod 11.

4. Find integers s and t such that $1 = 7 \cdot s + 11 \cdot t$. Show that s and t are not unique.

5. Prove that every integer that is a common multiple of every member of a finite set of integers is a multiple of the least common multiple of those integers.

6. Prove that for any three consecutive integers n, $n+1$, $n+2$ one must be divisible by 3.

7. Show that if a and b are positive integers, then $ab = \operatorname{lcm}(a, b) \cdot \gcd(a, b)$.

8. Suppose a and b are integers that divide the integer c. If a and b are relatively prime, show that ab divides c. Show, by example, that if a and b are not relatively prime, then ab need not divide c.

9. If a and b are integers and n is a positive integer, prove that $a \bmod n = b \bmod n$ if and only if n divides $a - b$.

10. Let $d = \gcd(a, b)$. If $a = da'$ and $b = db'$, show that $\gcd(a', b') = 1$.

11. Let n be a fixed positive integer greater than 1. If $a \bmod n = a'$ and $b \bmod n = b'$, prove that $(a+b) \bmod n = (a'+b') \bmod n$ and $(ab) \bmod n = (a'b') \bmod n$. (This exercise is referred to in Chapters 6, 8, 10, and 15.)

12. Let a and b be positive integers and let $d = \gcd(a, b)$ and $m = \operatorname{lcm}(a, b)$. If t divides both a and b, prove that t divides d. If s is a multiple of both a and b, prove that s is a multiple of m.

13. Let n and a be positive integers and let $d = \gcd(a, n)$. Show that the equation $ax \bmod n = 1$ has a solution if and only if $d = 1$. (This exercise is referred to in Chapter 2.)

14. Show that $5n + 3$ and $7n + 4$ are relatively prime for all n.

15. Suppose that m and n are relatively prime and r is any integer. Show that there are integers x and y such that $mx + ny = r$.

16. Let p, q, and r be primes other than 3. Show that 3 divides $p^2 + q^2 + r^2$.

17. Prove that every prime greater than 3 can be written in the form $6n + 1$ or $6n + 5$.

18. Determine $7^{1000} \bmod 6$ and $6^{1001} \bmod 7$.

19. Let a, b, s, and t be integers. If $a \bmod st = b \bmod st$, show that $a \bmod s = b \bmod s$ and $a \bmod t = b \bmod t$. What condition on s and t is needed to make the converse true? (This exercise is referred to in Chapter 8.)

20. If n is an integer greater than 1 and $(n - 1)! = 1 \bmod n$, prove that n is prime.

21. Show that $\gcd(a, bc) = 1$ if and only if $\gcd(a, b) = 1$ and $\gcd(a, c) = 1$. (This exercise is referred to in Chapter 8.)

22. Let $p_1, p_2, \ldots, p_n$ be primes. Show that $p_1 p_2 \cdots p_n + 1$ is divisible by none of these primes.

23. Prove that there are infinitely many primes. (*Hint:* Use Exercise 22.)

24. For any complex numbers z_1 and z_2, prove that $|z_1 z_2| = |z_1||z_2|$.

25. Give an "if and only if" statement that describes when the logic gate x NAND y modeled by $1 + xy$ is 1. Give an "if and only if" statement that describes when the logic gate x XNOR y modeled by $1 + x + y$ is 1.

26. For inputs of 0 and 1 and mod 2 arithmetic describe the output of the formula $z + xy + xz$ in the form "If $x \ldots$, else $\ldots$."

27. For every positive integer n, prove that a set with exactly n elements has exactly 2^n subsets (counting the empty set and the entire set).

28. Prove that $2^n 3^{2n} - 1$ is always divisible by 17.

29. Prove that there is some positive integer n such that n, $n+1, n+2, \ldots, n+200$ are all composite.

30. (Generalized Euclid's Lemma) If p is a prime and p divides $a_1 a_2 \ldots a_n$, prove that p divides a_i for some i.

31. Use the Generalized Euclid's Lemma (see Exercise 30) to establish the uniqueness portion of the Fundamental Theorem of Arithmetic.

32. What is the largest bet that cannot be made with chips worth \$7.00 and \$9.00? Verify that your answer is correct with both forms of induction.

33. Prove that the First Principle of Mathematical Induction is a consequence of the Well Ordering Principle.

34. The Fibonacci numbers are $1, 1, 2, 3, 5, 8, 13, 21, 34, \ldots$. In general, the Fibonacci numbers are defined by $f_1 = 1, f_2 = 1$, and for $n \geq 3$, $f_n = f_{n-1} + f_{n-2}$. Prove that the nth Fibonacci number f_n satisfies $f_n < 2^n$.

35. Prove by induction on n that for all positive integers $n, n^3 + (n+1)^3 + (n+2)^3$ is a multiple of 9.

36. Suppose that there is a statement involving a positive integer parameter n and you have an argument that shows that whenever the statement is true for a particular n it is also true for $n+2$. What remains to be done to prove the statement is true for every positive integer? Describe a situation in which this strategy would be applicable.

37. In the cut "As" from *Songs in the Key of Life*, Stevie Wonder mentions the equation $8 \times 8 \times 8 = 4$. Find all integers n for which this statement is true, modulo n.

38. Prove that for every integer $n, n^3 \bmod 6 = n \bmod 6$.

39. If it is 2:00 A.M. now, what time will it be 3736 hours from now?

40. Determine the check digit for a money order with identification number 7234541780.

41. Suppose that in one of the noncheck positions of a money order number, the digit 0 is substituted for the digit 9 or vice versa. Prove that this error will not be detected by the check digit. Prove that all other errors involving a single position are detected.

42. Suppose that a money order identification number and check digit of 21720421168 is erroneously copied as 27750421168. Will the check digit detect the error?

43. A transposition error involving distinct adjacent digits is one of the form $\ldots ab \ldots \to \ldots ba \ldots$ with $a \neq b$. Prove that the money order check-digit scheme will not detect such errors unless the check digit itself is transposed.

44. Explain why the check digit for a money order for the number N is the repeated decimal digit in the real number $N \div 9$.

45. The 10-digit International Standard Book Number (ISBN-10) $a_1a_2a_3a_4a_5a_6a_7a_8a_9a_{10}$ has the property $(a_1, a_2, \ldots, a_{10}) \cdot (10, 9, 8, 7, 6, 5, 4, 3, 2, 1) \bmod 11 = 0$. The digit a_{10} is the check digit. When a_{10} is required to be 10 to make the dot product 0, the character X is used as the check digit. Verify the check digit for the ISBN-10 assigned to this book.

46. Suppose that an ISBN-10 has a smudged entry where the question mark appears in the number 0-716?-2841-9. Determine the missing digit.

47. Suppose three consecutive digits abc of an ISBN-10 are scrambled as bca. Which such errors will go undetected?

48. Let S be the set of real numbers. If $a, b \in S$, define $a \sim b$ if $a - b$ is an integer. Show that $\sim$ is an equivalence relation on S. Describe the equivalence classes of S.

49. Let S be the set of integers. If $a, b \in S$, define aRb if $ab \geq 0$. Is R an equivalence relation on S?

50. Let S be the set of integers. If $a, b \in S$, define aRb if $a + b$ is even. Prove that R is an equivalence relation and determine the equivalence classes of S.

51. Complete the proof of Theorem 0.7 by showing that $\sim$ is an equivalence relation on S.

52. Prove that 3, 5, and 7 are the only three consecutive odd integers that are prime.

Computer Exercises

Computer exercises for this chapter are available at the website:

http://www.d.umn.edu/~jgallian

1 Introduction to Groups

And symmetry is a powerful guiding principle that has been used in creating these models [for quantum physics]. The more symmetrical a model is, the easier it is to analyze.

Edward Frenkel, *Love and Math*

Symmetry is a vast subject, significant in art and nature. Mathematics lies at its root, and it would be hard to find a better one on which to demonstrate the working of the mathematical intellect.

Hermann Weyl, *Symmetry*

Symmetries of a Square

Suppose we remove a square region from a plane, move it in some way, then put the square back into the space it originally occupied. Our goal in this chapter is to describe all possible ways in which this can be done. More specifically, we want to describe the possible relationships between the starting position of the square and its final position in terms of motions. However, we are interested in the net effect of a motion, rather than in the motion itself. Thus, for example, we consider a $90°$ rotation and a $450°$ rotation as equal, since they have the same net effect on every point. With this simplifying convention, it is an easy matter to achieve our goal.

To begin, we can think of the square region as being transparent (glass, say), with the corners marked on one side with the colors blue, white, pink, and green. This makes it easy to distinguish between motions that have different effects. With this marking scheme, we are now in a position to describe, in simple fashion, all possible ways in which a square object can be repositioned. See Figure 1.1. We now claim that any motion—no matter how complicated—is equivalent to one of these eight. To verify this claim, observe that the final position of the square is completely determined by the location and orientation (i.e., face up or face

R_0 = Rotation of 0° (no change in position)

R_{90} = Rotation of 90° (counterclockwise)

R_{180} = Rotation of 180°

R_{270} = Rotation of 270°

H = Flip about a horizontal axis

V = Flip about a vertical axis

D = Flip about the main diagonal

D' = Flip about the other diagonal

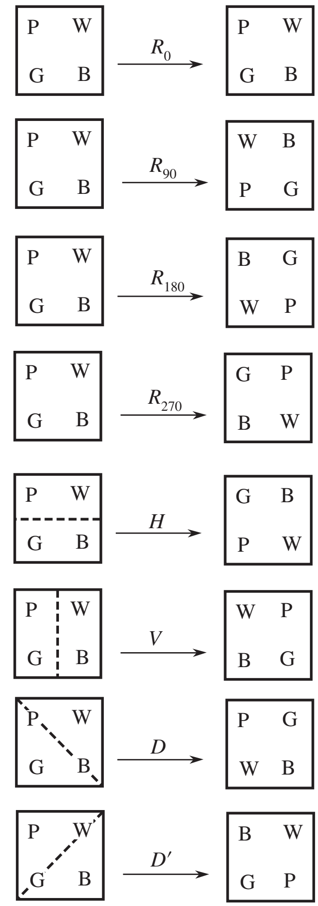
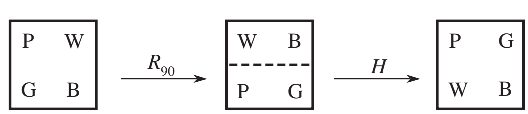


Figure 1.1 Symmetries of a square.

down) of any particular corner. But, clearly, there are only four locations and two orientations for a given corner, so there are exactly eight distinct final positions for the corner.

Let's investigate some consequences of the fact that every motion is equal to one of the eight listed in Figure 1.1. Suppose a square is repositioned by a rotation of 90° followed by a flip about the horizontal axis of symmetry.



Thus, we see that this pair of motions—taken together—is equal to the single motion D. This observation suggests that we can compose two motions to obtain a single motion. And indeed we can, since the eight motions may be viewed as functions from the square region to itself, and as such we can combine them using function composition.

With this in mind, we write $H\ R_{90} = D$ because in lower level math courses function composition $f \circ g$ means "g followed by f." The eight motions R_0, R_{90}, R_{180}, R_{270}, H, V, D, and D', together with the operation composition, form a mathematical system called the *dihedral group of order 8* (the order of a group is the number of elements it contains). It is denoted by D_4. Rather than introduce the formal definition of a group here, let's look at some properties of groups by way of the example D_4.

To facilitate future computations, we construct an *operation table, the Cayley table* (so named in honor of the prolific English mathematician Arthur Cayley, who first introduced them in 1854) for D_4 below. The circled entry represents the fact that $D = HR_{90}$. (In general, ab denotes the entry at the intersection of the row with a at the left and the column with b at the top.)

	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_0	R_0	R_{90}	R_{180}	R_{270}	H	V	D	D'
R_{90}	R_{90}	R_{180}	R_{270}	R_0	D'	D	H	V
R_{180}	R_{180}	R_{270}	R_0	R_{90}	V	H	D'	D
R_{270}	R_{270}	R_0	R_{90}	R_{180}	D	D'	V	H
H	H	D	V	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	D'	H	D	R_{180}	R_0	R_{270}	R_{90}
D	D	V	D'	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	H	D	V	R_{90}	R_{270}	R_{180}	R_0

Notice how orderly this table looks! This is no accident. Perhaps the most important feature of this table is that it has been completely filled in without introducing any new motions. Of course, this is because, as we have already pointed out, any sequence of motions turns out to be the same as one of these eight. Algebraically, this says that if A and B are in D_4, then so is AB. This property is called *closure*, and it is one of the requirements for a mathematical system to be a group. Next, notice that if A is any element of D_4, then $AR_0 = R_0A = A$. Thus, combining any element A on either side with R_0 yields A back again. An element R_0 with this property is called an *identity*, and every group must have one. Moreover, we see that for each element A in D_4, there is exactly one element B in D_4 such that $AB = BA = R_0$. In this case, B is said to be the *inverse* of A and vice versa. For example, R_{90} and R_{270} are inverses of each other, and H is its own inverse. The term *inverse* is a descriptive one, for if A and B are inverses of each other, then B "undoes" whatever A "does," in the sense that A and B taken together in either order produce R_0, representing no change. Another striking feature of the table is that every element of D_4 appears exactly once in each row and column. This feature is something that all groups must have, and, indeed, it is quite useful to keep this fact in mind when constructing the table in the first place.

Another property of D_4 deserves special comment. Observe that $HD \neq DH$ but $R_{90}R_{180} = R_{180}R_{90}$. Thus, in a group, ab may or may not be the same as ba. If it happens that $ab = ba$ for *all* choices of group elements a and b, we say the group is *commutative* or—better yet—*Abelian* (in honor of the great Norwegian mathematician Niels Abel). Otherwise, we say the group is *non-Abelian*.

Thus far, we have illustrated, by way of D_4, three of the four conditions that define a group—namely, closure, existence of an identity, and existence of inverses. The remaining condition required for a group is *associativity*; that is, $(ab)c = a(bc)$ for all a, b, c in the set. To be sure that D_4 is indeed a group, we should check this equation for each of the $8^3 = 512$ possible choices of a, b, and c in D_4. In practice, however, this is rarely done! Here, for example, we simply observe that the eight motions are functions and the operation is function composition. Then, since function composition is associative, we do not have to check the equations.

The Dihedral Groups

The analysis carried out above for a square can similarly be done for an equilateral triangle or regular pentagon or, indeed, any regular n-gon ($n \geq 3$). The corresponding group is denoted by D_n and is called the *dihedral group of order 2n.*

The dihedral groups arise frequently in art and nature. Many of the decorative designs used on floor coverings, pottery, and buildings have one of the dihedral groups as a group of symmetry. Corporation logos are rich sources of dihedral symmetry. Chrysler's logo has D_5 as a symmetry group, and that of Mercedes-Benz has D_3. The ubiquitous five-pointed star has symmetry group D_5.

The phylum Echinodermata contains many sea animals (such as starfish, sea cucumbers, feather stars, and sand dollars) that exhibit patterns with D_5 symmetry. Snowflakes have D_6 symmetry (see Exercise 19).

Chemists classify molecules according to their symmetry. Moreover, symmetry considerations are applied in orbital calculations, in determining energy levels of atoms and molecules, and in the study of molecular vibrations. The symmetry group of a pyramidal molecule such as ammonia (NH_3), depicted in Figure 1.2, is D_3.

Mineralogists determine the internal structures of crystals (i.e., rigid bodies in which the particles are arranged in three-dimensional discrete patterns—table salt and table sugar are two examples) by studying two-dimensional x-ray projections of the atomic makeup of the crystals. The symmetry present in the projections reveals the internal symmetry of the crystals themselves. Commonly occurring symmetry patterns are D_4 and D_6 (see Figure 1.3). Interestingly, it is mathematically impossible for a crystal to possess a D_n repeating symmetry pattern with $n = 5$ or $n > 6$.

The dihedral group of order $2n$ is often called the *group of symmetries of a regular n-gon.* A *plane symmetry* of a figure F in a plane is a function from the plane to itself that carries F onto F and preserves distances; that is, for any points p and q in the plane, the distance from the image of p to the image of q is the same as the distance from p to q. (The term *symmetry* is from the Greek word *symmetros*, meaning "of like measure.") The

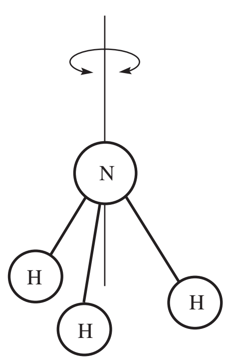


Figure 1.2 A pyramidal molecule with symmetry group D_3.

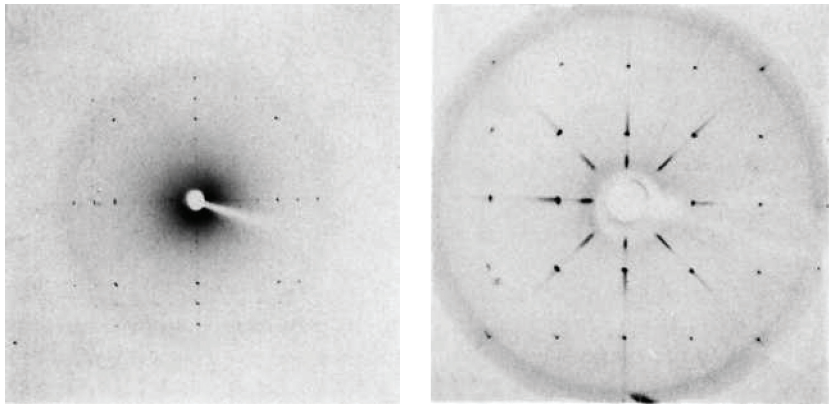


Figure 1.3 X-ray diffraction photos revealing D_4 symmetry patterns in crystals.

symmetry group of a plane figure is the set of all symmetries of the figure. Symmetries in three dimensions are defined analogously. Obviously, a rotation of a plane about a point in the plane is a symmetry of the plane, and a rotation about a line in three dimensions is a symmetry in three-dimensional space. Similarly, any translation of a plane or of three-dimensional space is a symmetry. A *reflection across a line* L is that function which leaves every point of L fixed and takes any point Q, not on L, to the point Q' so that L is the perpendicular bisector of the line segment joining Q and Q' (see Figure 1.4). Figure 1.4 illustrates the characteristic that distinguishes a reflection from a rotation. The reflected image of a clockwise spiral is a counterclockwise spiral and vice versa. In particular, rotations preserve orientation and reflections reverse orientation. We will use this fact often. Also note that reflections reverse handedness, which is an important fact in chemistry.

A reflection across a plane in three dimensions is defined analogously. Notice that the restriction of a 180° rotation about a line L in three dimensions to a plane containing L is a reflection across L in the plane. Thus, in the dihedral groups, the motions that we described as flips about axes of symmetry in three dimensions (e.g., H, V, D, D') are reflections across lines in two dimensions. Just as a reflection across a line is a plane symmetry that cannot be achieved by a physical motion of the plane in two dimensions, a reflection across a plane is a three-dimensional symmetry that cannot be achieved by a physical motion of three-dimensional space. A cup, for instance, has reflective symmetry across the plane

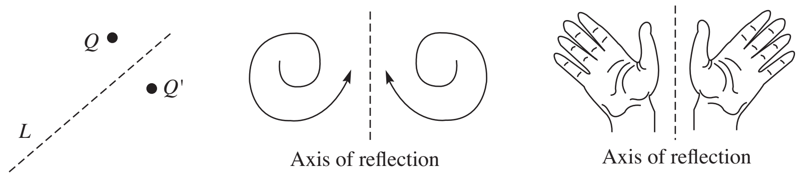


Figure 1.4 Reflected images.

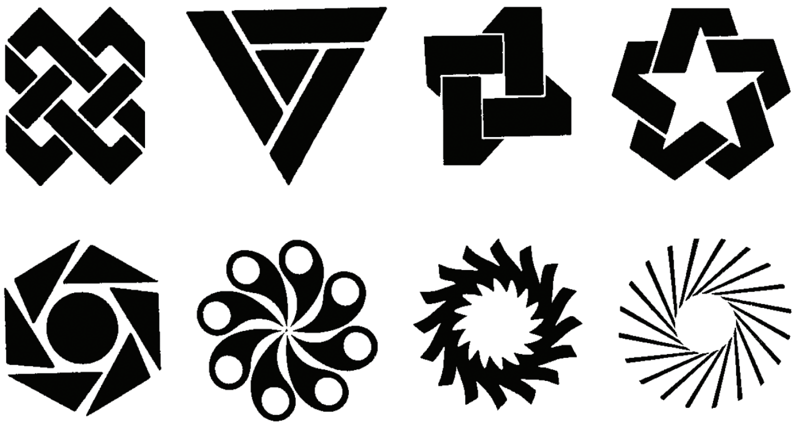


Figure 1.5 Logos with cyclic rotation symmetry groups.

bisecting the cup, but this symmetry cannot be duplicated with a physical motion in three dimensions.

Many objects and figures have rotational symmetry but not reflective symmetry. A symmetry group consisting of the rotational symmetries of $0°$, $360°/n$, $2(360°)/n, \ldots,$ $(n-1)360°/n$, and no other symmetries, is called a *cyclic rotation group of order* n and is denoted by $\langle R_{360/n} \rangle$. Cyclic rotation groups, along with dihedral groups, are favorites of artists, designers, and nature. Figure 1.5 illustrates with corporate logos the cyclic rotation groups of orders 2, 3, 4, 5, 6, 8, 16, and 20.

A study of symmetry in greater depth is given in Chapter 26.

Exercises

The only way to learn mathematics is to do mathematics.
Paul R. Halmos, *A* Hilbert *Space Problem Book*

1. With pictures and words, describe each symmetry in D_3 (the set of symmetries of an equilateral triangle).

2. Write out a complete Cayley table for D_3. Is D_3 Abelian?

3. In D_4, find all elements X such that

a. $X^3 = V$;

b. $X^3 = R_{90}$;

c. $X^3 = R_0$;

d. $X^2 = R_0$;

e. $X^2 = H$.

4. Describe in pictures or words the elements of D_5 (symmetries of a regular pentagon).

5. For $n \geq 3$, describe the elements of D_n. (*Hint:* You will need to consider two cases—n even and n odd.) How many elements does D_n have?

6. In D_n, explain geometrically why a reflection followed by a reflection must be a rotation.

7. In D_n, explain geometrically why a rotation followed by a rotation must be a rotation.

8. In D_n, explain geometrically why a rotation and a reflection taken together in either order must be a reflection.

9. Associate the number 1 with a rotation and the number -1 with a reflection. Describe an analogy between multiplying these two numbers and multiplying elements of D_n.

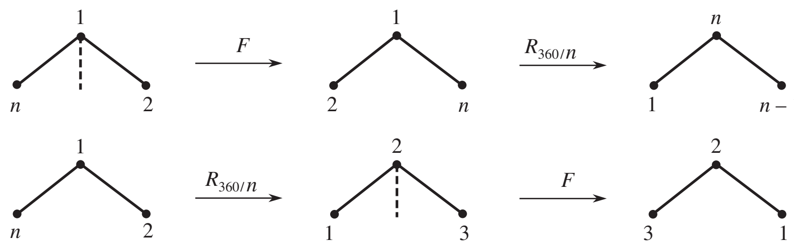
10. If r_1, r_2, and r_3 represent rotations from D_n and f_1, f_2, and f_3 represent reflections from D_n, determine whether $r_1 r_2 f_1 r_3 f_2 f_3 r_3$ is a rotation or a reflection.

11. Suppose that a, b, and c are elements of a dihedral group. Is $a^2 b^4 a c^5 a^3 c$ a rotation or a reflection? Explain your reasoning.

12. Suppose that an element X of a dihedral group is the product of m rotations and n reflections. Complete the following statement: X is a rotation if and only if ________________.

13. Find elements A, B, and C in D_4 such that $AB = BC$ but $A \neq C$. (Thus, "cross cancellation" is not valid.)

14. Explain what the following diagram proves about the group D_n.



15. Describe the symmetries of a nonsquare rectangle. Construct the corresponding Cayley table. Is this group Abelian? What property does this group have in common with the dihedral groups?

16. Describe the symmetries of a parallelogram that is neither a rectangle nor a rhombus. Describe the symmetries of a rhombus that is not a rectangle.

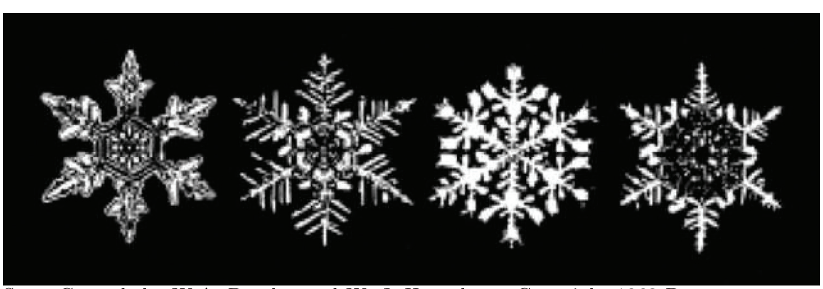
17. Describe the symmetries of a noncircular ellipse. Do the same for a hyperbola.

18. Consider an infinitely long strip of equally spaced H's:

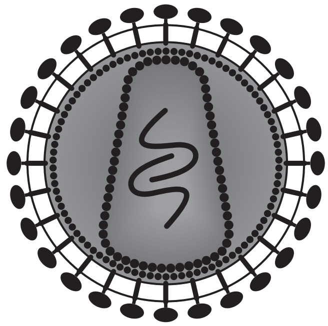
$$\cdots \text{H H H H} \cdots$$

Describe the symmetries of this strip. Is the group of symmetries of the strip Abelian?

19. For each of the snowflakes in the figure, find the symmetry group and locate the axes of reflective symmetry (disregard imperfections).



Photographs of snowflakes from the Bentley and Humphreys atlas.

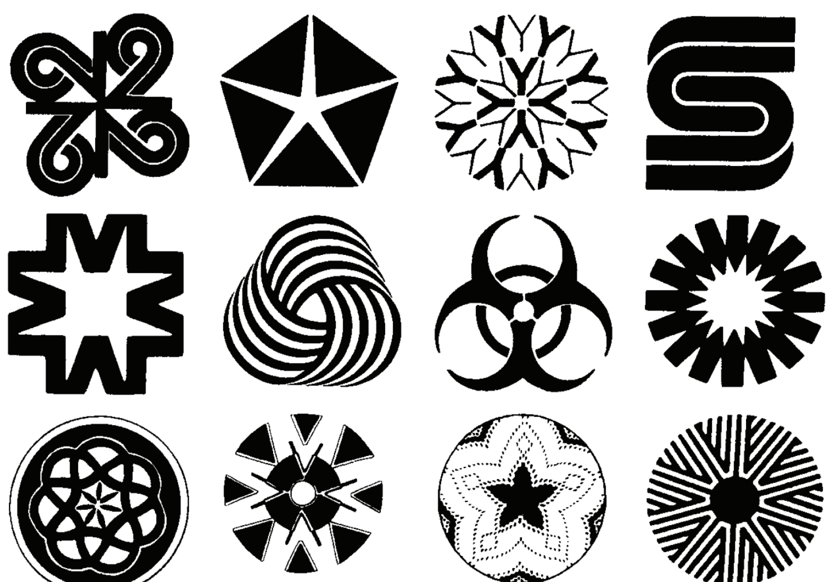


Courtesy of George V. Kelvin

20. Determine the symmetry group of the outer shell of the cross section of the human immunodeficiency virus (HIV) shown above.

21. Let X, Y, R_{90} be elements of D_4 with $Y \neq R_{90}$ and $X^2Y = R_{90}$. Determine Y. Show your reasoning.

22. For each design below, determine the symmetry group (ignore imperfections).

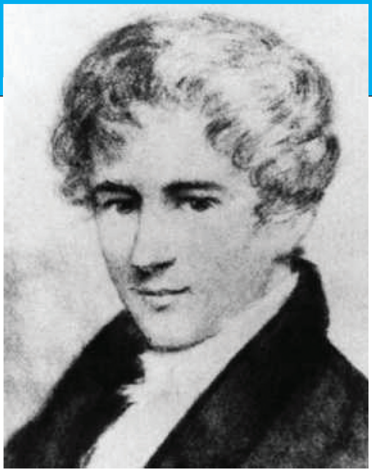


Joseph Gallian

23. Let n be a positive integer greater than or equal to 3. Prove that D_n contains R_{180} if and only if n is even.

24. If F is a reflection in the dihedral group D_n find all elements X in D_n such that $X^2 = F$ and all elements X in D_n such that $X^3 = F$.

Niels Abel



He [Abel] has left mathematicians something to keep them busy for five hundred years.

CHARLES HERMITE

A 500-kroner bank note first issued by Norway in 1948.

NIELS HENRIK ABEL, one of the foremost mathematicians of the 19th century, was born in Norway on August 5, 1802. At the age of 16, he began reading the classic mathematical works of Newton, Euler, Lagrange, and Gauss. When Abel was 18 years old, his father died, and the burden of supporting the family fell upon him. He took in private pupils and did odd jobs, while continuing to do mathematical research. At the age of 19, Abel solved a problem that had vexed leading mathematicians for hundreds of years. He proved that, unlike the situation for equations of degree 4 or less, there is no finite (closed) formula for the solution of the genral fifth-degree equation. Although Abel died long before the advent of the subjects that now make up abstract algebra, his solution to the quintic problem laid the groundwork for many of these subjects. Just when his work was beginning to receive the attention it deserved, Abel contracted tuberculosis. He died on April 6, 1829, at the age of 26.

In recognition of the fact that there is no Nobel Prize for mathematics, in 2002 Norway established the Abel Prize as the "Nobel Prize in mathematics" in honor of its native son. At approximately the \$1,000,000 level, the Abel Prize is now seen as an award equivalent to a Nobel Prize.

Abel prize winners John Thompson and Andrew Wiles are profiled in this book.

2 Groups

> Whenever groups disclose themselves, or could be introduced, simplicity crystallized out of comparative chaos.
>
> E. T. Bell, *Mathematics: Queen and Servant of Science*

> A good stock of examples, as large as possible, is indispensable for a thorough understanding of any concept, and when I want to learn something new, I make it my first job to build one.
>
> Paul R. Halmos

Definition and Examples of Groups

The term *group* was used by Galois around 1830 to describe sets of one-to-one functions on finite sets that could be grouped together to form a set closed under composition. As is the case with most fundamental concepts in mathematics, the modern definition of a group that follows is the result of a long evolutionary process. Although this definition was given by both Heinrich Weber and Walther von Dyck in 1882, it did not gain universal acceptance until the 20th century.

Definition Binary Operation
Let G be a set. A *binary operation* on G is a function that assigns each ordered pair of elements of G an element of G.

A binary operation on a set G, then, is simply a method (or formula) by which the members of an ordered pair from G combine to yield a new member of G. This condition is called *closure*. The most familiar binary operations are ordinary addition, subtraction, and multiplication of integers. Division of integers is not a binary operation on the integers because an integer divided by an integer need not be an integer.

The binary operations addition modulo n and multiplication modulo n on the set $\{0, 1, 2, \ldots, n - 1\}$, which we denote by Z_n, play an extremely important role in abstract algebra. In certain situations we will want to combine the elements of Z_n by addition modulo n only; in other situations we will want to use both addition modulo n and multiplication modulo n to combine the elements. It will be clear from the context whether we are using addition only or addition and multiplication. For example, when multiplying matrices with entries from Z_n, we will need both addition modulo n and multiplication modulo n.

Definition Group

Let G be a set together with a binary operation (usually called multiplication) that assigns to each ordered pair (a, b) of elements of G an element in G denoted by ab. We say G is a *group* under this operation if the following three properties are satisfied.

1. ***Associativity.* The operation is associative; that is, $(ab)c = a(bc)$ for all a, b, c in G.**
2. ***Identity.* There is an element e (called the *identity*) in G such that $ae = ea = a$ for all a in G.**
3. ***Inverses.* For each element a in G, there is an element b in G (called an *inverse* of a) such that $ab = ba = e$.**

In words, then, a group is a set together with an associative operation such that there is an identity, every element has an inverse, and any pair of elements can be combined without going outside the set. Be sure to verify closure when testing for a group (see Example 5). Notice that if a is the inverse of b, then b is the inverse of a.

If a group has the property that $ab = ba$ for every pair of elements a and b, we say the group is *Abelian*. A group is *non-Abelian* if there is some pair of elements a and b for which $ab \neq ba$. When encountering a particular group for the first time, one should determine whether or not it is Abelian.

Now that we have the formal definition of a group, our first keystone in learning about them is to build a good stock of examples. These examples will be used throughout the text to illustrate the theorems. (The best way to grasp the meat of a theorem is to see what it says in specific cases.) As we progress, the reader is bound to have hunches and conjectures that can be tested against the stock of examples. To develop a better understanding of the following examples, the reader should supply the missing details.

EXAMPLE 1 The set of integers Z (so denoted because the German word for numbers is *Zahlen*), the set of rational numbers Q (for quotient), and the set of real numbers $\mathbf{R}$ are all groups under ordinary addition. In each case, the identity is 0 and the inverse of a is $-a$. ∎

EXAMPLE 2 The set of integers under ordinary multiplication is not a group. Since the number 1 is the identity, property 3 fails. For example, there is no integer b such that $5b = 1$. ∎

EXAMPLE 3 The subset $\{1, -1, i, -i\}$ of the complex numbers is a group under complex multiplication. Note that -1 is its own inverse, whereas the inverse of i is $-i$, and vice versa. ∎

EXAMPLE 4 The set Q^+ of positive rationals is a group under ordinary multiplication. The inverse of any a is $1/a = a^{-1}$. ∎

EXAMPLE 5 The set S of positive irrational numbers together with 1 under multiplication satisfies the three properties given in the definition of a group but is not a group. Indeed, $\sqrt{2} \cdot \sqrt{2} = 2$, so S is not closed under multiplication. ∎

EXAMPLE 6 A rectangular array of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is called a 2×2 *matrix*. The set of all 2×2 matrices with real entries is a group under componentwise addition. That is,

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} + \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1 + a_2 & b_1 + b_2 \\ c_1 + c_2 & d_1 + d_2 \end{bmatrix}$$

The identity is $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, and the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is $\begin{bmatrix} -a & -b \\ -c & -d \end{bmatrix}$. ∎

EXAMPLE 7 The set $Z_n = \{0, 1, \ldots, n-1\}$ for $n \geq 1$ is a group under addition modulo n. For any $j > 0$ in Z_n, the inverse of j is $n - j$. This group is usually referred to as the *group of integers modulo n*. ∎

As we have seen, the real numbers, the 2×2 matrices with real entries, and the integers modulo n are all groups under the appropriate addition. But what about multiplication? In each case, the existence of some elements that do not have inverses prevents the set from being a group under the usual multiplication. However, we can form a group in each case by simply throwing out the rascals. Examples 8, 9, and 11 illustrate this.

EXAMPLE 8 The set $\mathbf{R}^*$ of nonzero real numbers is a group under ordinary multiplication. The identity is 1. The inverse of a is $1/a$. ■

EXAMPLE 9[1] The *determinant* of the 2×2 matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is the number $ad - bc$. If A is a 2×2 matrix, det A denotes the determinant of A. The set

$$GL(2, \mathbf{R}) = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| a, b, c, d \in \mathbf{R}, ad - bc \neq 0 \right\}$$

of 2×2 matrices with real entries and nonzero determinants is a non-Abelian group under the operation

$$\begin{bmatrix} a_1 & b_1 \\ c_1 & d_1 \end{bmatrix} \begin{bmatrix} a_2 & b_2 \\ c_2 & d_2 \end{bmatrix} = \begin{bmatrix} a_1a_2 + b_1c_2 & a_1b_2 + b_1d_2 \\ c_1a_2 + d_1c_2 & c_1b_2 + d_1d_2 \end{bmatrix}.$$

The first step in verifying that this set is a group is to show that the product of two matrices with nonzero determinants also has a nonzero determinant. This follows from the fact that for any pair of 2×2 matrices A and B, det $(AB) = (\det A)(\det B)$. Associativity can be verified by direct (but cumbersome) calculations. The identity is $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$; the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is

$$\begin{bmatrix} \dfrac{d}{ad - bc} & \dfrac{-b}{ad - bc} \\ \dfrac{-c}{ad - bc} & \dfrac{a}{ad - bc} \end{bmatrix}$$

(explaining the requirement in the definition of $GL(2, \mathbf{R})$ that $ad - bc \neq 0$). In particular, the determinant of a matrix determines if it has an inverse. Another useful fact about determinants is det $A^{-1} = (\det A)^{-1}$.

This very important non-Abelian group is called the *general linear group* of 2×2 matrices over $\mathbf{R}$. ■

EXAMPLE 10 The set of all 2×2 matrices with real entries is not a group under the operation defined in Example 9. Inverses do not exist when the determinant is 0. ■

[1] For simplicity, we have restricted our matrix examples to the 2×2 case. However, the examples in this chapter generalize to $n \times n$ matrices.

Now that we have shown how to make subsets of the real numbers and subsets of the set of 2×2 matrices into multiplicative groups, we next consider the integers under multiplication modulo n.

In Chapter 1 we introduced the dihedral groups by designating the four reflections in D_4 with the intuitive notions of horizental, vertical and diagonals axes and used a Cayley table to perform multiplication. For integers greater than 4 there is no natural analog for describing the reflections and a Cayley table is too cumbersome. Moreover, performing operations on regular n-gon for large n is impractical. The following example demonstrates properties of the dihedral groups that allows us to algebraically describe all dihedral groups and perform multiplication in a simple way.

EXAMPLE 11 Algebraic approach to the dihedral groups In D_n let R denote $R_{360/n}$ and let F be any reflection. Then

$$D_n = \{R_0, R, R^2, \ldots, R^{n-1}, F, RF, R^2F, \ldots, R^{n-1}F\}.$$

To do the multiplication observe that because every reflection is its own inverse we know $FF = R_0$ and $R^iFR^iF = R_0$. Rearranging we have $R^iF = F^{-1}R^{-i} = FR^{-i}$. This tells us that we can always "switch" R^i and F provided we change the sign of the exponent of R. In particular, this relationship serves as a commutativity-like property that allows us to rearrange the product of rotations and reflections. For instance, if $n = 10$, we have $R^{10} = R_0$ and $R^3FR^7F = R^3FFR^{-7} = R^3R^{-7} = R^{-4} = R^6$; $FR^4 = R^{-4}F = R^6F$; and $FR^{-3}F = R^3FF = R^3$. ∎

For real numbers a and b with $a \neq 0$ the equation $ax = b$ has the unique solution $x = a^{-1}b$. In contrast, for a and b with $a \neq b$ in Z_n the equation $ax = b$ can have no solution, a unique solution, or multiple solutions. The next example shows that elements a in Z_n for which $ax = 1$ have special significance.

EXAMPLE 12 **(L. EULER, 1761)** By Exercise 13 in Chapter 0, an integer a has a multiplicative inverse modulo n if and only if a and n are relatively prime. So, for any positive integer n, we define $U(n)$ to be the set of all positive integers less than n and relatively prime to n. Then $U(n)$ is a group under multiplication modulo n. (We leave it to the reader to check that this set is closed under this operation.)

For $n = 10$, we have $U(10) = \{1, 3, 7, 9\}$. The Cayley table for $U(10)$ is

mod 10	**1**	**3**	**7**	**9**
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

(Recall that ab mod n is the unique integer r with the property $a \cdot b = nq + r$, where $0 \leq r < n$ and $a \cdot b$ is ordinary multiplication.) In the case that n is a prime, $U(n) = \{1, 2, \ldots, n - 1\}$. ∎

In his classic book *Lehrbuch der Algebra*, published in 1895, Heinrich Weber gave an extensive treatment of the groups $U(n)$ and described them as the most important examples of finite Abelian groups.

EXAMPLE 13 The set $\{0, 1, 2, 3\}$ is not a group under multiplication modulo 4. Although 1 and 3 have inverses, the elements 0 and 2 do not. ∎

EXAMPLE 14 The set of integers under subtraction is not a group, since the operation is not associative. ∎

With the examples given thus far as a guide, it is wise for the reader to pause here and think of his or her own examples. Study actively! Don't just read along and be spoon-fed by the book.

EXAMPLE 15 The complex numbers $\mathbf{C} = \{a + bi \mid a, b \in \mathbf{R}, i^2 = -1\}$ are a group under the operation $(a + bi) + (c + di) = (a + c) + (b + d)i$. The inverse of $a + bi$ is $-a - bi$. The nonzero complex numbers $\mathbf{C}^*$ are a group under the operation $(a + bi)(c + di) = (ac - bd) + (ad + bc)i$ The inverse of $a + bi$ is $\dfrac{1}{a + bi} = \dfrac{1}{a + bi}\dfrac{a - bi}{a - bi} = \dfrac{1}{a^2 + b^2}a - \dfrac{1}{a^2 + b^2}bi$. ∎

EXAMPLE 16 For all integers $n \geq 1$, the set of complex nth roots of unity

$$\left\{\cos\frac{k \cdot 360^\circ}{n} + i \sin\frac{k \cdot 360^\circ}{n} \,\middle|\, k = 0, 1, 2, \ldots, n - 1\right\}$$

(i.e., complex zeros of $x^n - 1$) is a group under multiplication. (See DeMoivre's Theorem—Example 13 in Chapter 0.) Compare this group with the one in Example 3. ∎

Recall from Chapter 0 that the complex number $\cos\theta + i\sin\theta$ can be represented geometrically as the point ($\cos\theta$, $\sin\theta$) in a plane coordinatized by a real horizontal axis and a vertical imaginary axis, where θ is the angle formed by the line segment joining the origin and the point ($\cos\theta$, $\sin\theta$) and the positive real axis. Thus, the six complex zeros of $x^6 = 1$ are located at points around the circle of radius 1, 60° apart, as shown in Figure 2.1.

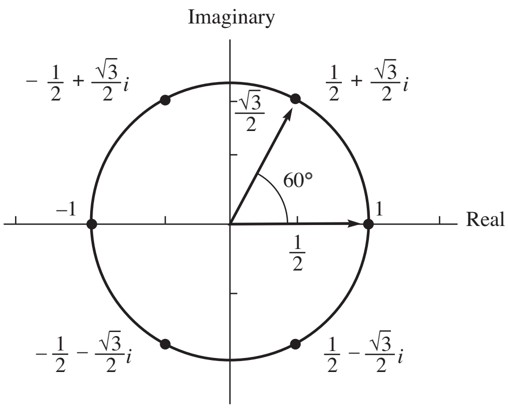


Figure 2.1 Zeros of $x^6 = 1$ on the unit circle.

EXAMPLE 17 The set $\mathbf{R}^n = \{(a_1, a_2, \ldots, a_n) | a_1, a_2, \ldots, a_n \in \mathbf{R}\}$ is a group under componentwise addition [i.e., $(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$]. ∎

EXAMPLE 18 The set of all 2×2 matrices with determinant 1 with entries from Q (rationals), $\mathbf{R}$ (reals), $\mathbf{C}$ (complex numbers), or Z_p (p a prime) is a non-Abelian group under matrix multiplication. This group is called the *special linear group* of 2×2 matrices over Q, $\mathbf{R}$, $\mathbf{C}$, or Z_p, respectively. If the entries are from F, where F is any of the above, we denote this group by $SL(2, F)$. For the group $SL(2, F)$, the formula given in Example 9 for the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ simplifies to $\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$. When the matrix entries are from Z_p, we use modulo p arithmetic to compute determinants, matrix products, and inverses. To illustrate the case $SL(2, Z_5)$, consider the element $A = \begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix}$. Then $\det A = (3 \cdot 4 - 4 \cdot 4) \bmod 5 = -4 \bmod 5 = 1$, and the inverse of A is $\begin{bmatrix} 4 & -4 \\ -4 & 3 \end{bmatrix} = \begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix}$. Note that

$\begin{bmatrix} 3 & 4 \\ 4 & 4 \end{bmatrix}\begin{bmatrix} 4 & 1 \\ 1 & 3 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ when the arithmetic is done modulo 5. ▮

Example 9 is a special case of the following general construction.

▮ EXAMPLE 19 Let F be any of Q, $\mathbf{R}$, $\mathbf{C}$, or Z_p (p a prime). The set $GL(2, F)$ of all 2×2 matrices with nonzero determinants and entries from F is a non-Abelian group under matrix multiplication. As in Example 18, when F is Z_p, modulo p arithmetic is used to calculate determinants, matrix products, and inverses. The formula given in Example 9 for the inverse of $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ remains valid for elements from $GL(2, Z_p)$, provided we interpret division by $ad - bc$ as multiplication by the inverse of $(ad - bc)$ modulo p. For example, in $GL(2, Z_7)$, consider $\begin{bmatrix} 4 & 5 \\ 6 & 3 \end{bmatrix}$. Then the determinant $(ad - bc)$ mod 7 is $(12 - 30)$ mod $7 = -18$ mod $7 = 3$ and the inverse of 3 is 5 [since $(3 \cdot 5)$ mod $7 = 1$]. So, the inverse of $\begin{bmatrix} 4 & 5 \\ 6 & 3 \end{bmatrix}$ is $\begin{bmatrix} 3 \cdot 5 & 2 \cdot 5 \\ 1 \cdot 5 & 4 \cdot 5 \end{bmatrix} = \begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix}$. [The reader should check that $\begin{bmatrix} 4 & 5 \\ 6 & 3 \end{bmatrix}\begin{bmatrix} 1 & 3 \\ 5 & 6 \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$ in $GL(2, Z_7)$]. ▮

The group $GL(n, F)$ is called the *general linear group* of $n \times n$ matrices over F.

▮ EXAMPLE 20 The set $\{1, 2, \ldots, n - 1\}$ is a group under multiplication modulo n if and only if n is prime. ▮

Table 2.1 summarizes many of the specific groups that we have presented thus far.

As the previous examples demonstrate, the notion of a group is a very broad one indeed. The goal of the axiomatic approach is to find properties general enough to permit many diverse examples having these properties and specific enough to allow one to deduce many interesting consequences.

The goal of abstract algebra is to discover truths about algebraic systems (i.e., sets with one or more binary operations) that are independent of the specific nature of the operations. All one knows or needs to know is that these operations, whatever they may be, have certain properties. We then seek to deduce consequences of these properties. This is why this branch of mathemat-

Table 2.1 Summary of Group Examples (F can be any of Q, R, C, or Z_p; L is a reflection).

Group	Operation	Identity	Form of Element	Inverse	Abelian
Z	Addition	0	k	$-k$	Yes
Q^+	Multiplication	1	m/n, $m, n > 0$	n/m	Yes
Z_n	Addition mod n	0	k	$n - k$	Yes
$\mathbf{R}^*$	Multiplication	1	x	$1/x$	Yes
$\mathbf{C}^*$	Multiplication	1	$a + bi$	$\frac{1}{a^2+b^2}a - \frac{1}{a^2+b^2}bi$	Yes
$GL(2, F)$	Matrix multiplication	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc \neq 0$	$\begin{bmatrix} \frac{d}{ad-bc} & \frac{-b}{ad-bc} \\ \frac{-c}{ad-bc} & \frac{a}{ad-bc} \end{bmatrix}$	No
$U(n)$	Multiplication mod n	1	k, $\gcd(k, n) = 1$	Solution to $kx \bmod n = 1$	Yes
$\mathbf{R}^n$	Componentwise addition	$(0, 0, \ldots, 0)$	$(a_1, a_2, \ldots, a_n)$	$(-a_1, -a_2, \ldots, -a_n)$	Yes
$SL(2, F)$	Matrix multiplication	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$, $ad - bc = 1$	$\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$	No
D_n	Composition	R_0	R_α, L	$R_{360-\alpha}$, L	No

ics is called *abstract* algebra. It must be remembered, however, that when a specific group is being discussed, a specific operation must be given (at least implicitly).

Elementary Properties of Groups

Now that we have seen many diverse examples of groups, we wish to deduce some properties that they share. The definition itself raises some fundamental questions. Every group has *an* identity. Could a group have more than one? Every group element has *an* inverse. Could an element have more than one? The examples suggest not. But examples can only suggest. One cannot prove that every group has a unique identity by looking at examples, because each example inherently has properties that may not be shared by all groups. We are forced to restrict ourselves to the properties that all groups have; that is, we must view groups as abstract entities rather than argue by example. The next three theorems illustrate the abstract approach.

Theorem 2.1 Uniqueness of the Identity

In a group G, there is only one identity element.

PROOF Suppose both e and e' are identities of G. Then,

1. $ae = a$ for all a in G, and
2. $e'a = a$ for all a in G.

The choices of $a = e'$ in (part 1) and $a = e$ in (part 2) yield $e'e = e'$ and $e'e = e$. Thus, e and e' are both equal to $e'e$ and so are equal to each other. ∎

Because of this theorem, we may unambiguously speak of "the identity" of a group and denote it by 'e' (because the German word for identity is *Einheit*).

Theorem 2.2 Cancellation

In a group G, the right and left cancellation laws hold; that is, $ba = ca$ implies $b = c$, and $ab = ac$ implies $b = c$.

PROOF Suppose $ba = ca$. Let a' be an inverse of a. Then multiplying on the right by a' yields $(ba)a' = (ca)a'$. Associativity yields $b(aa') = c(aa')$. Then $be = ce$ and, therefore, $b = c$ as desired. Similarly, one can prove that $ab = ac$ implies $b = c$ by multiplying by a' on the left. ∎

A consequence of the cancellation property is the fact that in a Cayley table for a group, each group element occurs exactly once in each row and column (see Exercise 33). Another consequence of the cancellation property is the uniqueness of inverses.

Theorem 2.3 Uniqueness of Inverses

For each element a in a group G, there is a unique element b in G such that $ab = ba = e$.

PROOF Suppose b and c are both inverses of a. Then $ab = e$ and $ac = e$, so that $ab = ac$. Canceling the a on both sides gives $b = c$, as desired. ∎

As was the case with the identity element, it is reasonable, in view of Theorem 2.3, to speak of "the inverse" of an element g of a group; in fact, we may unambiguously denote it by g^{-1}. This notation is suggested by that used for ordinary real numbers under multiplication. Similarly, when n is a positive integer, the associative law allows us to use g^n to denote the unambiguous product:

$$\underbrace{gg \cdots g.}_{n \text{ factors}}$$

We define $g^0 = e$. When n is negative, we define $g^n = (g^{-1})^{|n|}$ [e.g., $g^{-3} = (g^{-1})^3$]. Unlike for real numbers, in an abstract group we do not permit noninteger exponents such as $g^{1/2}$. With this notation, the familiar laws of exponents hold for groups; that is, for all integers m and n and any group element g, we have $g^m g^n = g^{m+1}$ and $(g^m)^n = g^{mn}$. Although the way one manipulates the group expressions $g^m g^n$ and $(g^m)^n$ coincides with the laws of exponents for real numbers, the laws of exponents fail to hold for expressions involving two group elements. Thus, for groups in general, $(ab)^2 = abab$ rather than a^2b^2. To remove parentheses in the expression $(ab)^{-2}$ we have $(ab)^{-2} = ((ab)^{-1})^2 = (b^{-1}a^{-1})^2 = b^{-1}a^{-1}b^{-1}a^{-1}$ $(ab)^n \neq a^n b^n$. (See Exercises 25 and 36.)

The important thing about the existence of a unique inverse for each group element a is that for every element b in the group there is a unique solution in the group of the equations $ax = b$ and $xa = b$. Namely, $x = a^{-1}b$ in the first case and $x = ba^{-1}$ in the second case. In contrast, in the set $\{0, 1, 2, 3, 4, 5\}$, the equation $2x = 4$ has the solutions $x = 2$ and $x = 5$ under the operation multiplication mod 6. However, this set is not a group under multiplication mod 6.

Also, one must be careful with this notation when dealing with a specific group whose binary operation is addition and is denoted by "+." In this case, the definitions and group properties expressed in multiplicative notation must be translated to additive notation. For example, the inverse of g is written as $-g$. Likewise, for example, g^3 means $g + g + g$ and is usually written as $3g$, whereas g^{-3} means $(-g) + (-g) + (-g)$ and is written as $-3g$. When additive notation is used, do not interpret "ng" as combining n and g under the group operation; n may not even be an element of the group! Table 2.2 shows the common notation and corresponding terminology for groups under multiplication and groups

Table 2.2

	Multiplicative Group		Additive Group
$a \cdot b$ or ab	Multiplication	$a + b$	Addition
e or 1	Identity or one	0	Zero
a^{-1}	Multiplicative inverse of a	$-a$	Additive inverse of a
a^n	Power of a	na	Multiple of a
ab^{-1}	Quotient	$a - b$	difference

under addition. As is the case for real numbers, we use $a - b$ as an abbreviation for $a + (-b)$.

Because of the associative property, we may unambiguously write the expression abc, for this can be reasonably interpreted as only $(ab)c$ or $a(bc)$, which are equal. In fact, by using induction and repeated application of the associative property, one can prove a general associative property that essentially means that parentheses can be inserted or deleted at will without affecting the value of a product involving any number of group elements. Thus,

$$a^2(bcdb^2) = a^2b(cd)b^2 = (a^2b)(cd)b^2 = a(abcdb)b,$$

and so on.

Although groups do not have the property that $(ab)^n = a^nb^n$, there is a simple relationship between $(ab)^{-1}$ and a^{-1} and b^{-1}.

Theorem 2.4 Socks–Shoes Property

For group elements a and b, $(ab)^{-1} = b^{-1}a^{-1}$.

PROOF Since $(ab)(ab)^{-1} = e$ and $(ab)(b^{-1}a^{-1}) = a(bb^{-1})a^{-1} = aea^{-1} = aa^{-1} = e$, we have by Theorem 2.3 that $(ab)^{-1} = b^{-1}a^{-1}$. ∎

Historical Note

We conclude this chapter with a bit of history concerning the noncommutativity of matrix multiplication. In 1925, quantum theory was replete with annoying and puzzling ambiguities. It was Werner Heisenberg who recognized the cause. He observed that the product of the quantum-theoretical analogs of the classical Fourier series did not necessarily commute. For all his boldness, this shook Heisenberg. As he later recalled [2, p. 94]:

> In my paper the fact that XY was not equal to YX was very disagreeable to me. I felt this was the only point of difficulty in the whole scheme, otherwise I would be perfectly happy. But this difficulty had worried me and I was not able to solve it.

Heisenberg asked his teacher, Max Born, if his ideas were worth publishing. Born was fascinated and deeply impressed by Heisenberg's new approach. Born wrote [1, p. 217]:

> After having sent off Heisenberg's paper to the *Zeitschrift für Physik* for publication, I began to ponder over his symbolic multiplication, and was soon so involved in it that I thought about it for the whole day and could hardly sleep at night. For I felt there was something fundamental behind it, the consummation of our endeavors of many years. And one morning, about the 10 July 1925, I suddenly saw light: Heisenberg's symbolic multiplication was nothing but the matrix calculus, well-known to me since my student days from Rosanes' lectures in Breslau.

Born and his student, Pascual Jordan, reformulated Heisenberg's ideas in terms of matrices, but it was Heisenberg who was credited with the formulation. In his autobiography, Born lamented [1, p. 219]:

> Nowadays the textbooks speak without exception of Heisenberg's matrices, Heisenberg's commutation law, and Dirac's field quantization.
>
> In fact, Heisenberg knew at that time very little of matrices and had to study them.

Upon learning in 1933 that he was to receive the Nobel Prize with Dirac and Schrödinger for this work, Heisenberg wrote to Born [1, p. 220]:

> If I have not written to you for such a long time, and have not thanked you for your congratulations, it was partly because of my rather bad conscience with respect to you. The fact that I am to receive the Nobel Prize alone, for work done in Göttingen in collaboration—you, Jordan, and I—this fact depresses

> me and I hardly know what to write to you. I am, of course, glad that our common efforts are now appreciated, and I enjoy the recollection of the beautiful time of collaboration. I also believe that all good physicists know how great was your and Jordan's contribution to the structure of quantum mechanics—and this remains unchanged by a wrong decision from outside. Yet I myself can do nothing but thank you again for all the fine collaboration, and feel a little ashamed.

The story has a happy ending, however, because Born received the Nobel Prize in 1954 for his fundamental work in quantum mechanics.

Exercises

"For example" is not proof.

JEWISH PROVERB

1. Which of the following binary operations are closed?

a. subtraction of positive integers
b. division of nonzero integers
c. function composition of polynomials with real coefficients
d. multiplication of 2×2 matrices with integer entries
e. exponentiation of integers

2. Which of the following binary operations are associative?

a. subtraction of integers
b. division of nonzero rationals
c. function composition of polynomials with real coefficients
d. multiplication of 2×2 matrices with integer entries
e. exponentiation of integers

3. Which of the following binary operations are commutative?

a. substraction of integers
b. division of nonzero real numbers
c. function composition of polynomials with real coefficients
d. multiplication of 2×2 matrices with real entries
e. exponentiation of integers

4. Which of the following sets are closed under the given operation?

a. $\{0, 4, 8, 12\}$ addition mod 16
b. $\{0, 4, 8, 12\}$ addition mod 15
c. $\{1, 4, 7, 13\}$ multiplication mod 15
d. $\{1, 4, 5, 7\}$ multiplication mod 9

5. In each case, find the inverse of the element under the given operation.

a. 13 in Z_{20}
b. 13 in $U(14)$
c. $n - 1$ in $U(n)$ $(n > 2)$
d. $3 - 2i$ in **C***, the group of nonzero complex numbers under multiplication

6. In each case, perform the indicated operation.

a. In **C***, $(7 + 5i)(-3 + 2i)$
b. In $GL(2, Z_{13})$, $\det\begin{bmatrix} 7 & 4 \\ 1 & 5 \end{bmatrix}$
c. In $GL(2, \mathbf{R})$, $\begin{bmatrix} 6 & 3 \\ 8 & 2 \end{bmatrix}^{-1}$
d. In $GL(2, Z_7)$, $\begin{bmatrix} 2 & 1 \\ 1 & 3 \end{bmatrix}^{-1}$

7. Which of the following sets are groups under matrix multplication?

a. $G_1 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in Z_4, \; ad - bc \neq 0 \text{ mod } 4 \right\}.$

b. $G_2 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \in Z, \; ad - bc \neq 0 \right\}.$

c. $G_3 = \left\{ \begin{bmatrix} a & b \\ c & d \end{bmatrix} \middle| \; a, b, c, d \text{ are positive rational numbers}, \; ad - bc \neq 0 \right\}.$

8. Does the set of integers under subtraction (i.e., $a - b$) have an identity element?

9. Show that the equation $5x = 3$ modulo 20 has no integer solution but $3x = 5$ does.

10. List the elements of $U(20)$ and find the inverse of each one.

11. For group elements a, b, and c provide a descriptive name for the property $(abc)^{-1} = c^{-1}b^{-1}a^{-1}$.

12. Give two reasons why the set of odd integers under addition is not a group.

13. Show that $\{1, 2, 3\}$ under multiplication modulo 4 is not a group but that $\{1, 2, 3, 4\}$ under multiplication modulo 5 is a group.

14. Show that the group $GL(2, \mathbf{R})$ of Example 9 is non-Abelian by exhibiting a pair of matrices A and B in $GL(2, \mathbf{R})$ such that $AB \neq BA$.

15. Let a belong to a group and $a^{12} = e$. Express the inverse of each of the elements a, a^6, a^8, and a^{11} in the form a^k for some positive integer k.

16. Show that the set $\{5, 15, 25, 35\}$ is a group under multiplication modulo 40. What is the identity element of this group? Can you see any relationship between this group and $U(8)$?

17. Translate each of the following multiplicative expressions into its additive counterpart. Assume that the operation is commutative.

a. a^2b^3
b. $a^{-2}(b^{-1}c)^2$
c. $(ab^2)^{-3}c^2 = e$

18. For group elements a, b, and c, express $(ab)^3$ and $(ab^{-2}c)^{-2}$ without parentheses.

19. Suppose that a and b belong to a group and $a^5 = e$ and $b^7 = e$. Write $a^{-2}b^{-4}$ and $(a^2b^4)^{-2}$ without using negative exponents.

20. List the members of $K = \{x^2 \mid x \in D_4\}$ and $L = \{x \in D_4 \mid x^2 = e\}$.

21. Prove that the set of all 2×2 matrices with entries from $\mathbf{R}$ and determinant $+1$ is a group under matrix multiplication.

22. For any integer $n > 2$, show that there are at least two elements in $U(n)$ that satisfy $x^2 = 1$.

23. An abstract algebra teacher intended to give a typist a list of nine integers that form a group under multiplication modulo 91. Instead, one of the nine integers was inadvertently left out, so that the list appeared as 1, 9, 16, 22, 53, 74, 79, 81. Which integer was left out? (This really happened!)

24. Let G be a group with the property that for any x, y, z in the group, $xy = zx$ implies $y = z$. Prove that G is Abelian. ("Left-right cancellation" implies commutativity.)

25. (Law of Exponents for Abelian Groups) Let a and b be elements of an Abelian group and let n be any integer. Show that $(ab)^n = a^n b^n$. Is this also true for non-Abelian groups?

26. (Socks–Shoes Property) Draw an analogy between the statement $(ab)^{-1} = b^{-1}a^{-1}$ and the act of putting on and taking off your socks and shoes. Find distinct nonidentity elements a and b from a non-Abelian group such that $(ab)^{-1} = a^{-1}b^{-1}$. Find an example that shows that in a group, it is possible to have $(ab)^{-2} \neq b^{-2}a^{-2}$. What would be an appropriate name for the group property $(abc)^{-1} = c^{-1}b^{-1}a^{-1}$?

27. Prove that a group G is Abelian if and only if $(ab)^{-1} = a^{-1}b^{-1}$ for all a and b in G.

28. Prove that in a group, $(a^{-1})^{-1} = a$ for all a.

29. For any elements a and b from a group and any integer n, prove that $(a^{-1}ba)^n = a^{-1}b^n a$.

30. If $a_1, a_2, \ldots, a_n$ belong to a group, what is the inverse of $a_1 a_2 \ldots a_n$?

31. The integers 5 and 15 are among a collection of 12 integers that form a group under multiplication modulo 56. List all 12.

32. For a positive integer n let S_n be the set of all 1-1 functions from $\{1, 2, \ldots, n\}$ to itself. What is the identity under the operation of function composition? Does every element of S_n have an inverse in S_n under this operation? Is S_n closed under this operation? Is S_n a group under this operation? How many elements are in S_n?

33. Prove that every group table is a *Latin square*[2]; that is, each element of the group appears exactly once in each row and each column.

34. Give an example of a group with exactly 105 elements. Give three examples of groups with 40 elements.

35. Prove that the set of all 3×3 matrices with real entries of

[2]Latin squares are useful in designing statistical experiments. There is also a close connection between Latin squares and finite geometries.

the form

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix}$$

is a group. (Multiplication is defined by

$$\begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & a' & b' \\ 0 & 1 & c' \\ 0 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 1 & a+a' & b'+ac'+b \\ 0 & 1 & c'+c \\ 0 & 0 & 1 \end{bmatrix}.$$

This group, sometimes called the *Heisenberg group* after the Nobel Prize–winning physicist Werner Heisenberg, is intimately related to the Heisenberg Uncertainty Principle of quantum physics.)

36. Prove that in a group, $(ab)^2 = a^2b^2$ if and only if $ab = ba$. Prove that in a group, $(ab)^{-2} = b^{-2}a^{-2}$ if and only if $ab = ba$.

37. Let G be a finite group and n an odd positive integer. Show that the number of elements x of G such that $x^n = e$ is odd. Show that the number of elements x of G such that $x^2 \neq e$ is even.

38. Let Q^+ denote the positive rational numbers. Explain why the correspondence from $Q^+ \times Q^+$ that takes $(\frac{a}{b}, \frac{c}{d})$ to $\frac{a+c}{b+d}$ is not a binary operation on Q^+. (That is, the correspondence is not well-defined.)

39. Suppose F_1 and F_2 are distinct reflections in a dihedral group D_n. Prove that $F_1F_2 \neq R_0$.

40. Suppose F_1 and F_2 are distinct reflections in a dihedral group D_n such that $F_1F_2 = F_2F_1$. Prove that $F_1F_2 = R_{180}$.

41. Let R be any fixed rotation and F any fixed reflection in a dihedral group. Prove that $R^kFR^k = F$.

42. Let R be any fixed rotation and F any fixed reflection in a dihedral group. Prove that $FR^kF = R^{-k}$. Why does this imply that D_n is non-Abelian?

43. Let R be some rotation in D_n and F some reflection in D_n. Express $R^6FRFR^{-3}FRF$ in the form R^k. Express $FR^4FR^5FR^2$ in the form R^kF.

44. Let R_α and R_β be rotations in D_n and F be a reflection in D_n. What rotation is $FR_\alpha FR_\beta$? What rotation is $R_\alpha FR_\beta F$?

45. Prove that if G is a group with the property that the square of every element is the identity, then G is Abelian. (This exercise is referred to in Chapter 25.)

46. In a finite group, show that the number of nonidentity elements that satisfy the equation $x^5 = e$ is a multiple of 4. If the stipulation that the group be finite is omitted, what can you say about the number of nonidentity elements that satisfy the equation $x^5 = e$?

47. List the six elements of $\mathrm{GL}(2, Z_2)$. Show that this group is non-Abelian by finding two elements that do not commute. (This exercise is referred to in Chapter 7.)

48. Prove the assertion made in Example 20 that the set $\{1, 2, . . . , n - 1\}$ is a group under multiplication modulo n if and only if n is prime.

49. Suppose the table below is a group table. Fill in the blank entries.

	e	***a***	***b***	***c***	***d***
e	e	—	—	—	—
a	—	b	—	—	e
b	—	c	d	e	—
c	—	d	—	a	b
d	—	—	—	—	—

Computer Exercises

Software for the computer exercises in this chapter is available at the website:

http://www.d.umn.edu/~jgallian

References

Max Born, *My Life: Recollections of a Nobel Laureate,* New York: Charles Scribner's Sons, 1978.

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3 Finite Groups; Subgroups

In our own time, in the period 1960–1980, we have seen particle physics emerge as the playground of group theory.

Freeman Dyson

What brought order and logic to the building blocks of matter ... was something called a "symmetry group"—a mathematical beast that Frenkel had never encountered in school. "This was a moment of epiphany," he recalls, a vision of "an entirely different world."

Jim Holt in *The New York Review of Books* December 5, 2013

Terminology and Notation

As we will soon discover, finite groups—that is, groups with finitely many elements—have interesting arithmetic properties. To facilitate the study of finite groups, it is convenient to introduce some terminology and notation.

Definition Order of a Group
The number of elements of a group (finite or infinite) is called its *order*. We will use $|G|$ to denote the order of G.

Thus, the group Z of integers under addition has infinite order, whereas the group $U(10) = \{1, 3, 7, 9\}$ under multiplication modulo 10 has order 4.

Definition Order of an Element
The *order* g in a group G is the smallest positive integer n such that $g^n = e$. (In additive notation, this would be $ng = 0$.) If no such integer exists, we say that g has *infinite order*. The order of an element g is denoted by $|g|$.

So, to find the order of a group element g, you need only compute the sequence of products g, g^2, g^3, ... , until you reach the identity for the first time. The exponent of this product (or coefficient if the operation is addition) is the order of g. If the identity never appears in the sequence, then g has infinite order.

EXAMPLE 1 Consider $U(15) = \{1, 2, 4, 7, 8, 11, 13, 14\}$ under multiplication modulo 15. This group has order 8. To find the order of the element 7, say, we compute the sequence $7^1 = 7$, $7^2 = 4$, $7^3 = 13$, $7^4 = 1$, so $|7| = 4$. To find the order of 11, we compute $11^1 = 11$, $11^2 = 1$, so $|11| = 2$. Similar computations show that $|1| = 1$, $|2| = 4$, $|4| = 2$, $|8| = 4$, $|13| = 4$, $|14| = 2$. [Here is a trick that makes these calculations easier. Rather than compute the sequence $13^1, 13^2, 13^3, 13^4$, we may observe that $13 = -2 \bmod 15$, so that $13^2 = (-2)^2 = 4, 13^3 = -2 \cdot 4 = -8, 13^4 = (-2)(-8) = 1$.][1] ∎

EXAMPLE 2 Consider Z_{10} under addition modulo 10. Since $1 \cdot 2 = 2$, $2 \cdot 2 = 4, 3 \cdot 2 = 6, 4 \cdot 2 = 8, 5 \cdot 2 = 0$, we know that $|2| = 5$. Similar computations show that $|0| = 1$, $|7| = 10$, $|5| = 2$, $|6| = 5$. (Here $2 \cdot 2$ is an abbreviation for $2 + 2, 3 \cdot 2$ is an abbreviation for $2 + 2 + 2$, etc.) ∎

EXAMPLE 3 Consider Z under ordinary addition. Here every nonzero element has infinite order, since the sequence a, $2a$, $3a$, . . . never includes 0 when $a \neq 0$. ∎

The perceptive reader may have noticed among our examples of groups in Chapter 2 that some are subsets of others with the same binary operation. The group $SL(2, \mathbf{R})$ in Example 18, for instance, is a subset of the group $GL(2, \mathbf{R})$ in Example 9. Similarly, the group of complex numbers $\{1, -1, i, -i\}$ under multiplication is a subset of the group described in Example 16 for n equal to any multiple of 4. This situation arises so often that we introduce a special term to describe it.

Definition Subgroup
If a subset H of a group G is itself a group under the operation of G, we say that H is a *subgroup* of G.

We use the notation $H \leq G$ to mean that H is a subgroup of G. If we want to indicate that H is a subgroup of G but is not

[1] The website http://www.google.com provides a convenient way to do modular arithmetic. For example, to compute 13^4 mod 15, just type "13^4 mod 15" in the search box.

equal to G itself, we write $H < G$. Such a subgroup is called a *proper subgroup*. The subgroup $\{e\}$ is called the *trivial subgroup* of G; a subgroup that is not $\{e\}$ is called a *nontrivial subgroup* of G.

Notice that Z_n under addition modulo n is *not* a subgroup of Z under addition, since addition modulo n is not the operation of Z.

Subgroup Tests

When determining whether or not a subset H of a group G is a subgroup of G, one need not directly verify the group axioms. The next three results provide simple tests that suffice to show that a subset of a group is a subgroup.

Theorem 3.1 One-Step Subgroup Test

Let G be a group and H a nonempty subset of G. If ab^{-1} is in H whenever a and b are in H, then H is a subgroup of G. (In additive notation, if $a - b$ is in H whenever a and b are in H, then H is a subgroup of G.)

PROOF Since the operation of H is the same as that of G, it is clear that this operation is associative. Next, we show that e is in H. Since H is nonempty, we may pick some x in H. Then, letting $a = x$ and $b = x$ in the hypothesis, we have $e = xx^{-1} = ab^{-1}$ is in H. To verify that x^{-1} is in H whenever x is in H, all we need to do is to choose $a = e$ and $b = x$ in the statement of the theorem. Finally, the proof will be complete when we show that H is closed; that is, if x, y belong to H, we must show that xy is in H also. Well, we have already shown that y^{-1} is in H whenever y is; so, letting $a = x$ and $b = y^{-1}$, we have $xy = x(y^{-1})^{-1} = ab^{-1}$ is in H. ∎

Although we have dubbed Theorem 3.1 the One-Step Subgroup Test, there are actually four steps involved in applying the theorem. (After you gain some experience, the first three steps will be routine.) Notice the similarity between the last three steps listed below and the three steps involved in the Second Principle of Mathematical Induction.

1. Identify the property P that distinguishes the elements of H; that is, identify a defining condition.
2. Prove that the identity has property P. (This verifies that H is nonempty.)
3. *Assume* that two elements a and b have property P.
4. Use the assumption that a and b have property P to show that ab^{-1} has property P.

The procedure is illustrated in Examples 4 and 5.

EXAMPLE 4 Let G be an Abelian group with identity e. Then $H = \{x \in G \mid x^2 = e\}$ is a subgroup of G. Here, the defining property of H is the condition $x^2 = e$. So, we first note that $e^2 = e$, so that H is nonempty. Now we assume that a and b belong to H. This means that $a^2 = e$ and $b^2 = e$. Finally, we must show that $(ab^{-1})^2 = e$. Since G is Abelian, $(ab^{-1})^2 = ab^{-1}ab^{-1} = a^2(b^{-1})^2 = a^2(b^2)^{-1} = ee^{-1} = e$. Therefore, ab^{-1} belongs to H and, by the One-Step Subgroup Test, H is a subgroup of G. ∎

In many instances, a subgroup will consist of all elements that have a particular form. Then the property P is that the elements have that particular form. This is illustrated in the following example.

EXAMPLE 5 Let G be an Abelian group under multiplication with identity e. Then $H = \{x^2 \mid x \in G\}$ is a subgroup of G. (In words, H is the set of all "squares.") Since $e^2 = e$, the identity has the correct form. Next, we write two elements of H in the correct form, say, a^2 and b^2. We must show that $a^2(b^2)^{-1}$ also has the correct form; that is, $a^2(b^2)^{-1}$ is the square of some element. Since G is Abelian, we may write $a^2(b^2)^{-1}$ as $(ab^{-1})^2$, which is the correct form. Thus, H is a subgroup of G. ∎

Beginning students often prefer to use the next theorem instead of Theorem 3.1.

Theorem 3.2 Two-Step Subgroup Test

Let G be a group and let H be a nonempty subset of G. If ab is in H whenever a and b are in H (H is closed under

the operation), and a^{-1} is in H whenever a is in H (H is closed under taking inverses), then H is a subgroup of G.

PROOF Since H is nonempty, the operation of H is associative, H is closed, and every element of H has an inverse in H, all that remains to show is that e is in H. To this end, let a belong to H. Then a^{-1} and $aa^{-1} = e$ are in H. ∎

When applying the Two-Step Subgroup Test, we proceed exactly as in the case of the One-Step Subgroup Test, except we use the assumption that a and b have property P to prove that ab has property P and that a^{-1} has property P.

EXAMPLE 6 Let G be an Abelian group. Then $H = \{x \in G \mid |x| \text{ is finite}\}$ is a subgroup of G. Since $e^1 = e, H \neq \theta$. To apply the Two-Step Subgroup Test we assume that a and b belong to H and prove that ab and a^{-1} belong to H. Let $|a| = m$ and $|b| = n$. Then, because G is Abelian, we have $(ab)^{mn} = (a^m)^n(b^n)^m = e^n e^m = e$. Thus, ab has finite order (this does not show that $|ab| = mn$). Moveover, $(a^{-1})^m = (a^m)^{-1} = e^{-1} = e$ shows that a^{-1} has finite order. ∎

We next illustrate how to use the Two-Step Subgroup Test by introducing an important technique for creating new subgroups of Abelian groups from existing ones. It is a preview of a method that will be extended to certain subgroups of non-Abelian groups in later chapters.

EXAMPLE 7 Let G be an Abelian group and H and K be subgroups of G. Then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G. First note that $e = ee$ belongs to HK because e is in both H and K. Now suppose that a and b are in HK. Then by definition of HK there are elements $h_1, h_2 \in H$ and $k_1, k_2 \in K$ such that $a = h_1k_1$ and $b = h_2k_2$. We must prove that $ab \in HK$ and $a^{-1} \in HK$. Observe that because G is Abelian and H and K are subgroups of G, we have $ab = h_1k_1h_2k_2 = (h_1h_2)(k_1k_2) \in HK$. Likewise, $a^{-1} = (h_1k_1)^{-1} = k_1^{-1}h_1^{-1} = h_1^{-1}k_1^{-1} \in HK$. ∎

How do you prove that a subset of a group is *not* a subgroup? Here are three possible ways, any one of which guarantees that the subset is not a subgroup:

1. Show that the identity is not in the set.
2. Exhibit an element of the set whose inverse is not in the set.
3. Exhibit two elements of the set whose product is not in the set.

EXAMPLE 8 Let G be the group of nonzero real numbers under multiplication, $H = \{x \in G \mid x = 1 \text{ or } x \text{ is irrational}\}$ and $K = \{x \in G \mid x \geq 1\}$. Then H is not a subgroup of G, since $\sqrt{2} \in H$ but $\sqrt{2} \cdot \sqrt{2} = 2 \notin H$. Also, K is not a subgroup, since $2 \in K$ but $2^{-1} \notin K$. ∎

When dealing with finite groups, it is easier to use the following subgroup test.

Theorem 3.3 Finite Subgroup Test

Let H be a nonempty finite subset of a group G. If H is closed under the operation of G, then H is a subgroup of G.

PROOF In view of Theorem 3.2, we need only prove that $a^{-1} \in H$ whenever $a \in H$. If $a = e$, then $a^{-1} = a$ and we are done. If $a \neq e$, consider the sequence $a, a^2, \ldots$. By closure, all of these elements belong to H. Since H is finite, not all of these elements are distinct. Say $a^i = a^j$ and $i > j$. Then, $a^{i-j} = e$; and since $a \neq e, i - j > 1$. Thus, $aa^{i-j-1} = a^{i-j} = e$ and, therefore, $a^{i-j-1} = a^{-1}$. But $i - j - 1 \geq 0$ implies $a^{i-j-1} \in H$ and we are done. ∎

Examples of Subgroups

The proofs of the next few theorems show how our subgroup tests work. We first introduce an important notation. For any element a from a group, we let $\langle a \rangle$ denote the set $\{a^n \mid n \in Z\}$. In particular, observe that the exponents of a include all negative integers as well as 0 and the positive integers (a^0 is defined to be the identity).

Theorem 3.4 $\langle a \rangle$ Is a Subgroup

Let G be a group, and let a be any element of G. Then, $\langle a \rangle$ is a subgroup of G.

PROOF Since $a \in \langle a \rangle$, $\langle a \rangle$ is not empty. Let a^n, $a^m \in \langle a \rangle$. Then, $a^n(a^m)^{-1} = a^{n-m} \in \langle a \rangle$; so, by Theorem 3.1, $\langle a \rangle$ is a subgroup of G. ∎

The subgroup $\langle a \rangle$ is called the *cyclic subgroup 5 of G generated by a*. In the case that $G = \langle a \rangle$, we say that G is *cyclic* and a is a *generator of G*. (A cyclic group may have many generators.) Notice that although the list $\ldots, a^{-2}, a^{-1}, a^0, a^1, a^2, \ldots$ has infinitely many entries, the set $\{a^n \mid n \in Z\}$ might have only finitely many elements. Also note that, since $a^i a^j = a^{i+j} = a^{j+i} = a^j a^i$, every cyclic group is Abelian.

EXAMPLE 9 In $U(10)$, $\langle 3 \rangle = \{3, 9, 7, 1\} = U(10)$, for $3^1 = 3, 3^2 = 9, 3^3 = 7, 3^4 = 1, 3^5 = 3^4 \cdot 3 = 1 \cdot 3, 3^6 = 3^4 \cdot 3^2 = 9, \ldots; 3^{-1} = 7$ (since $3 \cdot 7 = 1$), $3^{-2} = 9, 3^{-3} = 3, 3^{-4} = 1, 3^{-5} = 3^{-4} \cdot 3^{-1} = 1 \cdot 7, 3^{-6} = 3^{-4} \cdot 3^{-2} = 1 \cdot 9 = 9, \ldots$. ∎

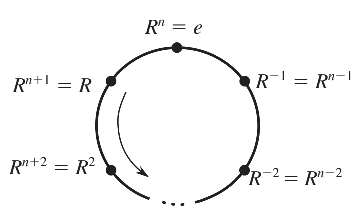
EXAMPLE 10 In Z_{10}, $\langle 2 \rangle = \{2, 4, 6, 8, 0\}$. Remember, a^n means na when the operation is addition. ∎

EXAMPLE 11 In Z, $\langle -1 \rangle = Z$. Here each entry in the list $\ldots, -2(-1), -1(-1), 0(-1), 1(-1), 2(-1), \ldots$ represents a distinct group element. ∎

EXAMPLE 12 In D_n, the dihedral group of order $2n$, let R denote a rotation of $360/n$ degrees. Then,

$$R^n = R_{360^\circ} = e, \quad R^{n+1} = R, \quad R^{n+2} = R^2, \ldots.$$

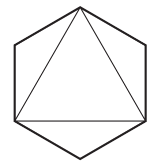
Similarly, $R^{-1} = R^{n-1}, R^{-2} = R^{n-2}, \ldots$, so that $\langle R \rangle = \{e, R, \ldots, R^{n-1}\}$. We see, then, that the powers of R "cycle back" periodically with period n. Visually, raising R to successive positive powers is the same as moving counterclockwise around the following circle one node at a time, whereas raising R to successive negative powers is the same as moving around the circle clockwise one node at a time.



∎

▮ EXAMPLE 13 D_3 is a Subgroup of D_6

Consider the equilateral triangle inscribed in the regular hexagon as shown in the figure below. Let F be a reflection that passes through one vertex of the triangle to the opposite vertex of the hexagon. Then $K = \{R_0, R_{120}, R_{240}, F, R_{120}F, R_{240}F\}$ is a subgroup of D_6.



▮

In Chapter 4 we will show that $|\langle a\rangle| = |a|$; that is, the order of the subgroup generated by a is the order of a itself. (Actually, the definition of $|a|$ was chosen to ensure the validity of this equation.)

For any element a of a group G, it is useful to think of $\langle a\rangle$ as the smallest subgroup of G containing a. This notion can be extended to any collection S of elements from a group G by defining $\langle S\rangle$ as the subgroup of G with the property that $\langle S\rangle$ contains S and if H is any subgroup of G containing S, then H also contains $\langle S\rangle$ Thus, $\langle S\rangle$ is the smallest subgroup of G that contains S. The set $\langle S\rangle$ is called *the subgroup generated by S*. We illustrate this concept in the next example.

▮ EXAMPLE 14 In Z_{20}, $\langle 8, 14\rangle = \{0, 2, 4, \ldots, 18\} = \langle 2\rangle$.
In Z, $\langle 8, 13\rangle = Z$.
In D_4, $\langle H, V\rangle = \{H, H^2, V, HV\} = \{R_0, R_{180}, H, V\}$.
In D_4, $\langle R_{90}, V\rangle = \{R_{90}, R_{90}^2, R_{90}^3, R_{90}^4, V, R_{90}V, R_{90}^2V, R_{90}^3V\} = D_4$.
In **R**, the group of real numbers under addition, $\langle 2, \pi, \sqrt{2}\rangle = \{2a + b\pi + c\sqrt{2} \mid a, b, c \in Z\}$.
In **C**, the group of complex numbers under addition, $\langle 1, i\rangle = \{a + bi \mid a, b \in Z\}$.
(This group is called the "Gaussian integers.")
In **C***, the group of nonzero complex numbers under multiplication, $\langle 1, i\rangle = \{1, -1, i, -i\} = \langle i\rangle$. ▮

We next consider one of the most important subgroups.

Definition Center of a Group
The *center*, $Z(G)$, of a group G is the subset of elements in G that commute with every element of G. In symbols,

$$Z(G) = \{a \in G \mid ax = xa \text{ for all } x \text{ in } G\}.$$

[The notation $Z(G)$ comes from the fact that the German word for center is *Zentrum*. The term was coined by J. A. de Séguier in 1904.]

Theorem 3.5 Center Is a Subgroup

The center of a group G is a subgroup of G.

PROOF For variety, we shall use Theorem 3.2 to prove this result. Clearly, $e \in Z(G)$, so $Z(G)$ is nonempty. Now, suppose $a, b \in Z(G)$. Then $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$ for all x in G; and, therefore, $ab \in Z(G)$.

Next, assume that $a \in Z(G)$. Then we have $ax = xa$ for all x in G. What we want is $a^{-1}x = xa^{-1}$ for all x in G. The desired equation can be obtained from the original one by multiplying it on the left and right by a^{-1}, like so:

$$\begin{aligned} a^{-1}(ax)a^{-1} &= a^{-1}(xa)a^{-1}, \\ (a^{-1}a)xa^{-1} &= a^{-1}x(aa^{-1}), \\ exa^{-1} &= a^{-1}xe, \\ xa^{-1} &= a^{-1}x. \end{aligned}$$

This shows that $a^{-1} \in Z(G)$ whenever a is. ∎

For practice, let's determine the centers of the dihedral groups.

EXAMPLE 15 For $n \geq 3$,

$$Z(D_n) = \begin{cases} \{R_0, R_{180}\} & \text{when } n \text{ is even,} \\ \{R_0\} & \text{when } n \text{ is odd.} \end{cases}$$

To verify this, first observe that since every rotation in D_n is a power of $R_{360/n}$, rotations commute with rotations. We now investigate when a rotation commutes with a reflection. Let R be any rotation in D_n and let F be any reflection in D_n. Observe that since RF is a reflection we have $RF = (RF)^{-1} = F^{-1}R^{-1} =$

FR^{-1}. Thus, it follows that R and F commute if and only if $FR = RF = FR^{-1}$. By cancellation, this holds if and only if $R = R^{-1}$. But $R = R^{-1}$ only when $R = R_0$ or $R = R_{180}$, and R_{180} is in D_n only when n is even. So, we have proved that $Z(D_n) = \{R_0\}$ when n is odd and $Z(D_n) = \{R_0, R_{180}\}$ when n is even. ∎

Although an element from a non-Abelian group does not necessarily commute with every element of the group, there are always some elements with which it will commute. For example, every element a commutes with all powers of a. This observation prompts the next definition and theorem.

Definition Centralizer of *a* in *G*
Let a be a fixed element of a group G. The *centralizer of a in G, $C(a)$*, is the set of all elements in G that commute with a. In symbols, $C(a) = \{g \in G \mid ga = ag\}$.

EXAMPLE 16 In D_4, we have the following centralizers:

$$\begin{aligned}
C(R_0) &= D_4 = C(R_{180}), \\
C(R_{90}) &= \{R_0, R_{90}, R_{180}, R_{270}\} = C(R_{270}), \\
C(H) &= \{R_0, H, R_{180}, V\} = C(V), \\
C(D) &= \{R_0, D, R_{180}, D'\} = C(D').
\end{aligned}$$

∎

Notice that each of the centralizers in Example 16 is actually a subgroup of D_4. The next theorem shows that this was not a coincidence.

Theorem 3.6 $C(a)$ Is a Subgroup

For each a in a group G, the centralizer of a is a subgroup of G.

PROOF A proof similar to that of Theorem 3.5 is left to the reader to supply (Exercise 47). ∎

Notice that for every element a of a group $G, Z(G) \subseteq C(a)$. Also, observe that G is Abelian if and only if $C(a) = G$ for all a in G.

Exercises

The purpose of proof is to understand, not to verify.

Arnold Ross

1. For each group in the following list, find the order of the group and the order of each element in the group. What relation do you see between the orders of the elements of a group and the order of the group? Z_{12}, $U(10)$, $U(12)$, $U(20)$, D_4
2. Let Q be the group of rational numbers under addition and let Q^* be the group of nonzero rational numbers under multiplication. In Q, list the elements in $\langle \frac{1}{2} \rangle$. In Q^*, list the elements in $\langle \frac{1}{2} \rangle$.
3. Let Q and Q^* be as in Exercise 2. Find the order of each element in Q and in Q^*.
4. For every group element a, prove that $|a| = |a^{-1}|$.
5. Let a belong to a group and $|a| = m$. If n is relatively prime to m, show that a can be written as the nth power of some element in the group.
6. Give an example of a group G under addition that has a nonempty subset H that is closed but is not a subgroup of G. Do the same for a group G under multiplication.
7. Without actually computing the orders, explain why the two elements in each of the following pairs of elements from Z_{30} must have the same order: $\{2, 28\}$, $\{8, 22\}$. Do the same for the following pairs of elements from $U(15)$: $\{2, 8\}, \{7, 13\}$.
8. In the group Z_{12}, find $|a|$, $|b|$, and $|a + b|$ for each case.

   a. $a = 6$, $b = 2$ **b.** $a = 3$, $b = 8$ **c.** $a = 5$, $b = 4$

   Do you see any relationship between $|a|$, $|b|$, and $|a + b|$?
9. If a, b, and c are group elements and $|a| = 6$, $|b| = 7$, express $(a^4c^{-2}b^4)^{-1}$ without using negative exponents.
10. Given that $\{e, a, a^2, b, ab, a^2b\}$ is a group where $|a| = 3$, $|b| = 2$ and $ba = a^2b$, determine which of the six elements is aba^2 and which is a^2bab.
11. Find a non-Abelian subgroup of order 6 in D_6.
12. Is $K = \{x^2 \mid x \in D_4\}$ is a subgroup of D_4? Is $K = \{x^3 \mid x \in D_3\}$ is a subgroup of D_3. What about D_6?

13. What can you say about a subgroup of D_4 that contains R_{270} and a reflection? What can you say about a subgroup of D_4 that contains H and D? What can you say about a subgroup of D_4 that contains H and V?

14. How many subgroups of order 4 does D_4 have?

15. Determine all elements of finite order in R^*, the group of nonzero real numbers under multiplication.

16. Complete the statement "A group element x is its own inverse if and only if $|x| =$ ________ ."

17. For any group elements a and x, prove that $|xax^{-1}| = |a|$. This exercise is referred to in Chapter 23.

18. Prove that if a is the only element of order 2 in a group, then a lies in the center of the group.

19. (1969 Putnam Competition) Prove that no group is the union of two proper subgroups. Does the statement remain true if "two" is replaced by "three"?

20. Let G be a group and n an odd positive integer with the property that $x^n = e$ for all x in G. If a and b belong to G and $a^2 = b^2$ prove that $a = b$.

21. For each divisor $k > 1$ of n, let $U_k(n) = \{x \in U(n) \mid x \bmod k = 1\}$. [For example, $U_3(21) = \{1, 4, 10, 13, 16, 19\}$ and $U_7(21) = \{1, 8\}$.] List the elements of $U_4(20)$, $U_5(20)$, $U_5(30)$, and $U_{10}(30)$. Prove that $U_k(n)$ is a subgroup of $U(n)$. (This exercise is referred to in Chapter 8.)

22. Suppose that a is a group element and $a^6 = e$. What are the possibilities for $|a|$? Provide reasons for your answer.

23. If a is a group element and a has infinite order, prove that $a^m \neq a^n$ when $m \neq n$.

24. For any group elements a and b, prove that $|ab| = |ba|$. Explain why this proves that $|abab| = |baba|$. Is it true that $|aba| = |bab|$?

25. If A belong to $GL(2, \mathbf{R})$ and $|A|$ is finite, what can you say about det A?

26. Let a and b be elements of a group and $|b| = 4$. If $a^n = b$, find a positive integer k such that $a^{-1} = a^k$.

27. Show that $U(14) = \langle 3 \rangle = \langle 5 \rangle$. [Hence, $U(14)$ is cyclic.] Is $U(14) = \langle 11 \rangle$? Show that $U(20) \neq \langle k \rangle$ for any k in $U(20)$. [Hence, $U(20)$ is not cyclic.]

28. Suppose n is an even positive integer and H is a subgroup of Z_n. Prove that either every member of H is even or exactly half of the members of H are even.

29. Let n be a positive even integer and let H be a subgroup of Z_n of odd order. Prove that every member of H is an even integer.

30. Prove that for every subgroup of D_n, either every member of the subgroup is a rotation or exactly half of the members are rotations.

31. Let H be a subgroup of D_n of odd order. Prove that every member of H is a rotation.

32. Prove that a group with two elements of order 2 that commute must have a subgroup of order 4.

33. For every even integer n, show that D_n has a subgroup of order 4.

34. Suppose that H is a proper subgroup of Z under addition and H contains 18, 30, and 40. Determine H.

35. Suppose that H is a proper subgroup of Z under addition and that H contains 12, 30, and 54. What are the possibilities for H?

36. Suppose that H is a subgroup of Z under addition and that H contains 2^{50} and 3^{50}. What are the possibilities for H?

37. Prove that the dihedral group of order 6 does not have a subgroup of order 4.

38. If H and K are subgroups of G, show that $H \cap K$ is a subgroup of G. (Can you see that the same proof shows that the intersection of any number of subgroups of G, finite or infinite, is again a subgroup of G?)

39. Let H_1 and H_2 be distinct subgroups of D_n of order 4. What subgroup is $H_1 \cap H_2$?

40. How many subgroups of order 6 does D_6 have? How many does D_{12} have? Generalize to D_n where n is a positive integer divisible by 6.

41. Let G be a group. Show that $Z(G) = \cap_{a \in G} C_{(a)}$. [This means the intersection of *all* subgroups of the form $C(a)$.]

42. Let G be a group, and let $a \in G$. Prove that $C(a) = C(a^{-1})$.

43. For any group element a and any integer k, show that $C(a) \subseteq C(a^k)$. Use this fact to complete the following statement: "In a group, if x commutes with a, then . . . ." Is the converse true?

44. If a and b are distinct group elements, prove that either $a^2 \neq b^2$ or $a^3 \neq b^3$.

45. Let S be a subset of a group and let H be the intersection of all subgroups of G that contain S.
a. Prove that $\langle S \rangle = H$.
b. If S is nonempty, prove that $\langle S \rangle = \{s_1^{n_1} s_2^{n_2} \cdots s_m^{n_m} \mid m \geq 1, s_i \in S, n_i \in Z\}$. (The s_i terms need not be distinct.)

46. In the group Z, find **a.** $\langle 8, 14 \rangle$, **b.** $\langle 8, 13 \rangle$, **c.** $\langle 6, 15 \rangle$, **d.** $\langle m, n \rangle$, **e.** $\langle 12, 18, 45 \rangle$.
In each part, find an integer k such that the subgroup is $\langle k \rangle$.

47. Prove Theorem 3.6.

48. If H is a subgroup of G, then by the *centralizer* $C(H)$ of H we mean the set $\{x \in G \mid xh = hx \text{ for all } h \in H\}$. Prove that $C(H)$ is a subgroup of G.

49. Must the centralizer of an element of a group be Abelian? Must the center of a group be Abelian?

50. Suppose a belongs to a group and $|a| = 5$. Prove that $C(a) = C(a^3)$. Find an element a from some group such that $|a| = 6$ and $C(a) \neq C(a^3)$.

51. Let G be an Abelian group with identity e and let n be some fixed integer. Prove that the set of all elements of G that satisfy the equation $x^n = e$ is a subgroup of G. Give an example of a group G in which the set of all elements of G that satisfy the equation $x^2 = e$ does not form a subgroup of G. (This exercise is referred to in Chapter 11.)

52. In each case, find elements a and b from a group such that $|a| = |b| = 2$ and **a.** $|ab| = 3$, **b.** $|ab| = 4$, **c.** $|ab| = 5$. Can you see any relationship among $|a|$, $|b|$, and $|ab|$?

53. Consider the element $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ in $SL(2, \mathbf{R})$. What is the order of A? If we view $A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ as a member of $SL(2, Z_p)$ (p is a prime), what is the order of A?

54. Consider the elements $A = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}$ from $SL(2, \mathbf{R})$. Find $|A|$, $|B|$, and $|AB|$. Does your answer surprise you?

55. Let a be a group element of order n, and suppose that d is a positive divisor of n. Prove that $|a^d| = n/d$.

56. Give an example of elements a and b from a group such that a has finite order, b has infinite order and ab has finite order.

57. Prove that a group of even order must have an odd number of elements of order 2.

58. For any positive integer n and any angle θ, show that in the group $SL(2, \mathbf{R})$,

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}^n = \begin{bmatrix} \cos n\theta & -\sin n\theta \\ \sin n\theta & \cos n\theta \end{bmatrix}$$

Use this formula to find the order of

$$\begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} \text{ and } \begin{bmatrix} \cos\sqrt{2}^\circ & -\sin\sqrt{2}^\circ \\ \sin\sqrt{2}^\circ & \cos\sqrt{2}^\circ \end{bmatrix}.$$

(Geometrically, $\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$ represents a rotation of the plane θ degrees.)

59. Let G be the symmetry group of a circle. Show that G has elements of every finite order as well as elements of infinite order.

60. In the group $\mathbf{R}^*$ find elements a and b such that $|a| = \infty$, $|b| = \infty$ and $|ab| = 2$.

61. Let G be the symmetry group of a circle. Explain why G contains D_n for all n.

62. Let G be the group of symmetries of a circle. Use the fact from plane geometry that if F and F' are reflections that intersect in an angle of θ degrees then FF' is a rotation of

2θ degrees to find reflections F and F' such that FF' has infinite order.

63. Let H be a subgroup of a finite group G. Suppose that g belongs to G and n is the smallest positive integer such that $g^n \in H$. Prove that n divides $|g|$.

64. Compute the orders of the following groups.
a. $U(3), U(4), U(12)$
b. $U(5), U(7), U(35)$
c. $U(4), U(5), U(20)$
d. $U(4), U(10), U(40)$
On the basis of your answers, make a conjecture about the relationship among $|U(r)|$, $|U(s)|$, and $|U(rs)|$.

65. Let $\mathbf{R}^*$ be the group of nonzero real numbers under multiplication and let $H = \{x \in \mathbf{R}^* \mid x^2 \text{ is rational}\}$. Given an example of an irrational number that is in H. Prove that H is a subgroup of $\mathbf{R}^*$. Can the exponent 2 be replaced by any positive integer and still have H be a subgroup? If $H = \{x \in \mathbf{R}^* \mid x^n \text{ is rational for some integer n}\}$ (n may vary with x) does your proof that H is a subgroup remain valid?

66. Find a noncyclic subgroup of order 4 in $U(40)$.

67. Suppose that a is an element of a group and $a^m = e$. Prove that $|a|$ divides m.

68. Let G be an Abelian group and $H = \{x \in G \mid x^n = e \text{ for some odd integer } n\}$ (n may vary with x). Prove that H is a subgroup of G.

69. Let G be an Abelian group and $H = \{x \in G \mid |x| \text{ is 1 or even}\}$. Give an example to show that H need not be a subgroup of G.

70. Prove that $H = \{x \in \mathbf{C}^* \mid x^n = \pm 1 \text{ for some integer } n\}$ (n may vary with x) is a subgroup of the nonzero complex numbers under multiplication. Let K be any subgroup of $\mathbf{C}^*$. Does your argument work when the condition $x^n = \pm 1$ is replaced by $x^n \in K$.

71. Let G be a group and let H be a subgroup of G. For any fixed x in G, define $xHx^{-1} = \{xhx^{-1} \mid h \in H\}$. Prove the following:

a. xHx^{-1} is a subgroup of G.

b. If H is cyclic, then xHx^{-1} is cyclic.

c. If H is Abelian, then xHx^{-1} is Abelian.

The group xHx^{-1} is called a *conjugate* of H. (Note that conjugation preserves structure.)

72. Let $G = GL(2, \mathbf{R})$ and $H = \{A \in G \mid \det A$ is a positive rational number$\}$. Prove that H is a subgroup of G.

73. Show that the set of all rational numbers that, when expressed in lowest terms, have an odd numerator and an odd denominator is a subgroup of Q^* under multiplication.

74. Let $H = \{A \in GL(2, \mathbf{R})$ and $H = \{A \in G \mid \det A$ is an integer power of $2\}$. Show that H is a subgroup of $GL(2, \mathbf{R})$.

75. Let H be a subgroup of $\mathbf{R}$ under addition. Let $K = \{2^a \mid a \in H\}$. Prove that K is a subgroup of $\mathbf{R}^*$ under multiplication.

76. Let G be a group of functions from $\mathbf{R}^*$ to $\mathbf{R}^*$, where the operation of G is multiplication of functions. Let $H = \{f \in G \mid f(2) = 1\}$. Prove that H is a subgroup of G. Can 2 be replaced by any real number?

77. Let $G = GL(2, \mathbf{R})$ and
$H = \left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| \, a \text{ and } b \text{ are nonzero integers} \right\}$ under the operation of matrix multiplication. Prove or disprove that H is a subgroup of $GL(2, \mathbf{R})$.

78. For any odd integer $n > 3$ give two examples of groups with exactly n elements of order 2.

79. Let $H = \{a + bi \mid a, b \in \mathbf{R},\ a^2 + b^2 = 1\}$. Prove or disprove that H is a subgroup of $\mathbf{C}^*$ under multiplication. Describe the elements of H geometrically.

80. Let G be a finite Abelian group and let a and b belong to G. Prove that the set $\langle a, b \rangle = \{a^i b^j \mid i, j \in Z\}$ is a subgroup of G. What can you say about $|\langle a, b \rangle|$ in terms of $|a|$ and $|b|$?

81. Let n be an integer greater than 1. Find a subgroup of $U(4n)$ of order 4. Is your subgroup cyclic?

82. Give an example to show that if H and K are subgroups of a group G the set $HK = \{hk \mid h \in H,\ k \in K\}$ need not be a subgroup of G.

83. Find an example of a group with elements a and b such that $|a| = 2$, $|b| = 10$, and $|ab| = 2$.

84. If p is an odd prime and n is a positive integer prove that $p^n - 1$ is the unique element in $U(p^n)$ of order 2.

85. If n is an integer at least 3, prove that $2^n - 1$ and $2^{n-1} \pm 1$ are the only elements in $U(2^n)$ of order 2.

86. For parts **a** and **b** below give examples of finite group and an infinite group for which each set is a proper subgroup. For parts **c** and **d** give finite examples.

a. $\{1, \ -1\}$, **b**. $\left\{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}, \begin{bmatrix}-1 & 0\\0 & -1\end{bmatrix}\right\}$, **c**. $\{1, \ 5\}$ **d**. $\{0, \ 3\}$

87. Let H be a subgroup of a group G. Prove that the set $HZ(G) = \{hz \mid h \in H, z \in Z(G\})$ is a subgroup of G. This exercise is referred to in this chapter.

88. If H and K are nontrivial subgroups of the rational numbers under addition, prove that $H \cap K$ is nontrivial.

89. Let H be a nontrival subgroup of the group of rational numbers under addition. Prove that H has a nontrivial proper subgroup.

90. Prove that a group of order n greater than 2 cannot have a subgroup of order $n - 1$.

91. If G is a finite group and x is an element of G give a probabilistic interpretation of $|C(x)|/|G|$. Prove that the probability that any two randomly chosen elements (they can be the same) from D_4 commute is greater than .5. What is the probability that any two elements from D_8 commute?

92. Let G be a finite group with more than one element. Show that G has an element of prime order.

Computer Exercises

Computer exercises for this chapter are available at the website:

http://www.d.umn.edu/~jgallian

4 Cyclic Groups

The notion of a "group," viewed only 30 years ago as the epitome of sophistication, is today one of the mathematical concepts most widely used in physics, chemistry, biochemistry, and mathematics itself.

Alexey Sosinsky, 1991

Indeed, group theory achieved precisely that–a unity and indivisibility of the patterns underlying a wide range of seemingly unrelated disciplines.

Mario Livio, *The Equation That Could Not Be Solved*

Properties of Cyclic Groups

Recall from Chapter 3 that a group G is called *cyclic* if there is an element a in G such that $G = \{a^n \mid n \in Z\}$. Such an element a is called a *generator* of G. In view of the notation introduced in the preceding chapter, we may indicate that G is a cyclic group generated by a by writing $G = \langle a \rangle$.

In this chapter, we examine cyclic groups in detail and determine their important characteristics. We begin with a few examples.

▮ **EXAMPLE 1** The set of integers Z under ordinary addition is cyclic. Both 1 and -1 are generators. (Recall that, when the operation is addition, 1^n is interpreted as

$$\underbrace{1 + 1 + \cdots + 1}_{n \text{ terms}}$$

when n is positive and as

$$\underbrace{(-1) + (-1) + \cdots + (-1)}_{|n| \text{ terms}}$$

when n is negative.) ▮

EXAMPLE 2 The set $Z_n = \{0, 1, \ldots, n-1\}$ for $n \geq 1$ is a cyclic group under addition modulo n. Again, 1 and $-1 = n-1$ are generators.
Unlike Z, which has only two generators, Z_n may have many generators (depending on which n we are given). ∎

EXAMPLE 3 $Z_8 = \langle 1 \rangle = \langle 3 \rangle = \langle 5 \rangle = \langle 7 \rangle$. To verify, for instance, that $Z_8 = \langle 3 \rangle$, we note that $\langle 3 \rangle = \{3, 3+3, 3+3+3, \ldots\}$ is the set $\{3, 6, 1, 4, 7, 2, 5, 0\} = Z_8$. Thus, 3 is a generator of Z_8. On the other hand, 2 is not a generator, since $\langle 2 \rangle = \{0, 2, 4, 6\} \neq Z_8$. ∎

EXAMPLE 4 (See Example 11 in Chapter 2.)

$U(10) = \{1, 3, 7, 9\} = \{3^0, 3^1, 3^3, 3^2\} = \langle 3 \rangle$. Also, $\{1, 3, 7, 9\} = \{7^0, 7^3, 7^1, 7^2\} = \langle 7 \rangle$. So both 3 and 7 are generators for $U(10)$. ∎

Quite often in mathematics, a "nonexample" is as helpful in understanding a concept as an example. With regard to cyclic groups, $U(8)$ serves this purpose; that is, $U(8)$ is not a cyclic group. How can we verify this? Well, note that $U(8) = \{1, 3, 5, 7\}$. But

$$\langle 1 \rangle = \langle 1 \rangle, \ \langle 3 \rangle = \{3, 1\}, \ \langle 5 \rangle = \{5, 1\}, \ \langle 7 \rangle = \{7, 1\},$$

so $U(8) \neq \langle a \rangle$ for any a in $U(8)$.

With these examples under our belts, we are now ready to tackle cyclic groups in an abstract way and state their key properties.

Theorem 4.1 Criterion for $a^i = a^j$

Let G be a group, and let a belong to G. If a has infinite order, then $a^i = a^j$ if and only if $i = j$. If a has finite order, say, n, then $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$ and $a^i = a^j$ if and only if n divides $i - j$.

PROOF If a has infinite order, there is no nonzero n such that a^n is the identity. Since $a^i = a^j$ implies $a^{i-j} = e$, we must have $i - j = 0$, and the first statement of the theorem is proved.

Now assume that $|a| = n$. We will prove that $\langle a \rangle = \{e, a, \ldots, a^{n-1}\}$. Certainly, the elements $e, a, \ldots, a^{n-1}$ are in $\langle a \rangle$.

Next suppose that a^k is an arbitrary member of $\langle a \rangle$. By the division algorithm, there exist integers q and r such that

$$k = nq + r \quad \text{with} \quad 0 \leq r < n.$$

Then $a^k = a^{nq+r} = a^{nq}a^r = (a^n)^q a^r = ea^r = a^r$, so that $a^k \in \{e, a, a^2, \ldots, a^{n-1}\}$. This proves that $\langle a \rangle = \{e, a, a^2, \ldots, a^{n-1}\}$.

Next, we assume that $a^i = a^j$ and prove that n divides $i - j$. We begin by observing that $a^i = a^j$ implies $a^{i-j} = e$. Again, by the division algorithm, there are integers q and r such that

$$i - j = nq + r \quad \text{with} \quad 0 \leq r < n.$$

Then $a^{i-j} = a^{nq+r}$, and therefore $e = a^{i-j} = a^{nq+r} = (a^n)^q a^r = e^q a^r = ea^r = a^r$. Since n is the least positive integer such that a^n is the identity, we must have $r = 0$, so that n divides $i - j$.

Conversely, if $i - j = nq$, then $a^{i-j} = a^{nq} = e^q = e$, so that $a^i = a^j$. ∎

Theorem 4.1 reveals the reason for the dual use of the notation and terminology for the order of an element and the order of a group.

Corollary 1 $|a| = |\langle a \rangle|$

For any group element a, $|a| = |\langle a \rangle|$.

One special case of Theorem 4.1 occurs so often that it deserves singling out.

Corollary 2 $a^k = e$ If and Only If $|a|$ Divides k

For any group element a, $a^k = e$ if and only if $|a|$ divides k.

PROOF We know from Theorem 4.1 that $a^k = e$ if and only if n divides $k - 0 = k$. ∎

In many situations the following reformulation of Corollary 2 is helpful.

Corollary 3 $a^k = e$ If and Only if k is a Multiple of $|a|$

For any group element a, $a^k = e$ if and only if k is a multiple of $|a|$.

Exercises 52 and 54 of Chapter 3 demonstrate that, in general, there is no relationship between $|ab|$ and $|a|$ and $|b|$. Let me say this in a slightly different way. Believe it or not, *knowing only* $|a|$ *and* $|b|$ *tells us nothing about* $|ab|$.

However we have the following useful fact. (See Exercise 23 for a stronger statement.)

Corollary 4 Relationship between $|ab|$ and $|a||b|$

If a and b belong to a finite group and $ab = ba$, then $|ab|$ divides $|a||b|$.

PROOF Let $|a| = m$ and $|b| = n$. Then $(ab)^{mn} = (a^m)^n(b^n)^m = e^n e^m = e$. So, by Corollary 2 of Theorem 4.1 we have that $|ab|$ divides mn. ∎

Theorem 4.1 and its corollaries for the case $|a| = 6$ are illustrated in Figure 4.1.

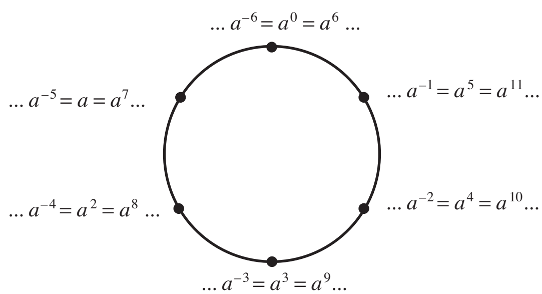


Figure 4.1 Powers of a for $|a| = 6$.

What is important about Theorem 4.1 in the finite case is that it says that multiplication in $\langle a \rangle$ is essentially done by *addition* modulo n. That is, if $(i + j)$ mod $n = k$, then $a^i a^j = a^k$. Thus, no matter what group G is, or how the element a is chosen, multiplication in $\langle a \rangle$ works the same as addition in Z_n whenever $|a| = n$. Similarly, if a has infinite order, then multiplication in

$\langle a \rangle$ works the same as addition in Z, since $a^i a^j = a^{i+j}$ and no modular arithmetic is done.

For these reasons, the cyclic groups Z_n and Z serve as prototypes for all cyclic groups, and algebraists say that there is essentially only one cyclic group of each order. What is meant by this is that, although there may be many different sets of the form $\{a^n \mid n \in Z\}$, there is essentially only one way to operate on these sets. Algebraists do not really care what the elements of a set are; they care only about the algebraic properties of the set—that is, the ways in which the elements of a set can be combined. We will return to this theme in the chapter on isomorphisms (Chapter 6).

The next theorem provides a simple method for computing $|a^k|$ knowing only $|a|$, and its first corollary provides a simple way to tell when $\langle a^i \rangle = \langle a^j \rangle$.

Theorem 4.2 $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$ and $|a^k| = n/\gcd(n,k)$

Let a be an element of order n in a group and let k be a positive integer. Then $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$ and $|a^k| = n/\gcd(n,k)$.

PROOF To simplify the notation, let $d = \gcd(n,k)$ and let $k = dr$. Since $a^k = (a^d)^r$, we have by closure that $\langle a^k \rangle \subseteq \langle a^d \rangle$. By Theorem 0.2 (the gcd theorem), there are integers s and t such that $d = ns + kt$. So,

$$a^d = a^{ns+kt} = a^{ns}a^{kt} = (a^n)^s(a^k)^t = e(a^k)^t = (a^k)^t \in \langle a^k \rangle.$$

This proves $\langle a^d \rangle \subseteq \langle a^k \rangle$. So, we have verified that $\langle a^k \rangle = \langle a^{\gcd(n,k)} \rangle$.

We prove the second part of the theorem by showing first that $|a^d| = n/d$ for any divisor d of n. Clearly, $(a^d)^{n/d} = a^n = e$, so that $|a^d| \leq n/d$. On the other hand, if i is a positive integer less than n/d, then $(a^d)^i \neq e$ by definition of $|a|$. We now apply this fact with $d = \gcd(n,k)$ to obtain $|a^k| = |\langle a^k \rangle| = |\langle a^{\gcd(n,k)} \rangle| = |\langle a^{\gcd(n,k)} | = n/\gcd(n,k)$. ∎

By doing simple arithmetic the next two examples illustrate how Theorem 4.2 allows us to easily list the elements of cyclic subgroups and compute the orders of elements of a cyclic group in cases where the elements are inconvenient to work with.

EXAMPLE 5 For $|a| = 30$ we find $\langle a^{26} \rangle, \langle a^{17} \rangle, \langle a^{18} \rangle$ and $|a^{26}|, |a^{17}|$, and $|a^{18}|$. Since gcd(30,26) $= 2$, we have $\langle a^{26} \rangle =$

$\langle a^2\rangle = \{e, a^2, a^4, a^6, \ldots, a^{28}\}$ and $|a^{26}| = |a^2| = 30/2 = 15$. Since $\gcd(30,17) = 1$, we have $\langle a^{17}\rangle = \langle a^1\rangle = \{e, a, a^2, a^3, \ldots, a^{29}\}$ and $|a^{17}| = |a^1| = 30/1 = 30$. Since $\gcd(30,18) = 6$, we have $\langle a^{18}\rangle = \langle a^6\rangle = \{e, a^6, a^{12}, a^{18}, a^{24}\}$ and $|a^{18}| = |a^6| = 30/6 = 5$. ∎

For large values of n and k we find $\gcd(n, k)$ by using the prime-power factorization of n and k.

EXAMPLE 6 For $|a| = 1000$ we find $\langle a^{140}\rangle, \langle a^{400}\rangle, \langle a^{62}\rangle$ and $|a^{140}|, |a^{400}|$, and $|a^{62}|$. Since $\gcd(1000, 140) = \gcd(2^3 5^3, 2^2 5 \cdot 7) = 2^2 5 = 20$ we have $\langle a^{140}\rangle = \langle a^{20}\rangle = \{e, a^{20}, a^{40}, a^{60}, \ldots, a^{980}\}$ and $|a^{140}| = |a^{20}| = 1000/20 = 50$. Since $\gcd(1000, 400) = \gcd(2^3 5^3, 2^4 5^2) = 2^3 5^2 = 200$ we have $\langle a^{400}\rangle = \langle a^{200}\rangle = \{e, a^{200}, a^{400}, a^{600}, a^{800}\}$ and $|a^{400}| = |a^{200}| = 1000/200 = 5$. Since $\gcd(1000,\ 62) = (2^3 5^3, 2 \cdot 31) = 2$ we have $\langle a^{62}\rangle = \langle a^2\rangle = \{e, a^2, a^4, a^6, \ldots, a^{998}\}$ and $|a^{62}| = |a^2| = 1000/2 = 500$. ∎

Theorem 4.2 establishes an important relationship between the order of an element in a finite cyclic group and the order of the group.

Corollary 1 Orders of Elements in Finite Cyclic Groups

In a finite cyclic group, the order of an element divides the order of the group.

Corollary 2 Criterion for $\langle a^i\rangle = \langle a^j\rangle$ and $|a^i| = |a^j|$

Let $|a| = n$. ***Then*** $\langle a^i\rangle = \langle a^j\rangle$ ***if and only if*** $\gcd(n, i) = \gcd(n, j)$, ***and*** $|a^i| = |a^j|$ ***if and only if*** $\gcd(n, i) = \gcd(n, j)$

PROOF Theorem 4.2 shows that $\langle a^i\rangle = \langle a^{\gcd(n,i)}\rangle$ and $\langle a^j\rangle = \langle a^{\gcd(n,j)}\rangle$, so that the proof reduces to proving that $\langle a^{\gcd(n,i)}\rangle = \langle a^{\gcd(n,j)}\rangle$ if and only if $\gcd(n,\ i) = \gcd(n,\ j)$. Certainly, $\gcd(n,\ i) = \gcd(n,\ j)$ implies that $\langle a^{\gcd(n,i)}\rangle = \langle a^{\gcd(n,j)}\rangle$. On the other hand, $\langle a^{\gcd(n,i)}\rangle = \langle a^{\gcd(n,j)}\rangle$ implies that $|a^{\gcd(n,i)}| = |a^{\gcd(n,j)}|$, so that by the second conclusion of Theorem 4.2, we have $n/\gcd(n, i) = n/\gcd(n, j)$, and therefore $\gcd(n, i)) = \gcd(n, j)$.

The second part of the corollary follows from the first part and Corollary 1 of Theorem 4.1. ∎

The next two corollaries are important special cases of the preceding corollary.

Corollary 3 Generators of Finite Cyclic Groups

Let $|a| = n$. Then $\langle a \rangle = \langle a^j \rangle$ if and only if $\gcd(n, j) = 1$, and $|a| = |\langle a^j \rangle|$ if and only if $\gcd(n, j) = 1$.

Corollary 4 Generators of Z_n

An integer k in Z_n is a generator of Z_n if and only if $\gcd(n, k) = 1$.

The value of Corollary 3 is that once one generator of a cyclic group has been found, all generators of the cyclic group can easily be determined. For example, consider the subgroup of all rotations in D_6. Clearly, one generator is R_{60}. And, since $|R_{60}| = 6$, we see by Corollary 3 that the only other generator is $(R_{60})^5 = R_{300}$. Of course, we could have readily deduced this information without the aid of Corollary 3 by direct calculations. So, to illustrate the real power of Corollary 3, let us use it to find all generators of the cyclic group $U(50)$. First, note that direct computations show that $|U(50)| = 20$ and that 3 is one of its generators. Thus, in view of Corollary 3, the complete list of generators for $U(50)$ is

$$\begin{array}{ll} 3 \bmod 50 = 3, & 3^{11} \bmod 50 = 47, \\ 3^3 \bmod 50 = 27, & 3^{13} \bmod 50 = 23, \\ 3^7 \bmod 50 = 37, & 3^{17} \bmod 50 = 13, \\ 3^9 \bmod 50 = 33, & 3^{19} \bmod 50 = 17. \end{array}$$

Admittedly, we had to do some arithmetic here, but it certainly entailed much less work than finding all the generators by simply determining the order of each element of $U(50)$ one by one.

The reader should keep in mind that Theorem 4.2 and its corollaries apply only to elements of finite order.

Classification of Subgroups of Cyclic Groups

The next theorem tells us how many subgroups a finite cyclic group has and how to find them.

Theorem 4.3 Fundamental Theorem of Cyclic Groups

Every subgroup of a cyclic group is cyclic. Moreover, if $|\langle a\rangle| = n$, then the order of any subgroup of $\langle a\rangle$ is a divisor of n; and, for each positive divisor k of n, the group $\langle a\rangle$ has exactly one subgroup of order k—namely, $\langle a^{n/k}\rangle$.

Before we prove this theorem, let's see what it means. Understanding what a theorem means is a prerequisite to understanding its proof. Suppose $G = \langle a\rangle$ and G has order 30. The first and second parts of the theorem say that if H is any subgroup of G, then H has the form $\langle a^{30/k}\rangle$ for some k that is a divisor of 30. The third part of the theorem says that G has one subgroup of each of the orders 1, 2, 3, 5, 6, 10, 15, and 30—and no others. The proof will also show how to find these subgroups.

PROOF Let $G = \langle a\rangle$ and suppose that H is a subgroup of G. We must show that H is cyclic. If it consists of the identity alone, then clearly H is cyclic. So we may assume that $H \neq \{e\}$. We now claim that H contains an element of the form a^t, where t is positive. Since $G = \langle a\rangle$, every element of H has the form a^t; and when a^t belongs to H with $t < 0$, then a^{-t} belongs to H also and $-t$ is positive. Thus, our claim is verified. Now let m be the least positive integer such that $a^m \in H$. By closure, $\langle a^m\rangle \subseteq H$. We next claim that $H = \langle a^m\rangle$. To prove this claim, it suffices to let b be an arbitrary member of H and show that b is in $\langle a^m\rangle$. Since $b \in G = \langle a\rangle$, we have $b = a^k$ for some k. Now, apply the division algorithm to k and m to obtain integers q and r such that $k = mq + r$ where $0 \le r < m$. Then $a^k = a^{mq+r} = a^{mq}a^r$, so that $a^r = a^{-mq}a^k$. Since $a^k = b \in H$ and $a^{-mq} = (a^m)^{-q}$ is in H also, $a^r \in H$. But, m is the *least* positive integer such that $a^m \in H$, and $0 \le r < m$, so r must be 0. Therefore, $b = a^k = a^{mq} = (a^m)^q \in \langle a^m\rangle$. This proves the assertion of the theorem that every subgroup of a cyclic group is cyclic.

To prove the next part of the theorem, suppose that $|\langle a\rangle| = n$ and H is any subgroup of $\langle a\rangle$. By the argument in the preceding paragraph and Theorem 4.2 we know that H is cyclic and has the form $\langle a^m\rangle$ where m divides n and $|a^m| = n/m$. Finally, observe that if k is any positive divisor of n then $|\langle a^{n/k}\rangle| = k$ and if K is any subgroup of $\langle a\rangle$ of order k then K has the form $\langle a^s\rangle$ where s divides n and $|\langle a^s\rangle| = n/s = k$. So, $s = n/k$. ∎

Returning for a moment to our discussion of the cyclic group $\langle a \rangle$, where a has order 30, we may conclude from Theorem 4.3 that the subgroups of $\langle a \rangle$ are precisely those of the form $\langle a^m \rangle$, where m is a divisor of 30. Moreover, if k is a divisor of 30, the subgroup of order k is $\langle a^{30/k} \rangle$. So the list of subgroups of $\langle a \rangle$ is:

$$\begin{array}{rcll}
\langle a \rangle & = & \{e, a, a^2, \ldots, a^{29}\} & \text{order } 30, \\
\langle a^2 \rangle & = & \{e, a^2, a^4, \ldots, a^{28}\} & \text{order } 15, \\
\langle a^3 \rangle & = & \{e, a^3, a^6, \ldots, a^{27}\} & \text{order } 10, \\
\langle a^5 \rangle & = & \{e, a^5, a^{10}, a^{15}, a^{20}, a^{25}\} & \text{order } 6, \\
\langle a^6 \rangle & = & \{e, a^6, a^{12}, a^{18}, a^{24}\} & \text{order } 5, \\
\langle a^{10} \rangle & = & \{e, a^{10}, a^{20}\} & \text{order } 3, \\
\langle a^{15} \rangle & = & \{e, a^{15}\} & \text{order } 2, \\
\langle a^{30} \rangle & = & \{e\} & \text{order } 1.
\end{array}$$

In general, if $\langle a \rangle$ has order n and k divides n, then $\langle a^{n/k} \rangle$ is the unique subgroup of order k.

Taking the group in Theorem 4.3 to be Z_n and a to be 1, we obtain the following important special case.

Corollary Subgroups of Z_n

For each positive divisor k of n, the set $\langle n/k \rangle$ is the unique subgroup of Z_n of order k; moreover, these are the only subgroups of Z_n.

EXAMPLE 7 The list of subgroups of Z_{30} is

$$\begin{array}{rcll}
\langle 1 \rangle & = & \{0, 1, 2, \ldots, 29\} & \text{order } 30, \\
\langle 2 \rangle & = & \{0, 2, 4, \ldots, 28\} & \text{order } 15, \\
\langle 3 \rangle & = & \{0, 3, 6, \ldots, 27\} & \text{order } 10, \\
\langle 5 \rangle & = & \{0, 5, 10, 15, 20, 25\} & \text{order } 6, \\
\langle 6 \rangle & = & \{0, 6, 12, 18, 24\} & \text{order } 5, \\
\langle 10 \rangle & = & \{0, 10, 20\} & \text{order } 3, \\
\langle 15 \rangle & = & \{0, 15\} & \text{order } 2, \\
\langle 30 \rangle & = & \{0\} & \text{order } 1.
\end{array}$$

∎

Theorems 4.2 and 4.3 provide a simple way to find all the generators of the subgroups of a finite cyclic group.

EXAMPLE 8 To find the generators of the subgroup of order 9 in Z_{36}, we observe that $36/9 = 4$ is one generator. To find the others, we have from Corollary 3 of Theorem 4.2 that they are all elements of Z_{36} of the form $4j$, where $\gcd(9, j) = 1$. Thus,

$$\langle 4 \cdot 1\rangle = \langle 4 \cdot 2\rangle = \langle 4 \cdot 4\rangle = \langle 4 \cdot 5\rangle = \langle 4 \cdot 7\rangle = \langle 4 \cdot 8\rangle.$$

In the generic case, to find all the subgroups of $\langle a\rangle$ of order 9 where $|a| = 36$, we have

$$\langle (a^4)^1\rangle = \langle (a^4)^2\rangle = \langle (a^4)^4\rangle = \langle (a^4)^5\rangle = \langle (a^4)^7\rangle = \langle (a^4)^8\rangle.$$

In particular, note that once you have the generator $a^{n/d}$ for the subgroup of order d where d is a divisor of $|a| = n$, all the generators of $\langle a^d\rangle$ have the form $(a^d)^j$ where $j \in U(d)$. ∎

By combining Theorems 4.2 and 4.3, we can easily count the number of elements of each order in a finite cyclic group. For convenience, we introduce an important number-theoretic function called the *Euler phi function*. Let $\phi(1) = 1$, and for any integer $n > 1$, let $\phi(n)$ denote the number of positive integers less than n and relatively prime to n. Notice that by definition of the group $U(n)$, $|U(n)| = \phi(n)$. The first 12 values of $\phi(n)$ are given in Table 4.1.

Table 4.1 Values of $\phi(n)$.

n	1	2	3	4	5	6	7	8	9	10	11	12
$\phi(n)$	1	1	2	2	4	2	6	4	6	4	10	4

Theorem 4.4 Number of Elements of Each Order in a Cyclic Group

If d is a positive divisor of n, the number of elements of order d in a cyclic group of order n is $\phi(d)$.

PROOF By Theorem 4.3, the group has exactly one subgroup of order d— call it $\langle a\rangle$. Then every element of order d also generates the subgroup $\langle a\rangle$ and, by Corollary 3 of Theorem 4.2, an element a^k generates $\langle a\rangle$ if and only if $\gcd(k, d) = 1$. The number of such elements is precisely $\phi(d)$. ∎

Notice that for a finite cyclic group of order n, the number of elements of order d for any divisor d of n depends only on d. Thus, Z_8, Z_{640}, and Z_{80000} each have $\phi(8) = 4$ elements of order 8.

Although there is no formula for the number of elements of each order for arbitrary finite groups, we still can say something important in this regard.

Corollary 4.1 Number of Elements of Order d in a Finite Group

In a finite group, the number of elements of order d is a multiple of $\phi(d)$.

PROOF If a finite group has no elements of order d, the statement is true, since $\phi(d)$ divides 0. Now suppose that $a \in G$ and $|a| = d$. By Theorem 4.4, we know that $\langle a \rangle$ has $\phi(d)$ elements of order d. If all elements of order d in G are in $\langle a \rangle$, we are done. So, suppose that there is an element b in G of order d that is not in $\langle a \rangle$. Then, $\langle b \rangle$ also has $\phi(d)$ elements of order d. This means that we have found $2\phi(d)$ elements of order d in G provided that $\langle a \rangle$ and $\langle b \rangle$ have no elements of order d in common. If there is an element c of order d that belongs to both $\langle a \rangle$ and $\langle b \rangle$, then we have $\langle a \rangle = \langle c \rangle = \langle b \rangle$, so that $b \in \langle a \rangle$, which is a contradiction. Continuing in this fashion, we see that the number of elements of order d in a finite group is a multiple of $\phi(d)$. ■

Theorem 4.4 together with the two number theory properties that for any prime p, $\phi(p^n) = p^n - p^{n-1}$ (see Exercise 85), and $\phi(p_1^{k_1} p_2^{k_2} \cdots p_m^{k_m}) = \phi(p_1^{k_1})\phi(p_2^{k_2}) \cdots \phi(p_m^{k_m})$ for distinct primes $p_1, p_2, \ldots, p_m$, greatly simplify the task of determining orders of element in $U(n)$ and whether or not $U(n)$ is cyclic.

EXAMPLE 9 Consider the element 3 in $U(50)$. Since $\phi(50) = \phi(2)\phi(5^2)$ we know that $|U(50)| = 1 \cdot (5^2 - 5) = 20$. So, $|3|$ must be 2, 4, 5, 10 or 20. Since $3^4 = 31$, $|3|$ is not 2 or 4. The observation that, modulo 50, $3^{10} = 3^5 \cdot 3^5 = 243 \cdot 243 = (-7)(-7) = 49$ rules out 5 and 10 as possible orders for $|3|$. So $|3| = 20$ and $U(50) = \langle 3 \rangle$. Likewise, to find $|2|$ in $U(13)$, which has order 12, we note that $2^4 = 3$ rules out orders 2 and 4 and $2^6 = 64 = 12$ rules out 3 and 6. So, $|2| = 12$.

On the other hand, for $U(n)$ $(n > 2)$ to be cyclic, we have from Theorem 4.3 that $n-1 = -1$ must be the unique element of order 2 in $U(n)$. But, because $9^2 = 1$ in $U(80)$, and $11^2 = 1$ in $U(120)$, neither is cyclic. Notice that we did not even have to know the orders of $U(80)$ and $U(120)$ to determine that they weren't cyclic. ∎

In Chapter 8 we will give a complete classification of U-groups which are cyclic.

The relationships among the various subgroups of a group can be illustrated with a *subgroup lattice* of the group. This is a diagram that includes all the subgroups of the group and connects a subgroup H at one level to a subgroup K at a higher level with a sequence of line segments if and only if H is a proper subgroup of K. Although there are many ways to draw such a diagram, the connections between the subgroups must be the same. Typically, one attempts to present the diagram in an eye-pleasing fashion. The lattice diagram for Z_{30} is shown in Figure 4.2. Notice that $\langle 10\rangle$ is a subgroup of both $\langle 2\rangle$ and $\langle 5\rangle$, but $\langle 6\rangle$ is not a subgroup of $\langle 10\rangle$.

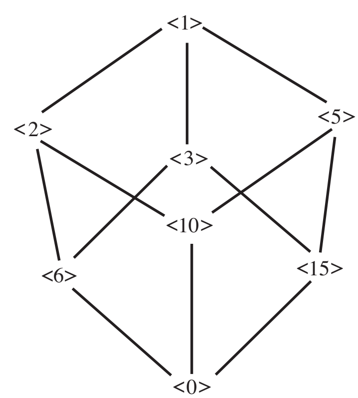


Figure 4.2 Subgroup lattice of Z_{30}.

The precision of Theorem 4.3 can be appreciated by comparing the ease with which we are able to identify the subgroups of Z_{30} with that of doing the same for, say, $U(30)$ or D_{30}. And these groups have relatively simple structures among noncyclic groups.

We will prove in Chapter 7 that a certain portion of Theorem 4.3 extends to arbitrary finite groups; namely, the order of a subgroup divides the order of the group itself. We will also see, however, that a finite group need not have exactly one subgroup corresponding to each divisor of the order of the group. For some divisors, there may be none at all, whereas for other divisors, there may be many. Indeed, D_4, the dihedral group of order 8, has five subgroups of order 2 and three of order 4.

One final remark about the importance of cyclic groups is appropriate. Although cyclic groups constitute a very narrow class of finite groups, we will see in Chapter 11 that they play the role of building blocks for all finite Abelian groups in much the same way that primes are the building blocks for the integers and that chemical elements are the building blocks for the chemical compounds.

Exercises

It is not unreasonable to use the hypothesis.

Arnold Ross

1. Find all generators of Z_6, Z_8, and Z_{20}.

2. Suppose that $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$ are cyclic groups of orders 6, 8, and 20, respectively. Find all generators of $\langle a \rangle$, $\langle b \rangle$, and $\langle c \rangle$.

3. List the elements of the subgroups $\langle 20 \rangle$ and $\langle 10 \rangle$ in Z_{30}. Let a be a group element of order 30. List the elements of the subgroups $\langle a^{20} \rangle$ and $\langle a^{10} \rangle$.

4. List the elements of the subgroups $\langle 3 \rangle$ and $\langle 15 \rangle$ in Z_{18}. Let a be a group element of order 18. List the elements of the subgroups $\langle a^3 \rangle$ and $\langle a^{15} \rangle$.

5. List the elements of the subgroups $\langle 3 \rangle$ and $\langle 7 \rangle$ in $U(20)$.

6. What do Exercises 3, 4, and 5 have in common? Try to make a generalization that includes these three cases.

7. Find an example of a noncyclic group, all of whose proper subgroups are cyclic.

8. Let a be an element of a group and let $|a| = 15$. Compute the orders of the following elements of G.
a. a^3, a^6, a^9, a^{12}
b. a^5, a^{10}
c. a^2, a^4, a^8, a^{14}

9. How many subgroups does Z_{20} have? List a generator for each of these subgroups. Suppose that $G = \langle a \rangle$ and $|a| = 20$. How many subgroups does G have? List a generator for each of these subgroups.

10. In Z_{24}, list all generators for the subgroup of order 8. Let $G = \langle a \rangle$ and let $|a| = 24$. List all generators for the subgroup of order 8.

11. Let G be a group and let $a \in G$. Prove that $\langle a^{-1} \rangle = \langle a \rangle$.

12. In Z, find all generators of the subgroup $\langle 3 \rangle$. If a has infinite order, find all generators of the subgroup $\langle a^3 \rangle$.

13. In Z, find a generator of the subgroup $\langle 10 \rangle \cap \langle 12 \rangle$. In general, what is a generator of the subgroup $\langle m \rangle \cap \langle n \rangle$? If a is a group element of infinite order, what is a generator of the subgroup $\langle a^m \rangle \cap \langle a^n \rangle$?

14. Suppose that a cyclic group G has exactly three subgroups: G itself, $\{e\}$, and a subgroup of order 7. What is $|G|$? What can you say if 7 is replaced with p where p is a prime?

15. Let G be an Abelian group and let $H = \{g \in G \mid |g| \text{ divides } 12\}$. Prove that H is a subgroup of G. Is there anything special about 12 here? Would your proof be valid if 12 were replaced by some other positive integer? State the general result.

16. Let a be an element of a group.
   a. Complete the statement: $|a| = |a^2|$ if and only if $|a| \ldots$
   b. Complete the statement: $|a^2| = |a^{12}|$ if and only if $\ldots$
   c. If $|a| = \infty$ complete the statement: $|a^i| = |a^j|$ if and only if $\ldots$
   d. If $|a| = \infty$ complete the statement: $\langle a^i \rangle = \langle a^j \rangle$ if and only if $\ldots$

17. Let a be a generator of a cyclic group G. If $|G| = n$ is odd and $\langle a^6 \rangle$ is a proper subgroup of $\langle a \rangle$ determine $|\langle a^6 \rangle|$.

18. Let a be a generator of a cyclic group G. If $|G| = n$ and $\langle a^{10} \rangle = \langle a^5 \rangle$ prove that n is odd.

19. Let n be an element of a group. If $|a^2| = 3$, what are the possibilities for $|a|$? What are they if $|a^2| = 4$?

20. Let p be prime and n a positive integer. How many cyclic subgroups does D_{p^n} have? If p and q are distinct primes how many cyclic subgroups does D_{pq} have?

21. Prove that if G is a group with the property that the square of every element is the identity then G is Abelian.

22. Determine $\phi(81)$, $\phi(60)$, $\phi(105)$ where ϕ is the Euler phi function.

23. Let a and b elements of a finite group and a and b commute. Prove that $|ab| = |a||b|$ if and only if $\gcd(|a|, |b|) = 1$.

24. Suppose that G is a group with more than one element and G has no proper, nontrivial subgroups. Prove that $|G|$ is prime. (Do not assume at the outset that G is finite.)

25. Let H be a subgroup of D_n of odd order. Use Exercise 31 in Chapter 3 to prove that every subgroup of H is cyclic.

26. Give infinitely many examples of Abelian groups that have exactly 2 elements of order 3. Do the same for infinitely many non-Abelian groups. Do your examples generalize to the case of exactly $p - 1$ elements of order p where p is an odd prime?

27. If a cyclic group has an element of infinite order, how many elements of finite order does it have?

28. Suppose that G is an Abelian group of order 35 and every element of G satisfies the equation $x^{35} = e$. Prove that G is cyclic. Does your argument work if 35 is replaced with 33?

29. Let G be a group and let a be an element of G.
a. If $a^{12} = e$, what can we say about the order of a?
b. If $a^m = e$, what can we say about the order of a?
c. Suppose that $|G| = 24$ and that G is cyclic. If $a^8 \neq e$ and $a^{12} \neq e$, show that $\langle a \rangle = G$.

30. Prove that a group of order 3 must be cyclic.

31. Let Z denote the group of integers under addition. Is every subgroup of Z cyclic? Why? Describe all the subgroups of Z. Let a be a group element with infinite order. Describe all subgroups of $\langle a \rangle$.

32. For any element a in any group G, prove that $\langle a \rangle$ is a subgroup of $C(a)$ (the centralizer of a).

33. If d is a positive integer, $d \neq 2$, and d divides n, show that the number of elements of order d in D_n is $\phi(d)$. How many elements of order 2 does D_n have?

34. Find all generators of Z. Let a be a group element that has infinite order. Find all generators of $\langle a \rangle$.

35. Prove that C^*, the group of nonzero complex numbers under multiplication, has a cyclic subgroup of order n for every positive integer n.

36. Supose that there are infinitely many subgroups H_1, H_2, H_3, ... of a group G such that $H_1 \subset H_2 \subset H_3 \subset \cdots$. Prove that $H = \cup_{i=1}^{\infty} H_i$ is a subgroup of G.

37. List all the elements of order 8 in $Z_{8000000}$. How do you know your list is complete? Let a be a group element such that $|a| = 8000000$. List all elements of order 8 in $\langle a \rangle$. How do you know your list is complete?

38. Suppose that G is a group with more than one element. If the only subgroups of G are $\{e\}$ and G, prove that G is cyclic and has prime order.

39. Let G be a finite group. Show that there exists a fixed positive integer n such that $a^n = e$ for all a in G. (Note that n is independent of a.)

40. Determine the subgroup lattice for Z_{12}. Generalize to Z_{p^2q}, where p and q are distinct primes.

41. Determine the subgroup lattice for Z_8. Generalize to Z_{p^n}, where p is a prime and n is some positive integer.

42. Prove that a finite group is the union of proper subgroups if and only if the group is not cyclic.

43. Show that the group of positive rational numbers under multiplication is not cyclic. Why does this prove that the group of nonzero rationals under multiplication is not cyclic?

44. Consider the set $\{4, 8, 12, 16\}$. Show that this set is a group under multiplication modulo 20 by constructing its Cayley table. What is the identity element? Is the group cyclic? If so, find all of its generators.

45. Give an example of a group that has exactly 7 subgroups (including the trivial subgroup and the group itself). Generalize to exactly n subgroups for any positive integer n.

46. Suppose that a and b belong to a group G, a and b commute, and $|a|$ and $|b|$ are finite. What are the possibilities for $|ab|$?

47. Suppose that a and b are group elements that commute. If $|a|$ is finite and $|b|$ infinite, prove that $|ab|$ has infinite order.

48. For a positive integer k and an Abelian group G prove that $G^k = \{x^k \mid x \in G\}$ is a subgroup of G. If G is Abelian and $|G| = 40$, prove that $G^3 = G$, $G^{15} = G^5$, and $G^{28} = G^4$. Generalize to the case that $|G| = n$ and k is any positive integer.

49. Let a belong to a group and $|a| = 100$. Find $|a^{98}|$ and $|a^{70}|$.

50. Let F and F' be distinct reflections in D_{21}. What are the possibilities for $|FF'|$?

51. Suppose that H is a cyclic subgroup of a group G and $|H| = 10$. If a belongs to G and a^6 belongs to H, what are the possibilities for $|a|$?

52. Which of the following numbers could be the exact number of elements of order 21 in a group: 21600, 21602, 21604?

53. If G is an infinite group, what can you say about the number of elements of order 8 in the group? Generalize.

54. If G is a cyclic group of order n, prove that for every element a in G, $a^n = e$.

55. For each positive integer n, prove that $\boldsymbol{C}^*$, the group of nonzero complex numbers under multiplication, has exactly $\phi(n)$ elements of order n.

56. Prove or disprove that $H = \{n \in Z \mid n \text{ is divisible by both } 8 \text{ and } 10\}$ is a subgroup of Z. What happens if "divisible by both 8 and 10" is changed to "divisible by 8 or 10?"

57. Suppose that G is a finite group with the property that every nonidentity element has prime order (e.g., D_3 and D_5). If $Z(G)$ is not trivial, prove that every nonidentity element of G has the same order.

58. Prove that an infinite group must have an infinite number of subgroups.

59. Let p be a prime. If a group has more than $p - 1$ elements of order p, why can't the group be cyclic?

60. Suppose that G is a cyclic group and that 6 divides $|G|$. How many elements of order 6 does G have? If 8 divides $|G|$, how many elements of order 8 does G have? If a is one element of order 8, list the other elements of order 8.

61. List all the elements of Z_{40} that have order 10. Let $|x| = 40$. List all the elements of $\langle x \rangle$ that have order 10.

62. Reformulate the corollary of Theorem 4.4 to include the case when the group has infinite order.

63. Determine the orders of the elements of D_{33} and how many there are of each.

64. When checking to see if $\langle 2 \rangle = U(25)$ explain why it is sufficient to check that $2^{10} \neq 1$ and $2^4 \neq 1$.

65. If G is an Abelian group and contains cyclic subgroups of orders 4 and 5, what other sizes of cyclic subgroups must G contain? Generalize.

66. If G is an Abelian group and contains cyclic subgroups of orders 4 and 6, what other sizes of cyclic subgroups must G contain? Generalize.

67. Prove that no group can have exactly two elements of order 2.

68. Given the fact that $U(49)$ is cyclic and has 42 elements, deduce the number of generators that $U(49)$ has without actually finding any of the generators.

69. Let a and b be elements of a group. If $|a| = 10$ and $|b| = 21$, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

70. Let a and b belong to a group. If $|a|$ and $|b|$ are relatively prime, show that $\langle a \rangle \cap \langle b \rangle = \{e\}$.

71. Let a and b belong to a group. If $|a| = 24$ and $|b| = 10$, what are the possibilities for $|\langle a \rangle \cap \langle b \rangle|$?

72. Let a and b belong to a group. If $|a| = 12$, $|b| = 22$, and $\langle a \rangle \cap \langle b \rangle \neq \{e\}$, prove that $a^6 = b^{11}$.

73. Let G be an Abelian group of order 15. Prove that G cannot have 14 elements of order 3.

74. Prove that Z_n has an even number of generators if $n > 2$. What does this tell you about $\phi(n)$?

75. If $|a^5| = 12$, what are the possibilities for $|a|$? If $|a^4| = 12$, what are the possibilities for $|a|$?

76. Suppose that $|x| = n$. Find a necessary and sufficient condition on r and s such that $\langle x^r \rangle \subseteq \langle x^s \rangle$.

77. Let a be a group element such that $|a| = 48$. For each part, find a divisor k of 48 such that
a. $\langle a^{21} \rangle = \langle a^k \rangle$,
b. $\langle a^{14} \rangle = \langle a^k \rangle$,
c. $\langle a^{18} \rangle = \langle a^k \rangle$.

78. Find a cyclic subgroup of D_{60} of order 4 and a non-cyclic subgroup of D_{60} of order 4.

79. Let G be an Abelian group and $H = \{x \in G \mid |x| \text{ is odd}\}$. Prove that H is a subgroup of G.

80. Let G be an Abelian group and $H = \{x \in G \mid |x| \text{ is 1 or even}\}$. Give an example to show that H need not be a subgroup of G.

81. Let G be a group and H a subgroup of G. Prove that $HZ(G) = \{hz \mid h \in H \;\; z \in Z(G)\}$ is a subgroup of G.

82. For every integer n greater than 2, prove that the group $U(n^2 - 1))$ is not cyclic.

83. For every integer $n > 2$, show that $U(n^2 - 1)$ has a subgroup of order 4.

84. Prove that $U(2^n)$ $(n \geq 3)$ is not cyclic.

85. Prove that for any prime p and positive integer n, $\phi(p^n) = p^n - p^{n-1}$.

86. Let ϕ be the Euler ϕ-function. For any positive integer n prove that $|U(|U(n)|)| = \phi(\phi(n))$.

87. Let ϕ be the Euler phi function. For each positive integer n prove that $n = \Sigma\phi(d)$ where the sum ranges over all positive divisors d of n.

88. If n is an even integer prove that $\phi(2n) = 2\phi(n)$.

89. (2008 GRE Practice Exam) If x is an element of a cyclic group of order 15 and exactly two of x^3, x^5 and x^9 are equal, determine $|x^{13}|$.

Computer Exercises

Computer exercises for this chapter are available at the website:

http://www.d.umn.edu/∼jgallian

James Joseph Sylvester

Stock Montage

I really love my subject.

J. J. SYLVESTER

James Joseph Sylvester was the most influential mathematician in America in the 19th century. Sylvester was born on September 3, 1814, in London and showed his mathematical genius early. At the age of 14, he studied under De Morgan and won several prizes for his mathematics, and at the unusually young age of 25, he was elected a fellow of the Royal Society.

After receiving B.A. and M.A. degrees from Trinity College in Dublin in 1841, Sylvester began a professional life that was to include academics, law, and actuarial careers. In 1876, at the age of 62, he was appointed to a prestigious position at the newly founded Johns Hopkins University. During his seven years at Johns Hopkins, Sylvester pursued research in pure mathematics with tremendous vigor and enthusiasm.

He also founded the *American Journal of Mathematics*, the first journal in America devoted to mathematical research. Sylvester returned to England in 1884 to a professorship at Oxford, a position he held until his death on March 15, 1897. Sylvester's major contributions to mathematics were in the theory of equations, matrix theory, determinant theory, and invariant theory (which he founded with Cayley. His writings and lectures—flowery and eloquent, pervaded with poetic flights, emotional expressions, bizarre utterances, and paradoxes—reflected the personality of this sensitive, excitable, and enthusiastic man.

5 Permutation Groups

Wigner's discovery about the electron permutation group was just the beginning. He and others found many similar applications and nowadays group theoretical methods—especially those involving characters and representations—pervade all branches of quantum mechanics.

George Mackey, *Proceedings of the American Philosophical Society*

Symmetry has been the scientists' pillar of fire, leading toward relativity and the standard model.

Mario Livio, *The Equation That Could Not Be Solved*

Definitions and Notation

At this point we know two families of finite Abelian groups, Z_n and $U(n)$, and two families of infinite non-Abelian groups, dihedral groups and matrix groups. In this chapter we introduce a family of non-Abelian groups of certain functions called permutation groups from a set A to itself. In the early and mid-19th century, groups of permutations were the only groups investigated by mathematicians. It was not until around 1850 that the notion of an abstract group was introduced by Cayley, and it took another quarter century before the idea firmly took hold. Permutation groups are important in the theory of groups and in computer science, physics, and chemistry.

Definitions Permutation of A, Permutation Group of A
A *permutation* of a set A is a function from A to A that is both one-to-one and onto. A *permutation group* of a set A is a set of permutations of A that forms a group under function composition.

Although groups of permutations of any nonempty set A of objects exist, we will focus on the case where A is finite. Furthermore, it is customary, as well as convenient, to take A to be a set of the form $\{1, 2, 3, \ldots, n\}$ for some positive integer n. Unlike in calculus, where most functions are defined on infinite sets and are given by formulas, in algebra, permutations of finite sets are usually given by an explicit listing of each element of the domain and its corresponding functional value. For example, we define a permutation α of the set $\{1, 2, 3, 4\}$ by specifying

$$\alpha(1) = 2, \quad \alpha(2) = 3, \quad \alpha(3) = 1, \quad \alpha(4) = 4.$$

A more convenient way to express this correspondence is to write α in array form as

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 1 & 4 \end{bmatrix}.$$

Here $\alpha(j)$ is placed directly below j for each j. Similarly, the permutation β of the set $\{1, 2, 3, 4, 5, 6\}$ given by

$$\beta(1) = 5, \quad \beta(2) = 3, \quad \beta(3) = 1, \quad \beta(4) = 6, \quad \beta(5) = 2, \quad \beta(6) = 4$$

is expressed in array form as

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{bmatrix}.$$

Composition of permutations expressed in array notation is carried out from right to left by going from top to bottom, then again from top to bottom. For example, let

$$\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 4 & 3 & 5 & 1 \end{bmatrix}$$

and

$$\gamma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{bmatrix},$$

then

$$\gamma\sigma = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ & \downarrow & & & \\ 5 & 4 & 1 & 2 & 3 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ \downarrow & & & & \\ 2 & 4 & 3 & 5 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 1 & 3 & 5 \end{bmatrix}$$

On the right we have 4 under 1, since $(\gamma\sigma)(1) = \gamma(\sigma(1)) = \gamma(2) = 4$, so $\gamma\sigma$ sends 1 to 4. The remainder of the bottom row $\gamma\sigma$ is obtained in a similar fashion.

We are now ready to give some examples of permutation groups.

EXAMPLE 1 Symmetric Group $\boldsymbol{S_3}$ Let S_3 denote the set of all one-to-one functions from $\{1, 2, 3\}$ to itself. Then S_3, under function composition, is a group with six elements. The six elements are

$$\varepsilon = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}, \quad \alpha = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{bmatrix}, \quad \alpha^2 = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 1 & 2 \end{bmatrix},$$

$$\beta = \begin{bmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{bmatrix}, \quad \alpha\beta = \begin{bmatrix} 1 & 2 & 3 \\ 2 & 1 & 3 \end{bmatrix}, \quad \alpha^2\beta = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix}.$$

Note that $\beta\alpha = \begin{bmatrix} 1 & 2 & 3 \\ 3 & 2 & 1 \end{bmatrix} = \alpha^2\beta \neq \alpha\beta$, so that S_3 is non-Abelian. ∎

The relation $\beta\alpha = \alpha^2\beta$ can be used to compute other products in S_3 without resorting to the arrays. For example, $\beta\alpha^2 = (\beta\alpha)\alpha = (\alpha^2\beta)\alpha = \alpha^2(\beta\alpha) = \alpha^2(\alpha^2\beta) = \alpha^4\beta = \alpha\beta$.

Example 1 can be generalized as follows.

EXAMPLE 2 Symmetric Group $\boldsymbol{S_n}$ Let $A = \{1, 2, \ldots, n\}$. The set of all permutations of A is called the *symmetric group of degree* n and is denoted by S_n. Elements of S_n have the form

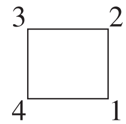
$$\alpha = \begin{bmatrix} 1 & 2 & \ldots & n \\ \alpha(1) & \alpha(2) & \ldots & \alpha(n) \end{bmatrix}.$$

It is easy to compute the order of S_n. There are n choices of $\alpha(1)$. Once $\alpha(1)$ has been determined, there are $n-1$ possibilities for $\alpha(2)$ [since α is one-to-one, we must have $\alpha(1) \neq \alpha(2)$]. After choosing $\alpha(2)$, there are exactly $n-2$ possibilities for $\alpha(3)$. Continuing along in this fashion, we see that S_n has $n(n-1)\cdots 3\cdot 2\cdot 1 = n!$ elements.[1] We leave it to the reader to prove that S_n is non-Abelian when $n \geq 3$ (Exercise 51). ∎

The symmetric groups are rich in subgroups. The group S_4 has 30 subgroups, and S_5 has well over 100 subgroups.

[1] The number of elements in S_{60} is roughly equal to the number of atoms in the universe!

EXAMPLE 3 Symmetries of a Square As a third example, we associate each motion in D_4 with the permutation of the locations of each of the four corners of a square. For example, if we label the four corner positions as in the figure below and keep these labels fixed for reference, we may describe a $90°$ counterclockwise rotation by the permutation



$$\rho = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{bmatrix},$$

whereas a reflection across a horizontal axis yields

$$\phi = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 1 & 4 & 3 \end{bmatrix}.$$

These two elements generate the entire group (i.e., every element is some combination of the ρ's and ϕ's).
When D_4 is represented in this way, we see that it is a subgroup of S_4. ∎

Cycle Notation

There is another notation commonly used to specify permutations. It is called *cycle notation* and was first introduced by the great French mathematician Cauchy in 1815. Cycle notation has theoretical advantages in that certain important properties of the permutation can be readily determined when cycle notation is used.

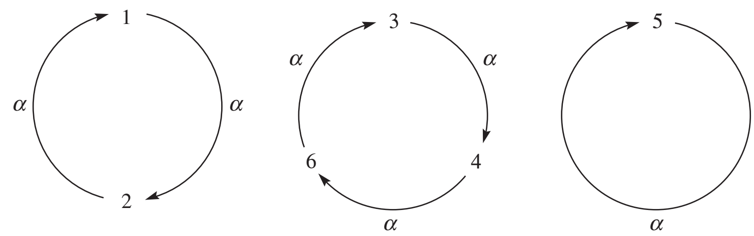
As an illustration of cycle notation, let us consider the permutation

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 4 & 6 & 5 & 3 \end{bmatrix}.$$

This assignment of values could be presented schematically as follows.

Although mathematically satisfactory, such diagrams are cumbersome. Instead, we leave out the arrows and simply write $\alpha = (1, 2)(3, 4, 6)(5)$. As a second example, consider

$$\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 5 & 3 & 1 & 6 & 2 & 4 \end{bmatrix}.$$



In cycle notation, β can be written (2, 3, 1, 5)(6, 4) or (4, 6)(3, 1, 5, 2), since both of these unambiguously specify the function β. An expression of the form $(a_1, a_2, \ldots, a_m)$ is called a *cycle of length* m or an m*-cycle.*

A multiplication of cycles can be introduced by thinking of a cycle as a permutation that fixes any symbol not appearing in the cycle. Thus, the cycle (4, 6) can be thought of as representing the permutation $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 2 & 3 & 6 & 5 & 4 \end{bmatrix}$. In this way, we can multiply cycles by thinking of them as permutations given in array form. Consider the following example from S_8. Let $\alpha = (13)(27)(456)(8)$ and $\beta = (1237)(648)(5)$. (When the domain consists of single-digit integers, it is common practice to omit the commas between the digits.) What is the cycle form of $\alpha\beta$? Of course, one could say that $\alpha\beta = (13)(27)(456)(8)(1237)(648)(5)$, but it is usually more desirable to express a permutation in a *disjoint* cycle form (i.e., the various cycles have no number in common). We proceed by treating each of the seven cycles of $\alpha\beta$ as a function from the set $\{1, 2 \ldots, 8\}$ to itself and use function composition, keeping in mind that function composition is done from right to left and that each cycle that does not contain a symbol fixes that symbol. So let's see what we get for $(\alpha\beta)(1)$. We observe that (5) fixes 1, then (648) fixes 1, then (1237) sends 1 to 2, then (8) fixes 2, then (456) fixes 2, then (27) sends 2 to 7, and, lastly, (13) fixes 7. So the net effect of $\alpha\beta$ is to send 1 to 7. Thus, we begin with $\alpha\beta = (17 \cdots) \cdots$. Pictorially, we see

$$1 \xrightarrow{(5)} 1 \xrightarrow{(648)} 1 \xrightarrow{(1237)} 2 \xrightarrow{(8)} 2 \xrightarrow{(456)} 2 \xrightarrow{(27)} 7 \xrightarrow{(13)} 7.$$

Now, repeating the entire process beginning with 7, we have, cycle by cycle, right to left,

$$7 \xrightarrow{(5)} 7 \xrightarrow{(648)} 7 \xrightarrow{(1237)} 1 \xrightarrow{(8)} 1 \xrightarrow{(456)} 1 \xrightarrow{(27)} 1 \xrightarrow{(13)} 3,$$

so that $\alpha\beta = (173\cdots)\cdots$. Ultimately, we have $\alpha\beta = (1732)(48)(56)$. The important thing to bear in mind when multiplying cycles is to "keep moving" from one cycle to the next from right to left.[2] (*Warning:* Some authors compose cycles from left to right. When reading another text, be sure to determine which convention is being used.)

To be sure you understand how to switch from one notation to the other and how to multiply permutations, we will do one more example of each.

If array notations for α and β, respectively, are

$$\begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 3 & 5 & 4 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 1 & 2 & 3 \end{bmatrix},$$

then, in cycle notation, $\alpha = (12)(3)(45), \beta = (153)(24)$, and $\alpha\beta = (12)(3)(45)(153)(24)$.

To put $\alpha\beta$ in disjoint cycle form, observe that (24) fixes 1; (153) sends 1 to 5; (45) sends 5 to 4; and (3) and (12) both fix 4. So, $\alpha\beta$ sends 1 to 4. Continuing in this way we obtain $\alpha\beta = (14)(253)$.

One can convert $\alpha\beta$ back to array form without converting each cycle of $\alpha\beta$ into array form by simply observing that (14) means 1 goes to 4 and 4 goes to 1; (253) means $2 \to 5, 5 \to 3, 3 \to 2$.

One final remark about cycle notation: Mathematicians prefer not to write cycles that have only one entry. In this case, it is understood that any missing element is mapped to itself. With this convention, the permutation α above can be written as (12)(45). Similarly,

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 2 & 4 & 1 & 5 \end{bmatrix}$$

can be written $\alpha = (134)$. Of course, the identity permutation consists only of cycles with one entry, so we cannot omit all of these! In this case, one usually writes just one cycle. For example,

$$\varepsilon = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 2 & 3 & 4 & 5 \end{bmatrix}$$

[2] This process is akin to evaluating a function such as $\sin(x^2+1)^3$ where x^2+1 takes 1 to 2, then the cube function takes 2 to 8, and the sine function takes 8 to sin 8.

can be written as $\varepsilon = (5)$ or $\varepsilon = (1)$. Just remember that missing elements are mapped to themselves.

Properties of Permutations

We are now ready to state several theorems about permutations and cycles. The proof of the first theorem is implicit in our discussion of writing permutations in cycle form.

Theorem 5.1 Products of Disjoint Cycles

Every permutation of a finite set can be written as a cycle or as a product of disjoint cycles.

PROOF Let α be a permutation on $A = \{1, 2, \ldots, n\}$. To write α in disjoint cycle form, we start by choosing any member of A, say a_1, and let

$$a_2 = \alpha(a_1), \quad a_3 = \alpha(\alpha(a_1)) = \alpha^2(a_1),$$

and so on, until we arrive at $a_1 = \alpha^m(a_1)$ for some m. We know that such an m exists because the sequence $a_1, \alpha(a_1), \alpha^2(a_1), \ldots$ must be finite; so there must eventually be a repetition, say $\alpha^i(a_1) = \alpha^j(a_1)$ for some i and j with $i < j$. Then $a_1 = \alpha^m(a_1)$, where $m = j - i$. We express this relationship among $a_1, a_2, \ldots, a_m$ as

$$\alpha = (a_1, a_2, \ldots, a_m) \cdots .$$

The three dots at the end indicate the possibility that we may not have exhausted the set A in this process. In such a case, we merely choose any element b_1 of A not appearing in the first cycle and proceed to create a new cycle as before. That is, we let $b_2 = \alpha(b_1), b_3 = \alpha^2(b_1)$, and so on, until we reach $b_1 = \alpha^k(b_1)$ for some k. This new cycle will have no elements in common with the previously constructed cycle. For, if so, then $\alpha^i(a_1) = \alpha^j(b_1)$ for some i and j. But then $\alpha^{i-j}(a_1) = b_1$, and therefore $b_1 = a_t$ for some t. This contradicts the way b_1 was chosen. Continuing this process until we run out of elements of A, our permutation will appear as

$$\alpha = (a_1, a_2, \ldots, a_m)(b_1, b_2, \ldots, b_k) \cdots (c_1, c_2, \ldots, c_s).$$

In this way, we see that every permutation can be written as a product of disjoint cycles. ∎

Theorem 5.2 Disjoint Cycles Commute

If the pair of cycles $\alpha = (a_1, a_2, \ldots, a_m)$ and $\beta = (b_1, b_2, \ldots, b_n)$ have no entries in common, then $\alpha\beta = \beta\alpha$.

PROOF For definiteness, let us say that α and β are permutations of the set

$$S = \{a_1, a_2, \ldots, a_m, b_1, b_2, \ldots, b_n, c_1, c_2, \ldots, c_k\},$$

where the c's are the members of S left fixed by both α and β (there may not be any c's). To prove that $\alpha\beta = \beta\alpha$, we must show that $(\alpha\beta)(x) = (\beta\alpha)(x)$ for all x in S. If x is one of the a elements, say a_i, then

$$(\alpha\beta)(a_i) = \alpha(\beta(a_i)) = \alpha(a_i) = a_{i+1},$$

since β fixes all a elements. (We interpret a_{i+1} as a_1 if $i = m$.) For the same reason,

$$(\beta\alpha)(a_i) = \beta(\alpha(a_i)) = \beta(a_{i+1}) = a_{i+1}.$$

Hence, the functions of $\alpha\beta$ and $\beta\alpha$ agree on the a elements. A similar argument shows that $\alpha\beta$ and $\beta\alpha$ agree on the b elements as well. Finally, suppose that x is a c element, say c_i. Then, since both α and β fix c elements, we have

$$(\alpha\beta)(c_i) = \alpha(\beta(c_i)) = \alpha(c_i) = c_i$$

and

$$(\beta\alpha)(c_i) = \beta(\alpha(c_i)) = \beta(c_i) = c_i.$$

This completes the proof. ∎

In demonstrating how to multiply cycles, we showed that the product (13)(27)(456)(8) (1237)(648)(5) can be written in disjoint cycle form as (1732)(48)(56). Is economy in expression the only advantage to writing a permutation in disjoint cycle form? No. The next theorem shows that the disjoint cycle form has the enormous advantage of allowing us to "eyeball" the order of the permutation.

Theorem 5.3 Order of a Permutation (Ruffini, 1799)

The order of a permutation of a finite set written in disjoint cycle form is the least common multiple of the lengths of the cycles.

PROOF First, observe that a cycle of length n has order n. (Verify this yourself.) Next, suppose that α and β are disjoint cycles of lengths m and n, and let k be the least common multiple of m and n. It follows from Theorem 4.1 that both α^k and β^k are the identity permutation ε and, since α and β commute, $(\alpha\beta)^k = \alpha^k\beta^k$ is also the identity. Thus, we know by Corollary 2 to Theorem 4.1 ($a^k = e$ implies that $|a|$ divides k) that the order of $\alpha\beta$—let us call it t—must divide k. But then $(\alpha\beta)^t = \alpha^t\beta^t = \varepsilon$. Since α and β are disjoint we know that α^t fixes the elements of β and β^t fixes the elements of α and it follows that $\alpha^t\beta^t$ fixes 1 through n only when both $\alpha^t = \varepsilon$ and $\beta^t = \varepsilon$. But then t is a common multiple of m and n. Because k is the least common multiple of m and n, we then have $k \leq t$, which gives us that $k = t$.

Thus far, we have proved that the theorem is true in the cases where the permutation is a single cycle or a product of two disjoint cycles. The general case involving more than two cycles can be handled in an analogous way. ∎

Theorem 5.3 is an powerful tool for calculating the orders of permutations and the number of permutations of a particular order. We demonstrate this in the next four examples.

EXAMPLE 4

$|(132)(45)| = 6$
$|(1432)(56)| = 4$
$|(123)(456)(78)| = 6$
$|(123)(145)| = |14523| = 5$ ∎

Arranging all possible disjoint cycle structures of elements of S_n according to the longest cycle lengths listed from left to right provides a systematic way of counting the number of elements in S_n of any particular order. There are two cases: permutations where the lengths of the disjoint cycles (ignoring 1-cycles) are distinct and permutations where there are at least two disjoint cycles (ignoring 1-cycles) of the same length. The two cases are illustrated in Examples 5, 6, and 7.

EXAMPLE 5 To determine the orders of the $7! = 5040$ elements of S_7, we need only consider the possible disjoint cycle structures of the elements of S_7. For convenience, we denote an n-cycle by $(\underline{n})$. Then, arranging all possible disjoint cycle structures of elements of S_7 according to longest cycle lengths left to right, we have $(\underline{7})$
$(\underline{6})\ (\underline{1})$
$(\underline{5})\ (\underline{2})$
$(\underline{5})\ (\underline{1})\ (\underline{1})$
$(\underline{4})\ (\underline{3})$
$(\underline{4})\ (\underline{2})\ (\underline{1})$
$(\underline{4})\ (\underline{1})\ (\underline{1})\ (\underline{1})$
$(\underline{3})\ (\underline{3})\ (\underline{1})$
$(\underline{3})\ (\underline{2})\ (\underline{2})$
$(\underline{3})\ (\underline{2})\ (\underline{1})\ (\underline{1})$
$(\underline{3})\ (\underline{1})\ (\underline{1})\ (\underline{1})\ (\underline{1})$
$(\underline{2})\ (\underline{2})\ (\underline{2})\ (\underline{1})$
$(\underline{2})\ (\underline{2})\ (\underline{1})\ (\underline{1})\ (\underline{1})$
$(\underline{2})\ (\underline{1})\ (\underline{1})\ (\underline{1})\ (\underline{1})\ (\underline{1})$
$(\underline{1})\ (\underline{1})\ (\underline{1})\ (\underline{1})\ (\underline{1})\ (\underline{1})\ (\underline{1})$.
Now, from Theorem 5.3 we see that the orders of the elements of S_7 are 7, 6, 10, 5, 12, 4, 3, 2, and 1. To do the same for the $10! = 3628800$ elements of S_{10} would be nearly as simple. ∎

EXAMPLE 6 We determine the number of elements in S_7 of order 12. By Theorems 5.2 and 5.3, we need only count the number of permutations with disjoint cycle form $(a_1a_2a_3a_4)(a_5a_6a_7)$. First consider the cycle $(a_1a_2a_3a_4)$. Although the number of ways to fill these slots is $7 \cdot 6 \cdot 5 \cdot 4$, this product counts the cycle $(a_1a_2a_3a_4)$ four times. For example, the 4-cycle (2741) can also be written as (7412), (4127), (1274) whereas the product $7 \cdot 6 \cdot 5 \cdot 4$ counts them as distinct. Likewise, the $3 \cdot 2 \cdot 1$ expressions for $(a_5a_6a_7)$ counts the cycles $(a_5a_6a_7), (a_6a_7a_5)$ and $(a_7a_5a_6)$ as distinct even though they are equal in S_7. Adjusting for these multiple countings, we have that there are $(7 \cdot 6 \cdot 5 \cdot 4)(3 \cdot 2 \cdot 1)/(4 \cdot 3) = 420$ elements of order 12 in S_7. ∎

EXAMPLE 7 We determine the number of elements in S_7 of order 3. By Theorem 5.3, we need only count the number of permutations of the form $(a_1a_2a_3)$ and $(a_1a_2a_3)(a_4a_5a_6)$. As in Example 6, there are $(7 \cdot 6 \cdot 5)/3 = 70$ elements of the form $(a_1a_2a_3)$. For elements of S_7 of the form $(a_1a_2a_3)(a_4a_5a_6)$ there are $(7 \cdot 6 \cdot 5)/3$

ways to create the first cycle and $(4 \cdot 3 \cdot 2)/3$ to create the second cycle but the product of $(7 \cdot 6 \cdot 5)/3$ and $(4 \cdot 3 \cdot 2)/3)$ counts $(a_1a_2a_3)(a_4a_5a_6)$ and $(a_4a_5a_6)(a_3a_2a_1)$ as distinct when they are equal group elements. Thus, the number of elements in S_7 of the form $(a_1a_2a_3)(a_4a_5a_6)$ is $(7 \cdot 6 \cdot 5)(4 \cdot 3 \cdot 2)/(3 \cdot 3 \cdot 2) = 280$. This gives us 350 elements of order 3 in S_7.
To count the number of elements in S_7 of the form say $(a_1a_2)(a_3a_4)(a_5a_6)$, we proceed as before to obtain $(7 \cdot 6)(5 \cdot 4)(3 \cdot 2)/(2 \cdot 2 \cdot 2 \cdot 3!) = 105$. The 3! term in the denominator appears because there are 3! ways the product of three 2-cycles can be written and each represents the same group element. ∎

As we will soon see, it is often greatly advantageous to write a permutation as a product of cycles of length 2—that is, as permutations of the form (ab) where $a \neq b$. Many authors call these permutations *transpositions*, since the effect of (ab) is to interchange or transpose a and b.

Example 8 and Theorem 5.4 show how this can always be done.

EXAMPLE 8

$$(12345) = (15)(14)(13)(12)$$

$$(1632)(457) = (12)(13)(16)(47)(45)$$

∎

Theorem 5.4 Product of 2-Cycles

Every permutation in $S_n, n > 1$, is a product of 2-cycles.

PROOF First, note that the identity can be expressed as $(12)(12)$, and so it is a product of 2-cycles. By Theorem 5.1, we know that every permutation can be written in the form

$$(a_1a_2 \cdots a_k)(b_1b_2 \cdots b_t) \cdots (c_1c_2 \cdots c_s).$$

A direct computation shows that this is the same as

$$(a_1a_k)(a_1a_{k-1}) \cdots (a_1a_2)(b_1b_t)(b_1b_{t-1}) \cdots (b_1b_2)$$

$$\cdots (c_1c_s)(c_1c_{s-1}) \cdots (c_1c_2).$$

This completes the proof. ∎

The decomposition of a permutation into a product of 2-cycles given in Example 8 and in the proof of Theorem 5.4 is not the only way a permutation can be written as a product of 2-cycles. Although the next example shows that even the *number* of 2-cycles may vary from one decomposition to another, we will prove in Theorem 5.5 (first proved by Cauchy) that there is one aspect of a decomposition that never varies.

EXAMPLE 9

$$(12345) = (54)(53)(52)(51)$$
$$(12345) = (54)(52)(21)(25)(23)(13)$$

∎

We isolate a special case of Theorem 5.5 as a lemma.

Lemma

If $\varepsilon = \beta_1\beta_2\cdots\beta_r$, where the β's are 2-cycles, then r is even.

PROOF Suppose that there is a product $\beta_1\beta_2\cdots\beta_r$ of an odd number of 2-cycles that is the the identity. Clearly, $r \neq 1$ since a 2-cycle is not the identity. Say $\beta_1 = (ab)$. Now observe that there must a 2-cycle β_i with $i > 1$ that contains a, for otherwise our product would take a to b. We may assume that our product $\beta_1\beta_2\cdots\beta_r$ is one for which such an i is minimum. We may further assume that among all such products our is one with the fewest number of a's as an entry in any cycle. If $i = 2$, then $\beta_1\beta_2$ has the form $(ab)(ab)$ or $(ab)(ac)$ where $c \neq b$ (we can lead with a in the second cycle because $(ca) = (ac)$). If $\beta_1\beta_2 = (ab)(ab) = \varepsilon$ then by deleting it we are left with a product with an odd number of 2-cycles that has fewer appearances of a's, which contradicts our choice of the product. If $\beta_1\beta_2 = (ab)(ac)$ where $c \neq b$ we observe that because $(ab)(ac) = (ac)(bc)$ we can replace $\beta_1\beta_2 = (ab)(ac)$ with $\beta_1'\beta_2' = (ac)(bc)$ to obtain a product of r 2-cycles equal to the identity with an a in the first cycle and one fewer a entry in the product $\beta_1'\beta_2'\cdots\beta_r$ than is in $\beta_1\beta_2\cdots\beta_r$. Since this gives us a contradiction, we may assume that $i > 2$. Next note that β_{i-1} does not contain a but does contain c for otherwise β_{i-1} and β_i are disjoint and therefore $\beta_1'\beta_2'\cdots\beta_{i-2}\beta_{i-2}\beta_{i+1}\cdots\beta_r$ is a product of r 2-cycles that is the identity, has an a in the first cycle, and the second appearance of an a entry occurs before the ith cycle in the product. This contradicts our choice of i. So, we know that $\beta_{i-1}\beta_i$ has the form $(dc)(ac)$ where $d \neq a$. Then, because $(dc)(ac) = (ad)(dc)$, we can replace $\beta_{i-1}\beta_i = (dc)(ac)$ in the product $\beta'_1\beta'_2 \cdots \beta_r$ with $\beta'_{i-1}\beta'_i = (ad)(dc)$ to obtain $\beta'_1\beta'_2 \cdots \beta'_{i-1}\beta'_i \cdots \beta_r$, which as before, is a product of r 2-cycles that is the identity and has an a in the first cycle and the second appearance of an a entry occurs before that ith cycle in the product. This contradiction finishes the proof. ▮

Theorem 5.5 Always Even or Always Odd

If a permutation α can be expressed as a product of an even (odd) number of 2-cycles, then every decomposition of α into a product of 2-cycles must have an even (odd) number of 2-cycles. In symbols, if

$$\alpha = \beta_1\beta_2 \cdots \beta_r \quad \text{and} \quad \alpha = \gamma_1\gamma_2 \cdots \gamma_s,$$

where the β's and the γ's are 2-cycles, then r and s are both even or both odd.

PROOF Observe that $\beta_1\beta_2 \cdots \beta_r = \gamma_1\gamma_2 \cdots \gamma_s$ implies

$$\begin{aligned} \varepsilon &= \gamma_1\gamma_2 \cdots \gamma_s\beta_r^{-1} \cdots \beta_2^{-1}\beta_1^{-1} \\ &= \gamma_1\gamma_2 \cdots \gamma_s\beta_r \cdots \beta_2\beta_1, \end{aligned}$$

since a 2-cycle is its own inverse. Thus, the lemma preceding Theorem 5.5 guarantees that $s + r$ is even. It follows that r and s are both even or both odd. ▮

Definitions Even and Odd Permutations

A permutation that can be expressed as a product of an even number of 2-cycles is called an *even* permutation. A permutation that can be expressed as a product of an odd number of 2-cycles is called an *odd* permutation.

Theorems 5.4 and 5.5 together show that every permutation can be unambiguously classified as either even or odd. The significance of this observation is given in Theorem 5.6.

Theorem 5.6 Even Permutations Form a Group

The set of even permutations in S_n forms a subgroup of S_n.

PROOF This proof is left to the reader. ∎

The subgroup of even permutations in S_n arises so often that we give it a special name and notation.

Definition Alternating Group of Degree n
The group of even permutations of n symbols is denoted by A_n and is called the *alternating group of degree n.*

The next result shows that exactly half of the elements of S_n $(n > 1)$ are even permutations.

Theorem 5.7 $|A_n|$

For $n > 1$, A_n has order $n!/2$.

PROOF For each odd permutation α, the permutation $(12)\alpha$ is even and, by the cancellation property in groups, $(12)\alpha \neq (12)\beta$ when $\alpha \neq \beta$. Thus, there are at least as many even permutations as there are odd ones. On the other hand, for each even permutation α, the permutation $(12)\alpha$ is odd and $(12)\alpha \neq (12)\beta$ when $\alpha \neq \beta$. Thus, there are at least as many odd permutations as there are even ones. It follows that there are equal numbers of even and odd permutations. Since $|S_n| = n!$, we have $|A_n| = n!/2$. ∎

The names for the symmetric group and the alternating group of degree n come from the study of polynomials over n variables. A *symmetric* polynomial in the variables $x_1, x_2, \ldots, x_n$ is one that is unchanged under any transposition of two of the variables. An *alternating* polynomial is one that changes signs under any transposition of two of the variables. For example, the polynomial $x_1x_2x_3$ is unchanged by any transposition of two of the three variables, whereas the polynomial $(x_1 - x_2)(x_1 - x_3)(x_2 - x_3)$ changes signs when any two of the variables are transposed. Since every member of the symmetric group is the product of transpositions, the symmetric polynomials are those that are unchanged by members of

Table 5.1 The Alternating Group A_4 of Even Permutations of $\{1, 2, 3, 4\}$.

In this table, the permutations of A_4 are designated as $\alpha_1, \alpha_2, \ldots, \alpha_{12}$ and an entry k inside the table represents α_k. For example, $\alpha_3\alpha_8 = \alpha_6$.

	α_1	α_2	α_3	α_4	α_5	α_6	α_7	α_8	α_9	α_{10}	α_{11}	α_{12}
$(1) = \alpha_1$	1	2	3	4	5	6	7	8	9	10	11	12
$(12)(34) = \alpha_2$	2	1	4	3	6	5	8	7	10	9	12	11
$(13)(24) = \alpha_3$	3	4	1	2	7	8	5	6	11	12	9	10
$(14)(23) = \alpha_4$	4	3	2	1	8	7	6	5	12	11	10	9
$(123) = \alpha_5$	5	8	6	7	9	12	10	11	1	4	2	3
$(243) = \alpha_6$	6	7	5	8	10	11	9	12	2	3	1	4
$(142) = \alpha_7$	7	6	8	5	11	10	12	9	3	2	4	1
$(134) = \alpha_8$	8	5	7	6	12	9	11	10	4	1	3	2
$(132) = \alpha_9$	9	11	12	10	1	3	4	2	5	7	8	6
$(143) = \alpha_{10}$	10	12	11	9	2	4	3	1	6	8	7	5
$(234) = \alpha_{11}$	11	9	10	12	3	1	2	4	7	5	6	8
$(124) = \alpha_{12}$	12	10	9	11	4	2	1	3	8	6	5	7

the symmetric group. Likewise, since any member of the alternating group is the product of an even number of transpositions, the alternating polynomials are those that are unchanged by members of the alternating group.

The alternating groups are among the most important examples of groups. The groups A_4 and A_5 will arise on several occasions in later chapters. In particular, A_5 has great historical significance.

A geometric interpretation of A_4 is given in Example 10, and a multiplication table for A_4 is given as Table 5.1.

▮ EXAMPLE 10 Rotations of a Tetrahedron

The 12 rotations of a regular tetrahedron can be conveniently described with the elements of A_4. The top row of Figure 5.1 illustrates the identity and three 180° "edge" rotations about axes joining midpoints of two edges. The second row consists of 120° "face" rotations about axes joining a vertex to the center of the opposite face. The third row consists of –120° (or 240°) "face" rotations. Notice that the four rotations in the second row can be obtained from those in the first row by left-multiplying the four in the first row by the rotation (123), whereas those in the third row can be obtained from those in the first row by left-multiplying the ones in the first row by (132). ▮

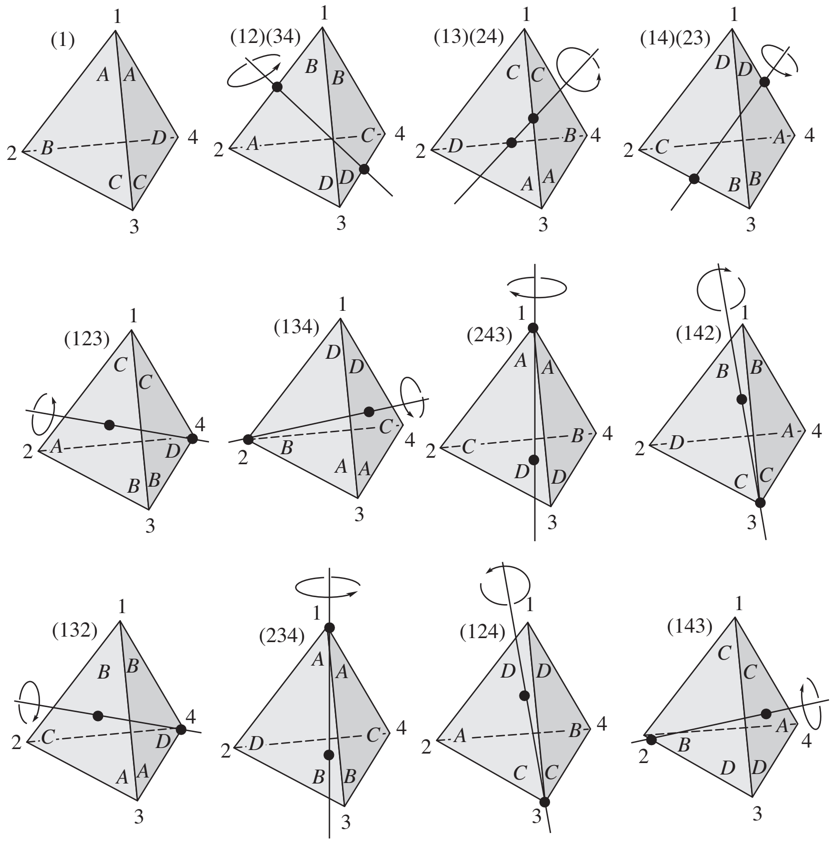


Figure 5.1 Rotations of a regular tetrahedron.

Many molecules with chemical formulas of the form AB_4, such as methane (CH_4) and carbon tetrachloride (CCl_4), have A_4 as their symmetry group. Figure 5.2 shows the form of one such molecule.

Many games and puzzles can be analyzed using permutations.

EXAMPLE 11 Encryption Using a Permutation

An interesting application of permutations is cryptography. Cryptography is the study of methods to make and break secret codes. The process of coding information to prevent unauthorized use is called encryption. Historically, encryption was used primarily for military and diplomatic transmissions. Today, encryption is essential for securing electronic transactions of all kinds. Cryptography is what allows you to have a Website safely receive your

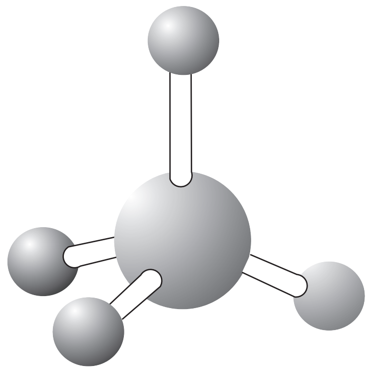


Figure 5.2 A tetrahedral AB_4 molecule.

credit card number. Cryptographic schemes prevent hackers from charging calls to your cell phone account.

Among the first known cryptosystems is the Caesar cipher, used by Julius Caesar to send messages to his troops. Caesar encrypted a message by replacing each letter with the letter three positions further in the alphabet with x, y and z wrapping around to a, b and c. Identifying the 26 letters of the alphabet with $0, 1, \ldots, 25$ in order, the Caesar method replaces letter i with letter $(i+3)$ mod 26. For example, the message ATTACK AT DAWN is encrypted as DWWDFN DW GDZQ. To decrypt the message one replaces letter i with letter $(i-3)$ mod 26.

Any permutation can be used as a cipher. To use the permutation $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 2 & 1 \end{bmatrix}$ to encrypt the message ATTACK AT DAWN we first break the message up into blocks of four letters each ignoring the spaces between the words to obtain the plaintext ATTA CKAT DAWN. (This has the added advantage of disguising the lengths of each word, which makes breaking the code more difficult.) We then reorder the four letters in each block in the same way α reordered the integers 1, 2, 3, and 4. That is, the first letter is put in the third position, the second letter is put in the fourth position, the third letter is put in the second position, and the fourth letter is put in the first position. Doing this for each block we have ATAT TACK NWDA. Of course one decrypts a message encrypted by using α by using α^{-1}.

To enhance security one would use a permutation of long length n and blocks of length other than n so that anyone not authorized to receive the encrypted message would not know the permutation length. In cases where the number of message characters is not divisible by the block length the sender simply fills out the last block with nonsense letters. For example, for the message RETREAT AT DUSK with block length 4 we could use RETR EATA TDUS KIUE. The recipient of the message will recognize the nonsense letters as padding needed to complete the last block.

Enigma machines were cipher devices used by the Germans in World War II (1939–1945). An Enigma machine had three to five wheels that would scramble the letters of a message. The machines were easy to use and offered a high degree of security when used properly. Although messages encoded with Enigma machines were difficult to break, operator negligence and the capture of a number of Enigma machines and the tables of wheel settings by the Allied forces allowed Polish and British cryptologists to break the code. ∎

Rubik's Cube

The Rubik's Cube made from 48 cubes called "facets" is the quintessential example of a group theory puzzle. It was invented in 1974 by the Hungarian Errő Rubik. By 2009 more than 350 million Rubik's Cubes had been sold. In 2020 the record time for solving it is under 4 seconds; under 17 seconds blindfolded. Although it was proved in 1995 that there was a starting configuration that required at least 20 moves to solve, it was not until 2010 that it was determined that every cube could be solved in at most 20 moves. This computer calculation utilized about 35 CPU-years donated by Google to complete. In early discussions about the minimum number of moves to solve the cube in the worst possible case, someone called it "God's number," and the name stuck. A history of the quest to find God's number is given at the website at **http://www.cube20.org/**.

The set of all configuration of the Rubik's Cube form a group of permutations of order 43,252,003,274,489,856,000. This order can be computed using GAP by labeling the faces of the cube as shown here.

			1	2	3						
			4	top	5						
			6	7	8						
9	10	11	17	18	19	25	26	27	33	34	35
12	left	13	20	front	21	28	right	29	36	rear	37
14	15	16	22	23	24	30	31	32	38	39	40
			41	42	43						
			44	bottom	45						
			46	47	48						

The group of permutations of the cube is generated by the following rotations of the six layers.

$$\text{top} = (1,3,8,6)(2,5,7,4)(9,33,25,17)(10,34,26,18)(11,35,27,19)$$
$$\text{left} = (9,11,16,14)(10,13,15,12)(1,17,41,40)(4,20,44,37)(6,22,46,35)$$
$$\text{front} = (17,19,24,22)(18,21,23,20)(6,25,43,16)(7,28,42,13)(8,30,41,11)$$
$$\text{right} = (25,27,32,30)(26,29,31,28)(3,38,43,19)(5,36,45,21)(8,33,48,24)$$
$$\text{rear} = (33,35,40,38)(34,37,39,36)(3,9,46,32)(2,12,47,29)(1,14,48,27)$$
$$\text{bottom} = (41,43,48,46)(42,45,47,44)(14,22,30,38)(15,23,31,39)(16,24,32,40)$$

A Check-Digit Scheme Based on D_5

In Chapter 0, we presented several schemes for appending a check digit to an identification number. Among these schemes, only the International Standard Book Number method was capable of detecting all single-digit errors and all transposition errors involving adjacent digits. However, recall that this success was achieved by introducing the alphabetical character X to handle the case where 10 was required to make the dot product 0 modulo 11.

In contrast, in 1969, J. Verhoeff devised a method utilizing the dihedral group of order 10 that detects all single-digit errors and all transposition errors involving adjacent digits without the necessity of avoiding certain numbers or introducing a new character. To describe this method, consider the permutation $\sigma = (01589427)(36)$ and the dihedral group of order 10 as represented in Table 5.2. (Here we use 0 through 4 for the rotations, 5 through 9 for the reflections, and $*$ for the operation of D_5.)

Verhoeff's idea was to view the digits 0 through 9 as the elements of the group D_5 and to replace ordinary addition with

Table 5.2 Multiplication for D_5.

$*$	**0**	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
0	0	1	2	3	4	5	6	7	8	9
1	1	2	3	4	0	6	7	8	9	5
2	2	3	4	0	1	7	8	9	5	6
3	3	4	0	1	2	8	9	5	6	7
4	4	0	1	2	3	9	5	6	7	8
5	5	9	8	7	6	0	4	3	2	1
6	6	5	9	8	7	1	0	4	3	2
7	7	6	5	9	8	2	1	0	4	3
8	8	7	6	5	9	3	2	1	0	4
9	9	8	7	6	5	4	3	2	1	0

calculations done in D_5. In particular, to any string of digits $a_1a_2\ldots a_{n-1}$, we append the check digit a_n so that $\sigma(a_1) * \sigma^2(a_2) * \cdots * \sigma^{n-2}(a_{n-2}) * \sigma^{n-1}(a_{n-1}) * \sigma^n(a_n) = 0$. [Here $\sigma^2(x) = \sigma(\sigma(x)), \sigma^3(x) = \sigma(\sigma^2(x))$, and so on.] Since σ has the property that $\sigma^i(a) \neq \sigma^i(b)$ if $a \neq b$, all single-digit errors are detected. Also, because

$$a * \sigma(b) \neq b * \sigma(a) \quad \text{if } a \neq b, \tag{5.1}$$

as can be checked on a case-by-case basis (see Exercise 71), it follows that all transposition errors involving adjacent digits are detected [since Equation (1) implies that $\sigma^i(a) * \sigma^{i+1}(b) \neq \sigma^i(b) * \sigma^{i+1}(a)$ if $a \neq b$].

From 1990 until 2002, the German government used a minor modification of Verhoeff's check-digit scheme to append a check digit to the serial numbers on German banknotes. Table 5.3 gives the values of the functions $\sigma, \sigma^2, \ldots, \sigma^{10}$ needed for the computations. [The functional value $\sigma^i(j)$ appears in the row labeled with σ^i and the column labeled j.] Since the serial numbers on the banknotes use 10 letters of the alphabet in addition to the 10 decimal digits, it is necessary to assign numerical values to the letters to compute the check digit. This assignment is shown in Table 5.4.

To any string of digits $a_1a_2\ldots a_{10}$ corresponding to a banknote serial number, the check digit a_{11} is chosen such that $\sigma(a_1) * \sigma^2(a_2) * \cdots * \sigma^9(a_9) * \sigma^{10}(a_{10}) * a_{11} = 0$ [instead of $\sigma(a_1) * \sigma^2(a_2) * \cdots * \sigma^{10}(a_{10}) * \sigma^{11}(a_{11}) = 0$ as in the Verhoeff scheme].

To trace through a specific example, consider the banknote (featuring the mathematician Gauss) shown in Figure 5.3 with the

Table 5.3 Powers of σ.

	0	**1**	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**
σ	1	5	7	6	2	8	3	0	9	4
σ^2	5	8	0	3	7	9	6	1	4	2
σ^3	8	9	1	6	0	4	3	5	2	7
σ^4	9	4	5	3	1	2	6	8	7	0
σ^5	4	2	8	6	5	7	3	9	0	1
σ^6	2	7	9	3	8	0	6	4	1	5
σ^7	7	0	4	6	9	1	3	2	5	8
σ^8	0	1	2	3	4	5	6	7	8	9
σ^9	1	5	7	6	2	8	3	0	9	4
σ^{10}	5	8	0	3	7	9	6	1	4	2

Table 5.4 Letter Values.

A	D	G	K	L	N	S	U	Y	Z
0	1	2	3	4	5	6	7	8	9

number AG8536827U7. To verify that 7 is the appropriate check digit, we observe that $\sigma(0) * \sigma^2(2) * \sigma^3(8) * \sigma^4(5) * \sigma^5(3) * \sigma^6(6) * \sigma^7(8) * \sigma^8(2) * \sigma^9(7) * \sigma^{10}(7) * 7 = 1 * 0 * 2 * 2 * 6 * 6 * 5 * 2 * 0 * 1 * 7 = 0$, as it should be. [To illustrate how to use the multiplication table for D_5, we compute $1 * 0 * 2 * 2 = (1 * 0) * 2 * 2 = 1 * 2 * 2 = (1 * 2) * 2 = 3 * 2 = 0$.]

Figure 5.3 German banknote with serial number AG8536827U and check digit 7.

One shortcoming of the German banknote scheme is that it does not distinguish between a letter and its assigned numerical value. Thus, a substitution of 7 for U (or vice versa) and the transposition of 7 and U are not detected by the check digit. Moreover, the banknote scheme does not detect all transpositions of adjacent characters involving the check digit itself. For example, the transposition of D and 8 in positions 10 and 11 is not detected. Both of these defects can be avoided by using the Verhoeff method with D_{18}, the dihedral group of order 36, to assign every letter and digit a distinct value together with an appropriate function σ. Using this method to append a check character, all single position errors and all transposition errors involving adjacent digits will be detected.

Exercises

My mind rebels at stagnation. Give me problems, give me work, give me the most abstruse cryptogram, or the most intricate analysis, and I am in my own proper atmosphere.

Sherlock Holmes, *The Sign of Four*

1. Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix} \quad \text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 1 & 2 & 4 & 3 & 5 \end{bmatrix}.$$

Compute each of the following.

a. α^{-1} **b.** $\beta\alpha$ **c.** $\alpha\beta$

2. Let

$$\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 2 & 3 & 4 & 5 & 1 & 7 & 8 & 6 \end{bmatrix}$$

$$\text{and} \quad \beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 \\ 1 & 3 & 8 & 7 & 6 & 5 & 2 & 4 \end{bmatrix}.$$

Write α, β, and $\alpha\beta$ as a products of disjoint cycles and as products of 2-cycles.

3. Write each of the following permutations as a product of disjoint cycles.

a. $(1235)(413)$ **b.** $(13256)(23)(46512)$ **c.** $(12)(13)(23)(142)$

4. Find the order of each of the following permutations.

a. (14) **b.** (147) **c.** (14762) **d.** $(a_1a_2 \ldots a_k)$

5. What is the order of each of the following permutations?

a. $(124)(357)$ **b.** $(124)(3567)$ **c.** $(124)(35)$
d. $(124)(357869)$ **e.** $(1235)(24567)$ **f.** $(345)(245)$

6. What is the order of each of the following permutations?

a. $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 5 & 4 & 6 & 3 \end{bmatrix}$ **b.** $\begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 7 & 6 & 1 & 2 & 3 & 4 & 5 \end{bmatrix}$

7. What is the order of the product of a pair of disjoint cycles of lengths 4 and 6? What about the product of three disjoint cycles of lengths 6, 8, and 10?

8. Determine whether the following permutations are even or odd.

a. (135) **b.** (1356) **c.** (13567) **d.** (12)(134)(152)
e. (1243)(3521)

9. Write $((14562)(2345)(136)(235))^{10}$ as a product of disjoint cycles.

10. Write (13)(1245)(13) and (24)(13456)(24) in disjoint cycle form. Give a simple description of how each product cycle compares with middle cycle in each product.

11. Let n be a positive integer. If n is odd, is an n-cycle an odd or an even permutation? If n is even, is an n-cycle an odd or an even permutation?

12. If α is even, prove that α^{-1} is even. If α is odd, prove that α^{-1} is odd.

13. Show that a function from a finite set S to itself is one-to-one if and only if it is onto. Is this true when S is infinite? (This exercise is referred to in Chapter 6.)

14. For $n \geq 4$ how many elements in S_n are there that map 1 to 2, 2 to 3, and 3 to 1? For $n \geq 5$, how many are in A_n?

15. Let α and β belong to S_n. Prove that $\alpha\beta$ is even if and only if α and β are both even or both odd.

16. Associate an even permutation with the number $+1$ and an odd permutation with the number -1. Draw an analogy between the result of multiplying two permutations and the result of multiplying their corresponding numbers $+1$ or -1.

17. If n is any integer and α is odd complete the following statement: α^n is odd if and only if ____________.

18. In S_n, let α be an r-cycle, β an s-cycle, and γ a t-cycle. Complete the following statements: $\alpha\beta$ is even if and only if $r + s$ is . . . ; $\alpha\beta\gamma$ is even if and only if $r + s + t$ is . . .

19. Let α and β belong to S_n. Prove that $\alpha^{-5}\beta\alpha^3$ is odd if and only if β is odd.

20. Let α and β belong to S_n. Prove that $\beta\alpha\beta^{-1}$ and α are both even or both odd.

21. Complete the following statement: A product of cycles is an even permutation if and only if the number of cycles of even length is ____________.

22. What is the smallest n for which S_n has an element of order 30? What about A_n?

23. What are the possible orders for the elements of S_6 and A_6? What about A_7? (This exercise is referred to in Chapter 24.)

24. Find an element in A_8 of order 15. Find an element in A_{12} of order 30.

25. Let $\beta = (1, 3, 5, 7, 9, 8, 6)(2, 4, 10)$. What is the smallest positive integer n for which $\beta^n = \beta^{-5}$?

26. What cycle is $(a_1 a_2 \cdots a_n)^{-1}$?

27. Show that if H is a subgroup of S_n, then either every member of H is an even permutation or exactly half of the members are even. (This exercise is referred to in Chapter 24.)

28. Suppose that H is a subgroup of S_n of odd order. Prove that H is a subgroup of A_n.

29. Give two reasons why the set of odd permutations in S_n is not a subgroup.

30. Let α and β belong to S_n. Prove that $\alpha^{-1}\beta^{-1}\alpha\beta$ is an even permutation.

31. How many elements are there of order 2 in A_8 that have the disjoint cycle form $(a_1 a_2)(a_3 a_4)(a_5 a_6)(a_7 a_8)$?

32. For $n \geq 5$, how many odd permutations of order 5 are in S_n?

33. How many elements in A_6 have order 2? How many elements in A_7 have order 6?

34. Let $H = \{x^2 \mid x \in A_4\}$. Prove that H is not a subgroup of A_4. (Compare with Example 5 of Chapter 3.)

35. What is the smallest n for which there is a solution in S_n to the equation $x^5 = (12345)$? Give an example of a solution. How many solutions are there for your n?

36. Let $\alpha = (1,3,5,7,9)(2,4,6)(8,10)$. If α^m is a 5-cycle, what can you say about m?

37. How many elements of order 4 does S_6 have? How many elements of order 2 does S_6 have?

38. Prove that (1234) is not the product of 3-cycles. Generalize.

39. Let $\beta \in S_7$ and suppose $\beta^4 = (2143567)$. Find β. What are the possibilities for β if $\beta \in S_9$?

40. Let $\beta = (123)(145)$. Write β^{99} in disjoint cycle form.

41. Let $(a_1a_2a_3a_4)$ and (a_5a_6) be disjoint cycles in S_{10}. Show that there is no element x in S_{10} such that $x^2 = (a_1a_2a_3a_4)(a_5a_6)$.

42. If α and β are distinct 2-cycles, what are the possibilities for $|\alpha\beta|$?

43. Let G be a group of permutations on a set X. Let $a \in X$ and define $\text{stab}(a) = \{\alpha \in G \mid \alpha(a) = a\}$. We call $\text{stab}(a)$ the *stabilizer of a in G* (since it consists of all members of G that leave a fixed). Prove that $\text{stab}(a)$ is a subgroup of G. (This subgroup was introduced by Galois in 1832.) This exercise is referred to in Chapter 7.

44. Let $H = \{\beta \in S_5 \mid \beta(1) = 1 \text{ and } \beta(3) = 3\}$. Prove that H is a subgroup of S_5. How many elements are in H? Is your argument valid when S_5 is replaced by S_n for $n \geq 3$? How many elements are in H when S_5 is replaced by A_n for $n \geq 4$?

45. For $n \geq 3$, let $H = \{\beta \in S_n \mid \beta(1) = 1 \text{ or } 2 \text{ and } \beta(2) = 1 \text{ or } 2\}$. Prove that H is a subgroup of S_n. Determine $|H|$.

46. Let α be a 12-cycle. Describe the cycle structure of $\alpha^2, \alpha^3, \alpha^4$ and α^6. More generally, if α is an n-cycle and k divides n describe the cycle structure of α^k.

47. Let α be a n-cycle and k a positive integer. Prove that α^k is an n-cycle if and only if $\gcd(n,k) = 1$.

48. In S_3, find elements α and β such that $|\alpha| = 2, |\beta| = 2$, and $|\alpha\beta| = 3$.

49. Find group elements α and β in S_5 such that $|\alpha| = 3, |\beta| = 3$, and $|\alpha\beta| = 5$.

50. Represent the symmetry group of an equilateral triangle as a group of permutations of its vertices (see Example 3).

51. Prove that S_n is non-Abelian for all $n \geq 3$. Prove that A_n is non-Abelian for all $n \geq 4$.

52. Show that in S_n $(n \geq 4)$, the equation $x^2 = (1234)$ has no solutions but in S_7 the equation $x^3 = (1234)$ three solutions.

53. Prove Theorem 5.6.

54. If (ab) and (cd) are distinct 2-cycles in S_n, prove that (ab) and (cd) commute if and only if they are disjoint.

55. In S_4, find a cyclic subgroup of order 4 and a noncyclic subgroup of order 4.

56. In S_{100} let $\alpha = (1, 2, 3, \ldots, 100)$ and $\beta = (1, 50, 100)$. What is $|\alpha\beta|$? Give an example of two elements in S_{100} that both have order 100 and whose product has order 3.

57. Viewing the members of D_4 as a group of permutations of a square labeled 1, 2, 3, 4 as described in Example 3, which geometric symmetries correspond to even permutations?

58. Viewing the members of D_5 as a group of permutations of a regular pentagon with consecutive vertices labeled 1, 2, 3, 4, 5, what geometric symmetry corresponds to the permutation (14253)? Which symmetry corresponds to the permutation (25)(34)?

59. Let n be an odd integer greater than 1. Viewing D_n as a group of permutations of a regular n-gon with consecutive vertices labeled $1, 2, \ldots, n$, explain why the rotation subgroup of D_n is a subgroup of A_n. Is D_5 a subgroup of A_5? If so, identify a reflection in D_5 that is in A_5. Why does this prove that D_5 is a subgroup of A_5? Is D_7 a subgroup of A_7? If so, identify a reflection in D_7 that is in A_7.

60. Show that every element in A_n for $n \geq 3$ can be expressed as a 3-cycle or a product of 3-cycles.

61. For $n \geq 4$ let B_n be the set of all elements in S_n that can be written as the product of an even number of 4-cycles. Prove that B_n is a subgroup of A_n.

62. Letting B_n be as defined in previous exercise prove that every 3-cycle in A_n $(n \geq 4)$ is the product of two 4-cycles.

63. Letting B_n $(n \geq 4)$ be the subgroup of A_n defined in Exercise 61, prove that $B_n = A_n$.

64. Let α_1, α_2 and α_3 be 2-cycles. Prove that $\alpha_1\alpha_2\alpha_3 \neq \epsilon$. Generalize.

65. Show that A_5 has 24 elements of order 5, 20 elements of order 3, and 15 elements of order 2. (This exercise is referred to in Chapter 24.)

66. Find a cyclic subgroup of A_6 that has order 4. Find a noncyclic subgroup of A_8 that has order 4.

67. Show that a permutation with odd order must be an even permutation.

68. Compute the order of each member of A_4. What arithmetic relationship do these orders have with the order of A_4?

69. For any odd integer n at least 3 write $(2, n)(3, n-1)\cdots((n+1)/2), ((n+3)/2)(12\cdots n)$ in disjoint cycle form. Notice that $|(2, n)(3, n-1)\cdots((n+1)/2), ((n+3)/2)| = 2$ and $|(12\cdots n)| = n$ and their product has order 2. Labeling the vertices of a regular n-gon with $1, 2, \ldots, n$ counterclockwise as in Example 3, which elements of D_n correspond to $(2, n)(3, n-1)\cdots((n+1)/2), ((n+3)/2)(12\cdots n)$ and $(12\cdots n)$?

70. Show that for $n \geq 3$, $Z(S_n) = \{\varepsilon\}$.

71. Verify the statement made in the discussion of the Verhoeff check digit scheme based on D_5 that $a * \sigma(b) \neq b * \sigma(a)$ for distinct a and b. Use this to prove that $\sigma^i(a) * \sigma^{i+1}(b) \neq \sigma^i(b) * \sigma^{i+1}(a)$ for all i. Prove that this implies that all transposition errors involving adjacent digits are detected.

72. Use the Verhoeff check-digit scheme based on D_5 to append a check digit to 45723.

73. Prove that every element of S_n $(n > 1)$ can be written as a product of elements of the form $(1k)$.

74. (Indiana College Mathematics Competition) A card-shuffling machine always rearranges cards in the same way relative to the order in which they were given to it. All of the hearts arranged in order from ace to king were put into the machine, and then the shuffled cards were put into the machine again to be shuffled. If the cards emerged in the order 10, 9, Q, 8, K, 3, 4, A, 5, J, 6, 2, 7, in what order were the cards after the first shuffle?

75. Determine integers n for which $H = \{\alpha \in A_n \mid \alpha^2 = \varepsilon\}$ is a subgroup of A_n.

76. Find five subgroups of S_5 of order 24.

77. Why does the fact that the orders of the elements of A_4 are 1, 2, and 3 imply that $|Z(A_4)| = 1$?

78. Let α belong to S_n. Prove that $|\alpha|$ divides $n!$.

79. Encrypt the message ATTACK POSTPONED using the permutation $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 1 & 5 & 3 & 4 \end{bmatrix}$.

80. The message VAADENWCNHREDEYA was encrypted using the permutation $\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{bmatrix}$. Decrypt it.

Computer Exercises

Computer exercises for this chapter are available at the website:

http://www.d.umn.edu/~jgallian

Augustin Cauchy

Stock Montage

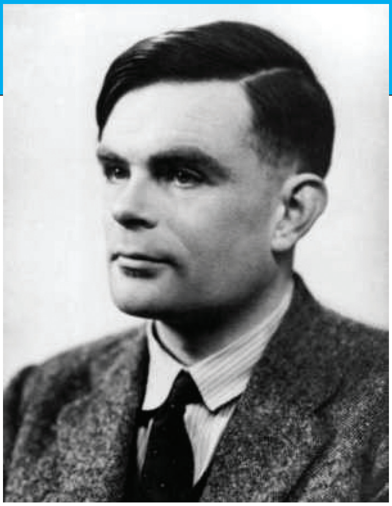
You see that little young man? Well! He will supplant all of us in so far as we are mathematicians.

Spoken by Lagrange to Laplace about the 11-year-old Cauchy

AUGUSTIN LOUIS CAUCHY was born on August 21, 1789, in Paris. By the time he was 11, both Laplace and Lagrange had recognized Cauchy's extraordinary talent for mathematics. In school he won prizes for Greek, Latin, and the humanities. At the age of 21, he was given a commission in Napoleon's army as a civil engineer. For the next few years, Cauchy attended to his engineering duties while carrying out brilliant mathematical research on the side.

In 1815, at the age of 26, Cauchy was made Professor of Mathematics at the École Polytechnique and was recognized as the leading mathematician in France. Cauchy and his contemporary Gauss were among the last mathematicians to know the whole of mathematics as known at their time, and both made important contributions to nearly every branch, both pure and applied, as well as to physics and astronomy. Cauchy introduced a new level of rigor into mathematical analysis. We owe our contemporary notions of limit and continuity to him. He gave the first proof of the Fundamental Theorem of Calculus. Cauchy was the founder of complex function theory and a pioneer in the theory of permutation groups and determinants. His total written output of mathematics fills 24 large volumes. He wrote more than 500 research papers after the age of 50. Cauchy died at the age of 67 on May 23, 1857.

Alan Turing



Courtesy of History/The Image Works

Every time you use a phone, or a computer, you use the ideas that Alan Turin invented. Alan discovered intelligence in computers, and today he surrounds us. A true hero of mankind.

ERIC E. SCHMIDT,
Executive Chairman,
Google

On Time magazine's list of the 100 most influential people of the 20th century was Alan Turing, a mathematician born in London on June 23, 1912. While a college student Turing developed ideas that would lay the foundation for theoretical computer science and artificial intelligence. After graduating from college Turing became a crucial member of a team of cryptologists working for the British government who successfully broke the Enigma codes.

Turing's life took a tragic turn in 1952 when he admitted that he had engaged in homosexual acts in his home, which was a felony in Britain at that time. As punishment, he was chemically castrated and subjected to estrogen treatments. Despondent by this treatment, he committed suicide by eating an apple laced with cyanide at the age of 41. Today Turing is widely honored for his fundamental contributions to computer science and his role in the defeat of Germany in World War II. Many rooms, lecture halls, and buildings at universities around the world have been named in honor of Turing. The annual award for contributions to the computing community, which is widely consider to be the equivalent to a Nobel Prize, is called the "Turing Award."

In 2013, Queen Elizabeth II granted Turing a pardon and issued a statement saying Turing's treatment was unjust and "Turing was an exceptional man with a brilliant mind who deserves to be remembered and recognized for his fantastic contribution to the war effort and his legacy to science."

6 Isomorphisms

Mathematics is the art of giving the same name to different things.

Henri Poincaré (1854–1912)

The basis for poetry and scientific discovery is the ability to comprehend the unlike in the like and the like in the unlike.

Jacob Bronowski

Motivation

Suppose an American and a German are asked to count a handful of objects. The American says, "One, two, three, four, five, . . . ," whereas the German says, "Eins, zwei, drei, vier, fünf, ..." Are the two doing different things? No. They are both counting the objects, but they are using different terminology to do so. Similarly, when one person says, "Two plus three is five" and another says, "Zwei und drei ist fünf," the two are in agreement on the *concept* they are describing, but they are using different terminology to describe the concept. An analogous situation often occurs with groups; the same group is described with different terminology. We have seen two examples of this so far. In Chapter 1, we described the symmetries of a square in geometric terms (e.g., R_{90}), whereas in Chapter 5 we described the *same* group by way of permutations of the corners. In both cases, the underlying group was the symmetries of a square. In Chapter 4, we observed that when we have a cyclic group of order n generated by a, the operation turns out to be essentially that of addition modulo n, since $a^r a^s = a^k$, where $k = (r + s) \bmod n$. For example, each of $U(43)$ and $U(49)$ is cyclic of order 42. So, each has the form $\langle a \rangle$, where $a^r a^s = a^{(r+s) \bmod 42}$.

Definition and Examples

In this chapter, we give a formal method for determining whether two groups defined in different terms are really the same. When this is the case, we say that there is an isomorphism between the two groups. This notion was first introduced by Galois about 190 years ago. The term *isomorphism* is derived from the Greek words *isos*, meaning "same" or "equal," and *morphe*, meaning "form." R. Allenby has colorfully defined an algebraist as "a person who can't tell the difference between isomorphic systems."

Definitions Group Isomorphism

An *isomorphism* ϕ from a group G to a group $\overline{G}$ is a one-to-one onto mapping (or function) from G to $\overline{G}$ that preserves the group operation. That is,

$$\phi(ab) = \phi(a)\phi(b) \qquad \text{for all } a, b \text{ in } G.$$

If there is an isomorphism from G onto $\overline{G}$, we say that G and $\overline{G}$ are *isomorphic* and write $G \approx \overline{G}$.

The definition of isomorphism ensures that if ϕ is an isomorphism from G to $\overline{G}$ then the operation table for $\overline{G}$ can be obtained from the operation table for G be replacing each entry in the table for G by $\phi(x)$. See Figure 6.1. Thus the groups differ in notation only.

G	–	–	–	b	–	–
–	–	–	–	–	–	–
–	–	–	–	–	–	–
a	–	–	–	ab	–	–
–	–	–	–	–	–	–

$\overline{G}$	–	–	–	$\phi(b)$	–	–
–	–	–	–	–	–	–
–	–	–	–	–	–	–
$\phi(a)$	–	–	–	$\phi(ab)$	–	–
–	–	–	–	–	–	–

Figure 6.1 Operations table for G and $\overline{G}$.

It is implicit in the definition of isomorphism that isomorphic groups have the same order. It is also implicit in the definition of isomorphism that the operation on the left side of the equal sign is that of G, whereas the operation on the right side is that of $\overline{G}$. The four cases involving $\cdot$ and $+$ are shown in Table 6.1.

There are four separate steps involved in proving that a group G is isomorphic to a group $\overline{G}$.

Table 6.1

G Operation	$\overline{G}$ Operation	Operation Preservation
$\cdot$	$\cdot$	$\phi(a \cdot b) = \phi(a) \cdot \phi(b)$
$\cdot$	$+$	$\phi(a \cdot b) = \phi(a) + \phi(b)$
$+$	$\cdot$	$\phi(a + b) = \phi(a) \cdot \phi(b)$
$+$	$+$	$\phi(a + b) = \phi(a) + \phi(b)$

Step 1 "Mapping." Define a candidate for the isomorphism; that is, define a function ϕ from G to $\overline{G}$.

Step 2 "1–1." Prove that ϕ is one-to-one; that is, assume that $\phi(a) = \phi(b)$ and prove that $a = b$.

Step 3 "Onto." Prove that ϕ is onto; that is, for any element $\overline{g}$ in $\overline{G}$, find an element g in G such that $\phi(g) = \overline{g}$.

Step 4 "O.P." Prove that ϕ is operation-preserving; that is, show that $\phi(ab) = \phi(a)\phi(b)$ for all a and b in G.

None of these steps is unfamiliar to you. The only one that may appear novel is the fourth one. It requires that one be able to obtain the same result by combining two elements and then mapping, or by mapping two elements and then combining them. Roughly speaking, this says that the two processes—operating and mapping—can be done in either order without affecting the result. This same concept arises in calculus when we say

$$\lim_{x \to a} (f(x) \cdot g(x)) = \lim_{x \to a} f(x) \lim_{x \to a} g(x)$$

or

$$\int_a^b (f + g)dx = \int_a^b f\,dx + \int_a^b g\,dx.$$

In linear algebra an invertible linear transformation from a vector space V onto a vector space W is a group isomorphism from V to W. (Every vector space is an Abelian group under vector addition.)

Before going any further, let's consider some examples.

▮ EXAMPLE 1 Let G be the real numbers under addition and let $\overline{G}$ be the positive real numbers under multiplication. Then G and

$\overline{G}$ are isomorphic under the mapping $\phi(x) = 2^x$. Certainly, ϕ is a function from G to $\overline{G}$. To prove that it is one-to-one, suppose that $2^x = 2^y$. Then $\log_2 2^x = \log_2 2^y$, and therefore $x = y$. For "onto," we must find for any positive real number y some real number x such that $\phi(x) = y$; that is, $2^x = y$. Well, solving for x gives $\log_2 y$. Finally,

$$\phi(x + y) = 2^{x+y} = 2^x \cdot 2^y = \phi(x)\phi(y)$$

for all x and y in G, so that ϕ is operation-preserving as well. ■

■ **EXAMPLE 2** Any infinite cyclic group is isomorphic to Z. Indeed, if a is a generator of the cyclic group, the mapping $a^k \to k$ is an isomorphism. Any finite cyclic group $\langle a \rangle$ of order n is isomorphic to Z_n under the mapping $a^k \to k \bmod n$. That these correspondences are functions and are one-to-one is the essence of Theorem 4.1. Obviously, the mappings are onto. That the mappings are operation-preserving follows from Exercise 11 in Chapter 0 in the finite case and from the definitions in the infinite case. ■

■ **EXAMPLE 3** The mapping from $\mathbf{R}$ under addition to itself given by $\phi(x) = x^3$ is *not* an isomorphism. Although ϕ is one-to-one and onto, it is not operation-preserving, since it is not true that $(x + y)^3 = x^3 + y^3$ for all x and y. ■

■ **EXAMPLE 4** $U(10) \approx Z_4$ and $U(5) \approx Z_4$. To verify this, one need only observe that both $U(10)$ and $U(5)$ are cyclic of order 4. Then appeal to Example 2. ■

■ **EXAMPLE 5** There is no isomorphism from Q, the group of rational numbers under addition, to Q^*, the group of nonzero rational numbers under multiplication. If ϕ were such a mapping, there would be a rational number a such that $\phi(a) = -1$. But then

$$-1 = \phi(a) = \phi\left(\frac{1}{2}a + \frac{1}{2}a\right) = \phi\left(\frac{1}{2}a\right)\phi\left(\frac{1}{2}a\right) = \left(\phi\left(\frac{1}{2}a\right)\right)^2.$$

However, no rational number squared is -1. ■

■ **EXAMPLE 6** Let $G = SL(2, \mathbf{R})$, the group of 2×2 real matrices with determinant 1. Let M be any 2×2 real matrix with determinant 1. Then we can define a mapping from G to G itself by $\phi_M(A) = MAM^{-1}$ for all A in G. To verify that ϕ_M is an isomorphism, we carry out the four steps.

Step 1 ϕ_M is a function from G to G. Here, we must show that $\phi_M(A)$ is indeed an element of G whenever A is. This follows from properties of determinants:

$$\det(MAM^{-1}) = (\det M)(\det A)(\det M)^{-1} = 1 \cdot 1 \cdot 1^{-1} = 1.$$

Thus, MAM^{-1} is in G.

Step 2 ϕ_M is one-to-one. Suppose that $\phi_M(A) = \phi_M(B)$. Then $MAM^{-1} = MBA^{-1}$ and, by left and right cancellation, $A = B$.

Step 3 ϕ_M is onto. Let B belong to G. We must find a matrix A in G such that $\phi_M(A) = B$. How shall we do this? If such a matrix A is to exist, it must have the property that $MAM^{-1} = B$. But this tells us exactly what A must be! For we can solve for A to obtain $A = M^{-1}BM$ and verify that $\phi_M(A) = MAM^{-1} = M(M^{-1}BM)M^{-1} = B$.

Step 4 ϕ_M is operation-preserving. Let A and B belong to G. Then,

$$\begin{aligned}\phi_M(AB) &= M(AB)M^{-1} = MA(M^{-1}M)BM^{-1} \\ &= (MAM^{-1})(MBM^{-1}) = \phi_M(A)\phi_M(B).\end{aligned}$$

The mapping ϕ_M is called *conjugation* by M. ∎

Properties of Isomorphisms

Our first two theorems give a catalog of properties of isomorphisms and isomorphic groups.

Theorem 6.1 Properties of Isomorphisms Acting on Elements

Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. Then

1. ***ϕ carries the identity of G to the identity of $\overline{G}$.***
2. ***For every integer n and for every group element a in G, $\phi(a^n) = [\phi(a)]^n$. (Additive form: $\phi(na) = n\phi(a)$.)***
3. ***For any elements a and b in G, a and b commute if and only if $\phi(a)$ and $\phi(b)$ commute.***
4. ***$G = \langle a \rangle$ if and only if $\overline{G} = \langle \phi(a) \rangle$.***

5. ***$|a| = |\phi(a)|$ for all a in G (isomorphisms preserve orders).***
6. ***For a fixed integer k and a fixed group element b in G, the equation $x^k = b$ has the same number of solutions in G as does the equation $x^k = \phi(b)$ in $\overline{G}$.***
7. ***If G is finite, then G and G have exactly the same number of elements of every order.***

PROOF We will restrict ourselves to proving only properties 1, 2, and 4, but observe that property 5 follows from properties 1 and 2, property 6 follows from property 2, and property 7 follows from property 5. For convenience, let us denote the identity in G by e and the identity in $\overline{G}$ by $\overline{e}$. Then, since $\phi(e) \in \overline{G}$ and $e = ee$, we have $\overline{e}\phi(e) = \phi(e) = \phi(ee) = \phi(e)\phi(e)$. Thus, by right cancellation, $\overline{e} = \phi(e)$. This proves property 1.

For positive integers, property 2 follows from the definition of an isomorphism and mathematical induction. If n is negative, then $-n$ is positive, and we have from property 1 and the observation about the positive integer case that $e = \phi(e) = \phi(g^n g^{-n}) = \phi(g^n)\phi(g^{-n}) = \phi(g^n)(\phi(g))^{-n}$. Thus, multiplying both sides on the right by $(\phi(g))^n$, we have $(\phi(g))^n = \phi(g^n)$. Property 1 takes care of the case $n = 0$.

To prove property 4, let $G = \langle a \rangle$ and note that, by closure, $\langle \phi(a) \rangle \subseteq \overline{G}$. Because ϕ is onto, for any element b in $\overline{G}$, there is an element a^k in G such that $\phi(a^k) = b$. Thus, $b = (\phi(a))^k$ and so $b \in \langle \phi(a) \rangle$. This proves that $\overline{G} = \langle \phi(a) \rangle$.

Now suppose that $\overline{G} = \langle \phi(a) \rangle$. Clearly, $\langle a \rangle \subseteq G$. For any element b in G, we have $\phi(b) \in \langle \phi(a) \rangle$. So, for some integer k we have $\phi(b) = (\phi(a))^k = \phi(a^k)$. Because ϕ is one-to-one, $b = a^k$. This proves that $\langle a \rangle = G$. ∎

Property 4 says that an isomorphism between two cyclic groups takes a generator to a generator.

Theorem 6.2 Properties of Isomorphisms Acting on Groups

Suppose that ϕ is an isomorphism from a group G onto a group $\overline{G}$. Then
1. ***ϕ^{-1} is an isomorphism from $\overline{G}$ onto G.***
2. ***G is Abelian if and only if $\overline{G}$ is Abelian.***

3. ***G is cyclic if and only if $\overline{G}$ is cyclic.***
4. ***If K is a subgroup of G, then $\phi(K) = \{\phi(k) \mid k \in K\}$ is a subgroup of $\overline{G}$.***
5. ***If $\overline{K}$ is a subgroup of $\overline{G}$, then $\phi^{-1}(\overline{K}) = \{g \in G \mid \phi(g) \in \overline{K}\}$ is a subgroup of G.***
6. ***$\phi(Z(G)) = Z(\overline{G})$.***

PROOF Properties 1 and 4 are left as exercises (Exercises 17 and 34). Properties 2 and 6 are a direct consequence of property 3 of Theorem 6.1. Property 3 follows from property 4 of Theorem 6.1 and property 1 of Theorem 6.2. Property 5 follows from properties 1 and 4. ■

Theorems 6.1 and 6.2 provide several convenient ways to prove that groups G and $\overline{G}$ are not isomorphic.

1. Observe that $|G| \neq |\overline{G}|$.
2. Observe that G or $\overline{G}$ is cyclic and the other is not.
3. Observe that G or $\overline{G}$ is Abelian and the other is not.
4. Show that largest order of any element in G is not the same as the largest order of any element in $\overline{G}$.
5. Show that the number of elements of some specific order in G (the smallest order greater than 1 is often the good choice) is not the same as the number of elements of that order in $\overline{G}$.

■ **EXAMPLE 7** Consider these three groups of order 12: Z_{12}, D_6, and A_4. A quick check shows that the largest order of any element in the three are 12, 6, and 3, respectively. So no two are isomorphic. Alternatively, the number of elements of order 2 in each is 1, 7, and 3. ■

■ **EXAMPLE 8** The group Q of rational numbers under addition is not isomorphic to the group Q^* of nonzero rational numbers under multiplication because every non-identity element of Q has infinite order (because $nx = 0$ if and only if $n = 0$) or $x = 0$ whereas in Q^*, $|-1| = 2$. ■

Theorems 6.1 and 6.2 show that isomorphic groups have many properties in common. Actually, the definition is precisely formulated so that isomorphic groups have *all* group theoretic properties in common.

By this we mean that if two groups are isomorphic, then any property that can be expressed in the language of group theory is true for one if and only if it is true for the other. This is why algebraists speak of isomorphic groups as "equal" or "the same." Admittedly, calling such groups equivalent, rather than the same, might be more appropriate, but we bow to long-standing tradition.

Besides providing a method for determining if two groups have identical group theoretical properties, isomorphisms allow us to deduce properties about a group G that is not convenient to work with by using a group $\overline{G}$ isomorphic to G that is easier to work with. Exercise 41 provides an example of a question about a particular group that seems difficult to answer that becomes easy when viewed as a question about an isomorphic version of the group. Many more examples will be given in Chapters 8 and 11. If G and $\overline{G}$ are isomorphic groups you can think of G as a camouflaged version of $\overline{G}$.

Automorphisms

Certain kinds of isomorphisms are referred to so often that they have been given special names.

Definition Automorphism
An isomorphism from a group G onto itself is called an automorphism of G.

The isomorphism in Example 6 is an automorphism of $SL(2, \mathbf{R})$. Two more examples follow.

EXAMPLE 9 The function ϕ from $\mathbf{C}$ to $\mathbf{C}$ given by $\phi(a + bi) = a - bi$ is an automorphism of the group of complex numbers under addition. The restriction of ϕ to $\mathbf{C}^*$ is also an automorphism of the group of nonzero complex numbers under multiplication. (See Exercise 39.) ∎

EXAMPLE 10 Let $\mathbf{R}^2 = \{(a, b) \mid a, b \in \mathbf{R}\}$. Then $\phi(a, b) = (b, a)$ is an automorphism of the group $\mathbf{R}^2$ under componentwise addition. Geometrically, ϕ reflects each point in the plane across the line $y = x$. In fact, any reflection across a line passing through the origin or any rotation of the plane about the origin is an automorphism of $\mathbf{R}^2$. ∎

More generally, every invertible linear transformation of a vector space V to itself is an automorphism of V.

The isomorphism in Example 6 is a particular instance of an automorphism that arises often enough to warrant a name and notation of its own.

Definition Inner Automorphism Induced by a

Let G be a group, and let $a \in G$. The function ϕ_a defined by $\phi_a(x) = axa^{-1}$ for all x in G is called the *inner automorphism of G induced by a*.

We leave it for the reader to show that ϕ_a is actually an automorphism of G. (Use Example 6 as a model.)

EXAMPLE 11 The action of the inner automorphism of D_4 induced by R_{90} is given in the following table. ∎

x	$\xrightarrow{\phi_{R_{90}}}$	$R_{90}\, x\, {R_{90}}^{-1}$
R_0	$\to$	$R_{90}R_0{R_{90}}^{-1} = R_0$
R_{90}	$\to$	$R_{90}R_{90}{R_{90}}^{-1} = R_{90}$
R_{180}	$\to$	$R_{90}R_{180}{R_{90}}^{-1} = R_{180}$
R_{270}	$\to$	$R_{90}R_{270}{R_{90}}^{-1} = R_{270}$
H	$\to$	$R_{90}H{R_{90}}^{-1} = V$
V	$\to$	$R_{90}V{R_{90}}^{-1} = H$
D	$\to$	$R_{90}D{R_{90}}^{-1} = D'$
D'	$\to$	$R_{90}D'{R_{90}}^{-1} = D$

When G is a group, we use $\mathrm{Aut}(G)$ to denote the set of all automorphisms of G and $\mathrm{Inn}(G)$ to denote the set of all inner automorphisms of G. The reason these sets are noteworthy is demonstrated by the next theorem.

Theorem 6.3 Aut(G) and Inn(G) are Groups[1]

The set of automorphisms of a group and the set of inner automorphisms of a group are both groups under the operation of function composition.

[1]The group $\mathrm{Aut}(G)$ was first studied by O. Hölder in 1893 and, independently, by E. H. Moore in 1894.

PROOF The proof of Theorem 6.3 is left as an exercise (Exercise 17). ■

The determination of $\text{Inn}(G)$ is routine. If $G = \{e, a, b, c \ldots.\}$, then $\text{Inn}(G) = \{\phi_e, \phi_a, \phi_b, \phi_c, \ldots\}$. This latter list may have duplications, however, since ϕ_a may be equal to ϕ_b even though $a \neq b$ (see Exercise 47). Thus, the only work involved in determining $\text{Inn}(G)$ is deciding which distinct elements give the distinct automorphisms. On the other hand, the determination of $\text{Aut}(G)$ is, in general, quite involved.

▮ EXAMPLE 12 $\text{Inn}(\boldsymbol{D_4})$

To determine $\text{Inn}(D_4)$, we first observe that the complete list of inner automorphisms is $\phi_{R_0}, \phi_{R_{90}}, \phi_{R_{180}}, \phi_{R_{270}}, \phi_H, \phi_V, \phi_D$, and $\phi_{D'}$. Our job is to determine the repetitions in this list. Since $R_{180} \in Z(D_4)$, we have $\phi_{R_{180}}(x) = R_{180}xR_{180}^{-1} = x$, so that $\phi_{R_{180}} = \phi_{R_0}$. Also, $\phi_{R_{270}}(x) = R_{270}xR_{270}^{-1} = R_{90}R_{180}xR_{180}^{-1}R_{90}^{-1} = R_{90}xR_{90}^{-1} = \phi_{R_{90}}(x)$, Similarly, since $H = R_{180}V$ and $D' = R_{180}D$, we have $\phi_H = \phi_V$ and $\phi_D = \phi_{D'}$. This proves that the previous list can be pared down to $\phi_{R_0}, \phi_{R_{90}}, \phi_H$, and ϕ_D. We leave it to the reader to show that these are distinct (Exercise 15). ■

The next example shows how the inner automorphisms of a group provide a convenient way to create multiple isomorphic subgroups of the group.

▮ EXAMPLE 13 Given the subgroup of S_4

$$H = \{(1), (1234), (13)(24), (1432), (12)(34), (24), (14)(23), (13)\}$$

we have the subgroups

$$(12)H(21) = \{(1), (1342), (14)(23), (1234), (12)(34), (14), (13)(24), (23)\}$$

and

$$(123)H(321) = \{(1), (1423), (12)(34), (1324), (14)(23), (34), (13)(24), (12)\}$$

of S_4 that are isomorphic to H. ■

▮ EXAMPLE 14 $\textbf{Aut}(\boldsymbol{Z_{10}})$

To compute $\text{Aut}(Z_{10})$, we try to discover enough information about an element α of $\text{Aut}(Z_{10})$ to determine how α must be defined. Because Z_{10} is so simple, this is not difficult to do. To begin with, observe that once we know $\alpha(1)$, we know $\alpha(k)$ for any k, because

$$\begin{aligned}\alpha(k) &= \underbrace{\alpha(1+1+\cdots+1)}_{k \text{ terms}} \\ &= \underbrace{\alpha(1)+\alpha(1)+\cdots+\alpha(1)}_{k \text{ terms}} = k\alpha(1).\end{aligned}$$

So, we need only determine the choices for $\alpha(1)$ that make α an automorphism of Z_{10}. Since property 5 of Theorem 6.1 tells us that $|\alpha(1)| = 10$, there are four candidates for $\alpha(1)$:

$$\alpha(1) = 1, \quad \alpha(1) = 3, \quad \alpha(1) = 7, \quad \alpha(1) = 9.$$

To distinguish among the four possibilities, we refine our notation by denoting the mapping that sends 1 to 1 by α_1, 1 to 3 by α_3, 1 to 7 by α_7, and 1 to 9 by α_9. So the only possibilities for $\text{Aut}(Z_{10})$ are α_1, α_3, α_7, and α_9. But are all these automorphisms? Clearly, α_1 is the identity. Let us check α_3. Since $x \bmod 10 = y \bmod 10$ implies $3x \bmod 10 = 3y \bmod 10$, α_3 is well defined. Moreover, because $a_3(1) = 3$ is a generator of Z_{10}, it follows that α_3 is onto (and, by Exercise 13 in Chapter 5, it is also one-to-one). Finally, since $\alpha_3(a+b) = 3(a+b) = 3a+3b = \alpha_3(a)+\alpha_3(b)$, we see that α_3 is operation-preserving as well. Thus, $\alpha_3 \in \text{Aut}(Z_{10})$. The same argument shows that α_7 and α_9 are also automorphisms.
This gives us the elements of $\text{Aut}(Z_{10})$ but not the structure. For instance, what is $\alpha_3\alpha_3$? Well, $(\alpha_3\alpha_3)(1) = \alpha_3(3) = 3 \cdot 3 = 9 = \alpha_9(1)$, so $\alpha_3\alpha_3 = \alpha_9$. Similar calculations show that ${\alpha_3}^3 = \alpha_7$ and ${\alpha_3}^4 = \alpha_1$, so that $|\alpha_3| = 4$. Thus, $\text{Aut}(Z_{10})$ is cyclic. Actually, the following Cayley tables reveal that $\text{Aut}(Z_{10})$ is isomorphic to $U(10)$.

$\boldsymbol{U(10)}$	**1**	**3**	**7**	**9**
1	1	3	7	9
3	3	9	1	7
7	7	1	9	3
9	9	7	3	1

$\textbf{Aut}(\boldsymbol{Z_{10}})$	$\boldsymbol{\alpha_1}$	$\boldsymbol{\alpha_3}$	$\boldsymbol{\alpha_7}$	$\boldsymbol{\alpha_9}$
$\boldsymbol{\alpha_1}$	α_1	α_3	α_7	α_9
$\boldsymbol{\alpha_3}$	α_3	α_9	α_1	α_7
$\boldsymbol{\alpha_7}$	α_7	α_1	α_9	α_3
$\boldsymbol{\alpha_9}$	α_9	α_7	α_3	α_1

∎

With Example 14 as a guide, we are now ready to tackle the group $\text{Aut}(Z_n)$. The result is particularly nice, since it relates the two kinds of groups we have most frequently encountered thus far—the cyclic groups Z_n and the U-groups $U(n)$.

Theorem 6.4 $\text{Aut}(Z_n) \approx U(n)$

For every positive integer n, Aut(Z_n) is isomorphic to $U(n)$.

PROOF As in Example 14, any automorphism α is determined by the value of $\alpha(1)$, and $\alpha(1) \in U(n)$. Now consider the correspondence from $\text{Aut}(Z_n)$ to $U(n)$ given by $T : \alpha \to \alpha(1)$. The fact that $\alpha(k) = k\alpha(1)$ implies that T is a one-to-one mapping. For if α and β belong to $\text{Aut}(Z_n)$ and $\alpha(1) = \beta(1)$, then $\alpha(k) = k\alpha(1) = k\beta(1) = \beta(k)$ for all k in Z_n, and therefore $\alpha = \beta$.

To prove that T is onto, let $r \in U(n)$ and consider the mapping α from Z_n to Z_n defined by $\alpha(s) = sr \pmod{n}$ for all s in Z_n. We leave it as an exercise to verify that α is an automorphism of Z_n (see Exercise 31). Then, since $T(\alpha) = \alpha(1) = r, T$ is onto $U(n)$.

Finally, we establish the fact that T is operation-preserving. Let $\alpha, \beta \in \text{Aut}(Z_n)$. We then have

$$
\begin{aligned}
T(\alpha\beta) &= (\alpha\beta)(1) = \alpha(\beta(1)) = \alpha(\underbrace{1 + 1 + \cdots + 1}_{\beta(1)}) \\
&= \underbrace{\alpha(1) + \alpha(1) + \cdots + \alpha(1)}_{\beta(1)} = \alpha(1)\beta(1). \\
&= T(\alpha)T(\beta).
\end{aligned}
$$

This completes the proof. ∎

Cayley's Theorem

Our final theorem is a classic result of Cayley. An important generalization of it will be given in Chapter 24.

Theorem 6.5 Cayley's Theorem (1854)

Every group is isomorphic to a group of permutations.

PROOF To prove this, let G be any group. We must find a group $\overline{G}$ of permutations that we believe is isomorphic to G. Since G is all we have to work with, we will have to use it to construct $\overline{G}$. For any g in G, define a function T_g from G to G by

$$T_g(x) = gx \qquad \text{for all } x \text{ in } G.$$

(In words, T_g is just multiplication by g on the left.) We leave it as an exercise (Exercise 37) to prove that T_g is a permutation on the set of elements of G. Now, let $\overline{G} = \{T_g \mid g \in G\}$. Then, $\overline{G}$ is a group under the operation of function composition. To verify this, we first observe that for any g and h in G we have $T_gT_h(x) = T_g(T_h(x)) = T_g(hx) = g(hx) = (gh)x = T_{gh}(x)$, so that $T_gT_h = T_{gh}$. From this it follows that T_e is the identity and $(T_g)^{-1} = T_{g^{-1}}$ (see Exercise 9). Since function composition is associative, we have verified all the conditions for $\overline{G}$ to be a group.

The isomorphism ϕ between G and $\overline{G}$ is now ready-made. For every g in G, define $\phi(g) = T_g$. If $T_g = T_h$, then $T_g(e) = T_h(e)$ or $ge = he$. Thus, $g = h$ and ϕ is one-to-one. By the way $\overline{G}$ was constructed, we see that ϕ is onto. The only condition that remains to be checked is that ϕ is operation-preserving. To this end, let a and b belong to G. Then

$$\phi(ab) = T_{ab} = T_aT_b = \phi(a)\phi(b).$$

■

The group $\overline{G}$ constructed previously is called the *left regular representation of G.*

EXAMPLE 15 For concreteness, let us calculate the left regular representation $\overline{U(12)}$ for $U(12) = \{1, 5, 7, 11\}$. Writing the permutations of $U(12)$ in array form, we have (remember, T_x is just multiplication by x)

$$T_1 = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 1 & 5 & 7 & 11 \end{bmatrix}, \quad T_5 = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 5 & 1 & 11 & 77 \end{bmatrix},$$

$$T_7 = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 7 & 11 & 1 & 5 \end{bmatrix}, \quad T_{11} = \begin{bmatrix} 1 & 5 & 7 & 11 \\ 11 & 7 & 5 & 1 \end{bmatrix}.$$

It is instructive to compare the Cayley tables for $U(12)$ and its left regular representation $\overline{U(12)}$.

$U(12)$	1	5	7	11
1	1	5	7	11
5	5	1	11	7
7	7	11	1	5
11	11	7	5	1

$\overline{U(12)}$	T_1	T_5	T_7	T_{11}
T_1	T_1	T_5	T_7	T_{11}
T_5	T_5	T_1	T_{11}	T_7
T_7	T_7	T_{11}	T_1	T_5
T_{11}	T_{11}	T_7	T_5	T_1

∎

It should be abundantly clear from these tables that $U(12)$ and $\overline{U(12)}$ are only notationally different.

▮ EXAMPLE 16 Writing the left regular representations for the permutations $T_{R_{270}}$ and T_H from D_4 in disjoint cycle form we have (see the Cayley table in Chapter 1):

$$T_{R_{270}} = (R_0 R_{270})(R_{90} R_0)(HD)(VD')$$
$$T_H = (R_0 H)(R_{90} D)(R_{180} V)(T_{270} D')$$

∎

Cayley's Theorem is important for two contrasting reasons. One is that it allows us to represent an abstract group in a concrete way. A second is that it shows that the present-day set of axioms we have adopted for a group is the correct abstraction of its much earlier predecessor—a group of permutations. Indeed, Cayley's Theorem tells us that abstract groups are not different from permutation groups. Rather, it is the viewpoint that is different. It is this difference of viewpoint that has stimulated the tremendous progress in group theory and many other branches of mathematics in the past 100 years.

It is sometimes very difficult to prove or disprove, whichever the case may be, that two particular groups are isomorphic. For example, it requires somewhat sophisticated techniques to prove the surprising fact that the group of real numbers under addition is isomorphic to the group of complex numbers under addition. Likewise, it is not easy to prove the fact that the group of nonzero complex numbers under multiplication is isomorphic to the group of complex numbers with absolute value of 1 under multiplication. In geometric terms, this says that, as groups, the punctured plane and the unit circle are isomorphic.

Exercises

Being a mathematician is a bit like being a manic depressive: you spend your life alternating between giddy elation and black despair.

Steven G. Krantz, *A Primer of Mathematical Writing*

1. Find an isomorphism from the group of integers under addition to the group of even integers under addition.
2. Find Aut(Z) and Aut(Z_6).
3. Let $\mathbf{R}^+$ be the group of positive real numbers under multiplication. Show that the mapping $\phi(x) = \sqrt{x}$ is an automorphism of $\mathbf{R}^+$.
4. Show that $U(8)$ is not isomorphic to $U(10)$. Show that U(8) is isomorphic to $U(12)$.
5. Let $G = \{0, \pm 2, \pm 4, \pm 6, \ldots\}$ and $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$. Prove that G and H are isomorphic groups under addition by defining a mapping that has the required properties. Does your isomorphism preserve multiplication? Generalize to the case when $G = \langle m \rangle$ and $H = \langle n \rangle$, where m and n are integers.
6. Prove that isomorphism is an equivalence relation. That is, for any groups G, H, and K

   $G \approx G$

   $G \approx H$ implies $H \approx G$

   $G \approx H$ and $H \approx K$ implies $G \approx K$.
7. Give three reasons why S_4 is not isomorphic to D_{12}.
8. Show that the mapping $a \to \log_{10} a$ is an isomorphism from $\mathbf{R}^+$ under multiplication to $\mathbf{R}$ under addition.
9. In the notation of Theorem 6.5, prove that T_e is the identity and that $(T_g)^{-1} = T_{g^{-1}}$.
10. Given that ϕ is a isomorphism from a group G under addition to a group $\overline{G}$ under addition, convert property 2 of Theorem 6.2 to additive notation.
11. Let G be a group under multiplication, $\overline{G}$ be a group under addition and ϕ be an isomorphism from G to $\overline{G}$. If $\phi(a) = \overline{a}$ and $\phi(b) = \overline{b}$, find an expression for $\phi(a^3 b^{-2})$ in terms of $\overline{a}$ and $\overline{b}$.

12. Let G be a group. Prove that the mapping $\alpha(g) = g^{-1}$ for all g in G is an automorphism if and only if G is Abelian.

13. If g and h are elements from a group, prove that $\phi_g\phi_h = \phi_{gh}$.

14. Prove or disprove that $U(6) \approx Z_6$.

15. Prove the assertion in Example 12 that the inner automorphisms $\phi_{R_0}, \phi_{R_{90}}, \phi_H$ and ϕ_D of D_4 are distinct.

16. Find two groups G and H such that $G \not\approx H$, but $\mathrm{Aut}(G) \approx \mathrm{Aut}(H)$.

17. If G is a group, prove that $\mathrm{Aut}(G)$ and $\mathrm{Inn}(G)$ are groups. (This exercise is referred to in this chapter.)

18. If a group G is isomorphic to H, prove that $\mathrm{Aut}(G)$ is isomorphic to $\mathrm{Aut}(H)$.

19. If ϕ_a is the inner automorphism of G induced by a (that is, $\phi_a(x) = axa^{-1}$ for all x in G), prove that $(\phi_a)^n = \phi_{a^n}$ for all integers n.

20. For any group G and inner automorphism ϕ_a of G prove $|\phi_a| = |a|$.

21. Suppose ϕ belongs to $\mathrm{Aut}(Z_n)$ and a is relatively prime to n. If $\phi(a) = b$, determine a formula for $\phi(x)$.

22. Let H be the subgroup of all rotations in D_n and let ϕ be an automorphism of D_n. Prove that $\phi(H) = H$. (In words, an automorphism of D_n carries rotations to rotations.)

23. Let $H = \{\beta \in S_5 \mid \beta(1) = 1\}$ and $K = \{\beta \in S_5 \mid \beta(2) = 2\}$. Prove that H is isomorphic to K. Is the same true if S_5 is replaced by S_n, where $n \geq 3$?

24. Show that Z has infinitely many subgroups isomorphic to Z.

25. Let n be an even integer greater than 2 and let ϕ be an automorphism of D_n. Determine $\phi(R_{180})$.

26. Suppose that G is an Abelian group and ϕ is an automorphism of G. Prove that $H = \{x \in G \mid \phi(x) = x^4\}$ is a subgroup of G. If $\phi(x) = x^{-1}$, what can you say about the order of elements of H? Does your proof work when 4 is replaced by any integer n?

27. Give an example of a cyclic group of smallest order that contains both a subgroup isomorphic to Z_{12} and a subgroup isomorphic to Z_{20}. Give an example of a dihedral group that has a subgroup isomorphic to Z_{12} and a subgroup isomorphic to Z_{20}. No need to prove anything but explain your reasoning.

28. Suppose that $\phi : Z_{20} \to Z_{20}$ is an automorphism and $\phi(5) = 5$. What are the possibilities for $\phi(x)$?

29. Identify a group G that has subgroups isomorphic to Z_n for all positive integers n.

30. Prove that every automorphism α of $\mathbf{R}^*$, the group of real numbers under multiplication, $\alpha(-1) = -1$.

31. Let $r \in U(n)$. Prove that the mapping α: $Z_n \to Z_n$ defined by $\alpha(s) = sr \bmod n$ for all s in Z_n is an automorphism of Z_n. (This exercise is referred to in this chapter.)

32. The group $\left\{ \begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix} \middle| \, a \in Z \right\}$ is isomorphic to what familiar group? What if Z is replaced by $\mathbf{R}$?

33. If ϕ and γ are isomorphisms from the cyclic group $\langle a \rangle$ to some group and $\phi(a) = \gamma(a)$, prove that $\phi = \gamma$.

34. Suppose that ϕ: $Z_{50} \to Z_{50}$ is an automorphism with $\phi(7) = 13$. Determine a formula for $\phi(x)$.

35. Find an isomorphism from Z_{10} to $U(11)$.

36. Let ϕ be an automorphism of Q^*. Explain why ϕ is completely determined by the values of $\phi(p)$ for every prime p.

37. Referring to Theorem 6.5, prove that T_g is indeed a permutation on the set G.

38. Prove or disprove that $U(20)$ and $U(24)$ are isomorphic.

39. Show that the mapping $\phi(a + bi) = a - bi$ is an automorphism of the group of complex numbers under addition. Show that ϕ preserves complex multiplication as well—that is, $\phi(xy) = \phi(x)\phi(y)$ for all x and y in $\mathbf{C}$. (This exercise is referred to in Chapter 15.)

40. Let

$$G = \{a + b\sqrt{2} \mid a, b \text{ are rational}\}$$

and

$$H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \middle| \, a, b \text{ are rational} \right\}.$$

Show that G and H are isomorphic under addition. Prove that G and H are closed under multiplication. Does your isomorphism preserve multiplication as well as addition? (G and H are examples of rings—a topic we will take up later.)

41. Prove that Z under addition is not isomorphic to Q under addition.

42. Explain why S_8 contains subgroups isomorphic to Z_{15}, $U(16)$, and D_8.

43. Let $\mathbf{C}$ be the complex numbers and

$$M = \left\{ \begin{bmatrix} a & -b \\ b & a \end{bmatrix} \,\middle|\, a, b \in \mathbf{R} \right\}.$$

Prove that $\mathbf{C}$ and $\boldsymbol{M}$ are isomorphic under addition and that $\mathbf{C}^*$ and $\boldsymbol{M}^*$, the nonzero elements of $\boldsymbol{M}$, are isomorphic under multiplication.

44. Let $\mathbf{R}^n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in \mathbf{R}\}$. Show that the mapping ϕ: $(a_1, a_2, \ldots, a_n) \to (-a_1, -a_2, \ldots, -a_n)$ is an automorphism of the group $\mathbf{R}^n$ under componentwise addition. This automorphism is called *inversion.* Describe the action of ϕ geometrically.

45. Consider the following statement: The order of a subgroup divides the order of the group. Suppose you could prove this for finite permutation groups. Would the statement then be true for all finite groups? Explain.

46. Suppose that G is a finite Abelian group and G has no element of order 2. Show that the mapping $g \to g^2$ is an automorphism of G. Show, by example, that there is an infinite Abelian group for which the mapping $g \to g^2$ is one-to-one and operation-preserving but not an automorphism.

47. Let G be a group and let $g \in G$. If $z \in Z(G)$, show that the inner automorphism induced by g is the same as the inner automorphism induced by zg (that is, that the mappings ϕ_g and ϕ_{zg} are equal).

48. Prove that $\mathbf{R}$ under addition is not isomorphic to $\mathbf{R}^*$ under multiplication.

49. Suppose that g and h induce the same inner automorphism of a group G. Prove that $h^{-1}g \in Z(G)$. Combine this exercise with the previous one to create a single "if and only if" theorem.

50. Give two examples of nonidentity automorphisms of S_6.

51. If α and β are elements in S_n $(n \geq 3)$, prove that $\phi_\alpha = \phi_\beta$ implies that $\alpha = \beta$. (Here, ϕ_α is the inner automorphism of S_n induced by α.)

52. Prove or disprove that the mapping ϕ from Q^+, the positive rational numbers under multiplication, to itself given by $\phi(x) = x^2$ is an automorphism.

53. Suppose the ϕ and γ are isomorphisms of some group G to the same group. Prove that $H = \{g \in G \mid \phi(g) = \gamma(g)\}$ is a subgroup of G.

54. Let G be a group. Complete the following statement: $|\text{Inn}(G)| = 1$ if and only if ________________.

55. In S_6 use the subgroup $H = \{(1), (135), (153), (35)(26), (15)(26), (13)(26)\}$ to find two other subgroups of order 6 in S_6.

56. Let ϕ be an automorphism of D_8. What are the possibilities for $\phi(R_{45})$?

57. Let ϕ be an automorphism of **C***, the group of nonzero complex numbers under multiplcation. Determine $\phi(-1)$. Determine the possibilities for $\phi(i)$.

58. If G is a non-Abelian group, prove that G has an automorphism that is not the identity.

59. Give three examples of groups of order 120, no two of which are isomophic. Explain why they are not isomorphic.

60. Let ϕ be an automorphism of D_4 such that $\phi(H) = D$. Find $\phi(V)$.

61. Suppose that ϕ is an automorphism of D_4 such that $\phi(R_{90}) = R_{270}$ and $\phi(V) = V$. Determine $\phi(D)$ and $\phi(H)$. Determine $\phi(D)$ and $\phi(H)$ if $\phi(R_{90}) = R_{90}$ and $\phi(V) = V$.

62. In $\text{Aut}(Z_9)$, let α_i denote the automorphism that sends 1 to i where $\gcd(i, 9) = 1$. Write α_5 and α_8 as permutations of $\{0, 1, \ldots, 8\}$ in disjoint cycle form. [For example, $\alpha_2 = (0)(124875)(36)$.]

63. Write the permutation corresponding to R_{90} in the left regular representation of D_4 in cycle form.

64. Suppose that the mapping $\phi(x) = x^2$ is an automorphism of a group G. What can you say about the order of elements in G? What can you say if $\phi(x) = x^n$ for some $n > 1$ is an automorphism of G?

65. Let F be a reflection in D_n. Explain why any automorphism α of D_n is completely determined by the choices for $\alpha(R_{360/n})$ and $\alpha(F)$. Explain why this means that $|\mathrm{Aut}(D_n)|$ is at most $n|U(n)|$. (With some extra effort one can show $|\mathrm{Aut}(D_n)| = n|U(n)|$).

66. Prove that for odd n $|\mathrm{Inn}(D_n)| = 2n$ and for even n $|\mathrm{Inn}(D_n)| = n$.

67. Let $H = \{\alpha \in A_5 \mid \alpha(3) = 3\}$. To what familiar group is H isomorphic? Let $H = \{\alpha \in A_7 \mid \alpha(1) = 1, \alpha(3) = 3, \alpha(5) = 5\}$. To what familiar group is H isomorphic?

68. Show that every automorphism ϕ of the rational numbers Q under addition to itself has the form $\phi(x) = x\phi(1)$.

69. Prove that Q^+, the group of positive rational numbers under multiplication, is isomorphic to a proper subgroup of itself.

70. Prove that Q, the group of rational numbers under addition, is not isomorphic to a proper subgroup of itself.

71. Prove that every automorphism of $\mathbf{R}^*$, the group of nonzero real numbers under multiplication, maps positive numbers to positive numbers and negative numbers to negative numbers.

72. Prove that Q^*, the group of nonzero rational numbers under multiplication, is not isomorphic to Q, the group of rational numbers under addition.

73. Give a group theoretic proof that Q under addition is not isomorphic to $\mathbf{R}^+$ under multiplication.

74. Prove or disprove that Q^* under multipliction is isomorphic to Q^+ under multipliction.

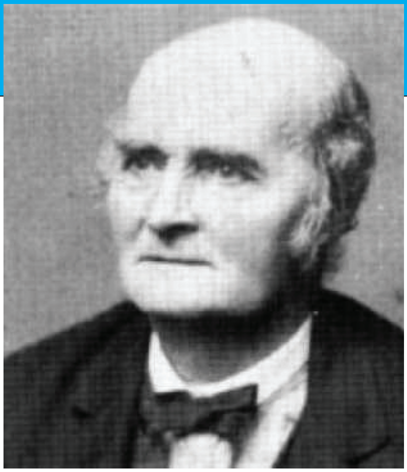
75. For all $n \geq 1$ prove that S_n is isomorphic to a subgroup of A_{n+2}. Why does this not contradict Theorem 5.5? This exercise is referred to in Chapter 24.

Computer Exercises

Software for the computer exercise in this chapter is available at the website:

http://www.d.umn.edu/~jgallian

Arthur Cayley



The Granger Collection, New York

Cayley is forging the weapons for future generations of physicists.

Peter Tait

Arthur Cayley was born on August 16, 1821, in England. His genius showed itself at an early age. He published his first research paper while an undergraduate of 20, and in the next year he published eight papers. While still in his early 20s, he originated the concept of n-dimensional geometry.

After graduating from Trinity College, Cambridge, Cayley stayed on for three years as a tutor. At the age of 25, he began a 14-year career as a lawyer. During this period, he published approximately 200 mathematical papers, many of which are now classics. In 1863, Cayley accepted the newly established Sadlerian professorship of mathematics at Cambridge University.

He spent the rest of his life in that position. One of his notable accomplishments was his role in the successful effort to have women admitted to Cambridge. Among Cayley's many innovations in mathematics were the notions of an abstract group and a group algebra, and the matrix concept. He made major contributions to geometry and linear algebra. Cayley and his lifelong friend and collaborator J. J. Sylvester were the founders of the theory of invariants, which was later to play an important role in the theory of relativity.

Cayley's collected works comprise 13 volumes, each about 600 pages in length. He died on January 26, 1895.

7 Cosets and Lagrange's Theorem

It might be difficult, at this point, for students to see the extreme importance of this result [Lagrange's Theorem]. As we penetrate the subject more deeply they will become more and more aware of its basic character.

I. N. Herstein, *Topics in Algebra*

Lagrange's theorem is extremely important and justly famous in group theory.

Norman J. Block, *Abstract Algebra with Applications*

Properties of Cosets

In this chapter, we will prove the single most important theorem in finite group theory—Lagrange's Theorem. In his book on abstract algebra, I. N. Herstein likened it to the ABC's for finite groups. But first we introduce a new and powerful tool for analyzing a group—the notion of a coset. This notion was invented by Galois in 1830, although the term was coined by G. A. Miller in 1910.

Definitions Coset of H in G

Let G be a group and let H be a nonempty subset of G. For any $a \in G$, the set $\{ah \mid h \in H\}$ is denoted by aH. Analogously, $Ha = \{ha \mid h \in H\}$ and $aHa^{-1} = \{aha^{-1} \mid h \in H\}$. When H is a subgroup of G, the set aH is called the left coset of H in G containing a, whereas Ha is called the right coset of H in G containing a. In this case, the element a is called the coset representative of aH (or Ha). We use $|aH|$ to denote the number of elements in the set aH, and $|Ha|$ to denote the number of elements in Ha.

EXAMPLE 1 Let $G = S_3$ and $H = \{(1), (13)\}$. Then the left cosets of H in G are:

$$\begin{aligned}
(1)H &= H, \\
(12)H &= \{(12), (12)(13)\} = \{(12), (132)\} = (132)H, \\
(13)H &= \{(13), (1)\} = H, \\
(23)H &= \{(23), (23)(13)\} = \{(23), (123)\} = (123)H.
\end{aligned}$$

EXAMPLE 2 Let $\mathcal{K} = \{R_0, R_{180}\}$, in D_4, the dihedral group of order 8. Then,

$$\begin{aligned}
R_0\mathcal{K} &= \mathcal{K}, \\
R_{90}\mathcal{K} &= \{R_{90}, R_{270}\} = R_{270}\mathcal{K}, \\
R_{180}\mathcal{K} &= \{R_{180}, R_0\} = \mathcal{K}, \\
V\mathcal{K} &= \{V, H\} = H\mathcal{K}, \\
D\mathcal{K} &= \{D, D'\} = D'\mathcal{K}.
\end{aligned}$$

EXAMPLE 3 Let $H = \{0, 3, 6\}$ in Z_9 under addition. In the case that the group operation is addition, we use the notation $a + H$ instead of aH. Then the cosets of H in Z_9 are

$$\begin{aligned}
0 + H &= \{0, 3, 6\} = 3 + H = 6 + H, \\
1 + H &= \{1, 4, 7\} = 4 + H = 7 + H, \\
2 + H &= \{2, 5, 8\} = 5 + H = 8 + H.
\end{aligned}$$

The three preceding examples illustrate a few facts about cosets that are worthy of our attention. First, cosets are usually not subgroups. Second, aH may be the same as bH, even though a is not the same as b. Third, since in Example 1 $(12)H = \{(12), (132)\}$ whereas $H(12) = \{(12), (123)\}$, aH need not be the same as Ha.

These examples and observations raise many questions. When does $aH = bH$? Do aH and bH have any elements in common? When does $aH = Ha$? Which cosets are subgroups? Why are cosets important? The next lemma and theorem answer these questions. (Analogous results hold for right cosets.)

Lemma 7.1 Properties of Cosets

Let H be a subgroup of G, and let a and b belong to G. Then,

1. $a \in aH$.
2. $aH = H$ *if and only if* $a \in H$.
3. $(ab)H = a(bH)$ *and* $H(ab) = (Ha)b$.
4. $aH = bH$ *if and only if* $a \in bH$.
5. $aH = bH$ *or* $aH \cap bH = \varnothing$.
6. $aH = bH$ *if and only if* $a^{-1}b \in H$.
7. $|aH| = |bH|$.
8. $aH = Ha$ *if and only if* $H = aHa^{-1}$.
9. aH *is a subgroup of* G *if and only if* $a \in H$.

PROOF

1. $a = ae \in aH$.

2. To verify property 2, we first suppose that $aH = H$. Then $a = ae \in aH = H$. Next, we assume that $a \in H$ and show that $aH \subseteq H$ and $H \subseteq aH$. The first inclusion follows directly from the closure of H. To show that $H \subseteq aH$, let $h \in H$. Then, since $a \in H$ and $h \in H$, we know that $a^{-1}h \in H$. Thus, $h = eh = (aa^{-1})h = a(a^{-1}h) \in aH$.

3. This follows directly from $(ab)h = a(bh)$ and $h(ab) = (ha)b$.

4. If $aH = bH$, then $a = ae \in aH = bH$. Conversely, if $a \in bH$ we have $a = bh$ where $h \in H$, and therefore $aH = (bh)H = b(hH) = bH$.

5. Property 5 follows directly from property 4, for if there is an element c in $aH \cap bH$, then $cH = aH$ and $cH = bH$.

6. Observe that $aH = bH$ if and only if $H = a^{-1}bH$. The result now follows from property 2.

7. To prove that $|aH| = |bH|$, it suffices to define a one-to-one mapping from aH onto bH. Obviously, the correspondence $ah \to bh$ maps aH onto bH. That it is one-to-one follows directly from the cancellation property.

8. Note that $aH = Ha$ if and only if $(aH)a^{-1} = (Ha)a^{-1} = H(aa^{-1}) = H$—that is, if and only if $aHa^{-1} = H$.

9. If aH is a subgroup, then it contains the identity e. Thus, $aH \cap eH \neq \varnothing$; and, by property 5, we have $aH = eH = H$. Thus, from property 2, we have $a \in H$. Conversely, if $a \in H$, then, again by property 2, $aH = H$. ∎

Although most mathematical theorems are written in symbolic form, one should also know what they say *in words.* In the preceding lemma, property 1 says simply that the left coset of H containing a does contain a. Property 2 says that the H "absorbs" an element if and only if the element belongs to H. Property 3 says that the left coset of H created by multiplying H on the left by ab is the same as the one created by multiplying H on the left by b then multiplying the resulting coset bH on the left by a (and analogously for multiplication on the right by ab). Property 4 shows that a left coset of H is uniquely determined by any one of its elements. In particular, any element of a left coset can be used to represent the coset. Property 5 says—and this is very important—that two left cosets of H are either identical or disjoint. Thus, a left coset of H is uniquely determined by any one of its elements. In particular, any element of a left coset can be used to represent the coset. Property 6 shows how we may transfer a question about equality of left cosets of H to a question about H itself and vice versa. Property 7 says that all left cosets of H have the same size. Property 8 is analogous to property 6 in that it shows how a question about the equality of the left and right cosets of H containing a is equivalent to a question about the equality of two subgroups of G. The last property of the lemma says that H itself is the only coset of H that is a subgroup of G.

Note that properties 1, 5, and 7 of the lemma guarantee that the left cosets of a subgroup H of G partition G into blocks of equal size. Indeed, we may view the cosets of H as a partitioning of G into equivalence classes under the equivalence relation defined by $a \sim b$ if $aH = bH$ (see Theorem 0.7).

Cosets of groups allow us to organize the group elements in a coherent way with every element of each coset sharing a special property. In particular, the subgroup H is often chosen so that the cosets partition the group in some highly desirable fashion. For example, if G is 3-space $\mathbf{R}^3$ and H is a plane through the origin, then the coset $(a, b, c) + H$ (addition is done componentwise) is the plane passing through the point (a, b, c) and parallel to H. Thus, the cosets of H constitute a partition of 3-space into planes parallel to H. If $G = GL(2, \mathbf{R})$ and $H = SL(2, \mathbf{R})$, then for any matrix A in G, the coset AH is the set of *all* 2×2 matrices with the same determinant as A. Thus,

$$\begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} H \quad \text{is the set of all } 2 \times 2 \text{ matrices of determinant } 2$$

and

$$\begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix} H \quad \text{is the set of all } 2 \times 2 \text{ matrices of determinant } -3.$$

Similarly, it follows from Example 16 of Chapter 2 and Property 7 of complex numbers in Chapter 0 that if $a + bi = \sqrt{a^2 + b^2}\,(\cos\theta + i\sin\theta)$ the set of n n^{th}-roots of $a+bi$ is the coset of $\langle \cos\frac{360°}{n} + i\sin\frac{360°}{n}\rangle$ that contains $\sqrt[n]{a^2+b^2}(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n})$.

Property 5 of the lemma is useful for actually finding the distinct cosets of a subgroup. We illustrate this in the next example.

EXAMPLE 4 To find the cosets of $H = \{1, 15\}$ in $G = U(32) = \{1, 3, 5, 7, 9, 11, 13, 15,$
$17, 19, 21, 23, 25, 27, 29, 31\}$, we begin with $H = \{1, 15\}$. We can find a second coset by choosing any element not in H, say 3, as a coset representative. This gives the coset $3H = \{3, 13\}$. We find our next coset by choosing a representative not already appearing in the two previously chosen cosets, say 5. This gives us the coset $5H = \{5, 11\}$. We continue to form cosets by picking elements from $U(32)$ that have not yet appeared in the previous cosets as representatives of the cosets until we have accounted for every element of $U(32)$. We then have the complete list of all distinct cosets of H. ∎

Lagrange's Theorem and Consequences

And now, the all-important Lagrange's theorem that has been around for more than 200 years—longer than group theory itself! (This theorem was not originally stated in group theoretic terms.) At this stage, it should come as no surprise.

Theorem 7.1 Lagrange's Theorem[1]: $|H|$ Divides $|G|$

If G is a finite group and H is a subgroup of G, then $|H|$ divides $|G|$. Moreover, the number of distinct left (right) cosets of H in G is $|G|/|H|$.

PROOF Let $a_1H, a_2H, \ldots, a_rH$ denote the distinct left cosets of H in G. Then, for each a in G, we have $aH = a_iH$ for some i.

[1]Lagrange stated his version of this theorem in 1770, but the first complete proof was given by Pietro Abbati some 30 years later.

Also, by property 1 of the lemma, $a \in aH$. Thus, each member of G belongs to one of the cosets a_iH. In symbols,

$$G = a_1H \cup \cdots \cup a_rH.$$

Now, property 5 of the lemma shows that this union is disjoint, so that

$$|G| = |a_1H| + |a_2H| + \cdots + |a_rH|.$$

Finally, since $|a_iH| = |H|$ for each i, we have $|G| = r|H|$. ∎

We pause to emphasize that Lagrange's Theorem is a subgroup candidate criterion; that is, it provides a list of candidates for the orders of the subgroups of a group. Thus, a group of order 12 may have subgroups of order 12, 6, 4, 3, 2, 1, but no others. *Warning!* The converse of Lagrange's Theorem is false. For example, a group of order 12 need not have a subgroup of order 6. We prove this in Example 5.

As we shall see in later chapters there are several theorems that guarantee the existence of subgroups of certain particular orders in finite groups.

A special name and notation have been adopted for the number of left (or right) cosets of a subgroup in a group. The *index* of a subgroup H in G is the number of distinct left cosets of H in G. This number is denoted by $|G:H|$. As an immediate consequence of the proof of Lagrange's Theorem, we have the following useful formula for the number of distinct left (or right) cosets of H in G.

Corollary 1 $|G:H| = |G|/|H|$

If G is a finite group and H is a subgroup of G, then $|G:H| = |G|/|H|$.

Corollary 2 $|a|$ Divides $|G|$

In a finite group, the order of each element of the group divides the order of the group.

PROOF Recall that the order of an element is the order of the subgroup generated by that element. ∎

Corollary 3 Groups of Prime Order Are Cyclic

Every group of prime order is isomorphic to Z_p.

PROOF Suppose that G has prime order and let $a \in G$ and $a \neq e$. Then, $|\langle a \rangle|$ divides p and $|\langle a \rangle| \neq 1$. Thus, $|\langle a \rangle| = |G|$ and the corollary follows from Example 2 of Chapter 6. ∎

Corollary 4 $a^{|G|} = e$

Let G be a finite group, and let $a \in G$. Then, $a^{|G|} = e$.

PROOF By Corollary 2, $|G| = |a|k$ for some positive integer k. Thus, $a^{|G|} = a^{|a|k} = e^k = e$. ∎

Corollary 5 Fermat's Little Theorem

For every integer a and every prime p, $a^p \bmod p = a \bmod p$.

PROOF By the division algorithm, $a = pm + r$, where $0 \leq r < p$. Thus, $a \bmod p = r$, and it suffices to prove that $r^p \bmod p = r$. If $r = 0$, the result is trivial, so we may assume that $r \in U(p)$. [Recall that $U(p) = \{1, 2, \ldots, p-1\}$ under multiplication modulo p.] Then, by the preceding corollary, $r^{p-1} \bmod p = 1$ and, therefore, $r^p \bmod p = r$. ∎

Fermat's Little Theorem has been used in conjunction with computers to test for primality of certain numbers. One case concerned the number $p = 2^{257} - 1$. If p is prime, then we know from Fermat's Little Theorem that $10^p \bmod p = 10 \bmod p$ and, therefore, $10^{p+1} \bmod p = 100 \bmod p$. Using multiple precision and a simple loop, a computer was able to calculate $10^{p+1} \bmod p = 10^{2^{257}} \bmod p$ in a few seconds. The result was not 100, and so p is not prime.

EXAMPLE 5 The Converse of Lagrange's Theorem Is False[2] The group A_4 of order 12 has no subgroups of order 6.

[2]The first counterexample to the converse of Lagrange's Theorem was given by Paolo Ruffini in 1799.

To verify this, recall that A_4 has eight elements of order 3 (α_5 through α_{12}, in the notation of Table 5.1) and suppose that H is a subgroup of order 6. Let a be any element of order 3 in A_4. If a is not in H, then $A_4 = H \cup aH$. But then a^2 is in H or a^2 is in aH. If a^2 is in H then so is $(a^2)^2 = a^4 = a$, so this case is ruled out. If a^2 is in aH, then $a^2 = ah$ for some h in H, but this also implies that a is in H. This argument shows that any subgroup of A_4 of order 6 must contain all eight elements of A_4 of order 3, which is absurd. ∎

Lagrange's Theorem demonstrates that the finiteness of a group imposes severe restrictions on the possible orders of subgroups. The next theorem is a counting technique that also places severe limits on the existence of certain subgroups in finite groups.

Theorem 7.2 $|HK| = |H||K|/|H \cap K|$

For two finite subgroups H and K of a group, define the set $HK = \{hk \mid h \in H, k \in K\}$. Then $|HK| = |H||K|/|H \cap K|$.

PROOF Although the set HK has $|H||K|$ products, not all of these products need represent distinct group elements. That is, we may have $hk = h'k'$ where $h \neq h'$ and $k \neq k'$. To determine $|HK|$, we must find the extent to which this happens. For every t in $H \cap K$, the product $hk = (ht)(t^{-1}k)$, so each group element in HK is represented by at least $|H \cap K|$ products in HK. But $hk = h'k'$ implies $t = h^{-1}h' = kk'^{-1} \in H \cap K$, so that $h' = ht$ and $k' = t^{-1}k$. Thus, each element in HK is represented by exactly $|H \cap K|$ products. So, $|HK| = |H||K|/|H \cap K|$. ∎

One might suspect that if H and K are subgroups of a group G then HK is a subgroup of G. Not so! (This is Exercise 6.)

Our next two examples illustrate how Theorem 7.2 limits the existence of certain subgroups in finite groups.

EXAMPLE 6 A group of order 75 can have at most one subgroup of order 25. (It is shown in Chapter 23 that every group of order 75 has a subgroup of order 25). To see that a group of order 75 cannot have two subgroups of order 25, suppose H and K are two such subgroups. Since $|H \cap K|$ divides $|H| = 25$ and $|H \cap K| = 1$ or 5 results in $|HK| = |H||K|/|H \cap K| = 25 \cdot 25/|H \cap K| = 625$.

or 125 elements, we have that $|H \cap K| = 25$ and therefore $H = K$. ∎

EXAMPLE 7 An Abelian group G of order 42 can have at most one subgroup of order 6. (It is shown in Chapter 11 that every Abelian group of order 42 is cyclic.) To verify this, suppose that H and K are two distinct subgroups of order 6. Then $|H \cap K| = 1$, 2 or 3 and, by Example 7 from Chapter 3, we have that HK is a subgroup of G of order $|HK| = 36/|H \cap K| = 36$, 18 or 12, none of which divides 42. This contradicts Lagrange's theorem. ∎

For any prime $p > 2$, we know that Z_{2p} and D_p are nonisomorphic groups of order $2p$. This naturally raises the question of whether there could be other possible groups of these orders. Remarkably, with just the simple machinery available to us at this point, we can answer this question.

Theorem 7.3 Classification of Groups of Order $2p$

Let G be a group of order $2p$, where p is a prime greater than 2. Then G is isomorphic to Z_{2p} or D_p.

PROOF We assume that G does not have an element of order $2p$ and show that $G \approx D_p$. We begin by first showing that G must have an element of order p. By our assumption and Lagrange's Theorem, any nonidentity element of G must have order 2 or p. Thus, to verify our assertion, we may assume that every nonidentity element of G has order 2. In this case, we have for all a and b in the group $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$, so that G is Abelian. Then, for any nonidentity elements $a, b \in G$ with $a \neq b$, the set $\{e, a, b, ab\}$ is closed and therefore is a subgroup of G of order 4. Since this contradicts Lagrange's Theorem, we have proved that G must have an element of order p; call it a.

Now let b be any element not in $\langle a \rangle$. Then by Lagrange's Theorem and our assumption that G does not have an element of order $2p$, we have that $|b| = 2$ or p. Because $|\langle a \rangle \cap \langle b \rangle|$ divides $|\langle a \rangle| = p$ and $\langle a \rangle \neq \langle b \rangle$ we have that $|\langle a \rangle \cap \langle b \rangle| = 1$. But then $|b| = 2$, for otherwise, by Theorem 7.2 $|\langle a \rangle \langle b \rangle| = |\langle a \rangle||\langle b \rangle| = p^2 > 2p = |G|$, which is impossible. So, any element of G not in $\langle a \rangle$ has order 2.

At this point we now know that $G = \langle a \rangle \cup \langle a \rangle b = \{e, a, a^2, \ldots, a^{p-1}, b, ab, a^2b, \ldots, a^{p-1}b\}$. To complete the proof all we need do is the prove that there is only one way that the multiplcation table for G can be constructed. (If there were two non-isomorphic noncyclic groups of order $2p$ their multiplication tables would differ.) Since $ab \notin \langle a \rangle$ we know $|ab| = 2$. Thus $abab = e$ and therefore $bab = a^{-1}$. Using the fact that $b^2 = e$ we have that $a^{-j} = (a^{-1})^j = (bab)^j$. In the Cayley table there are three kinds of products in G: $a^i a^j = a^{i+j}$; $a^i(a^j b) = a^{i+j}b$; and $(a^i b)(a^j b) = a^i(ba^j)b = a^i(a^{-j}b)b = a^{i-j}$. This shows that when $p > 2$ is a prime the mutiplication table for a group of order $2p$ is uniquely determined by the two conditions that G has an element of order $2p$ or does not have one. ∎

As an immediate corollary, we have that the non-Abelian groups S_3, the symmetric group of degree 3, and $\mathrm{GL}(2, Z_2)$, the group of 2×2 matrices with nonzero determinants with entries from Z_2 (see Example 19 and Exercise 47 in Chapter 2) are isomorphic to D_3.

An Application of Cosets to Permutation Groups

Lagrange's Theorem and its corollaries dramatically demonstrate the fruitfulness of the coset concept. We next consider an application of cosets to permutation groups.

> **Definition Definition Stabilizer of a Point**
> **Let G be a group of permutations of a set S. For each i in S, let $\mathrm{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$. We call $\mathrm{stab}_G(i)$ the *stabilizer of i in G*.**

The student should verify that $\mathrm{stab}_G(i)$ is a subgroup of G. (See Exercise 43 in Chapter 5.)

> **Definition Orbit of a Point**
> **Let G be a group of permutations of a set S. For each i in S, let $\mathrm{orb}_G(i) = \{\phi(i) \mid \phi \in G\}$. The set $\mathrm{orb}_G(i)$ is a subset of S called the *orbit of i under G*. We use $|\mathrm{orb}_G(i)|$ to denote the number of elements in $\mathrm{orb}_G(i)$.**

Example 8 should clarify these two definitions.

EXAMPLE 8 Let G be the following subgroup of S_8

$$\{(1), (132)(465)(78), (132)(465), (123)(456), \\ (123)(456)(78), (78)\}.$$

Then,

$\text{orb}_G(1) = \{1, 3, 2\}$, $\text{stab}_G(1) = \{(1), (78)\}$,
$\text{orb}_G(2) = \{2, 1, 3\}$, $\text{stab}_G(2) = \{(1), (78)\}$,
$\text{orb}_G(4) = \{4, 6, 5\}$, $\text{stab}_G(4) = \{(1), (78)\}$,
$\text{orb}_G(7) = \{7, 8\}$, $\text{stab}_G(7) = \{(1), (132)(465), (123)(456)\}$. ▮

▮ **EXAMPLE 9** We may view D_4 as a group of permutations of a square region. Figure 7.1(a) illustrates the orbit of the point p under D_4, and Figure 7.1(b) illustrates the orbit of the point q under D_4. Observe that $\text{stab}_{D_4}(p) = \{R_0, D\}$, whereas $\text{stab}_{D_4}(q) = \{R_0\}$. ▮

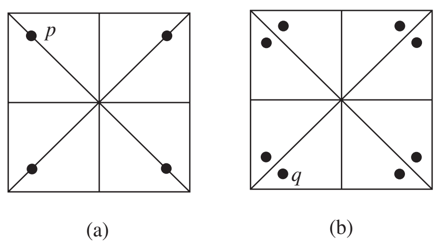


Figure 7.1

The preceding two examples also illustrate the following theorem.

▮ Theorem 7.4 Orbit-Stabilizer Theorem

Let G be a finite group of permutations of a set S. Then, for any i from S, $|G| = |\text{orb}_G(i)||\text{stab}_G(i)|$.

PROOF By Lagrange's Theorem, $|G|/|\text{stab}_G(i)|$ is the number of distinct left cosets of $\text{stab}_G(i)$ in G. Thus, it suffices to establish a one-to-one correspondence between the left cosets of $\text{stab}_G(i)$ and the elements in the orbit of i. To do this, we define a correspondence T by mapping the coset $\phi\text{stab}_G(i)$ to $\phi(i)$ under T. To show that T is a well-defined function, we must show that $\alpha\text{stab}_G(i) = \beta\text{stab}_G(i)$ implies $\alpha(i) = \beta(i)$. But $\alpha\text{stab}_G(i) = \beta\text{stab}_G(i)$ implies $\alpha^{-1}\beta \in \text{stab}_G(i)$, so that $(\alpha^{-1}\beta)(i) = i$ and, therefore, $\beta(i) = \alpha(i)$. Reversing the argument from the last step to the first step shows that T is also one-to-one. We conclude the proof by showing that T is onto $\text{orb}_G(i)$. Let $j \in \text{orb}_G(i)$. Then

$\alpha(i) = j$ for some $\alpha \in G$ and clearly $T(\alpha\mathrm{stab}_G(i)) = \alpha(i) = j$, so that T is onto. ∎

We leave as an exercise the proof of the important fact that the orbits of the elements of a set S under a group partition S (Exercise 67).

One might think that because Orbit-Stabilizer Theorem only applies to permutation groups that it is of limited value but keep in mind that Cayley's Theorem (Theorem 6.5) says that every group is isomorphic to a group of permutations.

The Rotation Group of a Cube and a Soccer Ball

It cannot be overemphasized that Theorem 7.4 and Lagrange's Theorem (Theorem 7.1) are *counting* theorems.[3] They enable us to determine the numbers of elements in various sets. To see how Theorem 7.4 works, we will determine the order of the rotation group of a cube and a soccer ball. That is, we wish to find the number of essentially different ways in which we can take a cube or a soccer ball in a certain location in space, physically rotate it, and then have it still occupy its original location.

EXAMPLE 10 Let G be the rotation group of a cube. Label the six faces of the cube 1 through 6. Since any rotation of the cube must carry each face of the cube to exactly one other face of the cube and different rotations induce different permutations of the faces, G can be viewed as a group of permutations on the set $\{1, 2, 3, 4, 5, 6\}$. Clearly, there is some rotation about a central horizontal or vertical axis that carries face number 1 to any other face, so that $|\mathrm{orb}_G(1)| = 6$. Next, we consider $\mathrm{stab}_G(1)$. Here, we are asking for all rotations of a cube that leave face number 1 where it is. Surely, there are only four such motions—rotations of $0°$, $90°$, $180°$, and $270°$—about the line perpendicular to the face and passing through its center (see Figure 7.2). Thus, by Theorem 7.4, $|G| = |\mathrm{orb}_G(1)|\,|\mathrm{stab}_G(1)| = 6 \cdot 4 = 24$. ∎

Now that we know how many rotations a cube has, it is simple to determine the actual structure of the rotation group of a cube. Recall that S_4 is the symmetric group of degree 4.

[3] People who don't count won't count" (Anatole France).

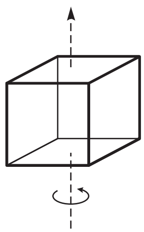


Figure 7.2 Axis of rotation of a cube.

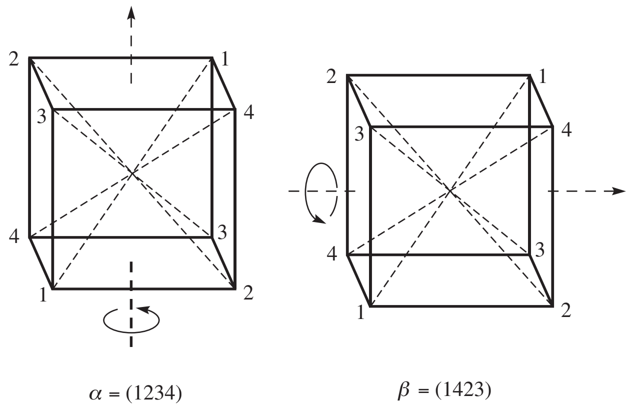


Figure 7.3

Theorem 7.5 The Rotation Group of a Cube

The group of rotations of a cube is isomorphic to S_4.

PROOF Since the group of rotations of a cube has the same order as S_4, we need only prove that the group of rotations is isomorphic to a subgroup of S_4. To this end, observe that a cube has four diagonals and that the rotation group induces a group of permutations on the four diagonals. But we must be careful not to assume that different rotations correspond to different permutations. To see that this is so, all we need do is show that all 24 permutations of the diagonals arise from rotations. Labeling the consecutive diagonals 1, 2, 3, and 4, it is obvious that there is a 90° rotation that yields the permutation $\alpha = (1234)$; another 90° rotation about an axis perpendicular to our first axis yields the permutation $\beta = (1423)$. See Figure 7.3. So, the group of permutations induced by the rotations contains the eight-element

subgroup $\{\varepsilon, \alpha, \alpha^2, \alpha^3, \beta^2, \beta^2\alpha, \beta^2\alpha^2, \beta^2\alpha^3\}$ (see Exercise 77) and $\alpha\beta$, which has order 3. Clearly, then, the rotations yield all 24 permutations, since the order of the rotation group must be divisible by both 8 and 3.

EXAMPLE 11 A traditional soccer ball has 20 faces that are regular hexagons and 12 faces that are regular pentagons. (The technical term for this solid is *truncated icosahedron*.) To determine the number of rotational symmetries of a soccer ball using Theorem 7.4, we may choose our set S to be the 20 hexagons or the 12 pentagons. Let us say that S is the set of 12 pentagons. Since any pentagon can be carried to any other pentagon by some rotation, the orbit of any pentagon is S. Also, there are five rotations that fix (stabilize) any particular pentagon. Thus, by the Orbit-Stabilizer Theorem, there are $12 \cdot 5 = 60$ rotational symmetries. (In case you are interested, the rotation group of a soccer ball is isomorphic to A_5.)

In 1985, chemists Robert Curl, Richard Smalley, and Harold Kroto caused tremendous excitement in the scientific community when they created a new form of carbon by using a laser beam to vaporize graphite. The structure of the new molecule was composed of 60 carbon atoms arranged in the shape of a soccer ball! Because the shape of the new molecule reminded them of the dome structures built by the architect R. Buckminster Fuller, Curl, Smalley, and Kroto named their discovery "buckyballs." Buckyballs are the roundest, most symmetric large molecules known. Group theory has been particularly useful in illuminating the properties of buckyballs, since the absorption spectrum of a molecule depends on its symmetries and chemists classify various molecular states according to their symmetry properties. The buckyball discovery spurred a revolution in carbon chemistry. In 1996, Curl, Smalley, and Kroto received the Nobel Prize in chemistry for their discovery.

In 2012 a study reported that carbon molecules in the shape of buckyballs may be the cause of mysterious bands seen in light across the Milky Way that have puzzled astronomers for nearly a century and that these molecules are common across the universe.

An Application of Cosets to the Rubik's Cube

Recall from Chapter 5 that in 2010 it was proved via a computer computation, which took 35 CPU-years to complete, that every Rubik's cube could be solved in at most 20 moves. To carry out this effort, the research team of Morley Davidson John Dethridge, Herbert Kociemba, and Tomas Rokicki applied a program of Rokicki, which built on early work of Kociemba, that checked the elements of the cosets of a subgroup H of order $(8! \cdot 8! \cdot 4!)/2 =$ 19,508,428,800 to see if each cube in a position corresponding to the elements in a coset could be solved within 20 moves. In the rare cases where Rokicki's program did not work, an alternate method was employed. Using symmetry considerations, they were able to reduce the approximately 2 billion cosets of H to about 56 million cosets for testing. Cosets played a role in this effort because Rokicki's program could handle the 19.5+ billion elements in the same coset in about 20 seconds.

Exercises

I don't know, Marge. Trying is the first step towards failure.

Homer Simpson

1. Let $H = \{0, \pm 3, \pm 6, \pm 9, \ldots\}$. Find all the left cosets of H in Z. Let n be a positive integer. Let $H = \{0, \pm n, \pm 2n, \pm 3n, \ldots\}$. Find all left cosets of H in Z. How many are there?

2. Rewrite the condition $a^{-1}b \in H$ given in property 6 of the lemma in this chapter in additive notation. Assume that the group is Abelian.

3. Let H be as in Exercise 1. Use Exercise 2 to decide whether or not the following cosets of H are the same.
a. $11 + H$ and $17 + H$ **b.** $-1 + H$ and $5 + H$ **c.** $7 + H$ and $23 + H$.

4. Find all of the left cosets of $\{1, 11\}$ in $U(30)$.

5. Let a belong to a group and $|a| = 30$. How many left cosets of $\langle a^5 \rangle$ in $\langle a \rangle$ are there? List them. Do the same for $\langle a^4 \rangle$ in $\langle a \rangle$.

6. Give an example of a group G and subgroups H and K such that $HK = \{hk \mid h \in H,\ k \in K\}$ is not a subgroup of G.

7. If H is a subgroup of Z and $\langle 3 \rangle \subseteq H \subseteq Z$, prove that $H = \langle 3 \rangle$ or $H = Z$.

8. Let a and b be elements of a group G and H and K be subgroups of G. If $aH = bK$, prove that $H = K$.

9. If H and K are subgroups of G and g belongs to G, show that $g(H \cap K) = gH \cap gK$.

10. Let a and b be nonidentity elements of different orders in a group G of order 155. Prove that the only subgroup of G that contains a and b is G itself.

11. Let H be a subgroup of $\mathbf{R}^*$, the group of nonzero real numbers under multiplication. If $\mathbf{R}^+ \subseteq H \subseteq \mathbf{R}^*$, prove that $H = \mathbf{R}^+$ or $H = \mathbf{R}^*$.

12. Let $\mathbf{C}^*$ be the group of nonzero complex numbers under multiplication and let $H = \{a + bi \in \mathbf{C}^* \mid a^2 + b^2 = 1\}$. Give a geometric description of the coset $(3 + 4i)H$. Give a geometric description of the coset $(c + di)H$.

13. Let G be a group of order 60. What are the possible orders for the subgroups of G?

14. Suppose that K is a proper subgroup of H and H is a proper subgroup of G. If $|K| = 42$ and $|G| = 420$, what are the possible orders of H?

15. Let G be a group with $|G| = pq$, where p and q are prime. Prove that every proper subgroup of G is cyclic.

16. Recall that, for any integer n greater than 1, $\phi(n)$ denotes the number of positive integers less than n and relatively prime to n. Prove that if a is any integer relatively prime to n, then $a^{\phi(n)} \bmod n = 1$.

17. Compute $5^{15} \bmod 7$ and $7^{13} \bmod 11$.

18. Use Corollary 2 of Lagrange's Theorem (Theorem 7.1) to prove that the order of $U(n)$ is even when $n > 2$.

19. Suppose G is a finite group of order n and m is relatively prime to n. If $g \in G$ and $g^m = e$, prove that $g = e$.

20. Suppose H and K are subgroups of a group G. If $|H| = 12$ and $|K| = 35$, find $|H \cap K|$. Generalize.

21. For any integer $n \geq 3$, prove that D_n has a subgroup of order 4 if and only if n is even.

22. Let H and K be subgroups of an Abelian group. If $|H| = 12$ and $|K| = 18$ prove that $H \cap K$ is cyclic. Does your proof generalize to the case where $|H \cap K|$ divides $2p$ where p is prime?

23. Suppose that G is an Abelian group with an odd number of elements. Show that the product of all of the elements of G is the identity.

24. Prove that every group of order 4 is Abelian.

25. Let G be a group of order 25. Prove that G is cyclic or $a^5 = e$ for all a in G. Generalize to any group of order p^n where p is prime.

26. What is the maximum number of elements of order 11 in a group of order 99?

27. Let $|G| = 33$. What are the possible orders for the elements of G? Show that G must have an element of order 3.

28. Suppose G is a non-Abelian group with $|G| = 8$. Prove that G has an element of order 4.

29. Prove that any group of order 55 must have exactly one subgroup of order 5 or exactly 11 subgroups of order 5.

30. Suppose G is a group of order 105 with the property that G has exactly one subgroup for each divisor of 105. Prove that G is cyclic.

31. Use Exercise 87 of Chapter 4 to prove the following converse of the Fundamental Theorem of Finite Cyclic Groups: If G is a finite group of order n with the property that G has exactly one subgroup of order d for each positive divisor d of n then G is cyclic.

32. Suppose that G is a group of order $2p$ where p is an odd prime. What are the possibilities for the number of elements of order 2?

33. Let G be a group with $|G| = 2m$ where m is odd. Prove that G has at most one subgroup of order m.

34. Determine all finite subgroups of $\mathbf{C}^*$, the group of nonzero complex numbers under multiplication.

35. Let H and K be subgroups of a finite group G with $H \subseteq K \subseteq G$. Prove that $|G{:}H| = |G{:}K||K{:}H|$.

36. If a finite group G has subgroups H and K such that $K \subseteq H \subseteq G$ with $[G : K] = p$ where p is prime, prove that $H = G$ or $H = K$.

37. Give an example of the dihedral group of smallest order that contains a subgroup isomorphic to Z_{12} and a subgroup isomorphic to Z_{20}. No need to prove anything, but explain your reasoning.

38. Let G be a group and $|G| = 21$. If $g \in G$ and $g^{14} = e$, what are the possibilities for $|g|$?

39. Suppose that a finite Abelian group G has at least three elements of order 3. Prove that 9 divides $|G|$.

40. Prove that if G is a finite group, the index of $Z(G)$ cannot be prime.

41. Let p be a prime and n be a positive integer. Prove that $a^{p^n} = a$ for every $a \in Z_p$.

42. Prove that a group of order 63 must have an element of order 3.

43. Let G be a group of order 100 that has a subgroup H of order 25. Prove that every element of G of order 5 is in H.

44. Let G be a group of order n and k be any integer relatively prime to n. Show that the mapping from G to G given by $g \to g^k$ is one-to-one. If G is also Abelian, show that the mapping given by $g \to g^k$ is an automorphism of G.

45. Let G be a finite group and d be the smallest positive integer such that $x^d = e$ for all x in G. Prove that d divides $|G|$.

46. Prove that every subgroup of D_n of odd order is cyclic.

47. In a finite Abelian group prove that the number of elements of order 2 has the form $2^n - 1$.

48. Prove that an Abelian group of order 15 is cyclic.

49. Show that in a group G of odd order, the equation $x^2 = a$ has a unique solution for all a in G.

50. Prove that an Abelian group of order 20 has an element of order 5. Does your proof generalize to prove that every Abelian group of order $4p^n$ where p is prime has a subgroup of order p?

51. Prove that a group of order np^m where p is prime and $n < p^m$ has at most one subgroup of order p^m.

52. Prove that a group G of order pq^n where p and q are primes and $p > q^n$ has an element of order q.

53. Suppose that a group of order 21 has exactly one subgroup of order 3. Prove that G is cyclic. Does your proof work for groups of order pq where $p < q$ are primes and there is a unique subgroups of order q?

54. Let G be a finite group with more than one element and let H be the intersection of all non-trivial subgroups of G. What can you say about the $|H|$?

55. Prove that a group that has more than one subgroup of order 5 must have order at least order 25. Generalize.

56. If a is an element finite order in a group G and ϕ_a is the inner automorphism of G induced by a (that is, $\phi_a(x) = axa^{-1}$ for all x in G), prove that $|\phi_a|$ divides $|a|$.

57. If G is a finite group with fewer than 100 elements and G has subgroups of orders 10 and 25, what is the order of G?

58. Why does the fact that A_4 has no subgroup of order 6 imply that $|Z(A_4)| = 1$?

59. Use properties of cosets to prove that for any subgroup H of S_n either every member of H is an even permutation or exactly half of the members is even. (See Exercise 27 in Chapter 5.)

60. Prove that A_5 has a subgroup of order 12.

61. Prove that A_5 has no subgroup of order 30.

62. Prove that A_5 has no subgroup of order 15 or 20.

63. Prove that S_5 has no subgroup of order 30.

64. Suppose that $\alpha \in S_n$ ($n \geq 3$) and $\alpha(1) = 3$. Let $H = \text{stab}(1)$. Prove that αH is the set of all elements of S_n that send 1 to 3. Generalize.

65. Prove that A_5 is the only subgroup of S_5 of order 60.

66. For which positive integers n is S_n isomorphic to a subgroup of D_{60}. No need to prove anything but give a reason why any specific n is or is not on your list.

67. Let G be a group of permutations of a set S. Prove that the orbits of the members of S constitute a partition of S. (This exercise is referred to in this chapter and in Chapter 27.)

68. The group D_4 acts as a group of permutations of the square regions shown below. (The axes of symmetry are drawn for reference purposes.) For each square region, locate the points in the orbit of the indicated point under D_4. In each case, determine the stabilizer of the indicated point.

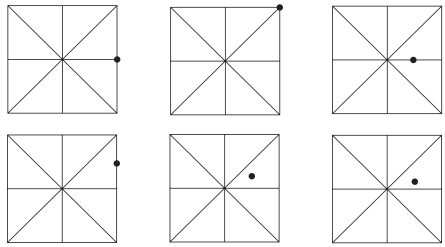
69. Let $G = \{(1), (12)(34), (1234)(56), (13)(24), (1432)(56), (56)(13), (14)(23), (24)(56)\}$.
a. Find the stabilizer of 1 and the orbit of 1.
b. Find the stabilizer of 3 and the orbit of 3.
c. Find the stabilizer of 5 and the orbit of 5.

70. Noting that the rotation group of a cube is isomorphic to S_4 and that S_4 has 8 elements of order 3, we know that there are 8 rotations of 120 degrees that are symmetries of the cube. Give a geometrical description of those 8 rotations.

71. Let $G = GL(2, \mathbf{R})$ and $H = SL(2, \mathbf{R})$. Let $A \in G$ and suppose that det $A = 2$. Prove that AH is the set of all 2×2 matrices in G that have determinant 2.

72. Let G be the group of rotations of a plane about a point P in the plane. Thinking of G as a group of permutations of the plane, describe the orbit of a point Q in the plane. (This is the motivation for the name "orbit.")

73. Let G be the rotation group of a cube. Label the faces of the cube 1 through 6, and let H be the subgroup of elements of G that carry face 1 to itself. If σ is a rotation that carries face 2 to face 1, give a physical description of the coset $H\sigma$.



74. A soccer ball has 20 faces that are regular hexagons and 12 faces that are regular pentagons. Use Theorem 7.4 to explain why a soccer ball cannot have a 60° rotational symmetry about a line through the centers of two opposite hexagonal faces.

75. Let $G = GL(2, \mathbf{R})$, the group of 2×2 matrices over $\mathbf{R}$ with nonzero determinant. Let H be the subgroup of matrices of determinant ± 1. If $a, b \in G$ and $aH = bH$, what can be said about $\det(a)$ and $\det(b)$? Prove or disprove the converse. [Determinants have the property that $\det(xy) = \det(x)\det(y)$.]

76. Calculate the orders of the following (refer to Figure 26.5 for illustrations).
a. The group of rotations of a regular tetrahedron (a solid with four congruent equilateral triangles as faces)
b. The group of rotations of a regular octahedron (a solid with eight congruent equilateral triangles as faces)
c. The group of rotations of a regular dodecahedron (a solid with 12 congruent regular pentagons as faces)
d. The group of rotations of a regular icosahedron (a solid with 20 congruent equilateral triangles as faces)

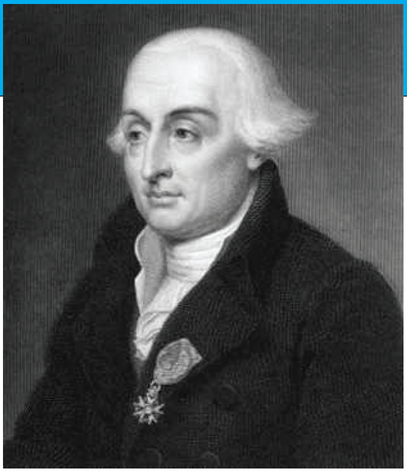
77. Prove that the eight-element set in the proof of Theorem 7.5 is a group.

Computer Exercises

A computer exercise for this chapter is available at the website:

http://www.d.umn.edu/~jgallian

Joseph Lagrange



World History Archive / Alamy

Lagrange is the Lofty Pyramid of the Mathematical Sciences.

NAPOLEON BONAPARTE

Joseph Louis Lagrange was born in Italy of French ancestry on January 25, 1736. He became captivated by mathematics at an early age when he read an essay by Halley on Newton's calculus. At the age of 19, he became a professor of mathematics at the Royal Artillery School in Turin. Lagrange made significant contributions to many branches of mathematics and physics, among them the theory of numbers, the theory of equations, ordinary and partial differential equations, the calculus of variations, analytic geometry, fluid dynamics, and celestial mechanics. His methods for solving third- and fourth-degree polynomial equations by radicals laid the groundwork for the group theoretic approach to solving polynomials taken by Galois.

Lagrange was a very careful writer with a clear and elegant style. At the age of 40, Lagrange was appointed head of the Berlin Academy, succeeding Euler. In offering this appointment, Frederick the Great proclaimed that the "greatest king in Europe" ought to have the "greatest mathematician in Europe" at his court. In 1787, Lagrange was invited to Paris by Louis XVI and became a good friend of the king and his wife, Marie Antoinette. In 1793, Lagrange headed a commission, which included Laplace and Lavoisier, to devise a new system of weights and measures. Out of this came the metric system. Late in his life he was made a count by Napoleon. Lagrange died on April 10, 1813.

8 External Direct Products

The universe is an enormous direct product of representations of symmetry groups.

Steven Weinberg[1]

In many areas of mathematics, there are ways of "building things up" and "breaking things down"

Norman J. Block, *Abstract Algebra with Applications*

Definition and Examples

In this chapter, we show how to piece together groups to make larger groups. In Chapter 9, we will show that we can often start with one large group and decompose it into a product of smaller groups in much the same way as a composite positive integer can be broken down into a product of primes. These methods will later be used to give us a simple way to construct all finite Abelian groups.

Definition External Direct Product
Let $G_1, G_2, \ldots, G_n$ be a finite collection of groups. The external direct product of $G_1, G_2, \ldots, G_n$, written as $G_1 \oplus G_2 \oplus \cdots \oplus G_n$, is the set of all n-tuples for which the ith component is an element of G_i and the operation is componentwise.

In symbols,

$$G_1 \oplus G_2 \oplus \cdots \oplus G_n = \{(g_1, g_2, \ldots, g_n) \mid g_i \in G_i\},$$

where $(g_1, g_2, \ldots, g_n)(g_1', g_2', \ldots, g_n')$ is defined to be $(g_1g_1', g_2g_2', \ldots, g_ng_n')$. It is understood that each product g_ig_i' is performed with the operation of G_i. Note that in the case that each G_i is finite, we have by properties of sets that $|G_1 \oplus G_2 \oplus \cdots \oplus G_n| = |G_1||G_2| \cdots |G_n|$. We leave it to the reader to show that the external direct product of groups is itself a group (Exercise 1).

This construction is not new to students who have had linear algebra or physics. Indeed, $\mathbf{R}^2 = \mathbf{R}\oplus\mathbf{R}$ and $\mathbf{R}^3 = \mathbf{R}\oplus\mathbf{R}\oplus\mathbf{R}$—the operation being componentwise addition. Of course, there is also scalar multiplication, but we ignore this for the time being, since we are interested only in the group structure at this point.

EXAMPLE 1

$$\begin{aligned} U(8) \oplus U(10) = \{&(1,1),(1,3),(1,7),(1,9),(3,1),(3,3),\\ &(3,7)(3,9)(5,1)(5,3)(5,7)(5,9),\\ &(7,1)(7,3)(7,7)(7,9)\}. \end{aligned}$$

The product (3, 7)(7, 9) = (5, 3), since the first components are combined by multiplication modulo 8, whereas the second components are combined by multiplication modulo 10. ∎

EXAMPLE 2

$$Z_2 \oplus Z_3 = \{(0,0),(0,1),(0,2),(1,0),(1,1),(1,2)\}.$$

Clearly, this is an Abelian group of order 6. Is this group related to another Abelian group of order 6 that we know, namely, Z_6? Consider the subgroup of $Z_2 \oplus Z_3$ generated by (1, 1). Since the operation in each component is addition, we have (1, 1) = (1, 1), 2(1, 1) = (0, 2), 3(1, 1) = (1, 0), 4(1, 1) = (0, 1), 5(1, 1) = (1, 2), and 6(1, 1) = (0, 0). Hence $Z_2 \oplus Z_3$ is cyclic. It follows that $Z_2 \oplus Z_3$ is isomorphic to Z_6. ∎

In Theorem 7.3 we classified the groups of order $2p$ where p is an odd prime. Now that we have defined $Z_2 \oplus Z_2$, it is easy to classify the groups of order 4.

EXAMPLE 3 Classification of Groups of Order 4

A group of order 4 is isomorphic to Z_4 or $Z_2 \oplus Z_2$. To verify this it suffices to show that for any non-cyclic group G of order 4 there is only one way to create an operation table for G. By Lagrange's Theorem the elements of G have order 1 or 2. Let a and b be distinct non-identity elements of G. By cancellation, $ab \neq a$ and $ab \neq b$. Moreover, $ab \neq e$, for otherwise $a = b^{-1} = b$. Thus $G = \{e, a, b, ab\}$. That the operation table is uniquely determined follows from the observation that $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$. ∎

We see from Examples 2 and 3 that in some cases $Z_m \oplus Z_n$ is isomorphic to Z_{mn} and in some cases it is not. Theorem 8.2 provides a simple characterization for when the isomorphism holds.

Properties of External Direct Products

Our first theorem gives a simple method for computing the order of an element in a direct product in terms of the orders of the component pieces.

Theorem 8.1 Order of an Element in a Direct Product

The order of an element in a direct product of a finite number of finite groups is the least common multiple of the orders of the components of the element. In symbols,

$$|(g_1, g_2, \ldots, g_n)| = \mathrm{lcm}(|g_1|, |g_2|, \ldots, |g_n|).$$

PROOF Denote the identity of G_i by e_i. Let $s = \mathrm{lcm}(|g_1|, |g_2|, \ldots, |g_n|)$ and $t = |(g_1, g_2, \ldots, g_n)|$. Because s is a multiple of each $|g_i|$ implies that $(g_1, g_2, \ldots, g_n)^s = (g_1^s, g_2^s, \ldots, g_n^s) = (e_1, e_2, \ldots, e_n)$, we know that $t \leq s$. On the other hand, from $(g_1^t, g_2^t, \ldots, g_n^t) = (g_1, g_2, \ldots, g_n)^t = (e_1, e_2, \ldots, e_n)$ we see that t is a common multiple of $|g_1|, |g_2|, \ldots, |g_n|$. Thus, $s \leq t$. ∎

The next three examples are applications of Theorem 8.1.

EXAMPLE 4 Examples of groups of order 100 include Z_{100}; $Z_{25} \oplus Z_2 \oplus Z_2$; $Z_5 \oplus Z_5 \oplus Z_4$; $Z_5 \oplus Z_5 \oplus Z_2 \oplus Z_2$; D_{50}; $D_{10} \oplus Z_5$; $D_5 \oplus Z_{10}$; and $D_5 \oplus D_5$. That these are not isomorphic is an easy consequence of Theorem 8.1. ∎

EXAMPLE 5 Let m and n be positive integers that are divisible by 5. We determine the number of elements of order 5 in $Z_m \oplus Z_n$. By Theorem 8.1, we need only count the number of elements (a, b) in $Z_m \oplus Z_n$ with the property that $5 = |(a, b)| = \mathrm{lcm}(|a|, |b|)$. Clearly this requires that $|a| = 1$ or 5 and $|b| = 1$ or 5, but not $(a, b) = (0, 0)$. Since both Z_m and Z_n each have a unique subgroup of order 5 there are exactly five choices for a and 5 choices for b and therefore 25 choices for (a, b), including (0,0). So, there are exactly 24 elements in $Z_m \oplus Z_n$ of order 5. ∎

The identical argument shows that if m and n be positive integers that are divisible by a prime p, then the number of elements of order p in $Z_m \oplus Z_n$ is $p^2 - 1$.

EXAMPLE 6 We determine the number of cyclic subgroups of order 10 in $Z_{150} \oplus Z_{50}$. We begin by counting the number of elements of order 10. For $10 = |(a, b)| = \text{lcm}(|a|, |b|)$ we must restrict $|a|$ and $|b|$ to 1, 2, 5, and 10. So, we know that that a must belong in the unique subgroup $\langle 15 \rangle$ of order 10 in Z_{150} and b must belong in the unique subgroup $\langle 5 \rangle$ of order 10 in Z_{50}. Thus, we know $(a, b) \in \langle 15 \rangle \oplus \langle 5 \rangle$, which is isomorphic to $Z_{10} \oplus Z_{10}$. Since the orders of the elements of $Z_{10} \oplus Z_{10}$ are 1, 2, 5, and 10, we can determine the number of elements of order 10 by subtracting from 100 the number of elements of orders 1, 2, and 5. From Example 5 we know that $Z_{10} \oplus Z_{10}$ has 24 elements of order 5 and 3 elements of order 2. So we get $100 - 24 - 3 - 1 = 72$ elements of order 10 in $Z_{150} \oplus Z_{50}$. Because each cyclic subgroup of order 10 has four elements of order 10 and no two of the cyclic subgroups can have an element of order 10 in common, there must be $72/4 = 18$ cyclic subgroups of order 10. (This method is analogous to determining the number of sheep in a flock by counting the number of legs and dividing by 4.) Imagine how much work it would be to do this problem without our theorems! ∎

EXAMPLE 7 We determine the number of elements of order 2 in $D_4 \oplus Z_4$. For $|(a, b)| = 2$, we can have $|a| = 1$ or 2 and $|b| = 1$ or 2, but not both with order 1. Counting the identity, there are 6 choices a and 2 choices for b for a total of 12 choices for (a, b). So, excluding the identity, we have that there are 11 elements of order 2. ∎

The direct product notation is convenient for specifying certain subgroups of a direct product.

EXAMPLE 8 For each divisor r of m and s of n, the group $Z_m \oplus Z_n$ has a subgroup isomorphic to $Z_r \oplus Z_s$ (see Exercise 19). To find a subgroup of, say, $Z_{30} \oplus Z_{12}$ isomorphic to $Z_6 \oplus Z_4$, we observe that $\langle 5 \rangle$ is a subgroup of Z_{30} of order 6 and $\langle 3 \rangle$ is a subgroup of Z_{12} of order 4, so $\langle 5 \rangle \oplus \langle 3 \rangle$ is the desired subgroup. ∎

The next theorem and its first corollary characterize those direct products of cyclic groups that are themselves cyclic.

Theorem 8.2 Criterion for $G \oplus H$ to be Cyclic

Let G and H be finite cyclic groups. Then $G \oplus H$ is cyclic if and only if $|G|$ and $|H|$ are relatively prime.

PROOF Let $|G| = m$ and $|H| = n$, so that $|G \oplus H| = mn$. To prove the first half of the theorem, we assume $G \oplus H$ is cyclic and show that m and n are relatively prime. Suppose that $\gcd(m, n) = d$ and (g, h) is a generator of $G \oplus H$. Since $(g, h)^{mn/d} = ((g^m)^{n/d}, (h^n)^{m/d}) = (e, e)$, we have $mn = |(g, h)| \leq mn/d$. Thus, $d = 1$.

To prove the other half of the theorem, let $G = \langle g \rangle$ and $H = \langle h \rangle$ and suppose $\gcd(m, n) = 1$. Then, $|(g, h)| = \operatorname{lcm}(m, n) = mn = |G \oplus H|$, so that (g, h) is a generator of $G \oplus H$. ■

As a consequence of Theorem 8.2 and an induction argument, we obtain the following extension of Theorem 8.2.

Corollary 1 Criterion for $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ to Be Cyclic

An external direct product $G_1 \oplus G_2 \oplus \cdots \oplus G_n$ of a finite number of finite cyclic groups is cyclic if and only if $|G_i|$ and $|G_j|$ are relatively prime when $i \neq j$.

Corollary 2 Criterion for $Z_{n_1 n_2 \cdots n_k} \approx Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$

Let $m = n_1 n_2 \cdots n_k$. Then Z_m is isomorphic to $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$ if and only if n_i and n_j are relatively prime when $i \neq j$.

By using the results above in an iterative fashion, one can express the same group (up to isomorphism) in many different forms. For example, we have

$$Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_5 \approx Z_2 \oplus Z_6 \oplus Z_5 \approx Z_2 \oplus Z_{30}.$$

Similarly,

$$Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_5 \approx Z_2 \oplus Z_6 \oplus Z_5$$
$$\approx Z_2 \oplus Z_3 \oplus Z_2 \oplus Z_5 \approx Z_6 \oplus Z_{10}.$$

Thus, $Z_2 \oplus Z_{30} \approx Z_6 \oplus Z_{10}$. Note, however, that, $Z_2 \oplus Z_{30} \not\approx Z_{60}$.

The Group of Units Modulo n as an External Direct Product

The U-groups provide a convenient way to illustrate the preceding ideas. We first introduce some notation. If $k > 1$ is proper divisor of a positive integer n, let

$$U_k(n) = \{x \in U(n) \mid x \bmod k = 1\}.$$

For example, $U_7(105) = \{1, 8, 22, 29, 43, 64, 71, 92\}$. It can be readily shown that $U_k(n)$ is indeed a subgroup of $U(n)$. (See Exercise 21 in Chapter 3.)

Theorem 8.3 $U(n)$ as an External Direct Product

Suppose s and t are relatively prime. Then $U(st)$ is isomorphic to the external direct product of $U(s)$ and $U(t)$. In short,

$$U(st) \approx U(s) \oplus U(t).$$

Moreover, $U_s(st)$ is isomorphic to $U(t)$ and $U_t(st)$ is isomorphic to $U(s)$.

PROOF An isomorphism from $U(st)$ to $U(s) \oplus U(t)$ is $x \to (x \bmod s, x \bmod t)$; an isomorphism from $U_s(st)$ to $U(t)$ is $x \to x \bmod t$; an isomorphism from $U_t(st)$ to $U(s)$ is $x \to x \bmod s$. We leave the verification that these mappings are operation-preserving, one-to-one, and onto to the reader. (See Exercises 11, 19, and 21 in Chapter 0.) ▮

As a consequence of Theorem 8.3, we have the following result.

Corollary

Let $m = n_1 n_2 \cdots n_k$, where $\gcd(n_i, n_j) = 1$ for $i \neq j$. Then,

$$U(m) \approx U(n_1) \oplus U(n_2) \oplus \cdots \oplus U(n_k).$$

To see how these results work, let's apply them to $U(105)$. We obtain

$$U(105) \approx U(7) \oplus U(15),$$
$$U(105) \approx U(21) \oplus U(5),$$
$$U(105) \approx U(3) \oplus U(5) \oplus U(7).$$

Moreover,
$U(7) \approx U_{15}(105) = \{1, 16, 31, 46, 61, 76\}$,
$U(15) \approx U_7(105) = \{1, 8, 22, 29, 43, 64, 71, 92\}$,
$U(21) \approx U_5(105) = \{1, 11, 16, 26, 31, 41, 46, 61, 71, 76, 86, 101\}$,
$U(5) \approx U_{21}(105) = \{1, 22, 43, 64\}$,
$U(3) \approx U_{35}(105) = \{1, 71\}$.

Since $|U(20)| = 8$ and $|U(10) \oplus U(2)| = 4$ we see that the condition that $\gcd(s, t) = 1$ in Theorem 8.3 is necessary.

Among all groups, surely the cyclic groups Z_n have the simplest structures and, at the same time, are the easiest groups with which to compute. Direct products of groups of the form Z_n are only slightly more complicated in structure and computability. Because of this, algebraists endeavor to describe a finite Abelian group as such a direct product. Indeed, we shall soon see that every finite Abelian group can be so represented. With this goal in mind, let us reexamine the U-groups. Using the corollary to Theorem 8.3 and the facts, first proved by Carl Gauss in 1801, that

$$U(2) \approx \{0\}, \qquad U(4) \approx Z_2, \qquad U(2^n) \approx Z_{2^{n-2}} \oplus Z_2 \qquad \text{for } n \geq 3,$$

and

$$U(p^n) \approx Z_{p^n - p^{n-1}} \qquad \text{for } p \text{ an odd prime},$$

we now can write any U-group as an external direct product of cyclic groups.[2] For example,

$$\begin{aligned} U(105) = U(3 \cdot 5 \cdot 7) &\approx U(3) \oplus U(5) \oplus U(7) \\ &\approx Z_2 \oplus Z_4 \oplus Z_6 \end{aligned}$$

and

$$\begin{aligned} U(144) &= U(16) \oplus U(9) \\ &\approx Z_4 \oplus Z_2 \oplus Z_6. \end{aligned}$$

What is the advantage of expressing the group $U(n)$ as an external direct product of groups of the form Z_m? Well, for one thing, we immediately see that $|U(105)| = 2 \cdot 4 \cdot 6 = 48$ and that $U(105)$ and $U(144)$ are isomorphic. Another is that from Theorem 8.1 we know that the orders of the elements in $U(105)$ are 1, 2, 3, 4, 6 and 12. Moreover, arguing as in Examples 5 and 6, we can determine that $U(105)$ has exactly 16 elements of order 12, say.

These calculations tell us more. Since $\text{Aut}(Z_{105})$ is isomorphic to $U(105)$, we also know that there are 16 automorphisms of Z_{105} of order 12. Imagine trying to deduce this information directly from $U(105)$ or, worse yet, from $\text{Aut}(Z_{105})$! These results beautifully illustrate the advantage of being able to represent a finite Abelian group as a direct product of cyclic groups. They also show the value of our theorems about $\text{Aut}(Z_n)$ and $U(n)$. After all, theorems are laborsaving devices. If you want to convince yourself of this, try to prove directly from the definitions that $\text{Aut}(Z_{105})$ has exactly 16 elements of order 12.

Here is a fun example that pulls together our results from Chapter 6 and Chapter 8.

EXAMPLE 9 Find $|(\text{Aut}(\text{Aut}(\text{Aut}(Z_{27})))|$. To this end we note that $\text{Aut}(\text{Aut}(\text{Aut}(Z_{27}))) \approx \text{Aut}(\text{Aut}(U(27))) \approx \text{Aut}(\text{Aut}(Z_{18})) \approx \text{Aut}(U(18)) \approx \text{Aut}(Z_6) \approx U(6) \approx Z_2$. ∎

Applications

We conclude this chapter with five applications of the material presented here—three to cryptography, the science of sending and deciphering secret messages, one to genetics, and one to electric circuits.

[2]To see that the number of integers from 1 to p^n that are relatively prime to p is $p^n - p^{n-1} = p^{n-1}(p-1)$ note that the condition excludes every pth element, which is $1/p$th of $p^n - p^{n-1}$.

Data Security

Because computers are built from two-state electronic components, it is natural to represent information as strings of 0s and 1s called *binary strings.* A binary string of length n can naturally be thought of as an element of $Z_2 \oplus Z_2 \oplus \cdots \oplus Z_2$ (n copies) where the parentheses and the commas have been deleted. Thus the binary string 11000110 corresponds to the element (1, 1, 0, 0, 0, 1, 1, 0) in $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$. Similarly, two binary strings $a_1a_2 \cdots a_n$ and $b_1b_2 \cdots b_n$ are added componentwise modulo 2 just as their corresponding elements in $Z_2 \oplus Z_2 \oplus \cdots \oplus Z_2$ are. For example,

$$11000111 + 01110110 = 10110001$$

and

$$10011100 + 10011100 + 00000000.$$

The fact that the sum of two binary sequences $a_1a_2 \cdots a_n + b_1b_2 \cdots b_n = 00 \cdots 0$ if and only if the sequences are identical is the basis for a data security system used to protect Internet transactions.

Suppose that you want to purchase an item from Amazon. Need you be concerned that a hacker will intercept your credit-card number during the transaction? As you might expect, your credit-card number is sent to Amazon in a way that protects the data. We explain one way to send credit-card numbers over the Web securely. When you place an order with Amazon, the company sends your computer a randomly generated string of 0's and 1's called a *key*. This key has the same length as the binary string corresponding to your credit-card number and the two strings are added (think of this process as "locking" the data). The resulting sum is then transmitted to Amazon. Amazon in turn adds the same key to the received string, which then produces the original string corresponding to your credit-card number (adding the key a second time "unlocks" the data).

To illustrate the idea, say you want to send an eight-digit binary string such as $s = 10101100$ to Amazon (actual credit-card numbers have very long strings) and Amazon sends your computer the key $k = 00111101$. Your computer returns the string $s + k = 10101100 + 00111101 = 10010001$ to Amazon, and Amazon adds k to this string to get $10010001 + 00111101 = 10101100$, which is the string representing your credit-card number. If someone intercepts the number $s + k = 10010001$ during transmission it is no value without knowing k.

The method is secure because the key sent by Amazon is randomly generated and used only one time.

Public Key Cryptography

Unlike auctions such as those on eBay, where each bid is known by everyone, a silent auction is one in which each bid is secret. Suppose that you wanted to use your Twitter account to run a silent auction. How could a scheme be devised so that users could post their bids in such a way that the amounts are intelligible only to the account holder? In the mid-1970s, Ronald Rivest, Adi Shamir, and Leonard Adleman devised an ingenious method that permits each person who is to receive a secret message to tell publicly how to scramble messages sent to him or her. And even though the method used to scramble the message is known publicly, only the person for whom it is intended will be able to unscramble the message. The idea is based on the fact that there exist efficient methods for finding very large prime numbers (say about 100 digits long) and for multiplying large numbers, but no one knows an efficient algorithm for factoring large integers (say about 200 digits long). The person who is to receive the message chooses a pair of large primes p and q and chooses an integer e (called the *encryption exponent*) with $1 < e < m$, where $m = \text{lcm}(p - 1, q - 1)$, such that e is relatively prime to m (any such e will do). This person calculates $n = pq$ (n is called the *key*) and announces that a message M is to be sent to him or her publicly as $M^e \bmod n$. Although e, n, and M^e are available to everyone, only the person who knows how to factor n as pq will be able to decipher the message.

To present a simple example that nevertheless illustrates the principal features of the method, say we wish to send the messages "YES." We convert the message into a string of digits by replacing A by 01, B by $02, \ldots, Z$ by 26, and a blank by 00. So, the message YES becomes 250519. To keep the numbers involved from becoming too unwieldy, we send the message in blocks of four digits and fill in with blanks when needed. Thus, the messages YES is represented by the two blocks 2505 and 1900. The person to whom the message is to be sent has picked two primes

p and q, say $p = 37$ and $q = 73$, and a number e that has no prime divisors in common with $\text{lcm}(p-1, q-1) = 72$, say $e = 5$, and has published $n = 37 \cdot 73 = 2701$ and $e = 5$ in a public forum. We will send the "scrambled" numbers $(2505)^5 \bmod 2701$ and $(1900)^5 \bmod 2701$ rather than 2505 and 1900, and the receiver will unscramble them. We show the work involved for us and the receiver only for the block 2505. We determine $(2505)^5 \bmod 2701 = 2415$ by using a modular arithmetic calculator such as the one at **planetcalc.com/8326/**.[3]

Thus, the number 2415 is sent to the receiver. Now the receiver must take this number and convert it back to 2505. To do so, the receiver takes the two factors of 2701, $p = 37$ and $q = 73$, and calculates the least common multiple of $p-1 = 36$ and $q-1 = 72$, which is 72. (This is where the knowledge of p and q is necessary.) Next, the receiver must find $e^{-1} = d$ (called the *decryption exponent*) in $U(72)$—that is, solve the equation $5 \cdot d = 1 \bmod 72$. This number is 29. See **https://www.dcode.fr/modular-inverse** or use a Google search box to compute 5^k for each divisor k of $|U(72)| = \phi(9) \cdot \phi(8) = 24$ starting with 2 until we reach $5^k \bmod 72 = 1$. Doing so, we obtain $5^6 \bmod 72 = 1$, which implies that $5^5 \bmod 72 = 29$ is 5^{-1} in $U(72)$.

Then the receiver takes the number received, 2415, and calculates $(2415)^{29} \bmod 2701 = 2505$, the encoded number. Thus, the receiver correctly determines the code for "YE." On the other hand, without knowing how pq factors, one cannot find the modulus (in our case, 72) that is needed to determine the decryption exponent d.

The procedure just described is called the *RSA public key encryption scheme* in honor of the three people (Rivest, Shamir, and Adleman) who discovered the method. It is widely used in conjunction with web servers and browsers, e-mail programs, remote login sessions, and electronic financial transactions. The algorithm is summarized below.

[3]Provided that the numbers are not too large, the Google search engine at **http://www.google.com** will do modular arithmetic. For example, entering 2505^2 mod 2701 in the search box yields 602. Be careful, however: Entering 2505^5 mod 2701 does not return a value, because 2505^5 is too large. Instead, we can use Google to compute smaller powers such as 2505^2 mod 2701 and 2505^3 mod 2701 (which yields 852) and then enter $(852 \times 602) \bmod 2701$.

Receiver

1. Pick very large primes p and q and compute $n = pq$.
2. Compute the least common multiple of $p - 1$ and $q - 1$; let us call it m.
3. Pick e relatively prime to m.
4. Find d such that $ed \bmod m = 1$.
5. Publicly announce n and e.

Sender

1. Convert the message to a string of digits.
2. Break up the message into uniform blocks of digits; call them $M_1, M_2, \ldots, M_k$. (The integer value of each M_i must be less than n. In practice, n is so large that this is not a concern.)
3. Check to see that the greatest common divisor of each M_i and n is 1. If not, n can be factored and our code is broken. (In practice, the primes p and q are so large that they exceed all M_i, so this step may be omitted.)
4. Calculate and send $R_i = M_i^e \bmod n$.

Receiver

1. For each received message R_i, calculate ${R_i}^d \bmod n$.
2. Convert the string of digits back to a string of characters.

Why does this method work? Well, we know that $U(n) \approx U(p) \oplus U(q) \approx Z_{p-1} \oplus Z_{q-1}$. Thus, an element of the form x^m in $U(n)$ corresponds under an isomorphism to one of the form (mx_1, mx_2) in $Z_{p-1} \oplus Z_{q-1}$. Since m is the least common multiple of $p - 1$ and $q - 1$, we may write $m = s(p-1)$ and $m = t(q-1)$ for some integers s and t. Then $(mx_1, mx_2) = (s(p-1)x_1, (q-1)x_2) = (0, 0)$ in $Z_{p-1} \oplus Z_{q-1}$, and it follows that $x^m = 1$ for all x in $U(n)$. So, because each message M_i is an element of $U(n)$ and e was chosen so that $ed = 1 + km$ for some k, we have, modulo n,

$$ {R_i}^d = ({M_i}^e)^d = {M_i}^{ed} = M_i^{1+km} = M_i(M_i^m)^k = M_i 1^k = M_i. $$

In 2002, Ronald Rivest, Adi Shamir, and Leonard Adleman received the Association for Computing Machinery A. M. Turing Award, which is considered the “Nobel Prize of computing,” for their contribution to public key cryptography.

Digital Signatures

With so many financial transactions now taking place electronically, the problem of authenticity is paramount. How is a stockbroker to know that an electronic message she receives that tells her to sell one stock and buy another actually came from her client? The technique used in public key cryptography allows for digital signatures as well. Let us say that person A wants to send a secret message to person B in such a way that only B can decode the message and B will know that only A could have sent it. Abstractly, let E_A and D_A denote the algorithms that A uses for encryption and decryption, respectively, and let E_B and D_B denote the algorithms that B uses for encryption and decryption, respectively. Here we assume that E_A and E_B are available to the public, whereas D_A is known only to A and D_B is known only to B, and that D_BE_B and E_AD_A applied to any message leaves the message unchanged. Then A sends a message M to B as $E_B(D_A(M))$ and B decodes the received message by applying the function E_AD_B to it to obtain

$$(E_AD_B)(E_B(D_A(M)) = E_A(D_BE_B)(D_A(M)) = E_A(D_A(M)) = M.$$

Notice that only A can execute the first step (i.e., create $D_A(M)$ and only B can implement the last step (i.e., apply E_AD_B to the received message).

Transactions using digital signatures became legally binding in the United States in October 2000.

Genetics[3]

The genetic code can be conveniently modeled using elements of $Z_4 \oplus Z_4 \oplus \cdots \oplus Z_4$, where we omit the parentheses and the commas and just use strings of 0's, 1's, 2's, and 3's and add componentwise modulo 4. A DNA molecule is composed of two long strands in the form of a double helix. Each strand is made up of strings of the four nitrogen bases adenine (A), thymine (T), guanine (G), and cytosine (C). Each base on one strand binds to a complementary base on the other strand. Adenine always is bound to thymine, and guanine always is bound to cytosine. To model this process, we identify A with 0, T with 2, G with 1, and C with 3. Thus,

[3]S. Washburn, T. Marlowe, and C. Ryan, Discrete Mathematics, Reading, MA: Addison-Wesley, 1999.

the DNA segment ACGTAACAGGA and its complement segment TGCATTGTCCT are denoted by 03120030110 and 21302212332. Noting that in $Z_4, 0+2=2, 2+2=0, 1+2=3$, and $3+2=1$, we see that adding 2 to elements of Z_4 interchanges 0 and 2 and 1 and 3. So, for any DNA segment $a_1a_2\cdots a_n$ represented by elements of $Z_4 \oplus Z_4 \oplus \cdots \oplus Z_4$, we see that its complementary segment is represented by $a_1a_2\cdots a_n + 22\cdots 2$.

Electric Circuits

Many homes have light fixtures that are operated by a pair of switches. They are wired so that when either switch is thrown, the light changes its status (from on to off or vice versa). Suppose the wiring is done so that the light is on when both switches are in the up position. We can conveniently think of the states of the two switches as being matched with the elements of $Z_2 \oplus Z_2$, with the two switches in the up position corresponding to (0, 0) and the two switches in the down position corresponding to (1, 1). Each time a switch is thrown, we add 1 to the corresponding component in the group $Z_2 \oplus Z_2$. We then see that the lights are on when the switches correspond to the elements of the subgroup $\langle(1,1)\rangle$ and are off when the switches correspond to the elements in the coset $(1, 0) + \langle(1,1)\rangle$. A similar analysis applies in the case of three switches, with the subgroup {(0, 0, 0), (1, 1, 0), (0, 1, 1), (1, 0, 1)} corresponding to the lights-on situation.

Exercises

What's the most difficult aspect of your life as a mathematician, Diane Maclagan, an assistant professor at Rutgers, was asked. "Trying to prove theorems," she said. And the most fun? "Trying to prove theorems."

1. Prove that the external direct product of any finite number of groups is a group. (This exercise is referred to in this chapter.)
2. Prove that (1,1) is an element of largest order in $Z_{n_1} \oplus Z_{n_2}$. State the general case.
3. Let G be a group with identity e_G and let H be a group with identity e_H. Prove that G is isomorphic to $G \oplus \{e_H\}$ and that H is isomorphic to $\{e_G\} \oplus H$.

4. Show that $G \oplus H$ is Abelian if and only if G and H are Abelian. State the general case.

5. Prove that $Z \oplus Z$ is not cyclic. Does your proof work for $Z \oplus G$ where G is any group with more than one element?

6. Prove, by comparing orders of elements, that $Z_8 \oplus Z_2$ is not isomorphic to $Z_4 \oplus Z_4$.

7. Prove that $G_1 \oplus G_2$ is isomorphic to $G_2 \oplus G_1$. State the general case.

8. Is $Z_3 \oplus Z_9$ isomorphic to Z_{27}? Why?

9. Give an example of an Abelian group of order 12 that has two subgroups of order 6. Generalize to the case that the group has order p^2m where p is prime, m is relatively prime to p, and the subgroup has order pm.

10. How many elements of order 9 does $Z_3 \oplus Z_9$ have? (Do not do this exercise by brute force.)

11. How many elements of order 4 does $Z_4 \oplus Z_4$ have? (Do not do this by examining each element.) Explain why $Z_4 \oplus Z_4$ has the same number of elements of order 4 as does $Z_{8000000} \oplus Z_{400000}$. Generalize to the case $Z_m \oplus Z_n$.

12. List the elements in the groups $U_5(35)$ and $U_7(35)$.

13. For each integer $n > 1$, give examples of two nonisomorphic groups of order n^2.

14. The dihedral group D_n of order $2n$ ($n \geq 3$) has a subgroup of n rotations and a subgroup of order 2. Explain why D_n cannot be isomorphic to the external direct product of two such groups.

15. Prove that the group of complex numbers under addition is isomorphic to $\mathbf{R} \oplus \mathbf{R}$.

16. Suppose that $G_1 \approx G_2$ and $H_1 \approx H_2$. Prove that $G_1 \oplus H_1 \approx G_2 \oplus H_2$. State the general case.

17. If $G \oplus H$ is cyclic, prove that G and H are cyclic. State the general case.

18. Find a cyclic subgroup of $Z_{40} \oplus Z_{30}$ of order 12 and a noncyclic subgroup of $Z_{40} \oplus Z_{30}$ of order 12.

19. If r is a divisor of m and s is a divisor of n, find a subgroup of $Z_m \oplus Z_n$ that is isomorphic to $Z_r \oplus Z_s$.

20. Find a subgroup of $Z_{12} \oplus Z_{18}$ that is isomorphic to $Z_9 \oplus Z_4$.

21. Let G and H be finite groups and $(g, h) \in G \oplus H$. State a necessary and sufficient condition for $\langle(g, h)\rangle = \langle g\rangle \oplus \langle h\rangle$.

22. Determine the number of elements of order 15 and the number of cyclic subgroups of order 15 in $Z_{30} \oplus Z_{20}$.

23. How many subgroups of order 3 are there in $Z_3 \oplus Z_3$? What about $Z_3 \oplus Z_3 \oplus Z_3$? What about $Z_3 \oplus Z_3 \oplus \cdots \oplus Z_3$ (n copies)?

24. Let m be an even integer at least 4 and n an odd integer at least 3. Determine an expression in m and n that gives the number of elements of order 2 in $D_m \oplus D_n$. If m is divisible by 4, give an expression for the number of elements of order 4 in $D_m \oplus D_n$.

25. Let M be the group of all real 2×2 matrices under addition. Let $N = \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R} \oplus \mathbf{R}$ under componentwise addition. Prove that M and N are isomorphic. What is the corresponding theorem for the group of $m \times n$ matrices under addition?

26. The group $S_3 \oplus Z_2$ is isomorphic to one of the following groups: $Z_{12}, Z_6 \oplus Z_2, A_4, D_6$. Determine which one by elimination.

27. Let G be a group, and let $H = \{(g, g) \mid g \in G\}$. Show that H is a subgroup of $G \oplus G$. (This subgroup is called the *diagonal* of $G \oplus G$.) When G is the set of real numbers under addition, describe $G \oplus G$. and H geometrically.

28. List six examples of non-Abelian groups of order 24.

29. Find all subgroups of order 3 in $Z_9 \oplus Z_3$.

30. Find all subgroups of order 4 in $Z_4 \oplus Z_4$.

31. What is the order of the largest cyclic subgroup of $Z_6 \oplus Z_{10} \oplus Z_{15}$? What is the order of the largest cyclic subgroup of $Z_{n_1} \oplus Z_{n_2} \oplus \cdots Z_{n_k}$?

32. How many elements of order 2 are in $Z_{2000000} \oplus Z_{4000000}$? Generalize.

33. Find two subgroups of $Z_{100} \oplus Z_{60}$ isomorphic to $Z_4 \oplus Z_3 \oplus Z_2$. No need to prove anything.

34. Find a subgroup of $Z_{100} \oplus D_4$ isomorphic to $Z_4 \oplus Z_4$.

35. Find a subgroup of $D_3 \oplus D_4$ isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$.

36. Find a subgroup of $Z_{12} \oplus Z_4 \oplus Z_{15}$ that has order 9.

37. Prove that $\mathbf{R}^* \oplus \mathbf{R}^*$ is not isomorphic to $\mathbf{C}^*$. (Compare this with Exercise 15.)

38. Let

$$H = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \middle| \, a, b \in Z_3 \right\}.$$

(See Exercise 35 in Chapter 2 for the definition of multiplication.) Show that H is an Abelian group of order 9. Is H isomorphic to Z_9 or to $Z_3 \oplus Z_3$?

39. Let $G = \{3^m 6^m \,|\, m, n \in Z\}$ under multiplication. Prove that G isomorphic to $Z \oplus Z$. Does your proof remain valid if $G = \{3^m 9^m \,|\, m, n \in Z\}$?

40. Let $(a_1, a_2, \ldots, a_n) \in G_1 \oplus G_2 \oplus \cdots \oplus G_n$. Give a necessary and sufficient condition for $|(a_1, a_2, \ldots, a_n| = \infty$.

41. Compare the number of elements of each order in D_6 with the number for each order in $D_3 \oplus Z_2$.

42. Prove or disprove that $U(40) \oplus Z_6$ is isomorphic to $U(72) \oplus Z_4$.

43. Prove that $\mathbf{C}^*$ has a subgroup isomorphic to $Z_m \oplus Z_n$ if and only if $\gcd(m, n) = 1$.

44. Let G be a group isomorphic to $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$. Let x be the product of all elements in G. Describe all possibilities for x.

45. If a group has exactly 24 elements of order 6, how many cyclic subgroups of order 6 does it have?

46. Give an example of an infinite group that has both a subgroup isomorphic to D_4 and a subgroup isomorphic to A_4.

47. Express $\text{Aut}(U(25))$ in the form $Z_m \oplus Z_n$.

48. Determine $\text{Aut}(Z_2 \oplus Z_2)$.

49. Suppose that $n_1, n_2, \ldots, n_k$ are positive even integers. How many elements of order 2 does $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$ have? How many are there if we drop the requirement that $n_1, n_2, \ldots, n_k$ must be even?

50. Is $Z_{10} \oplus Z_{12} \oplus Z_6 \approx Z_{60} \oplus Z_6 \oplus Z_2$? Is $Z_{10} \oplus Z_{12} \oplus Z_6 \approx Z_{15} \oplus Z_4 \oplus Z_{12}$?

51. **a.** How many isomorphisms are there from Z_{18} to $Z_2 \oplus Z_9$? Give a formula for two of the isomorphisms.
b. How many isomorphisms are there from Z_{18} to $Z_2 \oplus Z_3 \oplus Z_3$?

52. Suppose that ϕ is an isomorphism from $Z_3 \oplus Z_5$ to Z_{15} and $\phi((2,3)) = 2$. Find the element in $Z_3 \oplus Z_5$ that maps to 1.

53. If ϕ is an isomorphism from $Z_4 \oplus Z_3$ to Z_{12}, what is $\phi((2,0))$? What are the possibilities for $\phi((1,0))$? Give reasons for your answer.

54. Find a subgroup of $U(140)$ isomorphic to $Z_4 \oplus Z_6$.

55. Let (a, b) belong to $Z_m \oplus Z_n$. Prove that $|(a, b)|$ divides $\text{lcm}(m, n)$.

56. Let $G = \{ax^2 + bx + c \mid a, b, c \in Z_3\}$. Add elements of G as you would polynomials with integer coefficients, except use modulo 3 addition. Prove that G is isomorphic to $Z_3 \oplus Z_3 \oplus Z_3$. Generalize

57. Determine all cyclic groups that have exactly two generators.

58. Explain a way that a string of length n of the four nitrogen bases A, T, G, and C could be modeled with the external direct product of n copies of $Z_2 \oplus Z_2$.

59. Let p be a prime. Prove that $Z_p \oplus Z_p$ has exactly $p + 1$ subgroups of order p.

60. Give an example of an infinite non-Abelian group that has exactly six elements of finite order.

61. Give an example to show that there exists a group with elements a and b such that $|a| = \infty$, $|b| = \infty$, and $|ab| = 2$. Generalize to the case $|a| = \infty$, $|b| = \infty$, and $|ab| = n$, where n is any positive integer.

62. Express $U(165)$ as an external direct product of cyclic groups of the form Z_n.

63. Express $U(165)$ as an external direct product of U-groups in four different ways.

64. Express $U_9(72)$ and $U_4(300)$ as an external direct product of cyclic groups of the form Z_n.

65. Prove that for odd n, $U(2n)$ is isomorphic to $U(n)$.

66. If n is an integer at least 3, determine the number of elements of order 2 in $U(2^n)$.

67. Without doing any calculations in $\text{Aut}(Z_{105})$, determine how many elements of $\text{Aut}(Z_{105})$ have order 6. Without doing any calculations in $U(27)$, decide how many subgroups $U(27)$ has.

68. Write $\text{Aut}(\text{Aut}(Z_{50}))$ as an external direct product of groups of the form Z_n.

69. Prove that $A_4 \oplus Z_4$ is not isomorphic to $D_{12} \oplus Z_2$.

70. Prove that $D_8 \oplus D_3 \not\approx D_6 \oplus D_4$.

71. Give an example of an infinite group that has a subgroup isomorphic to D_4.

72. Let p and q be odd primes and let m and n be positive integers. Explain why $U(p^m) \oplus U(q^n)$ is not cyclic.

73. Use the results presented in this chapter to prove that $U(55)$ is isomorphic to $U(75)$.
Use the results presented in this chapter to prove that $U(144)$ is isomorphic to $U(140)$.

74. What is the largest order of any element in $U(900)$?

75. What are the two smallest positive integers n such that $\text{Aut}(Z_n)$ is not cyclic.

76. Find an integer n such that $U(n)$ is isomorphic to $Z_2 \oplus Z_4 \oplus Z_9$.

77. What is the smallest positive integer k such that $x^k = e$ for all x in $U(7 \cdot 17)$? Generalize to $U(pq)$ where p and q are distinct primes.

78. Is $U(200) \approx U(50) \oplus U(4)$? Is $U_{50}(200) \approx U(4)$? Why does your answer not contradict Theorem 8.3?

79. Let p be an odd prime and n be a positive integer. To what familiar group is $U_p(p^n)$ isomorphic?

80. Find the smallest positive integer n such that $x^n = 1$ for all x in $U(100)$. Show your reasoning.

81. Which of the following groups are cyclic? **a.** $U(35)$ **b.** $U_5(40)$ **c.** $U_8(40)$

82. Let $p_1, p_2, \ldots, p_k$ be distinct odd primes and $n_1, n_2, \ldots, n_k$ be positive integers. Determine the number of elements of order 2 in $U(p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k})$ How many are there in $U(2^n p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k})$ where n is at least 3?

83. Find a subgroup of $U(140)$ isomorphic to $U(28)$. Find a subgroup of $U(140)$ isomorphic to $Z_4 \oplus Z_6$.

84. Find all integers n such that $U(n)$ is isomorphic to a group of the form $Z_2 \oplus Z_2 \oplus \cdots \oplus Z_2$.

85. For relatively prime positive integers $s \leq n$ and $t \leq n$ show that $U_{st}(n) = U_s(n) \cap U_t(n)$.

86. Explain why S_{20} contains a subgroup isomorphic to $D_5 \oplus Z_2$.

87. Let $G = \left\{ \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} -1 & 0 \\ 0 & -1 \end{bmatrix} \right\}$. Is G isomorphic to Z_4 or $Z_2 \oplus Z_2$?

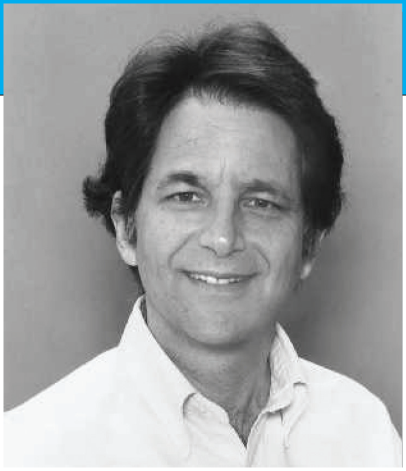
88. Let $G = \left\{ \begin{bmatrix} a & 0 & 0 \\ 0 & b & 0 \\ 0 & 0 & c \end{bmatrix} \middle| \; a, b, c \text{ are } 1 \text{ or } -1 \right\}$. Is G Abelian? Is G isomorphic to Z_8, $Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$?

89. Assuming that a message has been sent via the RSA scheme with $p = 37$, $q = 73$, and $e = 5$, decode the received message "34."

90. Using the RSA scheme with $p = 37$, $q = 73$, $e = 5$, and replacing the letters A, B, . . .,Z by 01, 02, . . . , 26, what number would be sent for the message "RL"?

91. Explain why the message YES cannot be sent using RSA scheme with $p = 31$ and $q = 73$ using blocks of length 4.

Leonard Adleman



Courtesy of Leonard Aldeman

For their ingenious contribution for making public-key cryptography useful in practice.

Citation for the ACM A. M. Turing Award

LEONARD ADLEMAN was born on December 31, 1945 in San Francisco, California. He received a B.A. degree in mathematics in 1968 and a Ph.D. degree in computer science in 1976 from the University of California, Berkeley. He spent 1976–1980 as professor of mathematics at the Massachusetts Institute of Technology where he met Ronald Rivest and Adi Shamir. Rivest and Shamir were at- tempting to devise a secure public key cryptosystem and asked Adleman if he could break their codes. Eventually, they invented what is now known as the RSA code that was simple to implement yet secure. In 1983, Adleman, Shamir, and Rivest formed the RSA Data Security company to license their algorithm. Their algorithm has become the primary cryptosystem used for security on the World Wide Web. They sold their company for $200 million in 1996.

In the early 1990s, Adleman became interested in trying to find out a way to use DNA as a computer. His pioneering work on this problem lead to the field now called "DNA computing."

Among his many honors are: the Association for Computing Machinery A. M. Turing Award, the Kanallakis Award for Theory and Practice, and election to the National Academy of Engineering, the American Academy of Arts and Sciences, and the National Academy of Sciences.

Adleman's current position is the Henry Salvatori Distinguished Chair in Computer Science and Professor of Computer Science and Biological Sciences at the University of Southern California, where he has been since 1980.

For more information on Adleman, visit: **http://www.nytimes.com/1994/12/13/science/scientist-at-work-leonard-adleman-hitting-the-high-spots-of-computer-theory.html**

9 Normal Subgroups and Factor Groups

It is tribute to the genius of Galois that he recognized that those subgroups for which the left and right cosets coincide are distinguished ones. Very often in mathematics the crucial problem is to recognize and to discover what are the relevant concepts; once this is accomplished the job may be more than half done.

I. N. Herstein, *Topics in Algebra*

[On the concept of 'group':] ... what a wealth, what a grandeur of thought may spring from what slight beginnings.

H. F. Baker

Normal Subgroups

As we saw in Chapter 7, if G is a group and H is a subgroup of G, it is not always true that $aH = Ha$ for all a in G. There are certain situations where this does hold, however, and these cases turn out to be of critical importance in the theory of groups. It was Galois, about 190 years ago, who first recognized that such subgroups were worthy of special attention.

Definition Normal Subgroup
A subgroup H of a group G is called a *normal* subgroup of G if $aH = Ha$ for all a in G. We denote this by $H \triangleleft G$.

You should think of a normal subgroup in this way: You can switch the order of a product of an element a from the group and an element h from the normal subgroup H, but you must "fudge" a bit on the element from the normal subgroup H by using some h' from H rather than h. That is, there is an element h' in H such that $ah = h'a$. Likewise, there is some h'' in H such that $ha = ah''$. (It is possible that $h' = h$ or $h'' = h$, but we may not assume this.)

To illustrate the importance of normality consider a situation where we have a group G generated by two elements a and b where $|a| = 3$ and $|b| = 2$ (one such example of this is D_3). If a and b commute then, because the set $\{e, a, a^2, b, ab, ab^2\}$ is closed, it is G itself. If not, then all we know about G is that the non-identity elements can be written as products of a's and b's with never more than two consecutive a's and never with two consecutive b's. That is, $e, a, b, a^2, ab, ba, a^2b, aba, ba^2, bab, a^2ba, aba^2, abab, ba^2b, baba, \ldots$. Some of these may be equal but we do not know which. The group may finite or infinite but we do not know which. In sharp contrast, if $H = \{e, a, a^2\}$ or $K = \{e, b\}$ is normal in G, then $G = \{e, a, a^2, b, ab, a^2b\} = HK$ (see Example 5). So, in this situation, and many others, normality is as good as commutativity.

There are several equivalent formulations of the definition of normality. We have chosen the one that is the easiest to use in applications. However, to *verify* that a subgroup is normal, it is usually better to use Theorem 9.1, which is a weaker version of property 8 of the lemma in Chapter 7. It allows us to substitute a condition about two subgroups of G for a condition about two cosets of G.

Theorem 9.1 Normal Subgroup Test

A subgroup H of G is normal in G if and only if $xHx^{-1} \subseteq H$ for all x in G.

PROOF If H is normal in G, then for any $x \in G$ and $h \in H$ there is an h' in H such that $xh = h'x$. Thus, $xhx^{-1} = h'$, and therefore $xHx^{-1} \subseteq H$.

Conversely, if $xHx^{-1} \subseteq H$ for all x, then, letting $x = a$, we have $aHa^{-1} \subseteq H$ or $aH \subseteq Ha$. On the other hand, letting $x = a^{-1}$, we have $a^{-1}H(a^{-1})^{-1} = a^{-1}Ha \subseteq H$ or $Ha \subseteq aH$. ■

EXAMPLE 1 Every subgroup of an Abelian group is normal. (In this case, $ah = ha$ for a in the group and h in the subgroup.) ■

EXAMPLE 2 The center $Z(G)$ of a group is always normal. [Again, $ah = ha$ for any $a \in G$ and any $h \in Z(G)$.] ■

EXAMPLE 3 The alternating group A_n of even permutations is a normal subgroup of S_n. [Note, for example, that for $(12) \in S_n$ and $(123) \in A_n$, we have $(12)(123) \neq (123)(12)$ but $(12)(123) = (132)(12)$ and $(132) \in A_n$.] See Exercise 9. ∎

EXAMPLE 4 Every subgroup of D_n consisting solely of rotations is normal in D_n. (For any rotation R and any reflection F, we have $FR = R^{-1}F$ and any two rotations commute.) ∎

The next example illustrates a way to use a normal subgroup to create new subgroups from existing ones.

EXAMPLE 5 Let H be a normal subgroup of a group G and K be any subgroup of G. Then $HK = \{hk \mid h \in H, k \in K\}$ is a subgroup of G. To verify this, note that $e = ee$ is in HK. Then for any $a = h_1k_1$ and $b = h_2k_2$, where h_1, h_2 are in H and k_1, k_2 are in K, there is an element h' in H such that $ab^{-1} = h_1k_1k_2^{-1}h_2^{-1} = h_1(k_1k_2^{-1})h_2^{-1} = (h_1h')(k_1k_2^{-1})$. So, ab^{-1} is in HK. ∎

Be careful not to assume that for any subgroups H and K of a group G, the set HK is a subgroup of G. See Exercise 6 in Chapter 7.

Combining Examples 4 and 5, we form a non-Abelian subgroup of D_8 of order 8.

EXAMPLE 6 In D_8, let $H = \{R_0, R_{90}, R_{180}, R_{270}\}$ and $K = \{R_0, F\}$, where F is any reflection. Then $HK = \{R_0, R_{90}, R_{180}, R_{270}, R_0F, R_{90}F, R_{180}F, R_{270}F\}$ is a subgroup of D_8. Note that HK is isomorphic to D_4. ∎

The next example is quite often quite useful for identifying certain normal subgroups.

EXAMPLE 7 If a group G has a unique subgroup H of some finite order, then H is normal in G. To see that this is so, observe that for any $g \in G$, the inner automorphism ϕ_g takes H to gHg^{-1} so gHg^{-1} is a subgroup of G and $|gHg^{-1}| = |H|$. ∎

EXAMPLE 8 The group $SL(2, \mathbf{R})$ of 2×2 matrices with determinant 1 is a normal subgroup of $GL(2, \mathbf{R})$, the group of 2×2 matrices with nonzero determinant. To verify this, we use the Normal Subgroup Test given in Theorem 9.1. Let $x \in GL(2, \mathbf{R}) = G, h \in SL(2, \mathbf{R}) = H$, and note that det $xhx^{-1} = (\det x)(\det h)(\det x)^{-1} = (\det x)(\det x)^{-1} = 1$. So, $xhx^{-1} \in H$, and, therefore, $xHx^{-1} \subseteq H$. ∎

EXAMPLE 9 Referring to the group table for A_4 given in Table 5.1, we may observe that $H = \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ is a normal subgroup of A_4, whereas $K = \{\alpha_1, \alpha_5, \alpha_9\}$ is *not* a normal subgroup of A_4. To see that H is normal, simply note that for any β in $A_4, \beta H \beta^{-1}$ is a subgroup of order 4 and H is the only subgroup of A_4 of order 4 (see Table 5.1). Thus, $\beta H \beta^{-1} = H$. In contrast, $\alpha_2 \alpha_5 \alpha_2^{-1} = \alpha_7$, so that $\alpha_2 K \alpha_2^{-1} \not\subseteq K$. ▮

Factor Groups

We have yet to explain why normal subgroups are of special significance. The reason is simple. When the subgroup H of G is normal, then the set of left (or right) cosets of H in G is itself a group—called the *factor group of G by H* (or the *quotient group of G by H*). Quite often, one can obtain information about a group by studying one of its factor groups. This method will be illustrated in the next section of this chapter.

Theorem 9.2 Factor Groups (O. Hölder, 1889)

Let G be a group and let H be a normal subgroup of G. The set $G/H = \{aH \mid a \in G\}$ is a group under the operation $(aH)(bH) = abH$.[1]

PROOF Our first task is to show that the operation is well-defined; that is, we must show that the correspondence defined above from $G/H \times G/H$ into G/H is actually a function. To do this, we assume that for some elements a, a', b, and b' from G, we have $aH = a'H$ and $bH = b'H$, and verify that $aHbH = a'Hb'H$. That is, verify that $abH = a'b'H$. (This shows that the definition of multiplication depends on only the cosets and not on the coset representatives.) From $aH = a'H$ and $bH = b'H$, we have $a' = ah_1$ and $b' = bh_2$ for some h_1, h_2 in H, and therefore $a'b'H = ah_1bh_2H = ah_1bH = ah_1Hb = aHb = abH$. Here we have made multiple use of associativity, property 2 of the lemma in Chapter 7, and the fact that $H \lhd G$. The rest is easy: $eH = H$ is the identity; $a^{-1}H$ is the inverse of aH; and $(aHbH)cH = (ab)HcH = (ab)cH = a(bc)H = aH(bc)H = aH(bHcH)$. This proves that G/H is a group. ▮

[1]The notation G/H was first used by C. Jordan.

Notice that the normality of H in G assures that the product of two left cosets aH and bH is a left coset of H is G, and since $aHbH$ contains ab, the product is abH.

Although it is merely a curiosity, we point out that the converse of Theorem 9.2 is also true; that is, if the correspondence $aHbH = abH$ defines a group operation on the set of left cosets of H in G, then H is normal in G. See Exercises 39 and 40.

The next few examples illustrate the factor group concept.

EXAMPLE 10 Let $4Z = \{0, \pm 4, \pm 8, \ldots\}$. To construct $Z/4Z$, we first must determine the left cosets of $4Z$ in Z. Consider the following four cosets:

$$\begin{aligned}
0 + 4Z &= 4Z = \{0, \pm 4, \pm 8, \ldots\},\\
1 + 4Z &= \{1, 5, 9, \ldots; -3, -7, -11, \ldots\},\\
2 + 4Z &= \{2, 6, 10, \ldots; -2, -6, -10, \ldots\},\\
3 + 4Z &= \{3, 7, 11, \ldots; -1, -5, -9, \ldots\}.
\end{aligned}$$

We claim that there are no others. For if $k \in Z$, then $k = 4q + r$, where $0 \leq r < 4$; and, therefore, $k + 4Z = r + 4q + 4Z = r + 4Z$. Now that we know the elements of the factor group, our next job is to determine the structure of $Z/4Z$. Its Cayley table is

	$\mathbf{0+4Z}$	$\mathbf{1+4Z}$	$\mathbf{2+4Z}$	$\mathbf{3+4Z}$
$\mathbf{0+4Z}$	$0+4Z$	$1+4Z$	$2+4Z$	$3+4Z$
$\mathbf{1+4Z}$	$1+4Z$	$2+4Z$	$3+4Z$	$0+4Z$
$\mathbf{2+4Z}$	$2+4Z$	$3+4Z$	$0+4Z$	$1+4Z$
$\mathbf{3+4Z}$	$3+4Z$	$0+4Z$	$1+4Z$	$2+4Z$

Clearly, then, $Z/4Z$ isomorphic to Z_4. More generally, if for any $n > 0$ we let $nZ = \{0, \pm n, \pm 2n, \pm 3n, \ldots\}$, then Z/nZ is isomorphic to Z_n. ∎

EXAMPLE 11 Let $G = Z_{18}$ and let $H = \langle 6 \rangle = \{0, 6, 12\}$. Then $G/H = \{0 + H, 1 + H, 2 + H, 3 + H, 4 + H, 5 + H\}$. To illustrate how the group elements are combined, consider $(5+H)+(4+H)$. This should be one of the six elements listed in the set G/H. Well, $(5 + H) + (4 + H) = 5 + 4 + H = 9 + H = 3 + 6 + H = 3 + H$, since H absorbs all multiples of 6. ∎

A few words of caution about notation are warranted here. When H is a normal subgroup of G, the expression $|aH|$ has two possible interpretations. One could be thinking of aH as a *set* of elements and $|aH|$ as the size of the set; or, as is more often the case, one could be thinking of aH as a group element of the factor group G/H and $|aH|$ as the order of the *element* aH in G/H. In Example 11, for instance, the *set* $3 + H$ has size 3, since $3 + H = \{3, 9, 15\}$. But the *group element* $3 + H$ has order 2, since $(3 + H) + (3 + H) = 6 + H = 0 + H$. As is usually the case when one notation has more than one meaning, the appropriate interpretation will be clear from the context.

■ **EXAMPLE 12** Let $\mathcal{K} = \{R_0, R_{180}\}$, and consider the factor group of the dihedral group D_4 (see the back inside cover for the multiplication table for D_4)

$$D_4/\mathcal{K} = \{\mathcal{K}, R_{90}\mathcal{K}, H\mathcal{K}, D\mathcal{K}\}.$$

The multiplication table for $D_4/\mathcal{K}$ is given in Table 9.1. (Notice that even though $R_{90}H = D'$, we have used $D\mathcal{K}$ in Table 9.1 for $R_{90}\mathcal{K}H\mathcal{K}$ because $D'\mathcal{K} = D\mathcal{K}$.)

Table 9.1

	$\mathcal{K}$	$R_{90}\mathcal{K}$	$H\mathcal{K}$	$D\mathcal{K}$
$\mathcal{K}$	$\mathcal{K}$	$R_{90}\mathcal{K}$	$H\mathcal{K}$	$D\mathcal{K}$
$R_{90}\mathcal{K}$	$R_{90}\mathcal{K}$	$\mathcal{K}$	$D\mathcal{K}$	$H\mathcal{K}$
$H\mathcal{K}$	$H\mathcal{K}$	$D\mathcal{K}$	$\mathcal{K}$	$R_{90}\mathcal{K}$
$D\mathcal{K}$	$D\mathcal{K}$	$H\mathcal{K}$	$R_{90}\mathcal{K}$	$\mathcal{K}$

$D_4/\mathcal{K}$ provides a good opportunity to demonstrate how a factor group of G is related to G itself. Suppose we arrange the heading of the Cayley table for D_4 in such a way that elements from the same coset of $\mathcal{K}$ are in adjacent columns (Table 9.2). Then, the multiplication table for D_4 can be blocked off into boxes that are cosets of $\mathcal{K}$, and the substitution that replaces a box containing the element x with the coset $x\mathcal{K}$ yields the Cayley table for $D_4/\mathcal{K}$. For example, when we pass from D_4 to $D_4/\mathcal{K}$, the box

$$\boxed{\begin{matrix} H & V \\ V & H \end{matrix}}$$

in Table 9.2 becomes the element $H\mathcal{K}$ in Table 9.1. Similarly, the box

$$\boxed{\begin{matrix} D & D' \\ D' & D \end{matrix}}$$

becomes the element $D\mathcal{K}$, and so on.

Table 9.2

	R_0	R_{180}	R_{90}	R_{270}	H	V	D	D'
R_0	R_0	R_{180}	R_{90}	R_{270}	H	V	D	D'
R_{180}	R_{180}	R_0	R_{270}	R_{90}	V	H	D'	D
R_{90}	R_{90}	R_{270}	R_{180}	R_0	D'	D	H	V
R_{270}	R_{270}	R_{90}	R_0	R_{180}	D	D'	V	H
H	H	V	D	D'	R_0	R_{180}	R_{90}	R_{270}
V	V	H	D'	D	R_{180}	R_0	R_{270}	R_{90}
D	D	D'	V	H	R_{270}	R_{90}	R_0	R_{180}
D'	D'	D	H	V	R_{90}	R_{270}	R_{180}	R_0

∎

In this way, one can see that the formation of a factor group G/H causes a systematic collapse of the elements of G. In particular, all the elements in the coset of H containing a collapse to the single group element aH in G/H.

In Chapter 11, we will prove that every finite Abelian group is isomorphic to a direct product of cyclic groups. In particular, an Abelian group of order 8 is isomorphic to one of $Z_8, Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$. In the next two examples, we examine Abelian factor groups of order 8 and determine the isomorphism type of each.

∎ **EXAMPLE 13** Let $G = U(32) = \{1, 3, 5, 7, 9, 11, 13, 15, 17, 19, 21, 23, 25, 27, 29, 31\}$ and $H = U_{16}(32) = \{1, 17\}$. Then G/H is an Abelian group of order $16/2 = 8$. Which of the three Abelian groups of order 8 is it—$Z_8, Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$? To answer this question, we need only determine the elements of G/H and their orders. Observe that the eight cosets

$$1H = \{1, 17\}, \quad 3H = \{3, 19\}, \quad 5H = \{5, 21\}, \quad 7H = \{7, 23\},$$
$$9H = \{9, 25\}, \quad 11H = \{11, 27\}, \quad 13H = \{13, 29\}, \quad 15H = \{15, 31\}$$

are all distinct, so that they form the factor group G/H. Clearly, $(3H)^2 = 9H \neq H$, and so $3H$ has order at least 4. Thus, G/H is not $Z_2 \oplus Z_2 \oplus Z_2$. On the other hand, $(7H)^2 = 49H = 17H = H$ and $(9H)^2 = 81H = 17H = H$ show that $7H$ and $9H$ are distinct elements of order 2. So G/H cannot be isomorphic to Z_8 either, since a cyclic group of even order has exactly one element of order 2 (Theorem 4.4). This proves that $U(32)/U_{16}(32) \approx Z_4 \oplus Z_2$, which (not so incidentally!) is isomorphic to $U(16)$. ▮

▮ **EXAMPLE 14** Let $G = Z_8 \oplus Z_4$ and let $H = \langle(2, 2)\rangle$ of G. Given that G/H is isomorphic to one of $Z_8, Z_4 \oplus Z_2, Z_2 \oplus Z_2 \oplus Z_2$, we can determine which one by elimination. First note that $H = \{(0, 0),\ (2, 2),\ (4, 0),\ (6, 2)\}$. Thus for any $(a, b) + H$ we have $((a, b) + H)^4 = (4a, 4b) + H) = (4, 0) + H$ if a is odd and $(0, 0) + H$ if a is even. Since H contains both (0,0) and (4,0) we have that $((a, b) + H)^4 = (4a, 4b) + H) = H$. Thus the maximum order of any element in G/H is 4. Since $((1, 0) + H)^2 = (2, 0) + H \neq H$ we know that $|(1, 0) + H| = 4$. Thus we have eliminated both Z_8 and $Z_2 \oplus Z_2 \oplus Z_2$. ▮

It is crucial to understand that when we factor out by a normal subgroup H, what we are essentially doing is defining every element in H to be the *identity*. Thus, in Example 12, we are making $R_{180}\mathcal{K} = \mathcal{K}$ the identity. Likewise, $R_{270}\mathcal{K} = R_{90}R_{180}\mathcal{K} = R_{90}\mathcal{K}$. Similarly, in Example 10, we are declaring any multiple of 4 to be 0 in the factor group $Z/4Z$. This is why $5+4Z = 1+4+4Z = 1+4Z$, and so on. In Example 13, we have $3H = 19H$, since $19 = 3 \cdot 17$ in $U(32)$ and going to the factor group makes 17 the identity. Algebraists often refer to the process of creating the factor group G/H as "killing" H.

Applications of Factor Groups

The next three theorems illustrate how knowledge of a factor group of G reveals information about G itself.

A natural consequence of the fact that the operation for a factor group G/H is inherited from the operation in G is that many properties of G are inherited by G/H and many properties of G can be deduced from properties of G/H. The importance of factor groups is that the structure of G/H is usually less complicated than that of G and yet G/H simulates G in many ways. Indeed, we may think of G/H as a less complicated approximation of G (similar to using the rational number 3.14 as an approximation of the irrational number π). A number of the relationships between a group and its factor groups are given in the exercises.

Theorem 9.3 G/Z Theorem

Let G be a group and let $Z(G)$ be the center of G. If $G/Z(G)$ is cyclic, then G is Abelian.

PROOF Since G is Abelian is equivalent to $Z(G) = G$, it suffices to show that the only element of $G/Z(G)$ is the identity coset $Z(G)$. To this end, let $G/Z(G) = \langle gZ(G) \rangle$ and let $a \in G$. Then there exists an integer i such that $aZ(G) = (gZ(G))^i = g^iZ(G)$. Thus, $a = g^iz$ for some z in $Z(G)$. Since both g^i and z belong to $C(g)$, so does a. Because a is an arbitrary element of G this means that every element of G commutes with g so $g \in Z(G)$. Thus, $gZ(G) = Z(G)$ is the only element of $G/Z(G)$. ∎

A few remarks about Theorem 9.3 are in order. First, our proof shows that a better result is possible: If G/H is cyclic, where H is a subgroup of $Z(G)$, then G is Abelian. Second, in practice, it is the contrapositive of the theorem that is most often used—that is, if G is non-Abelian, then $G/Z(G)$ is not cyclic. For example, it follows immediately from this statement and Lagrange's Theorem that a non-Abelian group of order pq, where p and q are primes, must have a trivial center. Third, if $G/Z(G)$ is cyclic, it must be trivial.

The next two examples demonstrate how information about a factor group G/H can be pulled back to gain information about G.

EXAMPLE 15 Let H be a normal subgroup of a group G and $\overline{K} = \{H, a_1H, a_2H, a_3H\}$ be a subgroup of the factor group G/H. Then the set $K = H \cup a_1H \cup a_2H \cup a_3H$ (called the "pull back" of $\overline{K}$ to G) is a subgroup of G of order $4|K|$. To see this let $a_0 = e$ and $a_i, a_j \in \{a_0, a_1, a_2, a_3\}$. Then for any $a_ih_i, a_jh_j \in K$ we have $(a_ih_i)(a_jh_j) \in a_iHa_jH = a_kH$ for some $k = 0, 1, 2, 3$ so $(a_ih_i)(a_jh_j)$ is in K. ∎

A similar argument applies to all factor groups by taking the K to be the union of all left cosets of H in G.

EXAMPLE 16 Suppose that G is a finite group and a factor group G/H has an element aH of order n. We show that G has an element of order n. Let $|a| = k$. Then in the group G/H we have $(aH)^k = a^kH = H$ and, by Corollary 3 of Theorem 4.1, we have that k is a multiple of n. Writing $k = mn$ we see that $(a^m)^n = a^k = e$ and n is the smallest such positive integer. ∎

Theorem 9.4 $G/Z(G) \approx \mathrm{Inn}(G)$

For any group $G, G/Z(G)$ is isomorphic to $\mathrm{Inn}(G)$.

PROOF Consider the correspondence from $G/Z(G)$ to $\mathrm{Inn}(G)$ given by $T : gZ(G) \to \phi_g$ [where, recall, $\phi_g(x) = gxg^{-1}$ for all x in G]. First, we show that T is well-defined. To do this, we assume that $gZ(G) = hZ(G)$ and verify that $\phi_g = \phi_h$. (This shows that the image of a coset of $Z(G)$ depends only on the coset itself and not on the element representing the coset.) From $gZ(G) = hZ(G)$, we have that $h^{-1}g$ belongs to $Z(G)$. Then, for all x in G, $h^{-1}gx = xh^{-1}g$. Thus, $gxg^{-1} = hxh^{-1}$ for all x in G, and, therefore, $\phi_g = \phi_h$. Reversing this argument shows that T is one-to-one, as well. Clearly, T is onto.

That T is operation-preserving follows directly from the fact that $\phi_g\phi_h = \phi_{gh}$ for all g and h in G. ∎

As an application of Theorems 9.3 and 9.4, we may easily determine $\mathrm{Inn}(D_6)$ without looking at $\mathrm{Inn}(D_6)$!

EXAMPLE 17 We know from Example 15 in Chapter 3 that $|Z(D_6)| = 2$. Thus, $|D_6/Z(D_6)| = 6$. So, by our classification of groups of order 6 (Theorem 7.3), we know that $\mathrm{Inn}(D_6)$ is isomorphic to D_3 or Z_6. Now, if $\mathrm{Inn}(D_6)$ were cyclic, then, by Theorem 9.4, $D_6/Z(D_6)$ would be also. But then, Theorem 9.3 would tell us that D_6 is Abelian. So, $\mathrm{Inn}(D_6)$ is isomorphic to D_3. ∎

The next theorem demonstrates one of the most powerful proof techniques available in the theory of finite groups—the combined use of factor groups and induction.

Theorem 9.5 Cauchy's Theorem for Abelian Groups

Let G be a finite Abelian group and let p be a prime that divides the order of G. Then G has an element of order p.

PROOF Clearly, this statement is true for the case in which G has order 2. We prove the theorem by using the Second Principle

of Mathematical Induction on $|G|$. That is, we assume that the statement is true for all Abelian groups with fewer elements than G and use this assumption to show that the statement is true for G as well. Certainly, G has elements of prime order, for if $|x| = m$ and $m = qn$, where q is prime, then $|x^n| = q$. So let x be an element of G of some prime order q, say. If $q = p$, we are finished; so assume that $q \neq p$. Since every subgroup of an Abelian group is normal, we may construct the factor group $\overline{G} = G/\langle x \rangle$. Then $\overline{G}$ is Abelian and p divides $|\overline{G}|$, since $|\overline{G}| = |G|/q$. By induction, then, $\overline{G}$ has an element of order p, and by Example 16, so does G.
∎

Internal Direct Products

As we have seen, the external direct product provides a method of putting groups together to get a larger group in such a way that we can determine many properties of the larger group from the properties of the component pieces. For example: If $G = H \oplus K$, then $|G| = |H||K|$; every element can be written uniquely in the form (h, k) when $h \in H$ and $k \in K$; $(h_1, k_1)(h_2, k_2) = (h_1h_2, k_1k_2)$; if $|h|$ and $|k|$ are finite, then $|(h, k)| = \text{lcm}(|h|, |k|)$; if H and K are Abelian, then G is Abelian; if H and K are cyclic and $|H|$ and $|K|$ are relatively prime, then $H \oplus K$ is cyclic. It would be quite useful to be able to reverse this process—that is, to be able to start with a large group G and break it down into a product of subgroups H and K in such a way that these same properties hold without the bother of the parentheses and commas. That is, if $G = HK$, then $|G| = |H||K|$; every element of G is uniquely written in the form hk where $h \in H$ and $\in K$; $h_1k_1h_2k_2 = h_1h_2k_1k_2$; if $|h|$ and $|k|$ are finite then $|hk| = \text{lcm}(|h|, |k|)$, and so on. In short, the group HK has the same properties as the group $H \oplus K$. It is occasionally possible to do this.

Definition Internal Direct Product of H and K
We say that G is the *internal direct product* of H and K and write $G = H \times K$ if H and K are normal subgroups of G and

$$G = HK \quad \text{and} \quad H \cap K = \{e\}.$$

The wording of the phrase "internal direct product" is easy to justify. We want to call G the internal direct product of H and K

if H and K are subgroups of G, and if G is naturally isomorphic to the external direct product of H and K. One forms the internal direct product by *starting* with a group G and then proceeding to find two subgroups H and K within G such that G is *isomorphic* to the external direct product of H and K. (The definition ensures that this is the case—see Theorem 9.6.) On the other hand, one forms an external direct product by *starting* with any two groups H and K, related or not, and proceeding to produce the larger group $H \oplus K$. The difference between the two products is that the internal direct product can be formed within G itself, using subgroups of G and the operation of G, whereas the external direct product can be formed with totally unrelated groups by creating a new set and a new operation. (See Figures 9.1 and 9.2.)

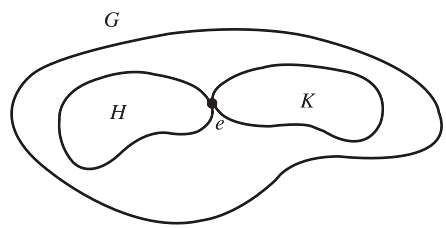


Figure 9.1 For the internal direct product, H and K must be subgroups of the same group.

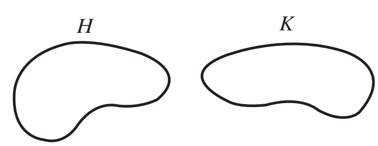


Figure 9.2 For the external direct product, H and K can be any groups.

Perhaps the following analogy with integers will be useful in clarifying the distinction between the two products of groups discussed in the preceding paragraph. Just as we may take any (finite) collection of integers and form their product, we may also take any collection of groups and form their external direct product. Conversely, just as we may start with a particular integer and express it as a product of certain of its divisors, we may be able to start with a particular group and factor it as an internal direct product of certain of its subgroups.

The next example recasts Theorem 8.3.

EXAMPLE 18 If s and t are relatively prime positive integers then $U(st) = U_s(st) \times U_t(st)$. ∎

EXAMPLE 19 D_6, the dihedral group of order 12, let F denote some reflection and let R_k denote a rotation of k degrees. Then,

$$D_6 = \{R_0, R_{120}, R_{240}, F, R_{120}F, R_{240}F\} \times \{R_0, R_{180}\}.$$

∎

Observe that Example 19 shows that D_6 is isomorphic to $D_3 \oplus Z_2$. (See Exercise 80.)

Students should be cautioned about the necessity of having all conditions of the definition of internal direct product satisfied to ensure that $HK \approx H \oplus K$. For example, if we take

$$G = S_3, \qquad H = \langle(123)\rangle, \qquad \text{and} \qquad K = \langle(12)\rangle,$$

then $G = HK$, and $H \cap K = \{(1)\}$. But G is *not* isomorphic to $H \oplus K$, since, by Theorem 8.2, $H \oplus K$, is cyclic, whereas S_3 is not. Note that K is not normal.

A group G can also be the internal direct product of a collection of subgroups.

Definition Internal Direct Product $H_1 \times H_2 \times \cdots \times H_n$

Let $H_1, H_2, \ldots, H_n$ be a finite collection of normal subgroups of G. We say that G is the *internal direct product* of $H_1, H_2, \ldots, H_n$ and write $G = H_1 \times H_2 \times \cdots \times H_n$, if

1. $G = H_1H_2 \cdots H_n = \{h_1h_2 \cdots h_n | h_i \in H_i\}$,

2. $(H_1H_2 \cdots H_i) \cap H_{i+1} = \{e\}$ for $i = 1, 2, \ldots, n-1$.

This definition is somewhat more complicated than the one given for two subgroups. The student may wonder about the motivation for it— that is, why should we want the subgroups to be normal and why is it desirable for each subgroup to be disjoint from the product of all previous ones? The reason is quite simple. We want the internal direct product to be isomorphic to the external direct product. As the next theorem shows, the conditions in the definition of internal direct product were chosen to ensure that the two products are isomorphic.

Theorem 9.6 $H_1 \times H_2 \times \cdots \times H_n \approx H_1 \oplus H_2 \oplus \cdots \oplus H_n$

If a group G is the internal direct product of a finite number of subgroups $H_1, H_2, \ldots, H_n$, then G is isomorphic to the external direct product of $H_1, H_2, \ldots, H_n$.

PROOF We first show that the normality of the H's together with the second condition of the definition guarantees that h's from different H_i's commute. For if $h_i \in H_i$ and $h_j \in H_j$ with $i \neq j$, then

$$(h_i h_j h_i^{-1}) h_j^{-1} \in H_j h_j^{-1} = H_j$$

and

$$h_i (h_j h_i^{-1} h_j^{-1}) \in h_i H_i = H_i.$$

Thus, $h_i h_j h_i^{-1} h_j^{-1} \in H_i \cap H_j = \{e\}$ (see Exercise 5), and, therefore, $h_i h_j = h_j h_i$. We next claim that each member of G can be expressed uniquely in the form $h_1 h_2 \cdots h_n$, where $h_i \in H_i$. That there is at least one such representation is the content of condition 1 of the definition. To prove uniqueness, suppose that $g = h_1 h_2 \cdots h_n$ and $g = h_1' h_2' \cdots h_n'$, where h_i and h_1' belong to H_i for $i = 1, \ldots, n$, Then, using the fact that the h's from different H_i's commute, we can solve the equation

$$h_1 h_2 \cdots h_n = h_1' h_2' \cdots h_n' \tag{9.1}$$

for $h_n' h_n^{-1}$ to obtain

$$h_n' h_n^{-1} = (h_1')^{-1} h_1 (h_2')^{-1} h_2 \cdots (h_{n-1}')^{-1} h_{n-1}.$$

But then

$$h_n' h_n^{-1} \in H_1 H_2 \cdots H_{n-1} \cap H_n = \{e\}.$$

so that $h_n' h_n^{-1} = e$ and, therefore, $h_n' = h_n$. At this point, we can cancel h_n and h_n' from opposite sides of the equal sign in Equation (9.1) and repeat the preceding argument to obtain $h_{n-1} = h_{n-1}'$. Continuing in this fashion, we eventually have $h_i = h_i'$ for $i = 1, \ldots, n$. With our claim established, we may now define a function ϕ from G to $H_1 \oplus H_2 \oplus \cdots \oplus H_n$ by $\phi(h_1 h_2 \cdots h_n) = (h_1, h_2, \cdots, h_n)$. We leave to the reader the easy verification that ϕ is an isomorphism. ∎

When we have a group $G = H \times K$ the essence of Theorem 9.6 is that in $H \oplus K$ the product $(h_1, k_1)(h_2, k_2) = (h_1 h_2, k_1 k_2)$

is the same as $h_1h_2k_1k_2$ in $H \times K$. So, the operation in $H \oplus K$ can be done inside HK by ignoring the parentheses and commas to separate the members of H and K.

The U-groups provide a convenient way to illustrate the preceding ideas and to clarify the distinction between internal and external direct products. It follows directly from Theorem 8.3, its corollary, and Theorem 9.6 that if $m = n_1n_2\cdots n_k$, where $\gcd(n_i, n_j) = 1$ for $i \neq j$, then

$$\begin{aligned} U(m) &= U_{m/n_1}(m) \times U_{m/n_2}(m) \times \cdots \times U_{m/n_k}(m) \\ &\approx U(n_1) \oplus U(n_2) \oplus \cdots \oplus U(n_k). \end{aligned}$$

Let us return to the examples given following Theorem 8.3.

$$\begin{aligned} U(105) &= U(15 \cdot 7) = U_{15}(105) \times U_7(105) \\ &= \{1, 16, 31, 46, 61, 76\} \times \{1, 8, 22, 29, 43, 64, 71, 92\} \\ &\approx U(7) \oplus U(15), \\ U(105) &= U(5 \cdot 21) = U_5(105) \times U_{21}(105) \\ &= \{1, 11, 16, 26, 31, 41, 46, 61, 71, 76, 86, 101\} \\ &\quad \times \{1, 22, 43, 64\} \approx U(21) \oplus U(5), \\ U(105) &= U(3 \cdot 5 \cdot 7) = U_{35}(105) \times U_{21}(105) \times U_{15}(105) \\ &= \{1, 71\} \times \{1, 22, 43, 64\} \times \{1, 16, 31, 46, 61, 76\} \\ &\approx U(3) \oplus U(5) \oplus U(7). \end{aligned}$$

The above ideas provide an easy way to find various subgroups of U-groups.

▮ **EXAMPLE 20** We find subgroups of $U(1000)$ orders 4, 100, and 200. From Theorem 8.3, we have $U(1000) \approx U(8) \oplus U(125) \approx U_{125}(1000) \oplus U_8(1000) \approx Z_2 \oplus Z_2 \oplus Z_{100}$. So subgroups of $U(1000)$ of orders 4 and 100 are $U_{125}(1000) = \{1, 251, 501, 751\}$ and $U_8(1000)$, respectively. Since 251 has order 2 the subgroup $\langle 251 \rangle U_8(1000)$ has order 200. ▮

As applications of Theorem 9.6 we next prove two classification theorems. But first we pause to review the classification theorems we have proved thus far. Note that Theorem 9.8 generalizes the last result on this list below to all primes p.

• Classification of subgroups of finite cyclic groups: There is exactly one subgroup for each divisor of the order of the group and no others.

• Classification of groups of prime order: Every group of prime order p is isomorphic to Z_p.
• Classification of groups of $2p$ where p is an odd prime: Every group of order $2p$ where p is prime is isomorphic to Z_{2p} or D_p.
• Classification of groups of 4: Every group of order 4 is isomorphic to Z_4 or $Z_2 \oplus Z_2$.

Theorem 9.7 Classification of finite Abelian groups of squarefree order

Every Abelian group of order $p_1p_2 \cdots p_k$ where the p_i are distinct primes is cyclic.

PROOF Let G be a group of order $p_1p_2 \cdots p_k$ where the p_i are distinct primes. By Cauchy's Theorem G has subgroups H_i of order p_i for $i = 1, 2, \ldots, p_k$. It follows from Theorem 7.2 and Lagrange's theorem that $G = H_1 \times H_2 \times \cdots \times H_k$. Then, by Theorem 9.6 and Corollary 1 of Theorem 8.2, we have that G is cyclic. ∎

Theorem 9.8 Classification of Groups of Order p^2

Every group of order p^2, where p is a prime, is isomorphic to Z_{p^2} or $Z_p \oplus Z_p$.

PROOF Let G be a group of order p^2, where p is a prime. If G has an element of order p^2, then G is isomorphic to Z_{p^2}. So, by Corollary 2 of Lagrange's Theorem, we may assume that every nonidentity element of G has order p. First we show that for any element a, the subgroup $\langle a \rangle$ is normal in G. If this is not the case, then there is an element b in G such that $b\langle a \rangle b^{-1} = \langle bab^{-1} \rangle \not\subseteq \langle a \rangle$. Then $\langle a \rangle$ and $\langle bab^{-1} \rangle$ are distinct subgroups of order p. Since $\langle a \rangle \cap \langle bab^{-1} \rangle$ is a subgroup of both $\langle a \rangle$ and $\langle bab^{-1} \rangle$, we have that $\langle a \rangle \cap \langle bab^{-1} \rangle = \{e\}$. From this it follows that the distinct left cosets of $\langle bab^{-1} \rangle$ are $\langle bab^{-1} \rangle, a\langle bab^{-1} \rangle, a^2\langle bab^{-1} \rangle, \ldots, a^{p-1}\langle bab^{-1} \rangle$. Since b^{-1} must lie in one of these cosets, we may write b^{-1} in the form $b^{-1} = a^i(bab^{-1})^j = a^iba^jb^{-1}$ for some i and j. Canceling the b^{-1} terms, we obtain $e = a^iba^j$ and therefore $b = a^{-1-j} \in \langle a \rangle$. This contradiction verifies our assertion that every subgroup of the form $\langle a \rangle$ is normal in G. To complete the proof, let x be any nonidentity element in G and y be any element of G not in $\langle x \rangle$.

Then, by comparing orders and using Theorem 9.6, we see that $G = \langle x \rangle \times \langle y \rangle \approx Z_p \oplus Z_p$. ■

As an immediate corollary of Theorem 9.8, we have the following important fact.

Corollary

If G is a group of order p^2, where p is a prime, then G is Abelian.

We mention in passing that if $G = H_1 \oplus H_2 \oplus \cdots \oplus H_n$, then G can be expressed as the internal direct product of subgroups isomorphic to $H_1, H_2, \ldots, H_n$. For example, if $G = H_1 \oplus H_2$, then $G = \overline{H}_1 \times \overline{H}_2$, where $\overline{H}_1 = H_1 \oplus \{e\}$ and $\overline{H}_2 = \{e\} \oplus H_2$.

The topic of direct products is one in which notation and terminology vary widely. Many authors use $H \times K$ to denote both the internal direct product and the external direct product of H and K, making no notational distinction between the two products. A few authors define only the external direct product. Many people reserve the notation $H \oplus K$ for the situation where H and K are Abelian groups under addition and call it the *direct sum* of H and K. In fact, we will adopt this terminology in the section on rings (Part 3), since rings are always Abelian groups under addition.

Exercises

The heart of mathematics is its problems.

Paul Halmos

1. Let $H = \{(1), (12)\}$. Is H normal in S_3?

2. Prove that A_n is normal in S_n.

3. In D_4, let $K = \{R_0, R_{90}, R_{180}, R_{270}\}$, Write HR_{90} in the form xH, where $x \in K$. Write DR_{270} in the form xD, where $x \in K$. Write $R_{90}V$ in the form Vx, where $x \in K$.

4. Write $(12)(13)(14)$ in the form $\alpha(12)$, where $\alpha \in A_4$. Write (1234) $(12)(23)$, in the form $\alpha(1234)$, where $\alpha \in A_4$.

5. Show that if G is the internal direct product of $H_1, H_2 \ldots, H_n$ and $i \neq j$ with $1 \leq i \leq n, 1 \leq j \leq n$, then $H_i \cap H_j = \{e\}$. (This exercise is referred to in this chapter.)

6. Let $H = \left\{ \begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in \mathbf{R}, ad \neq 0 \right\}$. Is H a normal subgroup of $GL(2, \mathbf{R})$?

7. Let $G = GL(2, \mathbf{R})$ and let K be a subgroup of $\mathbf{R}^*$. Prove that $H = \{A \in G \mid \det A \in K\}$ is a normal subgroup of G.

8. Viewing $\langle 3 \rangle$ and $\langle 12 \rangle$ as subgroups of Z, prove that $\langle 3 \rangle / \langle 12 \rangle$is isomorphic to Z_4. Similarly, prove that $\langle 8 \rangle / \langle 48 \rangle$ is isomorphic to Z_6. Generalize to arbitrary integers k and n.

9. Prove that if H has index 2 in G, then H is normal in G. (This exercise is referred to in Chapters 23 and 24 and this chapter.)

10. Let $H = \{(1), (12)(34)\}$ in A_4.
a. Show that H is not normal in A_4.
b. Referring to the multiplication table for A_4 in Table 5.1, show that, although $\alpha_6 H = \alpha_7 H$ and $\alpha_9 H = \alpha_{11} H$, it is not true that $\alpha_6 \alpha_9 H = \alpha_7 \alpha_{11} H$. Explain why this proves that the left cosets of H do not form a group under coset multiplication.

11. Prove that a factor group of a cyclic group is cyclic.

12. Prove that a factor group of an Abelian group is Abelian.

13. Let H be a normal subgroup of a finite group G and let a be an element of G. Complete the following statement: The order of the element aH in the factor group G/H is the smallest positive integer n such that a^n is ______________.

14. What is the order of the element $14 + \langle 8 \rangle$ in the factor group $Z_{24}/\langle 8 \rangle$?

15. In $U(100)$ let $H = \langle 21 \rangle = \{1, 21, 41, 61, 81\}$. In $U(100)/H$, find $|9H|$ and $|13H|$.

16. Recall that $Z(D_6) = \{R_0, R_{180}\}$. What is the order of the element $R_{60}Z(D_6)$ in the factor group $D_6/Z(D_6)$?

17. In the group $Q/\langle 3.5 \rangle$ find the unique element $a + \langle 3.5 \rangle$ such that $a + \langle 3.5 \rangle = 8 + \langle 3.5 \rangle$ where $0 < a < 3.5$.

18. Let k be a divisor of n. To what familiar group is $Z_n/\langle k \rangle$ isomorphic?

19. If H is a normal subgroup of G and $|H| = 2$, prove that H is contained in the center of G. Determine all normal subgroups of D_n of order 2.

20. Prove that $U(40)/U_8(40)$ is cyclic but $U(40)/U_5(40)$ is not cyclic.

21. Prove that an Abelian group of order 33 is cyclic. Does your proof hold when 33 is replaced by pq where p and q are distinct primes? Generalize to the case of an Abelian group of order $p_1p_2\cdots p_k$ where the p_i are distinct primes.

22. Prove that the group $(Z \oplus Z)/(\langle(2,2)\rangle)$ is generated by the two elements $(1,0)+\langle(2,2)\rangle$ and $(1,1)+\langle(2,2)\rangle$. That is, every element of $(Z \oplus Z)/(\langle(2,2)\rangle)$ can be written in the form $a(1,0)+\langle(2,2)\rangle+b(1,1)+\langle(2,2)\rangle$ where $b=0$ or 1. To which external direct product of cyclic groups is $(Z \oplus Z)/(\langle(2,2)\rangle)$ isomorphic? To which external direct product of cyclic groups is $(Z \oplus Z)/(\langle(n,n)\rangle)$ isomorphic?

23. How many elements are in the group $(Z \oplus Z)/(\langle(2,0)\rangle \oplus \langle(0,2)\rangle)$? Is this group cyclic?

24. The group $(Z_4 \oplus Z_{12})/\langle(2,2)\rangle$ is isomorphic to one of Z_8, $Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$. Determine which one by elimination.

25. Let $G=U(32)$ and $H=\{1,15\}$. The group G/H is isomorphic to one of Z_8, $Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$. Determine which one by elimination.

26. Let $H=\{1,17,41,49,73,89,97,113\}$ under multiplication modulo 120. Write H as a external direct product of groups of the form Z_{2^k}. Write H as an internal direct product of nontrivial subgroups.

27. Let $G=U(16)$, $H=\{1,15\}$, and $K=\{1,9\}$. Are H and K isomorphic? Are G/H and G/K isomorphic?

28. Let $G=Z_4 \oplus Z_4$, $H=\{(0,0),(2,0),(0,2),(2,2)\}$, and $K=\langle(1,2)\rangle$. Is G/H isomorphic to Z_4 or $Z_2 \oplus Z_2$? Is G/K isomorphic to Z_4 or $Z_2 \oplus Z_2$?

29. Explain why a non-Abelian group of order 8 cannot be the internal direct product of proper subgroups.

30. Express $U(165)$ as an internal direct product of proper subgroups in four different ways.

31. Let $\mathbf{R}^*$ denote the group of all nonzero real numbers under multiplication. Let $\mathbf{R}^+$ denote the group of positive real numbers under multiplication. Prove that $\mathbf{R}^*$ is the internal direct product of $\mathbf{R}^+$ and the subgroup $\{1,-1\}$.

32. If N is a normal subgroup of G and $|G/N| = m$, show that $x^m \in N$ for all x in G.

33. Let H and K be subgroups of a group G. If $G = HK$ and $g = hk$, where $h \in H$ and $k \in K$, is there any relationship among $|g|$, $|h|$, and $|k|$? What if $G = H \times K$?

34. In Z, let $H = \langle 5 \rangle$ and $K = \langle 7 \rangle$. Prove that $Z = HK$. Does $Z = H \times K$?

35. Let $G = \{3^a 6^b 10^c \mid a, b, c \in Z\}$ under multiplication and $H = \{3^a 6^b 12^c \mid a, b, c \in Z\}$ under multiplication. Prove that $G = \langle 3 \rangle \times \langle 6 \rangle \times \langle 10 \rangle$, whereas $H \neq \langle 3 \rangle \times \langle 6 \rangle \times \langle 12 \rangle$.

36. Determine all subgroups of $\mathbf{R}^*$ (nonzero reals under multiplication) of index 2.

37. Let G be a finite group and let H be a normal subgroup of G. Prove that for any $g \in G$, $|gH|$ divides $|g|$.

38. Prove that for every positive integer n, Q/Z has an element of order n.

39. Let H be a subgroup of a group G with the property that for all a and b in G, $aHbH = abH$. Prove that H is a normal subgroup of G.

40. Let in S_3 let $H = \{(1), (12\})$. Show that $(13)H(23)H \neq (13)(23)H$. (This proves that when H is not a normal subgroup of a group G, the product of two left cosets of H in G need not be a left coset of H in G.)

41. Show that Q, the group of rational numbers under addition, has no proper subgroup of finite index.

42. An element is called a *square* if it can be expressed in the form b^2 for some b. Suppose that G is an Abelian group and H is a subgroup of G. If every element of H is a square and every element of G/H is a square, prove that every element of G is a square. Does your proof remain valid when "square" is replaced by "nth power," where n is any integer?

43. Show, by example, that in a factor group G/H it can happen that $aH = bH$ but $|a| \neq |b|$.

44. Verify that the mapping defined at the end of the proof of Theorem 9.6 is an isomorphism.

45. Let H be any subgroup of rotations in D_n. Prove that H is a normal subgroup of D_n.

46. Show that D_{13} is isomorphic to $\mathrm{Inn}(D_{13})$.

47. Let H and K be subgroups of a group G. If $|H| = 63$ and $|K| = 45$, prove that $H \cap K$ is Abelian. Generalize.

48. If G is a group and $|G{:}Z(G)| = 4$, prove that $G/Z(G) \approx Z_2 \oplus Z_2$.

49. Suppose that G is a non-Abelian group of order p^3, where p is a prime, and $Z(G) \neq \{e\}$. Prove that $|Z(G)| = p$.

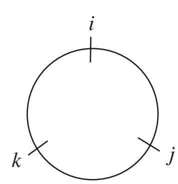
50. If $|G| = pq$, where p and q are primes that are not necessarily distinct, prove that $|Z(G)| = 1$ or pq.

51. Let H be a normal subgroup of G and K a subgroup of G that contains H. Prove that K is normal in G if and only if K/H is normal in G/H.

52. Let G be an Abelian group and let H be the subgroup consisting of all elements of G that have finite order. Prove that every nonidentity element in G/H has infinite order.

53. Determine all subgroups of $\mathbf{R}^*$ that have finite index.

54. Let $G = \{\pm 1, \pm i, \pm j, \pm k\}$, where $i^2 = j^2 = k^2 = -1, -i = (-1)i, 1^2 = (-1)^2 = 1, ij = -ji = k, jk = -kj = i$, and $ki = -ik = j$.
a. Show that $H = \{1, -1\} \triangleleft G$.
b. Construct the Cayley table for G/H. Is G/H isomorphic to Z_4 or $Z_2 \oplus Z_2$?
(The rules involving i, j, and k can be remembered by using the circle below.



Going clockwise, the product of two consecutive elements is the third one. The same is true for going counterclockwise, except that we obtain the negative of the third element. This group is called the *quaternions.* It was invented by William Hamilton in 1843. The quaternions are used to describe rotations in three-dimensional space, and they are used in physics. The quaternions can be used to extend the complex numbers in a natural way).

55. In D_4, let $K = \{R_0, D\}$ and let $L = \{R_0, D, D', R_{180}\}$. Show that $K \triangleleft L \triangleleft D_4$, but that K is not normal in D_4. (Normality is not transitive.)

56. Show that the intersection of two normal subgroups of G is a normal subgroup of G. Generalize.

57. Suppose that H is a normal subgroup of a finite group G and G/H is isomorphic to D_6. What can you say about the orders of the elements in G?

58. If N and M are normal subgroups of G, prove that NM is also a normal subgroup of G.

59. Let N be a normal subgroup of a group G. If N is cyclic, prove that every subgroup of N is also normal in G. (This exercise is referred to in Chapter 23.)

60. Without looking at inner automorphisms of D_n, determine the number of such automorphisms.

61. Let H be a normal subgroup of a finite group G and let $x \in G$. If $\gcd(|x|, |G/H|) = 1$, show that $x \in H$. (This exercise is referred to in Chapter 24.)

62. Let G be a group of order pm where p is prime and $p > m$. If H is a subgroup of G of order p, prove that H is normal.

63. If a group of order 24 has more than one subgroup of order 3 prove that none of them can be a normal subgroup.

64. Let G be a group and let G' be the subgroup of G generated by the set $S = \{x^{-1}y^{-1}xy \mid x, y \in G\}$.
a. Prove that G' is normal in G.
b. Prove that G/G' is Abelian.
c. If G/N is Abelian, prove that $G' \leq N$.
d. Prove that if H is a subgroup of G and $G' \leq H$, then H is normal in G.

65. Prove that the group $\mathbf{C}^*/\mathbf{R}^*$ has infinite order.

66. Suppose that a group G has a subgroup of order n. Prove that the intersection of all subgroups of G of order n is a normal subgroup of G.

67. If G is non-Abelian, show that $\mathrm{Aut}(G)$ is not cyclic.

68. Let $|G| = p^n m$, where p is prime and $\gcd(p, m) = 1$. Suppose that H is a normal subgroup of G of order p^n. If K is a subgroup of G of order p^k, show that $K \subseteq H$.

69. Let g be an element in a group G and H be a normal subgroup of G of order 2. If $|g| = 16$ prove that in the group G/H the element gH has order 8 or 16. What can you say about the order of gH in G/H if $|g| = 10$? What about $|g| = 2n$?

70. Prove that A_4 is the only subgroup of S_4 of order 12.

71. If $|G| = 30$ and $|Z(G)| = 5$, to what familiar group is $G/Z(G)$ isomorphic? To what familiar group is $G/Z(G)$ isomorphic if $|\mathrm{Z}(G)| = 3$? Generalize to the case that $|G| = 2pq$ where p and q are distinct odd primes.

72. If G is a group and $|G/Z(G)| = 2p$ where p is an odd prime, to what familiar group is $G/Z(G)$ isomorphic?

73. Prove that A_5 cannot have a normal subgroup of order 2.

74. Let G be a group and H be a subgroup of G of index 2. Show that H contains every element of G of odd order.

75. In $U(72)$ find subgroups of orders 4, 6, and 12.

76. In $U(70)$ let H be the subgroup $\{1, 11, 31, 41, 51, 61\}$. Find $|9H|$ and $|17H|$.

77. Prove that A_5 has no normal subgroup of order 12.

78. Explain why $U(60)$ has no subgroup H such that $U(60)/H$ is isomorphic to Z_8.

79. For the group $U(200)$ and the normal subgroup $H = \{1, 51, 101, 151\}$ explain why $153H = 3H$.

80. In the group $Q/\langle 3.5\rangle$, find $|2 + \langle 3.5\rangle|$.

81. Let g be an element in a group G and H be a normal subgroup of G. If $\gcd(|g|, |H|) = 1$, prove that $|gH| = |g|$.

82. Use the method described in Example 15 to find a subgroup of $U(80)$ of order 16.

83. If $G = \langle a_1, a_2, \ldots, a_n\rangle$ and H is a normal subgroup of G prove that $G/H = \langle a_1H, a_2H, \ldots, a_nH\rangle$.

84. For any integer $n \geq 3$ prove that D_{2n} can be expressed as an internal direct product of D_n and a subgroup of order 2 if and only if n is odd.

85. Suppose that G is an Abelian group and $H_1, H_2, \ldots, H_k$ are subgroups of G such that every element in G is uniquely expressible in the form $h_1h_2\cdots h_k$ where $h_i \in H_i$. Prove that $G = H_1 \times H_2 \times \cdots \times H_k$.

86. Let G be a group and let H be a subgroup of G. Define $N(H) = \{x \in G \mid xHx^{-1} = H\}$. Prove that $N(H)$ (called the *normalizer* of H in G) is a subgroup of G.

87. If a factor group $G/Z(G)$ has order p^2 for some prime p, to what familiar group is $G/Z(G)$ isomorphic?

88. Explain why there is no group G with the property that $G/Z(G)$ is isomorphic to the integers under addition.

89. Prove that if a group G of order $9m$ where m is relatively prime to 3 has a normal subgroup of order 9 it is the unique subgroup of order 9. Generalize to the case where the group has order pm where p is prime and p is relatively prime to m.

90. Let G be a group. Prove that $\mathrm{Inn}(G)$ is normal in $\mathrm{Aut}(G)$.

91. Let G be an Abelian group of order 2^n for some positive integer n. Prove that G is cyclic if and only if G has exactly one element of order 2. What is the appropriate generalization when 2 is replaced by any prime p?

92. Let G be a finite Abelian group of order mn where m and n are relatively prime. Let $G^m = \{x \in G \mid x^m = e\}$ and $G^n = \{x \in G \mid x^n = e\}$. Prove that $G = G^m \times G^n$.

93. Let $H = \{x^2 \mid x \in Q^*\}$. Prove that $|Q^* : H| = \infty$. What can you say about the orders of the elements in the factor group Q^*/H?

Évariste Galois

The Granger Collection, NYC

Galois at seventeen was making discoveries of epochal significance in the theory of equations, discoveries whose consequences are not yet exhausted after more than a century.

E. T. BELL, *Men of Mathematics*

ÉVARISTE GALOIS (pronounced gal-WAH) had at once an inspiring and tragically short life. He was born on October 25, 1811, near Paris. Although he had mastered the works of Legendre and Lagrange at age 15, Galois twice failed his entrance examination to the École Polytechnique. He did not know some basic mathematics, and he did mathematics almost entirely in his head, to the annoyance of the examiner.

At 18, Galois wrote his important research on the theory of equations and submitted it to the French Academy of Sciences for publication. The paper was given to Cauchy for refereeing. Cauchy, impressed by the paper, agreed to present it to the academy, but he never did. At the age of 19, Galois entered a paper of the highest quality in the competition for the Grand Prize in Mathematics, given by the French Academy of Sciences.

The paper was given to Fourier who died shortly thereafter. Galois's paper was never seen again. Galois spent most of the last year and a half of his life in prison for revolutionary political offenses. While in prison, he attempted suicide and prophesied that he would die in a duel. On May 30, 1832, Galois was shot in a duel; he died the next day at the age of 20.

Among the many concepts introduced by Galois are normal subgroups, isomorphisms, simple groups, finite fields, and Galois theory. His work provided a method for disposing of several famous constructability problems, such as trisecting an arbitrary angle and doubling a cube. In his book *Love and Math* Edward Frenkel wrote "His [Galois's] brilliant insight has forever changed the way people think about numbers and equations." Galois's entire works fill only 60 pages.

10 Group Homomorphisms

When it comes to laws, there is absolutely no doubt that symmetry and group theory are extremely *useful* concepts. Without the introduction of symmetry and the language of groups into particle physics the description of the elementary particles and their interactions would have been an intricate nightmare. Groups truly flesh out order and identify patterns like no other mathematical machinery.

Mario Livio, *The Equation That Couldn't be Solved*

In a certain sense the subject of group theory is built up out of three basic concepts: that of a homomorphism, that of a normal subgroup, and that of the factor group of a group by a normal subgroup.

I. N. Herstein, *Abstract Algebra, 3rd ed.*

Definition and Examples

In this chapter, we consider one of the most fundamental ideas of algebra—homomorphisms. The term *homomorphism* comes from the Greek words *homo*, "like," and *morphe,* "form." We will see that a homomorphism is a natural generalization of an isomorphism and that there is an intimate connection between factor groups of a group and homomorphisms of a group. The concept of group homomorphisms was introduced by Camille Jordan in 1870, in his influential book *Traité des substitutions*.

Definition **Group Homomorphism**

A *homomorphism* ϕ from a group G to a group $\overline{G}$ is a mapping from G into $\overline{G}$ that preserves the group operation; that is, $\phi(ab) = \phi(a)\phi(b)$ for all a, b in G.

Before giving examples and stating numerous properties of homomorphisms, it is convenient to introduce an important subgroup that is intimately related to the image of a homomorphism. (See property 4 of Theorem 10.1.)

Definition Kernel of a Homomorphism
The *kernel* of a homomorphism ϕ from a group G to a group with identity e is the set $\{x \in G \mid \phi(x) = e\}$. The kernel of ϕ is denoted by Ker ϕ.

EXAMPLE 1 Any isomorphism is a homomorphism that is also onto and one-to-one. The kernel of an isomorphism is the trivial subgroup. ∎

EXAMPLE 2 Let $\mathbf{R}^*$ be the group of nonzero real numbers under multiplication. Then the determinant mapping $A \to \det A$ is a homomorphism from $GL(2, \mathbf{R})$ to $\mathbf{R}^*$. The kernel of the determinant mapping is $SL(2, \mathbf{R})$. ∎

EXAMPLE 3 The mapping ϕ from $\mathbf{R}^*$ to $\mathbf{R}^*$, defined by $\phi(x) = |x|$, is a homomorphism with Ker $\phi = \{1, -1\}$. ∎

EXAMPLE 4 Let $\mathbf{R}[x]$ denote the group of all polynomials with real coefficients under addition. For any f in $\mathbf{R}[x]$, let f' denote the derivative of f. Then the mapping $f \to f'$ is a homomorphism from $\mathbf{R}[x]$ to itself. The kernel of the derivative mapping is the set of all constant polynomials. ∎

EXAMPLE 5 The mapping ϕ from Z to Z_n, defined by $\phi(m) = m \bmod n$, is a homomorphism (see Exercise 11 in Chapter 0). The kernel of this mapping is $\langle n \rangle$. ∎

EXAMPLE 6 Let i denote the complex number $\sqrt{-1}$. The set $Z[i] = \{a+bi \mid a, b \in Z\}$ under addition is subgroup of the complex numbers called the *Gaussian integers*. For a positive integer n the set $Z_n[i] = \{a + bi \mid a, b \in Z_n\}$ under addition modulo n is a group called the *Gaussian integers modulo n*. The mapping from $Z[i]$ onto $Z_n[i]$ defined by $\phi(a + bi) = a \bmod n + (b \bmod n)i$ is a homomorphism. The kernel of this mapping is $\{sn+tni \mid s, t \in Z\}$. ∎

Our algorithm for expressing $U(n)$ as an internal direct product of proper subgroups provides us with a simple way to define homomorphisms from $U(n)$ to $U(n)$ with a designated kernel (see Exercise 9).

EXAMPLE 7 Expressing $U(40) = U_5(40) \times U_8(40)$ in the form $U_5(40)U_8(40) = \{ab \mid a \in U_5(40),\ b \in U(_8(40)\}$ the mapping from $U(40)$ to $U(40)$ defined by $\phi(ab) = a$ is a homomorphism with kernel $U_8(40)$. The mapping from $U(40)$ to $U(40)$ defined by $\phi(ab) = b$ is a homomorphism with kernel $U_5(40)$. ∎

EXAMPLE 8 The mapping $\phi(x) = x^2$ from **R**, the real numbers under addition, to itself is not a homomorphism, since $\phi(a + b) = (a + b)^2 = a^2 + 2ab + b^2$, whereas $\phi(a) + \phi(b) = a^2 + b^2$. ∎

When defining a homomorphism from a group in which there are several ways to represent the elements, caution must be exercised to ensure that the correspondence is a function. (The term *well-defined* is often used in this context.) For example, since $3(x + y) = 3x + 3y$ in Z_6, one might believe that the correspondence $x + \langle 3\rangle \to 3x$ from $Z/\langle 3\rangle$ to Z_6 is a homomorphism. But it is not a function, since $0 + \langle 3\rangle = 3 + \langle 3\rangle$ in $Z/\langle 3\rangle$ but $3 \cdot 0 \neq 3 \cdot 3$ in Z_6.

For students who have had linear algebra, we remark that every linear transformation is a group homomorphism and the null-space is the same as the kernel. An invertible linear transformation is a group isomorphism.

Properties of Homomorphisms

Theorem 10.1 Properties of Elements Under Homomorphisms

Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let g be an element of G. Then

1. ***ϕ carries the identity of G to the identity of $\overline{G}$.***
2. ***$\phi(g^n) = \phi(g))^n$ for all n in Z.***
3. ***If $|g|$ is finite, then $|\phi(g)|$ divides $|g|$ and if $|G|$ is finite then $|\phi(g)|$ divides $|g|$ and $|\phi(G)|$.***
4. ***Ker ϕ is a subgroup of G.***
5. ***$\phi(a) = \phi(b)$ if and only if aKer $\phi = b$Ker ϕ.***
6. ***If $\phi(g) = g'$, then $\phi^{-1}(g') = \{x \in G | \phi(x) = g'\} = g$Ker ϕ.***

PROOF The proofs of properties 1 and 2 are identical to the proofs of properties 1 and 2 of isomorphisms in Theorem 6.2. To prove property 3, notice that properties 1 and 2 together with $g^n = e$ imply that $e = \phi(e) = \phi(g^n) = (\phi(g))^n$. So, by Corollary 2 to Theorem 4.1, we have $|\phi(g)|$ divides n. That $|\phi(g)|$ divides $|\phi(G)|$ when $G|$ is finite follows from Lagrange's theorem.

By property 1 we know that Ker ϕ is not empty. So, to prove property 4, we assume that $a, b \in$ Ker ϕ and show that $ab^{-1} \in$ Ker ϕ. Since $\phi(a) = e$ and $\phi(b) = e$, we have $\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \phi(a)(\phi(b))^{-1} = ee^{-1} = e$. So, $ab^{-1} \in$ Ker ϕ.

To prove property 5, first assume that $\phi(a) = \phi(b)$. Then $e = (\phi(b))^{-1}\phi(a) = \phi(b^{-1})\phi(a) = \phi(b^{-1}a)$, so that $b^{-1}a \in$ Ker ϕ. It now follows from property 6 of the lemma in Chapter 7 that bKer $\phi = a$Ker ϕ. Reversing this argument completes the proof.

To prove property 6, we must show that $\phi^{-1}(g') \subseteq g$Ker ϕ and that gKer $\phi \subseteq \phi^{-1}(g')$. For the first inclusion, let $x \in \phi^{-1}(g')$, so that $\phi(x) = g'$. Then $\phi(g) = \phi(x)$ and by property 5 we have gKer $\phi = x$Ker ϕ and therefore $x \in g$Ker ϕ. This completes the proof that $\phi^{-1}(g') \subseteq g$Ker ϕ. To prove that gKer $\phi \subseteq \phi^{-1}(g')$, suppose that $k \in$ Ker ϕ. Then $\phi(gk) = \phi(g)\phi(k) = g'e = g'$. Thus, by definition, $gk \in \phi^{-1}(g')$. ▮

Since homomorphisms preserve the group operation, it should not be a surprise that they preserve many group properties.

Theorem 10.2 Properties of Subgroups Under Homomorphisms

Let ϕ be a homomorphism from a group G to a group $\overline{G}$ and let H be a subgroup of G. Then

1. ***$\phi(H) = \{\phi(h) \mid h \in H\}$ is a subgroup of $\overline{G}$.***
2. ***If H is cyclic, then $\phi(H)$ is cyclic.***
3. ***If H is Abelian, then $\phi(H)$ is Abelian.***
4. ***If H is normal in G, then $\phi(H)$ is normal in $\phi(G)$.***
5. ***If $|\text{Ker } \phi| = n$, then ϕ is an n-to-1 mapping from G onto $\phi(G)$.***
6. ***If H is finite, then $|\phi(H)|$ divides $|H|$.***
7. ***$\phi(Z(G))$ is a subgroup of $Z(\phi(G))$.***
8. ***If $\overline{K}$ is a subgroup of $\overline{G}$ then $\phi^{-1}(\overline{K}) = \{k \in G \mid \phi(k) \in \overline{K}\}$ is a subgroup of G.***
9. ***If $\overline{K}$ is a normal subgroup of $\overline{G}$, then $\phi^{-1}(\overline{K}) = \{k \in G \mid \phi(k) \in \overline{K}\}$ is a normal subgroup of G.***
10. ***If ϕ is onto and Ker $\phi = \{e\}$, then ϕ is an isomorphism from G to $\overline{G}$.***

PROOF First note that the proofs of properties 1, 2, and 3 are identical to the proofs of properties 4, 3, 2, and 6, respectively, of Theorem 6.2, since those proofs use only the fact that an isomorphism is an operation-preserving mapping.

To prove property 4, let $\phi(h) \in \phi(H)$ and $\phi(g) \in \phi(G)$. Then $\phi(g)\phi(h)\phi(g)^{-1} = \phi(ghg^{-1}) \in \phi(H)$, since H is normal in G.

Property 5 follows directly from property 6 of Theorem 10.1 and the fact that all cosets of Ker $\phi = \phi^{-1}(e)$ have the same number of elements.

To prove property 6, let ϕ_H denote the restriction of ϕ to the elements of H. Then ϕ_H is a homomorphism from H onto $\phi(H)$. Suppose $|\text{Ker } \phi_H| = t$. Then, by property 5, ϕ_H is a t-to-1 mapping. So, $|\phi(H)|t = |H|$.

To prove property 8, we use the One-Step Subgroup Test. Clearly, $e \in \phi^{-1}(\overline{K})$, so that $\phi^{-1}(\overline{K})$ is not empty. Let $k_1, k_2 \in \phi^{-1}(\overline{K})$. Then, by the definition of $\phi^{-1}(\overline{K})$, we know that $\phi(k_1), \phi(k_2) \in \overline{K}$. Thus, $\phi(k_2)^{-1} \in \overline{K}$ as well and $\phi(k_1k_2^{-1}) = \phi(k_1)\phi(k_2)^{-1} \in \overline{K}$. So, by the definition of $\phi^{-1}(\overline{K})$, we have $k_1k_2^{-1} \in \phi^{-1}(\overline{K})$.

To prove property 9, we use the normality test given in Theorem 9.1. Note that every element in $x\phi^{-1}(\overline{K})x^{-1}$ has the form xkx^{-1}, where $\phi(k) \in \overline{K}$. Thus, since $\overline{K}$ is normal in $\overline{G}$, $\phi(xkx^{-1}) = \phi(x)\phi(k)(\phi(x))^{-1} \in \overline{K}$, and, therefore, $xkx^{-1} \in \phi^{-1}(\overline{K})$.

Finally, property 10 follows directly from property 5. ∎

A few remarks about Theorems 10.1 and 10.2 are in order. Students should remember the various properties of these theorems in words. For example, properties 2 and 3 of Theorem 10.2 say that the homomorphic image of a cyclic group is cyclic and the homomorphic image of an Abelian group is Abelian. Property 4 of Theorem 10.2 says that the homomorphic image of a normal subgroup of G is normal in the image of G. Property 5 of Theorem 10.2 says that if ϕ is a homomorphism from G to $\overline{G}$, then every element of $\overline{G}$ that gets "hit" by ϕ gets hit the same number of times as does the identity. The set $\phi^{-1}(g')$ defined in property 6 of Theorem 10.1 is called the *inverse image* of g' (or the *pullback* of g'). Note that the inverse image of an element is a coset of the kernel and that every element in that coset has the same image. Similarly, the set $\phi^{-1}(\overline{K})$ defined in property 8 of Theorem 10.2 is called the *inverse image* of $\overline{K}$ (or the *pullback* of $\overline{K}$).

Property 6 of Theorem 10.1 is reminiscent of something from linear algebra and differential equations. Recall that if x is a particular solution to a system of linear equations and S is the entire solution set of the corresponding homogeneous system of linear equations, then $x + S$ is the entire solution set of the nonhomogeneous system. In reality, this state- ment is just a special case of property 6. Properties 1 and 6 of Theorem 10.1 and property 5 of Theorem 10.2 are pictorially represented in Figure 10.1.

The special case of property 9 of Theorem 10.2, where $\overline{K} = \{e\}$, is of such importance that we single it out.

Corollary Kernels are Normal

Let ϕ be a group homomorphism from G to $\overline{G}$. Then* Ker *ϕ is a normal subgroup of G.

The next three examples illustrate several properties of Theorems 10.1 and 10.2.

EXAMPLE 9 Consider the mapping ϕ from $\mathbf{C}^*$ to $\mathbf{C}^*$ given by $\phi(x) = x^4$. Since $(xy)^4 = x^4y^4, \phi$ is a homomorphism. Clearly, Ker $\phi = \{x \mid x^4 = 1\} = \{1, -1, i - i\}$. So, by property 5 of Theorem 10.2, we know that ϕ is a 4-to-1 mapping. Now let's find all elements that map to, say, 2. Certainly, $\phi(\sqrt[4]{2}) = 2$. Then, by property 6 of Theorem 10.1, the set of all elements that map to 2 is $\sqrt[4]{2}$ Ker $\phi = \{\sqrt[4]{2}, -\sqrt[4]{2}, \sqrt[4]{2}i, -\sqrt[4]{2}i\}$. ▮

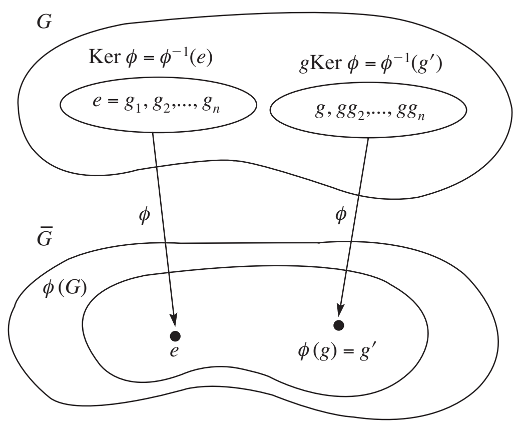


Figure 10.1 Homomorphism from G to $\overline{G}$.

Finally, we verify a specific instance of property 3 of Theorem 10.1 and of properties 2 and 6 of Theorem 10.2. Let $H = \langle \cos 30° + i \sin 30° \rangle$. It follows from DeMoivre's Theorem (Example 12 in Chapter 0) that $|H| = 12, \phi(H) = \langle \cos 120° + i \sin 120° \rangle$, and $|\phi(H)| = 3$.

EXAMPLE 10 Define ϕ: $Z_{12} \to Z_{12}$ by $\phi(x) = 3x$. To verify that ϕ is a homomorphism, we observe that in $Z_{12}, 3(a + b) = 3a + 3b$ (since the group operation is addition modulo 12). Direct calculations show that Ker $\phi = \{0, 4, 8\}$. Thus, we know from property 5 of Theorem 10.2 that ϕ is a 3-to-1 mapping. Since $\phi(2) = 6$, we have by property 6 of Theorem 10.1 that $\phi^{-1}(6) = 2 + \text{Ker } \phi = \{2, 6, 10\}$. Notice also that $\langle 2 \rangle$ is cyclic and $\phi(\langle 2 \rangle) = \{0, 6\}$ is cyclic. Moreover, $|2| = 6$ and $|\phi(2)| = |6| = 2$, so $|\phi(2)|$ divides $|2|$ in agreement with property 3 of Theorem 10.1. Letting $\overline{K} = \{0, 6\}$, we see that the subgroup $\phi^{-1}(\overline{K}) = \{0, 2, 4, 6, 8, 10\}$. This verifies property 8 of Theorem 10.2 in this particular case. ▮

The next example illustrates how one can easily determine all homomorphisms from a cyclic group to a cyclic group.

EXAMPLE 11 We determine all homomorphisms from Z_{12} to Z_{30}. By property 2 of Theorem 10.1, such a homomorphism is completely specified by the image of 1. That is, if 1 maps to a, then x maps to xa. Lagrange's Theorem and property 3 of Theorem 10.1 require that $|a|$ divide both 12 and 30. So, $|a| = 1$, 2, 3, or 6. Thus, $a = 0, 15, 10, 20, 5,$ or 25. This gives us a list of candidates for the homomorphisms. That each of these six possibilities yields an operation-preserving, well-defined function can now be verified by direct calculations. [Note that gcd(12, 30) = 6. This is not a coincidence!] ▮

EXAMPLE 12 The mapping from S_n to Z_2 that takes an even permutation to 0 and an odd permutation to 1 is a homomorphism. Figure 10.2 illustrates the telescoping nature of the mapping. ▮

The First Isomorphism Theorem

In Chapter 9, we showed that for a group G and a normal subgroup H, we could arrange the Cayley table of G into boxes that represented the cosets of H in G, and that these boxes then became a Cayley table for G/H. The next theorem shows that for any homomorphism ϕ of G and the normal subgroup Ker ϕ, the same process produces a Cayley table isomorphic to the homomorphic image of G. Thus, homomorphisms, like factor groups, cause a *systematic* collapse of a group to a simpler but closely related group. This can be likened to viewing a group through the reverse

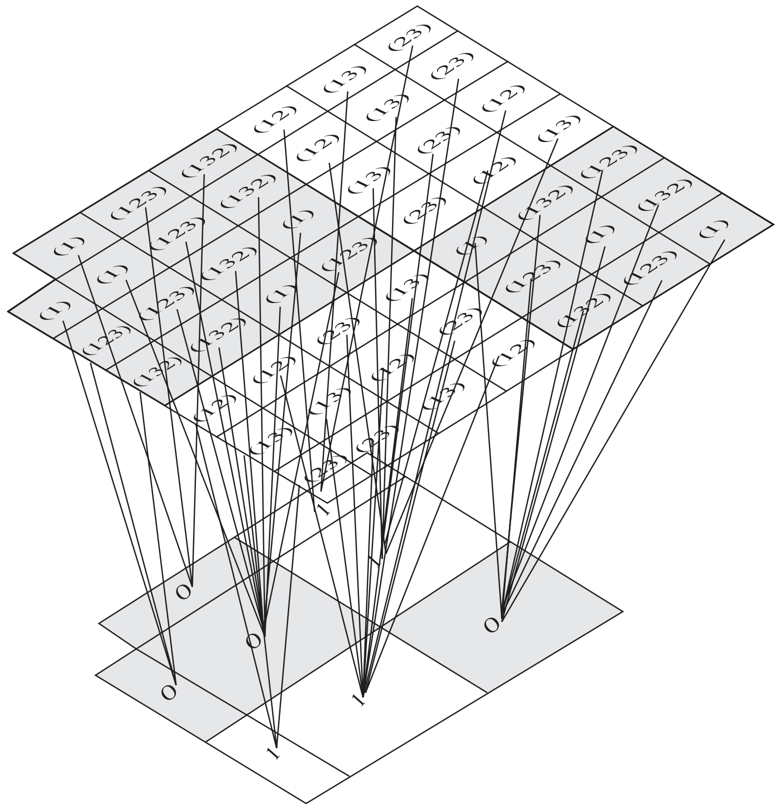


Figure 10.2 Homomorphism from S_3 to Z_2.

end of a telescope—the general features of the group are present, but the apparent size is diminished. The important relationship between homomorphisms and factor groups given below is often called the Fundamental Theorem of Group Homomorphisms.

Theorem 10.3 First Isomorphism Theorem (Jordan, 1870)

Let ϕ be a group homomorphism from G to $\overline{G}$. Then the mapping from $G/\mathrm{Ker}\ \phi$ to $\phi(G)$, given by $g\mathrm{Ker}\ \phi \to \phi(g)$, is an isomorphism. In symbols, $G/\mathrm{Ker}\ \phi \approx \phi(G)$.

PROOF Let us use ψ to denote the correspondence $g\mathrm{Ker}\ \phi \to \phi(g)$. That ψ is well-defined (i.e., the correspondence is independent of the particular coset representative chosen) and one-to-one follows directly from property 5 of Theorem 10.1. To show that ψ is operation-preserving, observe that $\psi(x\text{Ker }\phi\ y\text{Ker }\phi) = \psi(xy\text{Ker }\phi) = \phi(xy) = \phi(x)\ \phi(y) = \psi(x\text{Ker }\phi)\psi(y\text{Ker }\phi)$. ■

The beauty of Theorem 10.3 is that we can use homomorphisms to gain insight about a factor group and the group itself and we can use factors groups to gain insight about homomorphisms and homomorphic images. For example, noting that the subgroup of A_4 of order 4 given in Example 9 in Chapter 9 is normal in S_4 we can deduce that S_3 is the homomorphic image of S_4 (see Exercise 43). Similarly, Example 5 in Chapter 7 implies that Z_2 is not the homomorphic image of A_4.

The following useful corollary of Theorem 10.3 is immediate.

Corollary 1

If ϕ is a homomorphism from a finite group G to $\overline{G}$, then $|G|/|\text{Ker }\phi| = |\phi(G)|$.

The next corollary follows directly from Theorem 10.3, property 1 of Theorem 10.2, and Lagrange's Theorem.

Corollary 2

If ϕ is a homomorphism from a finite group G to $\overline{G}$, then $|\phi(G)|$ divides $|G|$ and $|\overline{G}|$.

EXAMPLE 13 To illustrate Theorem 10.3 and its proof, consider the homomorphism ϕ from D_4 to itself given by the following:

$$R_0,\ R_{180} \longrightarrow R_0$$
$$R_{90},\ R_{270} \longrightarrow H$$
$$H,\ V \longrightarrow R_{180}$$
$$D,\ D' \longrightarrow V.$$

Then $\text{Ker }\phi = \{R_0, R_{180}\}$, and the mapping ψ in Theorem 10.3 is $R_0\text{Ker }\phi \to R_0$, $R_{90}\text{Ker }\phi \to H$, $H\text{Ker }\phi \to R_{180}$, $D\text{Ker }\phi \to V$. It is straightforward to verify that the mapping ψ is an isomorphism. ■

Mathematicians often give a pictorial representation of Theorem 10.3, as follows:

$$\begin{tikzcd} G \arrow[rr, "\phi"] \arrow[dr, "\gamma"'] & & \phi(G) \\ & G/\mathrm{Ker}\,\phi \arrow[ur, "\psi"'] & \end{tikzcd}$$

where γ: $G \to G/\mathrm{Ker}\ \phi$ is defined as $\gamma(g) = g\mathrm{Ker}\ \phi$. The mapping γ is called the *natural mapping* from G to $G/\mathrm{Ker}\ \phi$. Our proof of Theorem 10.3 shows that $\psi\gamma = \phi$. In this case, one says that the preceding diagram is *commutative.*

As a consequence of Theorem 10.3, we see that all homomorphic images of G can be determined using G. We may simply consider the various factor groups of G. For example, we know that the homomorphic image of an Abelian group is Abelian because the factor group of an Abelian group is Abelian. We know that the number of homomorphic images of a cyclic group G of order n is the number of divisors of n, since there is exactly one subgroup of G (and therefore one factor group of G) for each divisor of n. (Be careful: The number of homomorphisms of a cyclic group of order n need not be the same as the number of divisors of n, since different homomorphisms can have the same image.)

▮ **EXAMPLE 14** For every positive integer n the homomorphisms in Examples 5 and 6 and Theorem 10.3 give us $Z/\langle n\rangle \approx Z_n$ and $Z[i]/\{sn + tni \mid s, t \in Z\} \approx Z_n[i]$. ▮

Theorem 10.3 provides a convenient way to indirectly prove that a factor group G/H is isomorphic to some group $\overline{G}$. Following are some examples.

▮ **EXAMPLE 15** Consider the normal subgroup $SL(2, \mathbf{R}) = \{A \in GL(2, \mathbf{R}) \mid \det A = 1\}$ of $GL(2, \mathbf{R})$. Then to prove that $GL(2, \mathbf{R})/SL(2, \mathbf{R}) \approx \mathbf{R}^*$ we observe that the mapping $\phi(A) = \det A$ from $GL(2, \mathbf{R})$ onto $\mathbf{R}^*$ is a homomorphism with $\mathrm{Ker}\ \phi = SL(2, \mathbf{R})$.

Likewise, for the normal subgroup $SL^{\pm}(2, \mathbf{R}) = \{A \in GL(2, \mathbf{R}) \mid \det A = \pm 1\}$ of $GL(2, \mathbf{R})$ the mapping $\phi(A) = (\det A)^2$ from $GL(2, \mathbf{R})$ onto $\mathbf{R}^+$ is a homomorphism with $\mathrm{Ker}\,\phi = SL^{\pm}(2, \mathbf{R})$. So, $GL(\mathbf{2}, \mathbf{R})/SL^{\pm}(2, \mathbf{R}) \approx \mathbf{R}^+$. ▮

▮ **EXAMPLE 16** For an Abelian group G and a positive integer k let G^k denote the subgroup $\{x^k \mid x \in G\}$ and $G^{(k)}$ the subgroup

$\{x \in G \mid x^k = e\}$. (See Examples 4 and 5 in Chapter 3). Then $G/G^{(k)} \approx G^k$. To verify this assertion note that the mapping ϕ from G onto G^k given by $\phi(x) = x^k$ is a homomorphism from G onto G^k with Ker $\phi = G^{(k)}$. ■

Our next example is a theorem that is used repeatedly in Chapters 23 and 24.

▮ EXAMPLE 17 (N/C Theorem)
Let H be a subgroup of a group G. Noting that the *normalizer* of H in G, $N(H) = \{x \in G \mid xHx^{-1} = H\}$, and the *centralizer* of H in G, $C(H) = \{x \in G \mid xhx^{-1} = h$ for all h in $H\}$, are subgroups of G, consider the mapping from $N(H)$ to Aut(H) given by $g \to \phi_g$, where ϕ_g is the inner automorphism of H induced by g [i.e., $\phi_g(h) = ghg^{-1}$ for all h in H]. This mapping is a homomorphism with kernel $C(H)$. So, by Theorem 10.3, $N(H)/C(H)$ is isomorphic to a subgroup of Aut(H). ■

As an application of the N/C Theorem, we will show that every group of order 35 is cyclic. The same proof works for any two primes p and q with $p < q$ where p does not divide $q - 1$ (see Exercise 71). For any two primes p and q with $p < q$ where p does divide $q - 1$ there is a unique (up to isomorphism) non-Abelian group of order pq.

▮ EXAMPLE 18 Let G be a group of order 35. By Lagrange's Theorem, every nonidentity element of G has order 5, 7, or 35. If some element has order 35, G is cyclic. So we may assume that all nonidentity elements have order 5 or 7. However, not all such elements can have order 5, since elements of order 5 come 4 at a time (if $|x| = 5$, then $|x^2| = |x^3| = |x^4| = 5$) and 4 does not divide 34. Similarly, since 6 does not divide 34, not all nonidentity elements can have order 7. So, G has elements of order 7 and order 5. Since G has an element of order 7, it has a subgroup of order 7. Let us call it H. In fact, H is the only sub- group of G of order 7, for if K is another subgroup of G of order 7, we have by Theorem 7.2 that $|HK| = |H||K|/|H \cap K| = 7 \cdot 7/1 = 49$. But, of course, this is impossible in a group of order 35. Since for every a in G, aHa^{-1} is also a subgroup of G of order 7, we must have $aHa^{-1} = H$. So, $N(H) = G$. Since H has prime order, it is cyclic and therefore Abelian. In particular, $C(H)$ contains H. So, 7 divides $|C(H)|$ and $|C(H)|$ divides 35. It follows, then, that

$C(H) = G$ or $C(H) = H$. If $C(H) = G$, then we may obtain an element x of order 35 by letting $x = hk$, where h is a nonidentity element of H and k has order 5. On the other hand, if $C(H) = H$, then $|C(H)| = 7$ and $|N(H)/C(H)| = 35/7 = 5$. However, 5 does not divide $|\mathrm{Aut}(H)| = |\mathrm{Aut}(Z_7)| = 6$. This contradiction shows that G is cyclic. ∎

The corollary of Theorem 10.2 says that the kernel of every homomorphism of a group is a normal subgroup of the group. We conclude this chapter by verifying that the converse of this statement is also true.

Theorem 10.4 Normal Subgroups Are Kernels

Every normal subgroup of a group G is the kernel of a homomorphism of G. In particular, a normal subgroup N is the kernel of the mapping $g \to gN$ from G to G/N.

PROOF Define γ: $G \to G/N$ by $\gamma(g) = gN$. (This mapping is called the *natural homomorphism* from G to G/N.) Then, $\gamma(xy) = (xy)N = xNyN = \gamma(x)\gamma(y)$. Moreover, $g \in \mathrm{Ker}\,\gamma$ if and only if $gN = \gamma(g) = N$, which is true if and only if $g \in N$ (see property 2 of the lemma in Chapter 7). ∎

EXAMPLE 19 In this example we show how to use the normal subgroup H of A_4 of order 4 given in Example 9 in Chapter 9 and Theorem 10.4 to construct a homomorphism from A_4 onto Z_3. By Theorem 10.4 the mapping from A_4 onto $A_4/H = \{H, (123)H, (123)^2H\}$ given by $\alpha \to \alpha H$ is a homomorphism. Since A_4/H is isomorphic to Z_3, it follows that the mapping from A_4 onto Z_3 given by $H \to 0, (123)H \to 1, (123)^2H \to 2$ is a homomorhism from A_4 onto Z_3. (Compare with Exercise 34.) ∎

Examples 14, 15, 16, and 17 illustrate the utility of the First Isomorphism Theorem. But what about homomorphisms in general? Why would one care to study a homomorphism of a group? The answer is that, just as was the case with factor groups of a group, homomorphic images of a group tell us *some* of the properties of the original group. One measure of the likeness of a group and its homomorphic image is the size of the kernel. If the kernel of the homomorphism of group G is the identity, then the image of G tells us everything (group theoretically) about G (the two being isomorphic). On the other hand, if the kernel of the homomorphism is G itself, then the image tells us nothing about G. Between these two extremes, some information about G is preserved and some is lost. The utility of a particular homomorphism lies in its ability to preserve the group properties we want, while losing some inessential ones. In this way, we have replaced G by a group less complicated (and therefore easier to study) than G; but, in the process, we have saved enough information to answer questions that we have about G itself. For example, if G is a group of order 60 and G has a homomorphic image of order 12 that is cyclic, then we know from properties 5, 8, and 9 of Theorem 10.2 that G has normal subgroups of orders 5, 10, 15, 20, 30, and 60.

The next two examples illustrate how one can use a homomorphism to simplify a problem.

EXAMPLE 20 Suppose we are asked to find an infinite group that is the union of three proper subgroups. Instead of attempting to do this directly, we first make the problem easier by finding a finite group that is the union of three proper subgroups. Since no cyclic group can be the union of proper subgroups the smallest candidate is a noncyclic group of order 4 group such as $U(8)$. Observing that $U(8)$ is the union of $H = \{1, 3\}, K = \{1, 5\}$, and $L = \{1, 7\}$ we have found our finite group. Now all we need do is think of an infinite group that has $U(8)$ as a homomorphic image and pull back H, K, and L, and our original problem is solved. Clearly, the mapping from $U(8) \oplus Z$ onto $U(8)$ given by $\phi(a, b) = a$ is such a mapping, and therefore $U(8) \oplus Z$ is the union of the proper subgroups $\phi^{-1}(H), \phi^{-1}(K)$ and $\phi^{-1}(L)$. ∎

EXAMPLE 21 The groups $Z \oplus Z$ and $Z \oplus Z \oplus Z$ are not isomorphic. To verify this suppose that α is an isomorphism from $Z \oplus Z$ onto $Z \oplus Z \oplus Z$. Let β be the homomorphism from $Z \oplus Z \oplus Z$ onto $Z_2 \oplus Z_2 \oplus Z_2$ given by $\beta(x, y, z) = (x \bmod 2,\ y \bmod 2,\ z \bmod 2)$. Then $\gamma = \beta\alpha$ is a homomorphism from $Z \oplus Z$ onto $Z_2 \oplus Z_2 \oplus Z_2$. Since $Z \oplus Z$ is generated by (1,0) and (0,1), $Z_2 \oplus Z_2 \oplus Z_2$ is generated by $\gamma(1, 0)$ and $\gamma(0, 1)$. But any subgroup generated by two elements of $Z_2 \oplus Z_2 \oplus Z_2$ has order at most 4. ∎

Although an isomorphism is a special case of a homomorphism, the two concepts have entirely different roles. Whereas isomorphisms allow us to look at a group in an alternative way, homomorphisms act as investigative tools. The following analogy between homomor- phisms and photography may be instructive.[1] A photograph of a person cannot tell us the person's exact height, weight, or age. Nevertheless, we *may* be able to decide from a photograph whether the person is tall or short, heavy or thin, old or young, male or female. In the same way, a homomorphic image of a group gives us *some* information about the group.

In certain branches of group theory, and especially in physics and chemistry, one often wants to know all homomorphic images of a group that are matrix groups over the complex numbers (these are called *group representations*). Here, we may carry our analogy with photography one step further by saying that this is like wanting photographs of a person from many different angles (front view, profile, head-to-toe view, close-up, etc.), as well as x-rays! Just as this composite information from the photographs reveals much about the person, several homomorphic images of a group reveal much about the group.

Exercises

The greater the difficulty, the more glory in surmounting it. Skillful pilots gain their reputation from storms and tempests.

Epicurus

1. Prove that the mapping given in Example 2 is a homomorphism.

2. Prove that the mapping given in Example 3 is a homomorphism.

3. Prove that the mapping given in Example 4 is a homomorphism.

4. Prove that the mapping given in Example 12 is a homomorphism.

5. Let $\mathbf{R}^*$ be the group of nonzero real numbers under multiplication, and let r be a positive integer. Show that the mapping that takes x to x^r is a homomorphism from $\mathbf{R}^*$ to $\mathbf{R}^*$ and determine the kernel. Which values of r yield an isomorphism?

6. Let G be the group of all polynomials with real coefficients under addition. For each f in G, let $\int f$ denote the antideriva-

[1]"All perception of truth is the detection of an analogy." Henry David Thoreau, *Journal*.

tive of f that passes through the point (0, 0). Show that the mapping $f \rightarrow \int f$ from G to G is a homomorphism. What is the kernel of this mapping? Is this mapping a homomorphism if $\int f$ denotes the antiderivative of f that passes through (0, 1)?

7. If ϕ is a homomorphism from G to H and σ is a homomorphism from H to K, show that $\sigma\phi$ is a homomorphism from G to K. How are Ker ϕ and Ker $\sigma\phi$ related? If ϕ and σ are onto and G is finite, describe [Ker $\sigma\phi$:Ker ϕ] in terms of $|H|$ and $|K|$.

8. Let G be a group of permutations. For each σ in G, define

$$\operatorname{sgn}(\sigma) = \begin{cases} +1 \text{ if } \sigma \text{ is an even permutation,} \\ -1 \text{ if } \sigma \text{ is an even permutation.} \end{cases}$$

Prove that sgn is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? Why does this homomorphism allow you to conclude that A_n is a normal subgroup of S_n of index 2? Why does this prove Exercise 27 of Chapter 5?

9. Prove that the mapping from $G \oplus H$ to G given by $(g, h) \rightarrow g$ is a homomorphism. What is the kernel? This mapping is called the *projection* of $G \oplus H$ onto G.

10. Let G be a subgroup of some dihedral group. For each x in G, define

$$\phi(x) = \begin{cases} +1 & \text{if } x \text{ is a rotation,} \\ -1 & \text{if } x \text{ is a reflection.} \end{cases}$$

Prove that ϕ is a homomorphism from G to the multiplicative group $\{+1, -1\}$. What is the kernel? Why does this prove Exercise 30 of Chapter 3?

11. Prove that $(Z \oplus Z)/(\langle(a, 0)\rangle \oplus \langle(0, b)\rangle)$ is isomorphic to $Z_a \oplus Z_b$.

12. Suppose that k is a divisor of n. Prove that $Z_n/\langle k\rangle \approx Z_k$.

13. Prove that $(A \oplus B)/(A \oplus \{e\}) \approx B$.

14. Explain why the correspondence $x \rightarrow 3x$ from Z_{12} to Z_{10} is not a homomorphism.

15. Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and

Ker $\phi = \{0, 10, 20\}$. If $\phi(23) = 9$, determine all elements that map to 9.

16. Prove that there is no homomorphism from $Z_8 \oplus Z_2$ onto $Z_4 \oplus Z_4$.

17. Prove that there is no homomorphism from $Z_{16} \oplus Z_2$ onto $Z_4 \oplus Z_4$.

18. Can there be a homomorphism from $Z_4 \oplus Z_4$ onto Z_8? Can there be a homomorphism from Z_{16} onto $Z_2 \oplus Z_2$? Can there be a homomorphism from Z_{18} onto $Z_3 \oplus Z_2$? Explain your answers.

19. Suppose that there is a homomorphism ϕ from Z_{17} to some group and that ϕ is not one-to-one. Determine ϕ.

20. How many homomorphisms are there from Z_{20} onto Z_8? How many are there to Z_8?

21. If ϕ is a homomorphism from Z_{30} onto a group of order 5, determine the kernel of ϕ.

22. Suppose that ϕ is a homomorphism from a finite group G onto $\overline{G}$ and that $\overline{G}$ has an element of order 8. Prove that G has an element of order 8. Generalize.

23. Let ϕ be a homomorphism from a finite group G to $\overline{G}$. If H is a subgroup of $\overline{G}$ give a formula for $|\phi^{-1}(H)|$ in terms of $|H|$ and ϕ.

24. Suppose that ϕ: $Z_{50} \to Z_{15}$ is a group homomorphism with $\phi(7) = 6$.

a. Determine $\phi(x)$.

b. Determine the image of ϕ.

c. Determine the kernel of ϕ.

d. Determine $\phi^{-1}(12)$. That is, determine the set of all elements that map to 12.

25. How many homomorphisms are there from Z_{20} onto Z_{10}? How many are there to Z_{10}?

26. Determine all homomorphisms from Z_4 to $Z_2 \oplus Z_2$.

27. Determine all homomorphisms from Z_n to itself.

28. Suppose that ϕ is a homomorphism from S_4 onto Z_2. Determine Ker ϕ. Determine all homomorphisms from S_4 to Z_2.

29. Suppose that there is a homomorphism from a finite group G onto Z_{10}. Prove that G has normal subgroups of indexes 2 and 5.

30. Suppose that ϕ is a homomorphism from a group G onto $Z_6 \oplus Z_2$ and that the kernel of ϕ has order 5. Explain why G must have normal subgroups of orders 5, 10, 15, 20, 30, and 60.

31. Suppose that ϕ is a homomorphism from $U(30)$ to $U(30)$ and that Ker $\phi = \{1, 11\}$. If $\phi(7) = 7$, find all elements $U(30)$ of that map to 7.

32. Suppose that ϕ is a homomorphism from Z_{30} to Z_{30} and that Ker $\phi = \{0, 5, 10, 15, 20, 25\}$. If $\phi(7) = 12$, Find all elements of Z_{30} that map to 24. Find all elements of Z_{30} that map to 6.

33. Suppose that ϕ is a homomorphism from $U(40)$ to $U(40)$ and that Ker $\phi = \{1,\ 9,\ 17,\ 33\}$. If $\phi(11) = 11$, find all elements of $U(40)$ that map to 11.

34. If ϕ is a homomorphism from A_4 onto a cyclic group G, prove that $|G| = 1$ or 3.

35. Prove that the mapping ϕ: $Z \oplus Z \to Z$ given by $(a,\ b) \to a - b$ is a homomorphism. What is the kernel of ϕ? Describe the set $\phi^{-1}(3)$ (i.e., all elements that map to 3).

36. Suppose that there is a homomorphism ϕ from $Z \oplus Z$ to a group G such that $\phi((3,\ 2)) = a$ and $\phi((2,\ 1)) = b$. Determine $\phi((4,\ 4))$ in terms of a and b. Assume that the operation of G is addition.

37. Let $H = \{z \in C^* \mid |z| = 1\}$. Prove that C^*/H is isomorphic to $\mathbf{R}^+$, the group of positive real numbers under multiplication. (Recall $|a + bi| = \sqrt{a^2 + b^2}$.)

38. Let α be a homomorphism from G_1 to H_1 and β be a homomorphism from G_2 to H_2. Determine the kernel of the homomorphism γ from $G_1 \oplus G_2$ to $H_1 \oplus H_2$ defined by $\gamma(g_1,\ g_2) = (\alpha(g_1),\ \beta(g_2))$.

39. Prove that the mapping $x \to x^6$ from $\mathbf{C}^*$ to $\mathbf{C}^*$ is a homomorphism. What is the kernel?

40. For each pair of positive integers m and n, we can define a homomorphism from Z to $Z_m \oplus Z_n$ by $x \to (x \bmod m,\ x$

mod n). What is the kernel when $(m, n) = (3, 4)$? What is the kernel when $(m, n) = (6, 4)$? Generalize.

41. If G is an Abelian group and ϕ is a homomorphism from G to some group, prove that $H = \{x \in G \mid \phi(x) = x^{-2}\}$ is a subgroup of G. If $\phi(x) = x^3$, what can you say about the orders of the elements of H?

42. If ϕ is a homomorphism from D_6 onto D_3, what is Ker ϕ?

43. Prove that there is no homomorphism from $Z_{16}, Z_8 \oplus Z_2$ or $Z_4 \oplus Z_4$ onto $Z_2 \oplus Z_2 \oplus Z_2$. Generalize to Z_{2^m} and $Z_{2^m} \oplus Z_{2^n}$ onto $Z_2 \oplus Z_2 \oplus Z_2$. Does your argument work for Z_{p^m} and $Z_{p^m} \oplus Z_{p^n}$ onto $Z_p \oplus Z_p \oplus Z_p$ where p is any prime?

44. Prove that is no homomorphism from S_3 onto Z_3. Prove that there is no homomorphism from S_4 onto Z_3.

45. Use the subgroup H defined in Example 9 in Chapter 9 to prove that there is a homomorphism from S_4 onto S_3. This exercise is referred to in this chapter.

46. Prove that every homomorphic image of $Z_m \oplus Z_n$ has the form $Z_s \oplus Z_t$ where s divides m and t divides n.

47. Let s and t be relatively prime positive integers. Prove that $U(st)/U_s(st)$ is isomorphic to $U(s)$.

48. If $G = \langle S \rangle$ and ϕ is a homomorphism from G to some group, prove that $\phi(G) = \langle \phi(S) \rangle$.

49. (Second Isomorphism Theorem) If K is a subgroup of G and N is a normal subgroup of G, prove that $K/(K \cap N)$ is isomorphic to KN/N.

50. (Third Isomorphism Theorem)

If M and N are normal subgroups of G and $N \leq M$, prove that $(G/N)/(M/N) \approx G/M$. Think of this as a form of "cancelling out" the N in the numerator and denominator.

51. Prove that the only homomorphism from A_4 to a finite group with order not divisible by 3 is the trivial mapping that takes every element to the identity.

52. Let k be a divisor of n. Consider the homomorphism from $U(n)$ to $U(k)$ given by $x \to x$ mod k. What is the relationship between this homomorphism and the subgroup $U_k(n)$ of $U(n)$?

53. Determine all homomorphic images of D_4 (up to isomorphism).

54. Let N be a normal subgroup of a finite group G. Use the theorems of this chapter to prove that the order of the group element gN in G/N divides the order of g.

55. Suppose that G is a finite group and that Z_{10} is a homomorphic image of G. What can we say about $|G|$? Generalize.

56. Suppose that Z_{10} and Z_{15} are both homomorphic images of a finite group G. What can be said about $|G|$? Generalize.

57. Suppose that for each prime p, Z_p is the homomorphic image of a group G. What can we say about $|G|$? Give an example of such a group.

58. (For students who have had linear algebra.) Suppose that x is a particular solution to a system of linear equations and that S is the entire solution set of the corresponding homogeneous system of linear equations. Explain why property 6 of Theorem 10.1 guarantees that $x + S$ is the entire solution set of the nonhomogeneous system. In particular, describe the relevant groups and the homomorphism between them.

59. Let N be a normal subgroup of a group G. Use property 8 of Theorem 10.2 to prove that every subgroup of G/N has the form H/N, where H is a subgroup of G. (This exercise is referred to in Chapter 11 and Chapter 23.)

60. Show that a homomorphism defined on a cyclic group is completely determined by its action on a generator of the group.

61. Use the First Isomorphism Theorem to prove Theorem 9.4.

62. Determine all homomorphisms from D_5 onto $Z_2 \oplus Z_2$. Determine all homomorphisms from D_5 to $Z_2 \oplus Z_2$.

63. Let $Z[x]$ be the group of polynomials in x with integer coefficients under addition. Prove that the mapping from $Z[x]$ into Z given by $f(x) \to f(3)$ is a homomorphism. Give a geometric description of the kernel of this homomorphism. Generalize.

64. Prove that the mapping from $\mathbf{R}$ under addition to $SL(2, \mathbf{R})$ that takes x to

$$\begin{bmatrix} \cos x & \sin x \\ -\sin x & \cos x \end{bmatrix}$$

is a group homomorphism. What is the kernel of the homomorphism?

65. Suppose there is a homomorphism ϕ from G onto $Z_2 \oplus Z_2$. Prove that G is the union of three proper normal subgroups.

66. If H and K are normal subgroups of G and $H \cap K = \{e\}$, prove that G is isomorphic to a subgroup of $G/H \oplus G/K$.

67. Find a homomorphism from $Z_8 \oplus Z_4$ onto $Z_4 \oplus Z_4$.

68. Suppose ϕ is a homomorphism from D_{12} onto D_3. What is $\phi(R_{180})$?

69. Does the argument given in Example 19 to prove every group of order 35 is cyclic remain valid for a group of order 55? If not, where does it fail?

70. If G is a non-Abelian group and $|G| = 55$, prove that G has exactly 11 subgroups of order 5 and they have the form a^iKa^{-i} for $i = 0, 1, 2, \ldots, 10$ for some element a in G and some subgroup K of G.

71. Prove that for any two primes p and q with $p < q$ where p does not divide $q - 1$, a group of order pq is cyclic.

72. Determine all homomorphisms from Z onto S_3. Determine all homomorphisms from Z to S_3.

73. Let G be an Abelian group. Determine all homomorphisms from S_3 to G.

74. If m and n are positive integers prove that the mapping from Z_m to Z_n given by $\phi(x) = x \bmod n$ is a homomorphism if and only if n divides m.

75. Prove that the mapping from $\mathbf{C}^*$ to $\mathbf{C}^*$ given by $\phi(x) = x^2$ is a homomorphism and that $\mathbf{C}^*/\{1, -1\}$ is isomorphic to $\mathbf{C}^*$. What happens if $\mathbf{C}^*$ is replaced by $\mathbf{R}^*$?

76. Let p be a prime. Determine the number of homomorphisms from $Z_p \oplus Z_p$ into Z_p.

Computer Exercises

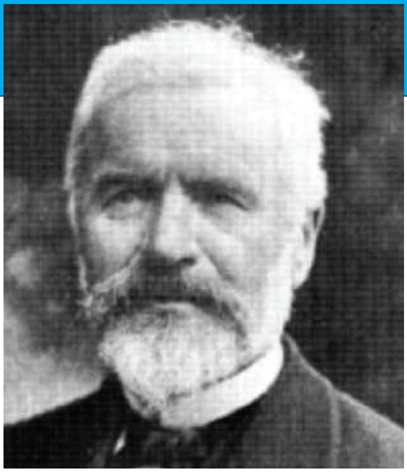
A computer exercise for this chapter is available at the website:

http://www.d.umn.edu/~jgallian

Camille Jordan

Although these contributions [to analysis and topology] would have been enough to rank Jordan very high among his mathematical contemporaries, it is chiefly as an algebraist that he reached celebrity when he was barely thirty; and during the next forty years he was universally regarded as the undisputed master of group theory.

J. DIEUDONNÉ, *Dictionary of Scientific Biography*



The Granger Collection, NYC

CAMILLE JORDAN was born into a well-to-do family on January 5, 1838, in Lyons, France. Like his father, he graduated from the École Polytechnique and became an engineer. Nearly all of his 120 research papers in mathematics were written before his retirement from engineering in 1885. From 1873 until 1912, Jordan taught simultaneously at the École Polytechnique and at the College of France. In the great French tradition, Jordan was a universal mathematician who published in nearly every branch of mathematics. Among the concepts named after him are the Jordan canonical form in matrix theory, the Jordan curve theorem from topology, and the Jordan–Hölder Theorem from group theory. His classic book *Traité des substitutions,* published in 1870, was the first to be devoted solely to group theory and its applications to other branches of mathematics. Another book that had great influence and set a new standard for rigor was his *Cours d'analyse.* This book gave the first clear definitions of the notions of volume and multiple integral. Nearly 100 years after this book appeared, the distinguished mathematician and mathematical historian B. L. van der Waerden wrote, "For me, every single chapter of the *Cours d'analyse* is a pleasure to read." Jordan died in Paris on January 22, 1922.

11 Fundamental Theorem of Finite Abelian Groups

By a small sample we may judge of the whole piece.

Miguel De Cervantes, *Don Quixote*

[On the concept of 'group':] . . . what a wealth, what a grandeur of thought may spring from what slight beginnings.

H. F. Baker

The Fundamental Theorem

In this chapter we present a theorem that describes to an algebraist's eye (i.e., up to isomorphism) all finite Abelian groups in a standardized way. Before giving the proof, which is long and difficult, we discuss some consequences of the theorem and its proof. The first proof of the theorem was given by Leopold Kronecker in 1858.

Theorem 11.1 Fundamental Theorem of Finite Abelian Groups

Every finite Abelian group is a direct product of cyclic groups of prime-power order. Moreover, the number of terms in the product and the orders of the cyclic groups are uniquely determined by the group.

Theorem 11.1 reduces questions about finite abelian groups to questions about cyclic groups, which when combined with the results from earlier chapters, usually yields complete answers to the questions.

Since a cyclic group of order n is isomorphic to Z_n, Theorem 11.1 shows that every finite Abelian group G is isomorphic to a group of the form

$$Z_{{p_1}^{n_1}} \oplus Z_{{p_2}^{n_2}} \oplus \cdots \oplus Z_{{p_k}^{n_k}},$$

where the p_i's are not necessarily distinct primes and the prime powers ${p_1}^{n_1}, {p_2}^{n_2}, \ldots, {p_k}^{n_k}$ are uniquely determined by G. Writing a group in this form is called *determining the isomorphism class* of G.

The Isomorphism Classes of Abelian Groups

The Fundamental Theorem is extremely powerful. As an application, we can use it as an algorithm for constructing all Abelian groups of any order. Let's look at groups whose orders have the form p^k, where p is prime and $k \leq 4$. In general, there is one group of order p^k for each set of positive integers whose sum is k (such a set is called a *partition* of k); that is, if k can be written as

$$k = n_1 + n_2 + \cdots + n_t,$$

where each n_i is a positive integer, then

$$Z_{p^{n_1}} \oplus Z_{p^{n_2}} \oplus \cdots \oplus Z_{p^{n_t}}$$

is an Abelian group of order p^k.

Order of G	Partitions of k	Possible direct products for G
p	1	Z_p
p^2	2	Z_{p^2}
	$1+1$	$Z_p \oplus Z_p$
p^3	3	Z_{p^3}
	$2+1$	$Z_{p^2} \oplus Z_p$
	$1+1+1$	$Z_p \oplus Z_p \oplus Z_p$
p^4	4	Z_{p^4}
	$3+1$	$Z_{p^3} \oplus Z_p$
	$2+2$	$Z_{p^2} \oplus Z_{p^2}$
	$2+1+1$	$Z_{p^2} \oplus Z_p \oplus Z_p$
	$1+1+1+1$	$Z_p \oplus Z_p \oplus Z_p \oplus Z_p$

Furthermore, the uniqueness portion of the Fundamental Theorem guarantees that distinct partitions of k yield distinct isomorphism classes. Thus, for example, $Z_9 \oplus Z_3$ is not isomorphic to $Z_3 \oplus Z_3 \oplus Z_3$. A reliable mnemonic for comparing external direct products is the cancellation property: If A is *finite,* then

$$A \oplus B \approx A \oplus C \quad \text{if and only if} \quad B \approx C.$$

Thus, $Z_4 \oplus Z_4$ is not isomorphic to $Z_4 \oplus Z_2 \oplus Z_2$, because Z_4 is not isomorphic to $Z_2 \oplus Z_2$.

To appreciate fully the potency of the Fundamental Theorem, contrast the ease with which the Abelian groups of order p^k, $k \leq 4$, were determined with the corresponding problem for non-Abelian groups. Even a description of the two non-Abelian groups of order 8 is a challenge (see Theorem 26.4), and a description of the nine non-Abelian groups of order 16 is well beyond the scope of this text.

Now that we know how to construct all the Abelian groups of prime-power order, we move to the problem of constructing all Abelian groups of a certain order n, where n has two or more distinct prime divisors. We begin by writing n in prime-power decomposition form $n = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$. Next, we individually form all Abelian groups of order ${p_1}^{n_1}$, then ${p_2}^{n_2}$, and so on, as described earlier. Finally, we form all possible external direct products of these groups. For example, let $n = 1176 = 2^3 \cdot 3 \cdot 7^2$. Then, the complete list of the distinct isomorphism classes of Abelian groups of order 1176 is

$$
\begin{aligned}
&Z_8 \oplus Z_3 \oplus Z_{49},\\
&Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_{49},\\
&Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_{49},\\
&Z_8 \oplus Z_3 \oplus Z_7 \oplus Z_7,\\
&Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_7 \oplus Z_7,\\
&Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_7 \oplus Z_7.
\end{aligned}
$$

If we are given any particular Abelian group G of order 1176, the question we want to answer about G is: Which of the preceding six isomorphism classes represents the structure of G? We can answer this question by comparing the orders of the elements of G with the orders of the elements in the six direct products, since it can be shown that two finite Abelian groups are isomorphic if and only if they have the same number of elements of each order. For instance, we could determine whether G has any elements of order 8. If so, then G must be isomorphic to the first or fourth group above, since these are the only ones with elements of order 8. To narrow G down to a single choice, we now need only check whether or not G has an element of order 49, since the first product above has such an element, whereas the fourth one does not.

In the following example the group is small enough that we can easily determine its isomorphism class from the orders of its elements.

▮ EXAMPLE 1 Let $G = \{1, 8, 12, 14, 18, 21, 27, 31, 34, 38, 44, 47, 51, 53, 57, 64\}$ under multiplication modulo 65. Since G has order 16, we know it is isomorphic to one of

$$Z_{16},$$
$$Z_8 \oplus Z_2,$$
$$Z_4 \oplus Z_4,$$
$$Z_4 \oplus Z_2 \oplus Z_2,$$
$$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2.$$

To decide which one, we can use an online modular arithmetic calculator such as **planetcalc.com/8326** to determine the orders of elements of G.

Element	1	8	12	14	18	21	27	31	34	38	44	47	51	53	57	64
Order	1	4	4	2	4	4	4	4	4	4	4	4	2	4	4	2

From the table of orders, we can instantly rule out all but $Z_4 \oplus Z_4$ and $Z_4 \oplus Z_2 \oplus Z_2$ as possibilities. Finally, we observe that since this latter group has a subgroup isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$, it has more than three elements of order 2, and therefore we must have $G \approx Z_4 \oplus Z_4$.

Expressing G as an internal direct product is even easier. Pick an element of maximum order, say the element 8. Then $\langle 8 \rangle$ is a factor in the product. Next, choose a second element, say a, so that a has order 4 and $\langle a \rangle \cap \langle 8 \rangle = \{1\}$. Since 12 has this property, we have $G = \langle 8 \rangle \times \langle 12 \rangle$. ▮

Example 1 illustrates how quickly and easily one can write an Abelian group as a direct product given the orders of the elements of the group. But calculating all those orders is certainly not an appealing prospect! The good news is that, in practice, a combination of theory and calculation of the orders of a few elements will usually suffice.

▮ EXAMPLE 2 Let $G = \{1, 8, 17, 19, 26, 28, 37, 44, 46, 53, 62, 64, 71, 73, 82, 89, 91, 98, 107, 109, 116, 118, 127, 134\}$ under multiplication modulo 135. Since G has order 24, it is isomorphic to one of

$$Z_8 \oplus Z_3 \approx Z_{24},$$
$$Z_4 \oplus Z_2 \oplus Z_3 \approx Z_{12} \oplus Z_2,$$
$$Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \approx Z_6 \oplus Z_2 \oplus Z_2.$$

Consider the element 8. Direct calculations show that $8^4 = 49$, $8^6 = 109$, and $8^{12} = 1$. So $|8| = 12$. But now we know G. Why? Clearly, $|8| = 12$ rules out the third group in the list. At the same time, $|109| = 2 = |134|$ (remember, $134 = -1 \bmod 135$) implies that G is not Z_{24} (see Theorem 4.4). Thus, $G \approx Z_{12} \oplus Z_2$, and $G = \langle 8 \rangle \times \langle 134 \rangle$. ∎

Rather than express an Abelian group as a direct product of cyclic groups of prime-power orders, it is often more convenient to combine the cyclic factors of relatively prime order, as we did in Example 2, to obtain a direct product of the form $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_k}$, where n_i divides n_{i-1}. For example, $Z_4 \oplus Z_4 \oplus Z_2 \oplus Z_9 \oplus Z_3 \oplus Z_5$ would be written as $Z_{180} \oplus Z_{12} \oplus Z_2$ (see Exercise 11).

As a consequence of the Fundamental Theorem of Finite Abelian Groups, we have the following corollary, which shows that the converse of Lagrange's Theorem is true for finite Abelian groups.

Corollary Existence of Subgroups of Abelian Groups

If m divides the order of a finite Abelian group G, then G has a subgroup of order m.

PROOF Suppose that G is an Abelian group of order n and m divides n. We induct on the order of G. The case where n or m is 1 is trivial. Let p be a prime that divides m. It follows from Theorem 11.1 and properties of cyclic groups that G has a subgroup K of order p. Then G/K is an Abelian group of order n/p and m/p divides $|G/K|$. By the Second Principle of Mathematical Induction G/K has a subgroup of the form H/K where H is a subgroup of G and $|H/K| = m/p$ (see Exercise 63 of Chapter 10). Then $|H| = (|H|/|K|)|K| = (m/p)p = m$. ∎

It is instructive to verify this corollary for a specific case. Let us say that G is an Abelian group of order 72 and we wish to produce a subgroup of order 12. According to the Fundamental Theorem, G is isomorphic to one of the following six groups:

$$\begin{array}{ll} Z_8 \oplus Z_9, & Z_8 \oplus Z_3 \oplus Z_3, \\ Z_4 \oplus Z_2 \oplus Z_9, & Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_3, \\ Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_9, & Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_3 \oplus Z_3. \end{array}$$

Obviously, $Z_8 \oplus Z_9 \approx Z_{72}$ and $Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_3 \approx Z_{12} \oplus Z_6$ both have a subgroup of order 12. To construct a subgroup of order 12 in $Z_4 \oplus Z_2 \oplus Z_9$, we simply piece together all of Z_4 and the subgroup of order 3 in Z_9; that is, $\{(a, 0, b) \mid a \in Z_4, b \in \{0, 3, 6\}\}$. A subgroup of order 12 in $Z_8 \oplus Z_3 \oplus Z_3$ is given by $\{(a, b, 0) \mid a \in \{0, 2, 4, 6\}, b \in Z_3\}$. An analogous procedure applies to the remaining cases and indeed to any finite Abelian group.

Proof of the Fundamental Theorem

Because of the length and complexity of the proof of the Fundamental Theorem of Finite Abelian Groups, we will break it up into a series of lemmas.

Lemma 1

Let G be a finite Abelian group of order $p^n m$, where p is a prime that does not divide m. Then $G = H \times K$, where $H = \{x \in G \mid x^{p^n} = e\}$ and $K = \{x \in G \mid x^m = e\}$. Moreover, $|H| = p^n$.

PROOF It is an easy exercise to prove that H and K are subgroups of G (see Exercise 51 in Chapter 3). Because G is Abelian, to prove that $G = H \times K$ we need only prove that $G = HK$ and $H \cap K = \{e\}$. Since we have $\gcd(m, p^n) = 1$, there are integers s and t such that $1 = sm + tp^n$. For any x in G, we have $x = x^1 = x^{sm+tp^n} = x^{sm}x^{tp^n}$ and, by Corollary 4 of Lagrange's Theorem (Theorem 7.1), $x^{sm} \in H$ and $x^{tp^n} \in K$. Thus, $G = HK$. Now suppose that some $x \in H \cap K$. Then $x^{p^n} = e = x^m$ and, by Corollary 2 of Theorem 4.1, $|x|$ divides both p^n and m. Since p does not divide m, we have $|x| = 1$ and, therefore, $x = e$.

To prove the second assertion of the lemma, note that $p^n m = |HK| = |H||K|/|H \cap K| = |H||K|$ (Theorem 7.2). It follows from Theorem 9.5 and Corollary 2 to Theorem 4.1 that p does not divide $|K|$ and therefore $|H| = p^n$. ∎

Given an Abelian group G with $|G| = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k}$, where the p's are distinct primes, we let $G(p_i)$ denote the set $\{x \in G \mid x^{p_i^{n_i}} = e\}$. It then follows immediately from Lemma 1 and induction that $G = G(p_1) \times G(p_2) \times \cdots \times G(p_k)$ and $|G(p_i)| = p_i^{n_i}$. Hence, we turn our attention to groups of prime-power order.

Lemma 2

Let G be an Abelian group of prime-power order and let a be an element of maximum order in G. Then G can be written in the form $\langle a\rangle \times K$.

PROOF We denote $|G|$ by p^n and induct on n. If $n = 1$, then $G = \langle a\rangle \times \langle e\rangle$. Now assume that the statement is true for all Abelian groups of order p^k, where $k < n$. Among all the elements of G, choose a of maximum order p^m. Then $x^{p^m} = e$ for all x in G. We may assume that $G \neq \langle a\rangle$, for otherwise there is nothing to prove. Now, among all the elements of G, choose b of smallest order such that $b \notin \langle a\rangle$. We claim that $\langle a\rangle \cap \langle b\rangle = \{e\}$. Since $|b^p| = |b|/p$, we know that $b^p \in \langle a\rangle$ by the manner in which b was chosen. Say $b^p = a^i$. Notice that $e = b^{p^m} = (b^p)^{p^{m-1}} = (a^i)^{p^{m-1}}$, so $|a^i| \leq p^{m-1}$. Thus, a^i is not a generator of $\langle a\rangle$ and, therefore by Corollary 3 to Theorem 4.2, $\gcd(p^m, i) \neq 1$. This proves that p divides i, so that we can write $i = pj$. Then $b^p = a^i = a^{pj}$. Consider the element $c = a^{-j}b$. Certainly, c is not in $\langle a\rangle$, for if it were, b would be, too. Also $c^p = a^{-jp}b^p = a^{-i}b^p = b^{-p}b^p = e$. Thus, we have found an element c of order p such that $c \notin \langle a\rangle$. Since b was chosen to have smallest order such that $b \in \langle a\rangle$, we conclude that b also has order p. It now follows that $\langle a\rangle \cap \langle b\rangle = \{e\}$, because any nonidentity element of the intersection would generate $\langle b\rangle$ and thus contradict $b \notin \langle a\rangle$.

Now consider the factor group $\overline{G} = G/\langle b\rangle$. To simplify the notation, we let $\overline{x}$ denote the coset $x\langle b\rangle$ in $\overline{G}$. If $|\overline{a}| < |a| = p^m$, then $\overline{a}^{p^{m-1}} = \overline{e}$. This means that $(a\langle b\rangle)^{p^{m-1}} = a^{p^{m-1}}\langle b\rangle = \langle b\rangle$, so that $a^{p^{m-1}} \in \langle a\rangle \cap \langle b\rangle = \{e\}$, contradicting the fact that $|a| = p^m$. Thus, $|\overline{a}| = |a| = p^m$, and therefore $\overline{a}$ is an element of maximum order in $\overline{G}$. By induction, we know that $\overline{G}$ can be written in the form $\langle\overline{a}\rangle \times \overline{K}$ for some subgroup $\overline{K}$ of $\overline{G}$. Let K be the pullback of $\overline{K}$ under the natural homomorphism from G to $\overline{G}$ (i.e., $K = \{x \in G \mid \overline{x} \in \overline{K}\}$). We claim that $\langle a\rangle \cap K = \{e\}$. For if $x \in \langle a\rangle \cap K$, then $\overline{x} \in \langle\overline{a}\rangle \cap \overline{K} = \{\overline{e}\} = \langle b\rangle$ and $x \in \langle a\rangle \cap \langle b\rangle = \{e\}$. It now follows from an order argument (see Exercise 37) that $G = \langle a\rangle K$, and therefore $G = \langle a\rangle \times K$. ∎

Lemma 2 and induction on the order of the group now give the following.

Lemma 3

A finite Abelian group of prime-power order is an internal direct product of cyclic groups.

Let us pause to determine where we are in our effort to prove the Fundamental Theorem of Finite Abelian Groups. The remark following Lemma 1 shows that $G = G(p_1) \times G(p_2) \times \cdots \times G(p_n)$, where each $G(p_i)$ is a group of prime-power order, and Lemma 3 shows that each of these factors is an internal direct product of cyclic groups. Thus, we have proved that G is an internal direct product of cyclic groups of prime-power order. All that remains to be proved is the uniqueness of the factors. Certainly the groups $G(p_i)$ are uniquely determined by G, since they comprise the elements of G whose orders are powers of p_i. So we must prove that there is only one way (up to isomorphism and rearrangement of factors) to write each $G(p_i)$ as an internal direct product of cyclic groups.

Lemma 4

Suppose that G is a finite Abelian group of prime-power order. If $G = H_1 \times H_2 \times \cdots \times H_m$ and $G = K_1 \times K_2 \times \cdots \times K_n$, where the H's and K's are nontrivial cyclic subgroups with $|H_1| \geq |H_2| \geq \cdots \geq |H_m|$ and $|K_1| \geq |K_2| \geq \cdots \geq |K_n|$, then $m = n$ and $|H_i| = |K_i|$ for all i.

PROOF We proceed by induction on $|G|$. Clearly, the case where $|G| = p$ is true. Now suppose that the statement is true for all Abelian groups of order less than $|G|$. For any Abelian group L, the set $L^p = \{x^p \mid x \in L\}$ is a subgroup of L (see Example 5 of Chapter 3) and, by Theorem 9.5, is a proper subgroup if p divides $|L|$. It follows that $G^p = {H_1}^p \times {H_2}^p \times \cdots \times {H_{m'}}^p$, and $G^p = {K_1}^p \times {K_2}^p \times \cdots \times {K_{n'}}^p$, where m' is the largest integer i such that $|H_i| > p$, and n' is the largest integer j such that $|K_j| > p$. (This ensures that our two direct products for G^p do not have trivial factors.) Since $|G^p| < |G|$, we have, by induction, $m' = n'$ and $|{H_i}^p| = |{K_i}^p|$ for $i = 1, \ldots, m'$. Since $|H_i| = p|{H_i}^p|$, this proves that $|H_i| = |K_i|$ for all $i = 1, \ldots, m'$. All that remains to be proved is that the number of H_i of order p equals the number of K_i of order p;

that is, we must prove that $m - m' = n - n'$ (since $n' = M'$). This follows directly from the facts that $|H_1||H_2| \cdots |H_{m'}|p^{m-m'} = |G| = |K_1||K_2| \cdots |K_{n'}|p^{n-n'}$, $|H_i| = |K_i|$, and $m' = n'$. ∎

Exercises

One problem after another presents itself and in the solving of them we can find our greatest pleasure.

Karl Menninger

1. What is the smallest positive integer n such that there are two nonisomorphic groups of order n? Name the two groups.
2. What is the smallest positive integer n such that there are three nonisomorphic Abelian groups of order n? Name the three groups.
3. What is the smallest positive integer n such that there are exactly four nonisomorphic Abelian groups of order n? Name the four groups.
4. Calculate the number of elements of order 2 in each of Z_{16}, $Z_8 \oplus Z_2$, $Z_4 \oplus Z_4$, and $Z_4 \oplus Z_2 \oplus Z_2$. Do the same for the elements of order 4.
5. Prove that any Abelian group of order 45 has an element of order 15. Does every Abelian group of order 45 have an element of order 9?
6. Show that there are two Abelian groups of order 108 that have exactly one subgroup of order 3.
7. Show that there are two Abelian groups of order 108 that have exactly four subgroups of order 3.
8. Show that there are two Abelian groups of order 108 that have exactly 13 subgroups of order 3.
9. Suppose that G is an Abelian group of order 120 and that G has exactly three elements of order 2. Determine the isomorphism class of G.
10. Find all Abelian groups (up to isomorphism) of order 360.
11. Prove that every finite Abelian group can be expressed as the (external) direct product of cyclic groups of orders $n_1, n_2, \ldots, n_t$, where n_{i+1} divides n_i for $i = 1, 2, \ldots, t - 1$. (This exercise is referred to in this chapter.)

12. Suppose that the order of some finite Abelian group is divisible by 10. Prove that the group has a cyclic subgroup of order 10.

13. Show, by example, that if the order of a finite Abelian group is divisible by 4, the group need not have a cyclic subgroup of order 4.

14. On the basis of Exercises 12 and 13, draw a general conclusion about the existence of cyclic subgroups of a finite Abelian group.

15. How many Abelian groups (up to isomorphism) are there
a. of order 6?
b. of order 15?
c. of order 42?
d. of order pq, where p and q are distinct primes?
e. of order pqr, where p, q, and r are distinct primes?
f. Generalize parts d and e.

16. How does the number (up to isomorphism) of Abelian groups of order n compare with the number (up to isomorphism) of Abelian groups of order m where
a. $n = 3^2$ and $m = 5^2$?
b. $n = 2^4$ and $m = 5^4$?
c. $n = p^r$ and $m = q^r$, where p and q are prime?
d. $n = p^r$ and $m = p^r q$, where p and q are distinct primes?
e. $n = p^r$ and $m = p^r q^2$, where p and q are distinct primes?

17. Up to isomorphism, how many additive Abelian groups of order 16 have the property that $x + x + x + x = 0$ for all x in the group?

18. Let $p_1, p_2, \ldots, p_n$ be distinct primes. Up to isomorphism, how many Abelian groups are there of order ${p_1}^4{p_2}^4 \ldots {p_n}^4$?

19. The symmetry group of a nonsquare rectangle is an Abelian group of order 4. Is it isomorphic to Z_4 or $Z_2 \oplus Z_2$?

20. Verify the corollary to the Fundamental Theorem of Finite Abelian Groups in the case that the group has order 1080 and the divisor is 180.

21. The set $\{1, 9, 16, 22, 29, 53, 74, 79, 81\}$ is a group under multiplication modulo 91. Determine the isomorphism class of this group.

22. Suppose that G is a finite Abelian group that has exactly one subgroup for each divisor of $|G|$. Show that G is cyclic.

23. Characterize those integers n such that the only Abelian groups of order n are cyclic.

24. Characterize those integers n such that any Abelian group of order n belongs to one of exactly four isomorphism classes.

25. Refer to Example 1 in this chapter and explain why it is unnecessary to compute the orders of the last five elements listed to determine the isomorphism class of G.

26. Let $G = \{1, 7, 17, 23, 49, 55, 65, 71\}$ under multiplication modulo 96. Express G as an external and an internal direct product of cyclic groups.

27. Let $G = \{1, 7, 43, 49, 51, 57, 93, 99, 101, 107, 143, 149, 151, 157, 193, 199\}$ under multiplication modulo 200. Express G as an external and an internal direct product of cyclic groups.

28. The set $G = \{1, 4, 11, 14, 16, 19, 26, 29, 31, 34, 41, 44\}$ is a group under multiplication modulo 45. Write G as an external and an internal direct product of cyclic groups of prime-power order.

29. Determine the isomorphism class of $(Z_{16} \oplus Z_{16})/\langle(2,2)\rangle$.

30. Let G be the subgroup of $U(200)$ of order 16. Determine the isomorphism class of G.

31. Suppose that G is an Abelian group of order 9. What is the maximum number of elements (excluding the identity) of which one needs to compute the order to determine the isomorphism class of G? What if G has order 18? What about 16?

32. Suppose that G is an Abelian group of order 16, and in computing the orders of its elements, you come across an element of order 8 and two elements of order 2. Explain why no further computations are needed to determine the isomorphism class of G.

33. Let G be an Abelian group of order 16. Suppose that there are elements a and b in G such that $|a| = |b| = 4$ and $a^2 \neq b^2$. Determine the isomorphism class of G.

34. Prove that an Abelian group of order 2^n $(n \geq 1)$ must have an odd number of elements of order 2.

35. Without using Lagrange's Theorem, show that an Abelian group of odd order cannot have an element of even order.

36. Let G be the group of all $n \times n$ diagonal matrices with ± 1 diagonal entries. What is the isomorphism class of G?

37. Prove the assertion made in the proof of Lemma 2 that $G = \langle a \rangle K$.

38. Suppose that G is a finite Abelian group. Prove that G has order p^n, where p is prime, if and only if the order of every element of G is a power of p.

39. Dirichlet's Theorem says that, for every pair of relatively prime integers a and b, there are infinitely many primes of the form $at + b$. Use Dirichlet's Theorem to prove that every finite Abelian group is isomorphic to a subgroup of a U-group.

40. Determine the isomorphism class of $\mathrm{Aut}(Z_2 \oplus Z_3 \oplus Z_5)$.

41. Give an example to show that Lemma 2 is false if G is non-Abelian.

42. Prove that an Abelian group of order 2^n is cyclic if and only if it it has exactly one element of order 2. Generalize to the case of an Abelian group of order p^n where p is prime.

43. Prove that every finite Abelian group of order at least 3 has a non-trivial automorphism.

44. Recall the exponent of a finite group G is the smallest positive integer n such that $x^n = e$ for all x in G. Prove that if G is a finite Abelian group the exponent of G is the largest order of any element in G. Give an example to show that this is not true for finite non-Abelian groups.

45. If G is a finite Abelian group such that number of solutions in G of $x^n = e$ is at most n for all positive integers n, prove that G is cyclic.

46. If H is a subgroup of a finite Abelian group of even order that contains all elements in G of even order, prove that $H = G$.

47. Verify the assertion made after Example 2 that the isomorphism class of the group in Example 2 can be determined by calculating only the orders of 8, 12, 18, 21 and 27.

48. Give an isomorphism class characterization of all finite Abelian groups with the property that every subgroup of the group is cyclic.

Computer Exercises

Computer exercises for this chapter are available at the website:

http://www.d.umn.edu/~jgallian

12 Introduction to Rings

Example is the school of mankind, and they will learn at no other.

Edmund Burke, *On a Regicide Peace*

. . . the source of all great mathematics is the special case, the concrete example. It is frequent in mathematics that every instance of a concept of seemingly great generality is in essence the same as a small and concrete special case.

Paul R. Halmos, *I Want to be a Mathematician*

Motivation and Definition

Many sets are naturally endowed with two binary operations: addition and multiplication. Examples that quickly come to mind are the integers, the integers modulo n, the real numbers, matrices, and polynomials. When considering these sets as groups, we simply used addition and ignored multiplication. In many instances, however, one wishes to take into account both addition and multiplication. One abstract concept that does this is the concept of a ring.[1] This notion was originated in the mid-19th century by Richard Dedekind, although its first formal abstract definition was not given until Abraham Fraenkel presented it in 1914.

Definition Ring

A *ring* R is a set with two binary operations, addition (denoted by $a + b$) and multiplication (denoted by ab), such that for all a, b, c in R:

1. $a + b = b + a$.

2. $(a + b) + c = a + (b + c)$.

3. There is an additive identity 0. That is, there is an element 0 in R such that $a + 0 = a$ for all a in R.

4. There is an element $-a$ in R such that $a + (-a) = 0$.

[1] The term *ring* was first applied in 1897 by the German mathematician David Hilbert (1862–1943).

5. $\boldsymbol{a(bc) = (ab)c}$.
6. $\boldsymbol{a(b + c) = ab + ac}$ **and** $\boldsymbol{(b + c)a = ba + ca}$.

So, a ring is an Abelian group under addition, also having an associative multiplication that is left and right distributive over addition. Note that multiplication need not be commutative. When it is, we say that the ring is *commutative*. Also, a ring need not have an identity under multiplication. A *unity* (or *identity*) in a ring is a nonzero element that is an identity under multiplication. A nonzero element of a commutative ring with unity need not have a multiplicative inverse. When it does, we say that it is a *unit* of the ring. Thus, a is a unit if a^{-1} exists.

The following terminology and notation are convenient. If a and b belong to a commutative ring R and a is nonzero, we say that a *divides* b (or that a is a *factor* of b) and write $a \mid b$, if there exists an element c in R such that $b = ac$. If a does not divide b, we write $a \nmid b$.

Recall that if a is an element from a group under the operation of addition and n is a positive integer, na means $a + a + \cdots + a$, where there are n summands. When dealing with rings, this notation can cause confusion, since we also use juxtaposition for the ring multiplication. When there is the potential for confusion, we will use $n \cdot a$ to mean $a + a + \cdots + a$ (n summands).

For an abstraction to be worthy of study, it must have many diverse concrete realizations. The following list of examples shows that the ring concept is pervasive.

Examples of Rings

▮ **EXAMPLE 1** The set Z of integers under ordinary addition and multiplication is a commutative ring with unity 1. The units of Z are 1 and -1. ▮

▮ **EXAMPLE 2** The set $Z_n = \{0, 1, \ldots, n - 1\}$ under addition and multiplication modulo n is a commutative ring with unity 1. The set of units is $U(n)$. ▮

▮ **EXAMPLE 3** The set $Z[x]$ of all polynomials in the variable x with integer coefficients under ordinary addition and multiplication is a commutative ring with unity $f(x) = 1$. ▮

EXAMPLE 4 The set $M_2(Z)$ of 2×2 matrices with integer entries is a noncommutative ring with unity $\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. The units of $M_2(Z)$ are the elements of A with $\det A = \pm 1$. (See Exercise 63.)

EXAMPLE 5 The set 2Z of even integers under ordinary addition and multiplication is a commutative ring without unity.

EXAMPLE 6 The set of all continuous real-valued functions of a real variable whose graphs pass through the point (1, 0) is a commutative ring without unity under the operations of pointwise addition and multiplication [i.e., the operations $(f + g)(a) = f(a) + g(a)$ and $(fg)(a) = f(a)g(a)$].

EXAMPLE 7 Let $R_1, R_2, \ldots, R_n$ be rings. We can use these to construct a new ring as follows. Let

$$R_1 \oplus R_2 \oplus \cdots \oplus R_n = \{(a_1, a_2, \ldots, a_n) \mid a_i \in R_i\}$$

and perform componentwise addition and multiplication; that is, define

$$(a_1, a_2, \ldots, a_n) + (b_1, b_2, \ldots, b_n) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and

$$(a_1, a_2, \ldots, a_n)(b_1, b_2, \ldots, b_n) = (a_1b_1, a_2b_2, \ldots, a_nb_n).$$

This ring is called the *direct sum* of $R_1, R_2, \ldots, R_n$.

Properties of Rings

Our first theorem shows how the operations of addition and multiplication intertwine. We use $b - c$ to denote $b + (-c)$.

Theorem 12.1 Rules of Multiplication

Let a, b, and c belong to a ring R. Then
1. $a0 = 0a = 0$.
2. $a(-b) = (-a)b = -(ab)$.
3. $(-a)(-b) = ab$.
4. $a(b - c) = ab - ac$ *and* $(b - c)a = ba - ca$.

Furthermore, if R has a unity element 1, then
5. $(-1)a = -a$.
6. $(-1)(-1) = 1$.

PROOF We will prove rules 1 and 2 and leave the rest as easy exercises (see Exercise 11). To prove statements such as those in Theorem 12.1, we need only "play off" the distributive property against the fact that R is a group under addition with additive identity 0. Consider rule 1. Clearly,

$$0 + a0 = a0 = a(0 + 0) = a0 + a0.$$

So, by cancellation, $0 = a0$. Similarly, $0a = 0$.

To prove rule 2, we observe that $a(-b) + ab = a(-b + b) = a0 = 0$. So, adding $-(ab)$ to both sides yields $a(-b) = -(ab)$. The remainder of rule 2 is done analogously. ∎

Recall that in the case of groups, the identity and inverses are unique. The same is true for rings, provided that these elements exist. The proofs are identical to the ones given for groups and therefore are omitted.

Theorem 12.2 Uniqueness of the Unity and Inverses

If a ring has a unity, it is unique. If a ring element has a multiplicative inverse, it is unique.

Many students have the mistaken tendency to treat a ring as if it were a group under *multiplication*. It is not. The two most common errors are the assumptions that ring elements have multiplicative inverses—they need not—and that a ring has a multiplicative identity—it need not. For example, if a, b, and c belong to a ring, $a \neq 0$ and $ab = ac$, we *cannot* conclude that $b = c$. Similarly, if $a^2 = a$, we *cannot* conclude that $a = 0$ or 1 (as is the case with real numbers). In the first place, the ring need not have multiplicative cancellation, and in the second place, the ring need not have a multiplicative identity. There is an important class of rings that contains Z and $Z[x]$ wherein multiplicative identities exist and for which multiplicative cancellation holds. This class is taken up in the next chapter.

Subrings

In our study of groups, subgroups played a crucial role. Subrings, the analogous structures in ring theory, play a much less prominent role than their counterparts in group theory. Nevertheless, subrings are important.

Definition Subring
A subset S of a ring R is a *subring of R* if S is itself a ring with the operations of R.

Just as was the case for subgroups, there is a simple test for subrings.

Theorem 12.3 Subring Test

A nonempty subset S of a ring R is a subring if S is closed under subtraction and multiplication—that is, if $a - b$ and ab are in S whenever a and b are in S.

PROOF Since addition in R is commutative and S is closed under subtraction, we know by the One-Step Subgroup Test (Theorem 3.1) that S is an Abelian group under addition. Also, since multiplication in R is associative as well as distributive over addition, the same is true for multiplication in S. Thus, the only condition remaining to be checked is that multiplication is a binary operation on S. But this is exactly what closure means. ∎

We leave it to the student to confirm that each of the following examples is a subring.

EXAMPLE 8 $\{0\}$ and R are subrings of any ring R. $\{0\}$ is called the *trivial* subring of R. ∎

EXAMPLE 9 $\{0, 2, 4\}$ is a subring of the ring Z_6, the integers modulo 6. Note that although 1 is the unity in Z_6, 4 is the unity in $\{0, 2, 4\}$. ∎

EXAMPLE 10 For each positive integer n, the set

$$nZ = \{0, \pm n, \pm 2n, \pm 3n, \ldots\}$$

is a subring of the integers Z. ∎

EXAMPLE 11 The set of Gaussian integers

$$Z[i] = \{a + bi \mid a, b \in Z\}$$

is a subring of the complex numbers **C**. ∎

EXAMPLE 12 Let R be the ring of all real-valued functions of a single real variable under pointwise addition and multiplication. The subset S of R of functions whose graphs pass through the origin forms a subring of R. ∎

EXAMPLE 13 The set

$$\left\{ \begin{bmatrix} a & 0 \\ 0 & b \end{bmatrix} \middle| a, b \in Z \right\}$$

of diagonal matrices is a subring of the ring of all 2×2 matrices over Z. ∎

We can picture the relationship between a ring and its various subrings by way of a subring lattice diagram. In such a diagram, any ring is a subring of all the rings that it is connected to by one or more upward lines. Figure 12.1 shows the relationships among some of the rings we have already discussed.

In the next several chapters, we will see that many of the fundamental concepts of group theory can be naturally extended to rings. In particular, we will introduce ring homomorphisms and factor rings.

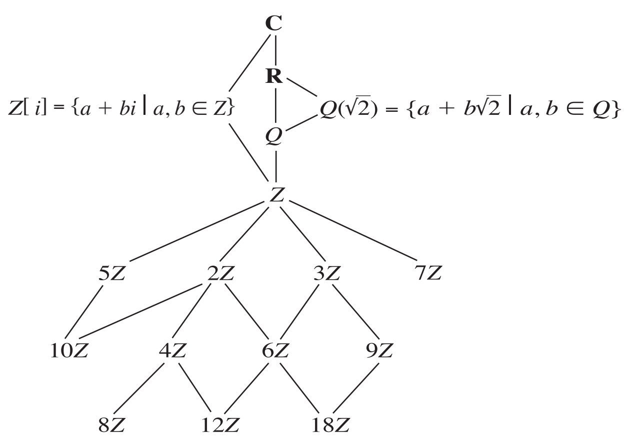


Figure 12.1 Partial subring lattice diagram of **C**.

In the next several chapters, we will see that many of the fundamental concepts of group theory can be naturally extended to rings. In particular, we will introduce ring homomorphisms and factors rings.

Exercises

There is no substitute for hard work.

Thomas Alva Edison, *Life*

1. Give an example of a finite noncommutative ring. Give an example of an infinite noncommutative ring that does not have a unity.

2. Show that the subring $R = \{0, 2, 4, 6, 8\}$ of Z_{10} has a unity and every nonzero element is a unit of R. Note that none of these elements is a unit in Z_{10}.

3. Give an example of a subset of a ring that is a subgroup under addition but not a subring.

4. Show, by example, that for fixed nonzero elements a and b in a ring, the equation $ax = b$ can have more than one solution. How does this compare with groups?

5. Which of the following subsets of $Z[x]$ are subrings?
a. All elements of $Z[x]$ that have all coefficients even.
b. $\{f(x) \in Z[x] \mid f'(0) = 0\}$. ($f'(x)$ is the derviative of $f(x)$).
c. All elements of $Z[x]$ whose coefficient of x^2 is 0.

6. Find an integer n that shows that the rings Z_n need not have the following properties that the ring of integers has.
a. $a^2 = a$ implies $a = 0$ or $a = 1$.
b. $ab = 0$ implies $a = 0$ or $b = 0$.
c. $ab = ac$ and $a \neq 0$ imply $b = c$.
Is the n you found prime?

7. Show that the three properties listed in Exercise 6 are valid for Z_p, where p is prime.

8. Show that a ring is commutative if it has the property that $ab = ca$ implies $b = c$ when $a \neq 0$. ("Outer cancellation implies commutativity.") Show the same is true if $ab = bc$ implies $a = c$ when $b \neq 0$. ("Inner cancellation implies commutativity.")

9. Prove that the intersection of any collection of subrings of a ring R is a subring of R.

10. Verify that Examples 8 through 13 in this chapter are as stated.

11. Prove rules 3 through 6 of Theorem 12.1.

12. Let a, b, and c be elements of a commutative ring, and suppose that a is a unit. Prove that b divides c if and only if ab divides c.

13. Describe all the subrings of the ring of integers.

14. Let a and b belong to a ring R and let m be an integer. Prove that $m \cdot (ab) = (m \cdot a)b = a(m \cdot b)$.

15. Show that if m and n are integers and a and b are elements from a ring, then $(m \cdot a)(n \cdot b) = (mn) \cdot (ab)$. (This exercise is referred to in Chapters 13 and 15.)

16. Show that if n is an integer and a is an element from a ring, then $n \cdot (-a) = -(n \cdot a)$.

17. Show that a ring that is cyclic under addition is commutative.

18. Let r be a fixed element of a ring R. Let $S = \{x \in R \mid rx = 0\}$. Show that S is a subring of R.

19. Let R be a ring. The *center of* R is the set $\{x \in R \mid ax = xa$ for all a in $R\}$. Prove that the center of a ring is a subring.

20. Find a strictly descending series of subrings $R_0 \subset R_1 \subset R_2 \cdots$ such that $R_0 = Z$ and $\cap_{i=0}^{\infty} R_i = \{0\}$.

21. Suppose that $R_1, R_2, \ldots, R_n$ are rings that contain nonzero elements. Show that $R_1 \oplus R_2 \ldots \oplus R_n$ has a unity if and only if each R_i has a unity.

22. Let R be a commutative ring with unity and let $U(R)$ denote the set of units of R. Prove that $U(R)$ is a group under the multiplication of R. (This group is called the *group of units of* R.)

23. Determine $U(Z[i])$ (see Example 11).

24. If $R_1, R_2, \ldots, R_n$ are commutative rings with unity, show that $U(R_1 \oplus R_2 \oplus \cdots \oplus R_n) = U(R_1) \oplus U(R_2) \oplus \cdots \oplus U(R_n)$.

25. Determine $U(Z[x])$. (This exercise is referred to in Chapter 17.)

26. Determine $U(\mathbf{R}[x])$.

27. Show that a unit of a ring divides every element of the ring.

28. In Z_6, show that $4 \mid 2$; in Z_8, show that $3 \mid 7$; in Z_{15}, show that $9 \mid 12$.

29. Suppose that a and b belong to a commutative ring R with unity. If a is a unit of R and $b^2 = 0$, show that $a + b$ is a unit of R.

30. Let a belong to a ring R with unity and suppose that $a^n = 0$ for some positive integer n. (Such an element is called *nilpotent*.) Prove that $1 - a$ has a multiplicative inverse R. [*Hint:* Consider $(1 - a)(1 + a + a^2 + \cdots + a^{n-1})$.]

31. Show that the nilpotent elements of a commutative ring form a subring.

32. Suppose that there is an integer $n > 1$ such that $x^n = x$ for all elements x of some ring. If m is a positive integer and $a^m = 0$ for some a, show that $a = 0$.

33. Give an example of ring elements a and b with the properties that $ab = 0$ but $ba \neq 0$.

34. Let n be an integer greater than 1. In a ring in which $x^n = x$, for all x, show that $ab = 0$ implies $ba = 0$.

35. Prove that for any integer $n > 1$, the set $R = \left\{ \begin{bmatrix} 0 & a \\ 0 & b \end{bmatrix} \,\middle|\, a, b \in Z \right\}$ is a non-commutative ring of order n^2. Generalize to non-commutative rings of order n^m for $m > 1$.

36. Suppose that R is a ring such that $x^3 = x$ for all x in R.
a. Prove that $6x = 0$ for all x in R.
b. Determine all positive integers n such that $x^3 = x$ for all x in Z_n.

37. Suppose that R is a ring such that $x^4 = x$ for all x in R. Prove that $2x = 0$ for all x in R.

38. Suppose that a belongs to a ring and $a^4 = a^2$. Prove that $a^{2n} = a^2$ for all $n \geq 1$.

39. Find an integer $n > 1$ such that $a^n = a$ for all a in Z_6. Do the same for Z_{10}. Show that no such n exists for Z_m when m is divisible by the square of some prime.

40. Let m and n be positive integers and let k be the least common multiple of m and n. Show that $mZ \cap nZ = kZ$.

41. Explain why every subgroup of Z_n under addition is also a subring of Z_n.

42. Is Z_6 a subring of Z_{12}?

43. Suppose that R is a ring with unity 1 and a is an element of R such that $a^2 = 1$. Let $S = \{ara \mid r \in R\}$. Prove that S is a subring of R. Does S contain 1?

44. Let $M_2(Z)$ be the ring of all 2×2 matrices over the integers and let $R = \left\{ \begin{bmatrix} a & a+b \\ a+b & b \end{bmatrix} \mid a, b \in Z \right\}$. Prove or disprove that R is a subring of $M_2(Z)$.

45. Let $M_2(Z)$ be the ring of all 2×2 matrices over the integers and let $R = \left\{ \begin{bmatrix} a & a-b \\ a-b & b \end{bmatrix} \mid a, b \in Z \right\}$. Prove or disprove that R is a subring of $M_2(Z)$.

46. Let $M_2(Z)$ be the ring of all 2×2 matrices over the integers and let $R = \left\{ \begin{bmatrix} a & a \\ b & b \end{bmatrix} \mid a, b \in Z \right\}$. Prove or disprove that R is a subring of $M_2(Z)$.

47. Prove or disprove that $R = \left\{ \left[\begin{array}{cc} 0 & a \\ 0 & 0 \end{array} \right] \middle| \ a \in Z \right\}$ and $S = \left\{ \left[\begin{array}{cc} 0 & 0 \\ a & 0 \end{array} \right] \middle| \ a \in Z \right\}$ are subrings of $M_2(Z)$. What is unusual about about R and S?

48. Prove or disprove that $R = \left\{ \left[\begin{array}{cc} a & 0 \\ 0 & 0 \end{array} \right] \middle| \ a \in Z \right\}$ and $S = \left\{ \left[\begin{array}{cc} 0 & 0 \\ 0 & a \end{array} \right] \middle| \ a \in Z \right\}$ are subrings of $M_2(Z)$.

49. Let $R = Z \oplus Z \oplus Z$ and $S = \{(a, b, c) \in R \mid a + b = c\}$. Prove or disprove that S is a subring of R.

50. Suppose that there is a positive even integer n such that $a^n = a$ for all elements a of some ring. Show that $-a = a$ for all a in the ring.

51. Let R be a ring with unity 1. Show that $S = \{n \cdot 1 \mid n \in Z\}$ is a subring of R.

52. Show that $2Z \cup 3Z$ is not a subring of Z.

53. Determine the smallest subring of Q that contains $2/3$. (That is, find the subring S with the property that S contains $2/3$ and, if T is any subring containing $2/3$, then T contains S.)

54. Let a, b, and c belong to a ring with unity. If a is a unit and $a = bc$, prove that b and c are units.

55. Let R be a ring. Prove that $a^2 - b^2 = (a + b)(a - b)$ for all a, b in R if and only if R is commutative.

56. Suppose that R is a ring and that $a^2 = a$ for all a in R. Show that R is commutative. [A ring in which $a^2 = a$ for all a is called a *Boolean* ring, in honor of the English mathematician George Boole] (1815–1864).]

57. Give an example of a Boolean ring with four elements. Give an example of an infinite Boolean ring.

58. If a, b, and c are elements of a ring, does the equation $ax + b = c$ always have a solution x? If it does, must the solution be unique? Answer the same questions given that a is a unit.

59. Let R and S be commutative rings. Prove that (a, b) is a zero-divisor in $R \oplus S$ if and only if a or b is a zero-divisor or exactly one of a or b is 0.

60. Show that $4x^2 + 6x + 3$ is a unit in $Z_8[x]$.

61. Let R be a commutative ring with more than one element. Prove that if for every nonzero element a of R we have $aR = R$, then R has a unity and every nonzero element has an inverse.

62. Find an example of a commutative ring R with unity such that $a, b \in R, a \neq b, a^n = b^n$, and $a^m = b^m$, where n and m are positive integers that are relatively prime. (Compare with Exercise 39, part b, in Chapter 13.)

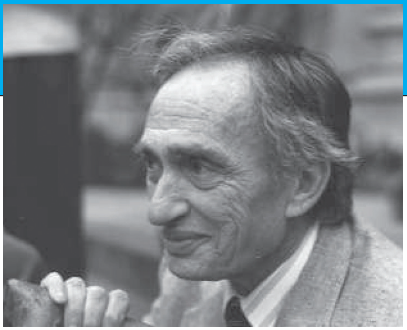
63. Prove that an element A in $M_2(Z)$ is a unit if and only if $\det A = \pm 1$.

Computer Exercises

Software for the computer exercises in this chapter is available at the website:

http://www.d.umn.edu/~jgallian

I. N. Herstein



Source: Archives of the Mathematisches

A whole generation of textbooks and an entire generation of mathematicians, myself included, have been profoundly influenced by that text [Herstein's *Topics in Algebra*].

GEORGIA BENKART

I. N. HERSTEIN was born on March 28, 1923, in Poland. His family moved to Canada when he was seven. He grew up in a poor and tough environment, on which he commented that in his neighborhood you became either a gangster or a college professor. During his school years he played football, hockey, golf, tennis, and pool. During this time he worked as a steeplejack and as a barber at a fair. Herstein received a B.S. degree from the University of Manitoba, an M.A. from the University of Toronto, and, in 1948, a Ph.D. degree from Indiana University under the supervision of Max Zorn. Before permanently settling at the University of Chicago in 1962, he held positions at the University of Kansas, the Ohio State University, the University of Pennsylvania, and Cornell University. Herstein wrote more than 100 research papers and a dozen books. Although his principal interest was noncommutative ring theory, he also wrote papers on finite groups, linear algebra, and mathematical economics. His textbook *Topics in Algebra,* first published in 1964, dominated the field for 20 years and has become a classic. Herstein had great influence through his teaching and his collaboration with colleagues. He had 30 Ph.D. students, and traveled and lectured widely. His nonmathematical interests included languages and art. He spoke Italian, Hebrew, Polish, and Portuguese. Herstein died on February 9, 1988, after a long battle with cancer.

13 Integral Domains

Don't just read it! Ask your own questions, look for your own examples, discover your own proofs. Is the hypothesis necessary? Is the converse true? What happens in the classical special case? Where does the proof use the hypothesis?

Paul Halmos

For those with eyes to see - groups, rings and fields are everywhere.

Hendrik W. Lenstra, Jr

Definition and Examples

To a certain degree, the notion of a ring was invented in an attempt to put the algebraic properties of the integers into an abstract setting. A ring is not the appropriate abstraction of the integers, however, for too much is lost in the process. Besides the two obvious properties of commutativity and existence of a unity, there is one other essential feature of the integers that rings in general do not enjoy—the cancellation property. In this chapter, we introduce integral domains—a particular class of rings that have all three of these properties. Integral domains play a prominent role in number theory and algebraic geometry.

Definition Zero-Divisors

A *zero-divisor* is a nonzero element a of a commutative ring R such that there is a nonzero element $b \in R$ with $ab = 0$.

Definition Integral Domain

An *integral domain* is a commutative ring with unity and no zero-divisors.

Thus, in an integral domain, a product is 0 only when one of the factors is 0; that is, $ab = 0$ only when $a = 0$ or $b = 0$. The following examples show that many familiar rings are integral domains and some familiar rings are not. For each example, the student should verify the assertion made.

EXAMPLE 1 The ring of integers is an integral domain. ∎

EXAMPLE 2 The ring of Gaussian integers $Z[i] = \{a + bi \mid a, b \in Z\}$ is an integral domain. ∎

EXAMPLE 3 The ring $Z[x]$ of polynomials with integer coefficients is an integral domain. ∎

EXAMPLE 4 The ring $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$ is an integral domain. ∎

EXAMPLE 5 The ring Z_p of integers modulo a prime p is an integral domain. ∎

EXAMPLE 6 The ring Z_n of integers modulo n is *not* an integral domain when n is not prime. ∎

EXAMPLE 7 The ring $M_2(Z)$ of 2×2 matrices over the integers is *not* an integral domain. ∎

EXAMPLE 8 $Z \oplus Z$ is *not* an integral domain. ∎

What makes integral domains particularly appealing is that they have an important multiplicative group theoretic property, in spite of the fact that the nonzero elements need not form a group under multiplication. This property is cancellation.

Theorem 13.1 Cancellation

Let a, b, and c belong to an integral domain. If $a \neq 0$ and $ab = ac$, then $b = c$.

PROOF From $ab = ac$, we have $a(b - c) = 0$. Since $a \neq 0$, we must have $b - c = 0$. ∎

Many authors prefer to define integral domains by the cancellation property—that is, as commutative rings with unity in which the cancellation property holds. This definition is equivalent to ours.

Fields

In many applications, a particular kind of integral domain called a *field* is necessary.

Definition Field
A *field* is a commutative ring with unity in which every nonzero element is a unit.

To verify that every field is an integral domain, observe that if a and b belong to a field with $a \neq 0$ and $ab = 0$, we can multiply both sides of the last expression by a^{-1} to obtain $b = 0$.

It is often helpful to think of ab^{-1} as a divided by b. With this in mind, a field can be thought of as simply an algebraic system that is closed under addition, subtraction, multiplication, and division (except by 0). We have had numerous examples of fields: the complex numbers, the real numbers, the rational numbers. The abstract theory of fields was initiated by Heinrich Weber in 1893. Groups, rings, and fields are the three main branches of abstract algebra. Theorem 13.2 says that, in the finite case, fields and integral domains are the same.

Theorem 13.2 Finite Integral Domains Are Fields

A finite integral domain is a field.

PROOF Let D be a finite integral domain with unity 1. Let a be any nonzero element of D. We must show that a is a unit. If $a = 1$, a is its own inverse, so we may assume that $a \neq 1$. Now consider the following sequence of elements of $D : a, a^2, a^3, \ldots$ Since D is finite, there must be two positive integers i and j such that $i > j$ and $a^i = a^j$. Then, by cancellation, $a^{i-j} = 1$. Since $a \neq 1$, we know that $i - j > 1$, and we have shown that a^{i-j-1} is the inverse of a. ∎

Corollary Z_p Is a Field

For every prime p, Z_p, the ring of integers modulo p is a field.

PROOF According to Theorem 13.2, we need only prove that Z_p has no zero-divisors. So, suppose that $a, b \in Z_p$ and $ab = 0$. Then $ab = pk$ for some integer k. But then, by Euclid's Lemma (see Chapter 0), p divides a or p divides b. Thus, in $Z_p, a = 0$ or $b = 0$. ∎

Putting the preceding corollary together with Example 6, we see that Z_n is a field if and only if n is prime. In Chapter 21, we will describe how all finite fields can be constructed. For now, we give one example of a finite field that is not of the form Z_p.

EXAMPLE 9 Field With Nine Elements
Let $Z_3[i] = \{a+bi \mid a, b \in Z_3\} = \{0, 1, 2, i, 1+i, 2+i, 2i, 1+2i, 2+2i\}$, where $i^2 = -1$. This is the ring of Gaussian integers modulo 3. Elements are added and multiplied as in the complex numbers, except that the coefficients are reduced modulo 3. In particular, $-1 = 2$. Table 13.1 is the multiplication table for the nonzero elements of $Z_3[i]$. Note that because 2 is the only element of order 2, $Z_3[i]^*$ is a cyclic group. ■

Table 13.1 Multiplication Table for $Z_3[i]^*$.

	$\mathbf{1}$	$\mathbf{2}$	$\boldsymbol{i}$	$\mathbf{1+i}$	$\mathbf{2+i}$	$\mathbf{2i}$	$\mathbf{1+2i}$	$\mathbf{2+2i}$
$\mathbf{1}$	1	2	i	$1+i$	$2+i$	$2i$	$1+2i$	$2+2i$
$\mathbf{2}$	2	1	$2i$	$2+2i$	$1+2i$	i	$2+i$	$1+i$
$\boldsymbol{i}$	i	$2i$	2	$2+i$	$2+2i$	1	$1+i$	$1+2i$
$\mathbf{1+i}$	$1+i$	$2+2i$	$2+i$	$2i$	1	$1+2i$	2	i
$\mathbf{2+i}$	$2+i$	$1+2i$	$2+2i$	1	i	$1+i$	$2i$	2
$\mathbf{2i}$	$2i$	i	1	$1+2i$	$1+i$	2	$2+2i$	$2+i$
$\mathbf{1+2i}$	$1+2i$	$2+i$	$1+i$	2	$2i$	$2+2i$	i	1
$\mathbf{2+2i}$	$2+2i$	$1+i$	$1+2i$	i	2	$2+i$	1	$2i$

EXAMPLE 10 Let $Z_5[i] = \{a+bi \mid a, b \in Z_5,\ i^2 = -1\}$. This ring has 25 elements but is not an integral domain because $(1+2i)(1-2i) = 1 - 4i^2 = 0$. (Compare with Exercises 43 and 61.) ■

EXAMPLE 11 Let $Q[\sqrt{2}] = \{a+b\sqrt{2} \mid a, b \in Q\}$. It is easy to see that $Q[\sqrt{2}]$ is a ring. Viewed as an element of $\mathbf{R}$, the multiplicative inverse of any nonzero element of the form $a+b\sqrt{2}$ is simply $1/(a+b\sqrt{2})$. To verify that $Q[\sqrt{2}]$ is a field, we must show that $1/(a+b\sqrt{2})$ can be written in the form $c+d\sqrt{2}$. In high school algebra, this process is called "rationalizing the denominator."

Specifically,

$$\frac{1}{a+b\sqrt{2}} = \frac{1}{a+b\sqrt{2}}\frac{a-b\sqrt{2}}{a-b\sqrt{2}} = \frac{a}{a^2-2b^2} - \frac{b}{a^2-2b^2}\sqrt{2}.$$

(Note that $a + b\sqrt{2} \neq 0$ guarantees that $a - b\sqrt{2} - 0$.) ∎

Characteristic of a Ring

Note that for any element x in $Z_3[i]$, we have $3x = x + x + x = 0$, since addition is done modulo 3. Similarly, in the subring $\{0, 3, 6, 9\}$ of Z_{12}, we have $4x = x + x + x + x = 0$ for all x. This observation motivates the following definition.

Definition Characteristic of a Ring
The *characteristic* of a ring R is the least positive integer n such that $nx = 0$ for all x in R. If no such integer exists, we say that R has characteristic 0. The characteristic of R is denoted by char R.

Thus, the ring of integers has characteristic 0, and Z_n has characteristic n. An infinite ring can have a nonzero characteristic. Indeed, the ring $Z_2[x]$ of all polynomials with coefficients in Z_2 has characteristic 2. (Addition and multiplication are done as for polynomials with ordinary integer coefficients except that the coefficients are reduced modulo 2.) When a ring has a unity, the task of determining the characteristic is simplified by Theorem 13.3.

Theorem 13.3 Characteristic of a Ring with Unity

Let R be a ring with unity 1. If 1 has infinite order under addition, then the characteristic of R is 0. If 1 has order n under addition, then the characteristic of R is n.

PROOF If 1 has infinite order, then there is no positive integer n such that $n \cdot 1 = 0$, so R has characteristic 0. Now suppose that 1 has additive order n. Then $n \cdot 1 = 0$, and n is the least positive integer with this property. So, for any x in R, we have

$$\begin{aligned} n \cdot x &= x + x + \cdots + x \,(n \text{ summands}) \\ &= 1x + 1x + \cdots + 1x \,(n \text{ summands}) \\ &= (1 + 1 + \cdots + 1)x \,(n \text{ summands}) \\ &= (n \cdot 1)x = 0x = 0. \end{aligned}$$

Thus, R has characteristic n. ■

In the case of an integral domain, the possibilities for the characteristic are severely limited.

Theorem 13.4 Characteristic of an Integral Domain

The characteristic of an integral domain is 0 or prime.

PROOF By Theorem 13.3, it suffices to show that if the additive order of 1 is finite, it must be prime. Suppose that 1 has order n and that $n = st$, where $1 \leq s, t \leq n$. Then, by Exercise 15 in Chapter 12,

$$0 = n \cdot 1 = (st) \cdot 1 = (s \cdot 1)(t \cdot 1).$$

So, $s \cdot 1 = 0$ or $t \cdot 1 = 0$. Since n is the least positive integer with the property that $n \cdot 1 = 0$, we must have $s = n$ or $t = n$. Thus, n is prime. ■

We conclude this chapter with a brief discussion of polynomials with coefficients from a ring—a topic we will consider in detail in later chapters. The existence of zero-divisors in a ring causes unusual results when one is finding zeros of polynomials with coefficients in the ring. Consider, for example, the equation $x^2 - 4x + 3 = 0$. In the integers, we could find all solutions by factoring

$$x^2 - 4x + 3 = (x - 3)(x - 1) = 0$$

and setting each factor equal to 0. But notice that when we say we can find *all* solutions in this manner, we are using the fact that the only way for a product to equal 0 is for one of the factors to be 0—that is, we are using the fact that Z is an integral domain. In Z_{12}, there are many pairs of nonzero elements whose products are $0 : 2 \cdot 6 = 0, 3 \cdot 4 = 0, 4 \cdot 6 = 0, 6 \cdot 8 = 0$, and so on. So, how do we find *all* solutions of $x^2 - 4x + 3 = 0$ in Z_{12}? The easiest way is simply to try every element! Upon doing so, we find four solutions: $x = 1, x = 3, x = 7$, and $x = 9$. Observe that we can find all solutions of $x^2 = 4x+3 = 0$ over Z_{11} or Z_{13}, say, by setting the two factors $x-3$ and $x-1$ equal to 0. Of course, the reason this works for these rings is that they are integral domains. Perhaps this will convince you that integral domains are particularly advantageous rings. Table 13.2 gives a summary of some of the rings we have introduced and their properties.

Table 13.2 Summary of Rings and Their Properties.

Ring	Form of Element	Unity	Commutative	Integral Domain	Field	Characteristic
Z	k	1	Yes	Yes	No	0
Z_n, n composite	k	1	Yes	No	No	n
Z_p, p prime	k	1	Yes	Yes	Yes	p
$Z[x]$	$a_nx^n + \cdots + a_1x + a_0$	$f(x) = 1$	Yes	Yes	No	0
$nZ, n > 1$	nk	None	Yes	No	No	0
$M_2(Z)$	$\begin{bmatrix} a & b \\ c & d \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$	No	No	No	0
$M_2(2Z)$	$\begin{bmatrix} 2a & 2b \\ 2c & 2d \end{bmatrix}$	None	No	No	No	0
$Z[i]$	$a + bi$	1	Yes	Yes	No	0
$Z_3[i]$	$a + bi; a, b \in Z_3$	1	Yes	Yes	Yes	3
$Z_5[i]$	$a + bi; a, b \in Z_5$	1	Yes	No	No	5
$Z[\sqrt{2}]$	$a + b\sqrt{2}; a, b \in Z$	1	Yes	Yes	No	0
$Q[\sqrt{2}]$	$a + b\sqrt{2}; a, b \in Q$	1	Yes	Yes	Yes	0
$Z \oplus Z$	(a, b)	(1, 1)	Yes	No	No	0

Exercises

It looked absolutely impossible. But it so happens that you go on worrying away at a problem in science and it seems to get tired, and lies down and lets you catch it.

William Lawrence Bragg[1]

1. Verify that Examples 1 through 8 are as claimed.

2. Which of Examples 1 through 5 are fields?

3. Show that a commutative ring with the cancellation property (under multiplication) has no zero-divisors.

4. List all zero-divisors in Z_{20}. Can you see a relationship between the zero-divisors of Z_{20} and the units of Z_{20}?

[1] Bragg, at age 24, won the Nobel Prize for the invention of x-ray crystallography. He remains the youngest person ever to receive the Nobel Prize.

5. Show that every nonzero element of Z_n is a unit or a zero-divisor.

6. Find a nonzero element in a ring that is neither a zero-divisor nor a unit.

7. Let R be a finite commutative ring with unity. Prove that every nonzero element of R is either a zero-divisor or a unit. What happens if we drop the "finite" condition on R?

8. Let $a \neq 0$ belong to a commutative ring. Prove that a is a zero-divisor if and only if $a^2b = 0$ for some $b \neq 0$.

9. Find elements a, b, and c in the ring $Z \oplus Z \oplus Z$ such that ab, ac, and bc are zero-divisors but abc is not a zero-divisor.

10. Describe all zero-divisors and units of $Z \oplus Q \oplus Z$.

11. Let d be an integer. Prove that $Z[\sqrt{d}] = \{a+b\sqrt{d} \mid a, b \in Z\}$ is an integral domain. (This exercise is referred to in Chapter 18.)

12. In Z_7, give a reasonable interpretation for the expressions $1/2, -2/3, \sqrt{-3}$, and $-1/6$.

13. Give an example of a commutative ring without zero-divisors that is not an integral domain.

14. Find two elements a and b in a ring such that both a and b are zero-divisors, $a + b \neq 0$, and $a + b$ is not a zero-divisor.

15. Let a belong to a ring R with unity and suppose that $a^n = 0$ for some positive integer n. (Such an element is called *nilpotent*.) Prove that $1 - a$ has a multiplicative inverse in R. [*Hint:* Consider $(1 - a)(1 + a + a^2 + \cdots + a^{n-1})$.]

16. Show that the nilpotent elements of a commutative ring form a subring.

17. Show that 0 is the only nilpotent element in an integral domain.

18. A ring element a is called an *idempotent* if $a^2 = a$. Prove that the only idempotents in an integral domain are 0 and 1.

19. Let a and b be idempotents in a commutative ring. Show that each of the following is also an idempotent: $ab, a - ab, a + b - ab, a + b - 2ab$. If a is an element in a ring with the property that $a^2 = -a$, what can you say about $-a$?

20. Show that Z_n has a nonzero nilpotent element if and only if n is divisible by the square of some prime.

21. Let R be the ring of real-valued continuous functions on $[-1, 1]$. Show that R has zero-divisors.

22. Prove that if a is a ring idempotent, then $a^n = a$ for all positive integers n.

23. Determine all ring elements that are both nilpotent elements and idempotents.

24. Find an idempotent in $Z_5[i] = \{a + bi \mid a, b \in Z_5\}$.

25. Use Table 13.1 to find a generator for the cyclic group $Z_3[i]^*$.

26. Find all units, zero-divisors, idempotents, and nilpotent elements in $Z_3 \oplus Z_6$.

27. Determine all elements of a ring that are both units and idempotents.

28. Let R be the set of all real-valued functions defined for all real numbers under function addition and multiplication.
a. Determine all zero-divisors of R.
b. Determine all nilpotent elements of R.
c. Show that every nonzero element is a zero-divisor or a.

29. (Subfield Test) Let F be a field and let K be a subset of F with at least two elements. Prove that K is a subfield of F if, for any a, b $(b \neq 0)$ in $K, a - b$ and ab^{-1} belong to K.

30. Let d be a positive integer. Prove that $Q[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in Q\}$ is a field.

31. Let R be a ring with unity 1. If the product of any pair of nonzero elements of R is nonzero, prove that $ab = 1$ implies $ba = 1$.

32. Let $R = \{0, 2, 4, 6, 8\}$ under addition and multiplication modulo 10. Prove that R is a field.

33. Formulate the appropriate definition of a subdomain (i.e., a "sub" integral domain). Let D be an integral domain with unity 1. Show that $P = \{n \cdot 1 \mid n \in Z\}$ (i.e., all integral multiples of 1) is a subdomain of D. Show that P is contained in every subdomain of D. What can we say about the order of P?

34. Prove that there is no integral domain with exactly six elements. Can your argument be adapted to show that there is no integral domain with exactly four elements? What about 15 elements? Use these observations to guess a general result about the number of elements in a finite integral domain.

35. Let F be a field of order 2^n. Prove that char $F = 2$. Does your proof remain valid when 2 is replaced by any prime p?

36. Determine all elements of an integral domain that are their own inverses under multiplication.

37. Characterize those integral domains for which 1 is the only element that is its own multiplicative inverse.

38. Determine all integers $n > 1$ for which $(n - 1)!$ is a zero-divisor in Z_n.

39. Suppose that a and b belong to an integral domain.
a. If $a^5 = b^5$ and $a^3 = b^3$, prove that $a = b$.
b. If $a^m = b^m$ and $a^n = b^n$, where m and n are positive integers that are relatively prime, prove that $a = b$.

40. Find an example of an integral domain and distinct positive integers m and n such that $a^m = b^m$ and $a^n = b^n$, but $a \neq b$.

41. Let F be a field of order 32. Show that the only subfields of F are F itself and $\{0, 1\}$.

42. Construct a multiplication table for $Z_2[i]$, the ring of Gaussian integers modulo 2. Is this ring a field? Is it an integral domain?

43. Show that $Z_7[\sqrt{3}] = \{a + b\sqrt{3} \mid a, b \in Z_7\}$ is a field. For any positive integer k and any prime p, determine a necessary and sufficient condition for $Z_p[\sqrt{k}] = \{a + b\sqrt{k} \mid a, b \in Z_p\}$ to be a field.

44. The nonzero elements of $Z_3[i]$ form an Abelian group of order 8 under multiplication. Is it isomorphic to Z_8, $Z_4 \oplus Z_2$, or $Z_2 \oplus Z_2 \oplus Z_2$?

45. Show that a finite commutative ring with no zero-divisors and at least two elements has a unity.

46. Suppose that a and b belong to a commutative ring and ab is a zero-divisor. Show that either a or b is a zero-divisor.

47. Suppose that R is a commutative ring without zero-divisors. Show that all the nonzero elements of R have the same additive order.

48. Suppose that R is a commutative ring without zero-divisors. Show that the characteristic of R is 0 or prime.

49. Let $x_1, x_2, \ldots, x_n$ belong to a commutative ring with prime characteristic p.
a. Show that $(x_1 + x_2 + \cdots + x_n)^p = x_1{}^p + x_2{}^p + \cdots + x_n{}^p$.
b. Show that for all positive integers m, $(x_1 + x_2 + \cdots + x_n)^{p^m} = x_1{}^{p^m} + x_2{}^{p^m} + \cdots + x_n{}^{p^m}$.
c. Find elements x and y in a ring of characteristic 4 such that $(x + y)^4 \neq x^4 + y^4$. (This exercise is referred to in Chapter 19.)

50. Let R be a commutative ring with unity 1 and prime characteristic. If $a \in R$ is nilpotent, prove that there is a positive integer k such that $(1 + a)^k = 1$.

51. Show that any finite field has order p^n, where p is a prime. *Hint:* Use facts about finite Abelian groups. (This exercise is referred to in Chapter 21.)

52. Give an example of an infinite integral domain that has characteristic 3.

53. Let R be a ring and let $M_2(R)$ be the ring of 2×2 matrices with entries from R. Explain why these two rings have the same characteristic.

54. Let R be a ring with m elements. Show that the characteristic of R divides m.

55. Explain why a finite ring must have a nonzero characteristic.

56. Find all solutions of $x^2 - x + 2 = 0$ over $Z_3[i]$. (See Example 9.)

57. Consider the equation $x^2 - 5x + 6 = 0$.
a. How many solutions does this equation have in Z_7?
b. Find all solutions of this equation in Z_8.
c. Find all solutions of this equation in Z_{12}.
d. Find all solutions of this equation in Z_{14}.

58. If R is a ring of characteristic $m > 0$ and S is a subring of R, what can you say about the characteristic of S?

59. Suppose that R is an integral domain in which $20 \cdot 1 = 0$ and $12 \cdot 1 = 0$. (Recall that $n \cdot 1$ means the sum $1 + 1 + \cdots + 1$ with n terms.) What is the characteristic of R?

60. In a commutative ring of characteristic 2, prove that the idempotents form a subring.

61. For a prime p prove that if $x^2 + 1 = 0$ has a solution in Z_p, then $Z_p[i] = \{a + bi \mid a, b \in Z_p\}$ is not a field.

62. Prove that any idempotent in a commutative ring with unity other than 0 or 1 is a zero divisor.

63. Show that any field of order 16 has no subfield of order 8.

64. Let F be a finite field with n elements. Prove that $x^{n-1} = 1$ for all nonzero x in F.

65. Let F be a field of prime characteristic p. Prove that $K = \{x \in F \mid x^p = x\}$ is a subfield of F.

66. Prove that the set of complex numbers $Q(i) = \{a+bi \mid a, b \in Q\}$ is a field. This field is called the *Gaussian rationals*.

67. Let $R = Z_2 \oplus Z_3 \oplus Z_4 \oplus \cdots$. What is the identity of R? What is char R? Prove that for each positive integer n, R has a subring that has characteristic n.

68. If a is an idempotent in a commutative ring R prove that a is the identity in the ideal $\langle a \rangle$.

69. Let S be a subring of a ring R and suppose that u_S is a unity in S and u_R is a unity in R and $u_S \neq u_R$. Prove that u_S and $u_R - u_S$ are zero divisors in R.

70. If a, b belong to an integral domain prove that $\langle a \rangle = \langle b \rangle$ if and only if $a = bu$ where u is a unit.

71. Suppose that F is a field with 27 elements. Show that for every element $a \in F$, $5a = -a$.

72. Suppose that a and b belong to a field of order 8 and that $a^2 + ab + b^2 = 0$. Prove that $a = 0$ and $b = 0$. Do the same when the field has order 2^n with n odd.

73. Let F be a field of characteristic 2 with more than two elements. Show that $(x + y)^3 \neq x^3 + y^3$ for some x and y in F.

Computer Exercises

Computer exercises for this chapter are available at the website:

http://www.d.umn.edu/~jgallian

14 Ideals and Factor Rings

Abstractness, sometimes hurled as a reproach at mathematics, is its chief glory and its surest title to practical usefulness. It is also the source of such beauty as may spring from mathematics.

E. T. Bell

The secret of science is to ask the right questions, and it is the choice of problem more than anything else that marks the man of genius in the scientific world.

Sir Henry Tizard In C. P. Snow, A postscript to *Science and Government*

Ideals

Normal subgroups play a special role in group theory—they permit us to construct factor groups. In this chapter, we introduce the analogous concepts for rings—ideals and factor rings.

Definition Ideal
A subring A of a ring R is called a (two-sided) *ideal* of R if for every $r \in R$ and every $a \in A$ both ra and ar are in A.

So, a subring A of a ring R is an ideal of R if A "absorbs" elements from R—that is, if $rA = \{ra \mid a \in A\} \subseteq A$ and $Ar = \{ar \mid a \in A\} \subseteq A$ for all $r \in R$.

An ideal A of R is called a *proper* ideal of R if A is a proper subset of R. In practice, one identifies ideals with the following test, which is an immediate consequence of the definition of ideal and the subring test given in Theorem 12.3.

Theorem 14.1 Ideal Test

A nonempty subset A of a ring R is an ideal of R if
1. *$a - b \in A$ whenever $a, b \in A$.*
2. *ra and ar are in A whenever $a \in A$ and $r \in R$.*

■ **EXAMPLE 1** For any ring R, $\{0\}$ and R are ideals of R. The ideal $\{0\}$ is called the *trivial* ideal. ■

■ **EXAMPLE 2** For any positive integer n, the set $nZ = \{0, \pm n, \pm 2n, \ldots\}$ is an ideal of Z. ■

■ **EXAMPLE 3** Let R be a commutative ring with unity and let $a \in R$. The set $\langle a \rangle = \{ra \mid r \in R\}$ is an ideal of R called the *principal ideal generated by* a. (Notice that $\langle a \rangle$ is also the notation we used for the cyclic subgroup generated by a. However, the intended meaning will always be clear from the context.) The assumption that R is commutative is necessary in this example. ■

■ **EXAMPLE 4** Let $\mathbf{R}[x]$ denote the set of all polynomials with real coefficients and let A denote the subset of all polynomials with constant term 0. Then A is an ideal of $\mathbf{R}[x]$ and $A = \langle x \rangle$. ■

■ **EXAMPLE 5** Let R be a commutative ring with unity and let $a_1, a_2, \ldots, a_n$ belong to R. Then $I = \langle a_1, a_2, \ldots, a_n \rangle = \{r_1a_1 + r_2a_2 + \cdots + r_na_n \mid r_i \in R\}$ is an ideal of R called the *ideal generated by* $a_1, a_2, \ldots, a_n$. The verification that I is an ideal is left as an easy exercise (Exercise 3). ■

■ **EXAMPLE 6** Let $Z[x]$ denote the ring of all polynomials with integer coefficients and let I be the subset of $Z[x]$ of all polynomials with even constant terms. Then I is an ideal of $Z[x]$ and $I = \langle x, 2 \rangle$ (see Exercise 51). ■

■ **EXAMPLE 7** Let R be the ring of all real-valued functions of a real variable. The subset S of all differentiable functions is a subring of R but not an ideal of R. ■

Factor Rings

Let R be a ring and let A be an ideal of R. Since R is a group under addition and A is a normal subgroup of R, we may form the factor group $R/A = \{r + A \mid r \in R\}$. The natural question at this point is: How may we form a ring of this group of cosets? The addition is already taken care of, and, by analogy with groups of cosets, we define the product of two cosets of $s + A$ and $t + A$ as $st + A$. The next theorem shows that this definition works as long as A is an ideal of R, and not just a subring of R.

Theorem 14.2 Existence of Factor Rings

Let R be a ring and let A be a subring of R. The set of cosets $\{r + A \mid r \in R\}$ is a ring under the operations $(s + A) + (t + A) = s + t + A$ and $(s + A)(t + A) = st + A$ if and only if A is an ideal of R.

PROOF We know that the set of cosets forms a group under addition. Once we know that multiplication is indeed a binary operation on the cosets, it is trivial to check that the multiplication is associative and that multiplication is distributive over addition. Hence, the proof boils down to showing that multiplication is well-defined if and only if A is an ideal of R. To do this, let us suppose that A is an ideal and let $s + A = s' + A$ and $t + A = t' + A$. Then we must show that $st + A = s't' + A$. Well, by definition, $s = s' + a$ and $t = t' + b$, where a and b belong to A. Then

$$st = (s' + a)(t' + b) = s't' + at' + s'b + ab,$$

and so

$$st + A = s't' + at' + s'b + ab + A = s't' + A,$$

since A absorbs $at' + s'b + ab$. Thus, multiplication is well-defined when A is an ideal.

On the other hand, suppose that A is a subring of R that is not an ideal of R. Then there exist elements $a \in A$ and $r \in R$ such that $ar \notin A$ or $ra \notin A$. For convenience, say $ar \notin A$. Consider the elements $a + A = 0 + A$ and $r + A$. Clearly, $(a + A)(r + A) = ar + A$ but $(0 + A)(r + A) = 0 \cdot r + A = A$. Since $ar + A \neq A$, the multiplication is not well-defined and the set of cosets is not a ring.

Let's look at a few factor rings. ■

EXAMPLE 8 $Z/4Z = \{0 + 4Z, 1 + 4Z, 2 + 4Z, 3 + 4Z\}$. To see how to add and multiply, consider $2 + 4Z$ and $3 + 4Z$.

$$(2 + 4Z) + (3 + 4Z) = 5 + 4Z = 1 + 4 + 4Z = 1 + 4Z,$$
$$(2 + 4Z)(3 + 4Z) = 6 + 4Z = 2 + 4 + 4Z = 2 + 4Z.$$

One can readily see that the two operations are essentially modulo 4 arithmetic. ■

EXAMPLE 9 $2Z/6Z = \{0 + 6Z, 2 + 6Z, 4 + 6Z\}$. Here the operations are essentially modulo 6 arithmetic. For example, $(4+6Z)+(4+6Z) = 2+6Z$ and $(4+6Z)(4+6Z) = 4+6Z$. ▮

Here is a noncommutative example of an ideal and factor ring.

EXAMPLE 10 Let $R = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \middle|\, a_i \in Z \right\}$ and let I be the subset of R consisting of matrices with even entries. It is easy to show that I is indeed an ideal of R (Exercise 25). Consider the factor ring R/I. The interesting question about this ring is: What is its size? We claim R/I has 16 elements; in fact, $R/I = \left\{ \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} + I \,\middle|\, r_i \in \{0, 1\} \right\}$. An example illustrates the typical situation. Which of the 16 elements is $\begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix} + I$? Well, observe that $\begin{bmatrix} 7 & 8 \\ 5 & -3 \end{bmatrix} + I = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + \begin{bmatrix} 6 & 8 \\ 4 & -4 \end{bmatrix} + I = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} + I$, since an ideal absorbs its own elements. In essence, we may replace even entries with 0 and odd entries with 1. The general case is left to the reader (see Exercise 27.) ▮

Example 10 demonstrates how we can use the particular properties of an ideal I of a ring R to simplify the form of the elements of the factor ring R/I. Following are more examples.

EXAMPLE 11 Consider the factor ring $R = Z_3[x]/\langle x^2 + 1\rangle$. To simplify the notation let $I = \langle x^2 + 1\rangle$. By definition the elements of R have the form $f(x) + I$ where $f(x)$ is a polynomial with coefficients from Z_3 but the important question is: What do the *distinct* cosets of R look like? The fact that $x^2 + 1 + I = 0 + I$ means that when *dealing with coset representatives* we may treat x^2+1 as equivalent to 0 and therefore $x^2 = -1$. For example, the coset $2x^2+x+1+I = -2+x+1+I = x-1+I$. Moreover, when dealing with coset representatives we have, $x^2 = -1$ implies that $x^3 = -x$ and $x^4 = 1$. So, $x^4+2x^3+x^2+2+I = 1-2x-1+2+I = -2x+2+I = x+2+I$ (because $-2 = 1$ in Z_3). In the same way, we have $x^5 = x$, $x^6 = x^2x^4 = -1$ and so on. Thus we see that $R = \{ax + b + I \mid a, b \in Z_3\}$. This means that R has order 9. We can make one more simplification by suppressing the use of I and just refer to the coset $ax + b + I$ as $ax + b$. All we need to keep in mind is that when we preform the product

$(ax+b)(cx+d) = acx^2+(ad+bc)x+bd$ we replace acx^2 with $-ac$. We can now ask if any particular non-zero element of R is a unit or a zero-divisor. Consider $x + 1$. Note that $(x + 1)^2 = x^2 + 2x + 1 = 2x$. Then $(x+1)^4 = (2x)^2 = 4x^2 = -1$. So, $x+1$ is a unit and $|x + 1| = 8$. This means the eight non-zero elements of R form a cyclic group and R is a field of order 9. ∎

Just for fun let's look at the factor ring $R = Z_5[x]/\langle x^2 + 1\rangle$.

EXAMPLE 12 Consider the factor ring $R = Z_5[x]/\langle x^2+1\rangle$. This time $|R| = 25$ and $(x + 1)^4 = (2x)^2 = 4x^2 = -4 = 1$. So, $x + 1$ is a unit and $|x + 1| = 4$. In contrast, $(x + 2)(x + 3) = x^2 + 1 = 0$, So, $x + 2$ and $x + 3$ are zero-divisors and R is not a field. ∎

A simple explantion for the opposite behaviors in the two previous examples is given in Chapter 16.

EXAMPLE 13 Consider the factor ring of the Gaussian integers $R = Z[i]/\langle 2 - i\rangle$. As always, we must first determine what the distinct elements of the factor ring look like. This time the elements of R have the form $a + bi + \langle 2 - i\rangle$, where a and b are integers, and when dealing with coset representatives, we may treat $\langle 2 - i\rangle$ as equivalent to 0, so that $2 = i$. For example, the coset $3 + 4i + \langle 2 - i\rangle = 3 + 8 + \langle 2 - i\rangle = 11 + \langle 2 - i\rangle$. Similarly, all the elements of R can be written in the form $a+\langle 2-i\rangle$, where a is an integer. But we can further reduce the set of distinct coset representatives by observing that when dealing with coset representatives, $2 = i$ implies (by squaring both sides) that $4 = -1$ or $5 = 0$. Thus, the coset $3+4i+\langle 2-i\rangle = 11+\langle 2-i\rangle = 1+5+5+\langle 2-i\rangle = 1+\langle 2-i\rangle$. In this way, we can show that every element of R is equal to one of the following cosets: $0+\langle 2-i\rangle$, $1+\langle 2-i\rangle$, $2+\langle 2-i\rangle$, $3+\langle 2-i\rangle$, $4 + \langle 2 - i\rangle$. Is any further reduction possible? To demonstrate that there is not, we will show that these five cosets are distinct. It suffices to show that $1 + \langle 2 - i\rangle$ has additive order 5. Since $5(1+\langle 2-i\rangle) = 5+\langle 2-i\rangle = 0+\langle 2-i\rangle$, $1+\langle 2-i\rangle$ has order 1 or 5. If the order is actually 1, then $1+\langle 2-i\rangle = 0+\langle 2-i\rangle$, so $1 \in \langle 2-i\rangle$. Thus $1 = (2 - i)(a + bi) = 2a + b + (-a + 2b)i$ for some integers a and b. But this equation implies that $1 = 2a + b$ and $0 = -a + 2b$, and solving these simultaneously yields $b = 1/5$, which is a contradiction. It should be clear that the ring R is essentially the field Z_5 in disguise. ∎

Examples 11 illustrates one of the most important applications of factor rings—the construction of rings with highly desirable properties. In particular, we shall show how one may use factor rings to construct integral domains and fields.

Prime Ideals and Maximal Ideals

Definitions Prime Ideal, Maximal Ideal
A *prime ideal* A of a commutative ring R is a proper ideal of R such that $a, b \in R$ and $ab \in A$ imply $a \in A$ or $b \in A$. A *maximal* ideal of a commutative ring R is a proper ideal of R such that, whenever B is an ideal of R and $A \subseteq B \subseteq R$, then $B = A$ or $B = R$.

So, the only ideal that properly contains a maximal ideal is the entire ring. The motivation for the definition of a prime ideal comes from the integers.

EXAMPLE 14 Let n be an integer greater than 1. Then, in the ring of integers, the ideal nZ is prime if and only if n is prime (Exercise 13). ($\{0\}$ is also a prime ideal of Z.) ∎

EXAMPLE 15 The lattice of ideals of Z_{36} (Figure 14.1) shows that only $\langle 2\rangle$ and $\langle 3\rangle$ are maximal ideals. ∎

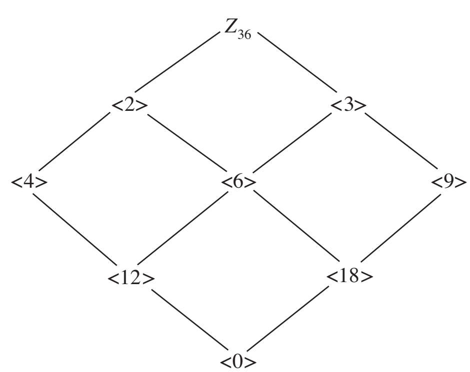


Figure 14.1 Lattice of ideals of Z_{36}.

EXAMPLE 16 The ideal $\langle x^2+1\rangle$ is maximal in $\mathbf{R}[x]$. To see this, assume that A is an ideal of $\mathbf{R}[x]$ that properly contains $\langle x^2+1\rangle$. We will prove that $A = \mathbf{R}[x]$ by showing that A contains some nonzero real number c. [This is the constant polynomial $h(x) = c$ for all x.] Then $1 = (1/c)c \in A$ and therefore, by Exercise 19, $A = \mathbf{R}[x]$. To this end, let $f(x) \in A$, but $f(x) \notin \langle x^2 + 1\rangle$. Then

$$f(x) = q(x)(x^2 + 1) + r(x),$$

where $r(x) \neq 0$ and the degree of $r(x)$ is less than 2. It follows that $r(x) = ax + b$, where a and b are not both 0, and

$$ax + b + r(x) = f(x) - q(x)(x^2 + 1) \in A.$$

Thus,

$$a^2x^2 - b^2 = (ax + b)(ax - b) \in A \quad \text{and} \quad a^2(x^2 + 1) \in A.$$

So,

$$0 \neq a^2 + b^2 = (a^2x^2 + a^2) - (a^2x^2 - b^2) \in A.$$ ∎

EXAMPLE 17 The ideal $\langle x^2 + 1\rangle$ is not prime in $Z_2[x]$, since it contains $(x + 1)^2 = x^2 + 2x + 1 = x^2 + 1$ but does not contain $x + 1$. ∎

The next two theorems are useful for determining whether a particular ideal is prime or maximal.

Theorem 14.3 *R/A* Is an Integral Domain If and Only If *A* Is Prime

Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is an integral domain if and only if A is prime.

PROOF Suppose that R/A is an integral domain and $ab \in A$. Then $(a + A)(b + A) = ab + A = A$, the zero element of the ring R/A. So, either $a + A = A$ or $b + A = A$; that is, either $a \in A$ or $b \in A$. Hence, A is prime.

To prove the other half of the theorem, we first observe that R/A is a commutative ring with unity for any proper ideal A. Thus, our task is simply to show that when A is prime, R/A has no zero-divisors. So, suppose that A is prime and $(a+A)(b+A) = 0 + A = A$. Then $ab \in A$ and, therefore, $a \in A$ or $b \in A$. Thus, one of $a + A$ or $b + A$ is the zero coset in R/A. ∎

For maximal ideals, we can do even better.

Theorem 14.4 R/A Is a Field If and Only If A Is Maximal

Let R be a commutative ring with unity and let A be an ideal of R. Then R/A is a field if and only if A is maximal.

PROOF Suppose that R/A is a field and B is an ideal of R that properly contains A. Let $b \in B$ but $b \notin A$. Then $b + A$ is a nonzero element of R/A and, therefore, there exists an element $c + A$ such that $(b + A) \cdot (c + A) = 1 + A$, the multiplicative identity of R/A. Since $b \in B$, we have $bc \in B$. Because

$$1 + A = (b + A)(c + A) = bc + A,$$

we have $1 - bc \in A \subset B$. So, $1 = (1 - bc) + bc \in B$. By Exercise 19, $B = R$. This proves that A is maximal.

Now suppose that A is maximal and let $b \in R$ but $b \notin A$. It suffices to show that $b + A$ has a multiplicative inverse. (All other properties for a field follow trivially.) Consider $B = \{br + a \mid r \in R, a \in A\}$. This is an ideal of R that properly contains A (Exercise 37). Since A is maximal, we must have $B = R$. Thus, $1 \in B$, say, $1 = bc + a'$, where $a' \in A$. Then

$$1 + A = bc + a' + A = bc + A = (b + A)(c + A).$$

From Theorems 14.3 and 14.4 and Examples 11 and 12 we see that the ideal $\langle x^2 + 1\rangle$ is maximal in $Z_3[x]$ but not prime in $Z_5[x]$.

When a commutative ring has a unity, it follows from Theorems 14.3 and 14.4 that a maximal ideal is a prime ideal. The next example shows that a prime ideal need not be maximal.

EXAMPLE 18 The ideal $\langle x\rangle$ is a prime ideal in $Z[x]$ but not a maximal ideal in $Z[x]$. To verify this, we begin with the observation that $\langle x\rangle = \{f(x) \in Z[x] \mid f(0) = 0\}$ (see Exercise 43). Thus, if $g(x)h(x) \in \langle x\rangle$, then $g(0)h(0) = 0$. And since $g(0)$ and $h(0)$ are integers, we have $g(0) = 0$ or $h(0) = 0$.

To see that $\langle x\rangle$ is not maximal, we simply note that $\langle x\rangle \subset \langle x, 2\rangle \subset Z[x]$ (see Exercise 51).

Exercises

One problem after another presents itself and in the solving of them we can find our greatest pleasure.

Karl Menninger

1. Verify that the set defined in Example 3 is an ideal.

2. Verify that the set A in Example 4 is an ideal and that $A = \langle x \rangle$.

3. Verify that the set I in Example 5 is an ideal and that if J is any ideal of R that contains $a_1, a_2, \ldots, a_n$, then $I \subseteq J$. (Hence, $\langle a_1, a_2, \ldots, a_n \rangle$ is the smallest ideal of R that contains $a_1, a_2, \ldots, a_n$.)

4. Find a subring of $Z \oplus Z$ that is not an ideal of $Z \oplus Z$.

5. Let $S = \{a + bi \mid a, b \in Z, b \text{ is even}\}$. Show that S is a subring of $Z[i]$, but not an ideal of $Z[i]$.

6. Find all maximal ideals in

   a. Z_8. **b.** Z_{10}. **c.** Z_{12}. **d.** Z_{30}.

7. Let $n = st$ where s and t are divisors of n greater that 1. Prove that $\langle s \rangle$ is a maximal ideal in Z_n if and only if s is prime.

8. Suppose that R is a ring of the form $\langle a \rangle$ for some a in R. Prove that the characteristic of R is the additive order of a.

9. Suppose that R is a commutative ring and a belongs to R. If $\{0\}$ is a maximal ideal of R prove that $aR = \{ar \mid r \in R\} = \{0\}$ or that a belongs to aR.

10. In $Z[x]$ prove that $\langle 2x, 3 \rangle = \langle x, 3 \rangle$.

11. Let a belong to a commutative ring R. Show that $aR = \{ar \mid r \in R\}$ is an ideal of R. If R is the ring of even integers, list the elements of $4R$.

12. Prove that the intersection of any set of ideals of a ring is an ideal.

13. If n is an integer greater than 1, show that $\langle n \rangle = nZ$ is a prime ideal of Z if and only if n is prime. (This exercise is referred to in this chapter.)

14. If A and B are ideals of a ring, show that the *sum* of A and B, $A + B = \{a + b \mid a \in A, b \in B\}$, is an ideal.

15. In the ring of integers, find a positive integer a such that

    a. $\langle a \rangle = \langle 2 \rangle + \langle 3 \rangle$.
    b. $\langle a \rangle = \langle 6 \rangle + \langle 8 \rangle$.
    c. $\langle a \rangle = \langle m \rangle + \langle n \rangle$.

16. If A and B are ideals of a ring, show that the *product* of A and B, $AB = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n \mid a_i \in A, b_i \in B, n \text{ a positive integer}\}$, is an ideal.

17. Find a positive integer a such that
a. $\langle a\rangle = \langle 3\rangle\langle 4\rangle$.
b. $\langle a\rangle = \langle 6\rangle\langle 8\rangle$.
c. $\langle a\rangle = \langle m\rangle\langle n\rangle$.

18. Let A and B be ideals of a ring. Prove that $AB \subseteq A \cap B$.

19. If A is an ideal of a ring R and 1 belongs to A, prove that $A = R$. (This exercise is referred to in this chapter.)

20. If A and B are ideals of a commutative ring R with unity and $A + B = R$, show that $A \cap B = AB$.

21. If an ideal I of a ring R contains a unit, show that $I = R$.

22. If R is a finite commutative ring with unity, prove that every prime ideal of R is a maximal ideal of R.

23. Give an example of a ring that has exactly two maximal ideals.

24. Suppose that R is a commutative ring and $|R| = 30$. If I is an ideal of R and $|I| = 10$, prove that I is a maximal ideal.

25. Let R and I be as described in Example 10. Prove that I is an ideal of R.

26. Prove that $I = \{f(x) \in Z[x] \mid f(1) \text{ is even}\}$ is an ideal of $Z[x]$.

27. Let $n > 1$ be an integer. For the ring R in Example 10 let $I = \left\{\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix} \,\middle|\, b_i \text{ is a multiple of } n\right\}$. Prove that I is an ideal of R. How many elements are in I?

28. In $M_2(Z) = \left\{\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \,\middle|\, a_i \in Z\right\}$ let $A = \left\{\begin{bmatrix} a_1 & a_2 \\ 0 & 0 \end{bmatrix} \,\middle|\, a_i \in Z\right\}$. Is A a subring of $M_2(Z)$? Is A an ideal of $M_2(Z)$?

29. Let $R = \left\{\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \,\middle|\, a, b, c \in \mathbf{R}\right\}$ and $I = \left\{\begin{bmatrix} 0 & x \\ 0 & y \end{bmatrix} \,\middle|\, x, y \in \mathbf{R}\right\}$. Prove that I is an ideal of R and R/I is field. What is the multiplicative inverse of $\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + I$?

30. Determine the order of $Z[i]/\langle i\rangle$.

31. Let $R = \{a + bi + \langle 2i\rangle \mid a, b \in Z\}$. List the distinct elements of R. Is R an intergal domain?

32. Use Exercise 24 in Chapter 0 to prove the assertion in Example 13 that 1 has order 5.

33. In $Z_5[x]$ let $I = \langle x^2 - 1\rangle$. Show that $R = Z_5[x]/I$ has exactly 8 zero-divisors without trying all possibilities. Explain why the other nonzero elements in R are the elements of $U(R)$, the group of units of R. Determine the isomorphism class of $U(R)$. (That is, find an external direct product of groups of the form Z_n that is isomorphic to $U(R)$.)

34. In Z, let $I = \langle 2\rangle$. Prove that $I[x]$ is not a maximal ideal of $Z[x]$ even though I is a maximal ideal of Z.

35. Verify the claim made in Example 10 about the size of R/I.

36. Give an example of a commutative ring that has a maximal ideal that is not a prime ideal.

37. Show that the set B in the latter half of the proof of Theorem 14.4 is an ideal of R. (This exercise is referred to in this chapter.)

38. Let $A = \{a_nx^n + \cdots + a_1x + a_0 \in Z[x] \mid \text{all } a_i \text{ are even }\}$. Find an $f(x)$ in $Z[x]$ such that $A = \langle f(x)\rangle$.

39. Prove that the only ideals of a field F are $\{0\}$ and F itself.

40. Let R be a commutative ring with unity. Suppose that the only ideals of R are $\{0\}$ and R. Show that R is a field.

41. Prove that every idempotent in a communitive ring with unity other than 0 or 1 is a zero divisor.

42. Show that $\mathbf{R}[x]/\langle x^2 + 1\rangle$ is a field.

43. In $Z[x]$, the ring of polynomials with integer coefficients, let $I = \{f(x) \in Z[x] \mid f(0) = 0\}$. Find a $g(x) \in Z[x]$ such that $I = \langle g(x)\rangle$. Is I maximal? (This exercise is referred to in this chapter and in Chapter 15.)

44. Show that $A = \{(3x, y) \mid x, y \in Z\}$ is a maximal ideal of $Z \oplus Z$. Generalize. What happens if $3x$ is replaced by $4x$? Generalize.

45. Let R be the ring of continuous functions from $\mathbf{R}$ to $\mathbf{R}$. Show that $I = \{f \in R \mid f(0) = 0\}$ is a maximal ideal of R.

46. Let $R = Z_8 \oplus Z_{30}$. Find all maximal ideals of R, and for each maximal ideal I, identify the size of the field R/I.

47. How many elements are in $Z[i]/\langle 3+i\rangle$? Give reasons for your answer.

48. In ring of Gaussian integers $Z[i] = \{a + bi \mid a, b \in Z\}$ let $I = \{a + bi \mid a, b \text{ are even }\}$. Prove that I is a ideal. How many elements are in $Z[i]/I$? Is $Z[i]/I$ a field? Is $Z[i]/I$ an integral domain?

49. In $Z \oplus Z$, let $I = \{(a, 0) \mid a \in Z\}$. Show that I is a prime ideal but not a maximal ideal.

50. Let R be a ring and let I be an ideal of R. Prove that the factor ring R/I is commutative if and only if $rs - sr \in I$ for all r and s in R.

51. In $Z[x]$, let $I = \{f(x) \in Z[x] \mid f(0) \text{ is an even integer}\}$. Prove that $I = \langle x, 2\rangle$. Is I a prime ideal of $Z[x]$? Is I a maximal ideal? How many elements does $Z[x]/I$ have? (This exercise is referred to in this chapter.)

52. In $Z[x]$ let $I = \{a_nx^n + \cdots + a_1x + 4a_0 \mid \text{ where all } a_i \text{ are even for } i > 0\}$. Express I in the form $\langle f_1(x), f_2(x)\rangle$ where $f_1(x)$ and $f_2(x)$ are specific elements in $Z[x]$.

53. Prove that the ideal $I = \langle x^2 + 1\rangle$ in $Z[x]$ is not maximal by exhibiting a proper ideal J of $Z[x]$ that properly contains I.

54. Prove that $I = \langle 2 + 2i\rangle$ is not a prime ideal of $Z[i]$. How many elements are in $Z[i]/I$? What is the characteristic of $Z[i]/I$?

55. In $Z_5[x]$, let $I = \langle x^2 + x + 2\rangle$. Find the multiplicative inverse of $2x + 3 + I$ in $Z_5[x]/I$.

56. Let R be a ring and let p be a fixed prime. Show that $I_p = \{r \in R \mid \text{ additive order of } r \text{ is a power of } p\}$ is an ideal of R.

57. An integral domain D is called a *principal ideal domain* if every ideal of D has the form $\langle a\rangle = \{ad \mid d \in D\}$ for some a in D. Show that Z is a principal ideal domain. (This exercise is referred to in Chapter 18.)

58. Let $R = \left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} \middle| \, a, b, d \in Z\right\}$ and $S = \left\{\begin{bmatrix} r & s \\ 0 & t \end{bmatrix} \middle| \, r, s, t \in Z, s \text{ is even}\right\}$. If S is an ideal of R, what can you say about r and t?

59. If R and S are principal ideal domains, prove that $R \oplus S$ is a principal ideal ring. (That is, every ideal of $R \oplus S$ has the form $\langle (r, s)\rangle$.)

60. In a principal ideal domain, show that every nontrivial prime ideal is a maximal ideal.

61. Let R be a commutative ring and let A be any subset of R. Show that the *annihilator* of $A, \mathrm{Ann}(A) = \{r \in R \,|\, ra = 0$ for all a in $A\}$, is an ideal.

62. Let R be a commutative ring and let A be any ideal of R. Show that the *nil radical* of $A, N(A) = \{r \in R \,|\, r^n \in A$ for some positive integer n(n depends on r)$\}$, is an ideal of R. [$N(\langle 0 \rangle)$ is called the *nil radical* of R.]

63. Let $R = Z_{27}$. Find
a. $N(\langle 0 \rangle)$. **b.** $N(\langle 3 \rangle)$. **c.** $N(\langle 9 \rangle)$.

64. Let $R = Z_{36}$. Find
a. $N(\langle 0 \rangle)$. **b.** $N(\langle 4 \rangle)$. **c.** $N(\langle 6 \rangle)$.

65. Let R be a commutative ring. Show that $R/N(\langle 0 \rangle)$ has no nonzero nilpotent elements.

66. Let A be an ideal of a commutative ring. Prove that $N(N(A)) = N(A)$.

67. Let $Z_2[x]$ be the ring of all polynomials with coefficients in Z_2 (i.e., coefficients are 0 or 1, and addition and multiplication of coefficients are done modulo 2). Show that $Z_2[x]/\langle x^2 + x + 1 \rangle$ is a field.

68. Let $R = \{a + bi + \langle 2i \rangle \mid a, b \in Z\}$. List the distinct elements of R. Is R an integral domain?

69. Show that $Z_3[x]/\langle x^2 + x + 1 \rangle$ is not a field.

70. Let R be a commutative ring without unity, and let $a \in R$. Describe the smallest ideal I of R that contains a (i.e., if J is any ideal that contains a, then $I \subseteq J$).

71. Let R be the ring of continuous functions from $\mathbf{R}$ to $\mathbf{R}$. Let $A = \{f \in R \,|\, f(0)$ is an even integer$\}$. Show that A is a subring of R, but not an ideal of R.

72. Show that $Z[i]/\langle 1 - i \rangle$ is a field. How many elements does this field have?

73. If R is a principal ideal domain and I is an ideal of R, prove that every ideal of R/I is principal (see Exercise 57).

74. How many elements are in $Z_5[i]/\langle 1 + i \rangle$?

75. Show, by example, that the intersection of two prime ideals need not be a prime ideal.

76. Let $\mathbf{R}$ denote the ring of real numbers. Determine all ideals of $\mathbf{R} \oplus \mathbf{R}$. What happens if $\mathbf{R}$ is replaced by any field F?

77. Find the characteristic of $Z[i]/\langle 2+i \rangle$.

78. Show that the characteristic of $Z[i]/\langle a+bi \rangle$ divides a^2+b^2.

79. Prove that the set of all polynomials whose coefficients are all even is a prime ideal in $Z[x]$.

80. Let $R = Z[\sqrt{-5}] = \{a + b\sqrt{-5} \mid a, b \in Z\}$ and $I = \{a + b\sqrt{-5} \mid a, b \in Z,\ a - b \text{ is even}\}$. Show that I is an ideal and it is maximal.

81. Let R be a commutative ring with unity that has the property that $a^2 = a$ for all a in R. Let I be a prime ideal in R. Show that $|R/I| = 2$.

82. Let R be a commutative ring with unity, and let I be a proper ideal with the property that every element of R that is not in I is a unit of R. Prove that I is the unique maximal ideal of R.

83. Let $I_0 = \{f(x) \in Z[x] \mid f(0) = 0\}$. For any positive integer n, show that there exists a sequence of strictly increasing ideals such that $I_0 \subset I_1 \subset I_2 \subset \cdots \subset I_n \subset Z[x]$.

Computer Exercises

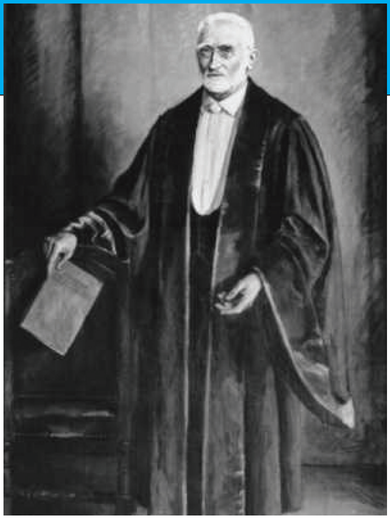
Computer exercises for this chapter are available at the website:

http://www.d.umn.edu/~jgallian

Richard Dedekind

Richard Dedekind was not only a mathematician, but one of the wholly great in the history of mathematics, now and in the past, the last hero of a great epoch, the last pupil of Gauss, for four decades himself a classic, from whose works not only we, but our teachers and the teachers of our teachers, have drawn.

EDMUND LANDAU,
Commemorative Address to the Royal Society of Göttingen



akg-images/Newscom

Richard Dedekind was born on October 6, 1831, in Brunswick, Germany, the birthplace of Gauss. Dedekind was the youngest of four children of a law professor. His early interests were in chemistry and physics, but he obtained a doctor's degree in mathematics at the age of 21 under Gauss at the University of Göttingen. Dedekind continued his studies at Göttingen for a few years, and in 1854 he began to lecture there.

Dedekind spent the years 1858–1862 as a professor in Zürich. Then he accepted a position at an institute in Brunswick where he had once been a student.

Although this school was less than university level, Dedekind remained there for the next 50 years. He died in Brunswick in 1916. During his career, Dedekind made numerous fundamental contributions to mathematics. His treatment of irrational numbers, "Dedekind cuts," put analysis on a firm, logical foundation. His work on unique factorization led to the modern theory of algebraic numbers. He was a pioneer in the theory of rings and fields. The notion of ideals as well as the term itself are attributed to Dedekind. Mathematics historian Morris Kline has called him "the effective founder of abstract algebra."

Emmy Noether


The Granger Collection, NY

. . . she discovered methods which have proved of enormous importance in the development of the present-day younger generation of mathematicians.

ALBERT EINSTEIN, *The New York Times*

EMMY NOETHER was born on March 23, 1882, in Germany. When she entered the University of Erlangen, she was one of only two women among the 1000 students. Noether completed her doctorate in 1907.

In 1916, Noether went to Göttingen and, under the influence of David Hilbert and Felix Klein, became interested in general relativity. While there, she made a major contribution to physics with her theorem that whenever there is a symmetry in nature, there is also a conservation law, and vice versa. In a 2012 issue of the New York Times science writer Ranson Stephens said "You can make a strong case that her theorem is the backbone on which all of modern physics is built." Hilbert tried unsuccessfully to obtain a faculty appointment at Göttingen for Noether, saying, "I do not see that the sex of the candidate is an argument against her admission as Privatdozent. After all, we are a university and not a bathing establishment."

It was not until she was 38 that Noether's true genius revealed itself. Over the next 13 years, she used an axiomatic method to develop a general theory of ideals and noncommutative algebras. With this abstract theory, Noether was able to weld together many important concepts. Her approach was even more important than the individual results. Hermann Weyl said of Noether, "She originated above all a new and epochmaking style of thinking in algebra."

With the rise of Hitler in 1933, Noether, a Jew, fled to the United States and took a position at Bryn Mawr College. She died suddenly on April 14, 1935, following an operation.

15 Ring Homomorphisms

If there is one central idea which is common to all aspects of modern algebra it is the notion of homomorphism.

I. N. Herstein, *Topics in Algebra*

In mathematics, functions are used, among other purposes, (1) to carry out a *matching up* of elements of one system with those of another; and (2) to transform a given system (or problem) into a simpler one.

Norman J. Block, *Abstract Algebra with Applications*

Definition and Examples

In our work with groups, we saw that one way to discover information about a group is to examine its interaction with other groups by way of homomorphisms. It should not be surprising to learn that this concept extends to rings with equally profitable results.

Just as a group homomorphism preserves the group operation, a ring homomorphism preserves the ring operations.

Definitions Ring Homomorphism, Ring Isomorphism

A *ring homomorphism* ϕ from a ring R to a ring S is a mapping from R to S that preserves the two ring operations; that is, for all a, b in R,

$$\phi(a + b) = \phi(a) + \phi(b) \qquad \text{and} \qquad \phi(ab) = \phi(a)\phi(b).$$

A ring homomorphism that is both one-to-one and onto is called a *ring isomorphism.*

As is the case for groups, in the preceding definition the operations on the left of the equal signs are those of R, whereas the operations on the right of the equal signs are those of S.

Again as with group theory, the roles of isomorphisms and homomorphisms are entirely distinct. An isomorphism is used to show that two rings are algebraically identical; a homomorphism is used to simplify a ring while retaining certain of its features.

A schematic representation of a ring homomorphism is given in Figure 15.1. The dashed arrows indicate the results of performing the ring operations.

The following examples illustrate ring homomorphisms. The reader should supply the missing details.

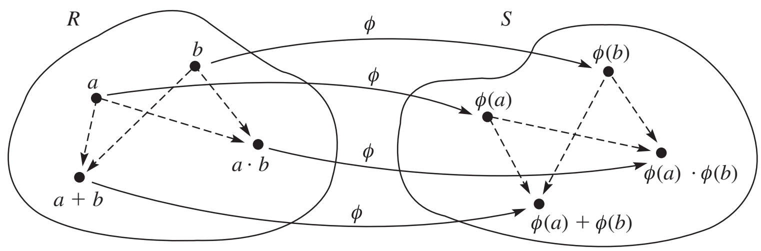


Figure 15.1 Actions of a homomorphism.

▮ **EXAMPLE 1** For any positive integer n, the mapping $k \to k$ mod n is a ring homomorphism from Z onto Z_n (see Exercise 9 in Chapter 0). This mapping is called the *natural homomorphism* from Z to Z_n. ▮

▮ **EXAMPLE 2** The mapping $a + bi \to a - bi$ is a ring isomorphism from the complex numbers onto the complex numbers (see Exercise 39 in Chapter 6). ▮

▮ **EXAMPLE 3** Let $\mathbf{R}[x]$ denote the ring of all polynomials with real coefficients. The mapping $f(x) \to f(1)$ is a ring homomorphism from $\mathbf{R}[x]$ onto $\mathbf{R}$. ▮

▮ **EXAMPLE 4** We determine all non-zero ring homomorphisms from Z_{10} to Z_{20} and from Z_{20} to Z_{10}. From properties of group homomorphisms we know that image of any ring non-zero homomorphism ϕ from Z_{10} to Z_{20} has order 10, 5, or 2. For any ring homomorphism ϕ from a ring R with unity 1 onto a ring S and $\phi(1) = a$ we have that $1 \cdot 1 = 1$ implies that $a \cdot a = a$. That is, a is an idempotent in S. If $|a| = 10$, from Corollary 4 of Theorem 4.2 we know that $a = 2,\ 2 \cdot 3 = 6,\ 2 \cdot 7 = 14$ or $2 \cdot 9 = 18$. Checking each case for $a^2 = a$ we see that none of these four possibilties pans out. If $|a| = 5$, then $a = 4,\ 8,\ 12$, or 16. Of these, only 16 is an idempotent. If $|a| = 2$, then $a = 10$ and but 10 is not an idempotent. So, the only possible non-zero ring homomorphism from Z_{10} to Z_{20} is $\phi(x) = 16x$. That this is well-defined and perserves both operations follows directly from the definitions. To determine that ring homomorphisms from Z_{20} to Z_{10} we let $\phi(1) = a$ where a is an idempotent in Z_{10}. Checking for idempotents in Z_{10} we find that they are 5 and 6. Since $|5| = 2$ and $|6| = 5$ and both 2 and 10 are divisors of 20 these images are candidates for $\phi(1)$ and indeed the mappings $\phi(x) = 5x$ and $\phi(x) = 6x$ are ring homomorphisms. ∎

∎ **EXAMPLE 5** We determine all ring homomorphisms from Z_{12} to Z_{30}. By Example 11 in Chapter 10, the only group homomorphisms from Z_{12} to Z_{30} are $x \to ax$, where $a = 0, 15, 10, 20, 5$, or 25. But, since $1 \cdot 1 = 1$ in Z_{12}, we must have $a \cdot a = a$ in Z_{30}. This requirement rules out 20 and 5 as possibilities for a. Finally, simple calculations show that each of the remaining four choices does yield a ring homomorphism. ∎

∎ **EXAMPLE 6** Let R be a commutative ring of characteristic 2. Then the mapping $a \to a^2$ is a ring homomorphism from R to R. ∎

∎ **EXAMPLE 7** Although $2Z$, the group of even integers under addition, is group-isomorphic to the group Z under addition, the ring $2Z$ is not ring-isomorphic to the ring Z. (Quick! What does Z have that $2Z$ doesn't?) ∎

Our next two examples are applications to number theory of the natural homomorphism given in Example 1.

∎ **EXAMPLE 8 Test for Divisibility by 9**

An integer n with decimal representation $a_k a_{k-1} \cdots a_0$ is divisible by 9 if and only if $a_k + a_{k-1} + \cdots + a_0$ is divisible by 9. To verify this, observe that $n = a_k 10^k + a_{k-1}10^{k-1} + \cdots + a_0$. Then, letting α denote the natural homomorphism from Z to Z_9 [in particular, $\alpha(10) = 1$], we note that n is divisible by 9 if and only if

$$\begin{aligned} 0 = \alpha(n) &= \alpha(a_k)(\alpha(10))^k + \alpha(a_{k-1})(\alpha(10))^{k-1} + \cdots + \alpha(a_0) \\ &= \alpha(a_k) + \alpha(a_{k-1}) + \cdots + \alpha(a_0) \\ &= \alpha(a_k + a_{k-1} + \cdots + a_0). \end{aligned}$$

But $\alpha(a_k + a_{k-1} + \cdots + a_0) = 0$ is equivalent to $a_k + a_{k-1} + \cdots + a_0$ being divisible by 9. ∎

The next example illustrates the value of the natural homomorphism given in Example 1.

EXAMPLE 9 Theorem of Gersonides

In 1844 Eugéne Charles Catalan conjectured that 2^3 and 3^2 is the only instance of two consecutive powers greater than 1 of natural numbers. That is, they are the only solution in the natural numbers of $x^m - y^n = 1$ where $m, n, x, y > 1$. This conjecture was proved in 2002 by Preda Mihăilescu. The special case where x and y are restricted to 2 and 3 was first proved by the Rabbi Gersonides in the fourteenth century who proved for $m, n > 1$ the only case when $2^m = 3^n \pm 1$ is for $(m, n) = (3, 2)$. To verify this is so for $2^m = 3^n + 1$, observe that for all n we have $3^n \bmod 8 = 3$ or 1. Thus, $3^n + 1 \bmod 8 = 4$ or 2. On the other hand, for $m > 2$, we have $2^m \bmod 8 = 0$. To handle the case where $2^m = 3^n - 1$, we first note that for all n, $3^n \bmod 16 = 3, 9, 11$, or 1, depending on the value of $n \bmod 4$. Thus, $(3^n - 1) \bmod 16 = 2, 8, 10$, or 0. Since $2^m \bmod 16 = 0$ for $m \geq 4$, we have ruled out the cases where $n \bmod 4 = 1, 2$, or 3. Because $3^{4k} \bmod 5 = (3^4)^k \bmod 5 = 1^k \bmod 5 = 1$, we know that $(3^{4k} - 1) \bmod 5 = 0$. But the only values for $2^m \bmod 5$ are $2, 4, 3$, and 1. This contradiction completes the proof. ∎

Properties of Ring Homomorphisms

Theorem 15.1 Properties of Ring Homomorphisms

Let ϕ be a ring homomorphism from a ring R to a ring S. Let A be a subring of R and let B be an ideal of S.

1. ***For any $r \in R$ and any positive integer n, $\phi(nr) = n\phi(r)$ and $\phi(r^n) = (\phi(r))^n$.***
2. ***$\phi(A) = \{\phi(a) \mid a \in A\}$ is a subring of S.***
3. ***If A is an ideal and ϕ is onto S, then $\phi(A)$ is an ideal.***
4. ***$\phi^{-1}(B) = \{r \in R \mid \phi(r) \in B\}$ is an ideal of R.***
5. ***If R is commutative, then $\phi(R)$ is commutative.***
6. ***If R has a unity 1, $S \neq \{0\}$, and ϕ is onto, then $\phi(1)$ is the unity of S and units in R map to units in S.***
7. ***ϕ is an isomorphism if and only if ϕ is onto and $\mathrm{Ker}\ \phi = \{r \in R \mid \phi(r) = 0\} = \{0\}$.***

8. ***If ϕ is an isomorphism from R onto S, then ϕ^{-1} is an isomorphism from S onto R.***

PROOF The proofs of these properties are similar to those given in Theorems 10.1 and 10.2 and are left as exercises (Exercise 1). ∎

The student should learn the various properties of Theorem 15.1 in words in addition to the symbols. Property 2 says that the homomorphic image of a subring is a subring. Property 4 says that the pullback of an ideal is an ideal, and so on.

The next three theorems parallel results we had for groups. The proofs are nearly identical to their group theory counterparts and are left as exercises (Exercises 2, 3, and 4).

Theorem 15.2 Kernels Are Ideals

Let ϕ be a ring homomorphism from a ring R to a ring S. Then Ker $\phi = \{r \in R \mid \phi(r) = 0\}$ is an ideal of R.

Theorem 15.3 First Isomorphism Theorem for Rings

Let ϕ be a ring homomorphism from R to S. Then the mapping from $R/$Ker ϕ to $\phi(R)$, given by $r + \text{Ker}\ \phi \to \phi(r)$, is an isomorphism. In symbols, $R/\text{Ker}\ \phi \approx \phi(R)$.

Theorem 15.4 Ideals Are Kernels

Every ideal of a ring R is the kernel of a ring homomorphism of R. In particular, an ideal A is the kernel of the mapping $r \to r + A$ from R to R/A.

The homomorphism from R to R/A given in Theorem 15.4 is called the *natural homomorphism* from R to R/A. Theorem 15.3 is often referred to as the Fundamental Theorem of Ring Homomorphisms.

In Example 18 in Chapter 14 we gave a direct proof that $\langle x \rangle$ is a prime ideal of $Z[x]$ but not a maximal ideal. In the following example we illustrate a better way to do this kind of problem.

EXAMPLE 10 Since the mapping ϕ from $Z[x]$ onto Z given by $\phi(f(x)) = f(0)$ is a ring homomorphism with Ker $\phi = \langle x \rangle$ (see Exercise 43 in Chapter 14), we have, by Theorem 15.3, $Z[x]/\langle x \rangle \approx Z$. And because Z is an integral domain but not a field, we know by Theorems 14.3 and 14.4 that the ideal $\langle x \rangle$ is prime but not maximal in $Z[x]$. ∎

Theorem 15.5 Homomorphism from Z to a Ring with Unity

Let R be a ring with unity 1. The mapping ϕ: $Z \to R$ given by $n \to n \cdot 1$ is a ring homomorphism.

PROOF Since the multiplicative group property $a^{m+n} = a^m a^n$ translates to $(m+n)a = ma + na$ when the operation is addition, we have $\phi(m+n) = (m+n) \cdot 1 = m \cdot 1 + n \cdot 1$. So, ϕ preserves addition.

That ϕ also preserves multiplication follows from Exercise 15 in Chapter 12, which says that $(m \cdot a)(n \cdot b) = (mn) \cdot (ab)$ for all integers m and n. Thus, $\phi(mn) = (mn) \cdot 1 = (mn) \cdot ((1)(1)) = (m \cdot 1)(n \cdot 1) = \phi(m)\phi(n)$. So, ϕ preserves multiplication as well. ∎

Corollary 1 A Ring with Unity Contains Z_n or Z

If R is a ring with unity and the characteristic of R is $n > 0$, then R contains a subring isomorphic to Z_n. If the characteristic of R is 0, then R contains a subring isomorphic to Z.

PROOF Let 1 be the unity of R and let $S = \{k \cdot 1 \mid k \in Z\}$. Theorem 15.5 shows that the mapping ϕ from Z to S given by $\phi(k) = k \cdot 1$ is a homomorphism, and by the First Isomorphism Theorem for rings, we have $Z/\text{Ker } \phi \approx S$. But, clearly, Ker $\phi = \langle n \rangle$, where n is the additive order of 1 and, by Theorem 13.3, n is also the characteristic of R. So, when R has characteristic n, $S \approx Z/\langle n \rangle \approx Z_n$. When R has characteristic 0, $S \approx Z/\langle 0 \rangle \approx Z$. ∎

Here is an example of the mapping in Corollary 1 where the unity of the ring is not immediately obvious.

EXAMPLE 11 Let $R = \{0, 2, 4, 6, 8\}$ under under addition and mulitiplication modulo 10. Note that 6 is the unity and, because

$5x = 0$ for all x, the characteristic of R is 5. So, the mapping $\phi(n) = n \cdot 6 = 6n$ is an isomorphism with Ker $\phi = \langle 5 \rangle$. Thus $R \approx Z/\langle 5 \rangle \approx Z_5$ and R is a field. ∎

Corollary 2 Z_m Is a Homomorphic Image of Z

For any positive integer m, the mapping of ϕ: $Z \rightarrow Z_m$ given by $x \rightarrow x$ mod m is a ring homomorphism.

PROOF This follows directly from the statement of Theorem 15.5, since in the ring Z_m, the integer x mod m is $x \cdot 1$. (For example, in Z_3, if $x = 5$, we have $5 \cdot 1 = 1 + 1 + 1 + 1 + 1 = 2$.) ∎

Corollary 3 A Field Contains Z_p or Q (Steinitz, 1910)

If F is a field of characteristic p, then F contains a subfield isomorphic to Z_p. If F is a field of characteristic 0, then F contains a subfield isomorphic to the rational numbers.

PROOF By Corollary 1, F contains a subring isomorphic to Z_p if F has characteristic p, and F has a subring S isomorphic to Z if F has characteristic 0. In the latter case, let

$$T = \{ab^{-1} \mid a, b \in S, b \neq 0\}.$$

Then T is isomorphic to the rationals (Exercise 63). ∎

Since the intersection of all subfields of a field is itself a subfield (Exercise 13), every field has a smallest subfield (i.e., a subfield that is contained in every subfield). This subfield is called the *prime subfield* of the field. It follows from Corollary 3 that the prime subfield of a field of characteristic p is isomorphic to Z_p, whereas the prime subfield of a field of characteristic 0 is isomorphic to Q. (See Exercise 75.)

The Field of Quotients

Although the integral domain Z is not a field, it is at least contained in a field—the field of rational numbers. And notice that the field of rational numbers is nothing more than quotients of integers. Can we mimic the construction of the rationals from the integers for other integral domains? Yes. The field constructed in Theorem 15.6 is called the *field of quotients of* D. Throughout the proof of Theorem 15.6, you should keep in mind that we are using the construction of the rationals from the integers as a model for our construction of the field of quotients of D.

Theorem 15.6 Field of Quotients

Let D be an integral domain. Then there exists a field F (called the field of quotients of D) that contains a subring isomorphic to D.

PROOF Let $S = \{(a, b) \mid a, b \in D, b \neq 0\}$. We define an equivalence relation on S by $(a, b) \equiv (c, d)$ if $ad = bc$. Now, let F be the set of equivalence classes of S under the relation $\equiv$ and denote the equivalence class that contains (x, y) by x/y. We define addition and multiplication on F by

$$a/b + c/d = (ad + bc)/(bd) \quad \text{and} \quad a/b \cdot c/d = (ac)/(bd).$$

(Notice that here we need the fact that D is an integral domain to ensure that multiplication is closed; that is, $bd \neq 0$ whenever $b \neq 0$ and $d \neq 0$.)

Since there are many representations of any particular element of F (just as in the rationals, we have $1/2 = 3/6 = 4/8$), we must show that these two operations are well-defined. To do this, suppose that $a/b = a'/b'$ and $c/d = c'/d'$, so that $ab' = a'b$ and $cd' = c'd$. It then follows that

$$\begin{aligned}(ad + bc)b'd' &= adb'd' + bcb'd' = (ab')dd' + (cd')bb' \\ &= (a'b)dd' + (c'd)bb' = a'd'bd + b'c'bd \\ &= (a'd' + b'c')bd.\end{aligned}$$

Thus, by definition, we have

$$(ad + bc)/(bd) = (a'd' + b'c')/(b'd'),$$

and, therefore, addition is well-defined. We leave the verification that multiplication is well-defined as an exercise (Exercise 63).

That F is a field is straightforward. Let 1 denote the unity of D. Then $0/1$ is the additive identity of F. The additive inverse of a/b is $-a/b$; the multiplicative inverse of a nonzero element a/b is b/a. The remaining field properties can be checked easily.

Finally, the mapping ϕ: $D \to F$ given by $x \to x/1$ is a ring isomorphism from D to $\phi(D)$ (see Exercise 7). ■

■ **EXAMPLE 12** Let $D = Z[x]$. Then the field of quotients of D is $\{f(x)/g(x) \mid f(x), g(x) \in D$, where $g(x)$ is not the zero polynomial$\}$. ■

When F is a field, the field of quotients of $F[x]$ is traditionally denoted by $F(x)$.

■ **EXAMPLE 13** Let p be a prime. Then $Z_p(x) = \{f(x)/g(x) \mid f(x), g(x) \in Z_p[x], g(x) \neq 0\}$ is an infinite field of characteristic p. ■

One point of clarification is warranted here. Whereas in the field of quotients $F(x)$ the element $f(x) = x$ has the inverse $g(x) = 1/x$, the F-valued function $f(x) = x$ does not have an inverse in any ring of polynomials functions that contains $F[x]$ since $1/x$ it is not defined at $x = 0$. Polynomials over a commutative ring R are abstract objects whereas polynomials that are used to define functions from R to R are a particular application of abstract polynomials. We will see in the next chapter that different polynomials can define the same function.

Exercises

We can work it out.

John Lennon and Paul McCartney,
"We Can Work It Out," single[1]

1. Prove Theorem 15.1.
2. Prove Theorem 15.2.
3. Prove Theorem 15.3.
4. Prove Theorem 15.4.

5. Show that the correspondence $x \to 5x$ from Z_5 to Z_{10} does not preserve addition.

6. Show that the correspondence $x \to 3x$ from Z_4 to Z_{12} does not preserve multiplication.

7. Show that the mapping $\phi\colon D \to F$ in the proof of Theorem 15.6 is a ring homomorphism.

8. Prove that every ring homomorphism ϕ from Z_n to itself has the form $\phi(x) = ax$, where $a^2 = a$.

9. Suppose that ϕ is a ring homomorphism from Z_m to Z_n. Prove that if $\phi(1) = a$, then $a^2 = a$. Give an example to show that the converse is false.

10. Prove that the ring R in Example 10 in Chapter 14 is isomorphic to the ring $S = \left\{ \begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix} \;\middle|\; a_i \in Z_2 \right\}$.

11. For the ring R in Example 11, determine the number of group isomorphisms and the number of ring isomorphisms from Z_5 to R.

12. **a.** Is the ring $2Z$ isomorphic to the ring $3Z$?
b. Is the ring $2Z$ isomorphic to the ring $4Z$?

13. Prove that the intersection of any collection of subfields of a field F is a subfield of F. (This exercise is referred to in this chapter.)

14. Let $Z_3[i] = \{a + bi \mid a, b \in Z_3\}$ (see Example 9 in Chapter 13). Show that the field $Z_3[i]$ is ring-isomorphic to the field $Z_3[x]/\langle x^2 + 1\rangle$.

15. Let

$$S = \left\{ \begin{bmatrix} a & b \\ -b & a \end{bmatrix} \;\middle|\; a, b \in \mathbf{R} \right\}.$$

Show that $\phi\colon \mathbf{C} \to S$ given by

$$\phi(a + bi) = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$$

is a ring isomorphism.

16. Let $Z[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Z\}$ and

$$H = \left\{ \begin{bmatrix} a & 2b \\ b & a \end{bmatrix} \;\middle|\; a, b \in Z \right\}.$$

Show that $Z[\sqrt{2}]$ and H are isomorphic as rings.

17. Consider the mapping from $M_2(Z)$ into Z given by $\begin{bmatrix} a & b \\ c & d \end{bmatrix} \to a.$
Prove or disprove that this is a ring homomorphism.

18. Let $\left\{ \begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \;\middle|\; a, b, c \in Z \right\}$. Prove or disprove that the mapping $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} \to a$ is a ring homomorphism.

19. Is the mapping from Z_5 to Z_{30} given by $x \to 6x$ a ring homomorphism? Note that the image of the unity is the unity of the image but not the unity of Z_{30}.

20. Is the mapping from Z_{10} to Z_{10} given by $x \to 2x$ a ring homomorphism?

21. Describe the kernel of the homomorphism given in Example 3.

22. To which familiar ring are the ones in Exercise 47 in Chapter 12 isomorphic?

23. Recall that a ring element a is called an idempotent if $a^2 = a$. Prove that a ring homomorphism carries an idempotent to an idempotent.

24. Determine all ring homomorphisms from Z_{25} to Z_{20}.

25. Determine all ring homomorphisms from Z_6 to Z_6. Determine all ring homomorphisms from Z_{20} to Z_{30}.

26. Determine all ring isomorphisms from Z_n to itself.

27. Determine all ring homomorphisms from Z to Z.

28. Suppose ϕ is a ring homomorphism from $Z \oplus Z$ into $Z \oplus Z$. What are the possibilities for $\phi((1, 0))$?

29. Determine all ring homomorphisms from $Z \oplus Z$ into $Z \oplus Z$.

30. In Z, let $A = \langle 2 \rangle$ and $B = \langle 8 \rangle$. Show that the group A/B is isomorphic to the group Z_4 but that the ring A/B is not ring-isomorphic to the ring Z_4.

31. Prove that $R[x]/\langle x^2 \rangle$ is ring isomorphic to $\left\{ \begin{bmatrix} a & b \\ 0 & a \end{bmatrix} \;\middle|\; a, b \in R \right\}$.

32. Prove that the ring $Z_3[x]/\langle x^2+1\rangle$ is isomorphic to the field $Z_3[i]$.

33. Given that ϕ is a ring homomorphism from $Z \oplus Z$ to $Z \oplus Z$ with $\phi((1,0)) = (0,1)$ and $\phi((1,1)) = (1,1)$ find a formula for $\phi((a,b))$.

34. Show that $(Z \oplus Z)/(\langle a\rangle \oplus \langle b\rangle)$ is ring-isomorphic to $Z_a \oplus Z_b$.

35. Determine all ring homomorphisms from $Z \oplus Z$ to Z.

36. Prove that the sum of the squares of three consecutive integers cannot be a square.

37. Let m be a positive integer and let n be an integer obtained from m by rearranging the digits of m in some way. (For example, 72345 is a rearrangement of 35274.) Show that $m-n$ is divisible by 9.

38. (Test for Divisibility by 11) Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 11 if and only if $a_0 - a_1 + a_2 - \cdots (-1)^k a_k$ is divisible by 11.

39. Show that the number 7,176,825,942,116,027,211 is divisible by 9 but not divisible by 11.

40. If m and n are positive integers, prove that the mapping from Z_m to Z_n given by $\phi(x) = x \bmod n$ is a ring homomorphism if and only if n divides m.

41. (Test for Divisibility by 3) Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 3 if and only if $a_k + a_{k-1} + \cdots + a_1 + a_0$ is divisible by 3.

42. (Test for Divisibility by 4) Let n be an integer with decimal representation $a_k a_{k-1} \cdots a_1 a_0$. Prove that n is divisible by 4 if and only if $a_1 a_0$ is divisible by 4.

43. For any integer $n > 1$, prove that $Z_n[x]/\langle x\rangle$ is isomorphic to Z_n.

44. For any integer $n > 1$, prove that $\langle x\rangle$ is a maximal ideal of $Z_n[x]$ if and only if n is prime.

45. Give an example of a ring homomorphism from a commutative ring R to a ring S that maps a zero-divisor in R to the unity of S.

46. Prove that any automorphism of a field F is the identity from the prime subfield to itself.

47. In your head, determine $(2\cdot 10^{75}+2)^{100}$ mod 3 and $(10^{100}+1)^{99}$ mod 3.

48. Determine all ring homomorphisms from Q to Q.

49. Let R and S be commutative rings with unity. If ϕ is a homomorphism from R onto S and the characteristic of R is nonzero, prove that the characteristic of S divides the characteristic of R.

50. Let R be a commutative ring of prime characteristic p. Show that the *Frobenius* map $x \to x^p$ is a ring homomorphism from R to R. If R is a field prove that the mapping is an isomorphism.

51. Is there a ring homomorphism from the reals to some ring whose kernel is the integers?

52. Show that a homomorphism from a field onto a ring with more than one element must be an isomorphism.

53. Suppose that R and S are commutative rings with unities. Let ϕ be a ring homomorphism from R onto S and let A be an ideal of S.
a. If A is prime in S, show that $\phi{-}1(A) = \{x \in R \mid \phi(x) \in A\}$ is prime in R.
b. If A is maximal in S, show that $\phi^{-1}(A)$ is maximal in R.

54. A commutative ring with unity is called a *principal ideal ring* if every ideal of the ring has the form $\langle a\rangle$. Show that the homomorphic image of a principal ideal ring is a principal ideal ring.

55. Let R and S be rings.
a. Show that the mapping from $R \oplus S$ onto R given by $(a, b) \to a$ is a ring homomorphism.
b. Show that the mapping from R to $R \oplus S$ given by $a \to (a, 0)$ is a one-to-one ring homomorphism.
c. Show that $R \oplus S$ is ring-isomorphic to $S \oplus R$.

56. Let R and S be communitive rings. If I is a maximal ideal of R, prove that $I \oplus S$ is a maximal ideal of $R \oplus S$.

57. Show that Z_{mn} is ring-isomorphic to $Z_m \oplus Z_n$ when m and n are relatively prime.

58. Show that if m and n are distinct positive integers, then mZ is not ring-isomorphic to nZ.

59. If ϕ is a ring automorphism of the field $Q[\sqrt[3]{2}] = \{r + s\sqrt[3]{2} + t(\sqrt[3]{2})^2 \mid r, s, t \in Q\}$ determine $\phi(\sqrt[3]{2})$.

60. Show that the only ring automorphism of the real numbers is the identity mapping.

61. Determine all ring homomorphisms from $\mathbf{R}$ to $\mathbf{R}$.

62. Suppose that n divides m and that a is an idempotent of Z_n (i.e., $a^2 = a$). Show that the mapping $x \to ax$ is a ring homomorphism from Z_m to Z_n. Show that the same correspondence need not yield a ring homomorphism if n does not divide m.

63. Show that the operation of multiplication defined in the proof of Theorem 15.6 is well-defined.

64. Let $Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Q\}$ and $Q[\sqrt{5}] = \{a + b\sqrt{5} \mid a, b \in Q\}$. Show that these two rings are not ring-isomorphic.

65. Let $Z[i] = \{a+bi \mid a, b \in Z\}$. Show that the field of quotients of $Z[i]$ is ring-isomorphic to $Q[i] = \{r + si \mid r, s \in Q\}$. (This exercise is referred to in Chapter 18.)

66. Let F be a field. Show that the field of quotients of F is ring-isomorphic to F.

67. Let D be an integral domain and let F be the field of quotients of D.

Show that if E is any field that contains D, then E contains a subfield that is ring-isomorphic to F. (Thus, the field of quotients of an integral domain D is the smallest field containing D.)

68. Explain why a commutative ring with unity that is not an integral domain cannot be contained in a field. (Compare with Theorem 15.6.)

69. Show that the relation $\equiv$ defined in the proof of Theorem 15.6 is an equivalence relation.

70. Give an example of a ring without unity that is contained in a field.

71. Prove that the set T in the proof of Corollary 3 to Theorem 15.5 is ring-isomorphic to the field of rational numbers.

72. For any nonzero complex number $a+bi$ show that $Z[i]/\langle a+bi\rangle$ is ring-isomorphic to $Z[i]/\langle a-bi\rangle$.

73. Let $f(x) \in \mathbf{R}[x]$. If $a+bi$ is a complex zero of $f(x)$ (here $i=\sqrt{-1}$), show that $a-bi$ is a zero of $f(x)$. (This exercise is referred to in Chapter 30.)

74. Let $R = \left\{ \begin{bmatrix} a & b \\ b & a \end{bmatrix} \mid a, b \in Z \right\}$, and let ϕ be the mapping that takes $\begin{bmatrix} a & b \\ b & a \end{bmatrix}$ to $a-b$.

a. Show that ϕ is a homomorphism.
b. Determine the kernel of ϕ.
c. Show that $R/\text{Ker}\ \phi$ is isomorphic to Z.
d. Is $\text{Ker}\ \phi$ a prime ideal?
e. Is $\text{Ker}\ \phi$ a maximal ideal?

75. Show that the prime subfield of a field of characteristic p is ring-isomorphic to Z_p and that the prime subfield of a field of characteristic 0 is ring-isomorphic to Q. (This exercise is referred to in this chapter.)

76. Let n be a positive integer. Show that there is a ring isomorphism from Z_2 to a subring of Z_{2n} if and only if n is odd.

16 Polynomial Rings

We lay down a fundamental principle of generalization by abstraction: The existence of analogies between central features of various theories implies the existence of a general theory which underlies the particular theories and unifies them with respect to those central features. . . .

E. H. Moore, (1862–1932)

Wit lies in recognizing the resemblance among things which differ and the difference between things which are alike.

Madame De Staël

Notation and Terminology

One of the mathematical concepts that students are most familiar with and most comfortable with is that of a polynomial. In high school, students study polynomials with integer coefficients, rational coefficients, real coefficients, and perhaps even complex coefficients. In earlier chapters of this book, we introduced something that was probably new—polynomials with coefficients from Z_n. Notice that all of these sets of polynomials are rings, and, in each case, the set of coefficients is also a ring. In this chapter, we abstract all of these examples into one.

Definition Ring of Polynomials over R

Let R be a commutative ring. The set of formal symbols

$$R[x] = \{a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0 \mid a_i \in R,\ n \text{ is a nonnegative integer}\}$$

is called the *ring of polynomials over R in the indeterminate x*. Two elements

$$a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$$

and

$$b_mx^m + b_{m-1}x^{m-1} + \cdots + b_1x + b_0$$

of $R[x]$ are considered equal if and only if $a_i = b_i$ for all nonnegative integers i. (Define $a_i = 0$ when $i > n$ and $b_i = 0$ when $i > m$.)

In this definition, the symbols $x, x^2, \ldots, x^n$ do not represent "unknown" elements or variables from the ring R. Rather, their purpose is to serve as convenient placeholders that separate the ring elements $a_n, a_{n-1}, \ldots, a_0$. We could have avoided the x's by defining a polynomial as an infinite sequence $a_0, a_1, a_2, \ldots, a_n, 0, 0, 0, \ldots$, but our method takes advantage of the student's experience in manipulating polynomials where x does represent a variable. The disadvantage of our method is that one must be careful not to confuse a polynomial with the function determined by a polynomial. For example, in $Z_3[x]$, the polynomials $f(x) = x$ and $g(x) = x^3$ determine the same function from Z_3 to Z_3, since $f(a) = g(a)$ for all a in Z_3.[1] But $f(x)$ and $g(x)$ are different elements of $Z_3[x]$. Also, in the ring $Z_n[x]$, be careful to reduce only the coefficients and not the exponents modulo n. For example, in $Z_3[x], 5x = 2x$, but $x^5 \neq x^2$.

To make $R[x]$ into a ring, we define addition and multiplication in the usual way.

Definition Addition and Multiplication in $R[x]$

Let R be a commutative ring and let

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0$$

and

$$g(x) = b_m x^m + b_{m-1} x^{m-1} + \cdots + b_1 x + b_0$$

belong to $R[x]$. Then

$$f(x) + g(x) = (a_s + b_s)x^s + (a_{s-1} + b_{s-1})x^{s-1} + \cdots + (a_1 + b_1)x + a_0 + b_0,$$

where s is the maximum of m and n, $a_i = 0$ for $i > n$, and $b_i = 0$ for $i > m$. Also,

$$f(x)g(x) = c_{m+n} x^{m+n} + c_{m+n-1} x^{m+n-1} + \cdots + c_1 x + c_0,$$

where

$$c_k = a_k b_0 + a_{k-1} b_1 + \cdots + a_1 b_{k-1} + a_0 b_k$$

for $k = 0, \ldots, m + n$.

[1] In general, given $f(x)$ in $R[x]$ and a in R, $f(a)$ means substitute a for x in the formula for $f(x)$. This substitution is a homomorphism from $R[x]$ to R.

Although the definition of multiplication might appear complicated, it is just a formalization of the familiar process of using the distributive property and collecting like terms. So, just multiply polynomials over a commutative ring R in the same way that polynomials are always multiplied. Here is an example.

Consider $f(x) = 2x^3 + x^2 + 2x + 2$ and $g(x) = 2x^2 + 2x + 1$ in $Z_3[x]$. Then, in our preceding notation, $a_5 = 0, a_4 = 0, a_3 = 2, a_2 = 1, a_1 = 2, a_0 = 2$, and $b_5 = 0, b_4 = 0, b_3 = 0, b_2 = 2, b_1 = 2, b_0 = 1$. Now, using the definitions and remembering that addition and multiplication of the coefficients are done modulo 3, we have

$$\begin{aligned} f(x) + g(x) &= (2+0)x^3 + (1+2)x^2 + (2+2)x + (2+1) \\ &= 2x^3 + 0x^2 + 1x + 0 \\ &= 2x^3 + x \end{aligned}$$

and

$$\begin{aligned} f(x) \cdot g(x) &= (0 \cdot 1 + 0 \cdot 2 + 2 \cdot 2 + 1 \cdot 0 + 2 \cdot 0 + 2 \cdot 0)x^5 \\ &\quad + (0 \cdot 1 + 2 \cdot 2 + 1 \cdot 2 + 2 \cdot 0 + 2 \cdot 0)x^4 \\ &\quad + (2 \cdot 1 + 1 \cdot 2 + 2 \cdot 2 + 2 \cdot 0)x^3 \\ &\quad + (1 \cdot 1 + 2 \cdot 2 + 2 \cdot 2)x^2 + (2 \cdot 1 + 2 \cdot 2)x + 2 \cdot 1 \\ &= x^5 + 0x^4 + 2x^3 + 0x^2 + 0x + 2 \\ &= x^5 + 2x^3 + 2. \end{aligned}$$

Our definitions for addition and multiplication of polynomials were formulated so that they are commutative and associative, and so that multiplication is distributive over addition. We leave the verification that $R[x]$ is a ring to the reader.

It is time to introduce some terminology for polynomials. If

$$f(x) = a_n x^n + a_{n-1} x^{n-1} + \cdots + a_1 x + a_0,$$

where $a_n \neq 0$, we say that $f(x)$ has *degree* n; the term a_n is called the *leading coefficient* of $f(x)$, and if the leading coefficient is the multiplicative identity element of R, we say that $f(x)$ is a *monic* polynomial. The polynomial $f(x) = 0$ has no degree. Polynomials of the form $f(x) = a_0$ are called *constant.* We often write deg $f(x) = n$ to indicate that $f(x)$ has degree n. As with polynomials with real coefficients, we may insert or delete terms of the form $0x^k$; $1x^k$ is the same as x^k; and $+(-a_k)x^k$ is the same as $-a_k x^k$.

Very often properties of R carry over to $R[x]$. Our first theorem is a case in point.

Theorem 16.1 D an Integral Domain Implies $D[x]$ an Integral Domain

> ***If D is an integral domain, then $D[x]$ is an integral domain.***

PROOF Since we already know that $D[x]$ is a ring, all we need to show is that $D[x]$ is commutative with a unity and has no zero-divisors. Clearly, $D[x]$ is commutative whenever D is. If 1 is the unity element of D, it is obvious that $f(x) = 1$ is the unity element of $D[x]$. Finally, suppose that

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0$$

and

$$g(x) = b_m x^m + b_{m-1}x^{m-1} + \cdots + b_0,$$

where $a_n \neq 0$ and $b_m \neq 0$. Then, by definition, $f(x)g(x)$ has leading coefficient $a_n b_m$ and, since D is an integral domain, $a_n b_m \neq 0$. ∎

The Division Algorithm and Consequences

One of the properties of integers that we have used repeatedly is the division algorithm: If a and b are integers and $b \neq 0$, then there exist unique integers q and r such that $a = bq + r$, where $0 \leq r < |b|$. The next theorem is the analogous statement for polynomials over a field.

Theorem 16.2 Division Algorithm for $F[x]$

> ***Let F be a field and let $f(x), g(x) \in F[x]$ with $g(x) \neq 0$. Then there exist unique polynomials $q(x)$ and $r(x)$ in $F[x]$ such that $f(x) = g(x)q(x) + r(x)$ and either $r(x) = 0$ or deg $r(x) <$ deg $g(x)$.***

PROOF We begin by showing the existence of $q(x)$ and $r(x)$. If $f(x) = 0$ or deg $f(x) < \deg g(x)$, we simply set $q(x) = 0$ and $r(x) = f(x)$. So, we may assume that $n = \deg f(x) \geq \deg g(x) =$

m and let $f(x) = a_nx^n + \cdots + a_0$ and $g(x) = b_mx^m + \cdots + b_0$. The idea behind this proof is to begin just as if you were going to "long divide" $g(x)$ into $f(x)$, then use the Second Principle of Mathematical Induction on deg $f(x)$ to finish up. Thus, resorting to long division, we let $f_1(x) = f(x) - a_nb_m^{-1}x^{n-m}g(x)$.[2] Then, $f_1(x) = 0$ or $\deg f_1(x) < \deg f(x)$; so, by our induction hypothesis, there exist $q_1(x)$ and $r_1(x)$ in $F[x]$ such that $f_1(x) = g(x)q_1(x) + r_1(x)$, where $r_1(x) = 0$ or $\deg r_1(x) < \deg g(x)$.
[Technically, we should get the induction started by proving the case in which deg $f(x) = 0$, but this is trivial.] Thus,

$$\begin{aligned} f(x) &= a_nb_m{}^{-1}x^{n-m}g(x) + f_1(x) \\ &= a_nb_m{}^{-1}x^{n-m}g(x) + q_1(x)g(x) + r_1(x) \\ &= [a_nb_m{}^{-1}x^{n-m} + q_1(x)]g(x) + r_1(x). \end{aligned}$$

So, the polynomials $q(x) = a_nb_m{}^{-1}x^{n-m} + q_1(x)$ and $r(x) = r_1(x)$ have the desired properties.

To prove uniqueness, suppose that $f(x) = g(x)q(x) + r(x)$ and $f(x) = g(x)\bar{q}(x) + \bar{r}(x)$, where $r(x) = 0$ or $\deg r(x) < \deg g(x)$ and $\bar{r}(x) = 0$ or $\deg \bar{r}(x) < deg\ g(x)$. Then, subtracting these two equations, we obtain

$$0 = g(x)[q(x) - \bar{q}(x)] + [r(x) - \bar{r}(x)]$$

or

$$\bar{r}(x) - r(x) = g(x)(q(x) - \bar{q}(x)).$$

[2]For example,

$$\begin{array}{r|l} & (3/2)x^2 \\ \hline 2x^2+2 & 3x^4 \qquad\quad + x + 1 \\ & 3x^4 + 3x^2 \\ \hline & \quad - 3x^2 + x + 1 \end{array}$$

So,

$$-3x^2 + x + 1 = 3x^4 + x + 1 - (3/2)x^2(2x^2 + 2)$$

In general,

$$\begin{array}{r|l} & a_nb_m{}^{-1}x^{n-m} \\ \hline b_mx^m + \cdots & a_nx^n + \cdots \\ & a_nx^n + \cdots \\ \hline & \qquad f_1(x) \end{array}$$

So,

$$f_1(x) = (a_nx^n + \cdots) - a_nb_m{}^{-1}x^{n-m}(b_mx^m + \cdots)$$

Thus, $\bar{r}(x) - r(x)$ is 0, or the degree of $\bar{r}(x) - r(x)$ is at least that of $g(x)$. Since the latter is clearly impossible, we have $\bar{r}(x) = r(x)$ and $q(x) = \bar{q}(x)$ as well. ∎

The polynomials $q(x)$ and $r(x)$ in the division algorithm are called the *quotient* and *remainder* in the division of $f(x)$ by $g(x)$. When the ring of coefficients of a polynomial ring is a field, we can use the long division process to determine the quotient and remainder.

EXAMPLE 1 To find the quotient and remainder upon dividing $f(x) = 3x^4 + x^3 + 2x^2 + 1$ by $g(x) = x^2 + 4x + 2$, where $f(x)$ and $g(x)$ belong to $Z_5[x]$, we may proceed by long division, provided we keep in mind that addition and multiplication are done modulo 5. Thus,

$$
\begin{array}{rrrrrr}
 & 3x^2 & +4x & & & \\ \cline{2-6}
x^2+4x+2\,\big) & 3x^4 & +x^3 & +2x^2 & & +1 \\
 & 3x^4 & +2x^3 & +x^2 & & \\ \cline{2-6}
 & & 4x^3 & +x^2 & & +1 \\
 & & 4x^3 & +x^2 & +3x & \\ \cline{2-6}
 & & & & 2x & +1
\end{array}
$$

So, $3x^2+4x$ is the quotient and $2x+1$ is the remainder. Therefore,

$$3x^4 + x^3 + 2x^2 + 1 = (x^2 + 4x + 2)(3x^2 + 4x) + 2x + 1.$$

∎

Let D be an integral domain. If $f(x)$ and $g(x) \in D[x]$, we say that $g(x)$ *divides* $f(x)$ in $D[x]$ [and write $g(x) \mid f(x)$] if there exists an $h(x) \in D[x]$ such that $f(x) = g(x)h(x)$. In this case, we also call $g(x)$ a *factor* of $f(x)$. An element a is a *zero* (or a *root*) of a polynomial $f(x)$ if $f(a) = 0$. [Recall that $f(a)$ means substitute a for x in the expression for $f(x)$.] When F is a field, $a \in F$, and $f(x) \in F[x]$, we say that a is a *zero of multiplicity* $k (k \geq 1)$ if $(x - a)^k$ is a factor of $f(x)$ but $(x - a)^{k+1}$ is not a factor of $f(x)$. With these definitions, we may now give several important corollaries of the division algorithm. No doubt you have seen these for the special case where F is the field of real numbers.

Corollary 1 Remainder Theorem

Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then $f(a)$ is the remainder in the division of $f(x)$ by $x - a$.

PROOF The proof of Corollary 1 is left as an exercise (Exercise 5).

Corollary 2 Factor Theorem

Let F be a field, $a \in F$, and $f(x) \in F[x]$. Then a is a zero of $f(x)$ if and only if $x - a$ is a factor of $f(x)$.

PROOF The proof of Corollary 2 is left as an exercise (Exercise 5).

Theorem 16.3 Polynomials of Degree n Have at Most n Zeros

A polynomial of degree n over a field has at most n zeros, counting multiplicity.

PROOF We proceed by induction on n. Clearly, a polynomial of degree 0 over a field has no zeros. Now suppose that $f(x)$ is a polynomial of degree n over a field and a is a zero of $f(x)$ of multiplicity k. Then, $f(x) = (x - a)^k q(x)$ and $q(a) \neq 0$; and, since $n = \deg f(x) = \deg(x - a)^k q(x) = k + \deg q(x)$, we have $k \leq n$ (see Exercise 19). If $f(x)$ has no zeros other than a, we are done. On the other hand, if $b \neq a$ and b is a zero of $f(x)$, then $0 = f(b) = (b - a)^k q(b)$, so that b is also a zero of $q(x)$ with the same multiplicity as it has for $f(x)$ (see Exercise 21). By the Second Principle of Mathematical Induction, we know that $q(x)$ has at most $\deg q(x) = n - k$ zeros, counting multiplicity. Thus, $f(x)$ has at most $k + n - k = n$ zeros, counting multiplicity.

We remark that Theorem 16.3 is not true for arbitrary polynomial rings. For example, the polynomial $x^2 + 7$ has 1, 3, 5 and 7 as zeros over Z_8. Lagrange was the first to prove Theorem 16.3 for polynomials in $Z_p[x]$.

EXAMPLE 2 The Complex Zeros of $X^n - 1$

We find all complex zeros of $x^n - 1$. Let $\omega = \cos(360°/n) +$

$i \sin(360°/n)$. It follows from DeMoivre's Theorem (see Example 11 in Chapter 0) that $\omega^n = 1$ and $\omega^k \neq 1$ for $1 \leq k < n$. Thus, each of $1, \omega, \omega^2, \ldots, \omega^{n-1}$ is a zero of $x^n - 1$ and, by Theorem 16.4, there are no others. ■

The complex number ω in Example 2 is called a *primitive nth root of unity.*

We conclude this chapter with an important theoretical application of the division algorithm, but first an important definition.

Definition Principal Ideal Domain (PID)
A *principal ideal domain* is an integral domain R in which every ideal has the form $\langle a \rangle = \{ra \mid r \in R\}$ for some a in R.

Theorem 16.4 $F[x]$ Is a PID

Let F be a field. Then $F[x]$ is a principal ideal domain.

PROOF By Theorem 16.1, we know that $F[x]$ is an integral domain. Now, let I be an ideal in $F[x]$. If $I = \{0\}$, then $I = \langle 0 \rangle$. If $I \neq \{0\}$, then among all the elements of I, let $g(x)$ be one of minimum degree. We will show that $I = \langle g(x) \rangle$. Since $g(x) \in I$, we have $\langle g(x) \rangle \subseteq I$. Now let $f(x) \in I$. Then, by the division algorithm, we may write $f(x) = g(x)q(x) + r(x)$, where $r(x) = 0$ or $\deg r(x) < \deg g(x)$. Since $r(x) = f(x) - g(x)q(x) \in I$, the minimality of $\deg g(x)$ implies that the latter condition cannot hold. So, $r(x) = 0$ and, therefore, $f(x) \in \langle g(x) \rangle$. This shows that $I \subseteq \langle g(x) \rangle$. ■

Because $Z[x]$ is an iconic integral domain of polynomials, it is natural to wonder if it is a principal ideal domain. The answer is no. The ideal of all elements in $Z[x]$ with even constant term is not generated by a single element. See Exercise 45.

The proof of Theorem 16.4 also establishes the following.

Theorem 16.5 Criterion for $I = \langle g(x) \rangle$

Let F be a field, I a nonzero ideal in $F[x]$, and $g(x)$ an element of $F[x]$. Then, $I = \langle g(x) \rangle$ if and only if $g(x)$ is a polynomial of minimum degree in I.

As an application of the First Isomorphism Theorem for Rings (Theorem 15.3) and Theorem 16.5, we verify the remark we made in Example 12 in Chapter 14 that the ring $\mathbf{R}[x]/\langle x^2+1\rangle$ is isomorphic to the ring of complex numbers.

EXAMPLE 3 Consider the homomorphism ϕ from $\mathbf{R}[x]$ onto $\mathbf{C}$ given by $f(x) \to f(i)$ (i.e., evaluate a polynomial in $\mathbf{R}[x]$ at i). Then $x^2+1 \in \operatorname{Ker} \phi$ and is clearly a polynomial of minimum degree in $\operatorname{Ker} \phi$. Thus, $\operatorname{Ker} \phi = \langle x^2+1\rangle$ and $\mathbf{R}[x]/\langle x^2+1\rangle$ is isomorphic to $\mathbf{C}$. ∎

Exercises

If I feel unhappy, I do mathematics to become happy. If I am happy, I do mathematics to keep happy.

Paul Turán

1. Let $f(x) = 4x^3 + 2x^2 + x + 3$ and $g(x) = 3x^4 + 3x^3 + 3x^2 + x + 4$, where $f(x), g(x) \in Z_5[x]$. Compute $f(x) + g(x)$ and $f(x) \cdot g(x)$.
2. In $Z_3[x]$, show that the distinct polynomials x^4+x and x^2+x determine the same function from Z_3 to Z_3.
3. Show that $x^2 + 3x + 2$ has four zeros in Z_6.
4. If R is a commutative ring, show that the characteristic of $R[x]$ is the same as the characteristic of R.
5. Prove that Theorem 16.2 and its corollaries are true for $D[x]$ where D is integral domain and $g(x)$ has a lead coefficient that is a unit.
6. List all the polynomials of degree 2 in $Z_2[x]$. Which of these are equal as functions from Z_2 to Z_2?
7. Prove that every element in the ring of polynomial functions from Z_3 to Z_3 under the usual definition of addition and multiplication can be written in the form $ax^2 + bx + c$ where $a, b, c \in Z_3$. Generalize to Z_p where p is prime. [Note that this ring is not $Z_3[x]$–see the discussion following the definition of $R[x]$.]
8. Prove that the ring of polynomial functions with coefficients from Z_3 and the usual definition of addition and multiplication is not an integral domain. Why does this statement not contradict Theorem 16.1?

9. Write $x^3 + x^2 - 2$ as a product of linear factors in $Z_3[x]$.

10. For any positive integer n, how many polynomials are there of degree n over Z_2? How many distinct polynomial functions from Z_2 to Z_2 are there?

11. Let $f(x) = 5x^4 + 3x^3 + 1$ and $g(x) = 3x^2 + 2x + 1$ in $Z_7[x]$. Determine the quotient and remainder upon dividing $f(x)$ by $g(x)$.

12. Find a polynomial with real coefficients that has i and $2+i$ as zeros.

13. If ϕ: $R \to S$ is a ring homomorphism, define $\bar{\phi}$: $R[x] \to S[x]$ by $(a_n x^n + \cdots + a_0) \to \phi(a_n)x^n + \cdots + \phi(a_0)$. Show that $\bar{\phi}$ is a ring homomorphism. (This exercise is referred to in Chapters 17 and 31.)

14. If the rings R and S are isomorphic, show that $R[x]$ and $S[x]$ are isomorphic. (The converse to not true.)

15. Prove that $Z_4[x]$ has infinitely many units and infinitely many nilpotent elements.

16. Let $f(x)$ and $g(x)$ be cubic polynomials with integer coefficients such that $f(a) = g(a)$ for four integer values of a. Prove that $f(x) = g(x)$. Generalize.

17. Show that the polynomial $2x+1$ in $Z_4[x]$ has a multiplicative inverse in $Z_4[x]$.

18. Show that Theorem 16.4 is false for any commutative ring that has a zero divisor.

19. (Degree Rule) Let D be an integral domain and $f(x), g(x) \in D[x]$. Prove that deg $(f(x) \cdot g(x)) =$ deg $f(x)$ + deg $g(x)$. Show, by example, that for a commutative ring R it is possible that deg $f(x)g(x) <$ deg $f(x)$+deg $g(x)$, where $f(x)$ and $g(x)$ are nonzero elements in $R[x]$. (This exercise is referred to in this chapter, Chapter 17, and Chapter 18.)

20. Prove that the only idempotent elements in $Z_4[x]$ are 0 and 1.

21. Let $f(x)$ belong to $F[x]$, where F is a field. Let a be a zero of $f(x)$ of multiplicity n, and write $f(x) = (x-a)^n q(x)$. If $b \neq a$ is a zero of $q(x)$, show that b has the same multiplicity as a zero of $q(x)$ as it does for $f(x)$. (This exercise is referred to in this chapter.)

22. Prove that $x^3 - x^2 - 4$ has a zero in every field.

23. Let R be a commutative ring. For any $r \in R$ prove that the mapping from $R[x]$ to R given by $\phi(f(x)) = f(r)$ is a ring homomorphism. This mapping is called the *evaluation homomorphism.*

24. If $f(x) \in \mathbf{C}[x]$ and $f(i) = 0$ prove that $f(i^3) = 0$.

25. For which primes p is $x - 2$ a factor of $f(x) = x^4 - 2x - 2$ over Z_p?

26. Let b be a unit in a commutative ring. If b a zero of $a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, prove that b^{-1} is a zero of $a_0 x^n + a_1 x^{n-1} + \cdots + a_{n-1}x + a_n$.

27. Prove that for any prime p the group $U(p)$ is cyclic.

28. Show that the ring $Z_4[x]$ has infinitely many elements with multiplicative order 2.

29. Give an example of be a commutative ring that has a polynomial of degree 1 that has five distinct zeros.

30. Let $n > 1$ be a non-prime integer. Show that there is a polynomial $f(x) \in Z_n[x]$ of degree $n - 1$ that has n distinct zeros. (Compare with Theorem 16.3.)

31. For any field F, recall that $F(x)$ denotes the field of quotients of the ring $F[x]$. Prove that there is no element in $F(x)$ whose square is x.

32. Give an example of a polynomial in $Z_5[x]$ of positive degree that has the property that $f(a) = 1$ for all $a \in Z_5$.

33. Let D be an integral domain. Prove that no nonconstant element in $D[x]$ can have a multiplicative inverse in $D[x]$.

34. Write $x^3 + 2x^2 - x - 2$ as a product of linear polynomials over Z_3.

35. Prove that $x^3 - 2x^2 - 9$ has a zero in every field.

36. Let F be a field and $f(x)$ and $g(x)$ be nonzero elements in $F[x]$. Prove that $\langle f(x)\rangle = \langle g(x)\rangle$ if and only if $f(x) = cg(x)$ for some c in F.

37. Let F be a field and let I be the ideal $\{f(x) \in F[x] \mid f(1) = 0 \text{ and } f(2) = 0\}$. Find an element $g(x) \in F[x]$ such that $I = \langle g(x)\rangle$.

38. If the only nilpotent element in Z_n is 0, prove that no element in $Z_n[x]$ of positive degree is an idempotent.
39. Determine all zeros of $x^9 + 1$ in $Z_{17}[x]$.
40. Prove that $Q[x]/\langle x^2 + 1\rangle$ isomorphic to the subfield of complex numbers $Q(i) = \{a + bi \mid a, b \in Q\}$.
41. Show that $x + 1$ is a factor of $f(x) = x^{25} + 1$ in Z_{37} but $(x + 1)^2$ is not.
42. Suppose the $f(x) \in Z_5[x]$ is a non-constant polynomial that has the property that $f(a) = 2$ for all $a \in Z_5$. What is the minimum possible degree for $f(x)$?
43. Let F be a field, and let $f(x)$ and $g(x)$ belong to $F[x]$ and not both zero. If there is no polynomial of positive degree in $F[x]$ that divides both $f(x)$ and $g(x)$ [in this case, $f(x)$ and $g(x)$ are said to be *relatively prime*], prove that there exist polynomials $h(x)$ and $k(x)$ in $F[x]$ with the property that $f(x)h(x) + g(x)k(x) = 1$. (This exercise is referred to in Chapter 19.)
44. If F is a field and $f(x)$ and $g(x)$ belong to $F[x]$ and are not both zero, a polynomial $d(x)$ is $F[x]$ is said to be a *greatest common divisor* of $f(x)$ and $g(x)$ if $d(x)$ divides both $f(x)$ and $g(x)$ and $d(x)$ has maximum degree among all such polynomials. Prove that for any $f(x)$ and $g(x)$ belong to $F[x]$, not both zero, $\langle f(x), g(x)\rangle = \langle d(x)\rangle$, where $d(x)$ is a greatest common divisor of $f(x)$ and $g(x)$. Prove that for any $f(x)$ and $g(x)$ belong to $F[x]$, are not both zero, there is a unique monic greatest common divisor of $f(x)$ and $g(x)$.
45. In $Z[x]$, prove that the ideal $I = \langle x, 2\rangle$ is not a principal ideal.
46. Let F be an infinite field and let $f(x), g(x) \in F[x]$. If $f(a) = g(a)$ for infinitely many elements a of F, show that $f(x) = g(x)$.
47. Let F be a field and let $p(x) \in F[x]$. If $f(x), g(x) \in F[x]$ and $\deg f(x) < \deg p(x)$ and $\deg g(x) < \deg p(x)$, show that $f(x) + \langle p(x)\rangle = g(x) + \langle p(x)\rangle$ implies $f(x) = g(x)$. (This exercise is referred to in Chapter 19.)
48. Find a polynomial with integer coefficients that has $1/2$ and $-1/3$ as zeros.

49. Prove that for any fixed positive integer n, the number of elements in a field that have multiplicative order at most n is at most $1 + 2 + \cdots + n$.

50. Let $f(x) \in \mathbf{R}[x]$. Suppose that $f(a) = 0$ but $f'(a) \neq 0$, where $f'(x)$ is the derivative of $f(x)$. Show that a is a zero of $f(x)$ of multiplicity 1.

51. Show that Corollary 2 of Theorem 16.2 is true over any commutative ring with unity.

52. Show that Theorem 16.3 is true for polynomials over integral domains.

53. Let F be a field and let

$$I = \{a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \mid a_n, a_{n-1}, \ldots, a_0 \in F \text{ and } a_n + a_{n-1} + \cdots + a_0 = 0\}.$$

Show that I is an ideal of $F[x]$ and find a generator for I.

54. Let F be a field and let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in F[x]$. Prove that $x - 1$ is a factor of $f(x)$ if and only if $a_n + a_{n-1} + \cdots + a_0 = 0$.

55. For any prime p prove that the ring $Z_{p^2}[x]$ has infinitely many units.

56. Find infinitely many polynomials $f(x)$ in $Z_3[x]$ such that $f(a) = 0$ for all a in Z_3.

57. For every prime p, show that

$$x^{p-1} - 1 = (x-1)(x-2)\cdots(x-(p-1))$$

in $Z_p[x]$.

58. (Wilson's Theorem) For every integer $n > 1$, prove that $(n-1)! \bmod n = n - 1$ if and only if n is prime.

59. For every prime p, show that $(p-2)! \bmod p = 1$.

60. Find the remainder upon dividing 98! by 101.

61. Find an element a in Z_5 such that $x + 4$ is a factor of $x^{48} + x^{21} + a$.

62. Let ϕ be the ring homomorphism from $Q[x]$ to Q given by $\phi(f(x)) = f(1)$. Find the monic polynomial $g(x)$ in $Q[x]$ such that Ker $\phi = \langle g(x) \rangle$. To which familiar ring is $Q[x]/\text{Ker } \phi$ isomorphic?

63. Give an example of a field that properly contains the field of complex numbers $\mathbf{C}$.

64. If I is an ideal of a ring R, prove that $I[x]$ is an ideal of $R[x]$.

65. Give an example of a commutative ring R with unity and a maximal ideal I of R such that $I[x]$ is not a maximal ideal of $R[x]$.

66. Let R be a commutative ring with unity. If I is a prime ideal of R, prove that $I[x]$ is a prime ideal of $R[x]$.

67. Show that 1 is the only solution of $x^{25} - 1 = 0$ in Z_{37}.

68. Prove that $Q[x]/\langle x^2 - 2\rangle$ is ring-isomorphic to $Q[\sqrt{2}] = \{a + b\sqrt{2} \mid a, b \in Q\}$.

69. Let $f(x) \in \mathbf{R}[x]$. If $f(a) = 0$ and $f'(a) = 0$ [$f'(a)$ is the derivative of $f(x)$ at a], show that $(x - a)^2$ divides $f(x)$.

70. Let F be a field and let $I = \{f(x) \in F[x] \mid f(a) = 0$ for all a in $F\}$. Prove that I is an ideal in $F[x]$. Prove that I is infinite when F is finite and $I = \{0\}$ when F is infinite. When F is finite, find a monic polynomial $g(x)$ such that $I = \langle g(x)\rangle$.

71. Let $g(x)$ and $h(x)$ belong to $Z[x]$ and let $h(x)$ be monic. If $h(x)$ divides $g(x)$ in $Q[x]$, show that $h(x)$ divides $g(x)$ in $Z[x]$. (This exercise is referred to in Chapter 31.)

72. Let R be a ring and x be an indeterminate. Prove that the rings $R[x]$ and $R[x^2]$ are ring-isomorphic.

73. Let $f(x)$ be a nonconstant element of $Z[x]$. Prove that $f(x)$ takes on infinitely many values in Z.

74. Let F be a field. Find a generator for the ideal $I = \{f(x) \in F(x) \mid f(0) = 0$ and $f(1) = 0\}$. Generalize.

75. Suppose that F is a field and there is a ring homomorphism from Z onto F. Show that F is isomorphic to Z_p for some prime p.

76. Let p be a prime, b belong to Z_p, and $f(x)$ belong to $Z_p[x]$. Prove that if $f(b) = 0$, then $f(b^p) = 0$. Generalize.

77. Suppose $f(x)$ is a polynomial with odd integer coefficients and even degree. Prove that $f(x)$ has no rational zeros.

78. Find the remainder when x^{51} is divided by $x + 4$ in $Z_7[x]$.

79. Let F be a field. Show that there exist $a, b \in F$ with the property that $x^2 + x + 1$ divides $x^{43} + ax + b$.

17 Factorization of Polynomials

Very early in our mathematical education—in fact in junior high school or early in high school itself—we are introduced to polynomials. For a seemingly endless amount of time we are drilled, to the point of utter boredom, in factoring them, multiplying them, dividing them, simplifying them. Facility in factoring a quadratic becomes confused with genuine mathematical talent.

I. N. Herstein, *Topics in Algebra*

The value of a principle is the number of things it will explain.

Ralph Waldo Emerson

Reducibility Tests

In high school, students spend much time factoring polynomials and finding their zeros. In this chapter, we consider the same problems in a more abstract setting.

To discuss factorization of polynomials, we must first introduce the polynomial analog of a prime integer.

Definitions Irreducible Polynomial, Reducible Polynomial
Let D be an integral domain. A polynomial $f(x)$ from $D[x]$ that is neither the zero polynomial nor a unit in $D[x]$ is said to be *irreducible over D* if, whenever $f(x)$ is expressed as a product $f(x) = g(x)h(x)$, with $g(x)$ and $h(x)$ from $D[x]$, then $g(x)$ or $h(x)$ is a unit in $D[x]$. A nonzero, nonunit element of $D[x]$ that is not irreducible over D is called *reducible over D*.

In the case that an integral domain is a field F, it is equivalent and more convenient to define a nonconstant $f(x) \in F[x]$ to be irreducible if $f(x)$ cannot be expressed as a product of two polynomials of lower degree.

EXAMPLE 1 The polynomial $f(x) = 2x^2 + 4$ is irreducible over Q but reducible over Z, since $2x^2 + 4 = 2(x^2 + 2)$ and neither 2 nor $x^2 + 2$ is a unit in $Z[x]$. ∎

EXAMPLE 2 The polynomial $f(x) = 2x^2 + 4$ is irreducible over $\mathbf{R}$ but reducible over $\mathbf{C}$. ∎

EXAMPLE 3 The polynomial $x^2 - 2$ is irreducible over Q but reducible over $\mathbf{R}$. ∎

EXAMPLE 4 The polynomial $x^2 + 1$ is irreducible over Z_3 but reducible over Z_5. ∎

In general, it is a difficult problem to decide whether or not a particular polynomial is reducible over an integral domain, but there are special cases when it is easy. Our first theorem is a case in point. It applies to the four preceding examples.

Theorem 17.1 Reducibility Test for Degrees 2 and 3

Let F be a field. If $f(x) \in F[x]$ and deg $f(x)$ is 2 or 3, then $f(x)$ is reducible over F if and only if $f(x)$ has a zero in F.

PROOF Suppose that $f(x) = g(x)h(x)$, where both $g(x)$ and $h(x)$ belong to $F[x]$ and have degrees less than that of $f(x)$. Since deg $f(x) = \deg g(x) + \deg h(x)$ (Exercise 19 in Chapter 16) and deg $f(x)$ is 2 or 3, at least one of $g(x)$ and $h(x)$ has degree 1. Say $g(x) = ax + b$. Then, clearly, $-a^{-1}b$ is a zero of $g(x)$ and therefore a zero of $f(x)$ as well.

Conversely, suppose that $f(a) = 0$, where $a \in F$. Then, by the Factor Theorem, we know that $x - a$ is a factor of $f(x)$ and, therefore, $f(x)$ is reducible over F. ∎

Theorem 17.1 is particularly easy to use when the field is Z_p, because in this case we can check for reducibility of $f(x)$ by simply testing to see if $f(a) = 0$ for $a = 0, 1, \ldots, p-1$. For example, since 2 is a zero of $x^2 + 1$ over $Z_5, x^2 + 1$ is reducible over Z_5. On the other hand, because neither 0, 1, nor 2 is a zero of $x^2 + 1$ over $Z_3, x^2 + 1$ is irreducible over Z_3.

Note that polynomials of degree larger than 3 may be reducible over a field even though they do not have zeros in the field. For example, in $Q[x]$, the polynomial x^4+2x^2+1 is equal to $(x^2+1)^2$, but has no zeros in Q.

Our next three tests deal with polynomials with integer coefficients. To simplify the proof of the first of these, we introduce some terminology and isolate a portion of the argument in the form of a lemma.

Definitions Content of a Polynomial, Primitive Polynomial
The *content* of a nonzero polynomial $a_nx^n+a_{n-1}x^{n-1}+\cdots+a_0$, where the a's are integers, is the greatest common divisor of the integers $a_n, a_{n-1}, \ldots, a_0$. A *primitive polynomial* is an element of $Z[x]$ with content 1.

Gauss's Lemma

The product of two primitive polynomials is primitive.

PROOF Let $f(x)$ and $g(x)$ be primitive polynomials, and suppose that $f(x)g(x)$ is not primitive. Let p be a prime divisor of the content of $f(x)g(x)$, and let $\bar{f}(x), \bar{g}(x)$, and $\overline{f(x)g(x)}$ be the polynomials obtained from $f(x), g(x)$, and $f(x)g(x)$ by reducing the coefficients modulo p. Then, $\bar{f}(x)$ and $\bar{g}(x)$ belong to the integral domain $Z_p[x]$ and $\bar{f}(x)\bar{g}(x) = \overline{f(x)g(x)} = 0$, the zero element of $Z_p[x]$ (see Exercise 55 in Chapter 16). Thus, $\bar{f}(x) = 0$ or $\bar{g}(x) = 0$. This means that either p divides every coefficient of $f(x)$ or p divides every coefficient of $g(x)$. Hence, either $f(x)$ is not primitive or $g(x)$ is not primitive. This contradiction completes the proof. ∎

Remember that the question of reducibility depends on which ring of coefficients one permits. Thus, x^2-2 is irreducible over Z but reducible over $Q[\sqrt{2}]$. In Chapter 19, we will prove that every polynomial of degree greater than 1 with coefficients from an integral domain is reducible over some field. Theorem 17.2 shows that in the case of polynomials irreducible over Z, this field must be larger than the field of rational numbers.

Theorem 17.2 Reducibility over Q Implies Reducibility over Z

Let $f(x) \in Z[x]$. If $f(x)$ is reducible over Q, then it is reducible over Z.

PROOF Suppose that $f(x) = g(x)h(x)$, where $g(x)$ and $h(x) \in Q[x]$. Clearly, we may assume that $f(x)$ is primitive because we can divide both $f(x)$ and $g(x)$ by the content of $f(x)$. Let a be the least common multiple of the denominators of the coefficients of $g(x)$, and b the least common multiple of the denominators of the coefficients of $h(x)$. Then $abf(x) = ag(x) \cdot bh(x)$, where $ag(x)$ and $bh(x) \in Z[x]$. Let c_1 be the content of $ag(x)$ and let c_2 be the content of $bh(x)$. Then $ag(x) = c_1g_1(x)$ and $bh(x) = c_2h_1(x)$, where both $g_1(x)$ and $h_1(x)$ are primitive, and $abf(x) = c_1c_2g_1(x)h_1(x)$. Since $f(x)$ is primitive, the content of $abf(x)$ is ab. Also, since the product of two primitive polynomials is primitive, it follows that the content of $c_1c_2g_1(x)h_1(x)$ is c_1c_2. Thus, $ab = c_1c_2$ and $f(x) = g_1(x)h_1(x)$, where $g_1(x)$ and $h_1(x) \in Z[x]$ and $\deg g_1(x) = \deg g(x)$ and $\deg h_1(x) = \deg h(x)$. ■

EXAMPLE 5 We illustrate the proof of Theorem 17.2 by tracing through it for the polynomial $f(x) = 6x^2 + x - 2 = (3x - 3/2)(2x + 4/3) = g(x)h(x)$. In this case we have $a = 2, b = 3, c_1 = 3, c_2 = 2, g_1(x) = 2x - 1$, and $h_1(x) = 3x + 2$, so that $2 \cdot 3(6x^2 + x - 2) = 3 \cdot 2(2x - 1)(3x + 2)$ or $6x^2 + x - 2 = (2x - 1)(3x + 2)$. ■

Irreducibility Tests

Theorem 17.1 reduces the question of irreducibility of a polynomial of degree 2 or 3 to one of finding a zero. The next theorem often allows us to simplify the problem even further.

Theorem 17.3 Mod p Irreducibility Test

Let p be a prime and suppose that $f(x) \in Z[x]$ with deg $f(x) \geq 1$. Let $\bar{f}(x)$ be the polynomial in $Z_p[x]$ obtained from $f(x)$ by reducing all the coefficients of $f(x)$ modulo p. If $\bar{f}(x)$ is irreducible over Z_p and deg $\bar{f}(x) =$ deg $f(x)$, then $f(x)$ is irreducible over Q.

PROOF It follows from the proof of Theorem 17.2 that if $f(x)$ is reducible over Q, then $f(x) = g(x)h(x)$ with $g(x)$, $h(x) \in Z[x]$, and both $g(x)$ and $h(x)$ have degree less than that of $f(x)$. Let $\bar{f}(x), \bar{g}(x)$, and $\bar{h}(x)$ be the polynomials obtained from $f(x)$, $g(x)$, and $h(x)$ by reducing all the coefficients modulo p. Since $\deg f(x) = \deg \bar{f}(x)$, we have $\deg \bar{g}(x) \leq \deg g(x) < \deg \bar{f}(x)$ and

$\deg \bar{h}(x) \leq \deg h(x) < \deg \bar{f}(x)$. But, $\bar{f}(x) = \bar{g}(x)\bar{h}(x)$, and this contradicts our assumption that $\bar{f}(x)$ is irreducible over Z_p. ▪

▮ **EXAMPLE 6** Let $f(x) = 21x^3 - 3x^2 + 2x + 9$. Then, over Z_2, we have $\bar{f}(x) = x^3 + x^2 + 1$ and, since $\bar{f}(0) = 1$ and $\bar{f}(1) = 1$, we see that $\bar{f}(x)$ is irreducible over Z_2. Thus, $f(x)$ is irreducible over Q. Notice that, over Z_3, $\bar{f}(x) = 2x$ is irreducible, but we may *not* apply Theorem 17.3 to conclude that $f(x)$ is irreducible over Q. ▮

Be careful not to use the converse of Theorem 17.3. If $f(x) \in Z[x]$ and $\bar{f}(x)$ is reducible over Z_p for some p, $f(x)$ may still be irreducible over Q. For example, consider $f(x) = 21x^3 - 3x^2 + 2x + 8$. Then, over Z_2, $\bar{f}(x) = x^3 + x^2 = x^2(x+1)$. This is a false positive because over Z_5, $\bar{f}(x)$ has no zeros and therefore is irreducible over Z_5. So, $f(x)$ is irreducible over Q. Note that this example shows that the Mod p Irreducibility Test may fail for some p and work for others. To conclude that a particular $f(x)$ in $Z[x]$ is irreducible over Q, all we need to do is find a single p for which the corresponding polynomial $\bar{f}(x)$ in Z_p is irreducible. However, this is not always possible, since $f(x) = x^4 + 1$ is irreducible over Q but reducible over Z_p for *every* prime p.

The Mod p Irreducibility Test can also be helpful in checking for irreducibility of polynomials of degree greater than 3 and polynomials with rational coefficients.

▮ **EXAMPLE 7** Let $f(x) = (3/7)x^4 - (2/7)x^2 + (9/35)x + 3/5$. We will show that $f(x)$ is irreducible over Q. First, let $h(x) = 35f(x) = 15x^4 - 10x^2 + 9x + 21$. Then $f(x)$ is irreducible over Q if $h(x)$ is irreducible over Z. Next, applying the Mod 2 Irreducibility Test to $h(x)$, we get $\bar{h}(x) = x^4 + x + 1$. Clearly, $\bar{h}(x)$ has no zeros in Z_2. Furthermore, $\bar{h}(x)$ has no quadratic factor in $Z_2[x]$ either. [For if so, the factor would have to be either $x^2 + x + 1$ or $x^2 + 1$. Long division shows that $x^2 + x + 1$ is not a factor, and $x^2 + 1$ cannot be a factor because it has a zero, whereas $\bar{h}(x)$ does not.] Thus, $\bar{h}(x)$ is irreducible over $Z_2[x]$. This guarantees that $h(x)$ is irreducible over Q. ▮

▮ **EXAMPLE 8** Let $f(x) = x^5 + 2x + 4$. Obviously, neither Theorem 17.1 nor the Mod 2 Irreducibility Test helps here. Let's try mod 3. Substitution of 0, 1, and 2 into $\bar{f}(x)$ does not yield 0, so there are no linear factors. But $\bar{f}(x)$ may have a quadratic factor. If so, we may assume it has the form $x^2 + ax + b$ where $b \neq 0$ (see Exercise

5). This gives six possibilities to check. We can immediately rule out each of the six that has a zero over Z_3, since $\bar{f}(x)$ does not have one. This leaves only x^2+1, x^2+x+2, and x^2+2x+2 to check. These are eliminated by long division. So, since $\bar{f}(x)$ is irreducible over Z_3, $f(x)$ is irreducible over Q. (Why is it unnecessary to check for cubic or fourth-degree factors?) ∎

Another important irreducibility test is the following one, credited to Ferdinand Eisenstein (1823–1852), a student of Gauss. The corollary was first proved by Gauss by a different method.

Theorem 17.4 Eisenstein's Criterion (1850)

Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in Z[x].$$

If there is a prime p such that $p \nmid a_n, p \mid a_{n-1}, \ldots, p \mid a_0$ and $p^2 \nmid a_0$, then $f(x)$ is irreducible over Q.

PROOF If $f(x)$ is reducible over Q, we know by Theorem 17.2 that there exist elements $g(x)$ and $h(x)$ in $Z[x]$ such that $f(x) = g(x)h(x)$, $1 \leq \deg g(x)$, and $1 \leq \deg h(x) < n$. Say $g(x) = b_r x^r + \cdots + b_0$ and $h(x) = c_s x^s + \cdots + c_0$. Then, since $p \mid a_0$, $p^2 \nmid a_0$, and $a_0 = b_0 c_0$, it follows that p divides one of b_0 and c_0 but not the other. Let us say $p \mid b_0$ and $p \nmid c_0$. Also, since $p \nmid a_n = b_r c_s$, we know that $p \nmid b_r$. So, there is a least integer t such that $p \nmid b_t$. Now, consider $a_t = b_t c_0 + b_{t-1}c_1 + \cdots + b_0 c_t$. By assumption, p divides at and, by choice of t, every summand on the right after the first one is divisible by p. Clearly, this forces p to divide $b_t c_0$ as well. This is impossible, however, since p is prime and p divides neither b_t nor c_0. ∎

Corollary Irreducibility of pth Cyclotomic Polynomial

For any prime p, the pth cyclotomic polynomial

$$\Phi_p(x) = \frac{x^p - 1}{x - 1} = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is irreducible over Q.

PROOF Let

$$f(x)=\Phi_p(x+1)=\frac{(x+1)^p-1}{(x+1)-1}=x^{p-1}+\binom{p}{1}x^{p-2}\binom{p}{2}x^{p-3}+\cdots+\binom{p}{1}.$$

Then, since $\binom{p}{k}=\frac{p!}{(p-k)!k!}$, every coefficient except that of x^{p-1} is divisible by p and the constant term is not divisible by p^2, by Eisenstein's Criterion, $f(x)$ is irreducible over Q. So, if $\Phi_p(x) = g(x)h(x)$ were a nontrivial factorization of $\Phi_p(x)$ over Q, then $f(x) = \Phi_p(x+1) = g(x+1) \cdot h(x+1)$ would be a nontrivial factorization of $f(x)$ over Q. Since this is impossible, we conclude that $\Phi_p(x)$ is irreducible over Q. ∎

EXAMPLE 9 The polynomial $3x^5 + 15x^4 - 20x^3 + 10x + 20$ is irreducible over Q because $5 \nmid 3$ and $25 \nmid 20$ but 5 does divide 15, -20, 10, and 20. ∎

The principal reason for our interest in irreducible polynomials stems from the fact that there is an intimate connection among them, maximal ideals, and fields. This connection is revealed in the next theorem and its first corollary.

Theorem 17.5 $\langle p(x)\rangle$ Is Maximal If and Only If $p(x)$ Is Irreducible

Let F be a field and let $p(x) \in F[x]$. Then $\langle p(x)\rangle$ is a maximal ideal in $F[x]$ if and only if $p(x)$ is irreducible over F.

PROOF Suppose first that $\langle p(x)\rangle$ is a maximal ideal in $F[x]$. Clearly, $p(x)$ is neither the zero polynomial nor a unit in $F[x]$, because neither $\{0\}$ nor $F[x]$ is a maximal ideal in $F[x]$. If $p(x) = g(x)h(x)$ is a factorization of $p(x)$ over F, then $\langle p(x)\rangle \subseteq \langle g(x)\rangle \subseteq F[x]$. Thus, $\langle p(x)\rangle = \langle g(x)\rangle$ or $F[x] = \langle g(x\rangle$. In the first case, we must have $\deg p(x) = \deg g(x)$. In the second case, it follows that $\deg g(x) = 0$ and, consequently, $\deg h(x) = \deg p(x)$. Thus, $p(x)$ cannot be written as a product of two polynomials in $F[x]$ of lower degree.

Now, suppose that $p(x)$ is irreducible over F. Let I be any ideal of $F[x]$ such that $\langle p(x)\rangle \subseteq I \subseteq F[x]$. Because $F[x]$ is a principal ideal domain, we know that $I = \langle g(x)\rangle$ for some $g(x)$ in $F[x]$. So, $p(x) \in \langle g(x)\rangle$ and, therefore, $p(x) = g(x)h(x)$, where $h(x) \in F[x]$. Since $p(x)$ is irreducible over F, it follows that either $g(x)$ is a constant or $h(x)$ is a constant. In the first case, we have $I = F[x]$; in the second case, we have $\langle p(x)\rangle = \langle g(x)\rangle = I$. So, $\langle p(x)\rangle$ is maximal in $F[x]$. ■

Corollary 1 $F[x]/\langle p(x)\rangle$ Is a Field

Let F be a field and $p(x)$ be an irreducible polynomial over F. Then $F[x]/\langle p(x)\rangle$ is a field.

PROOF This follows directly from Theorems 17.5 and 14.4. ■

The next corollary is a polynomial analog of Euclid's Lemma for primes (see Chapter 0).

Corollary 2 $p(x) \mid a(x)b(x)$ Implies $p(x) \mid a(x)$ or $p(x) \mid b(x)$

Let F be a field and let $p(x), a(x), b(x) \in F[x]$. If $p(x)$ is irreducible over F and $p(x) \mid a(x)b(x)$, then $p(x) \mid a(x)$ or $p(x) \mid b(x)$.

PROOF Since $p(x)$ is irreducible, $F[x]/\langle p(x)\rangle$ is a field and, therefore, an integral domain. From Theorem 14.3, we know that $\langle p(x)\rangle$ is a prime ideal, and since $p(x)$ divides $a(x)b(x)$, we have $a(x)b(x) \in \langle p(x)\rangle$. Thus, $a(x) \in \langle p(x)\rangle$ or $b(x) \in \langle p(x)\rangle$. This means that $p(x) \mid a(x)$ or $p(x) \mid b(x)$. ■

The next two examples put the theory to work.

■ EXAMPLE 10 Field With Eight Elements

We construct a field with eight elements. By Theorem 17.1 and Corollary 1 of Theorem 17.5, it suffices to find a cubic polynomial over Z_2 that has no zero in Z_2. By inspection, x^3+x+1 fills the bill. Thus, $Z_2[x]/\langle x^3+x+1\rangle = \{ax^2+bx+c+\langle x^3+x+1\rangle \mid a, b, c \in Z_2\}$ is a field with eight elements. For practice, let us do a few calculations in this field. Since the sum of two polynomials of the form ax^2+bx+c is another one of the same form, addition is easy. For example,

$$(x^2+x+1+\langle x^3+x+1\rangle) + (x^2+1+\langle x^3+x+1\rangle) \\ = x + \langle x^3+x+1\rangle.$$

Table 17.1 A Partial Multiplication Table for Example 10.

	$\mathbf{1}$	$\boldsymbol{x}$	$\boldsymbol{x+1}$	$\boldsymbol{x^2}$	$\boldsymbol{x^2+1}$	$\boldsymbol{x^2+x}$	$\boldsymbol{x^2+x+1}$
$\mathbf{1}$	1	x	$x+1$	x^2	x^2+1	x^2+x	x^2+x+1
$\boldsymbol{x}$	x	x^2	x^2+x	$x+1$	1	x^2+x+1	x^2+1
$\boldsymbol{x+1}$	$x+1$	x^2+x	x^2+1	x^2+x+1	x^2	1	x
$\boldsymbol{x^2}$	x^2	$x+1$	x^2+x+1	x^2+x	x	x^2+1	1
$\boldsymbol{x^2+1}$	x^2+1	1	x^2	x	x^2+x+1	$x+1$	x^2+x

On the other hand, multiplication of two coset representatives need not yield one of the original eight coset representatives:

$$\begin{aligned}&(x^2+x+1+\langle x^3+x+1\rangle)\cdot(x^2+1+\langle x^3+x+1\rangle)\\&\qquad=x^4+x^3+x+1+\langle x^3+x+1\rangle=x^4+\langle x^3+x+1\rangle\end{aligned}$$

(since the ideal absorbs the last three terms). How do we express this in the form $ax^2+bx+c+\langle x^3+x+1\rangle$? One way is to long divide x^4 by x^3+x+1 to obtain the remainder of x^2+x (just as one reduces $12+\langle 5\rangle$ to $2+\langle 5\rangle$ by dividing 12 by 5 to obtain the remainder 2). Another way is to observe that $x^3+x+1+\langle x^3+x+1\rangle=0+\langle x^3+x+1\rangle$ implies $x^3+\langle x^3+x+1\rangle=x+1+\langle x^3+x+1\rangle$. Thus, we may multiply both sides by x to obtain

$$x^4+\langle x^3+x+1\rangle=x^2+x+\langle x^3+x+1\rangle.$$

Similarly,

$$\begin{aligned}&(x^2+x+\langle x^3+x+1\rangle)\cdot(x+\langle x^3+x+1\rangle)\\&\qquad=x^3+x^2+\langle x^3+x+1\rangle\\&\qquad=x^2+x+1+\langle x^3+x+1\rangle.\end{aligned}$$

A partial multiplication table for this field is given in Table 17.1. To simplify the notation, we indicate a coset by its representative only. (Complete the table yourself. Keep in mind that x^3 can be replaced by $x+1$ and x^4 by x^2+x.)

∎

EXAMPLE 11 Since x^2+1 has no zero in Z_{11}, it is irreducible over Z_{11}. Thus, $Z_{11}[x]/\langle x^2+1\rangle$ is a field. Analogous to Example 11 in Chapter 14, $Z_{11}[x]/\langle x^2+1\rangle=\{ax+b+\langle x^2+1\rangle \mid a,b\in Z_{11}\}$. Thus, this field has 49 elements. ∎

Unique Factorization in $Z[x]$

As a further application of the ideas presented in this chapter, we next prove that $Z[x]$ has an important factorization property. In Chapter 18, we will study this property in greater depth. The first proof of Theorem 17.6 was given by Gauss. In reading this theorem and its proof, keep in mind that the units in $Z[x]$ are precisely $f(x) = 1$ and $f(x) = -1$ (see Exercise 25 in Chapter 12), the irreducible polynomials of degree 0 over Z are precisely those of the form $f(x) = p$ and $f(x) = -p$ where p is a prime, and every nonconstant polynomial from $Z[x]$ that is irreducible over Z is primitive (see Exercise 3).

Theorem 17.6 Unique Factorization in $Z[x]$

Every polynomial in $Z[x]$ that is not the zero polynomial or a unit in $Z[x]$ can be written in the form $b_1b_2 \cdots b_s p_1(x)p_2(x) \cdots p_m(x)$, where the bi's are irreducible polynomials of degree 0 and the $p_i(x)$'s are irreducible polynomials of positive degree. Furthermore, if

$$b_1b_2 \cdots b_s p_1(x)p_2(x) \cdots p_m(x) = c_1c_2 \cdots c_t q_1(x)q_2(x) \cdots q_n(x),$$

where the b_i's and c_i's are irreducible polynomials of degree 0 and the $p_i(x)$'s and $q_i(x)$'s are irreducible polynomials of positive degree, then $s = t, m = n$, and, after renumbering the c's and $q(x)$'s, we have $b_i = \pm c_i$ for $i = 1, \ldots, s$ and $p_i(x) = \pm q_i(x)$ for $i = 1, \ldots, m$.

PROOF Let $f(x)$ be a nonzero, nonunit polynomial from $Z[x]$. If $\deg f(x) = 0$, then $f(x)$ is constant and the result follows from the Fundamental Theorem of Arithmetic. If $\deg f(x) > 0$, let b denote the content of $f(x)$, and let $b_1b_2 \cdots b_s$ be the factorization of b as a product of primes. Then, $f(x) = b_1b_2 \cdots b_s f_1(x)$, where $f_1(x)$ belongs to $Z[x]$, is primitive and $\deg f_1(x) = \deg f(x)$. Thus, to prove the existence portion of the theorem, it suffices to show that a primitive polynomial $f(x)$ of positive degree can be written as a product of irreducible polynomials of positive degree. We proceed by induction on $\deg f(x)$. If $\deg f(x) = 1$, then $f(x)$ is already irreducible and we are done. Now suppose that every primitive polynomial of degree less than $\deg f(x)$ can be written as a product of irreducibles of positive degree. If $f(x)$ is irreducible, there is nothing to prove. Otherwise, $f(x) = g(x)h(x)$, where both $g(x)$ and $h(x)$ are primitive and have degree less than that of $f(x)$. Thus, by induction, both $g(x)$ and $h(x)$ can be written as a product of irreducibles of positive degree. Clearly, then, $f(x)$ is also such a product.

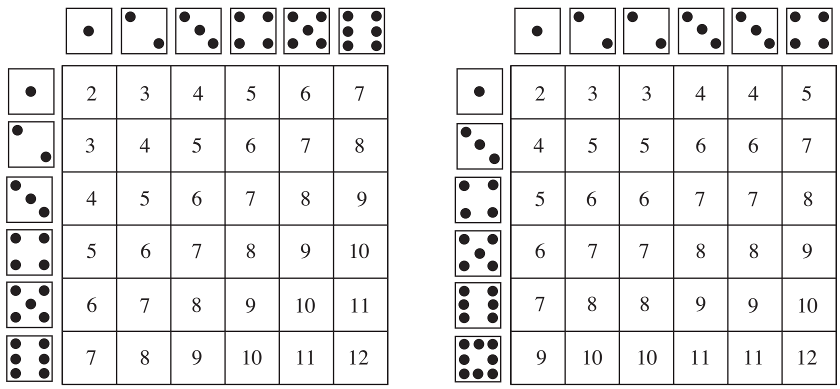
To prove the uniqueness portion of the theorem, suppose that

$$\begin{aligned} f(x) &= b_1 b_2 \cdots b_s p_1(x) p_2(x) \cdots p_m(x) \\ &= c_1 c_2 \cdots c_t q_1(x) q_2(x) \cdots q_n(x), \end{aligned}$$

where the b_i's and c_i's are irreducible polynomials of degree 0 and the $p_i(x)$'s and $q_i(x)$'s are irreducible polynomials of positive degree. Let $b = b_1 b_2 \cdots b_s$ and $c = c_1 c_2 \cdots c_t$. Since the $p(x)$'s and $q(x)$'s are primitive, it follows from Gauss's Lemma that $p_1(x)p_2(x) \cdots p_m(x)$ and $q_1(x)q_2(x) \cdots q_n(x)$ are primitive. Hence, both b and c must equal plus or minus the content of $f(x)$ and, therefore, are equal in absolute value. It then follows from the Fundamental Theorem of Arithmetic that $s = t$ and, after renumbering, $b_i = \pm c_i$ for $i = 1, 2, \ldots, s$. Thus, by canceling the constant terms in the two factorizations for $f(x)$, we have $p_1(x)p_2(x) \cdots p_m(x) = \pm q_1(x)q_2(x) \cdots q_n(x)$. Now, viewing the $p(x)$'s and $q(x)$'s as elements of $Q[x]$ and noting that $p_1(x)$ divides $q_1(x) \cdots q_n(x)$, it follows from Corollary 2 of Theorem 17.5 and induction (see Exercise 36) that $p_1(x) \mid q_i(x)$ for some i. By renumbering, we may assume $i = 1$. Then, since $q_1(x)$ is irreducible, we have $q_1(x) = (r/s)p_1(x)$, where $r, s \in Z$. However, because both $q_1(x)$ and $p_1(x)$ are primitive, we must have $r/s = \pm 1$. So, $q_1(x) = \pm p_1(x)$. Also, after canceling, we have $p_2(x) \cdots p_m(x) = \pm q_2(x) \cdots q_n(x)$. Now, we may repeat the argument above with $p_2(x)$ in place of $p_1(x)$. If $m < n$, after m such steps we would have 1 on the left and a nonconstant polynomial on the right. Clearly, this is impossible. On the other hand, if $m > n$, after n steps we would have ± 1 on the right and a nonconstant polynomial on the left—another impossibility. So, $m = n$ and $p_i(x) = \pm q_i(x)$ after suitable renumbering of the $q(x)$'s. ∎

Weird Dice: An Application of Unique Factorization

EXAMPLE 12 Consider an ordinary pair of dice whose faces are labeled 1 through 6. The probability of rolling a sum of 2 is $1/36$, the probability of rolling a sum of 3 is $2/36$, and so on. In a 1978 issue of *Scientific American*, Martin Gardner remarked that if one were to label the six faces of one cube with the integers 1, 2, 2, 3, 3, 4 and the six faces of another cube with the integers 1, 3, 4, 5, 6, 8, then the probability of obtaining any particular sum with these dice (called *Sicherman dice*) would be the same as the probability of rolling that sum with ordinary dice (i.e., $1/36$ for a 2, $2/36$ for a 3, and so on). See Figure 17.1. In this example, we show how the Sicherman labels can be derived, and that they are the only possible such labels besides 1 through 6. To do so, we utilize the fact that $Z[x]$ has the unique factorization property.



2	3	4	5	6	7
3	4	5	6	7	8
4	5	6	7	8	9
5	6	7	8	9	10
6	7	8	9	10	11
7	8	9	10	11	12

2	3	3	4	4	5
4	5	5	6	6	7
5	6	6	7	7	8
6	7	7	8	8	9
7	8	8	9	9	10
9	10	10	11	11	12

Figure 17.1

To begin, let us ask ourselves how we may obtain a sum of 6, say, with an ordinary pair of dice. Well, there are five possibilities for the two faces: $(5, 1), (4, 2), (3, 3), (2, 4)$, and $(1, 5)$. Next we consider the product of the two polynomials created by using the ordinary dice labels as exponents:

$$(x^6 + x^5 + x^4 + x^3 + x^2 + x)(x^6 + x^5 + x^4 + x^3 + x^2 + x).$$

Observe that we pick up the term x^6 in this product in precisely the following ways: $x^5 \cdot x^1, x^4 \cdot x^2, x^3 \cdot x^3, x^2 \cdot x^4, x^1 \cdot x^5$. Notice the correspondence between pairs of labels whose sums are 6 and pairs of terms whose products are x^6. This correspondence is one-to-one, and it is valid for all sums and all dice—including the Sicherman dice and any other dice that yield the desired probabilities. So, let $a_1, a_2, a_3, a_4, a_5, a_6$ and $b_1, b_2, b_3, b_4, b_5, b_6$ be any two lists of positive integer labels for the faces of a pair of cubes with the property that the probability of rolling any particular sum with these dice (let us call them *weird dice*) is the same as the probability of rolling that sum with ordinary dice labeled 1 through 6. Using our observation about products of polynomials, this means that

$$\begin{aligned}(x^6 + x^5 + x^4 + x^3 + x^2 + x)&(x^6 + x^5 + x^4 + x^3 + x^2 + x)\\ &= (x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6})\cdot\\ &\quad (x^{b_1} + x^{b_2} + x^{b_3} + x^{b_4} + x^{b_5} + x^{b_6}).\end{aligned} \tag{17.1}$$

Now all we have to do is solve this equation for the a's and b's. Here is where unique factorization in $Z[x]$ comes in. The polynomial $x^6 + x^5 + x^4 + x^3 + x^2 + x$ factors uniquely into irreducibles as

$$x(x + 1)(x^2 + x + 1)(x^2 - x + 1)$$

so that the left-hand side of Equation (1) has the irreducible factorization

$$x^2(x + 1)^2(x^2 + x + 1)^2(x^2 - x + 1)^2.$$

So, by Theorem 17.6, this means that these factors are the only possible irreducible factors of $P(x) = x^{a_1} + x^{a_2} + x^{a_3} + x^{a_4} + x^{a_5} + x^{a_6}$. Thus, $P(x)$ has the form

$$x^q(x + 1)^r(x^2 + x + 1)^t(x^2 - x + 1)^u,$$

where $0 \leq q, r, t, u \leq 2$.

To restrict further the possibilities for these four parameters, we evaluate $P(1)$ in two ways. $P(1) = 1^{a_1} + 1^{a_2} + \cdots + 1^{a_6} = 6$ and $P(1) = 1^q 2^r 3^t 1^u$. Clearly, this means that $r = 1$ and $t = 1$. What about q? Evaluating $P(0)$ in two ways shows that $q \neq 0$. On the other hand, if $q = 2$, the smallest possible sum one could roll with the corresponding labels for dice would be 3. Since this violates our assumption, we have now reduced our list of possibilities for $q, r, t,$ and u to $q = 1$, $r = 1$, $t = 1$, and $u = 0, 1, 2$. Let's consider each of these possibilities in turn.

When $u = 0, P(x) = x^4 + x^3 + x^3 + x^2 + x^2 + x$, so the die labels are 4, 3, 3, 2, 2, 1—a Sicherman die.

When $u = 1, P(x) = x^6 + x^5 + x^4 + x^3 + x^2 + x$, so the die labels are 6, 5, 4, 3, 2, 1—an ordinary die.

When $u = 2, P(x) = x^8 + x^6 + x^5 + x^4 + x^3 + x$, so the die labels are 8, 6, 5, 4, 3, 1—the other Sicherman die.

This proves that the Sicherman dice do give the same probabilities as ordinary dice *and* that they are the *only* other pair of dice that have this property. ∎

Exercises

No matter how good you are at something, there's always about a million people better than you.

Homer Simpson

1. Verify the assertion made in Example 2.
2. Suppose that D is an integral domain and F is a field containing D. If $f(x) \in D[x]$ and $f(x)$ is irreducible over F but reducible over D, what can you say about the factorization of $f(x)$ over D?
3. Show that a nonconstant polynomial from $Z[x]$ that is irreducible over Z is primitive. (This exercise is referred to in this chapter.)
4. Suppose that $f(x) = x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in Z[x]$. If r is rational and $x - r$ divides $f(x)$, show that r is an integer.
5. Let F be a field and let a be a nonzero element of F.
   a. If $af(x)$ is irreducible over F, prove that $f(x)$ is irreducible over F.
   b. If $f(ax)$ is irreducible over F, prove that $f(x)$ is irreducible over F.
   c. If $f(x + a)$ is irreducible over F, prove that $f(x)$ is irreducible over F.
   d. Use part c to prove that $8x^3 - 6x + 1$ is irreducible over Q. (This exercise is referred to in this chapter.)
6. Let F be a field and $f(x) \in F[x]$. Show that, as far as deciding upon the irreducibility of $f(x)$ over F is concerned, we may assume that $f(x)$ is monic. (This assumption is useful when one uses a computer to check for irreducibility.)
7. Suppose there is a real number r with the property that $r + 1/r$ is an odd integer. Prove that r is irrational.
8. Show that the equation $x^2 - 7y^2 = 3$ has no solutions in the integers.
9. Explain how the Mod p Irreducibility Test (Theorem 17.3) can be used to test members of $Q[x]$ for irreducibility.

10. Suppose that $f(x) \in Z_p[x]$ and $f(x)$ is irreducible over Z_p, where p is a prime. If $\deg f(x) = n$, prove that $Z_p[x]/\langle f(x)\rangle$ is a field with p^n elements.

11. Construct a field of order 25.

12. Construct a field of order 27.

13. Show that $x^3 + x^2 + x + 1$ is reducible over Q. Does this fact contradict the corollary to Theorem 17.4?

14. Determine which of the polynomials below is (are) irreducible over Q.
a. $x^5 + 9x^4 + 12x^2 + 6$
b. $x^4 + x + 1$
c. $x^4 + 3x^2 + 3$
d. $x^5 + 5x^2 + 1$
e. $(5/2)x^5 + (9/2)x^4 + 15x^3 + (3/7)x^2 + 6x + 3/14$

15. Referring to the field in Example 10 and denoting a coset by its representative only, find the representatives for $(x^2 + x)^2$ and $(x^2 + x)^{-1}$.

16. Let F denote the field in Example 10 and F^* be the multiplicative subgroup of F. Prove that for any element a of F^* other than the identity, the cyclic subgroup generated by a is F^*. Would the same be true for a field F if $|F| = 32$? What about $|F| = 16$?

17. What can you conclude about a polynomial $f(x)$ in $Z[x]$ if $f(x)$ is irreducible over $Z_p[x]$ for some prime p (i.e., when the coefficients of $f(x)$ are reduced modulo p)? What can you conclude about a polynomial $f(x)$ in $Z[x]$ if $f(x)$ is reducible over $Z_p[x]$ for every prime p (i.e., when the coefficients of $f(x)$ are reduced modulo p)?

18. Let $f(x) \in Z_2[x]$ and $\deg f(x) = 5$. If neither 0 or 1 is a zero of $f(x)$ explain that in order to prove that $f(x)$ is irreducible over Z_2 it suffices to show that $x^2 + x + 1$ is not a factor of $f(x)$.

19. For the field $Z_7[x]/I$ where $I = \langle x^2 + 2\rangle$, find the multiplicative orders of $x + I$ and $x + 1 + I$. Find the multiplicative inverse of $x + I$.

20. Let F be a field and $f(x) \in F[x]$ be reducible over F with $\deg f(x) > 1$. Prove that $F[x]/\langle f(x)\rangle$ is not an integral domain.

21. Show that $x^4 + 1$ is irreducible over Q.

22. Let $f(x) = ax^4 + bx^2 + c \in Z[x]$. If $p(x)$ is a factor of $f(x)$ in $Z[x]$ prove that $p(-x)$ is also a factor of $f(x)$ in $Z[x]$.

23. Let $f(x) = x^3 + 6 \in Z_7[x]$. Write $f(x)$ as a product of irreducible polynomials over Z_7.

24. Let $f(x) = x^3 + x^2 + x + 1 \in Z_2[x]$. Write $f(x)$ as a product of irreducible polynomials over Z_2.

25. Prove that $x^4 + 15x^3 + 7$ is irreducible over Q.

26. Find all zeros of $f(x) = 3x^2 + x + 4$ over Z_7 by substitution. Find all zeros of $f(x)$ by using the quadratic formula $(-b \pm \sqrt{b^2 - 4ac}) \cdot (2a)^{-1}$ (all calculations are done in Z_7). Do your answers agree? Should they? Find all zeros of $g(x) = 2x^2 + x + 3$ over Z_5 by substitution. Try the quadratic formula on $g(x)$. Do your answers agree? State necessary and sufficient conditions for the quadratic formula to yield the zeros of a quadratic from $Z_p[x]$, where p is a prime greater than 2.

27. Let p be a prime.
a. Show that the number of reducible polynomials over Z_p of the form $x^2 + ax + b$ is $p(p + 1)/2$.
b. Determine the number of reducible quadratic polynomials over Z_p.

28. Let p be a prime.
a. Determine the number of irreducible polynomials over Z_p of the form $x^2 + ax + b$.
b. Determine the number of irreducible quadratic monic polynomials over Z_p.

29. Show that for every prime p there exists a field of order p^2.

30. Prove that, for every positive integer n, there are infinitely many monic polynomials of degree n in $Z[x]$ that are irreducible over Q.

31. Show that the field given in Example 11 in this chapter is isomorphic to the field given in Example 9 in Chapter 13.

32. Let $f(x) \in Z_p[x]$. Prove that if $f(x)$ has no factor of the form $x^2 + ax + b$, then it has no quadratic factor over Z_p.

33. Find all monic irreducible polynomials of degree 2 over Z_3.

34. Given that π is not the zero of a nonzero polynomial with rational coefficients, prove that π^2 cannot be written in the form $a\pi + b$, where a and b are rational.

35. (Rational Root Theorem) Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in Z[x]$$

and $a_n \neq 0$. Prove that if r and s are relatively prime integers and $f(r/s) = 0$, then $r \mid a_0$ and $s \mid a_n$.

36. Let F be a field and let $p(x), a_1(x), a_2(x), \ldots, a_k(x) \in F[x]$, where $p(x)$ is irreducible over F. If $p(x) \mid a_1(x)a_2(x) \cdots a_k(x)$, show that $p(x)$ divides some $a_i(x)$. (This exercise is referred to in the proof of Theorem 17.6.)

37. Let F be a field and $p(x) \in F[x]$. Use Theorem 14.4 to prove that if $\langle p(x) \rangle$ is a maximal ideal in $F[x]$, then $p(x)$ is irreducible over F (see Theorem 17.5).

38. If p is a prime, prove that $x^{p-1} - x^{p-2} + x^{p-3} - \cdots - x + 1$ is irreducible over Q.

39. Let F be a field and let $p(x)$ be irreducible over F. If E is a field that contains F and there is an element a in E such that $p(a) = 0$, show that the mapping ϕ: $F[x] \to E$ given by $f(x) \to f(a)$ is a ring homomorphism with kernel $\langle p(x) \rangle$. (This exercise is referred to in Chapter 19.)

40. Prove that the ideal $\langle x^2+1 \rangle$ is prime in $Z[x]$ but not maximal in $Z[x]$.

41. Let F be a field and let $p(x)$ be irreducible over F. Show that $\{a + \langle p(x) \rangle \mid a \in F\}$ is a subfield of $F[x]/\langle p(x) \rangle$ isomorphic to F. (This exercise is referred to in Chapter 19.)

42. Let F be a field and let $f(x)$ be a polynomial in $F[x]$ that is reducible over F. Prove that $\langle f(x) \rangle$ is not a prime ideal in $F[x]$.

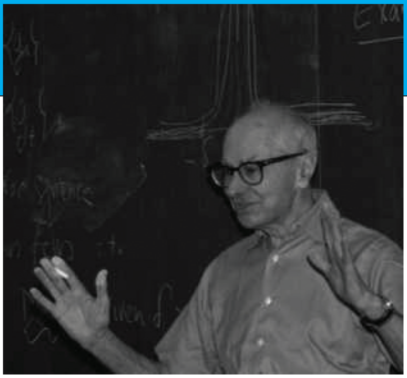
43. Example 1 in this chapter shows the converse of Theorem 17.2 is not true. That is, a polynomial $f(x)$ in $Z[x]$ can be reducible over Z but irreducible over Q. State a condition on $f(x)$ that makes the converse true.

Computer Exercises

A computer exercise for this chapter is available at the website:

http://www.d.umn.edu/~jgallian

Serge Lang



Courtesy of Bogdan Oporowski

> Lang's *Algebra* changed the way graduate algebra is taught .... It has affected all subsequent graduate-level algebra books.
>
> *Citation for the Steele Prize*

SERGE LANG was a prolific mathematician, inspiring teacher, and political activist. He was born near Paris on May 19, 1927. His family moved to Los Angeles when he was a teenager. Lang received a B.A. in physics from Caltech in 1946 and a Ph.D. in mathematics from Princeton in 1951 under Emil Artin. His first permanent position was at Columbia University in 1955, but in 1971 Lang resigned his position at Columbia as a protest against Columbia's handling of Vietnam antiwar protesters. He joined Yale University in 1972 and remained there until his retirement.

Lang made significant contributions to number theory, algebraic geometry, differential geometry, and analysis. He wrote more than 120 research articles and 60 books.

His most famous and influential book was his graduate-level *Algebra*. Lang was a prize-winning teacher known for his extraordinary devotion to students. Lang often got into heated discussions about mathematics, the arts, and politics. In one incident, he threatened to hit a fellow mathematician with a bronze bust for not conceding it was self-evident that the Beatles were greater musicians than Beethoven. Among Lang's honors were the Steele Prize for Mathematical Exposition from the American Mathematical Society, the Cole Prize in Algebra (see Chapter 24), and election to the National Academy of Sciences. Lang died on September 25, 2005, at the age of 78.

18 Divisibility in Integral Domains

Fundamental definitions do not arise at the start but at the end of the exploration, because in order to define a thing you must know what it is and what it is good for.

Hans Freudenthal, *Developments in Mathematical Education*

Give me a fruitful error anytime, full of seeds, bursting with its own corrections. You can keep your sterile truth for yourself.

Vilfredo Pareto

Irreducibles, Primes

In the preceding two chapters, we focused on factoring polynomials over the integers or a field. Several of those results—unique factorization in $Z[x]$ and the division algorithm for $F[x]$, for instance—are natural counterparts to theorems about the integers. In this chapter and the next, we examine factoring in a more abstract setting.

Definitions Associates, Irreducibles, Primes
Elements a and b of an integral domain D are called *associates* if $a = ub$, where u is a unit of D. A nonzero element a of an integral domain D is called an *irreducible* if a is not a unit and, whenever $b, c \in D$ with $a = bc$, then b or c is a unit. A nonzero element a of an integral domain D is called a *prime* if a is not a unit and $a \mid bc$ implies $a \mid b$ or $a \mid c$.

Roughly speaking, an irreducible is an element that can be factored only in a trivial way. Notice that an element a is a prime if and only if $\langle a\rangle$ is a prime ideal.

Relating the definitions above to the integers may seem a bit confusing, since in Chapter 0 we defined a positive integer to be a prime if it satisfies our definition of an irreducible, and we proved that a prime integer satisfies the definition of a prime in an integral domain (Euclid's Lemma). The source of the confusion is that in the case of the integers, the concepts of irreducibles and primes are equivalent, but in general, as we will soon see, they are not.

The distinction between primes and irreducibles is best illustrated by integral domains of the form $Z[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in Z\}$, where d is not 1 and is not divisible by the square of a prime. (These rings are of fundamental importance in number theory.) To analyze these rings, we need a convenient method of determining their units, irreducibles, and primes. To do this, we define a function N, called the *norm*, from $Z[\sqrt{d}]$ into the nonnegative integers by $N(a + b\sqrt{d}) = |\, a^2 - db^2\,|$. We leave it to the reader (Exercise 1) to verify the following four properties: $N(x) = 0$ if and only if $x = 0$; $N(xy) = N(x)N(y)$ for all x and y; x is a unit if and only if $N(x) = 1$; and, if $N(x)$ is prime, then x is irreducible in $Z[\sqrt{d}]$.

EXAMPLE 1 We exhibit an irreducible in $Z[\sqrt{-3}\,]$ that is not prime. Here, $N(a+b\sqrt{-3}) = a^2+3b^2$. Consider $1+\sqrt{-3}$. Suppose that we can factor this as xy, where neither x nor y is a unit. Then $N(xy) = N(x)N(y) = N(1 + \sqrt{-3}) = 4$, and it follows that $N(x) = 2$. But there are no integers a and b that satisfy $a^2+3b^2 = 2$. Thus, x or y is a unit and $1 + \sqrt{-3}$ is an irreducible. To verify that it is not prime, we observe that $(1+\sqrt{-3})(1+\sqrt{-3}) = 4 = 2\cdot 2$, so that $1+\sqrt{-3}$ divides $2\cdot 2$. On the other hand, for integers a and b to exist so that $2 = (1+\sqrt{-3})(a+b\sqrt{-3}) = (a-3b)+(a+b)\sqrt{-3}$, we must have $a - 3b = 2$ and $a + b = 0$, which is impossible. ∎

Showing that an element of a ring of the form $Z[\sqrt{d}\,]$ is irreducible is more difficult when $d > 1$. The next example illustrates one method of doing this. The example also shows that the converse of the fourth property above for the norm is not true. That is, it shows that x may be irreducible even if $N(x)$ is not prime.

EXAMPLE 2 The element 7 is irreducible in the ring $Z[\sqrt{5}\,]$. To verify this assertion, suppose that $7 = xy$, where neither x nor y is a unit. Then $49 = N(7) = N(x)N(y)$, and since x is not a unit, we cannot have $N(x) = 1$. This leaves only the case $N(x) = 7$. Let $x = a + b\sqrt{5}$. Then there are integers a and b satisfying $|a^2 - 5b^2| = 7$. This means that $a^2 - 5b^2 = \pm 7$. Keeping in mind that any integer solution of the equation $a^2 - 5b^2 = \pm 7$ would also be a solution in Z_n for every positive integer n, we have, modulo 5, that we need only check for $a^2 = 2$ and $a^2 = -2 = 3$.

Trying all five cases for a we see that there is no solution. ∎

Example 1 raises the question of whether or not there is an integral domain containing a prime that is not an irreducible. The answer: no.

Theorem 18.1 Prime Implies Irreducible

In an integral domain, every prime is an irreducible.

PROOF Suppose that a is a prime in an integral domain and $a = bc$. We must show that b or c is a unit. By the definition of prime, we know that $a \mid b$ or $a \mid c$. Say $at = b$. Then $1b = b = at = (bc)t = b(ct)$ and, by cancellation, $1 = ct$. Thus, c is a unit. ∎

Recall that a principal ideal domain is an integral domain in which every ideal has the form $\langle a \rangle$. The next theorem reveals a circumstance in which primes and irreducibles are equivalent.

Theorem 18.2 PID Implies Irreducible Equals Prime

In a principal ideal domain, an element is an irreducible if and only if it is a prime.

PROOF Theorem 18.1 shows that primes are irreducibles. To prove the converse, let a be an irreducible element of a principal ideal domain D and suppose that $a \mid bc$. We must show that $a \mid b$ or $a \mid c$. Consider the ideal $I = \{ax + by \mid x, y \in D\}$ and let $\langle d \rangle = I$. Since $a \in I$, we can write $a = dr$, and because a is irreducible, d is a unit or r is a unit. If d is a unit, then $I = D$ and we may write $1 = ax + by$. Then $c = acx + bcy$, and since a divides both terms on the right, a also divides c.

On the other hand, if r is a unit, then $\langle a \rangle = \langle d \rangle = I$, and, because $b \in I$, there is an element t in D such that $at = b$. Thus, a divides b. ∎

It is an easy consequence of the respective division algorithms for Z and $F[x]$, where F is a field, that Z and $F[x]$ are principal ideal domains (see Exercise 57 in Chapter 14 and Theorem 16.3). Our next example shows, however, that one of the most familiar rings is not a principal ideal domain.

EXAMPLE 3 We show that $Z[x]$ is not a principal ideal domain. Consider the ideal $I = \langle 2, x\rangle$. We claim that I is not of the form $\langle h(x)\rangle$. If this were so, there would be $f(x)$ and $g(x)$ in $Z[x]$ such that $2 = h(x)f(x)$ and $x = h(x)g(x)$, since both 2 and x belong to I. By the degree rule (Exercise 19 in Chapter 16), $0 = \deg 2 = \deg h(x) + \deg f(x)$, so that $h(x)$ is a constant polynomial. To determine which constant, we observe that $2 = h(1)f(1)$. Thus, $h(1) = \pm 1$ or ± 2. Since 1 is not in I, we must have $h(x) = \pm 2$. But then $x = \pm 2g(x)$, which is nonsense. ∎

We have previously proved that the integral domains Z and $Z[x]$ have important factorization properties: Every integer greater than 1 can be uniquely factored as a product of irreducibles (i.e., primes), and every nonzero, nonunit polynomial can be uniquely factored as a product of irreducible polynomials. It is natural to ask whether all integral domains have this property. The question of unique factorization in integral domains first arose with the efforts to solve a famous problem in number theory that goes by the name Fermat's Last Theorem.

Historical Discussion of Fermat's Last Theorem

There are infinitely many nonzero integers x, y, z that satisfy the equation $x^2 + y^2 = z^2$. But what about the equation $x^3 + y^3 = z^3$ or, more generally, $x^n + y^n = z^n$, where n is an integer greater than 2 and x, y, z are nonzero integers? Well, no one has ever found a single solution of this equation, and for more than three centuries many have tried to prove that there is none. The tremendous effort put forth by the likes of Euler, Legendre, Abel, Gauss, Dirichlet, Cauchy, Kummer, Kronecker, and Hilbert to prove that there are no solutions to this equation has greatly influenced the development of ring theory.

About a thousand years ago, Arab mathematicians gave an incorrect proof that there were no solutions when $n = 3$. The problem lay dormant until 1637, when the French mathematician Pierre de Fermat (1601–1665) wrote in the margin of a book, "... it is impossible to separate a cube into two cubes, a fourth power into two fourth powers, or, generally, any power above the second into two powers of the same degree: I have discovered a truly marvelous demonstration [of this general theorem] which this margin is too narrow to contain."

Because Fermat gave no proof, many mathematicians tried to prove the result. The case where $n = 3$ was done by Euler in 1770, although his proof was incomplete. The case where $n = 4$ is elementary and was done by Fermat himself. The case where $n = 5$ was done in 1825 by Dirichlet, who had just turned 20, and by Legendre, who was past 70. Since the validity of the case for a particular integer implies the validity for all multiples of that integer, the next case of interest was $n = 7$. This case resisted the efforts of the best mathematicians until it was done by Gabriel Lamé in 1839. In 1847, Lamé stirred excitement by announcing that he had completely solved the problem. His approach was to factor the expression $x^p + y^p$, where p is an odd prime, into

$$(x + y)(x + \alpha y) \cdots (x + \alpha^{p-1} y),$$

where α is the complex number $\cos(2\pi/p) + i \sin(2\pi/p)$. Thus, his factorization took place in the ring $Z[\alpha] = \{a_0 + a_1\alpha + \cdots + a_{p-1}\alpha^{p-1} \mid a_i \in Z\}$. But Lamé made the mistake of assuming that, in such a ring, factorization into the product of irreducibles is unique. In fact, three years earlier, Ernst Eduard Kummer had proved that this is not always the case. Undaunted by the failure of unique factorization, Kummer began developing a theory to "save" factorization by creating a new type of number. Within a few weeks of Lamé's announcement, Kummer had shown that Fermat's Last Theorem is true for all primes of a special type. This proved that the theorem was true for all exponents less than 100, prime or not, except for 37, 59, 67, and 74. Kummer's work has led to the theory of ideals as we know it today.

Over the centuries, many proposed proofs have not held up under scrutiny. The famous number theorist Edmund Landau received so many of these that he had a form printed with "On page _______, lines _______ to _______, you will find there is a mistake." Martin Gardner, "Mathematical Games" columnist of *Scientific American,* had postcards printed to decline requests from readers asking him to examine their proofs.

Recent discoveries tying Fermat's Last Theorem closely to modern mathematical theories gave hope that these theories might eventually lead to a proof. In March 1988, newspapers and scientific publications worldwide carried news of a proof by Yoichi Miyaoka (see Figure 18.1). Within weeks, however, Miyaoka's proof was shown to be invalid. In June 1993, excitement spread

Doubts about Fermat solution

Careful scrutiny of a recently proposed proof of Fermat's last theorem (SN: 3/19/88, p.180) has turned up several flaws that cast doubt on the proof's validity. Japanese mathematician Yoichi Miyaoka, who is presently working at the Max Planck Institute for Mathematics in Bonn, West Germany, last week admitted that his proof has a serious problem. He is now studying how to revise his proof.

"That do... irr... Ma...

Institut des Hautes Etudes ... near Paris, wh... discussing the ... Miyaoka. "But it ... means there's more ... do. It's a rather c... proof. If you change things in one part of ... proof, then all the other p... may be subject to change."

Fermat's last theorem concerns equations of the form xn + yn = ...

Fermat's last theorem: a promising approach.

The end of a centuries-long search for a proof of Fermat's last theorem, one of the most famous unsolved problems in mathematics, may at last be in sight. A Japanese mathematician, Yoichi Miyaoka of the Tokyo Metropolitan University, has proposed a proof for a key link in a chain of reasoning that establishes the theorem's truth. If Miyaoka's proof survives the mathematical community's intense scrutiny, then fermat's conjecture (as it ought to be called until a proof is firmly established) can truly be called a theorem.

Miyaoka's method builds on work done by several Russian mathematicians and links important ideas in three mathematical fields: number theory, algebra and geometry. Though highly technical, his argument fills fewer than a dozen manuscript pages -- short for such a significant mathematical proof. Miyaoka recently presented a sketch of his ideas at a seminar at the Max Planck Institute for Mathematics in Bonn, West Germany.

"It looks very nice," mathematician Don B. Zagier of the Max Planck Institute told Science News.

A curvy path leads to Fermat's last theorem.

After more than 300 years, Fermat's last theorem may finally live up to its common designation as a theorem. In a dramatic announcement that caught the mathematical community completely by surprise, Andrew Wiles of Princeton University revealed last week that he had proved major parts of a significant conjecture in number theory, These results, in turn, establish the truth of Fermat's famous, devilishly simple conjecture.

"It's an amazing piece of work," says Peter C. Sarnak, one of Wiles' Princeton colleagues. "The proof hasn't been totally checked, but it's very convincing?"

Pierre de Fermat's last theorem goes back to the 17th century, when the French jurist and mathematician asserted that for any whole number n greater than 2, the equation [x.sup.n]+[y.sup.n] = [z.sup.n] has no solution for which x, y, and z are all whole numbers greater than zero.

Fermat scribbled hi... a page in a mathem... Then, in a tantalizing... mathematicians for ce... that although he had ... theorem, he didn't have ...

After Fermat died, schol... of the proof in any o... mathematicians proved th... the exponent n = 3 and solved several other special cases. Last year, a massive computer-aided effort by J.P. Buhler of Reed College in Portland, Ore., and Richard E. Crandall of NeXT Computer Inc., in Redwood City, Calif., verified Fermat's last theorem for exponents up to 4 million.

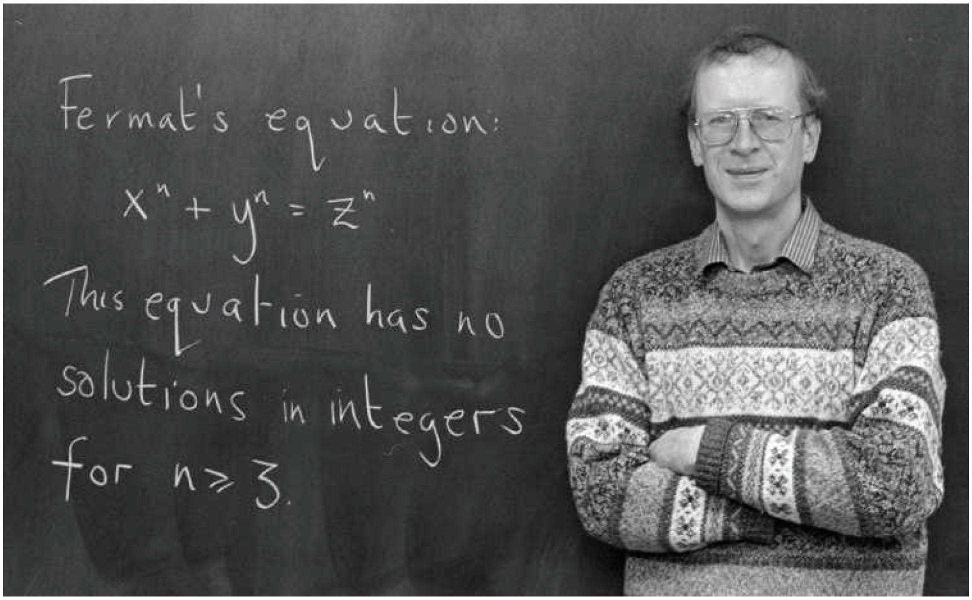
Meanwhile, mathematicians had picked up some valuable hints of a potential avenue to a general proof that the conjecture is true. In the mid-1980s, Gerhard Frey of the University of the Saarlands in Saarbrucken, Germany, unexpectedly uncovered an intriguing link between Fermat% conjecture and a seemingly

...'s problem into an area of ...es for which mathematicians had already developed a wide range of techniques for solving problems. A number of mathematicians, including Barry Mazur of Harvard University and Kenneth A. Ribet of the University of California, Berkeley, followed up Frey's surprising insight with additional results that ultimately tied Fermat's last theorem to a central conjecture in number theory (SN: 6/20/87, p.397).

Named for Japanese mathematician Yutaka Taniyama, this conjecture concerns certain characteristics of elliptic curves. A proof of this conjecture would automatically imply that Fermat's last theorem must be true.

Figure 18.1

Times Magazine, Science News #133. LA Times



AP Images/Charles Rex Arbogast

Figure 18.2 Andrew Wiles.

through the mathematics community with the announcement that Andrew Wiles of Princeton University had proved Fermat's Last Theorem (see Figure 18.2). The Princeton mathematics department chairperson was quoted as saying, "When we heard it, people starting walking on air." But guess what. Yes, you guessed it. Once again a proof did not hold up under scrutiny. This story does have a happy ending. The mathematical community has agreed on the validity of the revised proof given by Wiles and Richard Taylor in September of 1994[1].

In view of the fact that so many eminent mathematicians were unable to prove Fermat's Last Theorem, despite the availability of the vastly powerful theories, it seems highly improbable that Fermat had a correct proof. Most likely, he made the error that his successors made of assuming that the properties of integers, such as unique factorization, carry over to integral domains in general.

[1]Science writer Howard Bloom estimated that it took over thirteen million hours of people-work to come up with a proof of Fermat's Last Theorem.

Unique Factorization Domains

We now have the necessary terminology to formalize the idea of unique factorization.

Definition Unique Factorization Domain (UFD)

An integral domain D is a *unique factorization domain* if

1. **every nonzero element of D that is not a unit can be written as a product of irreducibles of D; and**
2. **the factorization into irreducibles is unique up to associates and the order in which the factors appear.**

Another way to formulate part 2 of this definition is the following: If ${p_1}^{n_1}{p_2}^{n_2}\cdots{p_r}^{n_r}$ and ${q_1}^{m_1}{q_2}^{m_2}\cdots{q_s}^{m_s}$ are two factorizations of some element as a product of irreducibles, where no two of the p_i's are associates and no two of the q_j's are associates, then $r = s$, each p_i is an associate of one and only one q_j, and $n_i = m_j$.

Of course, the Fundamental Theorem of Arithmetic tells us that the ring of integers is a unique factorization domain, and Theorem 17.6 says that $Z[x]$ is a unique factorization domain. In fact, as we shall soon see, most of the integral domains we have encountered are unique factorization domains.

Before proving our next theorem, we need the ascending chain condition for ideals.

Lemma Ascending Chain Condition for a PID

In a principal ideal domain, any strictly increasing chain of ideals $I_1 \subset I_2 \subset \cdots$ must be finite in length.

PROOF Let $I_1 \subset I_2 \subset \cdots$ be a chain of strictly increasing ideals in an integral domain D, and let I be the union of all the ideals in this chain. We leave it as an exercise (Exercise 3) to verify that I is an ideal of D.

Then, since D is a principal ideal domain, there is an element a in D such that $I = \langle a \rangle$. Because $a \in I$ and $I = \cup I_k, a$ belongs to some member of the chain, say $a \in I_n$. Clearly, then, for any member I_i of the chain, we have $I_i \subseteq I = \langle a \rangle \subseteq I_n$, so that I_n must be the last member of the chain. ∎

Theorem 18.3 PID Implies UFD

Every principal ideal domain is a unique factorization domain.

PROOF Let D be a principal ideal domain and let a_0 be any nonzero nonunit in D. We will show that a_0 is a product of irreducibles (the product might consist of only one factor). We begin by showing that a_0 has at least one irreducible factor. If a_0 is irreducible, we are done. Thus, we may assume that $a_0 = b_1a_1$, where neither b_1 nor a_1 is a unit and a_1 is nonzero. If a_1 is not irreducible, then we can write $a_1 = b_2a_2$, where neither b_2 nor a_2 is a unit and a_2 is nonzero. Continuing in this fashion, we obtain a sequence $b_1, b_2, \ldots$ of elements that are not units in D and a sequence $a_0, a_1, a_2, \ldots$ of nonzero elements of D with $a_n = b_{n+1}a_{n+1}$ for each n. Hence, $\langle a_0\rangle \subset \langle a_1\rangle \subset \cdots$ is a strictly increasing chain of ideals (see Exercise 5), which, by the preceding lemma, must be finite, say, $\langle a_0\rangle \subset \langle a_1\rangle \subset \cdots \subset \langle a_r\rangle$. In particular, a_r is an irreducible factor of a_0. This argument shows that every nonzero nonunit in D has at least one irreducible factor.

Now write $a_0 = p_1c_1$, where p_1 is irreducible and c_1 is not a unit. If c_1 is not irreducible, then we can write $c_1 = p_2c_2$, where p_2 is irreducible and c_2 is not a unit. Continuing in this fashion, we obtain, as before, a strictly increasing sequence $\langle a_0\rangle \subset \langle c_1\rangle \subset \langle c_2\rangle \subset \cdots$, which must end in a finite number of steps. Let us say that the sequence ends with $\langle c_s\rangle$. Then c_s is irreducible and $a_0 = p_1p_2\cdots p_sc_s$, where each p_i is also irreducible. This completes the proof that every nonzero nonunit of a principal ideal domain is a product of irreducibles.

It remains to be shown that the factorization is unique up to associates and the order in which the factors appear. To do this, suppose that some element a of D can be written

$$a = p_1p_2\cdots p_r = q_1q_2\cdots q_s,$$

where the p's and q's are irreducible and repetition is permitted. We use induction on r. If $r = 1$, then a is irreducible and, clearly, $s = 1$ and $p_1 = q_1$. So we may assume that any element that can be expressed as a product of fewer than r irreducible factors can be so expressed in only one way (up to order and associates). Since D is a principal ideal domain, by Theorem 18.2, each irreducible

pi in the product $p_1p_2 \cdots p_r$ is prime. Then because p_1 divides $q_1q_2 \cdots q_s, p_1$ must divide some q_i (see Exercise 33), say $p_1 \mid q_1$. Then, $q_1 = up_1$, where u is a unit of D. Since

$$up_1p_2 \cdots p_r = uq_1q_2 \cdots q_s = q_1(uq_2) \cdots q_s$$

and

$$up_1 = q_1,$$

we have, by cancellation,

$$p_2 \cdots p_r = (uq_2) \cdots q_s.$$

The induction hypothesis now tells us that these two factorizations are identical up to associates and the order in which the factors appear. Hence, the same is true about the two factorizations of a. ∎

In the existence portion of the proof of Theorem 18.3, the only way we used the fact that the integral domain D is a principal ideal domain was to say that D has the property that there is no infinite, strictly increasing chain of ideals in D. An integral domain with this property is called a *Noetherian domain*, in honor of Emmy Noether, who inaugurated the use of chain conditions in algebra. Noetherian domains are of the utmost importance in algebraic geometry. One reason for this is that, for many important rings R, the polynomial ring $R[x]$ is a Noetherian domain but not a principal ideal domain. One such example is $Z[x]$. In particular, $Z[x]$ shows that a UFD need not be a PID (see Example 3).

As an immediate corollary of Theorem 18.3, we have the following fact.

Corollary $F[x]$ Is a UFD

Let F be a field. Then $F[x]$ is a unique factorization domain.

PROOF By Theorem 16.4, $F[x]$ is a principal ideal domain. So, $F[x]$ is a unique factorization domain, as well. ∎

As an application of the preceding corollary, we give an elegant proof, due to Richard Singer, of Eisenstein's Criterion (Theorem 17.4).

EXAMPLE 4 Let

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 \in Z[x],$$

and suppose that p is prime such that

$$p \nmid a_n, p \mid a_{n-1}, \ldots, p \mid a_0 \quad \text{and} \quad p_2 \nmid a_0.$$

We will prove that $f(x)$ is irreducible over Q. If $f(x)$ is reducible over Q, we know by Theorem 17.2 that there exist elements $g(x)$ and $h(x)$ in $Z[x]$ such that $f(x) = g(x)h(x), 1 \leq \deg g(x) < n$, and $1 \leq \deg h(x) < n$. Let $\bar{f}(x), \bar{g}(x)$, and $\bar{h}(x)$ be the polynomials in $Z_p[x]$ obtained from $f(x), g(x)$, and $h(x)$ by reducing all coefficients modulo p. Then, since p divides all the coefficients of $f(x)$ except a_n, we have $\bar{a}_n x^n = \bar{f}(x) = \bar{g}(x) \cdot \bar{h}(x)$. Since Z_p is a field, $Z_p[x]$ is a unique factorization domain. Thus, $x \mid \bar{g}(x)$ and $x \mid \bar{h}(x)$. So, $\bar{g}(0) = \bar{h}(0) = 0$ and, therefore, $p \mid g(0)$ and $p \mid h(0)$. But then $p^2 \mid g(0)h(0) = f(0) = a_0$, which is a contradiction. ∎

Euclidean Domains

Another important kind of integral domain is a Euclidean domain.

> **Definition Euclidean Domain (ED)**
> **An integral domain D is called a *Euclidean domain* if there is a function d (called the *measure*) from the nonzero elements of D to the nonnegative integers such that**
> **1. $d(a) \leq d(ab)$ for all nonzero a, b in D; and**
> **2. if $a, b \in D, b \neq 0$, then there exist elements q and r in D such that $a = bq + r$, where $r = 0$ or $d(r) < d(b)$.**

EXAMPLE 5 The ring Z is a Euclidean domain with $d(a) = |a|$ (the absolute value of a). ∎

EXAMPLE 6 Let F be a field. Then $F[x]$ is a Euclidean domain with $d(f(x)) = \deg f(x)$ (see Theorem 16.2). ∎

Examples 5 and 6 illustrate just one of many similarities between the rings Z and $F[x]$. Additional similarities are summarized in Table 18.1.

EXAMPLE 7 The ring of Gaussian integers $Z[i] = \{a + bi \mid a, b \in Z\}$ is a Euclidean domain with $d(a + bi) = a^2 + b^2$. Unlike the previous two examples, in this example the function d does not

Table 18.1 Similarities Between Z and $F[x]$.

$\boldsymbol{Z}$		$\boldsymbol{F[x]}$
Euclidean domain: $d(a) = \|a\|$	$\leftrightarrow$	Euclidean domain: $d(f(x)) = \deg f(x)$
Units: a is a unit if and only if $\|a\| = 1$		Units: $f(x)$ is a unit if and only if $\deg f(x) = 0$
Division algorithm: For $a, b \in Z, b \neq 0$, there exist $q, r \in Z$ such that $a = bq + r, 0 \leq r < \|b\|$	$\leftrightarrow$	Division algorithm: For $f(x), g(x) \in F[x], g(x) \neq 0$, there exist $q(x), r(x) \in F[x]$ such that $f(x) = g(x)q(x) + r(x), 0 \leq \deg r(x) < \deg g(x)$ or $r(x) = 0$
PID: Every nonzero ideal $I = \langle a \rangle$, where $a \neq 0$ and $\|a\|$ is minimum	$\leftrightarrow$	PID: Every nonzero ideal $I = \langle f(x) \rangle$, where $\deg f(x)$ is minimum
Prime: No nontrivial factors	$\leftrightarrow$	Irreducible: No nontrivial factors
UFD: Every element is a "unique" product of primes	$\leftrightarrow$	UFD: Every element is a "unique" product of irreducibles

obviously satisfy the necessary conditions. That $d(x) \leq d(xy)$ for $x, y \in Z[i]$ follows directly from the fact that $d(xy) = d(x)d(y)$ (Exercise 7). To verify that condition 2 holds, observe that if $x, y \in Z[i]$ and $y \neq 0$, then $xy^{-1} \in Q[i]$, the field of quotients of $Z[i]$ (Exercise 65 in Chapter 15). Say $xy^{-1} = s + ti$, where $s, t \in Q$. Now let m be the integer nearest s, and let n be the integer nearest t. (These integers may not be uniquely determined, but that does not matter.) Thus, $|m - s| \leq 1/2$ and $|n - t| \leq 1/2$. Then

$$\begin{aligned} xy^{-1} &= s + ti = (m - m + s) + (n - n + t)i \\ &= (m + ni) + [(s - m) + (t - n)i]. \end{aligned}$$

So,

$$x = (m + ni)y + [(s - m) + (t - n)i]y.$$

We claim that the division condition of the definition of a Euclidean domain is satisfied with $q = m + ni$ and

$$r = [(s - m) + (t - n)i]y.$$

Clearly, q belongs to $Z[i]$, and since $r = x - qy$, so does r. Finally,

$$\begin{aligned} d(r) &= d([(s-m)+(t-n)i])d(y) \\ &= [(s-m)^2+(t-n)^2]d(y) \\ &\leq \left(\frac{1}{4}+\frac{1}{4}\right)d(y) < d(y). \end{aligned}$$

∎

Theorem 18.4 ED Implies PID

Every Euclidean domain is a principal ideal domain.

PROOF Let D be a Euclidean domain and I a nonzero ideal of D. Among all the nonzero elements of I, let a be such that $d(a)$ is a minimum. Then $I = \langle a \rangle$. For, if $b \in I$, there are elements q and r such that $b = aq + r$, where $r = 0$ or $d(r) < d(a)$. But $r = b - aq \in I$, so $d(r)$ cannot be less than $d(a)$. Thus, $r = 0$ and $b \in \langle a \rangle$. Finally, the zero ideal is $\langle 0 \rangle$. ∎

Although it is not easy to verify, we remark that there are principal ideal domains that are not Euclidean domains. The first such example was given by T. Motzkin in 1949.

As an immediate consequence of Theorems 18.3 and 18.4, we have the following important result.

Corollary ED Implies UFD

Every Euclidean domain is a unique factorization domain.

We may summarize our theorems and remarks as follows:

$$\text{ED} \Rightarrow \text{PID} \Rightarrow \text{UFD};$$
$$\text{UFD} \not\Rightarrow \text{PID} \not\Rightarrow \text{ED}.$$

(You can remember these implications by listing the types alphabetically.)

In Chapter 17, we proved that $Z[x]$ is a unique factorization domain. Since Z is a unique factorization domain, the next theorem is a broad generalization of this fact. The proof is similar to that of the special case, and we therefore omit it.

Theorem 18.5 D a UFD Implies $D[x]$ a UFD

If D is a unique factorization domain, then $D[x]$ is a unique factorization domain.

We conclude this chapter with an example of an integral domain that is not a unique factorization domain.

EXAMPLE 8 The ring $Z[\sqrt{-5}\,] = \{a+b\sqrt{-5} \mid a, b \in Z\}$ is an integral domain but not a unique factorization domain. It is straightforward that $Z[\sqrt{-5}\,]$ is an integral domain (see Exercise 11 in Chapter 13). To verify that unique factorization does not hold, we mimic the method used in Example 1 with $N(a+b\sqrt{-5}) = a^2+5b^2$. Since $N(xy) = N(x)N(y)$ and $N(x) = 1$ if and only if x is a unit (see Exercise 1), it follows that the only units of $Z[\sqrt{-5}\,]$ are ± 1. Now consider the following factorizations:

$$46 = 2 \cdot 23,$$
$$46 = (1 + 3\sqrt{-5})(1 - 3\sqrt{-5}).$$

We claim that each of these four factors is irreducible over $Z[\sqrt{-5}\,]$. Suppose that, say, $2 = xy$, where $x, y \in Z[\sqrt{-5}\,]$ and neither is a unit. Then $4 = N(2) = N(x)N(y)$ and, therefore, $N(x) = N(y) = 2$, which is impossible. Likewise, if $23 = xy$ were a nontrivial factorization, then $N(x) = 23$. Thus, there would be integers a and b such that $a^2 + 5b^2 = 23$. Clearly, no such integers exist. The same argument applies to $1 \pm 3\sqrt{-5}$. ∎

In light of Examples 7 and 8, one can't help but wonder for which $d < 0$ is $Z[\sqrt{d}\,]$ a unique factorization domain. The answer is only when $d = -1$ or -2. The case where $d = -1$ was first proved, naturally enough, by Gauss.

Exercises

I tell them that if they will occupy themselves with the study of mathematics they will find in it the best remedy against lust of the flesh.

Thomas Mann, *The Magic Mountain*

1. For the ring $Z[\sqrt{d}] = \{a + b\sqrt{d} \mid a, b \in Z\}$, where $d \neq 1$ and d is not divisible by the square of a prime, prove that the norm $N(a + b\sqrt{d}) = |a^2 - db^2|$ satisfies the four assertions made preceding Example 1. (This exercise is referred to in this chapter.)

2. In an integral domain, show that a and b are associates if and only if $\langle a \rangle = \langle b \rangle$.

3. Show that the union of a chain $I_1 \subset I_2 \subset \cdots$ of ideals of a ring R is an ideal of R. (This exercise is referred to in this chapter.)

4. In an integral domain, show that the product of an irreducible and a unit is an irreducible.

5. Suppose that a and b belong to an integral domain and $b \neq 0$. Show that $\langle ab \rangle$ is a proper subset of $\langle b \rangle$ if and only if a is not a unit. This exercise is referred to in this chapter.

6. Let D be an integral domain. Define $a \sim b$ if a and b are associates. Show that this defines an equivalence relation on D.

7. In the notation of Example 7, show that $d(xy) = d(x)d(y)$.

8. Let D be a Euclidean domain with measure d. Prove that u is a unit in D if and only if $d(u) = d(1)$.

9. Let D be a Euclidean domain with measure d. Show that if a and b are associates in D, then $d(a) = d(b)$.

10. Let D be a principal ideal domain and let $p \in D$. Prove that $\langle p \rangle$ is a maximal ideal in D if and only if p is irreducible.

11. Let d be an integer less than -1 that is not divisible by the square of a prime. Prove that the only units of $Z[\sqrt{d}\,]$ are $+1$ and -1.

12. Let D be a principal ideal domain. Show that every proper ideal of D is contained in a maximal ideal of D.

13. In $Z[\sqrt{-5}]$, show that 21 does not factor uniquely as a product of irreducibles.

14. Show that $1 - i$ is an irreducible in $Z[i]$.

15. Show that $Z[\sqrt{-6}\,]$ is not a unique factorization domain. (*Hint:* Factor 10 in two ways.) Why does this show that $Z[\sqrt{-6}\,]$ is not a principal ideal domain?

16. Give an example of a unique factorization domain with a subdomain that does not have a unique factorization.

17. In $Z[i]$, show that 3 is irreducible but 2 and 5 are not.

18. Prove that 7 is irreducible in $Z[\sqrt{6}\,]$, even though $N(7)$ is not prime.

19. Prove that if p is a prime in Z that can be written in the form $a^2 + b^2$, then $a + bi$ is irreducible in $Z[i]$. Find three primes that have this property and the corresponding irreducibles.

20. Prove that $Z[\sqrt{-3}\,]$ is not a principal ideal domain.

21. In $Z[\sqrt{-5}]$, prove that $1+3\sqrt{-5}$ is irreducible but not prime.

22. In $Z[\sqrt{5}\,]$, prove that both 2 and $1 + \sqrt{5}$ are irreducible but not prime.

23. Prove that $Z[\sqrt{5}\,]$ is not a unique factorization domain.

24. Let F be a field. Show that in $F[x]$ a non-zero prime ideal is a maximal ideal.

25. Trace through the argument given in Example 7 to find q and r in $Z[i]$ such that $3-4i = (2+5i)q+r$ and $d(r) < d(2+5i)$.

26. In $Z[\sqrt{2}] = \{a+b\sqrt{2} \mid a, b \in Z\}$, show that every element of the form $(3 + 2\sqrt{2})^n$ is a unit, where n is a positive integer.

27. If a and b belong to $Z[\sqrt{d}\,]$, where d is not divisible by the square of a prime and ab is a unit, prove that a and b are units.

28. For a commutative ring with unity we may define associates, irreducibles, and primes exactly as we did for integral domains. With these definitions, show that both 2 and 3 are prime in Z_{12} but 2 is irreducible and 3 is not.

29. Let n be a positive integer and p a prime that divides n. Prove that p is prime in Z_n. (See Exercise 28).

30. Let p be a prime divisor of a positive integer n. Prove that p is irreducible in Z_n if and only if p^2 divides n. (See Exercise 28).

31. Prove or disprove that if D is a principal ideal domain, then $D[x]$ is a principal ideal domain.

32. Determine the units in $Z[i]$.

33. Let p be a prime in an integral domain. If $p \mid a_1a_2 \cdots a_n$, prove that p divides some a_i. (This exercise is referred to in this chapter.)

34. Show that $3x^2 + 4x + 3 \in Z_5[x]$ factors as $(3x + 2)(x + 4)$ and $(4x + 1)\ (2x + 3)$. Explain why this does not contradict the corollary of Theorem 18.3.

35. Let D be a principal ideal domain and p an irreducible element of D. Prove that $D/\langle p\rangle$ is a field.

36. Show that an integral domain with the property that every strictly decreasing chain of ideals $I_1 \supset I_2 \supset \cdots$ must be finite in length is a field.

37. An ideal A of a commutative ring R with unity is said to be *finitely generated* if there exist elements $a_1, a_2, \ldots, a_n$ of A such that $A = \langle a_1, a_2, \ldots, a_n\rangle$. An integral domain R is said to satisfy the *ascending chain condition* if every strictly increasing chain of ideals $I_1 \subset I_2 \subset \cdots$ must be finite in length. Show that an integral domain R satisfies the ascending chain condition if and only if every ideal of R is finitely generated.

38. Prove or disprove that a subdomain of a Euclidean domain is a Euclidean domain.

39. Show that for any nontrivial ideal I of $Z[i]$, $Z[i]/I$ is finite.

40. Find the inverse of $1+\sqrt{2}$ in $Z[\sqrt{2}]$. What is the multiplicative order of $1+\sqrt{2}$?

41. In $Z[\sqrt{-7}\,]$, show that $N(6+2\sqrt{-7}) = N(1+3\sqrt{-7})$ but $6+2\sqrt{-7}$ and $1+3\sqrt{-7}$ are not associates.

42. Let $R = Z \oplus Z \oplus \cdots$ (the collection of all sequences of integers under componentwise addition and multiplication). Show that R has ideals $I_1, I_2, I_3, \ldots$ with the property that $I_1 \subset I_2 \subset I_3 \subset \cdots$. (Thus R does not have the ascending chain condition.)

43. Prove that in a unique factorization domain, an element is irreducible if and only if it is prime.

44. Let F be a field and let R be the integral domain in $F[x]$ generated by x^2 and x^3. (That is, R is contained in every integral domain in $F[x]$ that contains x^2 and x^3.) Show that R is not a unique factorization domain.

45. Prove that for every field F, there are infinitely many irreducible elements in $F[x]$.

46. Prove that $Z[\sqrt{-2}\,]$ and $Z[\sqrt{2}\,]$ are unique factorization domains. (*Hint*: Mimic Example 7 in Chapter 18.)

47. Express both 13 and $5+i$ as products of irreducibles from $Z[i]$.

48. Find a mistake in the statement shown in Figure 18.2.

49. Let I be a non-zero ideal in a principal ideal domain R. Prove that R/I has a finite number of ideals.

Computer Exercises

Software to determine when a positive integer n is prime in $Z[i]$ is available at the website:

http://www.d.umn.edu/~jgallian

Sophie Germain


Stock Montage

When a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men... succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of [number theory], then without doubt she must have the noblest courage, quite extraordinary talents and superior genius.

CARL FRIEDRICK GAUSS

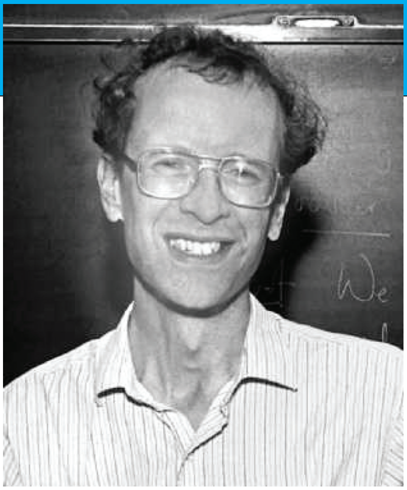
Sophie Germain was born in Paris on April 1, 1776. She educated herself by reading the works of Newton and Euler in Latin and the lecture notes of Lagrange. In 1804, Germain wrote to Gauss about her work in number theory but used the pseudonym Monsieur LeBlanc because she feared that Gauss would not take seriously the efforts of a woman. Gauss gave Germain's results high praise and a few years later, upon learning her true identity, wrote to her:

But how to describe to you my admiration and astonishment at seeing my esteemed correspondent Mr. LeBlanc metamorphose himself into this illustrious personage who gives such a brilliant example of what I would find it difficult to believe. A taste for the abstract sciences in general and above all the mysteries of numbers is excessively rare: it is not a subject which strikes everyone; the enchanting charms of this sublime science reveal themselves only to those who have the courage to go deeply into it. But when a person of the sex which, according to our customs and prejudices, must encounter infinitely more difficulties than men to familiarize herself with these thorny researches, succeeds nevertheless in surmounting these obstacles and penetrating the most obscure parts of them, then without doubt she must have the noblest courage, quite extraordinary talents, and a superior genius.[2]

Germain is best known for her work on Fermat's Last Theorem. She died on June 27, 1831, in Paris.

[2] Quote from *Math's Hidden Woman*, Nova Online, http://www.pbs.org/wgbh/nova/proof/germain.html (accessed Nov 5, 2008).

Andrew Wiles


Princeton University

For spectacular contributions to number theory and related fields, for major advances on fundamental conjectures, and for settling Fermat's Last Theorem.

Citation for the Wolf Prize

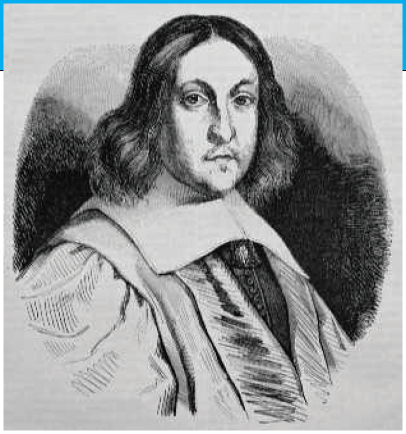
Postage stamp issued by the Czech Republic in honor of Fermat's Last Theorem.

In 1993, Andrew Wiles of Princeton electrified the mathematics community by announcing that he had proved Fermat's Last Theorem after seven years of effort. His proof, which ran 200 pages, relied heavily on ring theory and group theory. Because of Wiles's solid reputation and because his approach was based on deep results that had already shed much light on the problem, many experts in the field believed that Wiles had succeeded where so many others had failed. Wiles's achievement was reported in newspapers and magazines around the world. *The New York Times* ran a front-page story on it, and one TV network announced it on the evening news. Wiles even made *People* magazine's list of the 25 most intriguing people of 1993! In San Francisco a group of mathematicians rented a 1200-seat movie theater and sold tickets for $5.00 each for public lectures on the proof. Scalpers received as much as $25.00 a ticket for the sold-out event.

The bubble soon burst when experts had an opportunity to scrutinize Wiles's manuscript. By December, Wiles released a statement saying he was working to resolve a gap in the proof. In September of 1994, a paper by Wiles and Richard Taylor, a former student of his, circumvented the gap in the original proof. Since then, many experts have checked the proof and have found no errors. One mathematician was quoted as saying, "The exuberance is back." In 1997, Wiles's proof was the subject of a PBS *Nova* program.

Wiles was born in 1953 in Cambridge, England. He obtained his bachelor's degree at Oxford and his doctoral degree at Cambridge University in 1980. In 2011 he rejoined Oxford. Among his many high honors are the Cole Prize, the Fermat Prize, the Abel Prize, the Copley Medal, and a knighthood. In 2018 he was the first person ever appointed as a Regius Professor of Mathematics at Oxford.

Pierre de Fermat



Marzolino/Shutterstock.com

This theorem [Fermat's Little Theorem] is one of the great tools of modern number theory.

WILLIAM DUNHAM

Pierre de Fermat (pronounced Fair-mah) was born in Beaumont-de-Lomagne, France in August of 1601 and died in 1665. Fermat obtained a Bachelor's degree in civil law from the University of Orleans in 1631. While earning his living practicing law he did mathematics as a hobby. Rather than proving and publishing theorems he sent the statements of his results and questions to leading mathematicians. One of his important observations is that any prime of the form $4k + 1$ can be written as the sum of two squares in one and only one way, whereas a prime of the form $4k - 1$ cannot be written as the sum of two squares in any manner whatever. Mathematics historian William Dunham asserts that Fermat's discovery of this dichotomy among primes ranks as one of the landmarks of number theory.

Addressing Fermat's contributions to number theory André Weil wrote that "...what we possess of his methods for dealing with curves of genus 1 is remarkably coherent; it is still the foundation for the modern theory of such curves." A Wikipedia article on Fermat concluded with the statement "Fermat essentially created the modern theory of numbers." Beyond his contributions to number theory, Fermat found a law of optics and is considered as one of the founders of analytic geometry and probability theory. In 1989 the Institut de Mathétiques de Toulousem in France established the Fermat prize for research in fields in which Fermat made majors contributions. Among the recipients are Andrew Wiles and Richard Taylor.

19 Extension Fields

Viewed with perfect hindsight, there were many occasions during the history of algebra when new number systems had to be *created*, or *constructed*, in order to provide roots for certain polynomials.

Norman J. Block, *Abstract Algebra with Applications*

In many respects this [Kronecker's Theorem] is the fundamental theorem of algebra.

Richard A. Dean, *Elements of Abstract Algebra*

The Fundamental Theorem of Field Theory

In our work on rings, we came across a number of fields, both finite and infinite. Indeed, we saw that $Z_3[x]/\langle x^2+1\rangle$ is a field of order 9, whereas $\mathbf{R}[x]/\langle x^2+1\rangle$ is a field isomorphic to the complex numbers. In the next three chapters, we take up, in a systematic way, the subject of fields.

Definition Extension Field
A field E is an *extension field* of a field F if $F \subseteq E$ and the operations of F are those of E restricted to F.

Cauchy's observation in 1847 that $\mathbf{R}[x]/\langle x^2+1\rangle$ is a field that contains a zero of x^2+1 prepared the way for the following sweeping generalization of that fact.

Theorem 19.1 Fundamental Theorem of Field Theory (Kronecker's Theorem, 1887)

Let F be a field and let $f(x)$ be a nonconstant polynomial in $F[x]$. Then there is an extension field E of F in which $f(x)$ has a zero.

PROOF Since $F[x]$ is a unique factorization domain, $f(x)$ has an irreducible factor, say, $p(x)$. Clearly, it suffices to construct an extension field E of F in which $p(x)$ has a zero. Our candidate for E is $F[x]/\langle p(x)\rangle$. We already know that this is a field from Corollary 1 of Theorem 17.5. Also, since the mapping of ϕ: $F \to E$ given by $\phi(a) = a + \langle p(x)\rangle$ is one-to-one and preserves both operations, E has a subfield isomorphic to F. We may think of E as containing F if we simply identify the coset $a + \langle p(x)\rangle$ with its unique coset representative a that belongs to F [i.e., think of $a + \langle p(x)\rangle$ as just a and vice versa; see Exercise 41 in Chapter 17].

Finally, to show that $p(x)$ has a zero in E, write

$$p(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0.$$

Then, in $E, x + \langle p(x)\rangle$ is a zero of $p(x)$, because

$$\begin{aligned} p(x + \langle p(x)\rangle) &= a_n(x + \langle p(x)\rangle)^n + a_{n-1}(x + \langle p(x)\rangle)^{n-1} + \cdots + a_0 \\ &= a_n(x^n + \langle p(x)\rangle) + a_{n-1}(x^{n-1} + \langle p(x)\rangle) + \cdots + a_0 \\ &= a_n x^n + a_{n-1}x^{n-1} + \cdots + a_0 + \langle p(x)\rangle \\ &= p(x) + \langle p(x)\rangle = 0 + \langle p(x)\rangle. \end{aligned}$$

■

EXAMPLE 1 Let $f(x) = x^2 + 1 \in Q[x]$. Then, viewing $f(x)$ as an element of $E[x] = (Q[x]/\langle x^2 + 1\rangle)[x]$, we have

$$\begin{aligned} f(x + \langle x^2 + 1\rangle) &= (x + \langle x^2 + 1\rangle)^2 + 1 \\ &= x^2 + \langle x^2 + 1\rangle + 1 \\ &= x^2 + 1 + \langle x^2 + 1\rangle \\ &= 0 + \langle x^2 + 1\rangle. \end{aligned}$$

Of course, the polynomial $x^2 + 1$ has the complex number $\sqrt{-1}$ as a zero, but the point we wish to emphasize here is that we have constructed a field that contains the rational numbers and a zero for the polynomial $x^2 + 1$ by using only the rational numbers. No knowledge of complex numbers is necessary. Our method utilizes only the field we are given. ■

EXAMPLE 2 Let $f(x) = x^5 + 2x^2 + 2x + 2 \in Z_3[x]$. Then, the irreducible factorization of $f(x)$ over Z_3 is $(x^2 + 1)(x^3 + 2x + 2)$. So, to find an extension E of Z_3 in which $f(x)$ has a zero, we may take $E = Z_3[x]/\langle x^2 + 1\rangle$, a field with nine elements, or $E = Z_3[x]/\langle x^3 + 2x + 2\rangle$, a field with 27 elements. ∎

Since every integral domain is contained in its field of quotients (Theorem 15.6), we see that every nonconstant polynomial with coefficients from an integral domain always has a zero in some field containing the ring of coefficients. The next example shows that this is not true for commutative rings in general.

∎ **EXAMPLE 3** Let $f(x) = 2x + 1 \in Z_4[x]$. Then $f(x)$ has no zero in any ring containing Z_4 as a subring, because if β were a zero in such a ring, then $0 = 2\beta + 1$, and therefore $0 = 2(2\beta + 1) = 2(2\beta) + 2 = (2 \cdot 2)\beta + 2 = 0 \cdot \beta + 2 = 2$. But $0 \neq 2$ in Z_4. ∎

Splitting Fields

To motivate the next definition and theorem, let's return to Example 1 for a moment. For notational convenience, in $F = Q[x]/\langle x^2 + 1\rangle$, let α be the element $x + \langle x^2 + 1\rangle$ in F, and for any $a \in Q$, we denote the coset $a + \langle x^2 + 1\rangle$ by a. Then, because both α and $-\alpha$ are zeros of the polynomial $x^2 + 1$ in F it should be the case that $x^2 + 1$ factors as $(x - \alpha)(x + \alpha)$ in F. To see this, note that $\alpha^2 = (x + \langle x^2 + 1\rangle)^2 = x^2 + \langle x^2 + 1\rangle = x^2 + (1-1) + \langle x^2 + 1\rangle = x^2 + 1 - 1 + \langle x^2 + 1\rangle = -1 + \langle x^2 = 1\rangle = -1$. So, in F, $(x - \alpha)(x + \alpha) = x^2 - \alpha^2 = x^2 + 1$.
This shows that $x^2 + 1$ can be written as a product of linear factors in some extension of Q. That was easy and you might argue coincidental. The polynomial given in Example 2 presents a greater challenge. Is there an extension of Z_3 in which that polynomial factors as a product of linear factors? Yes, there is. But first some notation and a definition.

Let F be a field and let $a_1, a_2, \ldots, a_n$ be elements of some extension E of F. We use $F(a_1, a_2, \ldots, a_n)$ to denote the smallest subfield of E that contains F and the set $\{a_1, a_2, \ldots, a_n\}$. We leave it as an exercise (Exercise 37) to show that $F(a_1, a_2, \ldots, a_n)$ is the intersection of all subfields of E that contain F and the set $\{a_1, a_2, \ldots, a_n\}$. That is, if K is any field that contains F and the set $\{a_1, a_2, \ldots, a_n\}$ then K contains $F(a_1, a_2, \ldots, a_n)$.

Definitions Splitting Field

Let E be an extension field of F and let $f(x) \in F[x]$ with degree at least 1. We say that $f(x)$ *splits* in E if there are

elements $a \in F$ and $a_1, a_2, \ldots, a_n \in E$ such that

$$f(x) = a(x - a_1)(x - a_2) \cdots (x - a_n).$$

We call E a *splitting field for $f(x)$ over F* if $E = F(a_1, a_2, \ldots, a_n)$.

Note that a splitting field of a polynomial over a field depends not only on the polynomial but on the field as well. Indeed, a splitting field of $f(x)$ over F is just a smallest extension field of F in which $f(x)$ splits. The next example illustrates how a splitting field of a polynomial $f(x)$ over field F depends on F.

EXAMPLE 4 Consider the polynomial $f(x) = x^2 + 1 \in Q[x]$. Since $x^2 + 1 = (x + \sqrt{-1})(x - \sqrt{-1})$, we see that $f(x)$ splits in $\mathbf{C}$, but a splitting field over Q is $Q(i) = \{r + si \mid r, s \in Q\}$. A splitting field for $x^2 + 1$ over $\mathbf{R}$ is $\mathbf{C}$. Likewise, $x^2 - 2 \in Q[x]$ splits in $\mathbf{R}$, but a splitting field over Q is $Q(\sqrt{2}) = \{r + s\sqrt{2} \mid r, s \in Q\}$. ▮

There is a useful analogy between the definition of a splitting field and the definition of an irreducible polynomial. Just as it makes no sense to say "$f(x)$ is irreducible," it makes no sense to say "E is a splitting field for $f(x)$." In each case, the underlying field must be specified; that is, one must say "$f(x)$ is irreducible over F" and "E is a splitting field for $f(x)$ over F."

Our notation in Example 4 appears to be inconsistent with the notation that we used in earlier chapters. For example, we denoted the set $\{a + b\sqrt{2} \mid a, b \in Z\}$ by $Z[\sqrt{2}]$ and the set $\{a + b\sqrt{2} \mid a, b \in Q\}$ by $Q(\sqrt{2})$. The difference is that $Z[\sqrt{2}\ ]$ is merely a ring, whereas $Q(\sqrt{2})$ is a field. In general, parentheses are used when one wishes to indicate that the set is a field, although no harm would be done by using, say, $Q[\sqrt{2}]$ to denote $\{a + b\sqrt{2} \mid a, b \in Q\}$ if we were concerned with its ring properties only. Using parentheses rather than brackets simply conveys a bit more information about the set.

Theorem 19.2 Existence of Splitting Fields

Let F be a field and let $f(x)$ be a nonconstant element of $F[x]$. Then there exists a splitting field E for $f(x)$ over F.

PROOF We proceed by induction on deg $f(x)$. If deg $f(x) = 1$, then $f(x)$ is linear. Now suppose that the statement is true for all fields and all polynomials of degree less than that of $f(x)$. By Theorem 19.1, there is an extension E of F in which $f(x)$ has a zero, say, a_1. Then we may write $f(x) = (x - a_1)g(x)$, where $g(x) \in E[x]$. Since deg $g(x) <$ deg $f(x)$, by induction, there is a field K that contains E and all the zeros of $g(x)$, say, $a_2, \ldots, a_n$. Clearly, then, a splitting field for $f(x)$ over F is $F(a_1, a_2, \ldots, a_n)$. ■

■ **EXAMPLE 5** Consider

$$f(x) = x^4 - x^2 - 2 = (x^2 - 2)(x^2 + 1)$$

over Q. Obviously, the zeros of $f(x)$ in $\mathbf{C}$ are $\pm\sqrt{2}$ and $\pm i$. So a splitting field for $f(x)$ over Q is

$$\begin{aligned} Q(\sqrt{2}, i) = Q(\sqrt{2})(i) &= \{\alpha + \beta i \mid \alpha, \beta \in Q(\sqrt{2})\} \\ &= \{(a + b\sqrt{2}) + (c + d\sqrt{2})i \mid a, b, c, d \in Q\}. \end{aligned}$$

■

■ **EXAMPLE 6** Consider $f(x) = x^2 + x + 2$ over Z_3. Then $Z_3(i) = \{a + bi \mid a, b \in Z_3\}$ (see Example 9 in Chapter 13) is a splitting field for $f(x)$ over Z_3 because

$$f(x) = (x - (1 + i))(x - (1 - i)).$$

At the same time, we know by the proof of Kronecker's Theorem that the element $x + \langle x^2 + x + 2\rangle$ of

$$F = Z_3[x]/\langle x^2 + x + 2\rangle$$

is a zero of $f(x)$. Since $f(x)$ has degree 2, it follows from the Factor Theorem (Corollary 2 of Theorem 16.2) that the other zero of $f(x)$ must also be in F. Thus, $f(x)$ splits in F, and because F is a two-dimensional vector space over Z_3, we know that F is also a splitting field of $f(x)$ over Z_3. But how do we factor $f(x)$ in F? Factoring $f(x)$ in F is confusing because we are using the symbol x in two distinct ways: It is used as a placeholder to write the polynomial $f(x)$, and it is used to create the coset representatives of the elements of F. This confusion can be avoided by simply identifying the coset $1 + \langle x^2 + x + 2\rangle$ with the element

1 in Z_3 and denoting the coset $x + \langle x^2 + x + 2 \rangle$ by β. With this identification, the field $Z_3[x]/\langle x^2 + x + 2 \rangle$ can be represented as $\{0, 1, 2, \beta, 2\beta, \beta+1, 2\beta+1, \beta+2, 2\beta+2\}$. These elements are added and multiplied just as polynomials are, except that we use the observation that $x^2+x+2+\langle x^2+x+2 \rangle = 0$ implies that $\beta^2+\beta+2 = 0$, so that $\beta^2 = -\beta - 2 = 2\beta + 1$. For example, $(2\beta + 1)(\beta + 2) = 2\beta^2 + 5\beta + 2 = 2(2\beta + 1) + 5\beta + 2 = 9\beta + 4 = 1$. To obtain the factorization of $f(x)$ in F, we simply long divide, as follows:

$$
\begin{array}{r l}
 & x + (\beta + 1) \\ \cline{2-2}
x - \beta \big) & x^2 + x + 2 \\
 & x^2 - \beta x \\ \cline{2-2}
 & (\beta + 1)x + 2 \\
 & (\beta + 1)x - (\beta + 1)\beta \\ \cline{2-2}
 & (\beta + 1)\beta + 2 = \beta^2 + \beta + 2 = 0.
\end{array}
$$

So, $x^2 + x + 2 = (x - \beta)(x + \beta + 1)$. Thus, we have found two splitting fields for x^2+x+2 over Z_3, one of the form $F(a)$ and one of the form $F[x]/\langle p(x) \rangle$ [where $F = Z_3$ and $p(x) = x^2 + x + 2$]. ∎

The next theorem shows how the fields $F(a)$ and $F[x]/\langle p(x) \rangle$ are related in the case where $p(x)$ is irreducible over F and a is a zero of $p(x)$ in some extension of F.

Theorem 19.3 $F(a) \approx F[x]/\langle p(x) \rangle$

Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is a zero of $p(x)$ in some extension E of F, then $F(a)$ is isomorphic to $F[x]/\langle p(x) \rangle$. Furthermore, if deg $p(x) = n$, then every member of $F(a)$ can be uniquely expressed in the form

$$c_{n-1}a^{n-1} + c_{n-2}a^{n-2} + \cdots + c_1 a + c_0,$$

where $c_0, c_1, \ldots, c_{n-1} \in F$.

PROOF Consider the function ϕ from $F[x]$ to $F(a)$ given by $\phi(f(x)) = f(a)$. Clearly, ϕ is a ring homomorphism. We claim that Ker $\phi = \langle p(x) \rangle$. (This is Exercise 39 in Chapter 17.) Since $p(a) = 0$, we have $\langle p(x) \rangle \subseteq$ Ker ϕ. On the other hand, we know by Theorem 17.5 that $\langle p(x) \rangle$ is a maximal ideal in $F[x]$. So, because Ker $\phi \neq F[x]$ [it does not contain the constant polynomial

$f(x) = 1$], we have Ker $\phi = \langle p(x) \rangle$. At this point it follows from the First Isomorphism Theorem for Rings and Corollary 1 of Theorem 17.5 that $\phi(F[x])$ is a subfield of $F(a)$. Noting that $\phi(F[x])$ contains both F and a and recalling that $F(a)$ is the smallest such field, we have $F[x]/\langle p(x) \rangle \approx \phi(F[x]) = F(a)$.

The final assertion of the theorem follows from the fact that every element of $F[x]/\langle p(x) \rangle$ can be expressed uniquely in the form

$$c_{n-1}x^{n-1} + \cdots + c_0 + \langle p(x) \rangle,$$

where $c_0, \ldots, c_{n-1} \in F$ (see Exercise 47 in Chapter 16), and the natural isomorphism from $F[x]/\langle p(x) \rangle$ to $F(a)$ carries $c_k x^k + \langle p(x) \rangle$ to $c_k a^k$.

As an immediate corollary of Theorem 19.3, we have the following attractive result. ▮

Corollary $F(a) \approx F(b)$

Let F be a field and let $p(x) \in F[x]$ be irreducible over F. If a is a zero of $p(x)$ in some extension E of F and b is a zero of $p(x)$ in some extension E' of F, then the fields $F(a)$ and $F(b)$ are isomorphic.

PROOF From Theorem 19.3, we have $F(a) \approx F[x]/\langle p(x) \rangle \approx F(b)$. ▮

Recall that a basis for an n-dimensional vector space over a field F is a set of n vectors $v_1, v_2, \ldots, v_n$ with the property that every member of the vector space can be expressed uniquely in the form $a_1v_1 + a_2v_2 + \cdots + a_nv_n$, where the a's belong to F. So, in the language of vector spaces, the latter portion of Theorem 19.3 says that if a is a zero of an irreducible polynomial over F of degree n, then the set $\{1, a, \ldots, a^{n-1}\}$ is a basis for $F(a)$ over F.

Theorem 19.3 often provides a convenient way of describing the elements of a field.

EXAMPLE 7 Consider the irreducible polynomial $f(x) = x^6 - 2$ over Q. Since $\sqrt[6]{2}$ is a zero of $f(x)$, we know from Theorem 19.3 that the set $\{1, 2^{1/6}, 2^{2/6}, 2^{3/6}, 2^{4/6}, 2^{5/6}\}$ is a basis for $Q(\sqrt[6]{2})$ over Q. Thus,

$$Q(\sqrt[6]{2}) = \{a_0 + a_1 2^{1/6} + a_2 2^{2/6} + a_3 2^{3/6} + a_4 2^{4/6} + a_5 2^{5/6} \mid a_i \in Q\}.$$

This field is isomorphic to $Q[x]/\langle x^6 - 2 \rangle$. ▮

Theorem 19.3 provides us with two ways to look at fields constructed by extending a field F to a larger one that has the a zero of a polynomial $p(x)$ of degree n that is irreducible over F. Generally speaking, we use the generic approach $F[x]/\langle p(x)\rangle$ to do theoretical things such as proving existence and uniqueness of splitting fields and we use $F(a) = \{c_{n-1}a^{n-1} + c_{n-2}x^{n-2} + \cdots + c_1x + c_0 \mid c_0, c_1, \ldots, c_{n-1} \in F\}$ when we have a specific polynomial as in Example 7. The latter is more concrete and is notationally more convenient. We will choose the between them depending on the circumstances.

In 1882, Ferdinand von Lindemann (1852–1939) proved that π is not the zero of any polynomial in $Q[x]$. Because of this important result, Theorem 19.3 does not apply to $Q(\pi)$ (see Exercise 11). Fields of the form $F(a)$ where a is in some extension field of F but not the zero of an element of $F(x)$ are discussed in the next chapter.

In Example 6, we produced two splitting fields for the polynomial x^2+x+2 over Z_3. Likewise, it is an easy exercise to show that both $Q[x]/\langle x^2+1\rangle$ and $Q(i) = \{r+si \mid r, s \in Q\}$ are splitting fields of the polynomial $x^2 + 1$ over Q. But are these different-looking splitting fields algebraically different? Not really. We conclude our discussion of splitting fields by proving that splitting fields are unique up to isomorphism. To make it easier to apply induction, we will prove a more general result.

We begin by observing first that any ring isomorphism ϕ from F to F' has a natural extension from $F[x]$ to $F'[x]$ given by $c_nx^n + c_{n-1}x^{n-1} + \cdots + c_1x + c_0 \to \phi(c_n)x^n + \phi(c_{n-1})x^{n-1} + \cdots + \phi(c_1)x + \phi(c_0)$. Since this mapping agrees with ϕ on F, it is convenient and natural to use ϕ to denote this mapping as well.

Lemma

Let F be a field, let $p(x) \in F[x]$ be irreducible over F, and let a be a zero of $p(x)$ in some extension of F. If ϕ is a field isomorphism from F to F' and b is a zero of $\phi(p(x))$ in some extension of F', then there is an isomorphism from $F(a)$ to $F'(b)$ that agrees with ϕ on F and carries a to b.

PROOF First observe that since $p(x)$ is irreducible over F, $\phi(p(x))$ is irreducible over F'. It is straightforward to check that the mapping from $F[x]/\langle p(x)\rangle$ to $F'[x]/\langle \phi(p(x))\rangle$ given by

$$f(x) + \langle p(x)\rangle \to \phi(f(x)) + \langle \phi(p(x))\rangle$$

is a field isomorphism. By a slight abuse of notation, we denote this mapping by ϕ also. (If you object, put a bar over the ϕ.) From the proof of Theorem 19.3, we know that there is an isomorphism α from $F(a)$ to $F[x]/\langle p(x)\rangle$ that is the identity on F and carries a to $x + \langle p(x)\rangle$. Similarly, there is an isomorphism β from $F'[x]/\langle \phi(p(x))\rangle$ to $F'(b)$ that is the identity on F' and carries $x + \langle \phi(p(x))\rangle$ to b. Thus, $\beta\phi\alpha$ is the desired mapping from $F(a)$ to $F'(b)$. See Figure 19.1. ∎

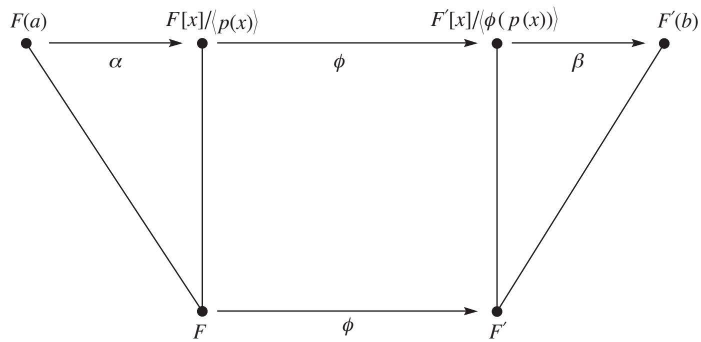


Figure 19.1 The mapping $\beta\phi\alpha$.

Theorem 19.4 Extending ϕ: $F \to F'$

Let ϕ be an isomorphism from a field F to a field F' and let $f(x) \in F[x]$. If E is a splitting field for $f(x)$ over F and E' is a splitting field for $\phi(f(x))$ over F', then there is an isomorphism from E to E' that agrees with ϕ on F.

PROOF We use induction on deg $f(x)$. If deg $f(x) = 1$, then $E = F$ and $E' = F'$, so that ϕ itself is the desired mapping. If deg $f(x) > 1$, let $p(x)$ be an irreducible factor of $f(x)$, let a be a zero of $p(x)$ in E, and let b be a zero of $\phi(p(x))$ in E'. By the preceding lemma, there is an isomorphism α from $F(a)$ to $F'(b)$ that agrees with ϕ on F and carries a to b. Now write $f(x) = (x - a)g(x)$, where $g(x) \in F(a)[x]$. Then E is a splitting field for $g(x)$ over $F(a)$ and E' is a splitting field for $\alpha(g(x))$ over $F'(b)$. Since deg

$g(x) < \deg f(x)$, there is an isomorphism from E to E' that agrees with α on $F(a)$ and therefore with ϕ on F. ∎

Corollary Splitting Fields Are Unique

Let F be a field and let $f(x) \in F[x]$. Then any two splitting fields of $f(x)$ over F are isomorphic.

PROOF Suppose that E and E' are splitting fields of $f(x)$ over F. The result follows immediately from Theorem 19.4 by letting ϕ be the identity from F to F. ∎

In light of the corollary above, we may refer to "the" splitting field of a polynomial over F without ambiguity.

Even though $x^6 - 2$ has a zero in $Q(\sqrt[6]{2})$, it does not split in $Q(\sqrt[6]{2})$. The splitting field is easy to obtain, however.

EXAMPLE 8 The Splitting Field of $x^n - a$ over Q

Let a be a positive rational number and let ω be a primitive nth root of unity (see Example 2 in Chapter 16). Then each of

$$a^{1/n}, \omega a^{1/n}, \omega^2 a^{1/n}, \ldots, \omega^{n-1} a^{1/n}$$

is a zero of $x^n - a$ in $Q(\sqrt[n]{a}, \omega)$. ∎

Zeros of an Irreducible Polynomial

Now that we know that every nonconstant polynomial over a field splits in some extension, we ask whether irreducible polynomials must split in some special way. Yes, they do. To discover how, we borrow something whose origins are in calculus.

Definition Derivative

Let $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$ belong to $F[x]$. The *derivative* of $f(x)$, denoted by $f'(x)$, is the polynomial $na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$ in $F[x]$.

Notice that our definition does not involve the notion of a limit. The standard rules for handling sums and products of functions in calculus carry over to arbitrary fields as well.

Lemma Properties of the Derivative

Let $f(x)$ and $g(x) \in F[x]$ and let $a \in F$. Then
1. $(f(x)+g(x))' = f'(x)+g'(x)$.
2. $(af(x))' = af'(x)$.
3. $(f(x)g(x))' = f(x)g'(x)+g(x)f'(x)$.

PROOF Properties 1 and 2 follow from straightforward applications of the definition. Using property 1 and induction on deg $f(x)$, property 3 reduces to the special case in which $f(x) = a_n x^n$. This also follows directly from the definition. ∎

Before addressing the question of the nature of the zeros of an irreducible polynomial, we establish a general result concerning zeros of multiplicity greater than 1. Such zeros are called *multiple* zeros.

Theorem 19.5 Criterion for Multiple Zeros

A polynomial $f(x)$ over a field F has a multiple zero in some extension E if and only if $f(x)$ and $f'(x)$ have a common factor of positive degree in $F[x]$.

PROOF If a is a multiple zero of $f(x)$ in some extension E, then there is a $g(x)$ in $E[x]$ such that $f(x) = (x-a)^2 g(x)$. Since $f'(x) = (x-a)^2 g'(x) + 2(x-a)g(x)$, we see that $f'(a) = 0$. Thus, $x - a$ is a factor of both $f(x)$ and $f'(x)$ in the extension E of F. Now if $f(x)$ and $f'(x)$ have no common divisor of positive degree in $F[x]$, there are polynomials $h(x)$ and $k(x)$ in $F[x]$ such that $f(x)h(x)+f'(x)k(x) = 1$ (see Exercise 43 in Chapter 16). Viewing $f(x)h(x)+f'(x)k(x)$ as an element of $E[x]$, we see also that $x-a$ is a factor of 1. Since this is nonsense, $f(x)$ and $f'(x)$ must have a common divisor of positive degree in $F[x]$.

Conversely, suppose that $f(x)$ and $f'(x)$ have a common factor of positive degree $F[x]$. Let a be a zero of the common factor. Then a is a zero of $f(x)$ and $f'(x)$. Since a is a zero of $f(x)$, there is a polynomial $q(x)$ such that $f(x) = (x-a)q(x)$. Then $f'(x) = (x-a)q'(x)+q(x)$ and $0 = f'(a) = q(a)$. Thus, $x-a$ is a factor of $q(x)$ and a is a multiple zero of $f(x)$. ∎

Theorem 19.6 Zeros of an Irreducible

Let $f(x)$ be an irreducible polynomial over a field F. If F has characteristic 0, then $f(x)$ has no multiple zeros. If F has characteristic $p \neq 0$, then $f(x)$ has a multiple zero only if it is of the form $f(x) = g(x^p)$ for some $g(x)$ in $F[x]$.

PROOF If $f(x)$ has a multiple zero, then, by Theorem 19.5, $f(x)$ and $f'(x)$ have a common divisor of positive degree in $F[x]$. Since the only divisor of positive degree of $f(x)$ in $F[x]$ is $f(x)$ itself (up to associates), we see that $f(x)$ divides $f'(x)$. Because a polynomial over a field cannot divide a polynomial of smaller degree, we must have $f'(x) = 0$.

Now what does it mean to say that $f'(x) = 0$? If we write $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, then $f'(x) = na_n x^{n-1} + (n-1)a_{n-1}x^{n-2} + \cdots + a_1$. Thus, $f'(x) = 0$ only when $ka_k = 0$ for $k = 1, \ldots, n$.

So, when char $F = 0$, we have $f(x) = a_0$, which is not an irreducible polynomial. This contradicts the hypothesis that $f(x)$ is irreducible over F. Thus, $f(x)$ has no multiple zeros.

When char $F = p \neq 0$, we have $a_k = 0$ when p does not divide k. Thus, the only powers of x that appear in the sum $a_n x^n + \cdots + a_1 x + a_0$ are those of the form $x^{pj} = (x^p)^j$. It follows that $f(x) = g(x^p)$ for some $g(x) \in F[x]$. [For example, if $f(x) = x^{4p} + 3x^{2p} + x^p + 1$, then $g(x) = x^4 + 3x^2 + x + 1$.] ▮

Theorem 19.6 shows that an irreducible polynomial over a field of characteristic 0 cannot have multiple zeros. The desire to extend this result to a larger class of fields motivates the following definition.

Definition Perfect Field
A field F is called *perfect* if F has characteristic 0 or if F has characteristic p and $F^p = \{a^p \mid a \in F\} = F$.

The most important family of perfect fields of characteristic p is the finite fields.

Theorem 19.7 Finite Fields Are Perfect

Every finite field is perfect.

PROOF Let F be a finite field of characteristic p. Consider the mapping ϕ from F to F defined by $\phi(x) = x^p$ for all $x \in F$. We claim that ϕ is a field automorphism. Obviously, $\phi(ab) = (ab)^p = a^p b^p = \phi(a)\phi(b)$. Moreover, $\phi(a+b) = (a+b)^p = a^p + \binom{p}{1} a^{p-1}b + \binom{p}{2} a^{p-2}b^2 + \cdots + \binom{p}{p-1} ab^{p-1} + b^p = a^p + b^p$, since each $\binom{p}{i}$ is divisible by p. Finally, since $x^p \neq 0$ when $x \neq 0$, Ker $\phi = \{0\}$. Thus, ϕ is one-to-one and, since F is finite, ϕ is onto. This proves that $F^p = F$. ∎

Theorem 19.8 Criterion for No Multiple Zeros

If $f(x)$ is an irreducible polynomial over a perfect field F, then $f(x)$ has no multiple zeros.

PROOF The case where F has characteristic 0 has been done. So let us assume that $f(x) \in F[x]$ is irreducible over a perfect field F of characteristic p and that $f(x)$ has multiple zeros. From Theorem 19.6 we know that $f(x) = g(x^p)$ for some $g(x) \in F[x]$, say, $g(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. Since $F^p = F$, each a_i in F can be written in the form b_i^p for some b_i in F. So, using Exercise 49a in Chapter 13, we have

$$\begin{aligned} f(x) = g(x^p) &= {b_n}^p x^{pn} + {b_{n-1}}^p x^{p(n-1)} + \cdots + {b_1}^p x^p + {b_0}^p \\ &= (b_n x^n + b_{n-1}x^{n-1} + \cdots + b_1 x + b_0)^p = (h(x))^p, \end{aligned}$$

where $h(x) \in F[x]$. But then $f(x)$ is not irreducible. ∎

The next theorem shows that when an irreducible polynomial does have multiple zeros, there is something striking about the multiplicities.

Theorem 19.9 Zeros of an Irreducible over a Splitting Field

Let $f(x)$ be an irreducible polynomial over a field F and let E be a splitting field of $f(x)$ over F. Then all the zeros of $f(x)$ in E have the same multiplicity.

PROOF Let a and b be distinct zeros of $f(x)$ in E. If a has multiplicity m, then in $E[x]$ we may write $f(x) = (x - a)^m g(x)$. It follows from the lemma preceding Theorem 19.4 and from Theorem 19.4 that there is a field isomorphism ϕ from E to itself that carries a to b and acts as the identity on F. Thus,

$$f(x) = \phi(f(x)) = (x - b)^m \phi(g(x)),$$

and we see that the multiplicity of b is greater than or equal to the multiplicity of a. By interchanging the roles of a and b, we observe that the multiplicity of a is greater than or equal to the multiplicity of b. So, we have proved that a and b have the same multiplicity. ∎

As an immediate corollary of Theorem 19.9 we have the following appealing result.

Corollary Factorization of an Irreducible over a Splitting Field

Let $f(x)$ be an irreducible polynomial over a field F and let E be a splitting field of $f(x)$. Then $f(x)$ has the form

$$a(x - a_1)^n (x - a_2)^n \cdots (x - a_t)^n,$$

where $a_1, a_2, \ldots, a_t$ are distinct elements of E and $a \in F$.

We conclude this chapter by giving an example of an irreducible polynomial over a field that does have a multiple zero. In particular, notice that the field we use is not perfect.

EXAMPLE 9 Let $F = Z_2(t)$ be the field of quotients of the ring $Z_2[t]$ of polynomials in the indeterminate t with coefficients from Z_2. (We must introduce a letter other than x, since the members of F are going to be our coefficients for the elements in $F[x]$.) Consider $f(x) = x^2 - t \in F[x]$. To see that $f(x)$ is irreducible over F, it suffices to show that it has no zeros in F. Well, suppose that $h(t)/k(t)$ is a zero of $f(x)$. Then $(h(t)/k(t))^2 = t$, and therefore $(h(t))^2 = t(k(t))^2$. But deg $(h(t))^2$ is even, whereas deg $tk(t)^2$ is odd. So, $f(x)$ is irreducible over F.
Finally, since t is a constant in $F[x]$ and the characteristic of F is 2, we have $f'(x) = 0$, so that $f'(x)$ and $f(x)$ have $f(x)$ as a common factor. So, by Theorem 19.5, $f(x)$ has a multiple zero in some extension of F. (Indeed, it has a single zero of multiplicity 2 in $K = F[x]/\langle x^2 - t\rangle$.) ∎

Exercises

I have yet to see any problem, however complicated, which, when you looked at it in the right way, did not become still more complicated.

Paul Anderson, *New Scientist*

1. Describe the elements of $Q(\sqrt[3]{5})$.

2. Show that $Q(\sqrt{2}, \sqrt{3}) = Q(\sqrt{2} + \sqrt{3})$.

3. Find the splitting field of $x^3 - 1$ over Q. Express your answer in the form $Q(a)$.

4. Let E be an extension field of F and let a and b belong to E. Prove that $F(a, b) = F(a)(b) = F(b)(a)$.

5. Find the splitting field of $x^4 + x^2 + 1 = (x^2 + x + 1)(x^2 - x + 1)$ over Q.

6. Prove that $Q(2^{1/3}) = Q(2^{2/3})$.

7. Let F be a field, and let a and b belong to F with $a \neq 0$. If c belongs to some extension field of F, prove that $F(c) = F(ac + b)$. (F "absorbs" its own elements.)

8. Let $F = Z_2$ and let $f(x) = x^3 + x + 1 \in F[x]$. Suppose that a is a zero of $f(x)$ in some extension field of F. How many elements does $F(a)$ have? Express each member of $F(a)$ in terms of a. Write out a complete multiplication table for $F(a)$.

9. Let $F(a)$ be the field described in Exercise 8. Express each of a^5, a^{-2}, and a^{100} in the form $c_2a^2 + c_1a + c_0$.

10. Let $F(a)$ be the field described in Exercise 8. Show that a^2 and $a^2 + a$ are zeros of $x^3 + x + 1$.

11. Describe the elements in $Q(\pi)$.

12. Let $F = Q(\pi^3)$. Find a basis for $F(\pi)$ over F.

13. Write $x^7 - x$ as a product of linear factors over Z_3. Do the same for $x^{10} - x$.

14. Find all ring automorphisms of $Q(\sqrt{5})$. Find all ring automorphisms of $Q(\sqrt[3]{5})$.

15. Let F be a field of characteristic p and let $f(x) = x^p - a \in F[x]$. Show that $f(x)$ is irreducible over F or $f(x)$ splits in F.

16. Suppose that β is a zero of $f(x) = x^4 + x + 1$ in some extension field E of Z_2. Write $f(x)$ as a product of linear factors in $E[x]$.

17. Find a, b, c in Q such that $(1+\sqrt[3]{4})/(2-\sqrt[3]{2}) = a+b\sqrt[3]{2}+c\sqrt[3]{4}$.
Note that such a, b, c exist, since
$(1+\sqrt[3]{4})/(2-\sqrt[3]{2}) \in Q(\sqrt[3]{2}) = \{a+b\sqrt[3]{2}+c\sqrt[3]{4} \mid a,b,c \in Q\}$.

18. Express $(3+4\sqrt{2})^{-1}$ in the form $a+b\sqrt{2}$, where $a,b \in Q$.

19. Show that $Q(4-i) = Q(1+i)$, where $i = \sqrt{-1}$.

20. Find a polynomial $p(x)$ in $Q[x]$ such that $Q(\sqrt{1+\sqrt{5}}\,)$ is ring-isomorphic to $Q[x]/\langle p(x)\rangle$.

21. Let $f(x) \in F[x]$ and let $a \in F$. Show that $f(x)$ and $f(x+a)$ have the same splitting field over F.

22. Recall that two polynomials $f(x)$ and $g(x)$ from $F[x]$ are said to be relatively prime if there is no polynomial of positive degree in $F[x]$ that divides both $f(x)$ and $g(x)$. Show that if $f(x)$ and $g(x)$ are relatively prime in $F[x]$, they are relatively prime in $K[x]$, where K is any extension field of F.

23. Determine all of the subfields of $Q(\sqrt{2})$.

24. Describe the elements of the extension field $Q(\sqrt[4]{2})$ over the field $Q(\sqrt{2})$.

25. What can you say about the order of the splitting field of $x^5+x^4+1 = (x^2+x+1)(x^3+x+1)$ over Z_2?

26. Find the splitting field of x^4+1 over Q.

27. Write x^3+2x+1 as a product of linear polynomials over some extension field of Z_3.

28. Express x^8-x as a product of irreducibles over Z_2.

29. Prove or disprove that $Q(\sqrt{3})$ and $Q(\sqrt{-3})$ are ring-isomorphic.

30. For any prime p, find a field of characteristic p that is not perfect.

31. If β is a zero of x^2+x+2 over Z_5, find the other zero.

32. Show that x^4+x+1 over Z_2 does not have any multiple zeros in any extension field of Z_2.

33. Show that $x^{21}+2x^8+1$ does not have multiple zeros in any extension of Z_3.

34. Show that $x^{19}+x^8+1$ has multiple zeros in some extension of Z_3.

35. Let F be a field of characteristic $p \neq 0$. Show that the polynomial $f(x) = x^{p^n} - x$ over F has distinct zeros.

36. Find the splitting field for $f(x) = (x^2 + x + 2)(x^2 + 2x + 2)$ over Z_3. What is the order of the splitting field? Write $f(x)$ as a product of linear factors over Z_3.

37. Let F be a field and E an extension field of F that contains $a_1, a_2, \ldots, a_n$. Prove that $F(a_1, a_2, \ldots, a_n)$ is the intersection of all subfields of E that contain F and the set $\{a_1, a_2, \ldots, a_n\}$. (This exercise is referred to in this chapter.)

38. Find the splitting field $x^4 - x^2 - 2$ over Z_3.

39. Suppose that $f(x)$ is a fifth-degree polynomial that is irreducible over Z_2. Prove that every nonidentity element is a generator of the cyclic group $(Z_2[x]/\langle f(x)\rangle^*$.

40. Show that $Q(\sqrt{7}, i)$ is the splitting field for $x^4 - 6x^2 - 7$.

41. Suppose that $p(x)$ is a quadratic polynomial with rational coefficients and is irreducible over Q. Show that $p(x)$ has two zeros in $Q[x]/\langle p(x)\rangle$.

42. If $p(x) \in F[x]$ and $\deg p(x) = n$, show that the splitting field for $p(x)$ has at most $n!$ basis elements over F.

43. Let p be a prime, $F = Z_p(t)$ (the field of quotients of the ring $Z_p[x]$) and $f(x) = x^p - t$. Prove that $f(x)$ is irreducible over F and has a multiple zero in $K = F[x]/\langle x^p - t\rangle$.

44. Let $f(x)$ be an irreducible polynomial over a field F. Prove that the number of distinct zeros of $f(x)$ in a splitting field divides $\deg f(x)$.

45. Let n be an even positive integer and F be a finite field of characteristic 2. Explain why the factorization $x^n - x$ into irreducibles over F cannot have the form
$x(x-1)(x+1)g_1(x)g_2(x)\cdots g_k(x)$.

46. Let F be a finite field of characteristic 2 and $f(x)$ belong to $F[x]$. If the factorization of $f(x)$ into irreducibles over F has the form $x(x-1)g_1(x)g_2(x)\cdots g_k(x)$ where the $g_i(x)$ are nonlinear, what is the maximum number of quadratic factors in the product?

47. What is the splitting field of $f(x) = x^3 - 2$ over $Q(\sqrt[3]{2})$? What is the splitting field over $Q(\sqrt{3}i)$?

48. Suppose that a is a zero of $x^3 - x - 1$ in some extension field of Z_3. Write $(a^2 + a)(a^2 + 1)$ in the form $a_2a^2 + a_1a + a_0$ where $a_2, a_1, a_0 \in Z_3$.

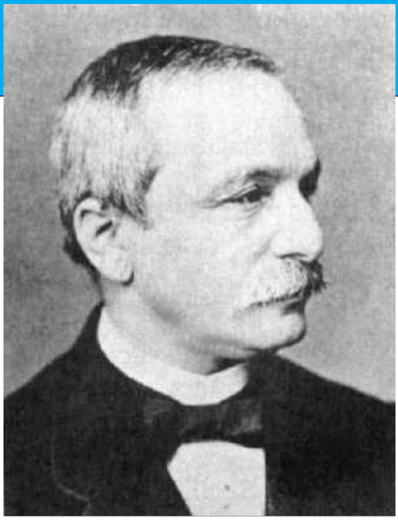
49. Does there exist an extension field $Z_5(\alpha)$ of Z_5 where α is not in Z_5 such that $Z_5(\alpha) = \{a\alpha+b \mid a, b \in Z_5\}$ and $\alpha^2 = 2\alpha+1$? Give a reason for your answer.

50. Does there exist an extension field $F(\alpha)$ of F where α is not in F such that $F(\alpha) = \{a\alpha + b \mid a, b \in F\}$ with $\alpha^2 = \alpha + 2$? Give a reason for your answer.

51. Suppose that $x^2 + 1$ and $x^2 + x + 1$ have no zeros in a field F. If α is a zero of $x^2 + 1$ in some extension field E of F and β is a zero of $x^2 + x + 1$ in E, prove that α is not in $F(\beta)$ and β is not in $F(\alpha)$.

52. Give infinitely many examples of extension fields $F_0, F_1, F_2, \ldots$ of Q in which $x^2 + 1$ splits.

Leopold Kronecker



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He [Kronecker] wove together the three strands of his greatest interests—the theory of numbers, the theory of equations and elliptic functions—into one beautiful pattern.

E. T. BELL

LEOPOLD KRONECKER was born on December 7, 1823, in Liegnitz, Prussia. As a schoolboy, he received special instruction from the great algebraist Kummer. Kronecker entered the University of Berlin in 1841 and completed his Ph.D. dissertation in 1845 on the units in a certain ring.

Kronecker devoted the years 1845–1853 to business affairs, relegating mathematics to a hobby. Thereafter, being well-off financially, he spent most of his time doing research in algebra and number theory. Kronecker was one of the early advocates of the abstract approach to algebra. He innovatively applied rings and fields in his investigations of algebraic numbers, established the Fundamental Theorem of Finite Abelian Groups, and was the first mathematician to master Galois's theory of fields.

Kronecker advocated constructive methods for all proofs and definitions. He believed that all mathematics should be based on relationships among integers. He went so far as to say to Lindemann who proved that π is transcendental, that irrational numbers do not exist. His most famous remark on the matter was "God made the integers, all the rest is the work of man." Henri Poincaré once remarked that Kronecker was able to produce fine work in number theory and algebra only by temporarily forgetting his own philosophy.

Kronecker died on December 29, 1891, at the age of 68.

20 Algebraic Extensions

All things are difficult before they are easy.

Thomas Fuller

Banach once told me, "Good mathematicians see analogies between theorems or theories, the very best ones see analogies between analogies."

S. M. Ulam, *Adventures of a Mathematician*

Characterization of Extensions

In Chapter 19, we saw that every element in the field $Q(\sqrt{2})$ has the particularly simple form $a + b\sqrt{2}$, where a and b are rational. On the other hand, the elements of $Q(\pi)$ have the more complicated form

$$(a_n\pi^n + a_{n-1}\pi^{n-1} + \cdots + a_0)/(b_m\pi^m + b_{m-1}\pi^{m-1} + \cdots + b_0),$$

where the a's and b's are rational. The fields of the first type have a great deal of algebraic structure. This structure is the subject of this chapter.

Definitions Types of Extensions

Let E be an extension field of a field F and let $a \in E$. We call a *algebraic over F* if a is the zero of some nonzero polynomial in $F[x]$. If a is not algebraic over F, it is called *transcendental over F*. An extension E of F is called an *algebraic* extension of F if every element of E is algebraic over F. If E is not an algebraic extension of F, it is called a *transcendental* extension of F. An extension of F of the form $F(a)$ is called a *simple* extension of F.

Leonhard Euler used the term *transcendental* for numbers that are not algebraic because "they transcended the power of algebraic methods." Although Euler made this distinction in 1744, it wasn't until 1844 that the existence of transcendental numbers over Q was proved by Joseph Liouville. Charles Hermite proved that e is transcendental over Q in 1873, and Lindemann showed that π is transcendental over Q in 1882. To this day, it is not known whether $\pi + e$ is transcendental over Q. With a precise definition of "almost all," it can be shown that almost all real numbers are transcendental over Q.

Theorem 20.1 shows why we make the distinction between elements that are algebraic over a field and elements that are transcendental over a field. Recall that $F(x)$ is the field of quotients of $F[x]$; that is,

$$F(x) = \{f(x)/g(x) \mid f(x), g(x) \in F[x], g(x) \neq 0\}.$$

Theorem 20.1 Characterization of Extensions

Let E be an extension field of the field F and let $a \in E$. If a is transcendental over F, then $F(a) \approx F(x)$. If a is algebraic over F, then $F(a) \approx F[x]/\langle p(x)\rangle$, where $p(x)$ is a polynomial in $F[x]$ of minimum degree such that $p(a) = 0$. Moreover, $p(x)$ is irreducible over F.

PROOF Consider the homomorphism $\phi: F[x] \to F(a)$ given by $f(x) \to f(a)$. If a is transcendental over F, then Ker $\phi = \{0\}$, and so we may extend ϕ to an isomorphism $\bar{\phi}: F(x) \to F(a)$ by defining $\bar{\phi}(f(x)/g(x)) = f(a)/g(a)$.

If a is algebraic over F, then Ker $\phi \neq \{0\}$; and, by Theorem 16.5, there is a polynomial $p(x)$ in $F[x]$ such that Ker $\phi = \langle p(x)\rangle$ and $p(x)$ has minimum degree among all nonzero elements of Ker ϕ. Thus, $p(a) = 0$ and, since $p(x)$ is a polynomial of minimum degree with this property, it is irreducible over F. ∎

The proof of Theorem 20.1 can readily be adapted to yield the next two results also. The details are left to the reader (see Exercise 1).

Theorem 20.2 Uniqueness Property

If a is algebraic over a field F, then there is a unique monic irreducible polynomial $p(x)$ in $F[x]$ such that $p(a) = 0$.

The polynomial with the property specified in Theorem 20.2 is called the *minimal polynomial for a over F*.

Theorem 20.3 Divisibility Property

Let a be algebraic over F, and let $p(x)$ be the minimal polynomial for a over F. If $f(x) \in F[x]$ and $f(a) = 0$, then $p(x)$ divides $f(x)$ in $F[x]$.

If E is an extension field of F, we may view E as a vector space over F (i.e., the elements of E are the vectors and the elements of F are the scalars). We are then able to use such notions as dimension and basis in our discussion.

Finite Extensions

Definitions Degree of an Extension
Let E be an extension field of a field F. We say that E *has degree n over F* and write $[E{:}F] = n$ if E has dimension n as a vector space over F. If $[E{:}F]$ is finite, E is called a *finite extension* of F; otherwise, we say that E is an *infinite extension* of F.

Figure 20.1 illustrates a convenient method of depicting the degree of a field extension over a field.

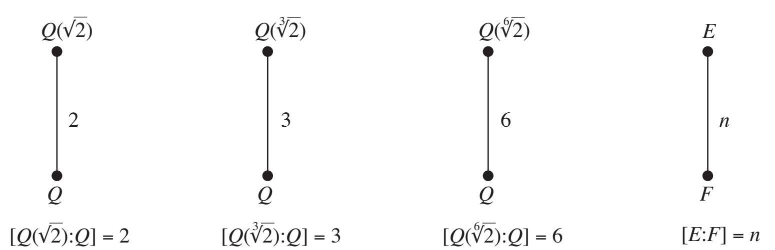


Figure 20.1 Finite extensions.

EXAMPLE 1 The field of complex numbers has degree 2 over the reals, since $\{1, i\}$ is a basis. The field of complex numbers is an infinite extension of the rationals. ∎

EXAMPLE 2 If a is algebraic over F and its minimal polynomial over F has degree n, then, by Theorem 19.3, we know that $\{1, a, \ldots, a^{n-1}\}$ is a basis for $F(a)$ over F; and, therefore, $[F(a){:}F] = n$. In this case, we say that a has *degree n over F*.of a over F ■

Theorem 20.4 Finite Implies Algebraic

If E is a finite extension of F, then E is an algebraic extension of F.

PROOF Suppose that $[E{:}F] = n$ and $a \in E$. Then the set $\{1, a, \ldots, a^n\}$ is linearly dependent over F; that is, there are elements $c_0, c_1, \ldots, c_n$ in F, not all zero, such that

$$c_n a^n + c_{n-1} a^{n-1} + \cdots + c_1 a + c_0 = 0.$$

Clearly, then, a is a zero of the nonzero polynomial

$$f(x) = c_n x^n + c_{n-1} x^{n-1} + \cdots + c_1 x + c_0.$$ ■

The converse of Theorem 20.4 is not true, for otherwise, the degrees of the elements of every algebraic extension of E over F would be bounded. But $Q(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots)$ is an algebraic extension of Q that contains elements of every degree over Q (see Exercise 3).

The next theorem is the field theory counterpart of Lagrange's Theorem for finite groups. Like all counting theorems, it has far-reaching consequences.

Theorem 20.5 $[K{:}F] = [K{:}E][E{:}F]$

Let K be a finite extension field of the field E and let E be a finite extension field of the field F. Then K is a finite extension field of F and $[K{:}F] = [K{:}E][E{:}F]$.

PROOF Let $X = \{x_1, x_2, \ldots, x_n\}$ be a basis for K over E, and let $Y = \{y_1, y_2, \ldots, y_m\}$ be a basis for E over F. It suffices to prove that

$$YX = \{y_j x_i \mid 1 \le j \le m, 1 \le i \le n\}$$

is a basis for K over F. To do this, let $a \in K$. Then there are elements $b_1, b_2, \ldots, b_n \in E$ such that

$$a = b_1 x_1 + b_2 x_2 + \ldots + b_n x_n$$

and, for each $i = 1, \ldots, n$, there are elements $c_{i1}, c_{i2}, \ldots, c_{im} \in F$ such that

$$b_i = c_{i1}y_1 + c_{i2}y_2 + \cdots + c_{im}y_m.$$

Thus,

$$a = \sum_{i=1}^{n} b_i x_i = \sum_{i=1}^{n} \left(\sum_{j=1}^{m} c_{ij} y_j \right) x_i = \sum_{i,j} c_{ij}(y_j x_i).$$

This proves that YX spans K over F.

Now suppose there are elements c_{ij} in F such that

$$0 = \sum_{i,j} c_{ij}(y_j x_i) = \sum_{i} \left(\sum_{j} (c_{ij} y_j) \right) x_i.$$

Then, since each $\sum_j c_{ij}y_j \in E$ and X is a basis for K over E, we have

$$\sum_j c_{ij} y_j = 0$$

for each i. But each $c_{ij} \in F$ and Y is a basis for E over F, so each $c_{ij} = 0$. This proves that the set YX is linearly independent over F. ■

Using the fact that for any field extension L of a field J, $[L{:}J] = n$ if and only if L is isomorphic to J^n as vector spaces (see Exercise 57), we may give a concise conceptual proof of Theorem 20.5, as follows. Let $[K{:}E] = n$ and $[E{:}F] = m$. Then $K \approx E^n$ and $E \approx F^m$, so that $K \approx E^n \approx (F^m)^n \approx F^{mn}$. Thus, $[K{:}F] = mn$.

The content of Theorem 20.5 can be pictured as in Figure 20.2. Examples 3, 4, and 5 show how Theorem 20.5 is often utilized.

EXAMPLE 3 Since $\{1, \sqrt{3}\}$ is a basis for $Q(\sqrt{3}, \sqrt{5})$ over $Q(\sqrt{5})$ (see Exercise 7) and $\{1, \sqrt{5}\}$ is a basis for $Q(\sqrt{5})$ over Q, the proof of Theorem 20.5 shows that $\{1, \sqrt{3}, \sqrt{5}, \sqrt{15}\}$ is a basis for $Q(\sqrt{3}, \sqrt{5})$ over Q. (See Figure 20.3.) ■

EXAMPLE 4 Consider $Q(\sqrt[3]{2}, \sqrt[4]{3})$. (See Figure 20.4) Then $[Q(\sqrt[3]{2}, \sqrt[4]{3}){:}Q] = 12$. For, clearly,

$$[Q(\sqrt[3]{2}, \sqrt[4]{3}){:}Q] = [Q(\sqrt[3]{2}, \sqrt[4]{3}){:}Q(\sqrt[3]{2})][Q(\sqrt[3]{2}){:}Q]$$

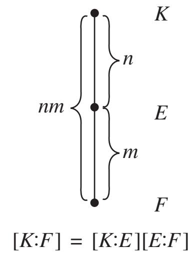


Figure 20.2 Extension of degree nm.

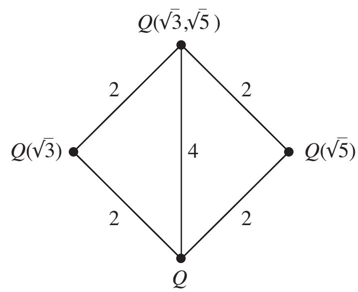


Figure 20.3 Extension of degree 4.

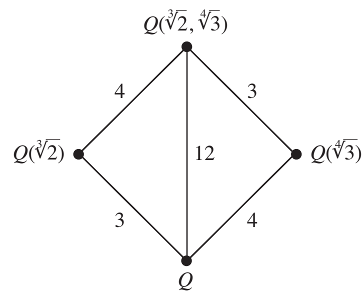


Figure 20.4 Extension of degree 12.

and

$$[Q(\sqrt[3]{2}, \sqrt[4]{3}):Q] = [Q(\sqrt[3]{2}, \sqrt[4]{3}):Q(\sqrt[4]{3})][Q(\sqrt[4]{3}):Q]$$

show that both $3 = [Q(\sqrt[3]{2}):Q]$ and $4 = [Q(\sqrt[4]{3}):Q]$ divide $[Q(\sqrt[3]{2}, \sqrt[4]{3}):Q]$. Thus, $[Q(\sqrt[3]{2}, \sqrt[4]{3}):Q] \geq 12$. On the other hand, $[Q(\sqrt[3]{2}, \sqrt[4]{3}):Q(\sqrt[3]{2})]$ is at most 4, since $\sqrt[4]{3}$ is a zero of $x^4 - 3 \in Q(\sqrt[3]{2})[x]$. Therefore,

$$[Q(\sqrt[3]{2}, \sqrt[4]{3}):Q] = [Q(\sqrt[3]{2}, \sqrt[4]{3}):Q(\sqrt[3]{2})][Q(\sqrt[3]{2}):Q] \leq 4 \cdot 3 = 12.$$

(See Figure 20.4.) ∎

Theorem 20.5 can sometimes be used to show that a field does not contain a particular element.

▮ **EXAMPLE 5** Recall from Example 7 in Chapter 17 that $h(x) = 15x^4 - 10x^2 + 9x + 21$ is irreducible over Q. Let β be a zero of $h(x)$ in some extension of Q. Then, even though we don't know what β is, we can still prove that $\sqrt[3]{2}$ is not an element of $Q(\beta)$. For, if so, then $Q \subset Q(\sqrt[3]{2}) \subseteq Q(\beta)$ and $4 = [Q(\beta):Q] = [Q(\beta):Q(\sqrt[3]{2})][Q(\sqrt[3]{2}):Q]$

implies that 3 divides 4. Notice that this argument cannot be used to show that $\sqrt{2}$ is not contained in $Q(\beta)$. ∎

EXAMPLE 6 Consider $Q(\sqrt{3}, \sqrt{5})$. We claim that $Q(\sqrt{3}, \sqrt{5}) = Q(\sqrt{3}+\sqrt{5})$. The inclusion $Q(\sqrt{3}+\sqrt{5}) \subseteq Q(\sqrt{3}, \sqrt{5})$ is clear. Now note that since

$$(\sqrt{3}+\sqrt{5})^{-1} = \frac{1}{\sqrt{3}+\sqrt{5}} \cdot \frac{\sqrt{3}-\sqrt{5}}{\sqrt{3}-\sqrt{5}} = -\frac{1}{2}(\sqrt{3}-\sqrt{5}),$$

we know that $\sqrt{3}-\sqrt{5}$ belongs to $Q(\sqrt{3}+\sqrt{5})$. It follows that $((\sqrt{3}+\sqrt{5})+(\sqrt{3}-\sqrt{5}))/2 = \sqrt{3}$ and $((\sqrt{3}-\sqrt{5})+(\sqrt{3}-\sqrt{5}))/2 = \sqrt{5}$ both belong to $Q(\sqrt{3}+\sqrt{5})$, and therefore $Q(\sqrt{3}, \sqrt{5}) \subseteq Q(\sqrt{3}+\sqrt{5})$. ∎

EXAMPLE 7 It follows from Example 6 and Theorem 19.3 that the minimal polynomial for $\sqrt{3}+\sqrt{5}$ over Q has degree 4. How can we find this polynomial? We begin with $x = \sqrt{3}+\sqrt{5}$. Then $x^2 = 3 + 2\sqrt{15} + 5$. From this we obtain $x^2 - 8 = 2\sqrt{15}$ and, by squaring both sides, $x^4 - 16x^2 + 64 = 60$. Thus, $\sqrt{3}+\sqrt{5}$ is a zero of $x^4 - 16x + 4$. We know that this is the minimal polynomial of $\sqrt{3}+\sqrt{5}$ over Q since it is monic and has degree 4. ∎

Example 6 shows that an extension obtained by adjoining two elements to a field can sometimes be obtained by adjoining a single element to the field. Our next theorem shows that, under certain conditions, this can always be done.

Theorem 20.6 Primitive Element Theorem (Steinitz, 1910)

If F is a field of characteristic 0, and a and b are algebraic over F, then there is an element c in $F(a, b)$ such that $F(a, b) = F(c)$.

PROOF Let $p(x)$ and $q(x)$ be the minimal polynomials over F for a and b, respectively. In some extension K of F, let $a_1, a_2, \ldots, a_m$ and $b_1, b_2, \ldots, b_n$ be the distinct zeros of $p(x)$ and $q(x)$, respectively, where $a = a_1$ and $b = b_1$. Among the infinitely many elements of F, choose an element d not equal to $(a_i - a)/(b - b_j)$ for all $i \geq 1$ and all $j > 1$. In particular, $a_i \neq a + d(b - b_j)$ for $j > 1$.

We shall show that $c = a + db$ has the property that $F(a, b) = F(c)$. Certainly, $F(c) \subseteq F(a, b)$. To verify that $F(a, b) \subseteq F(c)$, it suffices to prove that $b \in F(c)$, for then b, c, and d belong to $F(c)$ and $a = c - bd$. Consider the polynomials $q(x)$ and $r(x) = p(c - dx)$ [i.e., $r(x)$ is obtained by substituting $c - dx$ for x in $p(x)$] over $F(c)$. Since both $q(b) = 0$ and $r(b) = p(c - db) = p(a) = 0$, both $q(x)$ and $r(x)$ are divisible by the minimal polynomial $s(x)$ for b over $F(c)$ (see Theorem 20.3). Because $s(x) \in F(c)[x]$, we may complete the proof by proving that $s(x) = x - b$. Since $s(x)$ is a common divisor of $q(x)$ and $r(x)$, the only possible zeros of $s(x)$ in K are the zeros of $q(x)$ that are also zeros of $r(x)$. But $r(b_j) = p(c - db_j) = p(a + db - db_j) = p(a + d(b - b_j))$ and d was chosen such that $a + d(b - b_j) \neq a_i$ for $j > 1$. It follows that b is the only zero of $s(x)$ in $K[x]$ and, therefore, $s(x) = (x - b)^u$. Since $s(x)$ is irreducible and F has characteristic 0, Theorem 19.6 guarantees that $u = 1$. ∎

In the terminology introduced earlier, it follows from Theorem 20.6 and induction that any finite extension of a field of characteristic 0 is a simple extension. An element a with the property that $E = F(a)$ is called a *primitive element* of E.

Properties of Algebraic Extensions

Theorem 20.7 Algebraic over Algebraic Is Algebraic

If K is an algebraic extension of E and E is an algebraic extension of F, then K is an algebraic extension of F.

PROOF Let $a \in K$. It suffices to show that a belongs to some finite extension of F. Since a is algebraic over E, we know that a is the zero of some irreducible polynomial in $E[x]$, say, $p(x) = b_n x^n + \cdots + b_0$. Now we construct a tower of extension fields of F, as follows:

$$F_0 = F(b_0),$$
$$F_1 = F_0(b_1), \ldots, F_n = F_{n-1}(b_n).$$

In particular,

$$F_n = F(b_0, b_1, \ldots, b_n),$$

so that $p(x) \in F_n[x]$. Thus, $[F_n(a){:}F_n] = n$; and, because each b_i is algebraic over F, we know that each $[F_{i+1}{:}F_i]$ is finite. So,

$$[F_n(a){:}F] = [F_n(a){:}F_n][F_n{:}F_{n-1}] \cdots [F_1{:}F_0][F_0{:}F]$$

is finite. (See Figure 20.5.)

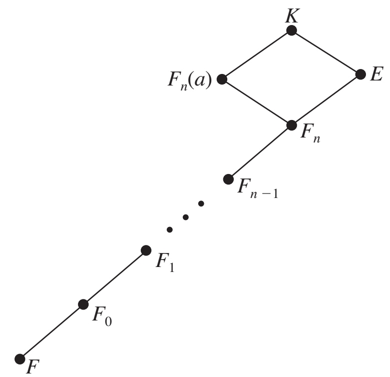


Figure 20.5 Tower of extensions fields.

Corollary Subfield of Algebraic Elements

Let E be an extension field of the field F. Then the set of all elements of E that are algebraic over F is a subfield of E.

PROOF Suppose that $a, b \in E$ are algebraic over F and $b \neq 0$. To show that $a + b$, $a - b$, ab, and a/b are algebraic over F, it suffices to show that $[F(a, b){:}F]$ is finite, since each of these four elements belongs to $F(a, b)$. But note that

$$[F(a, b){:}F] = [F(a, b){:}F(b)][F(b){:}F].$$

Also, since a is algebraic over F, it is certainly algebraic over $F(b)$. Thus, both $[F(a, b){:}F(b)]$ and $[F(b){:}F]$ are finite.

For any extension E of a field F, the subfield of E of the elements that are algebraic over F is called the *algebraic closure of F in E*.

One might wonder if there is such a thing as a maximal algebraic extension of a field F—that is, whether there is an algebraic extension E of F that has no proper algebraic extensions. For such an E to exist, it is necessary that every polynomial in $E[x]$ splits in E. Otherwise, it follows from Kronecker's Theorem that E would have a proper algebraic extension. This condition is also sufficient. If every member of $E[x]$ splits in E, and K is an algebraic extension of E, then every member of K is a zero of some element of $E[x]$. But the zeros of elements of $E[x]$ are in E. A field that has no proper algebraic extension is called *algebraically closed.* In 1910, Ernst Steinitz proved that every field F has a unique (up to isomorphism) algebraic extension that is algebraically closed. This field is called the *algebraic closure of F.* A proof of this result requires a sophisticated set theory background.

In 1799, Gauss, at the age of 22, proved that **C** is algebraically closed. This fact was considered so important at the time that it was called the Fundamental Theorem of Algebra. Over a 50-year period, Gauss found three additional proofs of the Fundamental Theorem. Today more than 100 proofs exist. In view of the ascendancy of abstract algebra in the 20th century, a more appropriate phrase for Gauss's result would be the Fundamental Theorem of Classical Algebra.

Exercises

It matters not what goal you seek
Its secret here reposes:
You've got to dig from week to week
To get Results or Roses.

Edgar Guest

1. Prove Theorem 20.2 and Theorem 20.3.

2. Let E be the algebraic closure of F. Show that every polynomial in $F[x]$ splits in E.

3. Prove that $Q(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}, \ldots)$ is an algebraic extension of Q but not a finite extension of Q. (This exercise is referred to in this chapter.)

4. Let E be an algebraic extension of F. If every polynomial in $F[x]$ splits in E, show that E is algebraically closed.

5. Suppose that F is a field and every irreducible polynomial in $F[x]$ is linear. Show that F is algebraically closed.

6. Suppose that $f(x)$ and $g(x)$ are irreducible over F and that $\deg f(x)$ and $\deg g(x)$ are relatively prime. If a is a zero of

$f(x)$ in some extension of F, show that $g(x)$ is irreducible over $F(a)$.

7. Let a and b belong to Q with $b \neq 0$. Show that $Q(\sqrt{a}) = Q(\sqrt{b})$ if and only if there exists some $c \in Q$ such that $a = bc^2$.

8. Find the degree and a basis for $Q(\sqrt{3}+\sqrt{5})$ over $Q(\sqrt{15})$. Find the degree and a basis for $Q(\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2})$ over Q.

9. If $[F(a):F] = 5$, find $[F(a^3):F]$. Does your argument apply equally well if a^3 is replaced with a^2 or a^4?

10. Suppose that E is an extension of F of prime degree. Show that, for every a in E, $F(a) = F$ or $F(a) = E$.

11. Let E and F be fields with $[E:F] = n$. If $f(x)$ is irreducible over F and the degree of $f(x)$ does not divide n, prove that $f(x)$ has no zero in E.

12. Let E be an extension field of a field F and $[E:F] = n$ where n is odd. If $a \in E$, prove that $F(a^2) = F(a)$.

13. Let F be a field, $f(x) \in F[x]$, and let a be a zero of $f(x)$ in some field extension of F. If b and c belong to F and $b \neq 0$, find a $g(x) \in F[x]$ that has $ab + c$ as a zero.

14. Let β be transcendental over F and n a positive integer. Determine $[F(\beta) : F(\beta^n)]$. For $n = 4$ write $3\beta^5 + a\beta^{-3}$ as a linear combination of basis elements with coefficients from $F(\beta^4)$.

15. If F is a field of characteristic 0 and $a_1, a_2, \ldots, a_n$ are algebraic over F, prove that there is an element c in $F(a_1, a_2, \ldots, a_n)$ such that $F(a_1, a_2, \ldots, a_n) = F(c)$.

16. Explain why $11\sqrt{12} - 7\sqrt{45}$ can be expressed in the form $a + b(\sqrt{3}+\sqrt{5}) +$
$c(\sqrt{3}+\sqrt{5})^2 + d(\sqrt{3}+\sqrt{5})^3$ where a, b, c and d are rational.

17. Determine $[Q(\sqrt[6]{2} : Q]$; $[Q(\sqrt[6]{2} : Q(\sqrt{2})]$; $[Q(\sqrt[6]{2} : Q(\sqrt[3]{2})]$.

18. Determine a basis for $Q(\sqrt[6]{2})$ over Q. Determine a basis for $Q(\sqrt[6]{2})$ over $Q(\sqrt{2})$. Determine a basis for $Q(\sqrt[6]{2})$ over $Q(\sqrt[3]{2})$.

19. How does the subfield lattice of $Q(\sqrt[3]{2}, \sqrt[4]{3})$ in Figure 20.4 compare with the subgroup lattice of Z_{12}?

20. If F is a field and $[F(a) : F] = n$, describe the elements in the subspace of $F(a)$ of dimension k where $1 \leq k \leq n$.

21. Let F be a field and $f(x)$ be the minimal polynomial in $F[x]$ for a. If b and c belong to F, find a minimal polynomial for $ab + c$ in $F[x]$.

22. Let $f(x) = x^3 - 2$. What is the deg $f(x)$ over Q, over $Q(\sqrt[3]{2})$, over $Q(\sqrt{3})$, over $Q(i)$?

23. Given that every element in $\mathbf{R}[x]$ splits in in $\mathbf{C}[x]$, prove that every element in $\mathbf{R}[x]$ that is irreducible over $\mathbf{R}$ is linear or quadratic.

24. Prove that the only subfield of $Q(\sqrt[3]{2}, \sqrt[4]{3})$ of degree 2 over Q is $Q(\sqrt{3})$.

25. Give an example of a field extension of Q that has exactly two proper subfields.

26. List five distinct proper subfields of $Q(\sqrt[3]{3}, \sqrt[3]{5})$.

27. Let E be a field extension of a field F. Prove that $[E : F] = 1$ if and only if $E = F$.

28. Let F, K and L be fields with $F \subseteq K \subseteq L$. If L is a finite extension of F and $[L : F] = [L : K]$, prove that $F = K$.

29. Without using the Primitive Element Theorem, prove that if $[K{:}F]$ is prime, then K has a primitive element.

30. Let a be a complex number that is algebraic over Q. Show that $\sqrt{a}$ is algebraic over Q. Does your argument show that $\sqrt[k]{a}$ is algebraic over Q for all positive integers $k \geq 2$? If so, why does this imply that $a^{m/n}$ is algebraic over Q for all positive integers m and n?

31. Let β be a zero of $f(x) = x^5 + 2x + 4$ (see Example 8 in Chapter 17). Show that none of $\sqrt{2}, \sqrt[3]{2}, \sqrt[4]{2}$ belongs to $Q(\beta)$.

32. Prove that $Q(\sqrt{2}, \sqrt[3]{2}) = Q(\sqrt[6]{2})$.

33. Let a and b be rational numbers. Show that $Q(\sqrt{a}, \sqrt{b}) = Q(\sqrt{a} + \sqrt{b})$.

34. Find the minimal polynomial for $\sqrt[3]{2} + \sqrt[3]{4}$ over Q.

35. Let K be an extension of F. Suppose that E_1 and E_2 are contained in K and are extensions of F. If $[E_1{:}F]$ and $[E_2{:}F]$ are both prime, show that $E_1 = E_2$ or $E_1 \cap E_2 = F$.

36. Find a polynomial $g(x)$ over $Q(\sqrt{3})$ of minimum degree such that $g(\sqrt[4]{3}) = 0$.

37. Suppose that a is algebraic over a field F. Show that a and $1 + a^{-1}$ have the same degree over F.

38. If ab is algebraic over F and $b \neq 0$, prove that a is algebraic over $F(b)$.

39. Let E be an algebraic extension of a field F. If R is a ring and $E \supseteq R \supseteq F$, show that R must be a field.

40. Prove that $\pi^2 - 1$ is algebraic over $Q(\pi^3)$.

41. If a is transcendental over F, show that every element of $F(a)$ that is not in F is transcendental over F.

42. Suppose that E is an extension of F and $a, b \in E$. If a is algebraic over F of degree m, and b is algebraic over F of degree n, where m and n are relatively prime, show that $[F(a, b){:}F] = mn$.

43. Let K be a field extension of F and let $a \in K$. Show that $[F(a){:}F(a^3)] \leq 3$. Find examples to illustrate that $[F(a){:}F(a^3)]$ can be 1, 2, or 3.

44. Find an example of a field F and elements a and b from some extension field such that $F(a, b) \neq F(a), F(a, b) \neq F(b)$, and $[F(a, b){:}F] < [F(a){:}F][F(b){:}F]$.

45. Let E be a finite extension of $\mathbf{R}$. Use the fact that $\mathbf{C}$ is algebraically closed to prove that $E = \mathbf{C}$ or $E = \mathbf{R}$.

46. Suppose that $[E{:}Q] = 2$. Show that there is an integer d such that $E = Q(\sqrt{d})$ where d is not divisible by the square of any prime.

47. Suppose that $p(x) \in F[x]$ and E is a finite extension of F. If $p(x)$ is irreducible over F, and deg $p(x)$ and $[E{:}F]$ are relatively prime, show that $p(x)$ is irreducible over E.

48. Let E be an extension field of F. Show that $[E{:}F]$ is finite if and only if $E = F(a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n$ are algebraic over F.

49. If α and β are real numbers and α and β are transcendental over Q, show that either $\alpha\beta$ or $\alpha + \beta$ is also transcendental over Q.

50. Let $f(x)$ be a nonconstant element of $F[x]$. If a belongs to some extension of F and $f(a)$ is algebraic over F, prove that a is algebraic over F.

51. Let $f(x) = ax^2 + bx + c \in Q[x]$. Find a primitive element for the splitting field for $f(x)$ over Q.

52. Let $f(x)$ and $g(x)$ be irreducible polynomials over a field F and let a and b belong to some extension E of F. If a is a zero of $f(x)$ and b is a zero of $g(x)$, show that $f(x)$ is irreducible over $F(b)$ if and only if $g(x)$ is irreducible over $F(a)$.

53. Let a be a complex zero of $x^3 - 1$. Prove that $Q(\sqrt{a}) = Q(a)$.

54. Let $f(x) \in F[x]$. If deg $f(x) = 2$ and a is a zero of $f(x)$ in some extension of F, prove that $F(a)$ is the splitting field for $f(x)$ over F.

55. If F is a field and the multiplicative group of nonzero elements of F is cyclic, prove that F is finite.

56. Prove that there are no positive rational numbers r, s and t such that $r^s = \pi^t$.

57. Prove that, if K is an extension field of F, then $[K{:}F] = n$ if and only if K is isomorphic to F^n as vector spaces. (This exercise is referred to in this chapter.)

58. Let a be a positive real number not in Q and let n be an integer greater than 1. Prove or disprove that $[Q(a^{1/n}){:}Q] = n$.

59. Let a and b belong to some finite extension field of F. Prove that $[F(a,b){:}F(a)] \leq [F(a,b){:}F]$.

60. Let a be a zero of an irreducible polynomial $c_{d-1}x^{d-1} + c_{d-2}x^{d-2} + \cdots + c_1x + c_0$ over a field F. Write a^{-1} as linear combination of the basis elements a, a, $a^2, \ldots,$ a^{d-1} with coefficients in F.

61. Let F be a field and K a splitting field for some nonconstant polynomial over F. Show that K is a finite extension of F.

62. Prove that **C** is not the splitting field of any polynomial in $Q[x]$.

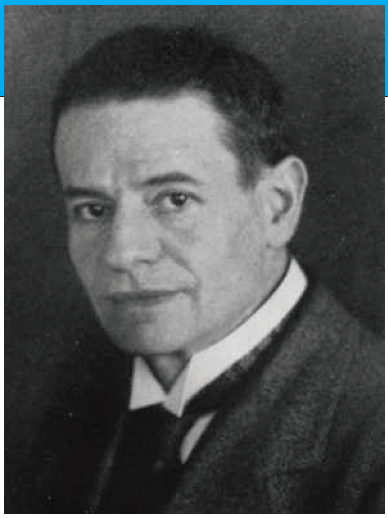
63. Prove that $\sqrt{2}$ is not an element of $Q(\pi)$. Generalize.

64. Let $\alpha = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7}$ and $\beta = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}$. Prove that β is not in $Q(\alpha)$.

65. Let m be a positive integer. If a is transcendental over a field F, prove that a^m is transcendental over F.

66. Suppose K is an extension of F of degree n. Prove that K can be written in the form $F(x_1, x_2, \ldots, x_n)$ for some $x_2, x_2, \ldots, x_n$ in K.

Ernst Steinitz



George M. Bergman, Archives of the Mathematisches

One cannot overestimate the importance of this paper [by Steinitz]. The appearance of this paper marks a turning point in the history of algebra of the 20th century.

BARTEL VAN DER WAERDEN

Ernst Steinitz was born in Laurahtte, Silesia, Germany (now Poland) on June 13, 1871. He received a Ph.D. at the University of Breslau in 1894. Steinitz's seminal work was a 1910 paper in which he was the first to give an abstract definition of the concept of a "field." Among the concepts he introduced there are: prime field, perfect fields, degree of an extension, and algebraic closure. In his classic textbook on Modern Algebra van der Waerden wrote: "Almost all the notions and facts about fields which we teach our students in such a course, are contained in Steinitz's paper." Hulmut Hasse wrote in his textbook on Higher Algebra:

"Every algebraist should have read at least once this basic original paper on field theory." In their book on the history of mathematics Bourbaki describe Steinitz's paper on fields as "a fundamental work which may be considered as the origin of today's concept of algebra." In addtion to field theory Steinitz made important contributions to theory of polyhedra, module theory, linear algebra, algebraic geometry and graph theory. Two of his famous theorems are the Steinitz replacement theorem for vector spaces and the Primitive Element Theorem. He died on September 29, 1928 in Kiel, Germany.

21 Finite Fields

Even though these numerical systems [finite fields] look very different from the numerical systems we are used to, such as the rational numbers, they have the same salient properties.

Edward Frenkel, *Love and Math*

This theory [of finite fields] is of considerable interest in its own right and it provides a particularly beautiful example of how the general theory of the preceding chapters fits together to provide a rather detailed description of all finite fields.

Richard A. Dean, *Elements of Abstract Algebra*

Classification of Finite Fields

In this, our final chapter on field theory, we take up one of the most beautiful and important areas of abstract algebra—finite fields. Finite fields were first introduced by Galois in 1830 in his proof of the unsolvability of the general quintic equation. When Cayley invented matrices a few decades later, it was natural to investigate groups of matrices over finite fields. To this day, matrix groups over finite fields are among the most important classes of groups. In the past 70 years, there have been important applications of finite fields in computer science, coding theory, information theory, and cryptography. Recently finite fields of orders 4 and 16 have been used in analyzing strings of DNA. But, besides the many uses of finite fields in pure and applied mathematics, there is yet another good reason for studying them. They are just plain fun!

The most striking fact about finite fields is the restricted nature of their order and structure. The proofs of our results will utilize properties of finite Abelian groups, integral domains, vector spaces, and finite extensions of fields.

Our first theorem characterizes the orders of finite fields.

Theorem 21.1 Classification of Finite Fields

For each prime p and each positive integer n, there is, up to isomorphism, a unique finite field of order p^n.

PROOF Consider the splitting field E of $f(x) = x^{p^n} - x$ over Z_p. We will show that $|E| = p^n$. Since $f(x)$ splits in E, we know that $f(x)$ has exactly p^n zeros in E, counting multiplicity. Moreover, by Theorem 19.5, every zero of $f(x)$ has multiplicity 1. Thus, $f(x)$ has p^n distinct zeros in E. On the other hand, the set of zeros of $f(x)$ in E is closed under addition, subtraction, multiplication, and division by nonzero elements (see Exercise 39), so that the set of zeros of $f(x)$ is itself an extension field of Z_p in which $f(x)$ splits. Thus, the set of zeros of $f(x)$ is E and, therefore, $|E| = p^n$.

To show that there is a unique field for each prime-power, suppose that K is any field of order p^n. Then K has a subfield isomorphic to Z_p (generated by 1), and, because the nonzero elements of K form a multiplicative group of order $p^n - 1$, every element of K is a zero of $f(x) = x^{p^n} - x$ (see Exercise 39). So, K must be a splitting field for $f(x)$ over Z_p. By the corollary to Theorem 19.4, there is only one such field up to isomorphism. ∎

The existence portion of Theorem 21.1 appeared in the works of Galois and Gauss in the first third of the 19th century. Rigorous proofs were given by Dedekind in 1857 and by Jordan in 1870 in his classic book on group theory. The uniqueness portion of the theorem was proved by E. H. Moore in an 1893 paper concerning finite groups. The mathematics historian E. T. Bell once said that this paper by Moore marked the beginning of abstract algebra in America.

Because there is only one field for each prime-power p^n, we may unambiguously denote it by $\mathrm{GF}(p^n)$, in honor of Galois, and call it the *Galois field of order* p^n. When discussing fields we sometimes use $\mathrm{GF}(p)$ interchangeably with Z_p.

Structure of Finite Fields

The next theorem tells us the additive and multiplicative group structure of a field of order p^n.

Theorem 21.2 Structure of Finite Fields

As a group under addition, $GF(p^n)$ is isomorphic to

$$\underbrace{Z_p \oplus Z_p \oplus \ldots \oplus Z_p}_{n\ factors}.$$

As a group under multiplication, the set of nonzero elements of $GF(p^n)$ is isomorphic to Z_{p^n-1} (and is, therefore, cyclic).

PROOF Since $\mathrm{GF}(p^n)$ has characteristic p (Theorem 13.3), every nonzero element of $\mathrm{GF}(p^n)$ has additive order p. Then by the Fundamental Theorem of Finite Abelian Groups, $\mathrm{GF}(p^n)$ under addition is isomorphic to a direct product of n copies of Z_p.

To see that the multiplicative group $\mathrm{GF}(p^n)^*$of nonzero elements of $\mathrm{GF}(p^n)$ is cyclic, we first note that by the Fundamental Theorem of Finite Abelian Groups (Theorem 11.1), $\mathrm{GF}(p^n)^*$ is isomorphic to a direct product of the form $Z_{n_1} \oplus Z_{n_2} \oplus \cdots \oplus Z_{n_m}$. If the orders of these components are pairwise relatively prime, then it follows from Corollary 1 of Theorem 8.2 that $\mathrm{GF}(p^n)^*$ is cyclic. Hence we may assume that there is an integer $d > 1$ that divides the orders of two of the components. From the Fundamental Theorem of Cyclic Groups (Theorem 4.3) we know that each of these components has a subgroup of order d. This means that $\mathrm{GF}(p^n)^*$ has two distinct subgroups of order d, call them H and K. But then every element of H and K is a zero of $x^d - 1$, which contradicts the fact that a polynomial of degree d over a field can have at most d zeros (Theorem 16.3). ∎

Some students misinterpret Theorem 21.2 to mean that $Z_p \oplus Z_p \oplus \cdots \oplus Z_p$ is a field of order p^n. Since any element of $Z_p \oplus Z_p \oplus \cdots \oplus Z_p$ that has at least one coordinate equal to 0 cannot have an inverse, it is not a field.

From the fact that $Z_p \oplus Z_p \oplus \cdots \oplus Z_p$ is a vector space over Z_p with $\{(1, 0, \ldots, 0), (0, 1, 0, \ldots, 0), \ldots, (0, 0, \ldots, 1)\}$ as a basis, we have the following useful and aesthetically appealing formula.

Corollary 1

$$[\mathbf{GF}(p^n){:}\mathbf{GF}(p)] = n$$

Corollary 2 GF(p^n) Contains an Element of Degree n

Let a be a generator of the group of nonzero elements of GF(p^n) under multiplication. Then a is algebraic over GF(p) of degree n.

PROOF Observe that $[\mathrm{GF}(p)(a) : \mathrm{GF}(p)] = [\mathrm{GF}(p^n) : \mathrm{GF}(p)] = n$. ∎

EXAMPLE 8 Let's examine the field GF(16) in detail. Since $x^4 + x + 1$ is irreducible over Z_2, we know by Theorem 21.1 that for any zero a of $x^4 + x + 1$ in GF(16) we have

$$\mathrm{GF}(16) \approx \{c_3a^3 + c_2a^2 + c_1a + c_0 \mid c_0, c_1, c_2, c_3, c_4 \in Z_2\}$$

where addition is done as in $Z_2(a)$. To determine which of the 16 elements in the field is, say, the product $(a^3+a^2+a+1)(a^3+a)$ we expand it in the usual way and note that the condition $a^4+a+1 = 0$ implies that

$$\begin{aligned} a^4 &= a + 1, \\ a^5 &= a^2 + a, \\ a^6 &= a^3 + a^2. \end{aligned}$$

Thus,

$$(a^3 + a^2 + a + 1)(a^3 + a) = a^6 + a^5 + a^4 + a^3 + a^4 + a^3 + a^2 + a$$

$$= a^6 + a^5 + a^2 + a = (a^3 + a^2) + (a^2 + a) + a^2 + a = a^3 + a^2.$$

A way to simplify the multiplication process is to make use of the fact that the nonzero elements of GF(16) form a cyclic group of order 15. To take advantage of this, we must first find a generator of this group. Since any element GF(16)* must have a multiplicative order that divides 15, all we need to do is find an element in GF(16)* whose order is not 1, 3, or 5. Our calculations have shown that a has these properties. So, we may think of GF(16) as the set $\{0, 1, a, a^2, \ldots, a^{14}\}$, where $a^{15} = 1$. This makes multiplication in GF(16) trivial, but, unfortunately, it makes addition more difficult. For example, $a^{10} \cdot a^7 = a^{17} = a^2$, but what is $a^{10} + a^7$? So, we face a dilemma. If we write the elements of GF(16) in the additive form $c_3a^3 + c_2a^2 + c_3a + c_4$, then addition is easy and multiplication is hard. On the other hand, if we write the elements of GF(16)* in the multiplicative form a^i, then multiplication is easy and addition is hard. Can we have the best of both? Yes, we can. All we need to do is use the relation $a^4 = a + 1$ to make a two-way conversion table, as in Table 21.1. So, we see from Table 21.1 that

$$a^{10} + a^7 = (a^2 + a + 1) + (a^3 + a + 1) = a^3 + a^2 = a^6$$

and

$$(a^3 + a^2 + 1)(a^3 + a^2 + a + 1) = a^{13} \cdot a^{12} = a^{25} = a^{10} = a^2 + a + 1.$$

∎

Don't be misled by the preceding example into believing that the element x is always a generator for the cyclic multiplicative group of nonzero elements. It is not. (See Exercise 33.) Although any two irreducible polynomials of the same degree over $Z_p[x]$ yield isomorphic fields, some are better than others for computational purposes.

EXAMPLE 9 Consider $f(x) = x^3 + x^2 + 1$ over Z_2. We will show how to write $f(x)$ as the product of linear factors. Let $F = Z_2[x]/\langle f(x) \rangle$ and let a be a zero of $f(x)$ in F. Then $|F| = 8$ and $|F^*| = 7$. So, by Corollary 2 to Theorem 7.1, we know that $|a| = 7$. Thus, by Theorem 19.3,

$$\begin{aligned} F &= \{0, 1, a, a^2, a^3, a^4, a^5, a^6\} \\ &= \{0, 1, a, a+1, a^2, a^2+a+1, a^2+1, a^2+a\}. \end{aligned}$$

We know that a is one zero of $f(x)$, and we can test the other elements of F to see if they are zeros. We can simplify the calculations by using the fact that $a^3 + a^2 + 1 = 0$ to make a conversion table for the two forms of writing the elements of F. Because char

Table 21.1 Conversion Table for Addition and Multiplication in GF(16).

Multiplicative Form to Additive Form		Additive Form to Multiplicative Form	
1	1	1	1
a	a	a	a
a^2	a^2	$a+1$	a^4
a^3	a^3	a^2	a^2
a^4	$a+1$	a^2+a	a^5
a^5	a^2+a	a^2+1	a^8
a^6	a^3+a^2	a^2+a+1	a^{10}
a^7	a^3+a+1	a^3	a^3
a^8	a^2+1	a^3+a^2	a^6
a^9	a^3+a	a^3+a	a^9
a^{10}	a^2+a+1	a^3+1	a^{14}
a^{11}	a^3+a^2+a	a^3+a^2+a	a^{11}
a^{12}	a^3+a^2+a+1	a^3+a^2+1	a^{13}
a^{13}	a^3+a^2+1	a^3+a+1	a^7
a^{14}	a^3+1	a^3+a^2+a+1	a^{12}

$F = 2$, we know that $a^3 = a^2 + 1$. Then,

$$\begin{aligned}
a^4 &= a^3 + a = (a^2+1) + a = a^2 + a + 1,\\
a^5 &= a^3 + a^2 + a = (a^2+1) + a^2 + a = a + 1,\\
a^6 &= a^2 + a,\\
a^7 &= 1.
\end{aligned}$$

Now let's see whether a^2 is a zero of $f(x)$.

$$\begin{aligned}
f(a^2) &= (a^2)^3 + (a^2)^2 + 1 = a^6 + a^4 + 1\\
&= (a^2+a) + (a^2+a+1) + 1 = 0.
\end{aligned}$$

So, yes, it is. Next we try a^3.

$$\begin{aligned}
f(a^3) &= (a^3)^3 + (a^3)^2 + 1 = a^9 + a^6 + 1\\
&= a^2 + (a^2+a) + 1 = a + 1 \neq 0.
\end{aligned}$$

Now a^4.

$$\begin{aligned}
f(a^4) &= (a^4)^3 + (a^4)^2 + 1 = a^{12} + a^8 + 1\\
&= a^5 + a + 1 = (a+1) + a + 1 = 0.
\end{aligned}$$

So, a^4 is our remaining zero. Thus, $f(x) = (x-a)(x-a^2)(x-a^4) = (x+a)(x+a^2)(x+a^4)$, since char $F = 2$.
We may check this factorization by expanding the product and using a conversion table to obtain $f(x) = x^3 + x^2 + 1$. ∎

Instead of testing for zeros one by one as we did in Example 2, our next theorem provides a simple method for finding all of the zeros of an irreducible polynomial over Z_p given a single zero.

Theorem 21.3 Zeros of an Irreducible over Z_p

Let $f(x) \in Z_p[x]$ be an irreducible polynomial over Z_p of degree d and let a be a zero of $f(x)$ in some extension field E of Z_p. Then $a, a^p, a^{p^2}, \ldots, a^{p^{d-1}}$ are the zeros of $f(x)$ and they are distinct.

PROOF The case that $a \in Z_p$ means that $d = 1$ and a is the only zero of $p(x)$. So we assume that $d \neq 1$. Observe that because $|Z_p^*| = p - 1$ for any element $c \in Z_p^*$ we have $c^{p-1} = 1$. Thus for all $c \in Z_p$ we have $c^p = c$. From this it follows that $c^{p^i} = c$ for all $c \in Z_p$ and all positive integers i (see Exercise 49 in Chapter 13). Let $f(x) = c_d x^d + c_{d-1}x^{d-1} + \cdots + c_1 x + c_0 \in Z_p[x]$. Then $f(a) = c_d a^d + c_{d-1}a^{d-1} + \cdots + c_1 a + c_0 = 0$. Applying the automorphism in Exercise 15 of GF(p^n) given by $\phi(x) = x^{p^i}$ we have $c_d^{p^i}(a^{p^i})^d + c_{n-1}^{p^i}(a^{p^i})^{d-1} + \cdots + c_1^{p^i}a^{p^i} + c_0^{p^i} = c_d(a^{p^i})^d + c_{n-1}(a^{p^i})^{d-1} + \cdots + c_1 a^{p^i} + c_0 = 0$.

To see that the zeros $a, a^p, a^{p^2}, \ldots, a^{p^{d-1}}$ are distinct suppose that $a^{p^m} = a^{p^n}$ for some $0 \le n < m < d$ and let $m - n = j$. Then $0 = a^{p^m} - a^{p^n} = a^{p^{n+j}} - a^{p^n} = (a^{p^n})^{p^j} - a^{p^n} = (a^{p^j} - a)^{p^n}$. Thus we have $a^{p^j} - a = 0$. Let $K = \{x \in Z_p(a) \mid x^{p^j} - x = 0\}$. Then K is a subfield of $Z_p(a)$ order at most p^j (here we use that $(Z_p(a))^*$ is cyclic) that contains both Z_p and a. Noting that $|Z_p(a)| = p^d$ we now have that $p^j \ge |K| \ge p^d$, which is impossible. ∎

EXAMPLE 10 Let a be a zero of $f(x) = x^3 + 2x + 1$ in GF(27). From Theorem 21.3 we know that the three zeros of $f(x)$ in GF(27) are a, a^3 and $a^{3^2} = a^9$. So, $f(x)$ factors over Z_3 as $(x-a)(x-a^3)(x-a^9)$. ∎

The contrast in ease in factoring the polynomial in Example 2 versus the one in Example 3 beautifully illustrates the power of Theorem 21.3.

As an immediate corollary of Theorem 21.3 we have the following useful result.

Corollary Splitting Field of an Irreducible Polynomial Over Z_p

If $f(x)$ is an irreducible polynomial over Z_p and a is a zero of $f(x)$ in some extension field of Z_p, then $Z_p(a)$ is the splitting field of $f(x)$ over Z_p.

Subfields of a Finite Field

Theorem 21.1 gives us a complete description of all finite fields. The following theorem gives us a complete description of all the subfields of a finite field. Notice the close analogy between this theorem and Theorem 4.3, which describes all the subgroups of a finite cyclic group.

Theorem 21.4 Subfields of a Finite Field

For each divisor m of n, $\mathrm{GF}(p^n)$ has a unique subfield of order p^m. Moreover, these are the only subfields of $\mathrm{GF}(p^n)$.

PROOF To show the existence portion of the theorem, suppose that m divides n. Then, since

$$p^n - 1 = (p^m - 1)(p^{n-m} + p^{n-2m} + \cdots + p^m + 1),$$

we see that $p^m - 1$ divides $p^n - 1$. For simplicity, write $p^n - 1 = (p^m - 1)t$. Let $K = \{x \in \mathrm{GF}(p^n) | x^{p^m} = x\}$. We leave it as an easy exercise for the reader to show that K is a subfield of $\mathrm{GF}(p^n)$ (Exercise 37). Since the polynomial $x^{p^m} - x$ has at most p^m zeros in $\mathrm{GF}(p^n)$, we have $|K| \leq p^m$. Let $\langle a \rangle = \mathrm{GF}(p^n)^*$. Then $|a^t| = p^m - 1$, and since $(a^t)^{p^m-1} = 1$, it follows that $a^t \in K$. So, K is a subfield of $\mathrm{GF}(p^n)$ of order p^m.

The uniqueness portion of the theorem follows from the observation that if $\mathrm{GF}(p^n)$ had two distinct subfields of order p^m, then the polynomial $x^{p^m} - x$ would have more than p^m zeros in $\mathrm{GF}(p^n)$. This contradicts Theorem 16.3.

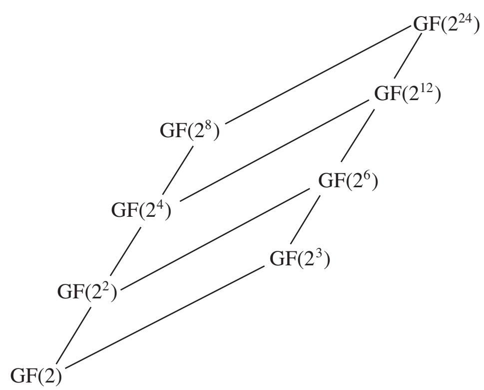
Finally, suppose that F is a subfield of $GF(p^n)$. Then F is isomorphic to $GF(p^m)$ for some m and, by Theorem 20.5,

$$\begin{aligned} n &= [\mathrm{GF}(p^n){:}\mathrm{GF}(p)] \\ &= [\mathrm{GF}(p^n){:}\mathrm{GF}(p^m)][\mathrm{GF}(p^m){:}\mathrm{GF}(p)] \\ &= [\mathrm{GF}(p^n){:}\mathrm{GF}(p^m)]m. \end{aligned}$$

Thus, m divides n. ■

As illustrated in the next three examples, Theorems 21.2 and 21.4, together with Theorem 4.3, make the task of finding the subfields of a finite field a simple exercise in arithmetic.

▮ EXAMPLE 11 The subfield lattice of $\mathrm{GF}(2^{24})$ is the following.



■

▮ EXAMPLE 12 Let F be the field of order 16 given in Example 1. Then there are exactly three subfields of F, and their orders are 2, 4, and 16. Obviously, the subfield of order 2 is $\{0, 1\}$ and the subfield of order 16 is F itself. To find the subfield of order 4, we merely observe that the three nonzero elements of this subfield must be the cyclic subgroup of $F^* = \langle a \rangle$ of order 3. So the subfield of order 4 is

$$\{0, 1, a^5, a^{10}\} = \{0, 1, a^2 + a, a^2 + a + 1\}.$$

■

EXAMPLE 13 If F is a field of order $3^6 = 729$ and a is a generator of F^*, then the subfields of F are

1. $\mathrm{GF}(3) = \{0\} \cup \langle a^{364} \rangle = \{0, 1, 2\}$,
2. $\mathrm{GF}(9) = \{0\} \cup \langle a^{91} \rangle$,
3. $\mathrm{GF}(27) = \{0\} \cup \langle a^{28} \rangle$,
4. $\mathrm{GF}(729) = \{0\} \cup \langle a \rangle$.

Theorem 21.4 provides us with the following a useful fact for determining information about the factorization of $x^{p^n} - x$ as a product of irreducible polynomials over Z_p.

Theorem 21.5 Degrees of Irreducible Factors of $x^{p^n} - x$ over Z_p

The degree of an irreducible factor of $x^{p^n} - x$ over Z_p divides n.

PROOF If $g(x)$ is an irreducible factor of $x^{p^n} - x$ over Z_p with degree d and $a \in \mathrm{GF}(p^n)$ is a zero of $g(x)$, then $|Z_p(a)| = p^d$. So, d is a divisor of n.

EXAMPLE 14 Let's look at the irreducible factorization of $x^{16} - x$ used to construct GF(16). Obviously, x and $x - 1$ are factors. By the corollary of Theorem 21.3 the other irreducible factors of $x^{16} - x$ have degrees 2 or 4. Fortunately, it is possible to determine these factors without resorting to long division. Since $x^{16} - x$ has no multiple zeros in GF(16), it has no irreducible factors with multiplicity greater than 1. Because the sum of the degrees of the nonlinear irreducible factors of $x^{16} - x$ must be 14, and 14 is not a multiple of 4, not the irreducible factors can be quartics. Noting that the only irreducible quadratic over Z_2 is $x^2 + x + 1$, we know the irreducible factorization consist of it and three quartics. Finally, we observe that any quartic irreducible factor must begin with x^4, end with 1, and have an odd number of terms in between (otherwise, 1 would be a zero). So, the remaining irreducibles factors must be $x^4 + x + 1, x^4 + x^2 + 1$, and $x^4 + x^3 + x^2 + x + 1$ because these are only possibilities that meet the two stated necessary conditions.

Exercises

No pressure, no diamonds.

Mary Case

1. Find [GF(729):GF(9)] and [GF(64):GF(8)].
2. If m divides n, show that $[\mathrm{GF}(p^n):\mathrm{GF}(p^m)] = n/m$.
3. Draw the lattice of subfields of GF(64).
4. Let a be a zero of $x^3 + x^2 + 1$ in some extension field of Z_2. Find the multiplicative inverse of $a + 1$ in $Z_2(\alpha)$. Solve the equation $(a + 1)x + a = a^2 + 1$ for an x of the form $c_2a^2 + c_1a + c_0$ where c_2, c_1 and c_0 are in Z_2.
5. What is the smallest field that has exactly three proper subfields?
6. Prove that every non-identity element in $\mathrm{GF}(32)^*$ is a generator of $\mathrm{GF}(32)^*$.
7. Let a be a zero of $f(x) = x^2 + 2x + 2$ in some extension field of Z_3. Find the other zero of $f(x)$ in $Z_3(a)$.
8. Let a be a zero of $f(x) = x^3 + x + 1$ in some extension field of Z_2. Find the other zeros of $f(x)$ in $Z_2(a)$.
9. For every positive integer i prove that the mapping from $\mathrm{GF}(p^n)$ to itself given by $\phi(x) = x^{p^i}$ is a field automorphism.
10. Let S be a finite subset of a field with at least one nonzero element. If S is closed under addition and multiplication prove that S is a field.
11. Show that 1 is the only element in $\mathrm{GF}(2^n)$ that is a zero of $x^2 + 1$.
12. How many elements of the cyclic group GF(81)* are generators?
13. Let $f(x)$ be a cubic irreducible over Z_2. Prove that the splitting field of $f(x)$ over Z_2 has order 8.
14. Prove that the rings $Z_3[x]/\langle x^2+x+2\rangle$ and $Z_3[x]/\langle x^2+2x+2\rangle$ are isomorphic. Explain why $Z_3[x]/\langle x^2 + x + 2\rangle$ is not ring-isomorphic to $Z_3[x]/\langle x^2 + 2x + 1\rangle$.
15. How many elements a in GF(64) have the property that $Z_2(a) =$GF(64)?
16. Given that a in GF(16) is a zero of the irreducible polynomial of degree 4 over Z_2, find the other three zeros. Given that a^7 in GF(16) is a zero of the irreducible polynomial of degree 4 over Z_2, find the other three zeros.

17. If p is a prime and a finite field F has subfields of order p^4 and p^6, what are the orders of other subfields that F must have?

18. If $g(x)$ is an irreducible factor of $x^8 - x$ over Z_2 and $\deg g(x) = m$, what are the possibilities for m?

19. Let a be an element in GF(8) other than 0 or 1. Prove that $Z_2(a) =$GF(8).

20. Let a be an element in GF(32) other than 0 or 1. Prove that a is the zero of a 5th degree irreducible polynomial over Z_2.

21. Let a be an element of GF(2^n). If $a^{16} = a^4$, prove that $a^4 = a$.

22. Let p be a prime and a be a zero of an irreducible polynomial $g(x)$ over Z_p of degree n. If b belongs to $Z_p(a)$ and b is a zero of $g(x)$ prove that $Z_p(b) = Z_p(a)$.

23. What are the degrees of the irreducible factors on $x^{27} - x$ over Z_3 and how many factors are of each degree?

24. Prove that in a finite field F of characteristic 2 every element can be written as a square of an element of F.

25. What is the smallest field that contains GF(p^n) as a proper subfield?

26. For any prime p prove that $x^{p^n} - x + 1$ has no zero in GF(p^n).

27. Let $F_1, F_2, F_3, \ldots$ be infinitely many fields with the property that $F_1 \subset F_2 \subset F_3 \subset \cdots$. Show that $F = \cup_{i=1}^{\infty} F_i$ is a field.

28. Determine the possible finite fields whose largest proper subfield is GF(2^5).

29. When $x^9 - x$ is expressed as a product of irreducibles over Z_3, what are the degrees of the factors and how many of each degree are there?

30. Find the smallest field that has exactly 6 subfields.

31. Show that x is a generator of the cyclic group $(Z_3[x]/\langle x^3 + 2x + 1\rangle)^*$.

32. Let GF$(5^n) = Z_5(a)$, where n is a positive integer. Find the smallest positive integer k such that $a^k = 2$.

33. Show that x is not a generator of the cyclic group $(Z_3[x]/\langle x^3 + 2x + 2\rangle)^*$. Find one such generator.

34. If $f(x)$ is a cubic irreducible polynomial over Z_3, prove that either x or $2x$ is a generator for the cyclic group $(Z_3[x]/\langle f(x)\rangle)^*$.

35. Prove the uniqueness portion of Theorem 21.3 using a group theoretic argument.

36. Suppose that a and b belong to GF(81)*, with $|a| = 5$ and $|b| = 16$. Show that ab is a generator of GF(81)*.

37. Show that the set K in the proof of Theorem 21.4 is a subfield. This exercise is referred to in the proof of Theorem 21.4.

38. Show that any finite subgroup of the multiplicative group of a field is cyclic.

39. Use a purely group theoretic argument to show that if F is a field of order p^n, then every element of F^* is a zero of $x^{p^n} - x$. (This exercise is referred to in the proof of Theorem 21.1.)

40. Draw the subfield lattices of GF(3^{18}) and of GF(2^{30}).

41. How does the subfield lattice of GF(2^{30}) compare with the subfield lattice of GF(3^{30})?

42. If $p(x)$ is a polynomial in $Z_p[x]$ with no multiple zeros, show that $p(x)$ divides $x^{p^n} - x$ for some n.

43. Suppose that p is a prime and $p \neq 2$. Let a be a nonsquare in Z_p—that is, a does not have the form b^2 for any b in Z_p. Show that a is a nonsquare in GF(p^n) if n is odd and that a is a square in GF(p^n) if n is even.

44. Let $f(x)$ be a cubic irreducible over Z_p, where p is a prime. Prove that the splitting field of $f(x)$ over Z_p has order p^3.

45. Suppose that F is a field of order 1024 and $F^* = \langle a\rangle$. List the elements of each subfield of F.

46. Show that no finite field is algebraically closed.

47. Suppose that F is a field of order 125 and $F^* = \langle a\rangle$. Show that $a^{62} = -1$.

48. Suppose that L and K are subfields of GF(p^n). If L has p^s elements and K has p^t elements, how many elements does $L \cap K$ have?

49. Show that a finite extension of a finite field is a simple extension.

50. Let F be a finite field of order q and let a be a nonzero element in F. If n divides $q-1$, prove that the equation $x^n = a$ has either no solutions in F or n distinct solutions in F.

51. Let a be a primitive element for the field $\mathrm{GF}(p^n)$, where p is an odd prime and n is a positive integer. Find the smallest positive integer k such that $a^k = p-1$.

52. In the field $\mathrm{GF}(p^n)$, show that for every positive divisor d of n, $x^{p^n} - x$ has an irreducible factor over $\mathrm{GF}(p)$ of degree d.

53. Prove that for every prime $p > 2$ a finite field F of order p^n has exactly $(p^n-1)/2$ elements that can be written as a square of an element of F.

54. Let a be a primitive element for the field $\mathrm{GF}(5^n)$, where n is a positive integer. Find the smallest positive integer k such that $a^k = 2$.

55. Let p be a prime such that $p \bmod 4 = 1$. How many elements of order 4 are in $\mathrm{GF}(p^n)^*$?

56. Let p be a prime such that $p \bmod 4 = 3$. How many elements of order 4 are in $\mathrm{GF}(p^n)^*$?

57. Let a be a zero of an irreducible quadratic polynomial over Z_5. Prove that there are elements b and c in $Z_5(a)$ such that $(3a+2)(ba+c) = 4a+1$.

58. If a is a zero of the irreducible polynomial x^4+x+1 over Z_2 prove that a is not a generator of $(Z_2[x]/\langle x^4+x+1\rangle)^*$.

59. If $g(x)$ is an irreducible polynomial over $\mathrm{GF}(4)$ of degree 5, what can you say about $\mathrm{GF}(4)[x]/\langle g(x)\rangle$?

60. If $f(x)$ is an irreducible non-linear irreducible polynomial over $\mathrm{GF}(8)$ that has a zero in $\mathrm{GF}(64)$, what is the degree of $f(x)$? Give an example of such a polynomial.

61. Let a be an element in $\mathrm{GF}(5^2)$ that is not in Z_5. Prove that $\mathrm{GF}(5^2)^* = \{1, 1+a, 1+a+a^2, 1+a+a^2+a^3, \ldots, 1+a+a^2+a^3+\cdots+a^{23}\}$. Does your proof generalize to $\mathrm{GF}(p^n)$ $(n \geq 2)$?

62. Let E be a field with 32 elements. If $a \in E$ and the minimal polynomial of a over Z_2 is $x^5+x^4+x^3+x^2+x+1$, find the other four zeros of $x^5+x^4+x^3+x^2+x+1$.

63. For a given finite field F_1 find infinitely many finite fields $F_2, F_3, F_4, \ldots$ such that $F_1 \subset F_2 \subset F_3 \subset F_4 \subset \cdots$.

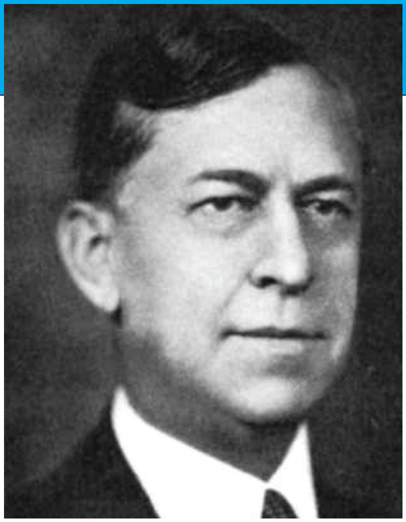
64. Without enumerating them by checking all 5th degree monic polynomials in $Z_3[x]$ for linear or quadratic factors, explain why there are exactly 44 monic irreducible polynomials of degree 5 over Z_3.

65. What field contains all finite fields with characteristic 2 as subfields?

66. Give an example of an infinite field of prime characteristic that contains exactly one finite subfield.

67. For a fixed prime p find infinitely many examples of finite fields $F_1, F_2, F_3, \ldots$ with the property that $\mathrm{GF}(p)$ is the only proper subfield of F_i for all i.

L. E. Dickson



Courtesy of the American Mathematical Society (www.ams.org)

One of the books [written by L. E. Dickson] is his major, three-volume *History of the Theory of Numbers* which would be a life's work by itself for a more ordinary man.

A. A. ALBERT, *Bulletin of the American Mathematical Society*

LEONARD EUGENE DICKSON was born in Independence, Iowa, on January 22, 1874. In 1896, he received the first Ph.D. to be awarded in mathematics at the University of Chicago. After spending a few years at the University of California and the University of Texas, he was appointed to the faculty at Chicago and remained there until his retirement in 1939.

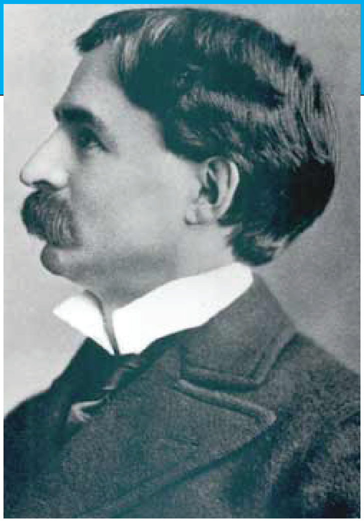
Dickson was one of the most prolific mathematicians of the 20th century, writing 267 research papers and 18 books. His principal interests were matrix groups, finite fields, algebra, and number theory. Dickson had a disdainful attitude toward applicable mathematics; he would often say, "Thank God that number theory is unsullied by any applications."

He also had a sense of humor. Dickson would often mention his honeymoon: "It was a great success," he said, "except that I only got two research papers written." Dickson received many honors in his career. He was the first to be awarded the prize from the American Association for the Advancement of Science for the most notable contribution to the advancement of science, and the first to receive the Cole Prize in algebra from the American Mathematical Society. The University of Chicago has research instructorships named after him. Dickson died on January 17, 1954.

E. H. Moore

Moore was an extraordinary genius, vivid, imaginative, sympathetic, foremost leader in freeing American mathematicians from dependence on foreign universities, and in building up a vigorous American School...

R. C. Archibald



Courtesy of the American Mathematical Society (www.ams.org)

E. H. Moore (Eliakim Hastings Moore) was born on January 26, 1862 in Marietta, Ohio. He received a B.A from Yale while earning prizes for mathematics, Latin, English, and astronomy. He was class valedictorian and Phi Beta Kappa. After obtaining a Ph.D. from Yale in 1895 he spent the seven years studying in Germany, tutoring at Yale, and teaching at Northwestern University. In 1892 he was appointed the department head at the newly founded University of Chicago and held that position until his retirement in 1931. Under his leadership, Chicago became the leading U. S. producer of Ph.D.'s in mathematics and the premier institution for research in mathematics. He was elected President of the American Mathematical Society and the National Academy of Science, as was four of his many Ph.D. students. He made significant contributions to field theory, algebraic geometry, and number theory. He served as President of American Association for the Advancement of Science and received six honorary degrees. In 2002, the American Mathematical Society established a research prize in his honor. He died on December 30, 1932.

22 Geometric Constructions

At the age of eleven, I began Euclid. . . . This was one of the great events of my life, as dazzling as first love.

Bertrand Russell

Meton: With the straight ruler I set to work to make the circle fourcornered.

Aristophanes (ca 444–380 BC)

Historical Discussion of Geometric Constructions

The ancient Greeks were fond of geometric constructions. They were especially interested in constructions that could be achieved using only a straightedge without markings and a compass. They knew, for example, that any angle can be bisected, and they knew how to construct an equilateral triangle, a square, a regular pentagon, and a regular hexagon. But they did not know how to trisect every angle or how to construct a regular seven-sided polygon (heptagon). Another problem that they attempted was the duplication of the cube—that is, given any cube, they tried to construct a new cube having twice the volume of the given one using only an unmarked straightedge and a compass. Legend has it that the ancient Athenians were told by the oracle at Delos that a plague would end if they constructed a new altar to Apollo in the shape of a cube with double the volume of the old altar, which was also a cube. Besides "doubling the cube," the Greeks also attempted to "square the circle"— to construct a square with area equal to that of a given circle. They knew how to solve all these problems using other means, such as a compass and a straightedge with two marks, or an unmarked straightedge and a spiral, but they could not achieve any of the constructions with a compass and an unmarked straightedge alone. These problems vexed mathematicians for over 2000 years. The resolution of these perplexities was made possible when they were transferred from questions of geometry to questions of algebra in the 19th century.

The first of the famous problems of antiquity to be solved was that of the construction of regular polygons. It had been known since Euclid that regular polygons with a number of sides of the form $2^k, 2^k \cdot 3, 2^k \cdot 5$, and $2^k \cdot 3 \cdot 5$ could be constructed, and it was believed that no others were possible. In 1796, while still a teenager, Gauss proved that the 17-sided regular polygon is constructible. In 1801, Gauss asserted that a regular polygon of n sides is constructible if and only if n has the form $2^k p_1 p_2 \cdots p_i$, where the p's are distinct primes of the form $2^{2^s} + 1$. We provide a proof of this statement in Theorem 31.5.

Thus, regular polygons with 3, 4, 5, 6, 8, 10, 12, 15, 16, 17, and 20 sides are possible to construct, whereas those with 7, 9, 11, 13, 14, 18, and 19 sides are not. How these constructions can be effected is another matter. One person spent 10 years trying to determine a way to construct the 65,537-sided polygon.

Gauss's result on the constructibility of regular n-gons eliminated another of the famous unsolved problems, because the ability to trisect a 60° angle enables one to construct a regular 9-gon. Thus, there is no method for trisecting a 60° angle with an unmarked straightedge and a compass. In 1837, Wantzel proved that it was not possible to double the cube. The problem of the squaring of a circle resisted all attempts until 1882, when Ferdinand Lindemann proved that π is transcendental, since, as we will show, all constructible numbers are algebraic.

Constructible Numbers

With the field theory we now have, it is an easy matter to solve the following problem: Given an unmarked straightedge, a compass, and a unit length, what other lengths can be constructed? To begin, we call a real number α *constructible* if, by means of an unmarked straightedge, a compass, and a line segment of length 1, we can construct a line segment of length $|\alpha|$ in a finite number of steps. It follows from plane geometry that if α and $\beta\,(\beta \neq 0)$ are constructible numbers, then so are $\alpha + \beta, \alpha - \beta, \alpha \cdot \beta$, and α/β. (See the exercises for hints.) Thus, the set of constructible numbers contains Q and is a subfield of the real numbers. What we desire is an algebraic characterization of this field. To derive such a characterization, let F be any subfield of the reals. Call the subset $\{(x, y) \in R^2 \mid x, y \in F\}$ of the real plane the *plane of* F, call any line joining two points in the plane of F a *line in* F, and call any circle whose center is in the plane of F and whose radius is in F a *circle in* F. Then a line in F has an equation of the form

$$ax + by + c = 0, \quad \text{where } a, b, c \in F,$$

and a circle in F has an equation of the form

$$x^2 + y^2 + ax + by + c = 0, \quad \text{where } a, b, c \in F.$$

In particular, note that to find the point of intersection of a pair of lines in F or the points of intersection of a line in F and a circle in F, one need only solve a linear or quadratic equation in F. We now come to the crucial question. Starting with points in the plane of some field F, which points in the real plane can be obtained with an unmarked straightedge and a compass? Well, there are only three ways to construct points, starting with points in the plane of F:

1. Intersect two lines in F.
2. Intersect a circle in F and a line in F.
3. Intersect two circles in F.

In case 1, we do not obtain any new points, because two lines in F intersect in a point in the plane of F. In case 2, the point of intersection is the solution to either a linear equation in F or a quadratic equation in F. So, the point lies in the plane of F or in the plane of $F(\sqrt{\alpha})$, where $\alpha \in F$ and α is positive. In case 3, no new points are obtained, because, if the two circles are given by $x^2 + y^2 + ax + by + c = 0$ and $x^2 + y^2 + a'x + b'y + c' = 0$, then we have $(a - a')x + (b - b')y + (c - c') = 0$, which is a line in F. So, case 3 reduces to case 2.

It follows, then, that the only points in the real plane that can be constructed from the plane of a field F are those whose coordinates lie in fields of the form $F(\sqrt{\alpha})$, where $\alpha \in F$ and α is positive. Of course, we can start over with $F_1 = F(\sqrt{\alpha})$ and construct points whose coordinates lie in fields of the form $F_2 = F_1(\sqrt{\beta})$, where $\beta \in F_1$ and β is positive. Continuing in this fashion, we see that a real number c is constructible if and only if there is a series of fields $Q = F_1 \subseteq F_2 \subseteq \cdots \subseteq F_n \subseteq \mathbf{R}$ such that $F_{i+1} = F_i(\sqrt{\alpha_i})$, where $\alpha_i \in F_i$ and $c \in F_n$. Since $[F_{i+1} : F_i] = 1$

or 2, we see by Theorem 20.5 that if c is constructible, then $[Q(c):Q] = 2^k$ for some nonnegative integer k.

We now dispatch the problems that plagued the Greeks. Consider doubling the cube of volume 1. The enlarged cube would have an edge of length $\sqrt[3]{2}$. But $[Q(\sqrt[3]{2}):Q] = 3$, so such a cube cannot be constructed.

Next consider the possibility of trisecting a 60° angle. If it were possible to trisect an angle of 60°, then cos 20° would be constructible. (See Figure 22.1.) In particular, $[Q(\cos 20°):Q] = 2^k$ for some k. Now, using the trigonometric identity $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, with $\theta = 20°$, we see that $1/2 = 4\cos^3 20° - 3\cos 20°$, so that cos 20° is a zero of $8x^3 - 6x - 1$. But, since $8x^3 - 6x - 1$ is irreducible over Q (see Exercise 13), we must also have $[Q(\cos 20°):Q] = 3$. This contradiction shows that trisecting a 60° angle is impossible.

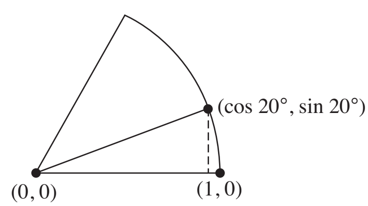


Figure 22.1 Sixty-degree angle.

The remaining problems are relegated to the reader as Exercises 14, 15, and 17.

Angle-Trisectors and Circle-Squarers

Down through the centuries, hundreds of people have claimed to have achieved one or more of the impossible constructions. In 1775, the Paris Academy, so overwhelmed with these claims, passed a resolution to no longer examine these claims or claims of machines purported to exhibit perpetual motion. Although it has been more than 100 years since the last of the constructions was shown to be impossible, there continues to be a steady parade of people who claim to have done one or more of them. Most of these people have heard that this is impossible but have refused to believe it. One person insisted that he could trisect any angle with a straightedge alone. Another found his trisection in 1973 after 12,000 hours of work. One got his from God. In 1971, a person with a Ph.D. in mathematics asserted that he had a valid trisection method. Many people have claimed the hat trick: trisecting the angle, doubling the cube, and squaring the circle. Two men who did this in 1961 succeeded in having their accomplishment noted in the *Congressional Record*. Occasionally, newspapers and magazines have run stories about "doing the impossible," often giving the impression that the construction may be valid. Many angle-trisectors and circle-squarers have had their work published at their own expense and distributed to colleges and universities. One had his printed in four languages!

Although he never claimed to have squared the circle, the incomparable Renaissance multifaceted genius Leonardo da Vinci made 169 attempts over many years at squaring the circle.

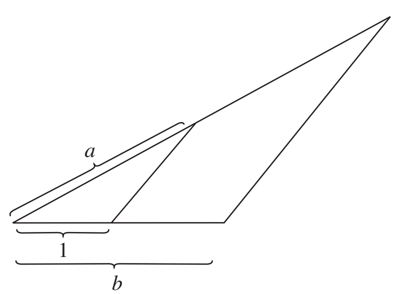
Exercises

Only prove to me that it is impossible, and I will set about it this very evening.

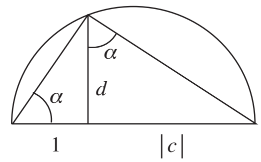
Spoken by a member of the audience after De Morgan gave a lecture on the impossibility of squaring the circle.

1. If a and b are constructible numbers and $a \geq b > 0$, give a geometric proof that $a + b$ and $a - b$ are constructible.

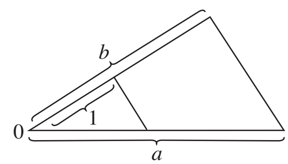
2. If a and b are constructible, give a geometric proof that ab is constructible. (*Hint:* Consider the following figure. Notice that all segments in the figure can be made with an unmarked straightedge and a compass.)



3. Prove that if c is a constructible number, then so is $\sqrt{|c|}$. (*Hint:* Consider the following semicircle with diameter $1 + |c|$.) (This exercise is referred to in Chapter 31.)



4. If a and b ($b \neq 0$) are constructible numbers, give a geometric proof that a/b is constructible. (*Hint:* Consider the following figure.)



5. Prove that $\sin\theta$ is constructible if and only if $\cos\theta$ is constructible.

6. Prove that an angle θ measured in degrees is constructible if and only if $\sin\theta$ is constructible.

7. Prove that $\cos 2\theta$ is constructible if and only if $\cos\theta$ is constructible.

8. Prove that $30°$ is a constructible angle.

9. Prove that a $45°$ angle can be trisected with an unmarked straightedge and a compass.

10. Prove that a $40°$ angle is not constructible.

11. Show that the point of intersection of two lines in the plane of a field F lies in the plane of F.

12. Show that the points of intersection of a circle in the plane of a field F and a line in the plane of F are points in the plane of F or in the plane of $F(\sqrt{\alpha})$, where $\alpha \in F$ and α is positive. Give an example of a circle and a line in the plane of Q whose points of intersection are not in the plane of Q.

13. Prove that $8x^3 - 6x - 1$ is irreducible over Q.

14. Use the fact that $8\cos^3(2\pi/7) + 4\cos^2(2\pi/7) - 4\cos(2\pi/7) - 1 = 0$ to prove that a regular seven-sided polygon is not constructible with an unmarked straightedge and a compass.

15. Show that a regular 9-gon cannot be constructed with an unmarked straightedge and a compass.

16. Show that if a regular n-gon is constructible, then so is a regular $2n$-gon.

17. (Squaring the Circle) Show that it is impossible to construct, with an unmarked straightedge and a compass, a square whose area equals that of a circle of radius 1. You may use the fact that π is transcendental over Q.

18. Use the fact that $4\cos^2(2\pi/5) + 2\cos(2\pi/5) - 1 = 0$ to prove that a regular pentagon is constructible.

19. Can the cube be "tripled"?

20. Can the cube be "quadrupled"?

21. Can the circle be "cubed"?

22. If a, b, and c are constructible, show that the real roots of $ax^2 + bx + c$ are constructible.

23 Sylow Theorems

Generally these three results are implied by the expression "Sylow's Theorem." All of them are of fundamental importance. In fact, if the theorems of group theory were arranged in order of their importance Sylow's Theorem might reasonably occupy the second place—coming next to Lagrange's Theorem in such an arrangement.

G. A. Miller, *Theory and Application of Finite Groups*

It is impossible to overstate the importance of Sylow's Theorem in the study of finite groups. Without it the subject would not get off the ground.

I. N. Herstein, *Abstract Algebra, 3rd ed.*

Conjugacy Classes

In this chapter, we derive several important arithmetic relationships between a group and certain of its subgroups. Recall from Chapter 7 that Lagrange's Theorem was proved by showing that cosets of a subgroup partition the group. Another fruitful method of partitioning the elements of a group is by way of conjugacy classes.

Definition Conjugacy Class of a

Let a and b be elements of a group G. We say that a and b are *conjugate* in G (and call b a *conjugate* of a) if $xax^{-1} = b$ for some x in G. The *conjugacy class of a* is the set $\text{cl}(a) = \{xax^{-1} | x \in G\}$.

We leave it to the reader (Exercise 1) to prove that conjugacy is an equivalence relation on G, and that the conjugacy class of a is the equivalence class of a under conjugacy. Thus, we may partition any group into disjoint conjugacy classes. Let's look at one example. In D_4 we have

$$\begin{aligned}\mathrm{cl}(H) = \{&R_0HR_0{}^{-1}, R_{90}HR_{90}{}^{-1}, R_{180}HR_{180}{}^{-1}, R_{270}HR_{270}{}^{-1},\\ &HHH^{-1}, VHV^{-1}, DHD^{-1}, D'HD'^{-1}\} = \{H, V\}.\end{aligned}$$

Similarly, one may verify that

$$\begin{aligned}\mathrm{cl}(R_0) &= \{R_0\},\\ \mathrm{cl}(R_{90}) &= \{R_{90}, R_{270}\} = \mathrm{cl}(R_{270}),\\ \mathrm{cl}(R_{180}) &= \{R_{180}\},\\ \mathrm{cl}(V) &= \{V, H\} = \mathrm{cl}(H),\\ \mathrm{cl}(D) &= \{D, D'\} = \mathrm{cl}(D').\end{aligned}$$

Theorem 23.1 gives an arithmetic relationship between the size of the conjugacy class of a and the size of $C(a)$, the centralizer of a.

Theorem 23.1 Number of Conjugates of *a*

Let G be a finite group and let a be an element of G. Then, $|\mathrm{cl}(a)| = |G : C(a)|$.

PROOF Consider the function T that sends the coset $xC(a)$ to the conjugate xax^{-1} of a. A routine calculation shows that T is well-defined, is one-to-one, and maps the set of left cosets onto the conjugacy class of a. Thus, the number of conjugates of a is the index of the centralizer of a. ∎

Corollary 1 $|\mathrm{cl}(a)|$ **Divides** $|G|$

In a finite group, $|\mathrm{cl}(a)|$ ***divides*** $|G|$.

The Class Equation

Since the conjugacy classes partition a group, the following important counting principle is a corollary to Theorem 23.1.

Corollary 2 Class Equation

For any finite group G,

$$|G| = \Sigma|G : C(a)|,$$

where the sum runs over one element a from each conjugacy class of G.

In finite group theory, counting principles such as this corollary are powerful tools.[1] Theorem 23.2 is the single most important fact about finite groups of prime-power order (a group of order p^n, where p is a prime, is called a *p-group*).

Theorem 23.2 *p*-Groups Have Nontrivial Centers

Let G be a nontrivial finite group whose order is a power of a prime p. Then $Z(G)$ has more than one element.

PROOF First observe that $\text{cl}(a) = \{a\}$ if and only if $a \in Z(G)$ (see Exercise 4). Thus, by culling out these elements, we may write the class equation in the form

$$|G| = |Z(G)| + \Sigma|G : C(a)|,$$

where the sum runs over representatives of all conjugacy classes with more than one element (this set may be empty). But $|G : C(a)| = |G|/|C(a)|$, so each term in $\sum |G : C(a)|$ has the form p^k with $k \geq 1$. Hence,

$$|G| - \Sigma|G : C(a)| = |Z(G)|,$$

where each term on the left is divisible by p. It follows, then, that p also divides $|Z(G)|$, and hence $|Z(G)| \neq 1$. ∎

Recall that a corollary of Theorem 9.7 is that groups of p^2 where p is a prime are Abelian. This result is also a corollary of Theorem 23.2.

[1] "Never underestimate a theorem that counts something." John Fraleigh, *A First Course in Abstract Algebra.*

Corollary Groups of Order p^2 Are Abelian

If $|G| = p^2$, where p is prime, then G is Abelian.

PROOF By Theorem 23.2 and Lagrange's Theorem, $|Z(G)| = p$ or p^2. If $|Z(G)| = p^2$, then $G = Z(G)$ and G is Abelian. If $|Z(G)| = p$, then $|G/Z(G)| = p$, so that $G/Z(G)$ is cyclic. But, then, by Theorem 9.3, G is Abelian. ∎

The Sylow Theorems

Now to the Sylow theorems.[2] Recall that the converse of Lagrange's Theorem is false; that is, if G is a group of order m and n divides m, G need *not* have a subgroup of order n. Our next theorem is a partial converse of Lagrange's Theorem. It, as well as Theorem 23.2, was first proved by the Norwegian mathematician Ludwig Sylow (1832–1918). Sylow's Theorem and Lagrange's Theorem are the two most important results in finite group theory.[3] The first gives a sufficient condition for the existence of subgroups, and the second gives a necessary condition.

Theorem 23.3 Existence of Subgroups of Prime-Power Order (Sylow's First Theorem, 1872)

Let G be a finite group and let p be a prime. If p^k divides $|G|$, then G has at least one subgroup of order p^k.

PROOF We proceed by induction on $|G|$. If $|G| = 1$, Theorem 23.3 is trivially true. Now assume that the statement is true for all groups of order less than $|G|$. If G has a proper subgroup H such that p^k divides $|H|$, then, by our inductive assumption, H has a subgroup of order p^k and we are done. Thus, we may henceforth assume that p^k does not divide the order of any proper subgroup of G . Next, consider the class equation for G in the form

$$|G| = |Z(G)| + \sum |G : C(a)|,$$

[2] "For a group theorist, Sylow's Theorem is such a basic tool, and so fundamental, that it is used almost without thinking, like breathing." Geoff Robinson
[3] My choice for the third most important result is the Fundamental Theorem of Finite Abelian Groups.

where we sum over a representative of each conjugacy class $\mathrm{cl}(a)$, where $a \notin Z(G)$. Since p^k divides $|G| = |G : C(a)||C(a)|$ and p^k does not divide $|C(a)|$, we know that p must divide $|G : C(a)|$ for all $a \notin Z(G)$. It then follows from the class equation that p divides $|Z(G)|$. The Fundamental Theorem of Finite Abelian Groups (Theorem 11.1), or Theorem 9.5, then guarantees that $Z(G)$ contains an element of order p, say x. Since x is in the center of G, $\langle x\rangle$ is a normal subgroup of G , and we may form the factor group $G/\langle x\rangle$. Now observe that p^{k-1} divides $|G/\langle x\rangle|$. Thus, by the induction hypothesis, $G/\langle x\rangle$ has a subgroup of order p^{k-1} and, by Exercise 63 in Chapter 10, this subgroup has the form $H/\langle x\rangle$, where H is a subgroup of G . Finally, note that $|H/\langle x\rangle| = p^{k-1}$ and $|\langle x\rangle| = p$ imply that $|H| = p^k$. Thus, we have produced a subgroup of order p^k, which contradicts our assumption that no such subgroup exists. Therefore, we must have originally had a subgroup of order p^k, and we can apply the induction hypothesis to that subgroup. ■

Let's be sure we understand exactly what Sylow's First Theorem means. Say we have a group G of order $2^3 \cdot 3^2 \cdot 5^4 \cdot 7$. Then Sylow's First Theorem says that G must have at least one subgroup of each of the following orders: 2, 4, 8, 3, 9, 5, 25, 125, 625, and 7. On the other hand, Sylow's First Theorem tells us nothing about the possible existence of subgroups of order 6, 10, 15, 30, or any other divisor of $|G|$ that has two or more distinct prime factors. Because certain subgroups guaranteed by Sylow's First Theorem play a central role in the theory of finite groups, they are given a special name.

Definition Sylow p-Subgroup

Let G be a finite group and let p be a prime. If p^k divides $|G|$ and p^{k+1} does not divide $|G|$, then any subgroup of G of order p^k is called a *Sylow p-subgroup of G*.[4]

So, returning to our group G of order $2^3 \cdot 3^2 \cdot 5^4 \cdot 7$, we call any subgroup of order 8 a Sylow 2-subgroup of G, any subgroup of order 625 a Sylow 5-subgroup of G, and so on. Notice that a Sylow p-subgroup of G is a subgroup whose order is the largest power of p consistent with Lagrange's Theorem.

Since any subgroup of order p is cyclic, we have the following generalization of Theorem 9.5, first proved by Cauchy in 1845. His proof ran nine pages!

Corollary Cauchy's Theorem

Let G be a finite group and let p be a prime that divides the order of G. Then G has an element of order p.

Sylow's First Theorem is so fundamental to finite group theory that many different proofs of it have been published over the years [our proof is essentially the one given by Georg Frobenius (1849–1917) in 1895]. Likewise, there are scores of generalizations of Sylow's Theorem.

Observe that the corollary to the Fundamental Theorem of Finite Abelian Groups and Sylow's First Theorem show that the converse of Lagrange's Theorem is true for all finite Abelian groups and all finite groups of prime-power order.

There are two more Sylow theorems that are extremely valuable tools in finite group theory. But first we introduce a new term and recall two concepts.

Definition Conjugate Subgroups
Let H and K be subgroups of a group G. We say that H and K are *conjugate* in G if there is an element g in G such that $H = gKg^{-1}$.

Recall from Chapter 7 that if G is a finite group of permutations on a set S and $i \in S$, then $\text{orb}_G(i) = \{\phi(i) \mid \phi \in G\}$ and $|\text{orb}_G(i)|$ divides $|G|$. For a subgroup H of a group G, the *normalizer of H in G*, is the subgroup $N(H) = \{x \in G \mid xHx^{-1} = H\}$.

Theorem 23.4 Sylow's Second Theorem

If H is a subgroup of a finite group G and $|H|$ is a power of a prime p, then H is contained in some Sylow p-subgroup of G.

PROOF Let K be a Sylow p-subgroup of G and let $C = \{K_1, K_2, \ldots, K_n\}$ with $K = K_1$ be the set of all conjugates of K in G. Since conjugation is an automorphism, each element of C is a Sylow p-subgroup of G. Let S_C denote the group of all permutations of C. For each $g \in G$, define $\phi_g : C \to C$ by $\phi_g(K_i) = gK_ig^{-1}$. It is easy to show that each $\phi_g \in S_C$.

Now define a mapping $T : G \to S_C$ by $T(g) = \phi_g$. Since $\phi_{gh}(K_i) = (gh)K_i(gh)^{-1} = g(hK_ih^{-1})g^{-1} = g\phi_h(K_i)g^{-1} = \phi_g(\phi_h(K_i)) = (\phi_g\phi_h)(K_i)$, we have $\phi_{gh} = \phi_g\phi_h$, and therefore T is a homomorphism from G to S_C.

Next consider $T(H)$, the image of H under T. Since $|H|$ is a power of p, so is $|T(H)|$ (see property 6 of Theorem 10.2). Thus, by the Orbit-Stabilizer Theorem (Theorem 7.4), for each $i, |\text{orb}_{T(H)}(K_i)|$ divides $|T(H)|$, so that $|\text{orb}_{T(H)}(K_i)|$ is a power of p. Now we ask: Under what condition does $|\text{orb}_{T(H)}(K_i)| = 1$? Well, $|\text{orb}_{T(H)}(K_i)| = 1$ means that $\phi_g(K_i) = gK_ig^{-1} = K_i$ for all $g \in H$; that is, $|\text{orb}_{T(H)}(K_i)| = 1$ if and only if $H \leq N(K_i)$. But the only elements of $N(K_i)$ that have orders that are powers of p are those of K_i (see Exercise 17). Thus, $|\text{orb}_{T(H)}(K_i)| = 1$ if and only if $H \leq K_i$.

So, to complete the proof, all we need to do is show that for some i, $|\text{orb}_{T(H)}(K_i)| = 1$. Analogous to Theorem 23.1, we have $|C| = |G : N(K)|$ (see Exercise 9). And since $|G : K| = |G : N(K)||N(K) : K|$ is not divisible by p, neither is $|C|$. Because the orbits partition $C, |C|$ is the sum of powers of p. If no orbit has size 1, then p divides each summand and, therefore, p divides $|C|$, which is a contradiction. Thus, there is an orbit of size 1, and the proof is complete. ∎

Theorem 23.5 Sylow's Third Theorem

Let p be a prime and let G be a group of order $p^k m$, where p does not divide m. Then the number n of Sylow p-subgroups of G is equal to 1 modulo p and divides m. Furthermore, any two Sylow p-subgroups of G are conjugate.

PROOF Let K be any Sylow p-subgroup of G and let $C = \{K_1, K_2, \ldots, K_n\}$, with $K = K_1$, be the set of all conjugates of K in G. We first prove that $n \bmod p = 1$.

Let S_C and T be as in the proof of Theorem 23.4. This time we consider $T(K)$, the image of K under T. As before, we have $|\text{orb}_{T(K)}(K_i)|$ is a power of p for each i and $|\text{orb}_{T(K)}(K_i)| = 1$ if and only if $K \leq K_i$. Thus, $|\text{orb}_{T(K)}(K_1)| = 1$ and $|\text{orb}_{T(K)}(K_i)|$ is a power of p greater than 1 for all $i \neq 1$. Since the orbits partition C, it follows that, modulo p, $n = |C| = 1$.

Next we show that every Sylow p-subgroup of G belongs to C. To do this, suppose that H is a Sylow p-subgroup of G that is not in C. Let S_C and T be as in the proof of Theorem 23.4, and this time consider $T(H)$. As in the previous paragraph, $|C|$ is the sum of the orbits' sizes under the action of $T(H)$. However, no orbit has size 1, since H is not in C. Thus, $|C|$ is a sum of terms each divisible by p, so that, modulo p, $n = |C| = 0$. This contradiction proves that H belongs to C, and that n is the number of Sylow p-subgroups of G.

Finally, that n divides m follows directly from the fact that $n = |G : N(K)|$ (see Exercise 9) and n is relatively prime to p. ■

It is convenient to let n_p denote the number of Sylow p-subgroups of a group. Observe that the first portion of Sylow's Third Theorem is a counting principle.[5] As an important consequence of Sylow's Third Theorem, we have the following corollary.

Corollary A Unique Sylow p-Subgroup Is Normal

A Sylow p-subgroup of a finite group G is a normal subgroup of G if and only if it is the only Sylow p-subgroup of G.

We illustrate Sylow's Third Theorem with two examples.

EXAMPLE 1 Consider the Sylow 2-subgroups of S_3. They are $\{(1), (12)\}$, $\{(1), (23)\}$, and $\{(1), (13)\}$. According to Sylow's Third Theorem, we should be able to obtain the latter two of these from the first by conjugation. Indeed,

$$(13)\{(1), (12)\}(13)^{-1} = \{(1), (23)\},$$

$$(23)\{(1), (12)\}(23)^{-1} = \{(1), (13)\}.$$ ■

EXAMPLE 2 Consider the Sylow 3-subgroups of A_4. They are $\{\alpha_1, \alpha_5, \alpha_9\}, \{\alpha_1, \alpha_6, \alpha_{11}\}, \{\alpha_1, \alpha_7, \alpha_{12}\}$, and $\{\alpha_1, \alpha_8, \alpha_{10}\}$. (See Table 5.1.) Then,

$$\alpha_2\{\alpha_1, \alpha_5, \alpha_9\}\alpha_2^{-1} = \{\alpha_1, \alpha_7, \alpha_{12}\},$$

$$\alpha_3\{\alpha_1, \alpha_5, \alpha_9\}\alpha_3^{-1} = \{\alpha_1, \alpha_8, \alpha_{10}\},$$

$$\alpha_4\{\alpha_1, \alpha_5, \alpha_9\}\alpha_4^{-1} = \{\alpha_1, \alpha_6, \alpha_{11}\}.$$

[5] "Whenever you can, count." Sir Francis Galton (1822–1911), *The World of Mathematics.*

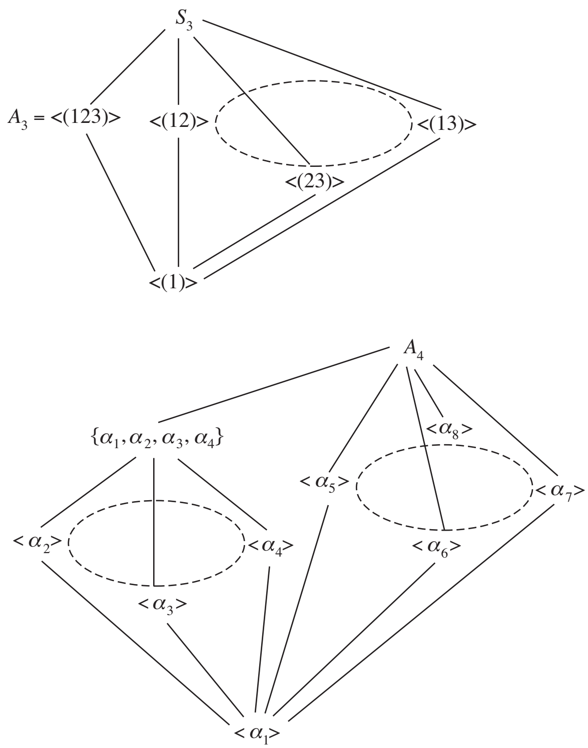


Figure 23.1 Lattices of subgroups for S_3 and A_4.

Thus, the number of Sylow 3-subgroups is 1 modulo 3, and the four Sylow 3-subgroups are conjugate. ∎

Figure 23.1 shows the subgroup lattices for S_3 and A_4. We have connected the Sylow p-groups with dashed circles to indicate that they belong to one orbit under conjugation. Notice that the three subgroups of order 2 in A_4 are contained in a Sylow 2-group, as required by Sylow's Second Theorem. As it happens, these three subgroups also belong to one orbit under conjugation, but this is not a consequence of Sylow's Third Theorem.

In contrast to the two preceding examples, observe that the dihedral group of order 12 has seven subgroups of order 2, but that conjugating $\{R_0, R_{180}\}$ does not yield any of the other six. (Why?)

Applications of Sylow Theorems

A few numerical examples will make the Sylow theorems come to life.

EXAMPLE 3 Say G is a group of order 40. What do the Sylow theorems tell us about G? A great deal! Since 1 is the only divisor of 40 that is congruent to 1 modulo 5, we know that G has exactly one subgroup of order 5, and therefore it is normal. Similarly, G has either one or five subgroups of order 8. If there is only one subgroup of order 8, it is normal. If there are five subgroups of order 8, none is normal and all five can be obtained by starting with any particular one, say H, and computing xHx^{-1} for various x's. Finally, if we let K denote the normal subgroup of order 5 and let H denote any subgroup of order 8, then $G = HK$. (See Example 5 in Chapter 9.) If H happens to be normal, we can say even more: $G = H \times K$. ∎

EXAMPLE 4 Consider a group of order 30. By Sylow's Third Theorem, it must have either one or six subgroups of order 5 and one or 10 subgroups of order 3. However, G cannot have both six subgroups of order 5 *and* 10 subgroups of order 3 (for then G would have more than 30 elements). Thus, the subgroup of order 3 is unique or the subgroup of order 5 is unique (or both are unique) and therefore is normal in G. It follows, then, that the product of a subgroup of order 3 and one of order 5 is a group of order 15 that is both cyclic (Exercise 35) and normal (Exercise 9 in Chapter 9) in G. [This, in turn, implies that *both* the subgroup of order 3 and the subgroup of order 5 are normal in G (Exercise 59 in Chapter 9).] So, if we let y be a generator of the cyclic subgroup of order 15 and let x be an element of order 2 (the existence of which is guaranteed by Cauchy's Theorem), we see that $G = \{x^i y^j \mid 0 \leq i \leq 1, 0 \leq j \leq 14\}$. ∎

EXAMPLE 5 We show that any group G of order 72 must have a proper, nontrivial normal subgroup. Our arguments are a preview of those in Chapter 24. By Sylow's Third Theorem, the number of Sylow 3-subgroups of G is equal to 1 mod 3 and divides 8. Thus, the number is 1 or 4. If there is only one, then it is normal by the corollary of Sylow's Third Theorem. Otherwise, let H and H' be two distinct Sylow 3-subgroups. By Theorem 7.2, we have that

$|HH'| = |H||H'|/|H \cap H'| = 81/|H \cap H'|$. Since $|G| = 72$ and $|H \cap H'|$ is a subgroup of H and H', we know that $|H \cap H'| = 3$. By the corollary to Theorem 23.2, $N(H \cap H')$ contains both H and H'. Thus, $|N(H \cap H')|$ divides 72, is divisible by 9, and has at least $|HH'| = 27$ elements. This leaves only 36 or 72 for $|N(H \cap H')|$. In the first case, we have from Exercise 9 of Chapter 9 that $N(H \cap H')$ is normal in G. In the second case, we have by definition that $H \cap H'$ is normal in G. ∎

Note that in these examples we were able to deduce all of this information from knowing only the order of the group—so many conclusions from one assumption! This is the beauty of finite group theory.

In Chapter 7 we saw that the only group (up to isomorphism) of prime order p is Z_p and the argument given in Example 18 of Chapter 10 (see Exercise 71 of Chapter 10) shows that for primes p and q with $p < q$, where p does not divide $q - 1$, the only group of order pq is Z_{pq} (up to isomorphism). As a further illustration of the power of the Sylow theorems, we next give another proof of the latter statement.[6]

Theorem 23.6 Cyclic Groups of Order pq

If G is a group of order pq, where p and q are primes, $p < q$, and p does not divide $q - 1$, then G is cyclic. In particular, G is isomorphic to Z_{pq}.

PROOF Let H be a Sylow p-subgroup of G and let K be a Sylow q-subgroup of G. Sylow's Third Theorem states that the number of Sylow p-subgroups of G is of the form $1 + kp$ and divides q. So $1 + kp = 1$ or $1 + kp = q$. Since p does not divide $q - 1$, we have that $k = 0$ and therefore H is the only Sylow p-subgroup of G.

Similarly, there is only one Sylow q-subgroup of G (see Exercise 25). Thus, by the corollary to Theorem 23.5, H and K are normal subgroups of G. Moreover, from Theorem 7.2 and Lagrange, we have $G = HK$ and $H \cap K = \{e\}$. This tells us that $G = H \times K$. Finally, by Theorem 8.2, $G \approx Z_p \oplus Z_q \approx Z_{pq}$. ∎

[6] It can be shown that for odd primes p and q such that $p < q$ and p divides $q - 1$ there is a non-Abelian group of order pq. See https://en.wikipedia.org/wiki/Semidirect_product.

Theorem 23.6 demonstrates the power of the Sylow theorems in classifying the finite groups whose orders have small numbers of prime factors. Similar results exist for groups of orders p^2q, p^2q^2, p^3, and p^4, where p and q are prime.

For your amusement, Figure 23.2 lists the number of nonisomorphic groups with order at most 100. Note in particular the large number of groups of order 64. Also observe that, generally speaking, it is not the size of the group that gives rise to a large number of groups of that size but the number of prime factors involved. In all, there are 1047 nonisomorphic groups with 100 or fewer elements. Contrast this with the fact that there are 49,487,365,422 groups of order $1024 = 2^{10}$. The number of groups of any order less than 2048 is given at **http://oeis.org/A000001/b000001.txt**.

As a final application of the Sylow theorems, you might enjoy seeing a determination of the groups of order 45, 66, and 255. In fact, our arguments serve as a good review of much of our work in group theory.

Order	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20
Number	1	1	1	2	1	2	1	5	2	2	1	5	1	2	1	14	1	5	1	5
Order	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40
Number	2	2	1	15	2	2	5	4	1	4	1	51	1	2	1	14	1	2	2	14
Order	41	42	43	44	45	46	47	48	49	50	51	52	53	54	55	56	57	58	59	60
Number	1	6	1	4	2	2	1	52	2	5	1	5	1	15	2	13	2	2	1	13
Order	61	62	63	64	65	66	67	68	69	70	71	72	73	74	75	76	77	78	79	80
Number	1	2	4	267	1	4	1	5	1	4	1	50	1	2	3	4	1	6	1	52
Order	81	82	83	84	85	86	87	88	89	90	91	92	93	94	95	96	97	98	99	100
Number	15	2	1	15	1	2	1	12	1	10	1	4	2	2	1	230	1	5	2	16

Figure 23.2 The number of groups of a given order up to 100.

EXAMPLE 6 Determination of the Groups of Order 45

Suppose that G is a group of order 45. Let H be a Sylow 3-subgroup of G and let K be a Sylow 5-subgroup of G. Since 1 is the only positive divisor of 45 that is equal to 1 modulo 5, we know from Sylow's Third Theorem and its corollary that K is normal in G. Similarly, H is normal in G. It follows, by the argument used in the proof of Theorem 23.6, that $G = H \times K$. Since both H and

K are Abelian, G is also Abelian. Thus, G is isomorphic to Z_{45} or $Z_3 \oplus Z_3 \oplus Z_5$. ∎

EXAMPLE 7 Determination of the Groups of Order 66

Suppose that G is a group of order 66. Let H be a Sylow 3-subgroup of G and let K be a Sylow 11-subgroup of G. Since 1 is the only positive divisor of 66 that is equal to 1 modulo 11, we know that K is normal in G. Thus, HK is a subgroup of G of order 33 (see Example 5 in Chapter 9 and Theorem 7.2). Since any group of order 33 is cyclic (Theorem 23.6), we may write $HK = \langle x \rangle$. Next, let $y \in G$ and $|y| = 2$. Since $\langle x \rangle$ has index 2 in G, we know it is normal. So $yxy^{-1} = x^i$ for some i from 1 to 32. Then, $yx = x^i y$ and, since every member of G is of the form $x^s y^t$, the structure of G is completely determined by the value of i. We claim that there are only four possibilities for i. To prove this, observe that $|x^i| = |x|$. Thus, i and 33 are relatively prime. But also, since y has order 2,

$$x = y^{-1}(yxy^{-1})y = y^{-1}x^i y = yx^i y^{-1} = (yxy^{-1})i = (x^i)i = x^{i^2}.$$

So $x^{i^2-1} = e$ and therefore 33 divides $i^2 - 1$. From this it follows that 11 divides $i \pm 1$, and therefore $i = 0 \pm 1, i = 11 \pm 1, i = 22 \pm 1$, or $i = 33 \pm 1$. Putting this together with the other information we have about i, we see that $i = 1, 10, 23$, or 32. This proves that there are at most four groups of order 66.

To prove that there are exactly four such groups, we simply observe that $Z_{66}, D_{33}, D_{11} \oplus Z_3$, and $D_3 \oplus Z_{11}$ each has order 66 and that no two are isomorphic. For example, $D_{11} \oplus Z_3$ has 11 elements of order 2, whereas $D_3 \oplus Z_{11}$ has only three elements of order 2. ∎

EXAMPLE 8 The Only Group of Order 255 is Z_{255}

Let G be a group of order $255 = 3 \cdot 5 \cdot 17$, and let H be a Sylow 17-subgroup of G. By Sylow's Third Theorem, H is the only Sylow 17-subgroup of G, so $N(H) = G$. By Example 17 in Chapter 10, $|N(H)/C(H)|$ divides $|\text{Aut}(H)| = |\text{Aut}(Z_{17})|$. By Theorem 6.5, $|\text{Aut}(Z_{17})| = |U(17)| = 16$. Since $|N(H)/C(H)|$ must divide 255 and 16, we have $|N(H)/C(H)| = 1$. Thus, $C(H) = G$. This means that every element of G commutes with every element of H, and, therefore, $H \subseteq Z(G)$. Thus, 17 divides $|Z(G)|$, which in turn divides 255. So $|Z(G)|$ is equal to 17, 51, 85, or 255 and $|G/Z(G)|$ is equal to 15, 5, 3, or 1. But the only groups of order 15, 5, 3, or 1 are the cyclic ones, so we know that $G/Z(G)$ is cyclic. Now the

G/Z Theorem (Theorem 9.3) shows that G is Abelian, and the Fundamental Theorem of Finite Abelian Groups tells us that G is cyclic. ∎

Exercises

I have always grown from my problems and challenges, from the things that don't work out. That's when I've really learned.

Carol Burnett

1. Show that conjugacy is an equivalence relation on a group.
2. If a is a group element, prove that every element in $\text{cl}(a)$ has the same order as a.
3. Let a be a group element of even order. Prove that a^2 is not in $\text{cl}(a)$.
4. Calculate all conjugacy classes for the quaternions (see Exercise 54, Chapter 9).
5. Show that the function T defined in the proof of Theorem 23.1 is well-defined, is one-to-one, and maps the set of left cosets onto the conjugacy class of a.
6. Show that $\text{cl}(a) = \{a\}$ if and only if $a \in Z(G)$.
7. Show that Z_2 is the only finite group that has exactly two conjugacy classes.
8. What can you say about the number of elements of order 7 in a group of order $168 = 8 \cdot 3 \cdot 7$?
9. Let H be a subgroup of a group G. Prove that the number of conjugates of H in G is $|G{:}N(H)|$. (This exercise is referred to in this chapter.)
10. Let H be a proper subgroup of a finite group G. Show that G is not the union of all conjugates of H.
11. If G is a group of odd order and $x \in G$, show that x^{-1} is not in $\text{cl}(x)$.
12. Determine the class equation for non-Abelian groups of orders 39 and 55.
13. Determine which of the equations below could be the class equation given in the proof of Theorem 23.2. For each part, provide your reasoning.
    a. $9 = 3 + 3 + 3$

b. $21 = 1 + 1 + 3 + 3 + 3 + 3 + 7$
c. $10 = 1 + 2 + 2 + 5$
d. $18 = 1 + 3 + 6 + 8$

14. Determine the class equation for D_6.

15. Suppose that G is a group of order 48. Show that the intersection of any two distinct Sylow 2-subgroups of G has order 8.

16. If a and b are group elements and i and k are positive integers, prove that $bab^{-1} = a^i$ implies that $b^k ab^{-k} = a^{i^k}$.

17. Let K be a Sylow p-subgroup of a finite group G. Prove that if $x \in N(K)$ and the order of x is a power of p, then $x \in K$. (This exercise is referred to in this chapter.)

18. Suppose that G is a group of the form $\{a^i b^j \mid i = 0, 1, \ldots, 6,\ j = 0, 1, 2\}$ and $ba = ab^2$. Express aba^2b in the form $a^i b^j$.

19. Suppose that G is a group and $|G| = p^n m$, where p is prime and $p > m$. Prove that a Sylow p-subgroup of G must be normal in G.

20. Let H be a Sylow p-subgroup of G. Prove that H is the only Sylow p-subgroup of G contained in $N(H)$.

21. Suppose that G is a group of order 168. If G has more than one Sylow 7-subgroup, exactly how many does it have?

22. Show that every group of order 56 has a proper nontrivial normal subgroup.

23. What is the smallest composite (i.e., nonprime and greater than 1) integer n such that there is a unique group of order n?

24. Let G be a noncyclic group of order 21. How many Sylow 3-subgroups does G have?

25. Let G be a group of order pq where p and q are distinct primes and $p < q$. Prove that the Sylow q-subgroup is normal in G. (This exercise is referred to in this chapter.)

26. Prove that none of S_5, S_6, S_7, A_5, A_6, and A_7 can have a normal subgroup of order 3 or 5.

27. What do the Sylow theorems tell you about any group of order 100?

28. Prove that a group of order 175 is Abelian.

29. Let G be a group with $|G| = 595 = 5 \cdot 7 \cdot 17$. Show that the Sylow 5-subgroup of G is normal in G and is contained in $Z(G)$.

30. One can construct a non-abelian group of order 21 of the form $\{a^i b^j\}$ where $|a| = 7$ and $|b| = 3$ and multiplication is defined using either the relation $ba = a^2 b$ or the relation $ba = a^4 b$. Use this information to find relationships among $|a|$, $|b|$, 2 and 4. One can construct a non-Abelian group of order 105 of the form $a^i b^j$ where $|a| = 35$ and $|b| = 3$ and multiplication is defined using either the relation $ba = a^{11} b$ or the relation $ba = a^{18} b$. Use this information to find relationships among $|a|, |b|, 11$, and 16.

31. Explain why there cannot exist a group G such that $[G : Z(G)] = 15$.

32. Let G be a finite group with more than one element and let H be the intersection of all non-trivial subgroups of G. What can you say about $|H|$?

33. Generalize the argument given in Example 6 to obtain a theorem about groups of order $p^2 q$, where p and q are distinct primes.

34. Prove that a group of order 375 has a subgroup of order 15.

35. Without using Theorem 23.6, prove that a group of order 15 is cyclic. (This exercise is referred to in the discussion about groups of order 30.)

36. Prove that a group of order 105 contains a normal subgroup of order 35.

37. Prove that a group of order 595 has a normal Sylow 17-subgroup.

38. Let G be a group of order 60. Show that G has exactly four elements of order 5 or exactly 24 elements of order 5. Which of these cases holds for A_5?

39. Show that the center of a group of order 60 cannot have order 4.

40. Suppose that G is a group of order 60 and G has a normal subgroup N of order 2. Show that

a. G has normal subgroups of orders 6, 10, and 30.

b. G has subgroups of orders 12 and 20.

c. G has a cyclic subgroup of order 30.

41. Let G be a group of order 60. If the Sylow 3-subgroup is normal, show that the Sylow 5-subgroup is normal.

42. Show that if G is a group of order 168 that has a normal subgroup of order 4, then G has a normal subgroup of order 28.

43. Suppose that p is prime and $|G| = p^n$. Show that G has normal subgroups of order p^k for all k between 1 and n (inclusive).

44. Suppose that G is a group of order p^n, where p is prime, and G has exactly one subgroup for each divisor of p^n. Show that G is cyclic.

45. Suppose that p is prime and $|G| = p^n$. If H is a proper subgroup of G, prove that $N(H) > H$. (This exercise is referred to in Chapter 24.)

46. If H is a finite subgroup of a group G and $x \in G$, prove that $|N(H)| = |N(xHx^{-1})|$.

47. Let H be a Sylow 3-subgroup of a finite group G and let K be a Sylow 5-subgroup of G. If 3 divides $|N(K)|$, prove that 5 divides $|N(H)|$.

48. Prove that a group of order 48 has a normal subgroup of order 16 or a normal subgroup of order 8.

49. Suppose that G is a finite group and G has a unique Sylow p-subgroup for each prime p. Prove that G is the internal direct product of its nontrivial Sylow p-subgroups. If each Sylow p-subgroup is cyclic, is G cyclic? If each Sylow p-subgroup is Abelian, is G Abelian?

50. Suppose that G is a finite group and G has exactly one subgroup for each divisor of $|G|$. Prove that G is cyclic.

51. Let G be a finite group and let H be a normal Sylow p-subgroup of G. Show that $\alpha(H) = H$ for all automorphisms α of G.

52. If H is a Sylow p-subgroup of a group, prove that $N(N(H)) = N(H)$.

53. Let p be a prime and H and K be Sylow p-subgroups of a group G. Prove that $|N(H)| = |N(K)|$.

54. Let G be a group of order p^2q^2, where p and q are distinct primes, $q \nmid p^2 - 1$, and $p \nmid q^2 - 1$. Prove that G is Abelian. List three pairs of primes that satisfy these conditions.

55. Let H be a normal subgroup of a group G. Show that H is the union of the conjugacy classes in G of the elements of H. Is this true when H is not normal in G?

56. Let G be a finite group and p be a prime that divides $|G|$. If H is a Sylow p-subgroup of $N(H)$, prove that H is a Sylow p-subgroup of G.

57. Show that a group of order 12 cannot have nine elements of order 2.

58. If $|G| = 36$ and G is non-Abelian, prove that G has more than one Sylow 2-subgroup or more than one Sylow 3-subgroup.

59. Let G be a non-Abelian group of order pq where p and q are primes and $p < q$. Prove that G has exactly $q + 1$ nontrivial proper subgroups.

60. Determine the groups of order 45.

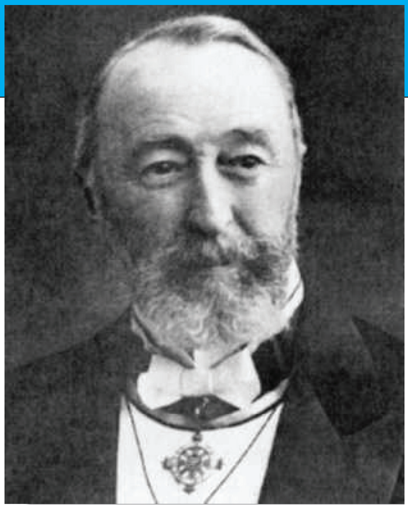
61. Explain why a group of order $4m$ where m is odd must have a sub- group isomorphic to Z_4 or $Z_2 \oplus Z_2$ but cannot have both a subgroup isomorphic to Z_4 and a subgroup isomorphic to $Z_2 \oplus Z_2$. Show that S_4 has a subgroup isomorphic to Z_4 and a subgroup isomorphic to $Z_2 \oplus Z_2$.

62. Let p be the smallest prime that divides the order of a finite group G. If H is a Sylow p-subgroup of G and is cyclic, prove that $N(H) = C(H)$.

63. Let G be a group of order $715 = 5 \cdot 11 \cdot 13$. Let H be a Sylow 13-subgroup of G and K be a Sylow 11-subgroup of G. Prove that H is contained in $Z(G)$. Can the argument you used to prove that H is contained in $Z(G)$ also be used to show that K is contained in $Z(G)$?

64. If H is a normal subgroup of a finite group G and $|H| = p^k$ for some prime p, show that H is contained in every Sylow p-subgroup of G.

Ludwig Sylow



Author: Wolfgang Gaschütz,
Source: Archives of the Mathematisches Forschungsinstitut Oberwolfach

Sylow's Theorem is 100 years old. In the course of a century this remarkable theorem has been the basis for the construction of numerous theories.

L. A. SHEMETKOV

LUDWIG SYLOW (pronounced "SEE-loe") was born on December 12, 1832, in Christiania (now Oslo), Norway. While a student at Christiania University, Sylow won a gold medal for competitive problem solving. In 1855, he became a high school teacher; despite the long hours required by his teaching duties, Sylow found time to study the papers of Abel. During the school year 1862–1863, Sylow received a temporary appointment at Christiania University and gave lectures on Galois theory and permutation groups. Among his students that year was the great mathematician Sophus Lie (pronounced "Lee"), after whom Lie algebras and Lie groups are named. From 1873 to 1881, Sylow, with some help from Lie, prepared a new edition of Abel's works.

In 1902, Sylow and Elling Holst published Abel's correspondence. Sylow's spectacular theorems came in 1872. Upon learning of Sylow's discovery, Jordan called it "one of the essential points in the theory of permutations." The results took on greater importance when the theory of abstract groups flowered in the late 19th century and early 20th century.

In 1869, Sylow was offered a professorship at Christiania University but turned it down. Upon Sylow's retirement from high school teaching at age 65, Lie mounted a successful campaign to establish a chair for Sylow at Christiania University. Sylow held this position until his death on September 7, 1918.

24 Finite Simple Groups

It is a widely held opinion that the problem of classifying finite simple groups is close to a complete solution. This will certainly be one of the great achievements of mathematics of this century.

Nathan Jacobson

It's supposed to be hard. If it wasn't hard, everyone would do it. The hard is what makes it great.

Jimmy Dugan from *A League of Their Own*

Historical Background

We now come to the El Dorado of finite group theory—the simple groups. Simple group theory is a vast and difficult subject; we call it the El Dorado of group theory because of the enormous effort put forth by hundreds of mathematicians over many years to discover and classify all finite simple groups. Let's begin our discussion with the definition of a simple group and some historical background.

Definition **Simple Group**
A group is *simple* if its only normal subgroups are the identity subgroup and the group itself.

The notion of a simple group was introduced by Galois about 190 years ago. The simplicity of A_5, the group of even permutations on five symbols, played a crucial role in his proof that there is not a solution by radicals of the general fifth-degree polynomial (i.e., there is no "quintic formula"). But what makes simple groups important in the theory of groups? They are important because they play a role in group theory somewhat analogous to that of primes in number theory or the elements in chemistry; that is, they serve as the building blocks for all groups. These building blocks may be determined in the following way. Given a finite group G, choose a proper normal subgroup G_1 of $G = G_0$ of largest order. Then the factor group G_0/G_1 is simple, and we next choose a proper normal subgroup G_2 of G_1 of largest order. Then G_1/G_2 is also simple, and we continue in this fashion until we arrive at $G_n = \{e\}$. The simple groups $G_0/G_1, G_1/G_2, \ldots, G_{n-1}/G_n$ are called the *composition factors* of G. More than 100 years ago, Jordan and Hölder proved that these factors are independent of the choices of the normal subgroups made in the process described. In a certain sense, a group can be reconstructed from its composition factors, and many of the properties of a group are determined by the nature of its composition factors. This and the fact that many questions about finite groups can be reduced (by induction) to questions about simple groups make clear the importance of determining all finite simple groups.

Just which groups are the simple ones? The Abelian simple groups are precisely Z_n, where $n = 1$ or n is prime. This follows directly from the corollary in Chapter 11. In contrast, it is extremely difficult to describe the non-Abelian simple groups. Their discovery began slowly and aimlessly. The best we can do here is to give a few examples and mention a few words about their discovery. It was Galois in 1831 who first observed that A_n is simple for all $n \geq 5$. The next discoveries were made by Jordan in 1870, when he found four infinite families of simple matrix groups over the field Z_p, where p is prime. One such family is the factor group $SL(n, Z_p)/Z(SL(n, Z_p))$, except when $n = 2$ and $p = 2$ or $p = 3$. Between the years 1892 and 1905, the American mathematician Leonard Dickson (see Chapter 21 for a biography) generalized Jordan's results to arbitrary finite fields and discovered several new infinite families of simple groups. About the same time, it was shown by G. A. Miller and F. N. Cole that a family of five groups first described by E. Mathieu in 1861 were in fact simple groups. Since these five groups were constructed by ad hoc methods that did not yield infinitely many possibilities, like A_n or the matrix groups over finite fields, they were called "sporadic."

The next important discoveries came in the 1950s. In that decade, many new infinite families of simple groups were found, and the initial steps down the long and winding road that led to the complete classification of all finite simple groups were taken. The first step was Richard Brauer's observation that the centralizer of an element of order 2 was an important tool for studying simple groups. A few years later, John Thompson, in his Ph.D. dissertation, introduced the crucial idea of studying the normalizers of various subgroups of prime-power order.

In the early 1960s came the momentous Feit–Thompson Theorem, which says that a non-Abelian simple group must have even order.[1] This property was first conjectured around 1900 by one of the pioneers of modern group theoretic methods, the Englishman William Burnside (see Chapter 27 for a biography). The proof of the Feit–Thompson Theorem filled an entire issue of a journal, 255 pages in all (see Figure 24.1). Writing in 2001 simple group theory expert Ronald Solomon said the theorem and its proof were "a moment in the evolution of finite group theory analogous to the emergence of fish onto dry land."

This result provided the impetus to classify the finite simple groups—that is, a program to discover all finite simple groups and *prove* that there are no more to be found. Throughout the 1960s, the methods introduced in the Feit–Thompson proof were generalized and improved with great success by many mathematicians. Moreover, between 1966 and 1975, 19 new sporadic simple groups were discovered. Despite many spectacular achievements, research in simple group theory in the 1960s was haphazard, and the decade ended with many people believing that the classification would never be completed. (The pessimists feared that the sporadic simple groups would foil all attempts. The anonymously written "song" on the next page captures the spirit of the times.) Others, more optimistic, were predicting that it would be accomplished in the 1990s.

[1]The Feit-Thompson Theorem is so famous that like Lagrange's Theorem or the Sylow's Theorems, most research papers do not even bother to provide citation information in the references.

—Oh, what are the orders of all simple groups?
I speak of the honest ones, not of the loops.
It seems that old Burnside their orders has guessed
Except for the cyclic ones, even the rest.
CHORUS: Finding all groups that are simple is no simple task.
Groups made up with permutes will produce some more:
For A_n is simple, if n exceeds 4.
Then, there was Sir Matthew who came into view
Exhibiting groups of an order quite new.
Still others have come on to study this thing. Of Artin and Chevalley now we shall sing. With matrices finite they made quite a list The question is: Could there be others they've missed?
Suzuki and Ree then maintained it's the case
That these methods had not reached the end of the chase.
They wrote down some matrices, just four by four.
That made up a simple group. Why not make more?
And then came the opus of Thompson and Feit
Which shed on the problem remarkable light. A group, when the order won't factor by two, Is cyclic or solvable. That's what is true.
Suzuki and Ree had caused eyebrows to raise, But the theoreticians they just couldn't faze.
Their groups were not new: if you added a twist,
You could get them from old ones with a flick of the wrist.
Still, some hardy souls felt a thorn in their side.
For the five groups of Mathieu all reason defied;
Not A_n, not twisted, and not Chevalley, They called them sporadic and filed them away.
Are Mathieu groups creatures of heaven or hell?
Zvonimir Janko determined to tell.
He found out [a new sporadic simple group] that nobody wanted to know: The masters had missed 1 7 5 5 6 0.
The floodgates were opened! New groups were the rage!
(And twelve or more sprouted, to greet the new age.)
By Janko and Conway and Fischer and Held, McLaughlin, Suzuki, and Higman, and Sims.
No doubt you noted the last lines don't rhyme.
Well, that is, quite simply, a sign of the time.
There's chaos, not order, among simple groups;
And maybe we'd better go back to the loops.

The 1970s began with Thompson receiving the Fields Medal

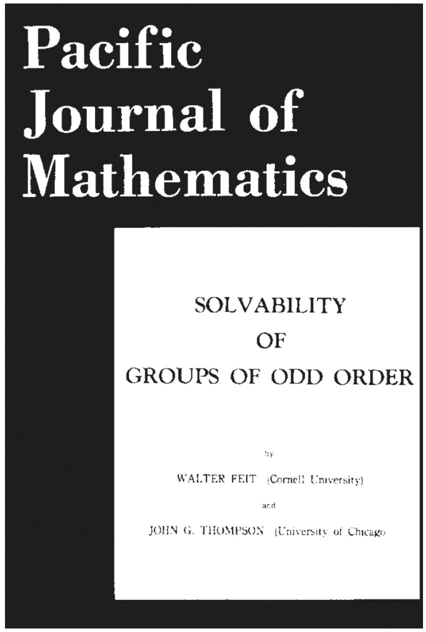


Figure 24.1 W. Feit and J. G. Thompson, "Solvability of Groups of Odd Order," Pacific Journal of Mathematics 13 (1963): 775–1029.

for his fundamental contributions to simple group theory. This honor is among the highest forms of recognition that a mathematician can receive (more information about the Fields Medal is given near the end of this chapter). Within a few years, three major events took place that ultimately led to the classification. First, Thompson published what is regarded as the single most important paper in simple group theory—the N-group paper. In this magnum opus, Thompson introduced many fundamental techniques and supplied a model for the classification of a broad family of simple groups. Second, Daniel Gorenstein produced an elaborate outline for the classification, which he delivered in a series of lectures at the University of Chicago in 1972. Here a program for the overall proof was laid out. The army of researchers now had a battle plan and a commander-in-chief. But this army still needed more and better weapons. Thus came the third critical development: the involvement of Michael Aschbacher. In a dazzling series of papers, Aschbacher combined his own insight with the methods of Thompson, which had been generalized throughout the 1960s, and a geometric approach pioneered by Bernd Fischer to achieve one brilliant result after another in rapid succession. In fact, so much progress was made by Aschbacher and others that by 1976, it was clear to nearly everyone involved that enough techniques had been developed to complete the classification. Only details remained.

The 1980s were ushered in with Aschbacher following in the footsteps of Feit and Thompson by winning the American Mathematical Society's Cole Prize in algebra (see the last section of this chapter).

A week later, Robert L. Griess made the spectacular announcement that he had constructed the "Monster."[2] The Monster is the largest of the sporadic simple groups. It is very, very, very large. In fact, it has vastly more elements than there are atoms on the earth! Its order is

808,017,424,794,512,875,886,459,904,961,710,757,005,754,368,000,000,000

(hence, the name). This is approximately 8×10^{53} (roughly the size of the simple group A_{44}). The Monster is a group of rotations in 196,883 dimensions. Thus, each element can be expressed as a $196{,}883 \times 196{,}883$ matrix.

At the annual meeting of the American Mathematical Society in 1981, Gorenstein announced that the "Twenty-Five Years' War" to classify all the finite simple groups was over. Group theorists at long last had a list of all finite simple groups and a proof that the list was complete. The proof was spread out over hundreds of papers—both published and unpublished—and ran more than 10,000 pages in length. Because of the proof's extreme length and complexity, and the fact that some key parts of it had not been published, there was some concern in the mathematics community that the classification was not a certainty. By the end of the decade, group theorists had concluded that there was indeed a gap in the unpublished work that would be difficult to rectify. In the mid-1990s, Aschbacher and Stephen Smith began work on this problem. And, lo and behold, at the annual meeting of the American Mathematical Society in 2004, Aschbacher announced

[2]The name was coined by John H. Conway. Griess called the group the "Friendly Giant." In 2010 the American Mathematical Society awarded Griess the Leroy P. Steele Seminal Contribution to Research Prize for his construction of the Monster.

the coup de grâce that he and Smith had completed the classification. Their monograph is over 1200 pages in length. Ronald Solomon, writing in *Mathematical Reviews*, called it "an amazing tour de force" and a "major milestone in the history of finite group theory." Aschbacher concluded his remarks by saying that he would not bet his house that the proof is now error free.

Several people who played a central role in the classification are working on a "second generation" proof that will be much shorter and more comprehensible.

The Herculean undertaking by group theorists from the 1950s through the mid 2000s to discover and classify the finite simple groups has parallels in both chemistry and physics. The end result is analogous to the completion of the periodic table for chemical elements and the effort is reminiscent of the quest of physicists from the 1920s through the 1970s to develop quantum field theory. An image "periodic table" for finite simple groups is availabe at https://irandrus.files.wordpress.com/2012/06/periodic-table-of-groups.pdf

Nonsimplicity Tests

In view of the fact that simple groups are the building blocks for all groups, it is surprising how scarce the non-Abelian simple groups are. For example, A_5 is the only one whose order is less than 168; there are only five non-Abelian simple groups of order less than 1000 and only 56 of order less than 1,000,000. In this section, we give a few theorems that are useful in proving that a particular integer is not the order of a non-Abelian simple group. Our first such result is an easy arithmetic test that comes from combining Sylow's Third Theorem and the fact that groups of prime-power order have nontrivial centers.

Theorem 24.1 Sylow Test for Nonsimplicity

Let n be a positive integer that is not prime, and let p be a prime divisor of n. If 1 is the only divisor of n that is equal to 1 modulo p, then there does not exist a simple group of order n.

PROOF If n is a prime-power, then a group of order n has a nontrivial center and, therefore, is not simple. If n is not a prime-power, then every Sylow subgroup is proper, and, by Sylow's Third Theorem, we know that the number of Sylow p-subgroups of a group of order n is equal to 1 modulo p and divides n. Since 1 is the only such number, the Sylow p-subgroup is unique, and therefore, by the corollary to Sylow's Third Theorem, it is normal. ∎

How good is this test? Well, applying this criterion to all the nonprime integers between 1 and 200 would leave only the following integers as possible orders of finite non-Abelian simple groups: 12, 24, 30, 36, 48, 56, 60, 72, 80, 90, 96, 105, 108, 112, 120, 132, 144, 150, 160, 168, 180, and 192. (In fact, computer experiments have revealed that for large intervals, say, 500 or more, this test eliminates more than 90% of the nonprime integers as possible orders of simple groups.

Our next test rules out 30, 90, and 150.

Theorem 24.2 2· Odd Test

> ***An integer of the form $2 \cdot n$, where n is an odd number greater than 1, is not the order of a simple group.***

PROOF Let G be a group of order $2n$, where n is odd and greater than 1. Recall from the proof of Cayley's Theorem (Theorem 6.5) that the mapping $g \to T_g$ is an isomorphism from G to a permutation group on the elements of G [where $T_g(x) = gx$ for all x in G]. Since $|G| = 2n$, Cauchy's Theorem guarantees that there is an element g in G of order 2. Then, when the permutation T_g is written in disjoint cycle form, each cycle must have length 1 or 2; otherwise, $|g| \neq 2$. But T_g can contain no 1-cycles, because the 1-cycle (x) would mean $x = T_g(x) = gx$, so $g = e$. Thus, in cycle form, T_g consists of exactly n transpositions, where n is odd. Therefore, T_g is an odd permutation. This means that the set of even permutations in the image of G is a normal subgroup of index 2. (See Exercise 27 in Chapter 5 and Exercise 9 in Chapter 9.) Hence, G is not simple. ∎

The next theorem is a broad generalization of Cayley's Theorem. We will make heavy use of its two corollaries.

Theorem 24.3 Generalized Cayley Theorem

Let G be a group and let H be a subgroup of G. Let S be the group of all permutations of the left cosets of H in G. Then there is a homomorphism from G into S whose kernel lies in H and contains every normal subgroup of G that is contained in H.

PROOF For each $g \in G$, define a permutation T_g of the left cosets of H by $T_g(xH) = gxH$. As in the proof of Cayley's Theorem, it is easy to verify that the mapping of $\alpha : g \to T_g$ is a homomorphism from G into S.

Now, if $g \in \operatorname{Ker} \alpha$, then T_g is the identity map, so $H = T_g(H) = gH$, and, therefore, g belongs to H. Thus, $\operatorname{Ker} \alpha \subseteq H$. On the other hand, if K is normal in G and $K \subseteq H$, then for any $k \subseteq K$ and any x in G, there is an element k' in K such that $kx = xk'$. Thus,

$$T_k(xH) = kxH = xk'H = xH$$

and, therefore, T_k is the identity permutation. This means that $k \in \operatorname{Ker} \alpha$. We have proved, then, that every normal subgroup of G contained in H is also contained in $\operatorname{Ker} \alpha$. ■

EXAMPLE 1 We illustrate the Generalized Cayley's Theorem with D_4 and the subgroup $\mathcal{H} = \{R_0, H\}$. Here S is the group of 24 permutations of the four cosets: $\mathcal{H} = \{R_0, H\}, R_{90}\mathcal{H} = \{R_{90}, D'\}, R_{180}\mathcal{H} = \{R_{180}, V\}, R_{270}\mathcal{H} = \{R_{270}, D\}$.

In matrix form the images of R_{90} and $\mathcal{H}$ are:

$$T_{R_{90}} = \begin{bmatrix} \mathcal{H} & R_{90}\mathcal{H} & R_{180}\mathcal{H} & R_{270}\mathcal{H} \\ R_{90}\mathcal{H} & R_{180}\mathcal{H} & R_{270}\mathcal{H} & \mathcal{H} \end{bmatrix} \text{ and}$$

$$T_H = \begin{bmatrix} \mathcal{H} & R_{90}\mathcal{H} & R_{180}\mathcal{H} & R_{270}\mathcal{H} \\ H\mathcal{H} & HR_{90}\mathcal{H} & HR_{180}\mathcal{H} & HR_{270}\mathcal{H} \end{bmatrix}$$

$$= \begin{bmatrix} \mathcal{H} & R_{90}\mathcal{H} & R_{180}\mathcal{H} & R_{270}\mathcal{H} \\ \mathcal{H} & R_{270}\mathcal{H} & R_{180}\mathcal{H} & R_{90}\mathcal{H} \end{bmatrix}.$$

In cycle form (we insert commas for clarity) the images of R_{90} and $\mathcal{H}$ are:

$T_{R_{90}} = (\mathcal{H}, R_{90}\mathcal{H}, R_{180}\mathcal{H}, R_{270}\mathcal{H})$ and $T_H = (\mathcal{H})(R_{180}\mathcal{H})(R_{90}\mathcal{H}, R_{270}\mathcal{H}) = (R_{90}\mathcal{H}, R_{270}\mathcal{H})$. ■

As a consequence of Theorem 24.3, we obtain the following very powerful arithmetic test for nonsimplicity.

Corollary 1 Index Theorem

If G is a finite group and H is a proper subgroup of G such that $|G|$ does not divide $|G : H|!$, then H contains a nontrivial normal subgroup of G. In particular, G is not simple.

PROOF Let α be the homomorphism given in Theorem 24.3. Then Ker α is a normal subgroup of G contained in H, and $G/\text{Ker } \alpha$ is isomorphic to a subgroup of S. Thus, $|G/\text{Ker } \alpha| = |G|/|\text{Ker } \alpha|$ divides $|S| = |G : H|!$. Since $|G|$ does not divide $|G : H|!$, the order of Ker α must be greater than 1. ∎

Corollary 2 Embedding Theorem

If a finite non-Abelian simple group G has a subgroup of index n, then G is isomorphic to a subgroup of A_n.

PROOF Let H be the subgroup of index n, and let S_n be the group of all permutations of the n left cosets of H in G. By the Generalized Cayley Theorem, there is a nontrivial homomorphism from G into S_n. Since G is simple and the kernel of a homomorphism is a normal subgroup of G, we see that the mapping from G into S_n is one-to-one, so that G is isomorphic to some subgroup of S_n. Recall from Exercise 27 in Chapter 5 that any subgroup of S_n consists of even permutations only or half even and half odd. If G were isomorphic to a subgroup of the latter type, the even permutations would be a normal subgroup of index 2 (see Exercise 9 in Chapter 9), which would contradict the fact that G is simple. Thus, G is isomorphic to a subgroup of A_n. ∎

Using the Index Theorem with the largest Sylow subgroup for H reduces our list of possible orders of non-Abelian simple groups still further. For example, let G be any group of order $80 = 16 \cdot 5$. We may choose H to be a subgroup of order 16. Since 80 is not a divisor of 5!, there is no simple group of order 80. The same argument applies to 12, 24, 36, 48, 96, 108, 160, and 192, leaving only 56, 60, 72, 105, 112, 120, 132, 144, 168, and 180 as possible orders of non-Abelian simple groups up to 200. Let's consider these orders. Quite often we may use a counting argument to eliminate an integer. Consider 56. By Sylow's Third Theorem, we know that a simple group of order $56 = 8 \cdot 7$ would contain eight Sylow 7-subgroups and seven Sylow 2-subgroups. Now, any two Sylow p-subgroups that have order p must intersect in only the identity. So the union of the eight Sylow 7-subgroups yields 48 elements of order 7, and the union of any two Sylow 2-subgroups gives at least $8 + 8 - 4 = 12$ new elements. But there are only 56 elements in all. This contradiction shows that there is not a simple group of order 56. An analogous argument also eliminates the integers 105 and 132.

So, our list of possible orders of non-Abelian simple groups up to 200 is down to 60, 72, 112, 120, 144, 168, and 180. Of these, 60 and 168 do correspond to simple groups. The others can be eliminated with a bit of razzle-dazzle.

The easiest case to handle is $112 = 2^4 \cdot 7$. Suppose there were a simple group G of order 112. A Sylow 2-subgroup of G must have index 7. So, by the Embedding Theorem, G is isomorphic to a subgroup of A_7. But 112 does not divide $|A_7|$, which is a contradiction.

Another easy case is 72. This case was done in Example 5 in Chapter 23 but we eliminate it using the Index Theorem. Recall from Exercise 9 in Chapter 23 that if we denote the number of Sylow p-subgroups of a group G by n_p, then $n_p = |G : N(H)|$, where H is any Sylow p-subgroup of G, and $n_p \bmod p = 1$. It follows, then, that in a simple group of order 72, we have $n_3 = 4$, which is impossible, since 72 does not divide 4!

Next consider the possibility of a simple group G of order $144 = 9 \cdot 16$. By the Sylow theorems, we know that $n_3 = 4$ or 16 and $n_2 \geq 3$. The Index Theorem rules out the case where $n_3 = 4$, so we know that there are 16 Sylow 3-subgroups. Now, if every pair of Sylow 3-subgroups had only the identity in common, a straightforward counting argument would produce more than 144 elements. So, let H and H' be a pair of Sylow 3-subgroups whose intersection has order 3. Then $H \cap H'$ is a subgroup of both H and H' and, by the corollary to Theorem 23.2 (or by Exercise 45 in Chapter 23), we see that $N(H \cap H')$ must contain both H and H' and, therefore, the set HH'. (HH' need not be a subgroup.)

Thus,

$$|N(H \cap H')| \geq |HH'| = \frac{|H||H'|}{|H \cap H'|} = \frac{9 \cdot 9}{3} = 27.$$

Now, we have three arithmetic conditions on $k = |N(H \cap H')|$. We know that 9 divides $k; k$ divides 144; and $k \geq 27$. Clearly, then, $k \geq 36$, and so $|G : N(H \cap H')| \leq 4$. The Index Theorem now gives us the desired contradiction.

Finally, suppose that G is a non-Abelian simple group of order $180 = 2^2 \cdot 3^2 \cdot 5$. Then $n_5 = 6$ or 36 and $n_3 = 10$ ($n_3 = 4$ is ruled out by the Index Theorem). First, assume that $n_5 = 36$. Then G has $36 \cdot 4 = 144$ elements of order 5. Now, if each pair of the Sylow 3-subgroups intersects in only the identity, then there are 80 more elements in the group, which is a contradiction. So, we may assume that there are two Sylow 3-subgroups L_3 and L_3' whose intersection has order 3. Then, as was the case for order 144, we have

$$|N(L_3 \cap L_3')| \geq |L_3 L_3'| = \frac{9 \cdot 9}{3} = 27.$$

Thus, $|N(L_3 \cap L_3')| = 9 \cdot k$, where $k \geq 3$ and k divides 20. Clearly, then, $|N(L_3 \cap L_3')| \geq 36$ and therefore $|G : N(L_3 \cap L_3')| \leq 5$.

The Index Theorem now gives us another contradiction. Hence, we may assume that $n_5 = 6$. In this case, we let H be the normalizer of a Sylow 5-subgroup of G. By Sylow's Third Theorem, we have $6 = |G : H|$, so that $|H| = 30$. In Chapter 23, we proved that every group of order 30 has an element of order 15. On the other hand, since $n_5 = 6$, G has a subgroup of index 6 and the Embedding Theorem tells us that G is isomorphic to a subgroup of A_6. But A_6 has no element of order 15. (See Exercise 23 in Chapter 5.)

Unfortunately, the argument for 120 is fairly long and complicated. However, no new techniques are required to do it. We leave this as an exercise (Exercise 17). Some hints are given in the answer section.

The Simplicity of A_5

Once 120 has been disposed of, we will have shown that the only integers between 1 and 200 that can be the orders of non-Abelian simple groups are 60 and 168. For completeness, we will now prove that A_5, which has order 60, is a simple group. A similar argument can be used to show that the factor group $SL(2, Z_7)/Z(SL(2, Z_7))$

is a simple group of order 168. (See Exercise 33.) [This group is denoted by $PSL(2, Z_7)$.]

If A_5 had a nontrivial proper normal subgroup H, then $|H|$ would be equal to 2, 3, 4, 5, 6, 10, 12, 15, 20, or 30. By Exercise 65 in Chapter 5, A_5 has 24 elements of order 5, 20 elements of order 3, and no elements of order 15. Now, if $|H|$ is equal to 3, 6, 12, or 15, then $|A_5/H|$ is relatively prime to 3, and by Exercise 61 in Chapter 9, H would have to contain all 20 elements of order 3. If $|H|$ is equal to 5, 10, or 20, then $|A_5/H|$ is relatively prime to 5, and, therefore, H would have to contain the 24 elements of order 5. If $|H| = 30$, then $|A_5/H|$ is relatively prime to both 3 and 5, and so H would have to contain all the elements of orders 3 and 5. Finally, if $|H| = 2$ or $|H = 4$, then $|A_5/H| = 30$ or $|A_5/H| = 15$. But we know from our results in Chapter 23 that any group of order 30 or 15 has an element of order 15. However, since A_5 contains no such element, neither does A_5/H. This proves that A_5 is simple.

The simplicity of A_5 was known to Galois in 1830, although the first formal proof was done by Jordan in 1870. A few years later, Felix Klein showed that the group of rotations of a regular icosahedron is simple and, therefore, isomorphic to A_5 (see Exercise 27). Since then it has frequently been called the *icosahedral group*. Klein was the first to prove that there is a simple group of order 168.

The problem of determining which integers in a certain interval are possible orders for finite simple groups goes back to 1892, when Hölder went up to 200. His arguments for the integers 144 and 180 alone used up 10 pages. By 1975, this investigation had been pushed to well beyond 1,000,000. Of course, now that all finite simple groups have been classified, this problem is merely a historical curiosity.

Putting together that Cayley's Theorem tells us that every group is isomorphic to a subgroup of S_n for some n, and the fact that A_n is simple for all $n \geq 5$, we have from Exercise 73 in Chapter 6, which says S_n is a subgroup of A_{n+2}, that every group is isomorphic to a subgroup of a finite simple group.

The Fields Medal

Among the highest awards for mathematical achievement is the Fields Medal. Two to four such awards are bestowed at the opening session of the International Congress of Mathematicians, held once every four years. Although the Fields Medal is considered by many mathematicians to be the equivalent of the Nobel Prize, there are great differences between these awards. Besides the huge disparity in publicity and monetary value associated with the two honors, the Fields Medal is restricted to those under 40 years of age.[3] This tradition stems from John Charles Fields's stipulation, in his will establishing the medal, that the awards should be "an encouragement for further achievement." This restriction precluded Andrew Wiles from winning the Fields Medal for his proof of Fermat's Last Theorem.

More details about the Fields Medal can be found at **http://www. wikipedia.com**.

The Fields Medal

Three-minute video clips of the recipients of the 2014 and 2018 Fields medal winners talking about their work are available at: https://www.youtube.com/results?search_query=2014+fields+

[3]"Take the sum of human achievement in action, in science, in art, in literature—subtract the work of the men above forty, and while we should miss great treasures, even priceless treasures, we would practically be where we are today. . . . The effective, moving, vitalizing work of the world is done between the ages of twenty-five and forty." Sir William Osler (1849–1919), *Life of Sir William Osler,* Vol. I, Chap. 24 (The Fixed Period).

medal
https://www.youtube.com/results?search_query=2018+fields+medalists

The Cole Prize

Approximately every five years, beginning in 1928 (changed to every three years in 2003), the American Mathematical Society awards one or two Cole Prizes for research in algebra and one or two Cole Prizes for research in algebraic number theory. The prize was founded in honor of Frank Nelson Cole on the occasion of his retirement as secretary of the American Mathematical Society. In view of the fact that Cole was one of the first people interested in simple groups, it is interesting to note that no fewer than six recipients of the prize—Dickson, Chevalley, Brauer, Feit, Thompson, and Aschbacher—have made fundamental contributions to simple group theory at some time in their careers. Recently the time between Cole Prizes was reduced to three years.

Exercises

If you don't learn from your mistakes, there's no sense making them.
Herbert V. Prochnow

1. Prove that there is no simple group of order $210 = 2 \cdot 3 \cdot 5 \cdot 7$.

2. Prove that there is no simple group of order $280 = 2^3 \cdot 5 \cdot 7$.

3. Prove that there is no simple group of order $216 = 2^3 \cdot 3^3$.

4. Prove that there is no simple group of order $300 = 2^2 \cdot 3 \cdot 5^2$.

5. Prove that there is no simple group of order $525 = 3 \cdot 5^2 \cdot 7$.

6. Prove that there is no simple group of order $315 = 3^2 \cdot 5 \cdot 7$.

7. Prove that there is no simple group of order $528 = 2^4 \cdot 3 \cdot 11$.

8. Prove that there is no simple group of order $540 = 2^2 \cdot 3^3 \cdot 5$.

9. Prove that there is no simple group of order $396 = 2^2 \cdot 3^2 \cdot 11$.

10. Prove that there is no simple group of order n, where $201 \leq n \leq 235$ and n is not prime.

11. Without using the Generalized Cayley Theorem or its corollaries, prove that there is no simple group of order 112.

12. Use the simplicity of A_n for $n \geq 5$ to prove that for $n \geq 5$, A_n is the unique subgroup of S_n of index 2.

13. Use the simplicity of A_n for $n \geq 5$ to prove that for $n \geq 5$, A_n is a proper subgroup of A_{n+1} of maximum order.

14. Show that there is no simple group of order pqr, where p, q, and r are primes (p, q, and r need not be distinct).

15. Show that A_5 does not contain a subgroup of order 30, 20, or 15.

16. Prove that that A_6 has no subgroup of order 120, 90, or 72.

17. Prove that there is no simple group of order $120 = 2^3 \cdot 3 \cdot 5$. (This exercise is referred to in this chapter.)

18. Prove that if G is a finite group and H is a proper normal subgroup of largest order, then G/H is simple.

19. Suppose that H is a subgroup of a finite group G and that $|H|$ and $(|G : H| - 1)!$ are relatively prime. Prove that H is normal in G. What does this tell you about a subgroup of index 2 in a finite group?

20. Suppose that p is the smallest prime that divides $|G|$. Show that any subgroup of index p in G is normal in G.

21. Prove that the only nontrivial proper normal subgroup of S_5 is A_5. (This exercise is referred to in Chapter 30.)

22. Prove that a simple group of order 60 has a subgroup of order 6 and a subgroup of order 10.

23. Prove that A_4 is not simple.

24. Show that the permutations (12) and (12345) generate S_5.

25. Suppose that a subgroup H of S_5 contains a 5-cycle and a 2-cycle. Show that $H = S_5$. (This exercise is referred to in Chapter 30.)

26. Suppose that G is a finite simple group and contains subgroups H and K such that $|G : H|$ and $|G : K|$ are prime. Show that $|H| = |K|$.

27. Show that (up to isomorphism) A_5 is the only simple group of order 60. (This exercise is referred to in this chapter.)

28. Prove that a simple group cannot have a subgroup of index 4.

29. Prove that there is no simple group of order p^2q, where p and q are odd primes and $q > p$.

30. If a simple group G has a subgroup K that is a normal subgroup of two distinct maximal subgroups, prove that $K = \{e\}$.

31. Show that a finite group of even order that has a cyclic Sylow 2-subgroup is not simple.

32. Show that S_5 does not contain a subgroup of order 40 or 30.

33. Show that $PSL(2, Z_7) = SL(2, Z_7)/Z(SL(2, Z_7))$, which has order 168, is a simple group. (This exercise is referred to in this chapter.)

34. Prove that if there is a non-trivial homomorphism from a finite group G to S_n where $|G| > n!$, then G is not simple.

35. Use the techniques from this chapter to prove that a group of order 36 has a normal subgroup of order 3 or a normal subgroup of order 9. Does your argument generalize to prove the statement: A group of order $4p^n$ where p is an odd prime has a normal subgroup of order p^{n-1} or a normal subgroup of order p^n?

36. Let G be a group of order 24. Use the techniques from this chapter to prove that G has a normal subgroup of order 8 or a normal subgroup of order 4. Does your argument generalize to prove the statement: A group of order p^nq where p and q are distinct primes with $p^2 > q$ has a normal subgroup of order p^{n-1} or p^n?

37. Use Theorem 24.3 to prove that if G is a finite group and H is a subgroup of G with $|G : H| = p$, where p is the smallest prime divisor of G, then H is normal in G.

Michael Aschbacher

Courtesy of Pam Aschbacher

Fresh out of graduate school, he [Aschbacher] had just entered the field, and from that moment he became the driving force behind my program. In rapid succession he proved one astonishing theorem after another. Although there were many other major contributors to this final assault, Aschbacher alone was responsible for shrinking my projected 30-year timetable to a mere 10 years.

DANIEL GORENSTEIN,
Scientific American

MICHAEL ASCHBACHER was born on April 8, 1944, in Little Rock, Arkansas. Shortly after his birth, his family moved to Illinois, where his father was a professor of accounting and his mother was a high school English teacher. When he was nine years old, his family moved to East Lansing, Michigan; six years later, they moved to Los Angeles.

After high school, Aschbacher enrolled at the California Institute of Technology. In addition to his schoolwork, he passed the first four actuary exams and was employed for a few years as an actuary, full-time in the summers and part-time during the academic year. Two of the Caltech mathematicians who influenced him were Marshall Hall and Donald Knuth. In his senior year, Aschbacher took abstract algebra but showed little interest in the course. Accordingly, he received a grade of C.

In 1966, Aschbacher went to the University of Wisconsin for a Ph.D. degree. He completed his dissertation in 1969, and, after spending one year as an assistant professor at the University of Illinois, he returned to Caltech and quickly moved up to the rank of professor.

Aschbacher's dissertation work in the area of combinatorial geometries had led him to consider certain group theoretic questions. Gradually, he turned his attention more and more to purely group theoretic problems, particularly those bearing on the classification of finite simple groups. The 1980 Cole Prize Selection Committee said of one of his papers, "[It] lifted the subject to a new plateau and brought the classification within reach." Aschbacher has been elected to the National Academy of Sciences, the American Academy of Sciences, and the vice presidency of the American Mathematical Society. In 2011, Aschbacher received the $75,000 Rolf Schock Prize from the Royal Swedish Academy of Sciences for "his fundamental contributions to one of the largest mathematical projects ever, the classification of finite simple groups." In 2012, he shared the $100,000 Wolf Prize for his work in the theory of finite groups and shared the American Mathematical Society's Steele Prize for Exposition.

Daniel Gorenstein

Courtesy of the Clark University Archives

Gorenstein was one of the most influential mathematicians of the last few decades.

MICHAEL ASCHBACHER,
Notices of the American Mathematical Society

Daniel Gorenstein was born in Boston on January 1, 1923. Upon graduating from Harvard in 1943 during World War II, Gorenstein was offered an instructorship at Harvard to teach mathematics to army personnel. After the war ended, he began graduate work at Harvard. He received his Ph.D. degree in 1951, working in algebraic geometry under Oscar Zariski. It was in his dissertation that he introduced the class of rings that is now named after him. In 1951, Gorenstein took a position at Clark University in Worcester, Massachusetts, where he stayed until moving to Northeastern University in 1964. From 1969 until his death on August 26, 1992, he was at Rutgers University.

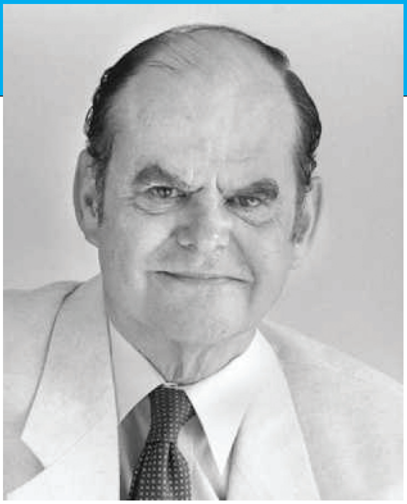
In 1957, Gorenstein switched from algebraic geometry to finite groups, learning the basic material from I. N. Herstein while collaborating with him over the next few years. A milestone in Gorenstein's development as a group theorist came during 1960–1961, when he was invited to participate in a "Group Theory Year" at the University of Chicago.

It was there that Gorenstein, assimilating the revolutionary techniques then being developed by John Thompson, began his fundamental work that contributed to the classification of finite simple groups.

Through his pioneering research papers, his dynamic lectures, his numerous personal contacts, and his influential book on finite groups, Gorenstein became the leader in the 25-year effort, by hundreds of mathematicians, that led to the classification of the finite simple groups.

Among the honors received by Gorenstein are the Steele Prize from the American Mathematical Society and election to membership in the National Academy of Sciences and the American Academy of Arts and Sciences.

John Thompson



Courtesy of University of Florida

There seemed to be no limit to his power.
DANIEL GORENSTEIN

JOHN G. THOMPSON was born on October 13, 1932, in Ottawa, Kansas. In 1951, he entered Yale University as a divinity student, but he switched to mathematics in his sophomore year. In 1955, he began graduate school at the University of Chicago, he obtained his Ph.D. degree four years later. After one year on the faculty at Harvard, Thompson returned to Chicago. He remained there until 1968, when he moved to Cambridge University in England. In 1993, Thompson accepted an appointment at the University of Florida.

Thompson's brilliance was evident early. In his dissertation, he verified a 50-year-old conjecture about finite groups possessing a certain kind of automorphism. (An article about his achievement appeared in *The New York Times.*) The novel methods Thompson used in his dissertation foreshadowed the revolutionary ideas he would later introduce in the Feit–Thompson paper and the classification of minimal simple groups (simple groups that contain no proper non-Abelian simple subgroups). The assimilation and extension of Thompson's methods by others throughout the 1960s and 1970s ultimately led to the classification of finite simple groups.

In the late 1970s, Thompson made significant contributions to coding theory, the theory of finite projective planes, and the theory of modular functions. His work on Galois groups is considered the most important in the field in the last half of the 20th century.

Among Thompson's many honors are the Cole Prize in algebra and the Fields Medal. He was elected to the National Academy of Sciences in 1967, the Royal Society of London in 1979, the Sylvester Medal in 1985, the Wolf Prize and the Poincaré Prize in 1992, the American Academy of Arts and Sciences in 1998, the National Medal of Science in 2000, and the De Morgan Medal in 2013. In 2008, he was a cowinner of the \$1,000,000 Abel Prize given by the Norwegian Academy of Science and Letters.

25 Generators and Relations

> One cannot escape the feeling that these mathematical formulae have an independent existence and an intelligence of their own, that they are wiser than we are, wiser even than their discoverers, that we get more out of them than we originally put into them.
>
> Heinrich Hertz

> I presume that to the unintiated the formulae will appear cold and cheerless.
>
> Benjamin Pierce

Motivation

In this chapter, we present a convenient way to define a group with certain prescribed properties. Simply put, we begin with a set of elements that we want to generate the group, and a set of equations (called *relations*) that specify the conditions that these generators are to satisfy. Among all such possible groups, we will select one that is as large as possible. This will uniquely determine the group up to isomorphism.

To provide motivation for the theory involved, we begin with a concrete example. Consider D_4, the group of symmetries of a square. Recall that $R = R_{90}$ and H, a reflection across a horizontal axis, generate the group. Observe that R and H are related in the following ways:

$$R^4 = H^2 = (RH)^2 = R_0 \quad \text{(the identity).} \tag{25.1}$$

Other relations between R and H, such as $HR = R^3H$ and $RHR = H$, also exist, but they can be derived from those given in Equation (1). For example, $(RH)^2 = R_0$ yields $HR = R^{-1}H^{-1}$, and $R^4 = H^2 = R_0$ yields $R^{-1} = R^3$ and $H^{-1} = H$. So, $HR = R^3H$. In fact, every relation between R and H can be derived from those given in Equation (1).

Thus, D_4 is a group that is generated by a pair of elements a and b subject to the relations $a^4 = b^2 = (ab)^2 = e$ and such that all other relations between a and b can be derived from these relations. This last stipulation is necessary because the subgroup $\{R_0, R_{180}, H, V\}$ of D_4 is generated by the two elements $a = R_{180}$ and $b = H$ that satisfy the relations $a^4 = b^2 = (ab)^2 = e$. However, the "extra" relation $a^2 = e$ satisfied by this subgroup cannot be derived from the original ones (since $R_{90}^2 \neq R_0$). It is natural to ask whether this description of D_4 applies to some other group as well. The answer is no. Any other group generated by two elements α and β satisfying only the relations $\alpha^4 = \beta^2 = (\alpha\beta)^2 = e$, and those that can be derived from these relations, is isomorphic to D_4.

Similarly, one can show that the group $Z_4 \oplus Z_2$ is generated by two elements a and b such that $a^4 = b^2 = e$ and $ab = ba$, and any other relation between a and b can be derived from these relations. The purpose of this chapter is to show that this procedure can be reversed; that is, we can begin with any set of generators and relations among the generators and construct a group that is uniquely described by these generators and relations, subject to the stipulation that all other relations among the generators can be derived from the original ones.

Definitions and Notation

We begin with some definitions and notation. For any set $S = \{a, b, c, \ldots\}$ of distinct symbols, we create a new set $S^{-1} = \{a^{-1}, b^{-1}, c^{-1}, \ldots\}$ by replacing each x in S by x^{-1}. Define the set $W(S)$ to be the collection of all formal finite strings of the form $x_1x_2 \cdots x_k$, where each $x_i \in S \cup S^{-1}$. The elements of $W(S)$ are called *words from S*. We also permit the string with no elements to be in $W(S)$. This word is called the *empty word* and is denoted by e.

We may define a binary operation on the set $W(S)$ by juxtaposition; that is, if $x_1x_2 \cdots x_k$ and $y_1y_2 \cdots y_t$ belong to $W(S)$, then so does $x_1x_2 \cdots x_ky_1y_2 \cdots y_t$. Observe that this operation is associative and the empty word is the identity. Also, notice that a word such as aa^{-1} is not the identity, because we are treating the elements of $W(S)$ as formal symbols with no implied meaning.

At this stage we have everything we need to make a group out of $W(S)$ except inverses. Here a difficulty arises, since it seems reasonable that the inverse of the word ab, say, should be $b^{-1}a^{-1}$. But $abb^{-1}a^{-1}$ is not the empty word! You may recall that we faced a similar obstacle long ago when we carried out the construction of the field of quotients of an integral domain. There we had formal symbols of the form a/b and we wanted the inverse of a/b to be b/a. But their product, $ab/(ba)$, was a formal symbol that was not the same as the formal symbol $1/1$, the identity. So, we proceed here as we did there—by way of equivalence classes.

Definition Equivalence Classes of Words
For any pair of elements u and v of $W(S)$, we say that u is related to v if v can be obtained from u by a finite sequence of insertions or deletions of words of the form xx^{-1} or $x^{-1}x$, where $x \in S$.

We leave it as an exercise to show that this relation is an equivalence relation on $W(S)$. (See Exercise 1.)

EXAMPLE 1 Let $S = \{a, b, c\}$. Then $acc^{-1}b$ is equivalent to ab; $aab^{-1}bbaccc^{-1}$ is equivalent to $aabac$; the word $a^{-1}aabb^{-1}a^{-1}$ is equivalent to the empty word; and the word $ca^{-1}b$ is equivalent to $cc^{-1}caa^{-1}a^{-1}bbca^{-1}ac^{-1}b^{-1}$. Note, however, that $cac^{-1}b$ is not equivalent to ab. ∎

Free Group

Theorem 25.1 Equivalence Classes Form a Group

Let S be a set of distinct symbols. For any word u in $W(S)$, let $\bar{u}$ denote the set of all words in $W(S)$ equivalent to u (i.e., $\bar{u}$ is the equivalence class containing u). Then the set of all equivalence classes of elements of $W(S)$ is a group under the operation $\bar{u} \cdot \bar{v} = \overline{uv}$.

PROOF This proof is left to the reader. ∎

The group defined in Theorem 25.1 is called a *free group on S*. Theorem 25.2 shows why free groups are important.

Theorem 25.2 Universal Mapping Property

Every group is a homomorphic image of a free group.

PROOF Let G be a group and let S be a set of generators for G. (Such a set exists, because we may take S to be G itself.) Now let F be the free group on S. Unfortunately, since our notation for any word in $W(S)$ also denotes an element of G, we have created a notational problem for our selves. So, to distinguish between these two cases, we will denote the word $x_1x_2\cdots x_n$ in $W(S)$ by $(x_1x_2\cdots x_n)_F$ and the product $x_1x_2\cdots x_n$ in G by $(x_1x_2\cdots x_n)_G$. As before, $\overline{x_1x_2\cdots x_n}$ denotes the equivalence class in F containing the word $(x_1x_2 ... x_n)_F$. Notice that $\overline{x_1x_2\cdots x_n}$ and $(x_1x_2\cdots x_n)_G$ are entirely different elements, since the operations on F and G are different.

Now consider the mapping from F into G given by

$$\phi\overline{(x_1x_2\cdots x_n)} = (x_1x_2\cdots x_n)_G.$$

[All we are doing is taking a product in F and viewing it as a product in G. For example, if G is the cyclic group of order 4 generated by a, then

$$\phi\overline{(aaaaa)} = (aaaaa)_G = a.]$$

Clearly, ϕ is well-defined, for inserting or deleting expressions of the form xx^{-1} or $x^{-1}x$ in elements of $W(S)$ corresponds to inserting or deleting the identity in G. To check that ϕ is operation-preserving, observe that

$$\begin{aligned}\phi\overline{(x_1x_2\cdots x_n)}\ \overline{(y_1y_2\cdots y_m)} &= \phi\overline{(x_1x_2\cdots x_ny_1y_2\cdots y_m)} \\ &= (x_1x_2\cdots x_ny_1y_2\cdots y_m)_G \\ &= (x_1x_2\cdots x_n)_G(y_1y_2\cdots y_m)_G.\end{aligned}$$

Finally, ϕ is onto G because S generates G. ■

The following corollary is an immediate consequence of Theorem 25.2 and the First Isomorphism Theorem for Groups.

■ Corollary Universal Factor Group Property

Every group is isomorphic to a factor group of a free group.

Generators and Relations

We have now laid the foundation for defining a group by way of generators and relations. Before giving the definition, we will illustrate the basic idea with an example.

EXAMPLE 2 Let F be the free group on the set $\{a, b\}$ and let N be the smallest normal subgroup of F containing the set $\{a^4, b^2, (ab)^2\}$. We will show that F/N is isomorphic to D_4. We begin by observing that the mapping ϕ from F onto D_4, which takes a to R_{90} and b to H (horizontal reflection), defines a homomorphism whose kernel contains N. Thus, $F/\text{Ker } \phi$ is isomorphic to D_4. On the other hand, we claim that the set

$$K = \{N, aN, a^2N, a^3N, bN, abN, a^2bN, a^3bN\}$$

of left cosets of N is F/N itself. To see this, notice that every member of F/N can be generated by starting with N and successively multiplying on the left by various combinations of a's and b's. So, it suffices to show that K is closed under multiplication on the left by a and b. It is trivial that K is closed under left multiplication by a. For b, we will do only one of the eight cases. The others can be done in a similar fashion. Consider $b(aN)$. Since $b^2, abab, a^4 \in N$ and $Nb = bN$, we have $baN = baNb^2 = babNb = a^{-1}(abab)Nb = a^{-1}Nb = a^{-1}a^4Nb = a^3Nb = a^3bN$. Upon completion of the other cases (Exercise 3), we know that F/N has at most eight elements. At the same time, we know that $F/\text{Ker } \phi$ has exactly eight elements. Since $F/\text{Ker } \phi$ is a factor group of F/N [indeed, $F/\text{Ker } \phi \approx (F/N)/(\text{Ker } \phi/N)$], it follows that F/N also has eight elements and $F/N = F/\text{Ker } \phi \approx D_4$. ∎

Definitions Generators and Relations

Let G be a group generated by some subset $A = \{a_1, a_2, \ldots, a_n\}$ and let F be the free group on A. Let $W = \{w_1, w_2, \ldots, w_t\}$ be a subset of F and let N be the smallest normal subgroup of F containing W. We say that G is *given by the generators $a_1, a_2, \ldots, a_n$ and the relations* $w_1 = w_2 = \cdots = w_t = e$ if there is an isomorphism from F/N onto G that carries a_iN to a_i.

The notation for this situation is

$$G = \langle a_1, a_2, \ldots, a_n \mid w_1 = w_2 = \cdots = w_t = e \rangle.$$

As a matter of convenience, we have restricted the number of generators and relations in our definition to be finite. This restriction is not necessary, however. Also, it is often more convenient to write a relation in implicit form. For example, the relation $a^{-1}b^{-3}ab = e$ is often written as $ab = b^3a$. In practice, one does not bother writing down the normal subgroup N that contains the relations. Instead, one just manipulates the generators and treats anything in N as the identity, as our notation suggests. Rather than saying that G is given by

$$\langle a_1, a_2, \ldots, a_n \mid w_1 = w_2 = \cdots = w_t = e \rangle,$$

many authors prefer to say that G has the *presentation*

$$\langle a_1, a_2, \ldots, a_n \mid w_1 = w_2 = \cdots = w_t = e \rangle.$$

Notice that a free group is "free" of relations; that is, the equivalence class containing the empty word is the only relation. We mention in passing the fact that a subgroup of a free group is also a free group. Free groups are of fundamental importance in a branch of algebra known as combinatorial group theory.

EXAMPLE 3 The discussion in Example 2 can now be summed up by writing

$$D_4 = \langle a, b \mid a^4 = b^2 = (ab)^2 = e \rangle.$$

■

EXAMPLE 4 The group of integers is the free group on one letter; that is, $Z \approx \langle a \rangle$. (This is the only nontrivial Abelian group that is free.) ■

The next theorem formalizes the argument used in Example 2 to prove that the group defined there has eight elements.

Theorem 25.3 Dyck's Theorem (1882)

Let

$$\boldsymbol{G = \langle a_1, a_2, \ldots, a_n \mid w_1 = w_2 = \cdots = w_t = e \rangle}$$

and let

$$\bar{G} = \langle a_1, a_2, \ldots, a_n \mid w_1 = w_2 = \cdots = w_t = w_{t+1} = \cdots = w_{t+k} = e \rangle.$$

Then $\bar{G}$ is a homomorphic image of G.

PROOF See Exercise 5. ∎

In words, Theorem 25.3 says that if you start with generators and relations for a group G and create a group $\bar{G}$ by imposing additional relations, then $\bar{G}$ is a homomorphic image of G.

Corollary Largest Group Satisfying Defining Relations

If K is a group satisfying the defining relations of a finite group G and $|K| \geq |G|$, then K is isomorphic to G.

PROOF See Exercise 5. ∎

EXAMPLE 5 Quaternions Consider the group $G = \langle a, b \mid a^2 = b^2 = (ab)^2 \rangle$. What does G look like? Formally, of course, G is isomorphic to F/N, where F is free on $\{a, b\}$ and N is the smallest normal subgroup of F containing $b^{-2}a^2$ and $(ab)^{-2}a^2$. Now, let $H = \langle b \rangle$ and $S = \{H, aH\}$. Then, just as in Example 2, it follows that S is closed under multiplication by a and b from the left. So, as in Example 2, we have $G = H \cup aH$. Thus, we can determine the elements of G once we know exactly how many elements there are in H. (Here again, the three relations come in.) To do this, first observe that $b^2 = (ab)^2 = abab$ implies $b = aba$. Then $a^2 = b^2 = (aba)(aba) = aba^2ba = ab^4a$ and therefore $b^4 = e$. Hence, H has at most four elements, and therefore G has at most eight—namely, $e, b, b^2, b^3, a, ab, ab^2$, and $ab3$. It is conceivable, however, that not all of these eight elements are distinct. For example, $Z_2 \oplus Z_2$ satisfies the defining relations and has only four elements. Perhaps it is the largest group satisfying the relations. How can we show that the eight elements listed above are distinct? Well, consider the group $\bar{G}$ generated by the matrices

$$A = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix},$$

where $i = \sqrt{-1}$. Direct calculations show that in $\bar{G}$, the elements $e, B, B^2, B^3, A, AB, AB^2$, and AB^3 are distinct and that $\bar{G}$ satisfies the relations $A^2 = B^2 = (AB)^2$. So, it follows from the corollary to Dyck's Theorem that $\bar{G}$ is isomorphic to G and therefore G has order 8. ■

The next example illustrates why, in Examples 2 and 5, it is necessary to show that the eight elements listed for the group are distinct.

▌EXAMPLE 6 Let

$$G = \langle a, b \mid a^3 = b^9 = e, a^{-1}ba = b^{-1}\rangle.$$

Once again, we let $H = \langle b\rangle$ and observe that $G = H \cup aH \cup a^2H$. Thus,

$$G = \{a^ib^j \mid 0 \le i \le 2,\ 0 \le j \le 8\},$$

and therefore G has at most 27 elements. But this time we will not be able to find some concrete group of order 27 satisfying the same relations that G does, for notice that $b^{-1} = a^{-1}ba$ implies

$$b = (a^{-1}ba)^{-1} = a^{-1}b^{-1}a.$$

Hence,

$$\begin{aligned} b = ebe = a^{-3}ba^3 = a^{-2}(a^{-1}ba)a^2 &= a^{-2}b^{-1}a^2 \\ &= a^{-1}(a^{-1}b^{-1}a)a = a^{-1}ba = b^{-1}. \end{aligned}$$

So, the original three relations imply the additional relation $b^2 = e$. But $b^2 = e = b^9$ further implies $b = e$. It follows, then, that G has at most three distinct elements—namely, e, a, and a^2. But Z_3 satisfies the defining relations with $a = 1$ and $b = 0$. So, $|G| = 3$. ■

We hope Example 6 convinces you of the fact that, once a list of the elements of the group given by a set of generators and relations has been obtained, one must further verify that this list has no duplications. Typically, this is accomplished by exhibiting a specific group that satisfies the given set of generators and relations and that has the same size as the list. Obviously, experience plays a role here.

Here is a fun example adapted from [1].

EXAMPLE 7 Let G be the group with the 26 letters of the alphabet as generators. For relations we take strings $A = B$, where A and B are words in some fixed reference, say [2], and have the same pronunciation but different meanings (such words are called homophones). For example, $buy = by = bye$, $hour = our$, $lead = led$, $whole = hole$. From these strings and cancellation, we obtain $u = e = h = a = w = \theta$ (θ is the identity string). With these examples in mind, we ask, What is the group given by these generators and relations? Surprisingly, the answer is the infinite cyclic group generated by v. To verify this, one must show that every letter except v is equivalent to θ and that there are no two homophones that contain a different number of v's. The former can easily be done with common words. For example, from $inn = in, plumb = plum$, and $knot = not$, we see that $n = b = k = \theta$. From $too = to$ we have $o = \theta$. That there are no two homophones in [2] that have a different number of v's can be verified by simply checking all cases. In contrast, the reference *Handbook of Homophones* by W. C. Townsend (see **http://members.peak.org/~jeremy/dictionaryclassic/chapters/homophones.php**) lists $felt/veldt$ as homophones. Of course, including these makes the group trivial. ∎

Classification of Groups of Order Up to 15

The next theorem illustrates the utility of the ideas presented in this chapter.

Theorem 25.4 Classification of Groups of Order 8 (Cayley, 1859)

Up to isomorphism, there are only five groups of order 8: $Z_8, Z_4 \oplus Z_2, Z_2 \oplus Z_2 \oplus Z_2, D_4$, and the quaternions.

PROOF The Fundamental Theorem of Finite Abelian Groups takes care of the Abelian cases. Now, let G be a non-Abelian group of order 8. Also, let $G_1 = \langle a, b \mid a^4 = b^2 = (ab)^2 = e\rangle$ and let $G_2 = \langle a, b \mid a^2 = b^2 = (ab)^2\rangle$. We know from the preceding examples that G_1 is isomorphic to D_4 and G_2 is isomorphic to the quaternions. Thus, it suffices to show that G must satisfy the defining relations for G_1 or G_2. It follows from Exercise 45 in Chapter 2 and Lagrange's Theorem that G has an element of order 4; call it a. Then, if b is any element of G not in $\langle a \rangle$, we know that

$$G = \langle a \rangle \cup \langle a \rangle b = \{e, a, a^2, a^3, b, ab, a^2b, a^3b\}.$$

Consider the element b^2 of G. Which of the eight elements of G can it be? Not b, ab, a^2b, or a^3b, by cancellation. Not a, for b^2 commutes with b and a does not. Not a^3, for the same reason. Thus, $b^2 = e$ or $b^2 = a^2$. Suppose $b^2 = e$. Since $\langle a \rangle$ is a normal subgroup of G, we know that $bab^{-1} \in \langle a \rangle$. From this and the fact that $|bab^{-1}| = |a|$, we then conclude that $bab^{-1} = a$ or $bab^{-1} = a^{-1}$. The first relation would mean that G is Abelian, so we know that $bab^{-1} = a^{-1}$. But then, since $b^2 = e$, we have $(ab)^2 = e$, and therefore G satisfies the defining relations for G_1.

Finally, if $b^2 = a^2$ holds instead of $b^2 = e$, we can use $bab^{-1} = a^{-1}$ to conclude that $(ab)^2 = a(bab^{-1})b^2 = aa^{-1}b^2 = b^2$, and therefore G satisfies the defining relations for G_2. ∎

The classification of the groups of order 8, together with our results on groups of order $p^2, 2p$, and pq from Chapter 23, allows us to classify the groups of order up to 15, with the exception of those of order 12. We already know four groups of order 12—namely, $Z_{12}, Z_6 \oplus Z_2, D_6$, and A_4. An argument along the lines of Theorem 25.4 can be given to show that there is only one more group of order 12. This group, called the *dicyclic group of order 12* and denoted by Q_6, has presentation $\langle a, b \mid a^6 = e, a^3 = b^2, b^{-1}ab = a^{-1}\rangle$. Table 25.1 lists the groups of order at most 15. We use Q_4 to denote the quaternions (see Example 5 in this chapter).

Characterization of Dihedral Groups

As another nice application of generators and relations, we will now give a characterization of the dihedral groups that has been known for more than 100 years. For $n \geq 3$, we have used D_n to denote the group of symmetries of a regular n-gon. Imitating Example 2, one can show that $D_n \approx \langle a, b \mid a^n = b^2 = (ab)^2 = e\rangle$ (see Exercise 9). By analogy, these generators and relations serve to define D_1 and D_2 also. (These are also called dihedral groups.) Finally, we define the infinite dihedral group D_∞ as $\langle a, b \mid a^2 = b^2 = e\rangle$. The elements of D_∞ can be listed as $e, a, b, ab, ba, (ab)a, (ba)b, (ab)^2, (ba)^2, (ab)^2a, (ba)^2b, (ab)^3, (ba)^3, \ldots$.

Table 25.1 Classification of Groups of Order Up to 15.

Order	Abelian Groups	Non-Abelian Groups
1	Z_1	
2	Z_2	
3	Z_3	
4	$Z_4, Z_2 \oplus Z_2$	
5	Z_5	
6	Z_6	D_3
7	Z_7	
8	$Z_8, Z_4 \oplus Z_2, Z_2 \oplus Z_2 \oplus Z_2$	D_4, Q_4
9	$Z_9, Z_3 \oplus Z_3$	
10	Z_{10}	$D5$
11	Z_{11}	
12	$Z_{12}, Z_6 \oplus Z_2$	D_6, A_4, Q_6
13	Z_{13}	
14	Z_{14}	D_7
15	Z_{15}	

Theorem 25.5 Characterization of Dihedral Groups

Any group generated by a pair of elements of order 2 is dihedral.

PROOF Let G be a group generated by a pair of distinct elements of order 2, say, a and b. We consider the order of ab. If $|ab| = \infty$, then G is infinite and satisfies the relations of D_∞. We will show that G is isomorphic to D_∞. By Dyck's Theorem, G is isomorphic to some factor group of D_∞, say, D_∞/H. Now, suppose $h \in H$ and $h \neq e$. Since every element of D_∞ has one of the forms $(ab)^i, (ba)^i, (ab)^i a$, or $(ba)^i b$, by symmetry, we may assume that $h = (ab)^i$ or $h = (ab)^i a$. If $h = (ab)^i$, we will show that D_∞/H satisfies the relations for D_i given in Exercise 9. Since $(ab)^i$ is in H, we have

$$H = (ab)^i H = (abH)^i,$$

so that $(abH)^{-1} = (abH)^{i-1}$. But

$$(ab)^{-1}H = b^{-1}a^{-1}H = baH,$$

and it follows that

$$aHabHaH = a^2HbHaH = eHbaH = baH = (abH)^{-1}.$$

Thus,

$$D_\infty/H = \langle aH, bH\rangle = \langle aH, abH\rangle$$

(see Exercise 7), and D_∞/H satisfies the defining relations for D_i (use Exercise 9 with $x = aH$ and $y = abH$). In particular, G is finite—an impossibility.

If $h = (ab)^i a$, then

$$H = (ab)^i aH = (ab)^i HaH,$$

and therefore

$$(abH)^i = (ab)^i H = (aH)^{-1} = a^{-1}H = aH.$$

It follows that

$$\langle aH, bH\rangle = \langle aH, abH\rangle \subseteq \langle abH\rangle.$$

However,

$$(abH)^{2i} = (aH)^2 = a^2H = H,$$

so that D_∞/H is again finite. This contradiction forces $H = \{e\}$ and G to be isomorphic to D_∞.

Finally, suppose that $|ab| = n$. Since $G = \langle a, b\rangle = \langle a, ab\rangle$, we can show that G is isomorphic to D_n by proving that $b(ab)b = (ab)^{-1}$, which is the same as $ba = (ab)^{-1}$ (see Exercise 9). But $(ab)^{-1} = b^{-1}a^{-1} = ba$, since a and b have order 2. ■

We conclude this chapter by commenting on the advantages and disadvantages of using generators and relations to define groups. The principal advantage is that in many situations—particularly in knot theory, algebraic topology, and geometry—groups defined by way of generators and relations arise in a natural way. Within group theory itself, it is often convenient to construct examples and counterexamples with generators and relations. Among the disadvantages of defining a group by generators and relations is the fact that it is often difficult to decide whether or not the group is finite, or even whether or not a particular element is the identity. Furthermore, the same group can be defined with entirely different sets of generators and relations, and, given two groups defined by generators and relations, it is often extremely difficult to decide whether or not these two groups are isomorphic. Nowadays, these questions are frequently tackled with the aid of a computer.

Exercises

It don't come easy.

Ringo Starr, "It Don't Come Easy," single

1. Let S be a set of distinct symbols. Show that the relation defined on $W(S)$ in this chapter is an equivalence relation.
2. Let n be an even integer. Prove that $D_n/Z(D_n)$ is isomorphic to $D_{n/2}$.
3. Verify that the set K in Example 2 is closed under multiplication on the left by b.
4. Show that $\langle a, b \mid a^5 = b^2 = e, ba = a^2b\rangle$ is isomorphic to Z_2.
5. Prove Theorem 25.3 and its corollary.
6. Let G be the group $\{\pm 1, \pm i, \pm j, \pm k\}$ with multiplication defined as in Exercise 54 in Chapter 9. Show that G is isomorphic to $\langle a, b \mid a^2 = b^2 = (ab)^2\rangle$. (Hence, the name "quaternions.")
7. In any group, show that $\langle a, b\rangle = \langle a, ab\rangle$. (This exercise is referred to in the proof of Theorem 25.5.)
8. Let $\alpha = (12)(34)$ and $\beta = (24)$. Show that the group generated by α and β is isomorphic to D_4.
9. Prove that $G = \langle x, y \mid x^2 = y^n = e, xyx = y^{-1}\rangle$ is isomorphic to D_n. (This exercise is referred to in the proof of Theorem 25.5.)
10. What is the minimum number of generators needed for $Z_2 \oplus Z_2 \oplus Z_2$? Find a set of generators and relations for this group.
11. Suppose that $x^2 = y^2 = e$ and $yz = zxy$. Show that $xy = yx$.
12. Let $G = \langle a, b \mid a^2 = b^4 = e, ab = b^3a\rangle$.
    a. Express $a^3b^2abab^3$ in the form b^ia^j, where $0 \leq i \leq 1$ and $0 \leq j \leq 3$.
    b. Express b^3abab^3a in the form b^ia^j, where $0 \leq i \leq 1$ and $0 \leq j \leq 3$.
13. Let $G = \langle a, b \mid a^2 = b^2 = (ab)^2\rangle$.
    a. Express b^2abab^3 in the form b^ia^j.
    b. Express b^3abab^3a in the form b^ia^j.

14. Let G be the group defined by the following table. Show that G is isomorphic to D_n.

	1	**2**	**3**	**4**	**5**	**6**	$\cdots$	**2n**
1	1	2	3	4	5	6	$\cdots$	$2n$
2	2	1	$2n$	$2n-1$	$2n-2$	$2n-3$	$\cdots$	3
3	3	4	5	6	7	8	$\cdots$	2
4	4	3	2	1	$2n$	$2n-1$	$\cdots$	5
5	5	6	7	8	9	10	$\cdots$	4
6	6	5	4	3	2	1	$\cdots$	7
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$
2n	$2n$	$2n-1$	$2n-2$	$2n-3$	$2n-4$	$2n-5$	$\cdots$	1

15. Let $G = \langle x, y \mid x = (xy)^3, y = (xy)^4 \rangle$. To what familiar group is G isomorphic?

16. Let $G = \langle z \mid z^6 = 1 \rangle$ and $H = \langle x, y \mid x^2 = y^3 = 1, xy = yx \rangle$. Show that G and H are isomorphic.

17. Let $G = \langle x, y \mid x^8 = y^2 = e, yxyx^3 = e \rangle$. Show that $|G| \leq 16$. Assuming that $|G| = 16$, find the center of G and the order of xy.

18. Prove that every element of Q_6, the dicyclic group of order 12 as defined in this chapter, can be written in the form a^i or $a^i b$ where $0 \leq i < 6$ and that $a^3 = b^2$ is the unique element of order 2.

19. Prove that every element of Q_6, the dicyclic group of order 12 as defined in this chapter, can be written in the form a^i or $a^i b$ where $0 \leq i < 6$ and that $a^3 = b^2$ is the unique element of order 2.

20. Let $G = \langle s, t \mid sts = tst \rangle$. Show that the permutations (23) and (13) satisfy the defining relations of G. Explain why this proves that G is non Abelian.

21. In $D_{12} = \langle x, y \mid x^2 = y^{12} = e, xyx = y^{-1} \rangle$, prove that the subgroup $H = \langle x, y^3 \rangle$ (which is isomorphic to D_4) is not a normal subgroup.

22. Let $G = \langle x, y \mid x^{2n} = e, x^n = y^2, y^{-1}xy = x^{-1} \rangle$. Show that $Z(G) = \{e, x^n\}$. Assuming that $|G| = 4n$, show that $G/ZG)$ is isomorphic to D_n. (The group G is called the *dicyclic* group of order $4n$.)

23. Let $G = \langle a, b \mid a^6 = b^3 = e, b^{-1}ab = a^3 \rangle$. How many elements does G have? To what familiar group is G isomorphic?

24. Let $G = \langle x, y \mid x^4 = y^4 = e, xyxy^{-1} = e\rangle$. Show that $|G| \leq 16$. Assuming that $G = 16$, find the center of G and show that $G/\langle y^2\rangle$ is isomorphic to D_4.

25. Determine the orders of the elements of D_∞.

26. Let $G = \left\{ \begin{bmatrix} 1 & a & b \\ 0 & 1 & c \\ 0 & 0 & 1 \end{bmatrix} \;\middle|\; a, b, c \in Z_2 \right\}$. Prove that G is isomorphic to D_4.

27. Let $G = \langle a, b, c, d \mid ab = c, bc = d, cd = a, da = b\rangle$. Determine $|G|$.

28. Let $G = \langle a, b \mid a^2 = e, b^2 = e, aba = bab\rangle$. To what familiar group is G isomorphic?

29. Let $G = \langle a, b \mid a^3 = e, b^2 = e, aba^{-1}b^{-1} = e\rangle$. To what familiar group is G isomorphic?

30. Give an example of a non-Abelian group that has exactly three elements of finite order.

31. Referring to Example 7 in this chapter, show as many letters as you can that are equivalent to θ.

32. Suppose that a group of order 8 has exactly five elements of order 2. Identify the group.

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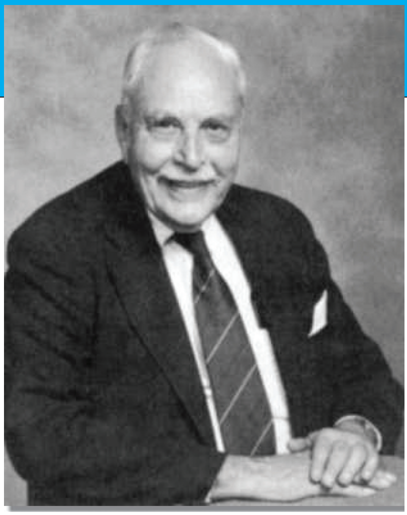
Marshall Hall, Jr.



Courtesy of Jonathan Hall

Professor Hall was a mathematician in the broadest sense of the word but with a predilection for group theory, geometry and combinatorics.

HANS ZASSENHAUS,
Notices of the American Mathematical Society

Marshall Hall, Jr., was born on September 17, 1910, in St. Louis, Missouri. He demonstrated interest in mathematics at the age of 11 when he constructed a seven-place table of logarithms for the positive integers up to 1000. He completed a B.A. degree in 1932 at Yale. After spending a year at Cambridge University, where he worked with Philip Hall, Harold Davenport, and G. H. Hardy, he returned to Yale for his Ph.D. degree. At the outbreak of World War II, he joined Naval Intelligence and had significant success in deciphering both the Japanese codes and the German Enigma messages. These successes helped to turn the tide of the war. After the war, Hall had faculty appointments at the Ohio State University, Caltech, and Emory University. He died on July 4, 1990.

Hall's highly regarded books on group theory and combinatorial theory are classics. His mathematical legacy includes more than 120 research papers on group theory, coding theory, and design theory. His 1943 paper on projective planes ranks among the most cited papers in mathematics. Several fundamental concepts as well as a sporadic simple group are identified with Hall's name. One of Hall's most celebrated results is his solution to the "Burnside Problem" for exponent 6—that is, his proof that a finitely generated group in which the order of every element divides 6 must be finite. Hall influenced both John Thompson and Michael Aschbacher, two of finite group theory's greatest contributors. It was Hall who suggested Thompson's Ph.D. dissertation problem. Hall's Ph.D. students at Caltech included Donald Knuth and Robert McEliece.

26 Symmetry Groups

Physicists have exalted symmetry to the position of *the* central concept in their attempts to organize and explain an otherwise bewildering and complex universe.

Mario Livio, *The Equation That Could Not Be Solved*

I'm not good at math, but I do know that the universe is formed with mathematical principles whether I understand them or not, and I am going to let that guide me.

Bob Dylan, *Chronicles, Volume One*

Isometries

In the early chapters of this book, we briefly discussed symmetry groups. In this chapter and the next, we examine this fundamentally important concept in some detail. It is convenient to begin such a discussion with the definition of an isometry (from the Greek *isometros*, meaning "equal measure") in $\mathbf{R}^n$.

Definition Isometry
An *isometry* of n-dimensional space $\mathbf{R}^n$ is a function from $\mathbf{R}^n$ onto $\mathbf{R}^n$ that preserves distance.

In other words, a function T from $\mathbf{R}^n$ onto $\mathbf{R}^n$ is an isometry if, for every pair of points p and q in $\mathbf{R}^n$, the distance from $T(p)$ to $T(q)$ is the same as the distance from p to q. With this definition, we may now make precise the definition of the symmetry group of an n-dimensional figure.

Definition Symmetry Group of a Figure in $\mathbf{R}^n$
Let F be a set of points in $\mathbf{R}^n$. The *symmetry group of F* in $\mathbf{R}^n$ is the set of all isometries of $\mathbf{R}^n$ that carry F onto itself. The group operation is function composition.

It is important to realize that the symmetry group of an object depends not only on the object, but also on the space in which we view it. For example, the symmetry group of a line segment in $\mathbf{R}^1$ has order 2, the symmetry group of a line segment considered as a set of points in $\mathbf{R}^2$ has order 4, and the symmetry group of a line segment viewed as a set of points in $\mathbf{R}^3$ has infinite order (see Exercise 9).

Although we have formulated our definitions for all finite dimensions, our chief interest will be the two-dimensional case. It has been known since 1831 that every isometry of $\mathbf{R}^2$ is one of four types: rotation, reflection, translation, and glide-reflection. Rotation about a point in a plane needs no explanation. A *reflection across a line* L is that transformation that leaves every point of L fixed and takes every point Q, not on L, to the point Q' so that L is the perpendicular bisector of the line segment from Q to Q' (see Figure 26.1). The line L is called the *axis of reflection.* In an xy-coordinate plane, the transformation $(x, y) \to (x, -y)$ is a reflection across the x-axis, whereas $(x, y) \to (y, x)$ is a reflection across the line $y = x$. Some authors call an axis of reflective symmetry L a *mirror* because L acts like a two-sided mirror; that is, the image of a point Q in a mirror placed on the line L is, in fact, the image of Q under the reflection across the line L. Reflections are called *opposite* isometries because they reverse orientation. For example, the reflected image of a clockwise spiral is a counterclockwise spiral. Similarly, the reflected image of a right hand is a left hand. (See Figure 26.1.)

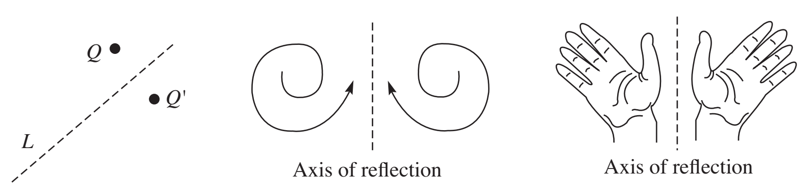


Figure 26.1 Reflected images.

A *translation* is simply a function that carries all points the same distance in the same direction. For example, if p and q are points in a plane and T is a translation, then the two directed line segments joining p to $T(p)$ and q to $T(q)$ have the same length and direction. A *glide-reflection* is the product of a translation and a reflection across the line containing the translation line segment. This line is called the *glide-axis*. In Figure 26.2, the arrow gives the direction and length of the translation, and is contained in the axis of reflection. A glide-reflection is also an opposite isometry. Successive footprints in wet sand are related by a glide-reflection.

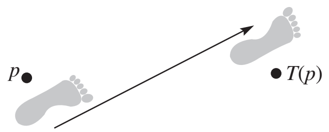


Figure 26.2 Glide-reflection

Classification of Finite Plane Symmetry Groups

Our first goal in this chapter is to classify all finite plane symmetry groups. As we have seen in earlier chapters, the dihedral group D_n is the plane symmetry group of a regular n-gon. (For convenience, call D_2 the plane symmetry group of a nonsquare rectangle and D_1 the plane symmetry group of the letter "V." In particular, $D_2 \approx Z_2 \oplus Z_2$ and $D_1 \approx Z_2$.) The cyclic groups Z_n are easily seen to be plane symmetry groups also. Figure 26.3 is an illustration of an organism whose plane symmetry group consists of four rotations and is isomorphic to Z_4. The surprising fact is that the cyclic groups and dihedral groups are the only finite plane symmetry groups. The famous mathematician Hermann Weyl attributes the following theorem to Leonardo da Vinci (1452–1519).

From Symmetry in Science and Art by A.V. Shubnikov and V.A. Kopstik, Copyright 1974 Plenum Publishing Co.

Figure 26.3 Aurelia insulinda, an organism whose plane symmetry group is Z_4.

Theorem 26.1 Finite Symmetry Groups in the Plane

The only finite plane symmetry groups are Z_n and D_n.

PROOF Let G be a finite plane symmetry group of some figure. We first observe that G cannot contain a translation or a glide-reflection, because in either case G would be infinite. Now observing that the composition of two reflections preserves orientation, we know that such a composition is a translation or rotation. When the two reflections have parallel axes of reflection, there is no fixed point so the composition is a translation. Thus, every two reflections in G have reflection axes that intersect in some point. We claim that all reflections intersect in the same point. Suppose that f and f' are two distinct reflections in G. Then because ff' preserves orientation, we know that ff' is a rotation. We use the fact from geometry that a finite group of rotations must have a common center, say P. This means that any two reflections must intersect at point P. So, we have shown that all the elements of G have the common fixed point P.

For convenience, let us denote a rotation about P of σ degrees by R_σ. Now, among all rotations in G, let β be the smallest positive angle of rotation. (Such an angle exists, since G is finite and R_{360} belongs to G.) We claim that every rotation in G is some power of R_β. To see this, suppose that R_σ is in G. We may assume $0° < \sigma \leq 360°$. Then, $\beta \leq \sigma$ and there is some integer t such that $t\beta \leq \sigma < (t+1)\beta$. But, then $R_{\sigma - t\beta} = R_\sigma(R_\beta)^{-t}$ is in G and $0 \leq \sigma - t\beta < \beta$. Since β represents the smallest positive angle of rotation among the elements of G, we must have $\sigma - t\beta = 0$, and therefore, $R_\sigma = (R_\beta)^t$. This verifies the claim.

For convenience, let us say that $|R_\beta| = n$. Now, if G has no reflections, we have proved that $G = \langle R_\beta \rangle \approx Z_n$. If G has at least one reflection, say f, then

$$f, fR_\beta, f(R_\beta)^2, \ldots, f(R_\beta)^{n-1}$$

are also reflections. Furthermore, this is the entire set of reflections of G. For if g is any reflection in G, then fg is a rotation, and so $fg = (R_\beta)^k$ for some k. Thus, $g = f^{-1}(R_\beta)^k = f(R_\beta)^k$. So

$$G = \{R_0, R_\beta, (R_\beta)^2, \ldots, (R_\beta)^{n-1}, f, fR_\beta, f(R_\beta)^2, \ldots, f(R_\beta)^{n-1}\},$$

and G is generated by the pair of reflections f and fR_β. Hence, by our characterization of the dihedral groups (Theorem 25.5), G is the dihedral group D_n. ∎

Classification of Finite Groups of Rotations in $\mathbf{R}^3$

One might think that the set of all possible finite symmetry groups in three dimensions would be much more diverse than is the case for two dimensions. Surprisingly, this is not the case. For example, moving to three dimensions introduces only three new groups of rotations. This observation was first made by the physicist and mineralogist Auguste Bravais in 1849, in his study of possible structures of crystals.

Theorem 26.2 Finite Groups of Rotations in $\mathbf{R}^3$

Up to isomorphism, the finite groups of rotations in R^3 are Z_n, D_n, A_4, S_4, and A_5.

Theorem 26.2, together with the Orbit-Stabilizer Theorem (Theorem 7.4), makes easy work of determining the group of rotations of an object in $\mathbf{R}^3$.

EXAMPLE 8 We determine the group G of rotations of the cuboctahedron in Figure 26.4, which is composed of six congruent squares and eight congruent equilateral triangles. We begin by singling out any one of the squares. Obviously, there are four rotations that map this square to itself, and the designated square can be rotated to the location of any of the other five. So, by the Orbit-Stabilizer Theorem (Theorem 7.4), the rotation group has order $4 \cdot 6 = 24$. By Theorem 26.2, G is one of Z_{24}, D_{12}, and S_4. But each of the first two groups has exactly two elements of order 4, whereas G has more than two. So, G is isomorphic to S_4. ∎

The group of rotations of a tetrahedron (the *tetrahedral group*) is isomorphic to A_4; the group of rotations of a cube or an octahedron (the *octahedral group*) is isomorphic to S_4; the group of rotations of a dodecahedron or an icosahedron (the *icosahedral group*) is isomorphic to A_5. These five solids are illustrated in Figure 26.5.

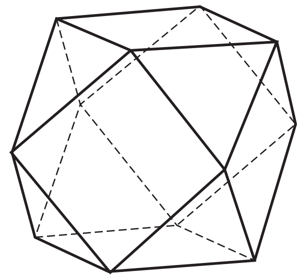


Figure 26.4 Cuboctahedron.

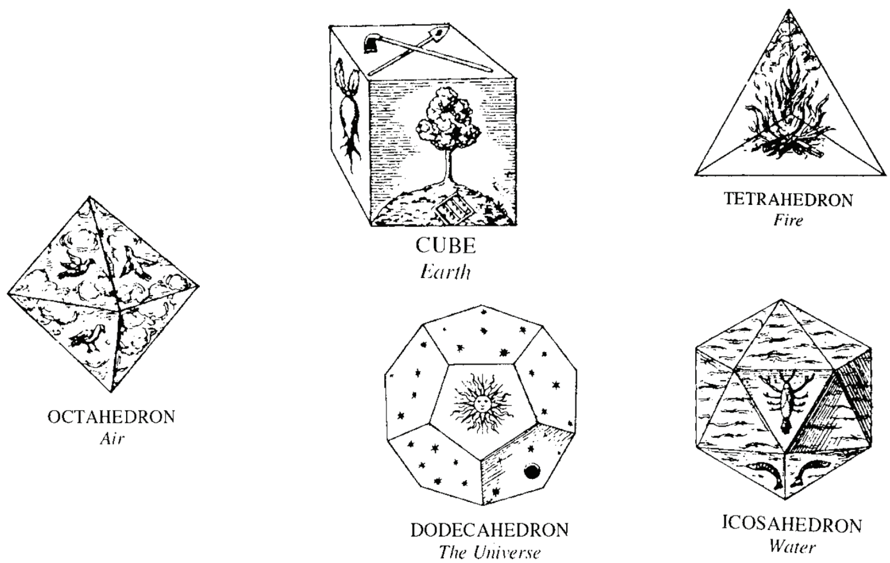


Figure 26.5 The five regular solids as depicted by Johannes Kepler in *Harmonices Mundi, Book II* (1619)

Exercises

Perhaps the most valuable result of all education is the ability to make yourself do the thing you have to do, when it ought to be done, whether you like it or not .

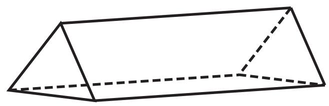
Thomas Henry Huxley, "Technical Education"

1. Show that an isometry of $\mathbf{R}^n$ is one-to-one.

2. Show that the translations of $\mathbf{R}^n$ form a group.

3. Exhibit a plane figure whose plane symmetry group is Z_5.

4. Show that the group of rotations in $\mathbf{R}^3$ of a 3-prism (i.e., a prism with equilateral ends, as in the following figure) is isomorphic to D_3.



5. What is the order of the (entire) symmetry group in $\mathbf{R}^3$ of a 3-prism?

6. What is the order of the symmetry group in $\mathbf{R}^3$ of a 4-prism (a box with square ends that is not a cube)?

7. What is the order of the symmetry group in $\mathbf{R}^3$ of an n-prism?

8. Show that the symmetry group in $\mathbf{R}^3$ of a box of dimensions $2'' \times 3'' \times 4''$ is isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$.

9. Describe the symmetry group of a line segment viewed as
a. a subset of $\mathbf{R}^1$.
b. a subset of $\mathbf{R}^2$.
c. a subset of $\mathbf{R}^3$.
(This exercise is referred to in this chapter.)

10. (From the "Ask Marilyn" column in *Parade Magazine*, December 11, 1994.)[1] The letters of the alphabet can be sorted into the following categories:
a. FGJLNPQRSZ
b. BCDEK
c. AMTUVWY
d. HIOX
What defines the categories?

11. Exactly how many elements of order 4 does the group in Example 1 have?

12. Why is inversion [that is, $\phi(x, y) = (-x, -y)$,] not listed as one of the four kinds of isometries in R^2?

13. Explain why inversion through a point in $\mathbf{R}^3$ cannot be realized by a rotation in $\mathbf{R}^3$.

14. Reflection across a line L in $\mathbf{R}^3$ is the isometry that takes each point Q to the point Q' with the property that L is a perpendicular bisector of the line segment joining Q and Q'. Describe a rotation that has this same effect.

15. In $\mathbf{R}^2$, a rotation fixes a point; in $\mathbf{R}^3$, a rotation fixes a line. In $\mathbf{R}^4$, what does a rotation fix? Generalize these observations to $\mathbf{R}^n$.

16. Show that an isometry of a plane preserves angles.

17. Show that an isometry of a plane is completely determined by the image of three noncollinear points.

18. Suppose that an isometry of a plane leaves three noncollinear points fixed. Which isometry is it?

19. Suppose that an isometry of a plane fixes exactly one point. What type of isometry must it be?

20. Suppose that A and B are rotations of 180° about the points a and b, respectively. What is A followed by B? How is the composite motion related to the points a and b?

27 Symmetry and Counting

Let us pause to slake our thirst one last time at symmetry's bubbling spring.

Timothy Ferris, *Coming of Age in the Milky Way*

Whenever you can, count.

Francis Galton, (1822–1911)

Motivation

Permutation groups naturally arise in many situations involving symmetric designs or arrangements. Consider, for example, the task of coloring the six vertices of a regular hexagon so that three are black and three are white. Figure 27.1 shows the $\binom{6}{3} = 20$ possibilities. However, if these designs appeared on one side of hexagonal ceramic tiles, it would be nonsensical to count the designs shown in Figure 27.1(a) as different, since all six designs shown there can be obtained from one of them by rotating. (A manufacturer would make only one of the six.) In this case, we say that the designs in Figure 27.1(a) are *equivalent* under the group of rotations of the hexagon. Similarly, the designs in Figure 27.1(b) are equivalent under the group of rotations, as are the designs in Figure 27.1(c) and those in Figure 27.1(d). And, since no design from Figure 27.1(a)–(d) can be obtained from a design in a different part by rotation, we see that the designs within each part of the figure are equivalent to each other but nonequivalent to any design in another part of the figure. However, the designs in Figure 27.1(b) and (c) are equivalent under the dihedral group D_6, since the designs in Figure 27.1(b) can be reflected to yield the designs in Figure 27.1(c). For example, for purposes of arranging three black beads and three white beads to form a necklace, the designs shown in Figure 27.1(b) and (c) would be considered equivalent.

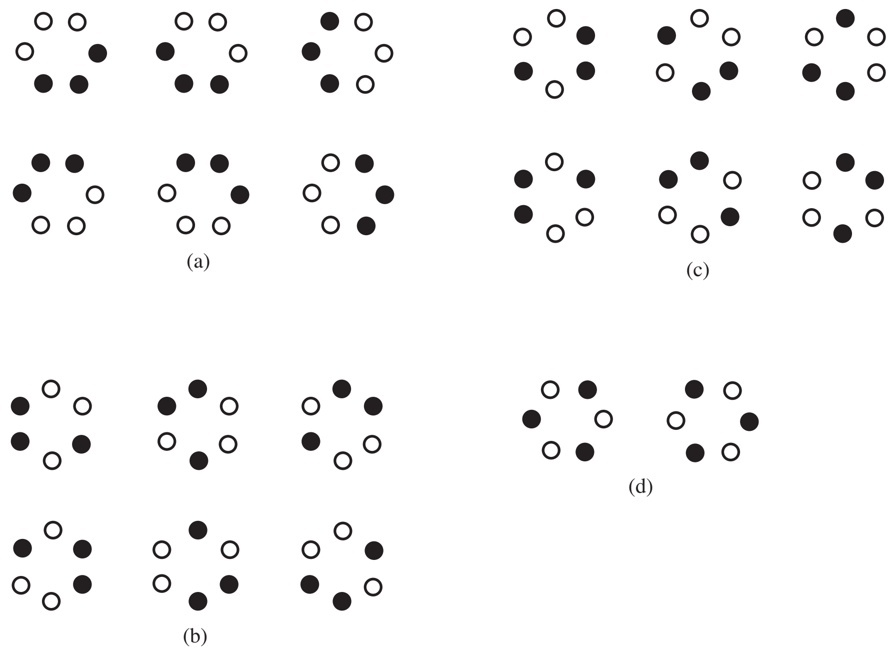


Figure 27.1 Necklaces with three black beads and three white beads.

In general, we say that two designs (arrangements of beads) A and B are *equivalent under a group* G of permutations of the arrangements if there is an element ϕ in G such that $\phi(A) = B$. That is, two designs are equivalent under G if they are in the same orbit of G. It follows, then, that the number of nonequivalent designs under G is simply the number of orbits of designs under G. (The set being permuted is the set of all possible designs or arrangements.)

Notice that the designs in Figure 27.1 divide into four orbits under the group of rotations but only three orbits under the group D_6, since the designs in Figure 27.1(b) and (c) form a single orbit under D_6. Thus, we could obtain all 20 tile designs from just four tiles, but we could obtain all 20 necklaces from just three of them.

Burnside's Theorem

Although the problems we have just posed are simple enough to solve by observation, more complicated ones require a more sophisticated approach. Such an approach was provided by Georg

Frobenius in 1887. Frobenius's theorem did not become widely known until it appeared in the classic book on group theory by William Burnside in 1911. By an accident of history, Frobenius's theorem has come to be known as Burnside's Theorem. Before stating this theorem, we recall some notation introduced in Chapter 7 and introduce new notation. If G is a group of permutations on a set S and $i \in S$, then $\mathrm{stab}_G(i) = \{\phi \in G \mid \phi(i) = i\}$ and $\mathrm{orb}_G(i) = \{\phi(i) \mid \phi \in G\}$. For any set X, we use $|X|$ to denote the number of elements in X.

Definitions Elements Fixed by ϕ

For any group G of permutations on a set S and any ϕ in G, we let $\mathrm{fix}(\phi) = \{i \in S \mid \phi(i) = i\}$. This set is called the *elements fixed by ϕ* (or more simply, "fix of ϕ").

Theorem 27.1 Burnside's Theorem

If G is a finite group of permutations on a set S, then the number of orbits of elements of S under G is

$$\frac{1}{|G|}\sum_{\phi \in G} |\mathrm{fix}(\phi)|.$$

PROOF Let n denote the number of pairs (ϕ, i), with $\phi \in G, i \in S$, and $\phi(i) = i$. We begin by counting these pairs in two ways. First, for each particular ϕ in G, the number of such pairs is exactly $|\mathrm{fix}(\phi)|$. So,

$$n = \sum_{\phi \in G} |\mathrm{fix}(\phi)|.$$

Second, for each particular i in S, observe that $|\mathrm{stab}_G(i)|$ is exactly the number of pairs (ϕ, i) for which $\phi(i) = i$. So,

$$n = \sum_{i \in S} |\mathrm{stab}_G(i)|.$$

It follows from Exercise 67 in Chapter 7 that if s and t are in the same orbit of G, then $\mathrm{orb}_G(s) = \mathrm{orb}_G(t)$, and thus by the Orbit-Stabilizer Theorem (Theorem 7.4) we have $|\mathrm{stab}_G(s)| = |G|/|\mathrm{orb}_G(s)| = |G|/|\mathrm{orb}_G(t)| = |\mathrm{stab}_G(t)$. So, if we choose $s \in S$ and sum over $\mathrm{orb}_G(s)$, we have

Table 27.1

Element	Number of Designs Fixed by Element
Identity	20
Rotation of 60°	0
Rotation of 120°	2
Rotation of 180°	0
Rotation of 240°	2
Rotation of 300°	0

$$\sum_{t\in \mathrm{orb}_G(s)} |\mathrm{stab}_G(t)| = |\mathrm{orb}_G(S)||\mathrm{stab}_G(S)| = |G|.$$

Finally, by summing over all the elements of G, one orbit at a time, it follows from Equations (1), (2), and (3) that

$$\sum_{\phi\in G} |\mathrm{fix}(\phi)| = \sum_{i\in S} |\mathrm{stab}(i)| = |G| \cdot (\text{number of orbits})$$

and the result follows. ∎

Applications

To illustrate how to apply Burnside's Theorem, let us return to the ceramic tile and necklace problems. In the case of counting hexagonal tiles with three black vertices and three white vertices, the objects being permuted are the 20 possible designs, whereas the group of permutations is the group of six rotational symmetries of a hexagon. Obviously, the identity fixes all 20 designs. We see from Figure 27.1 that rotations of 60°, 180°, or 300° fix none of the 20 designs. Finally, Figure 27.2 shows $\mathrm{fix}(\phi)$ for the rotations of 120° and 240°. These data are collected in Table 27.1.

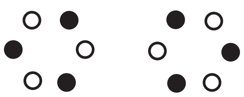


Figure 27.2 Tile designs fixed by 120° rotation and 240° rotation.

So, applying Burnside's Theorem, we obtain the number of orbits under the group of rotations as

$$\frac{1}{6}(20 + 0 + 2 + 0 + 2 + 0) = 4.$$

Now let's use Burnside's Theorem to count the number of necklace arrangements consisting of three black beads and three white beads. (For the purposes of analysis, we may arrange the beads in the shape of a regular hexagon.) For this problem, two arrangements are equivalent if they are in the same orbit under D_6. Figure 27.3 shows the arrangements fixed by a reflection across a diagonal. Table 27.2 summarizes the information needed to apply Burnside's Theorem.

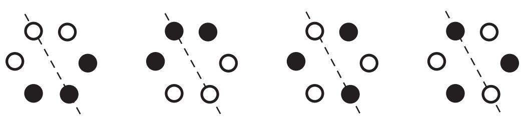


Figure 27.3 Bead arrangements fixed by the reflection across a diagonal.

Table 27.2

Type of Element	Number of Elements of This Type	Number of Arrangements Fixed by Type of Element
Identity	1	20
Rotation of order 2 (180°)	1	0
Rotation of order 3 (120° or 240°)	2	2
Rotation of order 6 (60° or 300°)	2	0
Reflection across diagonal	3	4
Reflection across side bisector	3	0

So, there are

$$\frac{1}{12}(1 \cdot 20 + 1 \cdot 0 + 2 \cdot 2 + 2 \cdot 0 + 3 \cdot 4 + 3 \cdot 0) = 3$$

nonequivalent ways to string three black beads and three white beads on a necklace.

Now that we have gotten our feet wet on a few easy problems, let's try a more difficult one. Suppose that we have the colors red (R), white (W), and blue (B) that can be used to color the edges of a regular tetrahedron (see Figure 5.1). First, observe that there are $3^6 = 729$ colorings without regard to equivalence. How shall we decide when two colorings of the tetrahedron are nonequivalent? Certainly, if we were to pick up a tetrahedron colored in a certain manner, rotate it, and put it back down, we would think of the tetrahedron as being positioned differently rather than as being colored differently (just as if we picked up a die labeled in the usual way and rolled it, we would not say that the die is now differently labeled). So, our permutation group for this problem is just the group of 12 rotations of the tetrahedron shown in Figure 5.1 and is isomorphic to A_4. (The group consists of the identity; eight elements of order 3, each of which fixes one vertex; and three elements of order 2, each of which fixes no vertex.) Every rotation permutes the 729 colorings, and to apply Burnside's Theorem we must determine the size of fix(ϕ) for each of the 12 rotations of the group.

Clearly, the identity fixes all 729 colorings. Next, consider the element (234) of order 3, shown in the bottom row, second from the left in Figure 5.1. Suppose that a specific coloring is fixed by this element (i.e., the tetrahedron appears to be colored the same before and after this rotation). Since (234) carries edge 12 to edge 13, edge 13 to edge 14, and edge 14 to edge 12, these three edges must agree in color (edge ij is the edge joining vertex i and vertex j). The same argument shows that the three edges 23, 34, and 42 also must agree in color. So, $|\text{fix}(234)| = 3^2$, since there are three choices for each of these two sets of three edges. The nine columns in Table 27.3 show the possible colorings of the two sets of three edges. The analogous analysis applies to the other seven elements of order 3.

Now consider the rotation (12)(34) of order 2. (See the second tetrahedron in the top row in Figure 5.1.) Since edges 12 and 34 are fixed, they may be colored in any way and will appear the same after the rotation (12)(34). This yields $3 \cdot 3$ choices for those two edges. Since edge 13 is carried to edge 24, these two edges must agree in color. Similarly, edges 23 and 14 must agree. So, we have three choices for the pair of edges 13 and 24 and three choices for the pair of edges 23 and 14. This means that we have

Table 27.3 Nine Colorings Fixed by (234).

Edge	Colorings								
12	R	R	R	W	W	W	B	B	B
13	R	R	R	W	W	W	B	B	B
14	R	R	R	W	W	W	B	B	B
23	R	W	B	W	R	B	B	R	W
34	R	W	B	W	R	B	B	R	W
24	R	W	B	W	R	B	B	R	W

$3 \cdot 3 \cdot 3 \cdot 3$ ways to color the tetrahedron that will be equivalent under (12)(34). (Table 27.4 gives the complete list of 81 colorings.) So, $|\text{fix}((12)(34))| = 3^4$, and the other two elements of order 2 yield the same results.

Now that we have analyzed the three types of group elements, we can apply Burnside's Theorem. In particular, the number of distinct colorings of the edges of a tetrahedron with three colors is

$$\frac{1}{12}(1 \cdot 3^6 + 8 \cdot 3^2 + 3 \cdot 3^4) = 87.$$

Surely it would be a difficult task to solve this problem without Burnside's Theorem.

Just as surely, you are wondering who besides mathematicians are interested in counting problems such as the ones we have discussed. Well, chemists are. Indeed, one set of benzene derivatives can be viewed as six carbon atoms arranged in a hexagon with one of the three radicals NH_2, COOH, or OH attached at each carbon atom. See Figure 27.4 for one example.

So Burnside's Theorem enables a chemist to determine the number of benzene molecules (see Exercise 4). Another kind of molecule considered by chemists is visualized as a regular tetrahedron with a carbon atom at the center and any of the four radicals $HOCH_2$ (hydroxymethyl), C_2H_5 (ethyl), Cl (chlorine), or H (hydrogen) at the four vertices. Again, the number of such molecules can be easily counted using Burnside's Theorem.

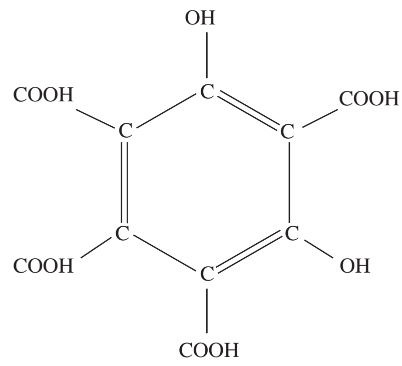


Figure 27.4 Benzene derivative.

Table 27.4 81 Colorings Fixed by (12)(34) (X and Y can be any of R, W, and B.)

Edge	Colorings								
12	X	X	X	X	X	X	X	X	X
34	Y	Y	Y	Y	Y	Y	Y	Y	Y
13	R	R	R	W	W	W	B	B	B
24	R	R	R	W	W	W	B	B	B
23	R	W	B	W	R	B	B	R	W
14	R	W	B	W	R	B	B	R	W

Group Action

Our informal approach to counting the number of objects that are considered nonequivalent can be made formal as follows. If G is a group and S is a set of objects, we say that G *acts on* S if there is a homomorphism γ from G to sym(S), the group of all permutations on S. (The homomorphism is sometimes called the *group action.*) For convenience, we denote the image of g under γ as γ_g. Then two objects x and y in S are viewed as equivalent under the action of G if and only if $\gamma_g(x) = y$ for some g in G.

Notice that when γ is one-to-one, the elements of G may be regarded as permutations on S. On the other hand, when γ is not one-to-one, the elements of G may still be regarded as permutations on S, but there are distinct elements g and h in G such that γ_g and γ_h induce the same permutation on S [that is, $\gamma_g(x) = \gamma_h(x)$ for all x in S]. Thus, a group acting on a set is a natural generalization of the permutation group concept.

As an example of group action, let S be the two diagonals of a square and let G be D_4, the group of symmetries of the square. Then γ_{R_0}, $\gamma_{R_{180}}$, γ_D, $\gamma_{D'}$ are the identity; $\gamma_{R_{90}}$, $\gamma_{R_{270}}$, γ_H, γ_V interchange the two diagonals; and the mapping $g \to \gamma_g$ from D_4 to $\text{sym}(S)$ is a group homomorphism. As a second example, note that $GL(n, F)$, the group of invertible $n \times n$ matrices with entries from a field F, acts on the set S of $n \times 1$ column vectors with entries from F by multiplying the vectors on the left by the matrices. In this case, the mapping $g \to \gamma_g$ from $GL(n, F)$ to $\text{sym}(S)$ is a one-to-one homomorphism.

We have used group actions several times in this text without calling them that. The proof of Cayley's Theorem (Theorem 6.5) has a group G acting on the elements of G; the proofs of Sylow's Second Theorem and Third Theorem (Theorems 23.4 and 23.5) have a group acting on the set of conjugates of a Sylow p-subgroup; and the proof of the Generalized Cayley Theorem (Theorem 24.3) has G acting on the left cosets of a subgroup H.

Exercises

The greater the difficulty, the more glory in surmounting it.

Epicurus

1. Determine the number of ways in which the four corners of a square can be colored with two colors. (It is permissible to use a single color on all four corners.)

2. Determine the number of different necklaces that can be made using 13 white beads and 3 black beads.

3. Determine the number of ways in which the vertices of an equilateral triangle can be colored with five colors so that at least two colors are used.

4. A benzene molecule can be modeled as six carbon atoms arranged in a regular hexagon in a plane. At each carbon atom, one of three radicals NH_2, COOH, or OH can be attached. How many such compounds are possible? (Make no distinction between single and double bonds between the atoms.)

5. Suppose that in Exercise 4 we permit only NH_2 and COOH for the radicals. How many compounds are possible?

6. Determine the number of ways in which the faces of a regular dodecahedron (regular 12-sided solid) can be colored with three colors.

7. Determine the number of ways in which the edges of a square can be colored with six colors so that no color is used on more than one edge.

8. Determine the number of ways in which the edges of a square can be colored with six colors with no restriction placed on the number of times a color can be used.

9. Determine the number of different 11-bead necklaces that can be made using two colors.

10. Determine the number of ways in which the faces of a cube can be colored with three colors.

11. Suppose a cake is cut into six identical pieces. How many ways can we color the cake with n colors assuming that each piece receives one color?

12. How many ways can the five points of a five-pointed crown be painted if three colors of paint are available?

13. Let G be a finite group and let $\mathrm{sym}(G)$ be the group of all permutations on G. For each g in G, let ϕ_g denote the element of $\mathrm{sym}(G)$ defined by $\phi_g(x) = gxg^{-1}$ for all x in G. Show that G acts on itself under the action $g \to \phi_g$. Give an example in which the mapping $g \to \phi_g$ is not one-to-one.

14. Let G be a finite group, let H be a subgroup of G, and let S be the set of left cosets of H in G. For each g in G, let γ_g denote the element of $\mathrm{sym}(S)$ defined by $\gamma_g(xH) = gxH$. Show that G acts on S under the action $g \to \gamma_g$.

15. For a fixed square, let L_1 be the perpendicular bisector of the top and bottom of the square and let L_2 be the perpendicular bisector of the left and right sides. Show that D_4 acts on $\{L_1, L_2\}$ and determine the kernel of the mapping $g \to \gamma_g$.

William Burnside

By permission of the Master and Fellows of Pembroke College in the University of Oxford

In one of the most abstract domains of thought, he [Burnside] has systematized and amplified its range so that, there, his work stands as a landmark in the widening expanse of knowledge. Whatever be the estimate of Burnside made by posterity, contemporaries salute him as a Master among the mathematicians of his own generation.

A. R. FORSYTH

WILLIAM BURNSIDE was born on July 2, 1852, in London. After graduating from Cambridge University in 1875, Burnside was appointed lecturer at Cambridge, where he stayed until 1885. He then accepted a position at the Royal Naval College at Greenwich and spent the rest of his career in that post.

Burnside wrote more than 150 research papers in many fields. He is best remembered, however, for his pioneering work in group theory and his classic book *Theory of Groups*, which first appeared in 1897. Because of Burnside's emphasis on the abstract approach, many consider him to be the first pure group theorist. One mark of greatness in a mathematician is the ability to pose important and challenging problems—problems that open up new areas of research for future generations. Here, Burnside excelled. It was he who first conjectured that a group G of odd order has a series of normal subgroups, $G = G_0 \geq G_1 \geq G_2 \geq \cdots \geq G_n = \{e\}$, such that G_i/G_{i+1} is Abelian. This extremely important conjecture was finally proved more than 50 years later by Feit and Thompson in a 255-page paper (see Chapter 24 for more on this). In 1994, Efim Zelmanov received the Fields Medal for his work on a variation of one of Burnside's conjectures.

Burnside was elected a Fellow of the Royal Society and awarded two Royal Medals. He served as president of the Council of the London Mathematical Society and received its De Morgan Medal. Burnside died on August 21, 1927.

28 Cayley Digraphs of Groups

The important thing in science is not so much to obtain new facts as to discover new ways of thinking about them.

Sir William Lawrence Bragg, *Beyond Reductionism*

The changing of a vague difficulty into a specific, concrete form is a very essential element in thinking.

J. P. Morgan

Motivation

In this chapter, we introduce a graphical representation of a group given by a set of generators and relations. The idea was originated by Cayley in 1878. Although this topic is not usually covered in an abstract algebra book, we include it for five reasons: It provides a method of visualizing a group; it connects two important branches of modern mathematics—groups and graphs; it has many applications to computer science; it gives a review of some of our old friends—cyclic groups, dihedral groups, direct products, and generators and relations; and, most importantly, it is fun!

Intuitively, a directed graph (or digraph) is a finite set of points, called *vertices*, and a set of arrows, called *arcs*, connecting some of the vertices. Although there is a rich and important general theory of directed graphs with many applications, we are interested only in those that arise from groups.

The Cayley Digraph of a Group

Definition **Cayley Digraph of a Group**

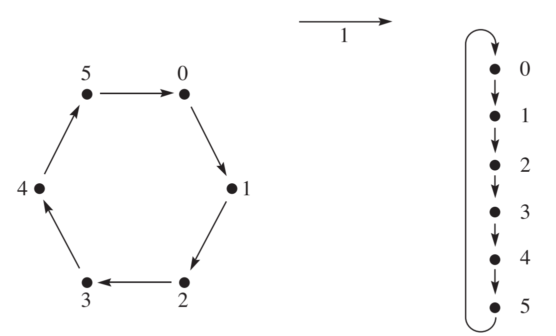
Let G be a finite group and let S be a set of generators for G. We define a digraph $\mathrm{Cay}(S{:}G)$, called the *Cayley digraph of G with generating set S*, as follows:

1. **Each element of G is a vertex of Cay(S:G).**
2. **For x and y in G, there is an arc from x to y if and only if $xs = y$ for some $s \in S$.**

To tell from the digraph which particular generator connects two vertices, Cayley proposed that each generator be assigned a color, and that the arrow joining x to xs be colored with the color assigned to s. He called the resulting figure the *color graph of the group.* This terminology is still used occasionally. Rather than use colors to distinguish the different generators, we will use solid arrows, dashed arrows, and dotted arrows. In general, if there is an arc from x to y, there need not be an arc from y to x. An arrow emanating from x and pointing to y indicates that there is an arc from x to y.

Following are numerous examples of Cayley digraphs. Note that there are several ways to draw the digraph of a group given by a particular generating set. However, it is not the appearance of the digraph that is relevant but the manner in which the vertices are connected. These connections are uniquely determined by the generating set. Thus, distances between vertices and angles formed by the arcs have no significance. (In the digraphs below, a headless arrow joining two vertices x and y indicates that there is an arc from x to y and an arc from y to x. This occurs when the generating set contains both an element and its inverse. For example, a generator of order 2 is its own inverse.)

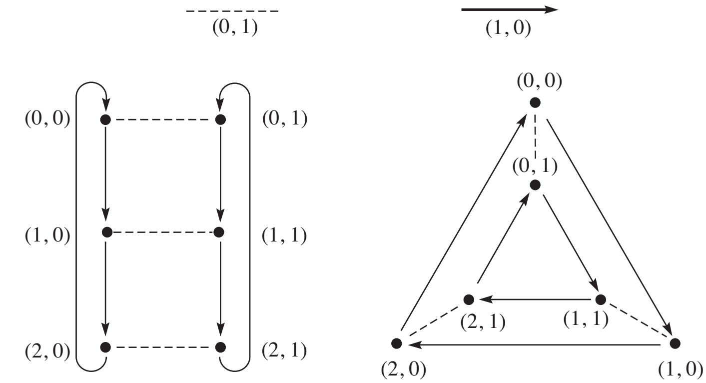
EXAMPLE 1 $Z_6 = \langle 1 \rangle$.



Cay({1}:Z_6)　　Cay({1}:Z_6)

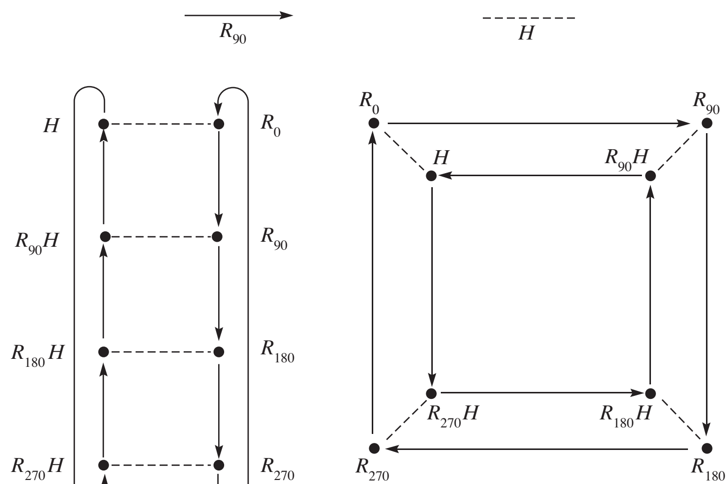
■

EXAMPLE 2 $Z_3 \oplus Z_2 = \langle (1, 0), (0, 1) \rangle$.



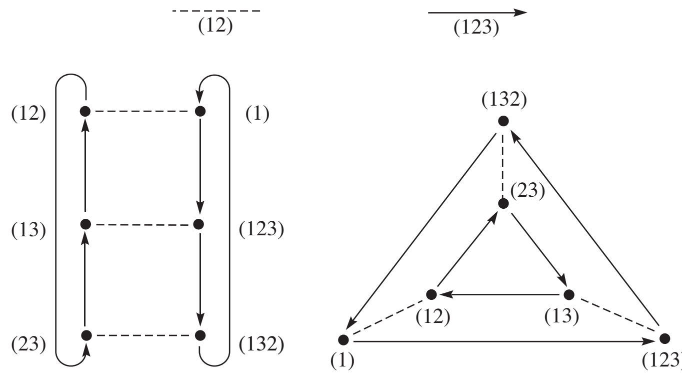
Cay({(1, 0), (0, 1)}:$Z_3 \oplus Z_2$) Cay({(1, 0), (0, 1)}:$Z_3 \oplus Z_2$)

EXAMPLE 3 $D_4 = \langle R_{90}, H \rangle$.



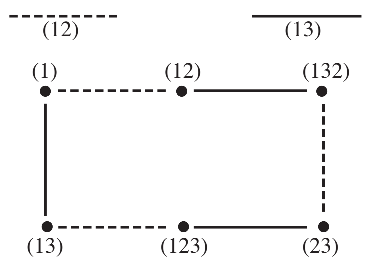
Cay({R_{90}, H}:D_4) Cay({R_{90}, H}:D_4)

EXAMPLE 4 $S_3 = \langle (12), (123) \rangle$.



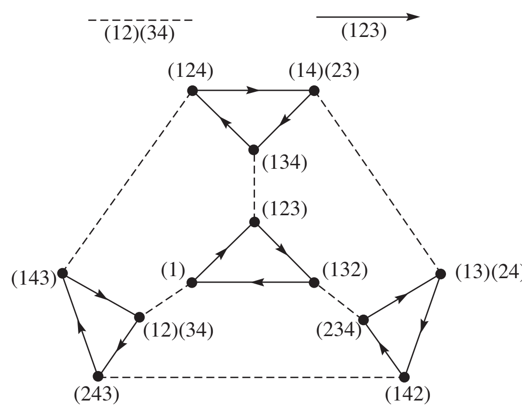
Cay({(12), (123)}:S_3) Cay({(12), (123)}:S_3)

EXAMPLE 5 $S_3 = \langle (12), (13) \rangle$.



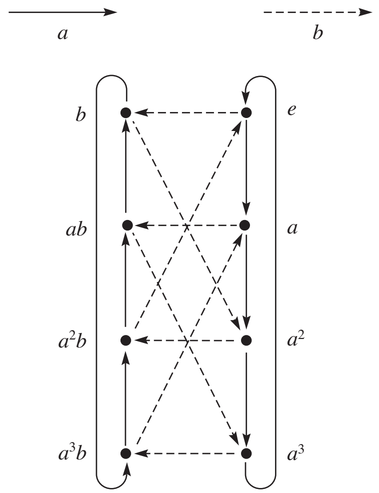
Cay({(12), (13)}:S_3)

EXAMPLE 6 $A_4 = \langle (12)(34), (123) \rangle$.



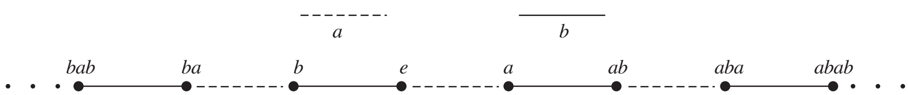
Cay({(12)(34), (123)}:A_4)

EXAMPLE 7 $Q_4 = \langle a, b \mid a^4 = e, a^2 = b^2, b^{-1}ab = a^3 \rangle$.



Cay({a, b}:Q_4)

EXAMPLE 8 $D_\infty = \langle a, b \mid a^2 = b^2 = e \rangle$.



Cay({a, b}:D_∞)

The Cayley digraph provides a quick and easy way to determine the value of any product of the generators and their inverses. Consider, for example, the product ab^3ab^{-2} from the group given in Example 7. To reduce this to one of the eight elements used to label the vertices, we need only begin at the vertex e and follow the arcs from each vertex to the next as specified in the given product. Of course, b^{-1} means traverse the b arc in reverse. (Observations such as $b^{-3} = b$ also help.) Tracing the product through, we obtain b. Similarly, one can verify or discover other relations among the generators.

Hamiltonian Circuits and Paths

Now that we have these directed graphs, what is it that we care to know about them? One question about directed graphs that has been the object of much research was popularized by the Irish mathematician Sir William Hamilton in 1859, when he invented a puzzle called "Around the World." His idea was to label the 20 vertices of a regular dodecahedron with the names of famous cities. One solves this puzzle by starting at any particular city (vertex) and traveling "around the world," moving along the arcs in such a way that each other city is visited exactly once before returning to the original starting point. One solution to this puzzle is given in Figure 28.1, where the vertices are visited in the order indicated.

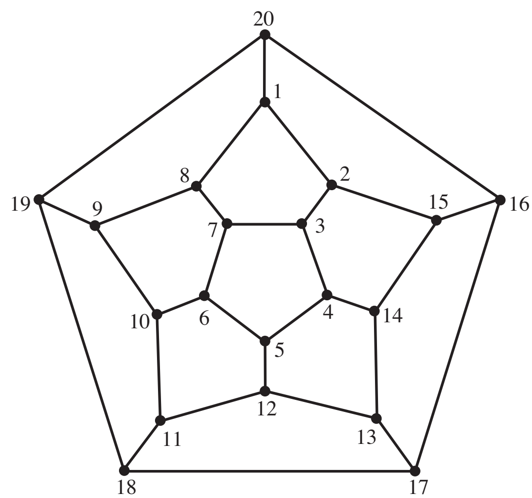


Figure 28.1 Around the World.

Obviously, this idea can be applied to any digraph; that is, one starts at some vertex and attempts to traverse the digraph by moving along arcs in such a way that each vertex is visited exactly once before returning to the starting vertex. (To go from x to y, there must be an arc from x to y.) Such a sequence of arcs is called a *Hamiltonian circuit* in the digraph. A sequence of arcs that passes through each vertex exactly once without returning to the starting point is called a *Hamiltonian path.* In the rest of this chapter, we concern ourselves with the existence of Hamiltonian circuits and paths in Cayley digraphs.

Figures 28.2 and 28.3 show a Hamiltonian path for the digraph given in Example 2 and a Hamiltonian circuit for the digraph given in Example 7, respectively.

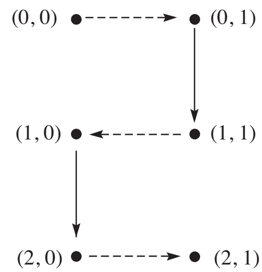


Figure 28.2 Hamiltonian path in $\mathrm{Cay}(\{(1,0),(0,1)\} : Z_3 \oplus Z_2)$ from (0, 0) to (2, 1).

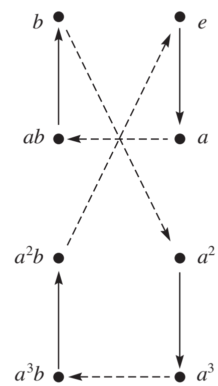


Figure 28.3 Hamiltonian circuit in $\mathrm{Cay}(\{a, b\} : Q_4)$.

Is there a Hamiltonian circuit in

$$\mathrm{Cay}(\{(1,0),(0,1)\} : Z_3 \oplus Z_2)?$$

More generally, let us investigate the existence of Hamiltonian circuits in

$$\mathrm{Cay}(\{(1,0),(0,1)\} : Z_m \oplus Z_n),$$

where m and n are relatively prime and both are greater than 1. Visualize the Cayley digraph as a rectangular grid coordinatized with $Z_m \oplus Z_n$, as in Figure 28.4. Suppose there is a Hamiltonian circuit in the digraph and (a, b) is some vertex from which the circuit exits horizontally. (Clearly, such a vertex exists.) Then the circuit must exit $(a-1, b+1)$ horizontally also, for otherwise the circuit passes through $(a, b+1)$ twice—see Figure 28.5. Repeating this argument again and again, we see that the circuit exits horizontally from each of the vertices $(a, b), (a-1, b+1), (a-2, b+2), \ldots,$

which is just the coset $(a, b) + \langle(-1, 1)\rangle$. But when m and n are relatively prime, $\langle(-1, 1)\rangle$ is the entire group. Obviously, there cannot be a Hamiltonian circuit consisting entirely of horizontal moves. Let us record what we have just proved.

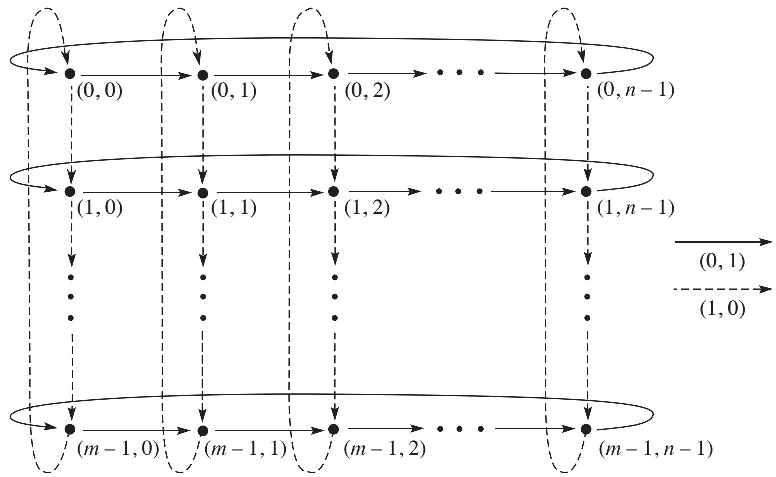


Figure 28.4 $\mathrm{Cay}(\{(1,0),(0,1)\} : Z_m \oplus Z_n)$.

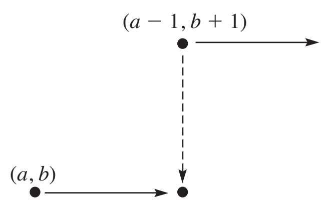


Figure 28.5

Theorem 28.1 A Necessary Condition

> **$\mathrm{Cay}(\{(1,0),(0,1)\} : Z_m \oplus Z_n)$** ***does not have a Hamiltonian circuit when m and n are relatively prime and greater than 1.***

What about when m and n are not relatively prime? In general, the answer is somewhat complicated, but the following special case is easy to prove.

Theorem 28.2 A Sufficient Condition

Cay$(\{(1,0),(0,1)\} : Z_m \oplus Z_n)$ *has a Hamiltonian circuit when n divides m.*

PROOF Say $m = kn$. Then we may think of $Z_m \oplus Z_n$ as k blocks of size $n \times n$. (See Figure 28.6 for an example.) Start at (0, 0) and cover the vertices of the top block as follows. Use the generator (0, 1) to move horizontally across the first row to the end. Then use the generator (1, 0) to move vertically to the point below, and cover the remaining points in the second row by moving horizontally. Keep this process up until the point $(n-1, 0)$—the lower lefthand corner of the first block—has been reached. Next, move vertically to the second block and repeat the process used in the first block. Keep this up until the bottom block is covered. Complete the circuit by moving vertically back to (0, 0). ∎

Notice that the circuit given in the proof of Theorem 28.2 is easy to visualize but somewhat cumbersome to describe in words. A much more convenient way to describe a Hamiltonian path or circuit is to specify the starting vertex and the sequence of generators in the order in which they are to be applied. In Example 5, for instance, we may start at (1) and alternate the generators (12) and (13) until we return to (1). In Example 3, we may start at R_0 and successively apply $R_{90}, R_{90}, R_{90}, H, R_{90}, R_{90}, R_{90}, H$. When k is a positive integer and $a, b, \ldots, c$ is a sequence of group elements, we use $k*(a, b, \ldots, c)$ to denote the concatenation of k copies of the sequence $(a, b, \ldots, c)$. Thus, $2*(R_{90}, R_{90}, R_{90}, H)$ and $2*(3*R_{90}, H)$ both mean $R_{90}, R_{90}, R_{90}, H, R_{90}, R_{90}, R_{90}, H$. With this notation, we may conveniently denote the Hamiltonian circuit given in Theorem 28.2 as

$$m * ((n-1) * (0,1), (1,0)).$$

We leave it as an exercise (Exercise 11) to show that if $x_1, x_2, \ldots, x_n$ is a sequence of generators determining a Hamiltonian circuit starting at some vertex, then the same sequence determines a Hamiltonian circuit for any starting vertex.

From Theorem 28.1, we know that there are some Cayley digraphs of Abelian groups that do not have any Hamiltonian circuits. But Theorem 28.3 shows that each of these Cayley digraphs does have a Hamiltonian path. There are some Cayley digraphs

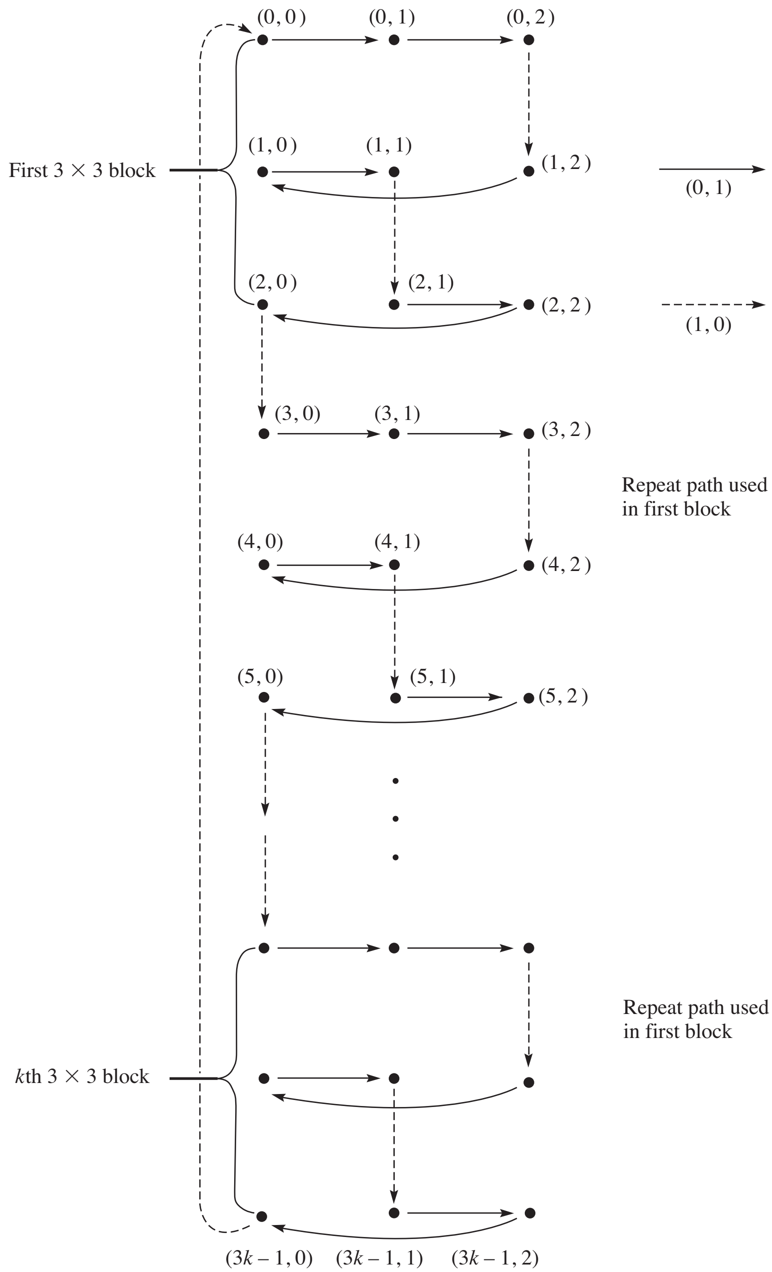


Figure 28.6 $\mathrm{Cay}(\{(1,0),(0,1)\} : Z_{3k} \oplus Z_3)$.

for *non-Abelian* groups that do not even have Hamiltonian paths, but we will not discuss them here.

Theorem 28.3 Abelian Groups Have Hamiltonian Paths

Let G be a finite Abelian group, and let S be any generating set for G. Then Cay$(S{:}G)$ has a Hamiltonian path.

PROOF We use induction on $|S|$. If $|S| = 1$, say, $S = \{a\}$, then the digraph is just a circle labeled with $e, a, a^2, \ldots, a^{m-1}$, where $|a| = m$. Obviously, there is a Hamiltonian path for this case. Now assume that $|S| > 1$. Choose some $s \in S$. Let $T = S - \{s\}$—that is, T is S with s removed—and set $H = \langle s_1, s_2, \ldots, s_{n-1}\rangle$ where $S = \{s_1, s_2, \ldots, s_n\}$ and $s = s_n$. (Notice that H may be equal to G.)

Because $|T| < |S|$ and H is a finite Abelian group, the induction hypothesis guarantees that there is a Hamiltonian path $(a_1, a_2, \ldots, a_k)$ in Cay$(T : H)$. We will show that

$$(a_1, a_2, \ldots, a_k, s, a_1, a_2, \ldots, a_k, s, \ldots, a_1, a_2, \ldots, a_k, s, a_1, a_2, \ldots, a_k),$$

where $a_1, a_2, \ldots, a_k$ occurs $|G|/|H|$ times and s occurs $|G|/|H| - 1$ times, is a Hamiltonian path in Cay$(S{:}G)$.

Since $S = T \cup \{s\}$ and T generates H, the coset Hs generates the factor group G/H. (Since G is Abelian, this group exists.) Hence, the cosets of H are $H, Hs, Hs^2, \ldots, Hs^n$, where $n = |G|/|H| - 1$. Starting from the identity element of G, the path given by $(a_1, a_2, \ldots, a_k)$ visits each element of H exactly once [because $(a_1, a_2, \ldots, a_k)$ is a Hamiltonian path in Cay$(T{:}H)$]. The generator s then moves us to some element of the coset Hs. Starting from there, the path $(a_1, a_2, \ldots, a_k)$ visits each element of Hs exactly once. Then, s moves us to the coset Hs^2, and we visit each element of this coset exactly once. Continuing this process, we successively move to $Hs^3, Hs^4, \ldots, Hs^n$, visiting each vertex in each of these cosets exactly once. Because each vertex of Cay$(S{:}G)$ is in exactly one coset Hs^i, this implies that we visit each vertex of Cay$(S{:}G)$ exactly once. Thus, we have a Hamiltonian path. ∎

We next look at Cayley digraphs with three generators.

EXAMPLE 9 Let

$$D_3 = \langle r, f \,|\, r^3 = f^2 = e, rf = fr^2\rangle.$$

Then a Hamiltonian circuit in

$$\text{Cay}(\{(r,0),(f,0),(e,1)\} : D_3 \oplus Z_6)$$

is given in Figure 28.7. ■

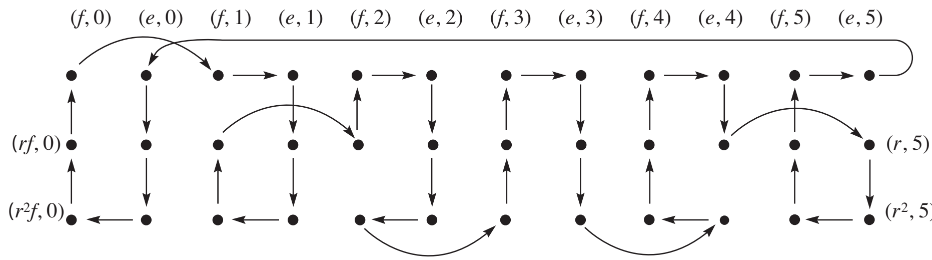


Figure 28.7 $\text{Cay}(\{(r,0),(f,0),(e,1)\} : D_3 \oplus Z_6)$.

Although it is not easy to prove, it is true that

$$\text{Cay}(\{(r,0),(f,0),(e,1)\} : D_n \oplus Z_m)$$

has a Hamiltonian circuit for all n and m. Example 10 shows the circuit for this digraph when m is even.

■ **EXAMPLE 10** Let

$$D_n = \langle r, f | r^n = f^2 = e, rf = fr^{-1} \rangle.$$

Then a Hamiltonian circuit in

$$\text{Cay}(\{(r,0),(f,0),(e,1)\} : D_n \oplus Z_m)$$

with m even is traced in Figure 28.8. The sequence of generators that traces the circuit is

$$m * [(n-1) * (r,0), (f,0), (n-1) * (r,0), (e,1)].$$ ■

Some Applications

Cayley digraphs are natural models for interconnection networks in computer designs, and Hamiltonicity is an important property in relation to sorting algorithms on such networks. One particular Cayley digraph that is used to design and analyze interconnection networks of parallel machines is the symmetric group S_n with the

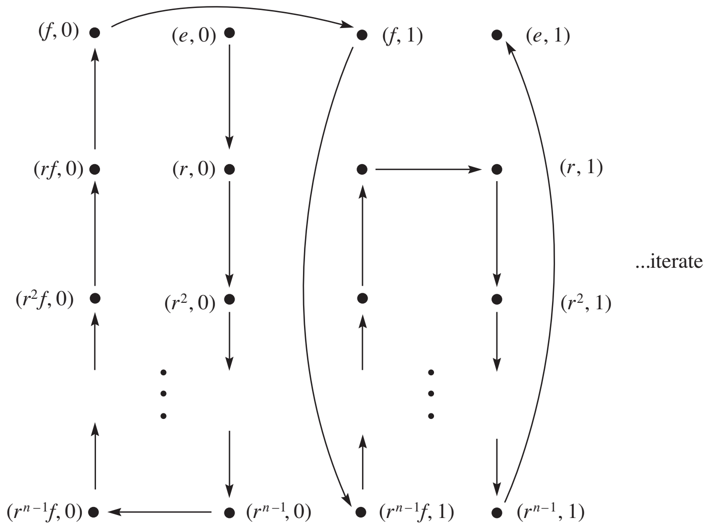


Figure 28.8 $\mathrm{Cay}(\{(r, 0), (f, 0), (e, 1)\} : D_n \oplus Z_m)$.

set of all transpositions as the generating set. Hamiltonian paths and circuits in Cayley digraphs arise in a variety of group theory contexts. A Hamiltonian path in a Cayley digraph of a group is simply an ordered listing of the group elements without repetition. The vertices of the digraph are the group elements, and the arcs of the path are generators of the group. In 1948, R. A. Rankin used these ideas (although not the terminology) to prove that certain bell-ringing exercises could not be done by the traditional methods employed by bell ringers. In 1981, Hamiltonian paths in Cayley digraphs were used in an algorithm for creating computer graphics of Escher-type repeating patterns in the hyperbolic plane. This program can produce repeating hyperbolic patterns in color from among various infinite classes of symmetry groups. The program has now been improved so that the user may choose from many kinds of color symmetry. The 2003 Mathematics Awareness Month poster featured one such image (see **mathaware.org/mam/03**). Two Escher drawings and their computer-drawn counterparts are given in Figures 28.9 through 28.12.

In this chapter, we have shown how one may construct a directed graph from a group. It is also possible to associate a group—called the *automorphism group*—with every directed graph. In fact, several of the 26 sporadic simple groups were first constructed in this way.

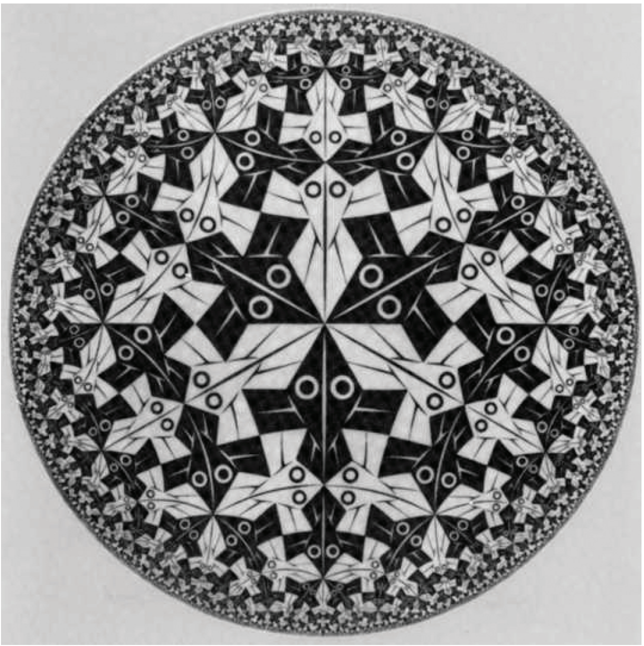
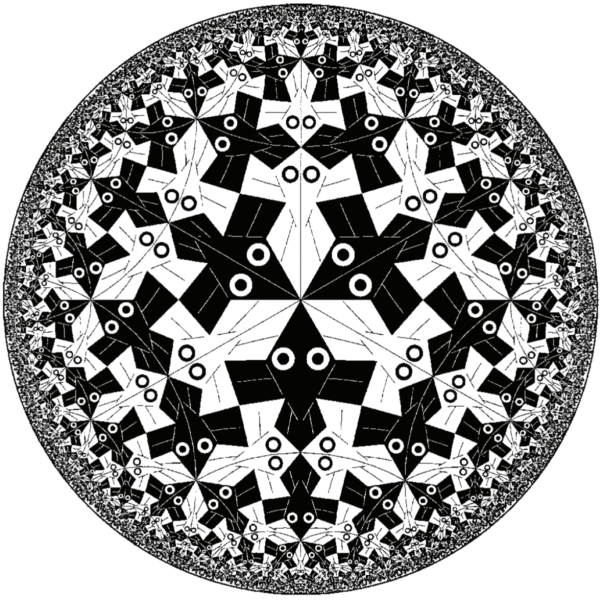


Figure 28.9 M. C. Escher's *Circle Limit I*.



Courtesy of Douglas Dunham.

Figure 28.10 A computer duplication of the pattern of M. C. Escher's *Circle Limit I* . The program used a Hamiltonian path in a Cayley digraph of the underlying symmetry group.

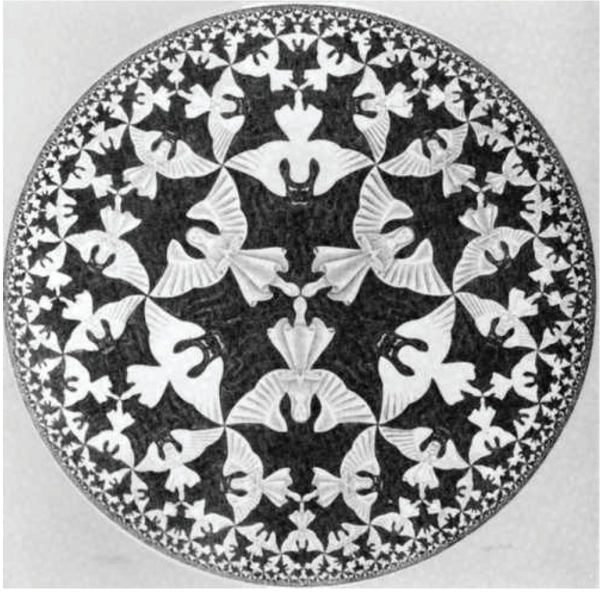
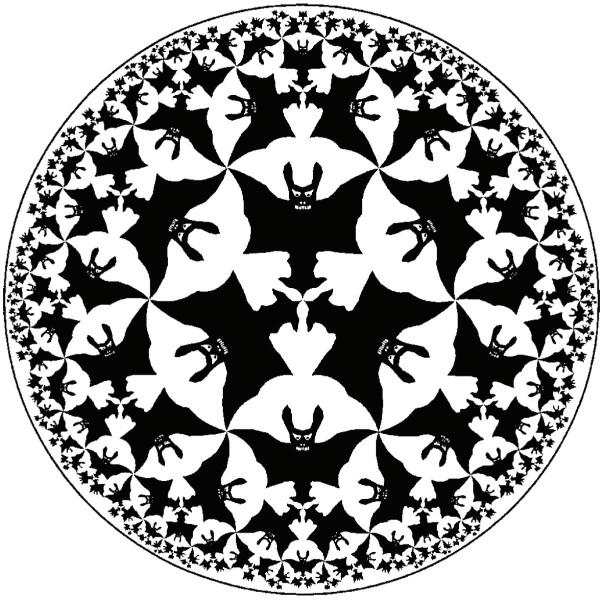


Figure 28.11 M. C. Escher's *Circle Limit IV*.



Courtesy of Douglas Dunham.

Figure 28.12 A computer drawing inspired by the pattern of M. C. Escher's *Circle Limit IV*. The program used a Hamiltonian path in a Cayley digraph of the underlying symmetry group.

Exercises

A mathematician is a machine for turning coffee into theorems.
Alfréd Rényi

1. Find a Hamiltonian circuit in the digraph given in Example 7 different from the one in Figure 28.3.

2. Find a Hamiltonian circuit in
$$\mathrm{Cay}(\{(a,0),(b,0),(e,1)\} : Q_4 \oplus Z_2).$$

3. Find a Hamiltonian circuit in
$$\mathrm{Cay}(\{(a,0),(b,0),(e,1)\} : Q_4 \oplus Z_m)$$
where m is even.

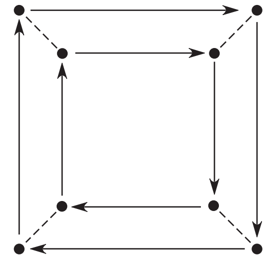
4. Write the sequence of generators for each of the circuits found in Exercises 1, 2, and 3.

5. Use the Cayley digraph in Example 7 to evaluate the product $a^3ba^{-1}ba^3b^{-1}$.

6. Let x and y be two vertices of a Cayley digraph. Explain why two paths from x to y in the digraph yield a group relation—that is, an equation of the form $a_1a_2\cdots a_m = b_1b_2\cdots b_n$, where the a_i's and b_j's are generators of the Cayley digraph.

7. Use the Cayley digraph in Example 7 to verify the relation $aba^{-1}b^{-1}a^{-1}b^{-1} = a^2ba^3$.

8. Identify the following Cayley digraph of a familiar group.



9. Let $D_4 = \langle r, f \mid r^4 = e = f^2, rf = fr^{-1}\rangle$. Verify that
$$6 * (3 * (r,0),(f,0),3 * (r,0),(e,1))$$
is a Hamiltonian circuit in
$$\mathrm{Cay}(\{(r,0),(f,0),(e,1)\} : D_4 \oplus Z_6).$$

10. Draw a picture of Cay($\{2, 5\} : Z_8$).

11. If $s_1, s_2, \ldots, s_n$ is a sequence of generators that determines a Hamiltonian circuit beginning at some vertex, explain why the same sequence determines a Hamiltonian circuit beginning at any point. (This exercise is referred to in this chapter.)

12. Show that the Cayley digraph given in Example 7 has a Hamiltonian path from e to a.

13. Show that there is no Hamiltonian path in

$$\text{Cay}(\{(1,0),(0,1)\} : Z_3 \oplus Z_2)$$

from (0, 0) to (2, 0).

14. Draw Cay($\{2, 3\} : Z_6$). Is there a Hamiltonian circuit in this digraph?

15. **a.** Let G be a group of order n generated by a set S. Show that a sequence $s_1, s_2, \ldots, s_{n-1}$ of elements of S is a Hamiltonian path in Cay(S:G) if and only if, for all i and j with $1 \leq i \leq j < n$, we have $s_i s_{i+1} \ldots s_j \neq e$.

b. Show that the sequence $s_1 s_2 \cdots s_n$ is a Hamiltonian circuit if and only if $s_1 s_2 \cdots s_n = e$, and that whenever $1 \leq i \leq j < n$, we have $s_i s_{i+1} \cdots s_j \neq e$.

16. Let $D_4 = \langle a, b | a^2 = b^2 = (ab)^4 = e \rangle$. Draw Cay($\{a, b\}$:$D_4$). Why is it reasonable to say that this digraph is undirected?

17. Let D_n be as in Example 10. Show that $2 * [(n-1) * r, f]$ is a Hamiltonian circuit in Cay($\{r, f\}$:D_n).

18. Let $Q_8 = \langle a, b \mid a^8 = e, a^4 = b^2, b^{-1}ab = a^{-1} \rangle$. Find a Hamiltonian circuit in Cay($\{a, b\}$:Q_8).

19. Let Q_8 be as in Exercise 18. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a,0),(b,0),(e,1)\}:Q_8 \oplus Z_5).$$

20. Prove that the Cayley digraph given in Example 6 does not have a Hamiltonian circuit. Does it have a Hamiltonian path?

21. Find a Hamiltonian circuit in

$$\text{Cay}(\{(R_{90},0),(H,0),(R_0,1)\}:D_4 \oplus Z_3).$$

Does this circuit generalize to the case $D_{n+1} \oplus Z_n$ for all $n \geq 3$?

22. Let Q_8 be as in Exercise 18. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a,0),(b,0),(e,1)\}{:}Q_8 \oplus Z_m) \text{ for all even } m.$$

23. Let Q_4 be as in Example 7. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a,0),(b,0),(e,1)\}{:}Q_4 \oplus Z_3).$$

24. Let Q_4 be as in Example 7. Find a Hamiltonian circuit in

$$\text{Cay}(\{(a,0),(b,0),(e,1)\}{:}Q_4 \oplus Z_m) \text{ for all odd } m \geq 3.$$

25. Write the sequence of generators that describes the Hamiltonian circuit in Example 9.

26. Let D_n be as in Example 10. Find a Hamiltonian circuit in

$$\text{Cay}(\{(r,0),(f,0),(e,1)\}{:}D_4 \oplus Z_5).$$

Does your circuit generalize to the case $D_n \oplus Z_{n+1}$ for all $n \geq 4$?

27. Prove that Cay$(\{(0,1),(1,1)\}{:}Z_m \oplus Z_n)$ has a Hamiltonian circuit for all m and n greater than 1.

28. Suppose that a Hamiltonian circuit exists for Cay$(\{(1,0),(0,1)\}{:}Z_m \oplus Z_n)$ and that this circuit exits from vertex (a, b) vertically. Show that the circuit exits from every member of the coset $(a,b) + \langle(1,-1)\rangle$ vertically.

29. Let $D_2 = \langle r, f \mid r^2 = f^2 = e, rf = fr^{-1}\rangle$. Find a Hamiltonian circuit in Cay$(\{(r,0),(f,0),(e,1)\}{:}D_2 \oplus Z_3)$.

30. Let Q_8 be as in Exercise 18. Find a Hamiltonian circuit in Cay$(\{(a,0),(b,0),(e,1)\}{:}Q_8 \oplus Z_3)$.

31. In Cay$(\{(1,0),(0,1)\}{:}Z_4 \oplus Z_5)$, find a sequence of generators that visits exactly one vertex twice and all others exactly once and returns to the starting vertex.

32. In Cay$(\{(1,0),(0,1)\}{:}Z_4 \oplus Z_5)$, find a sequence of generators that visits exactly two vertices twice and all others exactly once and returns to the starting vertex.

33. Find a Hamiltonian circuit in Cay$(\{(1,0),(0,1)\}{:}Z_4 \oplus Z_6)$.

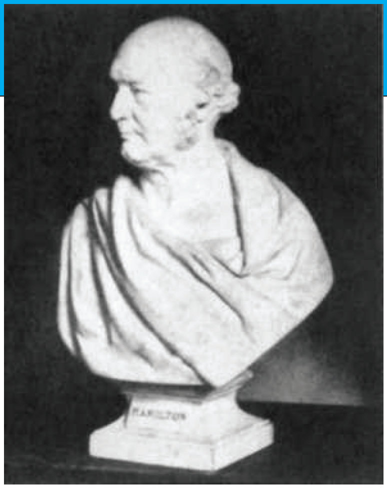
34. Let G be the digraph obtained from $\text{Cay}(\{(1,0),(0,1)\}:Z_3 \oplus Z_5)$ by deleting the vertex (0, 0). [Also, delete each arc to or from (0, 0).] Prove that G has a Hamiltonian circuit.

35. Prove that the digraph obtained from $\text{Cay}(\{(1,0),(0,1)\}:Z_4 \oplus Z_7)$ by deleting the vertex (0, 0) has a Hamiltonian circuit.

36. Let G be a finite group generated by a and b. Let $s_1, s_2, \ldots, s_n$ be the arcs of a Hamiltonian circuit in the digraph $\text{Cay}(\{a,b\}:G)$. We say that the vertex $s_1 s_2 \cdots s_i$ *travels by* a if $s_{i+1} = a$. Show that if a vertex x travels by a, then every vertex in the coset $x\langle ab^{-1}\rangle$ travels by a.

37. A finite group is called *Hamiltonian* if all of its subgroups are normal. (One non-Abelian example is Q_4.) Show that Theorem 28.3 can be generalized to include all Hamiltonian groups.

William Rowan Hamilton



Stock Montage

After Isaac Newton, the greatest mathematician of the English-speaking peoples is William Rowan Hamilton.

SIR EDMUND WHITTAKER,
Scientific American

WILLIAM ROWAN HAMILTON was born on August 3, 1805, in Dublin, Ireland. At three, he was skilled at reading and arithmetic. At five, he read and translated Latin, Greek, and Hebrew; at 14, he had mastered 14 languages, including Arabic, Sanskrit, Hindustani, Malay, and Bengali.

In 1833, Hamilton provided the first modern treatment of complex numbers. In 1843, he made what he considered his greatest discovery—the algebra of quaternions. The quaternions represent a natural generalization of the complex numbers with three numbers i, j, and k whose squares are -1. With these, rotations in three and four dimensions can be algebraically treated.

Of greater significance, however, is the fact that the quaternions are noncommutative under multiplication. This was the first ring to be discovered in which the commutative property does not hold. The essential idea for the quaternions suddenly came to Hamilton after 15 years of fruitless thought! Today Hamilton's name is attached to several concepts, such as the Hamiltonian function, which represents the total energy in a physical system; the Hamilton–Jacobi differential equations; and the Cayley–Hamilton Theorem from linear algebra. He also coined the terms *vector*, *scalar*, and *tensor*.

In his later years, Hamilton was plagued by alcoholism. He died on September 2, 1865, at the age of 60.

Paul Erdős

Paul Erdős is a socially helpless Hungarian who has thought about more mathematical problems than anyone else in history.

The Atlantic

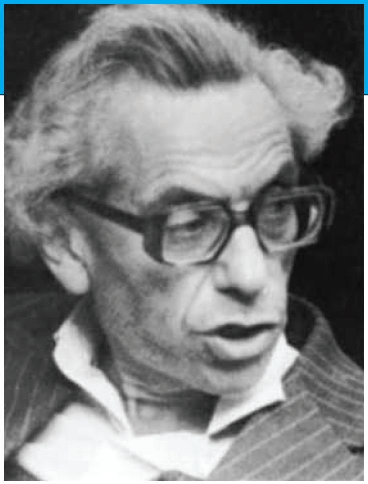


Photo by Seymour Schuster, Carleton College

Paul Erdős (pronounced AIR-dish) was one of the best-known and most highly respected mathematicians of the 20th century. Unlike most of his contemporaries, who have concentrated on theory building, Erdős focused on problem solving and problem posing. The problems and methods of solution of Erdős—like those of Euler, whose solutions to special problems pointed the way to much of the mathematical theory we have today—have helped pioneer new theories, such as combinatorial and probabilistic number theory, combinatorial geometry, probabilistic and transfinite combinatorics, and graph theory.

Erdős was born on March 26, 1913, in Hungary. Both of his parents were high school mathematics teachers. Erdős, a Jew, left Hungary in 1934 at the age of 21 because of the rapid rise of anti-Semitism in Europe. For the rest of his life he traveled incessantly, rarely pausing more than a month in any one place, giving lectures for small honoraria and staying with fellow mathematicians. He had little property and no fixed address.

All that he owned he carried with him in a medium-sized suitcase, frequently visiting as many as 15 places in a month. His motto was, "Another roof, another proof." Even in his eighties, he put in 19-hour days doing mathematics. Erdős wrote more than 1500 research papers. He coauthored papers with more than 500 people. These people are said to have Erdős number 1. People who do not have Erdős number 1, but who have written a paper with someone who does, are said to have Erdős number 2, and so on, inductively.

Erdős received the Cole Prize in number theory from the American Mathematical Society, the Wolf Prize for lifelong contributions, and was elected to the U.S. National Academy of Sciences. Erdős died of a heart attack on September 20, 1996.

29 Introduction to Algebraic Coding Theory

There is no branch of mathematics, however abstract, which may not some day be applied to phenomena of the real world.

Nikolai Lobatchevsky

Damn it, if the machine can detect an error, why can't it locate the position of the error and correct it?

Richard W. Hamming

Motivation

One of the most interesting and important applications of finite fields has been the development of algebraic coding theory. This theory, which originated in the late 1940s, was created in response to practical communication problems. (Algebraic coding has nothing to do with secret codes.) Algebraic codes are now used in compact disc and DVD players, fax machines, digital televisions, and bar code scanners, and are essential to computer maintenance.

To motivate this theory, imagine that we wish to transmit one of two possible signals to a spacecraft approaching Mars. If the proposed landing site appears unfavorable, we will command the craft to orbit the planet; otherwise, we will command the craft to land. The signal for orbiting will be a 0, and the signal for landing will be a 1. But it is possible that some sort of interference (called *noise*) could cause an incorrect message to be received. To decrease the chance of this happening, redundancy is built into the transmission process. For example, if we wish the craft to orbit Mars, we could send five 0's. The craft's onboard computer is programmed to take any five-digit message received and decode the result by majority rule. So, if 00000 is sent and 10001 is received, the computer decides that 0 was the intended message. Notice that, for the computer to make the wrong decision, at least three errors must occur during transmission. If we assume that errors occur independently, it is less likely that three errors will occur than that two or fewer errors will occur. For this reason, this decision process is frequently called the *maximum-likelihood decoding* procedure. Our particular situation is illustrated in Figure 29.1. The general coding procedure is illustrated in Figure 29.2.

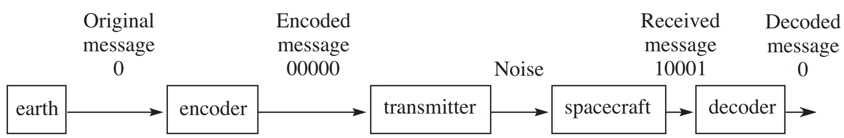


Figure 29.1 Encoding and decoding by fivefold repetition.

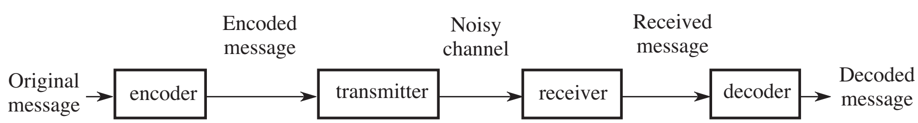


Figure 29.2 General encoding–decoding.

In practice, the means of transmission are telephone, radiowave, microwave, or even a magnetic disk. The noise might be human error, crosstalk, lightning, thermal noise, or deterioration of a disk. Throughout this chapter, we assume that errors in transmission occur independently. Different methods are needed when this is not the case.

Now, let's consider a more complicated situation. This time, assume that we wish to send a sequence of 0's and 1's of length 500. Further, suppose that the probability that an error will be made in the transmission of any particular digit is .01. If we send this message directly, without any redundancy, the probability that it will be received error-free is $(.99)^{500}$, or approximately .0066.

On the other hand, if we adopt a threefold repetition scheme by sending each digit three times and decoding each block of three digits received by majority rule, we can do much better. For example, the sequence 1011 is encoded as 111000111111. If the received message is 011000001110, the decoded message is 1001. Now, what is the probability that our 500-digit message will be error-free? Well, if a 1, say, is sent, it will be decoded as a 0 if and only if the block received is 001, 010, 100, or 000. The probability that this will occur is

$$\begin{aligned}(.01)(.01)(.99) + (.01)(.99)(.01) + (.99)(.01)(.01) + (.01)(.01)(.01) \\ = (.01)^2[3(.99) + .01] \\ = .000298 < .0003.\end{aligned}$$

Thus, the probability that any particular digit in the sequence will be decoded correctly is greater than .9997, and it follows that the probability that the entire 500-digit message will be decoded correctly is greater than $(.9997)^{500}$, or approximately .86—a dramatic improvement over .0066.

This example illustrates the three basic features of a code. There is a set of messages, a method of encoding these messages, and a method of decoding the received messages. The encoding procedure builds some redundancy into the original messages; the decoding procedure corrects or detects certain prescribed errors. Repetition codes have the advantage of simplicity of encoding and decoding, but they are too inefficient. In a fivefold repetition code, 80% of all transmitted information is redundant. The goal of coding theory is to devise message encoding and decoding methods that are reliable, efficient, and reasonably easy to implement.

Before plunging into the formal theory, it is instructive to look at a sophisticated example.

EXAMPLE 1 Hamming (7, 4) Code

This time, our message set consists of all possible 4-tuples of 0's and 1's (i.e., we wish to send a sequence of 0's and 1's of length 4). Encoding will be done by viewing these messages as 1×4 matrices with entries from Z_2 and multiplying each of the 16 messages on the right by the matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

(All arithmetic is done modulo 2.) The resulting 7-tuples are called *code words.* (See Table 29.1.)

Notice that the first four digits of each code word constitute just the original message corresponding to the code word. The last three digits of the code word constitute the redundancy features. For this code, we use the *nearest-neighbor* decoding method

Table 29.1

Message	Encoder G	Code Word	Message	Encoder G	Code Word
0000	$\rightarrow$	0000000	0110	$\rightarrow$	0110010
0001	$\rightarrow$	0001011	0101	$\rightarrow$	0101110
0010	$\rightarrow$	0010111	0011	$\rightarrow$	0011100
0100	$\rightarrow$	0100101	1110	$\rightarrow$	1110100
1000	$\rightarrow$	1000110	1101	$\rightarrow$	1101000
1100	$\rightarrow$	1100011	1011	$\rightarrow$	1011010
1010	$\rightarrow$	1010001	0111	$\rightarrow$	0111001
1001	$\rightarrow$	1001101	1111	$\rightarrow$	1111111

(which, in the case that the errors occur independently, is the same as the maximum-likelihood decoding procedure). For any received word v, we assume that the word sent is the code word v' that differs from v in the fewest number of positions. If the choice of v' is not unique, we can decide not to decode or arbitrarily choose one of the code words closest to v. (The first option is usually selected when retransmission is practical.)

Once we have decoded the received word, we can obtain the message by deleting the last three digits of v'. For instance, suppose that 1000 were the intended message. It would be encoded and transmitted as $u = 1000110$. If the received word were $v = 1100110$ (an error in the second position), it would still be decoded as u, since v and u differ in only one position, whereas v and any other code word would differ in at least two positions. Similarly, the intended message 1111 would be encoded as 1111111. If, instead of this, the word 0111111 were received, our decoding procedure would still give us the intended message 1111. ∎

The code in Example 1 is one of an infinite class of important codes discovered by Richard Hamming in 1948. The Hamming codes are the most widely used codes.

The Hamming (7, 4) encoding scheme can be conveniently illustrated with the use of a Venn diagram, as shown in Figure 29.3. Begin by placing the four message digits in the four overlapping regions I, II, III, and IV, with the digit in position 1 in region I, the digit in position 2 in region II, and so on. For regions V, VI, and VII, assign 0 or 1 so that the total number of 1s in each circle is even.

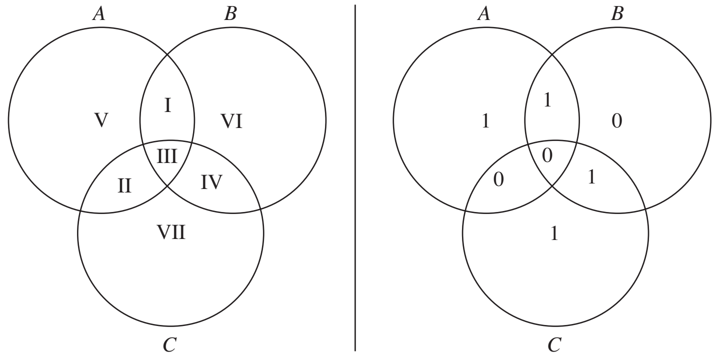
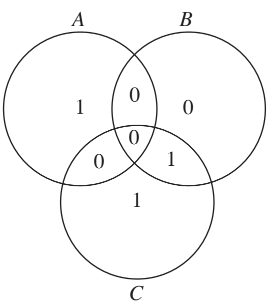


Figure 29.3 Venn diagram of the message 1001 and the encoded message 1001101.

Consider the Venn diagram of the received word 0001101:



How may we detect and correct an error? Well, observe that each of the circles A and B has an odd number of 1s. This tells us that something is wrong. At the same time, we note that circle C has an even number of 1s. Thus, the portion of the diagram that is in both A and B but not in C is the source of the error. See Figure 29.4.

Quite often, codes are used to detect errors rather than correct them. This is especially appropriate when it is easy to retransmit a message. If a received word is not a code word, we have detected an error. For example, computers are designed to use a parity check for numbers. Inside the computer, each number is represented by a string of 0's and 1's. If there is an even number of 1's in this representation, a 0 is attached to the string; if there is an odd number of 1's in the representation, a 1 is attached to the string.

Thus, each number stored in the computer memory has an even number of 1's. Now, when the computer reads a number from memory, it performs a parity check. If the read number has an odd number of 1's, the computer will know that an error has been made, and it will reread the number. Note that an even number of errors will not be detected by a parity check.

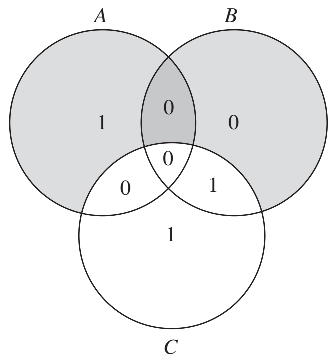


Figure 29.4 Circles A and B but not C have wrong parity.

The methods of error detection introduced in Chapters 0 and 5 are based on the same principle. An extra character is appended to a string of numbers so that a particular condition is satisfied. If we find that such a string does not satisfy that condition, we know that an error has occurred.

Linear Codes

We now formalize some of the ideas introduced in the preceding discussion.

Definitions Linear Code

An (n, k) *linear code* over a finite field F is a k-dimensional subspace V of the vector space

$$F^n = \underbrace{F \oplus F \oplus \cdots \oplus F}_{n \text{ copies}}$$

over F. The members of V are called the *code words*. When F is Z_2, the code is called *binary*.

One should think of an (n, k) linear code over F as a set of n-tuples from F, where each n-tuple has two parts: the message part, consisting of k digits; and the redundancy part, consisting of the remaining $n-k$ digits. Note that an (n, k) linear code over a finite field F of order q has q^k code words, since every member of the code is uniquely expressible as a linear combination of the k basis vectors with coefficients from F. The set of q^k code words is closed under addition and scalar multiplication by members of F. Also, since errors in transmission may occur in any of the n positions, there are q^n possible vectors that can be received. Where there is no possibility of confusion, it is customary to denote an n-tuple $(a_1, a_2, \ldots, a_n)$ more simply as $a_1a_2 \cdots a_n$, as we did in Example 1.

EXAMPLE 2 The set

$$\{0000000, 0010111, 0101011, 1001101, \\ 1100110, 1011010, 0111100, 1110001\}$$

is a (7, 3) binary code. ∎

EXAMPLE 3 The set $\{0000, 0101, 1010, 1111\}$ is a (4, 2) binary code. ∎

Although binary codes are by far the most important ones, other codes are occasionally used.

EXAMPLE 4 The set

$$\{0000, 0121, 0212, 1022, 1110, 1201, 2011, 2102, 2220\}$$

is a (4, 2) linear code over Z_3. A linear code over Z_3 is called a *ternary code.* ∎

To facilitate our discussion of the error-correcting and error-detecting capability of a code, we introduce the following terminology.

Definitions Hamming Distance, Hamming Weight

The *Hamming distance* between two vectors in F^n is the number of components in which they differ. The *Hamming weight* of a vector is the number of nonzero components of the vector. The *Hamming weight* of a linear code is the minimum weight of any nonzero vector in the code.

We will use $d(u, v)$ to denote the Hamming distance between the vectors u and v, and $\mathrm{wt}(u)$ for the Hamming weight of the vector u.

EXAMPLE 5 Let $s = 0010111, t = 0101011, u = 1001101$, and $v = 1101101$. Then, $d(s,t) = 4, d(s,u) = 4, d(s,v) = 5, d(u,v) = 1$; and $\text{wt}(s) = 4$, $\text{wt}(t) = 4$, $\text{wt}(u) = 4$, $\text{wt}(v) = 5$. ■

The Hamming distance and Hamming weight have the following important properties.

Theorem 29.1 Properties of Hamming Distance and Hamming Weight

For any vectors u, v, and w, $d(u,v) \le d(u,w) + d(w,v)$ and $d(u,v) = \text{wt}(u - v)$.

PROOF To prove that $d(u,v) = \text{wt}(u - v)$, simply observe that both $d(u,v)$ and $\text{wt}(u-v)$ equal the number of positions in which u and v differ. To prove that $d(u,v) \le d(u,w) + d(w,v)$, note that if u and v differ in the ith position and u and w agree in the ith position, then w and v differ in the ith position. ■

With the preceding definitions and Theorem 29.1, we can now explain why the codes given in Examples 1, 2, and 4 will correct any single error, but the code in Example 3 will not.

Theorem 29.2 Correcting Capability of a Linear Code

If the Hamming weight of a linear code is at least $2t + 1$, then the code can correct any t or fewer errors. Alternatively, the same code can detect any $2t$ or fewer errors.

PROOF We will use nearest-neighbor decoding; that is, for any received vector v, we will assume that the corresponding code word sent is a code word v' such that the Hamming distance $d(v, v')$ is a minimum. (If there is more than one such v', we do not decode.) Now, suppose that a transmitted code word u is received as the vector v and that at most t errors have been made in transmission. Then, by the definition of distance between u and v, we have $d(u,v) \le t$. If w is any code word other than u, then $w - u$ is a nonzero code word. Thus, by assumption,

$$2t + 1 \le \text{wt}(w - u) = d(w,u) \le d(w,v) + d(v,u) \le d(w,v) + t,$$

and it follows that $t + 1 \le d(w,v)$. So, the code word closest to the received vector v is u, and therefore v is correctly decoded as u.

To show that the code can detect $2t$ errors, we suppose that a transmitted code word u is received as the vector v and that at least one error, but no more than $2t$ errors, was made in transmission. Because only code words are transmitted, an error will be detected whenever a received word is not a code word. But v cannot be a code word, since $d(v, u) \leq 2t$, whereas we know that the minimum distance between distinct code words is at least $2t + 1$. ■

■ **EXAMPLE 6** Since the binary code $\{000, 001, 010, 100, 110, 101, 011, 111\}$ has weight $1 = 2t+1$, it will not detect any error ($t = 0$).

Since the binary code $\{000, 0101, 1010, 1111\}$ has weight $2 = 2t + 1$, it will not correct every 1 error ($t = 1/2$) but it will detect any 1 error.

Since the binary code $\{00000, 10011, 01010, 11001, 00101, 10110, 01111, 11100\}$ has weight $3 = 2t + 1$, it will correct any 1 error ($t = 1$) or it will detect any 2 or fewer errors.

Since the binary code $\{0000000, 0010111, 0101011, 1001101, 1100110, 1011010, 0111100, 1110001\}$ has weight $4 = 2t + 1$, it will correct any 1 error ($t = 3/2$) or it will detect any 3 or fewer errors. ■

Theorem 29.2 is often misinterpreted to mean that a linear code with Hamming weight $2t + 1$ can correct any t errors *and* detect any $2t$ or fewer errors simultaneously. This is not the case. The user must choose one or the other role for the code. Consider, for example, the Hamming (7, 4) code given in Table 29.1. By inspection, the Hamming weight of the code is $3 = 2 \cdot 1 + 1$, so we may elect either to correct any single error or to detect any one or two errors. To understand why we can't do both, consider the received word 0001010. The intended message could have been 0000000, in which case two errors were made (like-wise for the intended messages 1011010 and 0101110), or the intended message could have been 0001011, in which case one error was made. But there is no way for us to know which of these possibilities occurred. If our choice were error correction, we would assume—perhaps mistakenly—that 0001011 was the intended message. If our choice were error detection, we simply would not decode. (Typically, one would request retransmission.)

On the other hand, if we write the Hamming weight of a linear code in the form $2t + s + 1$, we can correct any t errors *and* detect any $t + s$ or fewer errors. Thus, for a code with Hamming weight 5, our options include the following:

1. Detect any four errors ($t = 0, s = 4$).
2. Correct any one error and detect any two or three errors ($t = 1, s = 2$).
3. Correct any two errors ($t = 2, s = 0$).

EXAMPLE 7 Since the Hamming weight of the linear code given in Example 2 is 4, it will correct any single error and detect any two errors ($t = 1, s = 1$) or detect any three errors ($t = 0, s = 3$). ∎

It is natural to wonder how the matrix G used to produce the Hamming code in Example 1 was chosen. Better yet, in general, how can one find a matrix G that carries a subspace V of F^k to a subspace of F^n in such a way that for any k-tuple v in V, the vector vG will agree with v in the first k components and build in some redundancy in the last $n - k$ components? Such a matrix is a $k \times n$ matrix of the form

$$\left[\begin{array}{cccc|ccc} 1 & 0 & \cdots & 0 & a_{11} & \cdots & a_{1n-k} \\ 0 & 1 & \cdots & 0 & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ \cdot & \cdot & & \cdot & \cdot & & \cdot \\ 0 & 0 & \cdots & 1 & a_{k1} & \cdots & a_{kn-k} \end{array}\right]$$

where the a_{ij}'s belong to F. A matrix of this form is called the *standard generator matrix* (or *standard encoding matrix*) for the resulting code.

Any $k \times n$ matrix whose rows are linearly independent will transform F^k to a k-dimensional subspace of F^n that could be used to build redundancy, but using the standard generator matrix has the advantage that the original message constitutes the first k components of the transformed vectors. An (n, k) linear code in which the k information digits occur at the beginning of each code word is called a *systematic code.* Schematically, we have the following.

message	$\longrightarrow$	message	+	redundant digits
k digits		k digits	+	$n - k$ digits

Notice that, by definition, a standard generator matrix produces a systematic code.

EXAMPLE 8 From the set of messages

$$\{000, 001, 010, 100, 110, 101, 011, 111\},$$

we may construct a (6, 3) linear code over Z_2 with the standard generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 1 \end{bmatrix}.$$

The resulting code words are given in Table 29.2. Since the minimum weight of any nonzero code word is 3, this code will correct any single error or detect any double error. ∎

Table 29.2

Message	Encoder G	Code Word
000	$\to$	000000
001	$\to$	001111
010	$\to$	010101
100	$\to$	100110
110	$\to$	110011
101	$\to$	101001
011	$\to$	011010
111	$\to$	111100

EXAMPLE 9 Here we take a set of messages as

$$\{00, 01, 02, 10, 11, 12, 20, 21, 22\},$$

and we construct a (4, 2) linear code over Z_3 with the standard generator matrix

$$G = \begin{bmatrix} 1 & 0 & 2 & 1 \\ 0 & 1 & 2 & 2 \end{bmatrix}.$$

The resulting code words are given in Table 29.3. Since the minimum weight of the code is 3, it will correct any single error or detect any double error. ∎

Table 29.3

Message	Encoder G	Code Word
00	$\rightarrow$	0000
01	$\rightarrow$	0122
02	$\rightarrow$	0211
10	$\rightarrow$	1021
11	$\rightarrow$	1110
12	$\rightarrow$	1202
20	$\rightarrow$	2012
21	$\rightarrow$	2101
22	$\rightarrow$	2220

Parity-Check Matrix Decoding

Now that we can conveniently encode messages with a standard generator matrix, we need a convenient method for decoding the received messages. Unfortunately, this is not as easy to do; however, in the case where at most one error per code word has occurred, there is a fairly simple method for decoding. (When more than one error occurs in a code word, our decoding method fails.)

To describe this method, suppose that V is a systematic linear code over the field F given by the standard generator matrix $G = [I_k|A]$, where I_k represents the $k \times k$ identity matrix and A is the $k \times (n-k)$ matrix obtained from G by deleting the first k columns of G. Then, the $n \times (n-k)$ matrix

$$H = \left[\frac{-A}{I_{n-k}}\right],$$

where $-A$ is the negative of A and I_{n-k} is the $(n-k) \times (n-k)$ identity matrix, is called the *parity-check matrix* for V. (In the literature, the transpose of H is called the parity-check matrix, but H is much more convenient for our purposes.) The decoding procedure is:

1.For any received word w, compute wH.

2. If wH is the zero vector, assume that no error was made.

3. If there is exactly one instance of a nonzero element $s \in F$ and a row i of H such that wH is s times row i, assume that the sent word was $w - (0 \ldots s \ldots 0)$, where s occurs in the ith component. If there is more than one such instance, do not decode.

3′. When the code is binary, category 3 reduces to the following: If wH is the ith row of H for exactly one i, assume that an error was made in the ith component of w. If wH is more than one row of H, do not decode.
4. If wH does not fit into either category 2 or category 3, we know that at least two errors occurred in transmission and we do not decode.

EXAMPLE 10 Consider the Hamming (7, 4) code given in Example 1. The generator matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

and the corresponding parity-check matrix is

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

Now, if the received vector is $v = 0000110$, we find $vH = 110$. Since this is the first row of H and no other row, we assume that an error has been made in the first position of v. Thus, the transmitted code word is assumed to be 1000110, and the corresponding message is assumed to be 1000. Similarly, if $w = 1011111$ is the received word, then $wH = 101$, and we assume that an error has been made in the second position. So, we assume that 1111111 was sent and that 1111 was the intended message. If the encoded message 1001101 is received as $z = 1001011$ (with errors in the fifth and sixth positions), we find $zH = 110$. Since this matches the first row of H, we decode z as 0001011 and incorrectly assume that the message 0001 was intended. On the other hand, nearest-neighbor decoding would yield the same incorrect result. ∎

Notice that when only one error was made in transmission, the parity-check decoding procedure gave us the originally intended message. We will soon see under what conditions this is true, but first we need an important fact relating a code given by a generator matrix and its parity-check matrix.

Lemma 29.1 Orthogonality Relation

Let C be a systematic (n, k) linear code over F with a standard generator matrix G and parity-check matrix H. Then, for any vector v in F^n, we have $vH = 0$ (the zero vector) if and only if v belongs to C.

PROOF First note that, since H has rank $n - k$, we may think of H as a linear transformation from F^n onto F^{n-k}. Therefore, it follows from the dimension theorem for linear transformations that $n = n - k +$ dim (Ker H), so that Ker H has dimension k. (Alternatively, one can use a group theory argument to show that $|\text{Ker } H| = |F|^k$.) Then, since the dimension of C is also k, it suffices to show that $C \subseteq \text{Ker } H$. To do this, let $G = [I_k|A]$, so that $H = \left[\frac{-A}{I_{n-k}}\right]$. Then,

$$GH = [I_k|A]\left[\frac{-A}{I_{n-k}}\right] = -A + A = [0] \quad \text{(the zero matrix)}.$$

Now, by definition, any vector v in C has the form mG, where m is a message vector. Thus, $vH = (mG)H = m(GH) = m[0] = 0$ (the zero vector). ∎

Because of the way H was defined, the parity-check matrix method correctly decodes any received word in which no error has been made. But it will do more.

Theorem 29.3 Parity-Check Matrix Decoding

Parity-check matrix decoding will correct any single error if and only if the rows of the parity-check matrix are nonzero and no one row is a scalar multiple of any other row.

PROOF For simplicity's sake, we prove only the binary case. In this special situation, the condition on the rows is that they are nonzero and distinct. So, let H be the parity-check matrix, and let's assume that this condition holds for the rows. Suppose that the transmitted code word w was received with only one error, and that this error occurred in the ith position. Denoting the vector that has a 1 in the ith position and 0's elsewhere by e_i, we may write the received word as $w + e_i$. Now, using the Orthogonality Lemma, we obtain

$$(w + e_i)H = wH + e_iH = 0 + e_iH = e_iH.$$

But this last vector is precisely the ith row of H. Thus, if there was exactly one error in transmission, we can use the rows of the parity-check matrix to identify the location of the error, provided that these rows are distinct. (If two rows, say, the ith and jth, are the same, we know that the error occurred in either the ith position or the jth position, but we do not know in which.)

Conversely, suppose that the parity-check matrix method correctly decodes all received words in which at most one error has been made in transmission. If the ith row of the parity-check matrix H were the zero vector and if the code word $u = 0 \cdots 0$ were received as e_i, we would find $e_iH = 0 \cdots 0$, and we would erroneously assume that the vector e_i was sent. Thus, no row of H is the zero vector. Now, suppose that the ith row of H and the jth row of H are equal and $i \neq j$. Then, if some code word w is transmitted and the received word is $w + e_i$ (i.e., there is a single error in the ith position), we find

$$(w + e_i)H = wH + e_iH = i\text{th row of } H = j\text{th row of } H.$$

Thus, our decoding procedure tells us not to decode. This contradicts our assumption that the method correctly decodes all received words in which at most one error has been made. ∎

Coset Decoding

There is another convenient decoding method that utilizes the fact that an (n, k) linear code C over a finite field F is a subgroup of the additive group of $V = F^n$. This method was devised by David Slepian in 1956 and is called *coset decoding* (or *standard decoding*). To use this method, we proceed by constructing a table, called a *standard array*. The first row of the table is the set C of code words, beginning in column 1 with the identity $0 \cdots 0$. To form additional rows of the table, choose an element v of V not listed in the table thus far. Among all the elements of the coset $v + C$, choose one of minimum weight, say, v'. Complete the next row of the table by placing under the column headed by the code word c the vector $v' + c$. Continue this process until all the vectors in V have been listed in the table. [Note that an (n, k) linear code over a field with q elements will have $|V : C| = q^{n-k}$ rows.] The words in the first column are called the *coset leaders.*The decoding procedure is simply to decode any received word w as the code word at the head of the column containing w.

EXAMPLE 11 Consider the (6, 3) binary linear code

$$C = \{000000, 100110, 010101, 001011, 110011, 101101, 011110, 111000\}.$$

The first row of a standard array is just the elements of C. Obviously, 100000 is not in C and has minimum weight among the elements of $100000 + C$, so it can be used to lead the second row. Table 29.4 is the completed table.

Table 29.4 A Standard Array for a (6, 3) Linear Code.

Coset Leaders	Words						
000000	100110	010101	001011	110011	101101	011110	111000
100000	000110	110101	101011	010011	001101	111110	011000
010000	110110	000101	011011	100011	111101	001110	101000
001000	101110	011101	000011	111011	100101	010110	110000
000100	100010	010001	001111	110111	101001	011010	111100
000010	100100	010111	001001	110001	101111	011100	111010
000001	100111	010100	001010	110010	101100	011111	111001
100001	000111	110100	101010	010010	001100	111111	011001

If the word 101001 is received, it is decoded as 101101, since 101001 lies in the column headed by 101101. Similarly, the received word 011001 is decoded as 111000. ■

Recall that the first method of decoding that we introduced was the nearest-neighbor method; that is, any received word w is decoded as the code word c such that $d(w, c)$ is a minimum, provided that there is only one code word c such that $d(w, c)$ is a minimum. The next result shows that in this situation, coset decoding is the same as nearest-neighbor decoding.

Theorem 29.4 Coset Decoding Is Nearest-Neighbor Decoding

In coset decoding, a received word w is decoded as a code word c such that $d(w, c)$ is a minimum.

PROOF Let C be a linear code, and let w be any received word. Suppose that v is the coset leader for the coset $w + C$. Then, $w + C = v + C$, so $w = v + c$ for some c in C. Thus, using coset decoding, w is decoded as c. Now, if c' is any code word, then $w - c' \in w + C = v + C$, so that $\mathrm{wt}(w - c') \geq \mathrm{wt}(v)$, since the coset leader v was chosen as a vector of minimum weight among the members of $v + C$.

Therefore,

$$d(w, c') = \mathrm{wt}(w - c') \geq \mathrm{wt}(v) = \mathrm{wt}(w - c) = d(w, c).$$

So, using coset decoding, w is decoded as a code word c such that $d(w, c)$ is a minimum. ∎

Recall that in our description of nearest-neighbor decoding, we stated that if the choice for the nearest neighbor of a received word v is not unique, then we can decide not to decode or to decode v arbitrarily from among those words closest to v. In the case of coset decoding, the decoded value of v is always uniquely determined by the coset leader of the row containing the received word. Of course, this decoded value may not be the word that was sent.

When we know a parity-check matrix for a linear code, coset decoding can be considerably simplified.

Definition Syndrome

If an (n, k) linear code over F has parity-check matrix H, then, for any vector u in F^n, the vector uH is called the *syndrome*[1] of u.

The importance of syndromes stems from the following property.

Theorem 29.5 Same Coset—Same Syndrome

Let C be an (n, k) linear code over F with a parity-check matrix H. Then, two vectors of F^n are in the same coset of C if and only if they have the same syndrome.

PROOF Two vectors u and v are in the same coset of C if and only if $u - v$ is in C. So, by the Orthogonality Lemma, u and v are in the same coset if and only if $0 = (u - v)H = uH - vH$. ▮

We may now use syndromes for decoding any received word w:

1. Calculate wH, the syndrome of w.
2. Find the coset leader v such that $wH = vH$.
3. Assume that the vector sent was $w - v$.

With this method, we can decode any received word with a table that has only two rows—one row of coset leaders and another row with the corresponding syndromes.

▮ **EXAMPLE 12** Consider the code given in Example 11. The parity-check matrix for this code is

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

The list of coset leaders and corresponding syndromes is the following. So, to decode the received word $w = 101001$, we compute

Coset leader	000000	100000	010000	001000	000100	000010	000001	100001
Syndromes	000	110	101	011	100	010	001	111

$wH = 100$. Since the coset leader $v = 000100$ has 100 as its syndrome, we assume that $w - 000100 = 101101$ was sent. If the received word is $w' = 011001$, we compute $w'H = 111$ and assume $w' - 100001 = 111000$ was sent because 100001 is the coset leader with syndrome 111. Notice that these answers are in agreement with those obtained by using the standard-array method of Example 11. ▮

The term *syndrome* is a descriptive term. In medicine, it is used to designate a collection of symptoms that typify a disorder. In coset decoding, the syndrome typifies an error pattern.

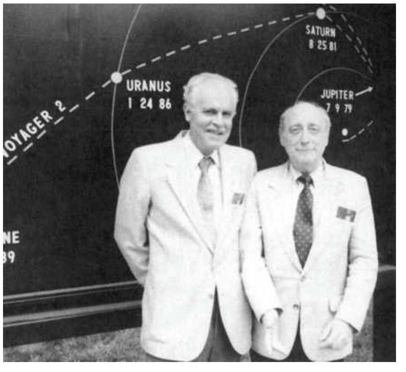
In this chapter, we have presented algebraic coding theory in its simplest form. A more sophisticated treatment would make substantially greater use of group theory, ring theory, and especially finite-field theory. For example, Gorenstein (see Chapter 24 for a biography) and Zierler, in 1961, made use of the fact that the multiplicative subgroup of a finite field is cyclic. They associated each digit of certain codes with a field element in such a way that an algebraic equation would be derived whose zeros determined the locations of the errors.

In some instances, two error-correcting codes are employed. The European Space Agency space probe Giotto, which came within 370 miles of the nucleus of Halley's Comet in 1986, had two error-correcting codes built into its electronics. One code checked for independently occurring errors, and another—a so-called Reed–Solomon code—checked for bursts of errors. Giotto achieved an error-detection rate of 0.999999. Reed–Solomon codes are also used on compact discs. They can correct thousands of consecutive errors.

Historical Note

The Ubiquitous Reed–Solomon Codes[2]

Irving Reed and Gustave Solomon monitor the encounter of *Voyager II* with Neptune at the Jet Propulsion Laboratory in 1989.



SIAM NEWS January 1993, Volume 25, Number 1, pages 1 and 11

We conclude this chapter with an adapted version of an article by Barry A. Cipra about the Reed–Solomon codes [1]. It was the first in a series of articles called "Mathematics That Counts" in *SIAM News,* the news journal of the Society for Industrial and Applied

[2]Adapted version of an article called, "The Ubiquitous Reed-Solomon Codes" in *SIAM News,* the news journal of the Society for Industrial and Applied Mathematics, by Barry A. Cipra. Reprinted from *SIAM News*, Volume 26-1, January 1993.

Mathematics. The articles highlight developments in mathematics that have led to products and processes of substantial benefit to industry and the public.

In this "Age of Information," no one need be reminded of the importance not only of speed but also of accuracy in the storage, retrieval, and transmission of data. Machines *do* make errors, and their non-man-made mistakes can turn otherwise flawless programming into worthless, even dangerous, trash. Just as architects design buildings that will remain standing even through an earthquake, their computer counterparts have come up with sophisticated techniques capable of counteracting digital disasters.

The idea for the current error-correcting techniques for everything from computer hard disk drives to CD players was first introduced in 1960 by Irving Reed and Gustave Solomon, then staff members at MIT's Lincoln Laboratory. . . .

"When you talk about CD players and digital audio tape and now digital television, and various other digital imaging systems that are coming—all of those need Reed–Solomon [codes] as an integral part of the system," says Robert McEliece, a coding theorist in the electrical engineering department at Caltech.

Why? Because digital information, virtually by definition, consists of strings of "bits"—0's and 1's—and a physical device, no matter how capably manufactured, may occasionally confuse the two. *Voyager II*, for example, was transmitting data at incredibly low power—barely a whisper—over tens of millions of miles. Disk drives pack data so densely that a read/write head can (almost) be excused if it can't tell where one bit stops and the next 1 (or 0) begins. Careful engineering can reduce the error rate to what may sound like a negligible level—the industry standard for hard disk drives is 1 in 10 billion—but given the volume of information processing done these days, that "negligible" level is an invitation to daily disaster. Error-correcting codes are a kind of safety net—mathematical insurance against the vagaries of an imperfect material world.

In 1960, the theory of error-correcting codes was only about a decade old. The basic theory of reliable digital communication had been set forth by Claude Shannon in the late 1940s. At the same time, Richard Hamming introduced an elegant approach to single-error correction and double-error detection. Through the 1950s, a number of researchers began experimenting with a variety of error-correcting codes. But with their SIAM journal paper, McEliece says, Reed and Solomon "hit the jackpot."

The payoff was a coding system based on groups of bits—such as bytes—rather than individual 0's and 1's. That feature makes Reed–Solomon codes particularly good at dealing with "bursts" of errors: six consecutive bit errors, for example, can affect at most two bytes. Thus, even a double-error-correction version of a Reed–Solomon code can provide a comfortable safety factor. . . .

Mathematically, Reed–Solomon codes are based on the arithmetic of finite fields. Indeed, the 1960 paper begins by defining a code as "a mapping from a vector space of dimension m over a finite field K into a vector space of higher dimension over the same field." Starting from a "message" $(a_0, a_1, \ldots, a_{m-1})$, where each a_k is an element of the field K, a Reed–Solomon code produces $(P(0), P(g), P(g^2), \ldots, P(g^{N-1}))$, where N is the number of elements in K, g is a generator of the (cyclic) group of nonzero elements in K, and $P(x)$ is the polynomial $a_0 + a_1x + \cdots + a_{m-1}x^{m-1}$. If N is greater than m, then the values of P overdetermine the polynomial, and the properties of finite fields guarantee that the coefficients of P—that is, the original message–can be recovered from any m of the values . . . .

In today's byte-sized world, for example, it might make sense to let K be the field of order 2^8, so that each element of K corresponds to a single byte (in computerese, there are four bits to a nibble and two nibbles to a byte). In that case, $N = 2^8 = 256$, and hence messages up to 251 bytes long can be recovered even if two errors occur in transmitting the values $P(0), P(g), \ldots, P(g^{255})$. That's a lot better than the 1255 bytes required by the say-everything-five-times approach.

Despite their advantages, Reed–Solomon codes did not go into use immediately—they had to wait for the hardware technology to catch up. "In 1960, there was no such thing as fast digital electronics"—at least not by today's standards, says McEliece. The Reed–Solomon paper "suggested some nice ways to process data, but nobody knew if it was practical or not, and in 1960 it probably wasn't practical."

But technology did catch up, and numerous researchers began to work on implementing the codes. . . . Many other bells and whistles (some of fundamental theoretic significance) have also been added. Compact discs, for example, use a version of a Reed–Solomon code.

Reed was among the first to recognize the significance of abstract algebra as the basis for error-correcting codes. "In hindsight it seems obvious," he told *SIAM News.* However, he added, "coding theory was not a subject when we published that paper." The two authors knew they had a nice result; they didn't know what impact the paper would have.

Three decades later, the impact is clear. The vast array of applications, both current and pending, has settled the question of the practicality and significance of Reed–Solomon codes. "It's clear they're practical, because everybody's using them now," says Elwyn Berkekamp. Billions of dollars in modern technology depend on ideas that stem from Reed and Solomon's original work. In short, says McEliece, "it's been an extraordinarily influential paper."

Exercises

The New Testament offers the basis for modern computer coding theory, in the form of an affirmation of the binary number system.

"But let your communication be yea, yea; nay, nay: for whatsoever is more than these cometh of evil."

Anonymous

1. Find the Hamming weight of each code word in Table 29.1.
2. Find the Hamming distance between the following pairs of vectors: $\{1101, 0111\}, \{0220, 1122\}, \{11101, 00111\}$.
3. Referring to Example 1, use the nearest-neighbor method to decode the received words 0000110 and 1110100.
4. For any vector space V and any u, v, w in F^n, prove that the Hamming distance has the following properties.
   - **a.** $d(u, v) = d(v, u)$ (symmetry).
   - **b.** $d(u, v) = 0$ if and only if $u = v$.
   - **c.** $d(u, v) = d(u + w, v + w)$ (translation invariance).
5. Determine the (6, 3) binary linear code with generator ma-

trix

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 1 & 1 & 0 \end{bmatrix}.$$

6. Show that for binary vectors, $\text{wt}(u + v) \geq \text{wt}(u) = \text{wt}(v)$ and equality occurs if and only if for all i the ith component of u is 1 whenever the ith component of v is 1.

7. If the minimum weight of any nonzero code word is 2, what can we say about the error-detecting capability of the code?

8. Suppose that C is a linear code with Hamming weight 3 and that C' is one with Hamming weight 4. What can C' do that C can't?

9. Let C be a binary linear code. Show that the code words of even weight form a subcode of C. (A *subcode* of a code is a subset of the code that is itself a code.)

10. Let

$$C = \{0000000, 1110100, 0111010, 0011101, 1001110, \\ 0100111, 1010011, 1101001\}.$$

What is the error-correcting capability of C? What is the error-detecting capability of C?

11. Suppose that the parity-check matrix of a binary linear code is

$$H = \begin{bmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Can the code correct any single error?

12. Use the generator matrix

$$G = \begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & 2 & 1 \end{bmatrix}$$

to construct a (4, 2) ternary linear code. What is the parity-check matrix for this code? What is the error-correcting capability of this code? What is the error-detecting capability of this code? Use parity-check decoding to decode the received word 1201.

13. Find all code words of the (7, 4) binary linear code whose generator matrix is

$$G = \begin{bmatrix} 1 & 0 & 0 & 0 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 \end{bmatrix}.$$

Find the parity-check matrix of this code. Will this code correct any single error?

14. Show that in a binary linear code, either all the code words end with 0, or exactly half end with 0. What about the other components?

15. Suppose that a code word v is received as the vector u. Show that coset decoding will decode u as the code word v if and only if $u - v$ is a coset leader.

16. Consider the binary linear code

$$C = \{00000, 10011, 01010, 11001, 00101, 10110, 01111, 11100\}.$$

Construct a standard array for C. Use nearest-neighbor decoding to decode 11101 and 01100. If the received word 11101 has exactly one error, can we determine the intended code word? If the received word 01100 has exactly one error, can we determine the intended code word?

17. Construct a (6, 3) binary linear code with generator matrix

$$G = \begin{bmatrix} 1 & 0 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 & 0 & 1 \end{bmatrix}.$$

Decode each of the received words

$$001001, 011000, 000110, 100001$$

by the following methods:
a. Nearest-neighbor method.
b. Parity-check matrix method.
c. Coset decoding using a standard array.
d. Coset decoding using the syndrome method.

18. Suppose that the minimum weight of any nonzero code word in a linear code is 6. Discuss the possible options for error correction and error detection.

19. Using the code and the parity-check matrix given in Example 10, show that parity-check matrix decoding cannot detect any multiple errors (i.e., two or more errors).

20. Suppose that the last row of a standard array for a binary linear code is

10000 00011 11010 01001 10101 00110 11111 01100.

Determine the code.

21. How many code words are there in a (6, 4) ternary linear code? How many possible received words are there for this code?

22. If the parity-check matrix for a binary linear code is

$$H = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}.$$

will the code correct any single error? Why?

23. Suppose that the parity-check matrix for a ternary code is

$$H = \begin{bmatrix} 2 & 1 \\ 2 & 2 \\ 1 & 2 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}.$$

Can the code correct all single errors? Give a reason for your answer.

24. Prove that for nearest-neighbor decoding, the converse of Theorem 29.2 is true.

25. Can a (6, 3) binary linear code be double-error-correcting using the nearest-neighbor method? Do not assume that the code is systematic.

26. Prove that there is no 2×5 standard generator matrix G that will produce a (5, 2) linear code over Z_3 capable of detecting all possible triple errors.

27. Why can't the nearest-neighbor method with a (4, 2) binary linear code correct all single errors?

28. Suppose that one row of a standard array for a binary code is

000100 110000 011110 111101 101010 001001 100111 010011.

Determine the row that contains 100001.

29. Use the field $F = Z_2[x]/\langle x^2 + x + 1\rangle$ to construct a (5, 2) linear code that will correct any single error.

30. Find the standard generator matrix for a (4, 2) linear code over Z_3 that encodes 20 as 2012 and 11 as 1100. Determine the entire code and the parity-check matrix for the code. Will the code correct all single errors?

31. Assume that C is an (n, k) binary linear code and that, for each position $i = 1, 2, \ldots, n$, the code C has at least one vector with a 1 in the ith position. Show that the average weight of a code word is $n/2$.

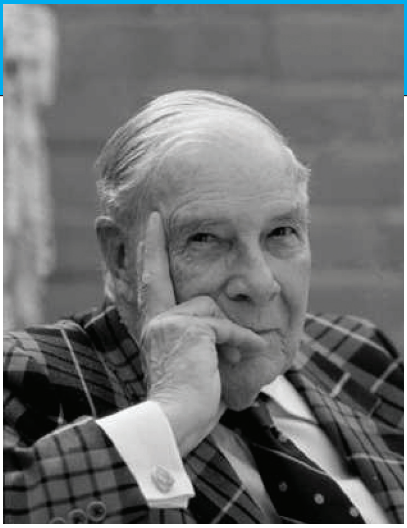
32. Let C be an (n, k) linear code over F such that the minimum weight of any nonzero code word is $2t + 1$. Show that not every vector of weight $t+1$ in F^n can occur as a coset leader.

33. Let C be an (n, k) binary linear code over $F = Z_2$. If $v \in F^n$ but $v \notin C$, show that $C \cup (v + C)$ is a linear code.

34. Let C be a binary linear code. Show that either every member of C has even weight or exactly half the members of C have even weight. (Compare with Exercise 27 in Chapter 5.)

35. Let C be an (n, k) linear code. For each i with $1 \leq i \leq n$, let $C_i = \{v \in C |$ the ith component of v is $0\}$. Show that C_i is a subcode of C.

Richard W. Hamming



Courtesy of Louis Bachrach

For introduction of error-correcting codes, pioneering work in operating systems and programming languages, and the advancement of numerical computation.

Citation for the Piore Award, 1979

Richard W. Hamming was born in Chicago, Illinois, on February 11, 1915. He graduated from the University of Chicago with a B.S. degree in mathematics. In 1939, he received an M.A. degree in mathematics from the University of Nebraska and, in 1942, a Ph.D. in mathematics from the University of Illinois.

During the latter part of World War II, Hamming was at Los Alamos, where he was involved in computing atomic-bomb designs. In 1946, he joined Bell Telephone Laboratories, where he worked in mathematics, computing, engineering, and science. In 1950, Hamming published his famous paper on error-detecting and error-correcting codes. This work started a branch of information theory.

The Hamming codes are used in many modern computers. Hamming's work in the field of numerical analysis has also been of fundamental importance.

Hamming received numerous prestigious awards, including the Turing Prize from the Association for Computing Machinery, the Piore Award from the Institute of Electrical and Electronics Engineers (IEEE), and the Oender Award from the University of Pennsylvania. In 1986 the IEEE Board of Directors established the Richard W. Hamming Medal "for exceptional contributions to information sciences, systems and technology" and named Hamming as its first recipient. Hamming died of a heart attack on January 7, 1998, at age 82.

Jessie MacWilliams

Courtesy Walter MacWilliams

She was a mathematician who was instrumental in developing the mathematical theory of error-correcting codes from its early development and whose Ph.D. thesis includes one of the most powerful theorems in coding theory.

VERA PLESS, *SIAM News*

An important contributor to coding theory was Jessie MacWilliams. She was born in 1917 in England. After studying at Cambridge University, MacWilliams came to the United States in 1939 to attend Johns Hopkins University. After one year at Johns Hopkins, she went to Harvard for a year.

In 1955, MacWilliams became a programmer at Bell Labs, where she learned about coding theory. Although she made a major discovery about codes while a programmer, she could not obtain a promotion to a math research position without a Ph.D. degree. She completed some of the requirements for the Ph.D. while working full-time at Bell Labs and looking after her family. She then returned to Harvard for a year (1961–1962), where she finished her degree.

Interestingly, both MacWilliams and her daughter Ann were studying mathematics at Harvard at the same time. MacWilliams returned to Bell Labs, where she remained until her retirement in 1983. The Institute of Electrical and Electronics Engineers published an issue of its journal *IEEE on Information Theory Transactions* containing papers dedicated to her in 1983. While at Bell Labs, she made many contributions to the subject of error-correcting codes, including *The Theory of Error-Correcting Codes,* written jointly with Neil Sloane. One of her results of great theoretical importance is known as the MacWilliams Identity. She died on May 27, 1990, at the age of 73.

Vera Pless

Vera Pless is a leader in the field of coding theory.

Courtesy Vera Pless

VERA PLESS was born on March 5, 1931, to Russian immigrants on the West Side of Chicago. She accepted a scholarship to attend the University of Chicago at age 15. The program at Chicago emphasized great literature but paid little attention to physics and mathematics. At age 18, with no more than one pre-calculus course in mathematics, she entered the prestigious graduate program in mathematics at Chicago, where, at that time, there were no women on the mathematics faculty or even women colloquium speakers. After passing her master's exam, she took a job as a research associate at Northwestern University while pursuing a Ph.D. there. In 1957, she obtained her degree. Over the next several years, Pless stayed at home to raise her children while teaching part-time at Boston University.

When she decided to work full-time, she found that women were not welcome at most colleges and universities. One person told her out-right, "I would never hire a woman." Fortunately, there was an Air Force Lab in the area that had a group working on error-correcting codes. Although she had never even heard of coding theory, she was hired because of her background in algebra. When the lab discontinued basic research, she took a position as a research associate at MIT in 1972. In 1975, she went to the University of Illinois–Chicago, where she remained until her retirement.

During her career, Pless wrote more than 100 research papers, authored a widely used textbook on coding theory, and had 11 Ph. D. students. Pless died on March 2, 2020 at the age of 88.

30 An Introduction to Galois Theory

It [Galois's work] is now considered as one of the pillars of modern mathematics.

Edward Frenkel, *Love and Math*

Today 'Galois groups' are ubiquitous in the literature, and the group idea has proved to be perhaps the most versatile in all mathematics, clarifying many a deep mystery. "When in doubt," the great André Weil advised, look for the group. "That's the *cherchez la femme* of mathematics."

Jim Holt, *The New York Review of Books*, December 5, 2013

Fundamental Theorem of Galois Theory

The Fundamental Theorem of Galois Theory is one of the most elegant theorems in mathematics. Look at Figures 30.1 and 30.2. Figure 30.1 depicts the lattice of subgroups of the group of field automorphisms of $Q(\sqrt[4]{2}, i)$. The integer along an upward lattice line from a group H_1 to a group H_2 is the index of H_1 in H_2. Figure 30.2 shows the lattice of subfields of $Q(\sqrt[4]{2}, i)$. The integer along an upward line from a field K_1 to a field K_2 is the degree of K_2 over K_1. Notice that the lattice in Figure 30.2 is the lattice of Figure 30.1 turned upside down. This is only one of many relationships between these two lattices. The Fundamental Theorem of Galois Theory relates the lattice of subfields of an algebraic extension E of a field F to the subgroup structure of the group of automorphisms of E that send each element of F to itself. This relationship was discovered in the process of attempting to solve a polynomial equation $f(x) = 0$ by radicals.

Before we can give a precise statement of the Fundamental Theorem of Galois Theory, we need some terminology and notation.

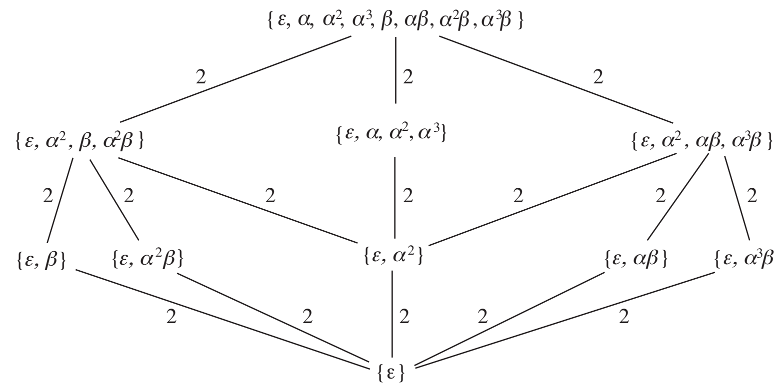


Figure 30.1 Lattice of subgroups of the group of field automorphisms of $Q(\sqrt[4]{2}, i)$, where $\boldsymbol{\alpha}$: $i \to \boldsymbol{i}$ and $\sqrt[4]{2} \to -i\sqrt[4]{2}$, $\boldsymbol{\beta}$: $i \to -i$, and $\sqrt[4]{2} \to \sqrt[4]{2}$.

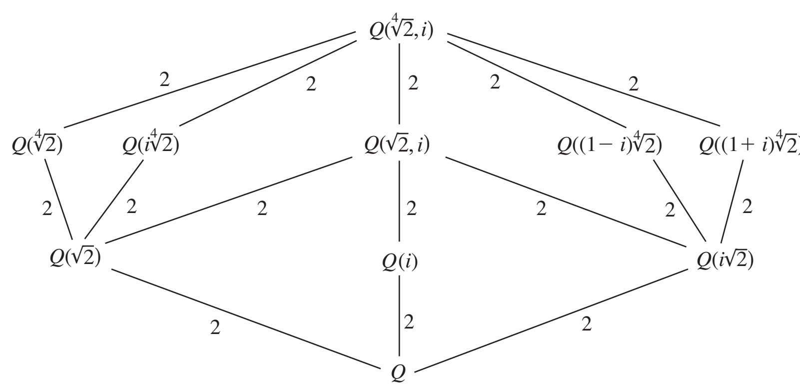


Figure 30.2 Lattice of subfields of $Q(\sqrt[4]{2}, i)$.

Definitions Automorphism, Galois Group, Fixed Field of *H*

Let E be an extension field of the field F. An *automorphism of E* is a ring isomorphism from E onto E. The *Galois group* of E over F, $\mathrm{Gal}(E/F)$, is the set of all automorphisms of E that take every element of F to itself. If H is a subgroup of $\mathrm{Gal}(E/F)$, the set

$$E_H = \{x \in E \mid \phi(x) = x \text{ for all } \phi \in H\}$$

is called the *fixed field of H*.

It is easy to show that the set of automorphisms of E forms a group under composition. We leave as exercises (Exercises 3 and 5) the verifications that the automorphism group of E fixing F

is a subgroup of the automorphism group of E and that, for any subgroup H of $\mathrm{Gal}(E/F)$, the fixed field E_H of H is a subfield of E. Be careful not to misinterpret $\mathrm{Gal}(E/F)$ as something that has to do with factor rings or factor groups. It does not.

The following examples will help you assimilate these definitions. In each example, we simply indicate how the automorphisms are defined. We leave to the reader the verifications that the mappings are indeed automorphisms.

EXAMPLE 1 Consider the extension $Q(\sqrt{2})$ of Q. Since

$$Q(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in Q\}$$

and any automorphism of a field containing Q must act as the identity on Q (Exercise 1), an automorphism ϕ of $Q(\sqrt{2})$ is completely determined by $\phi(\sqrt{2})$. Thus,

$$2 = \phi(2) = \phi(\sqrt{2}\sqrt{2}) = (\phi(\sqrt{2}))^2,$$

and therefore $\phi(\sqrt{2}) = \pm\sqrt{2}$. This proves that the group $\mathrm{Gal}(Q(\sqrt{2})/Q)$ has two elements, the identity mapping and the mapping that sends $a + b\sqrt{2}$ to $a - b\sqrt{2}$. ▮

EXAMPLE 2 Consider the extension $Q(\sqrt[3]{2})$ of Q. An automorphism ϕ of $Q(\sqrt[3]{2})$ is completely determined by $\phi(\sqrt[3]{2})$. By an argument analogous to that in Example 1, we see that $\phi(\sqrt[3]{2})$ must be a cube root of 2. Since $Q(\sqrt[3]{2})$ is a subset of the real numbers and $\sqrt[3]{2}$ is the only real cube root of 2, we must have $\phi(\sqrt[3]{2}) = \sqrt[3]{2}$. Thus, ϕ is the identity automorphism and $\mathrm{Gal}(Q(\sqrt[3]{2})/Q)$ has only one element. Obviously, the fixed field of $\mathrm{Gal}(Q\sqrt[3]{2}/Q)$ is $Q(\sqrt[3]{2})$. ▮

EXAMPLE 3 Consider the extension $Q(\sqrt[4]{2}, i)$ of $Q(i)$. Any automorphism ϕ of $Q(\sqrt[4]{2}, i)$ fixing $Q(i)$ is completely determined by $\phi(\sqrt[4]{2})$. Since

$$2 = \phi(2) = \phi((\sqrt[4]{2})^4) = (\phi(\sqrt[4]{2}))^4,$$

we see that $\phi(\sqrt[4]{2})$ must be a fourth root of 2. Thus, there are at most four possible automorphisms of $Q(\sqrt[4]{2}, i)$ fixing $Q(i)$. If we define an automorphism α such that $\alpha(i) = i$ and $\alpha(\sqrt[4]{2}) = i\sqrt[4]{2}$, then $\alpha \in \mathrm{Gal}(Q(\sqrt[4]{2}, i)/Q(i))$ and α has order 4. Thus, $\mathrm{Gal}(Q(\sqrt[4]{2}, i)/Q(i))$ is a cyclic group of order 4. The fixed field of $\{\varepsilon, \alpha^2\}$ (where ε is the identity automorphism) is $Q(\sqrt{2}, i)$. The lattice of subgroups of $\mathrm{Gal}(Q(\sqrt[4]{2}, i)/Q(i))$ and the lattice of subfields of $Q(\sqrt[4]{2}, i)$ containing $Q(i)$ are shown in Figure 30.3. As in Figures 30.1 and 30.2, the integers along the lines of the group lattice represent the index of a subgroup in the group above it, and the integers along the lines of the field lattice represent the degree of the extension of a field over the field below it. ∎

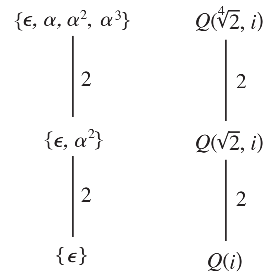


Figure 30.3 Lattice of subgroups of Gal $(Q(\sqrt[4]{2}, i)/Q(i))$ and lattice of subfields of $Q(\sqrt[4]{2}, i)$ containing $Q(i)$.

EXAMPLE 4 Consider the extension $Q(\sqrt{3}, \sqrt{5})$ of Q. Since

$$Q(\sqrt{3}, \sqrt{5}) = \{a + b\sqrt{3} + c\sqrt{5} + d\sqrt{3}\sqrt{5} \mid a, b, c, d \in Q\},$$

any automorphism ϕ of $Q(\sqrt{3}, \sqrt{5})$ is completely determined by the two values $\phi(\sqrt{3})$ and $\phi(\sqrt{5})$. This time there are four automorphisms.

ε	α	β	$\alpha\beta$
$\sqrt{3} \to \sqrt{3}$	$\sqrt{3} \to -\sqrt{3}$	$\sqrt{3} \to \sqrt{3}$	$\sqrt{3} \to -\sqrt{3}$
$\sqrt{5} \to \sqrt{5}$	$\sqrt{5} \to \sqrt{5}$	$\sqrt{5} \to -\sqrt{5}$	$\sqrt{5} \to -\sqrt{5}$

Obviously, $\mathrm{Gal}(Q(\sqrt{3}, \sqrt{5})/Q)$ is isomorphic to $Z_2 \oplus Z_2$. The fixed field of $\{\varepsilon, \alpha\}$ is $Q(\sqrt{5})$, the fixed field of $\{\varepsilon, \beta\}$ is $Q(\sqrt{3})$, and the fixed field of $\{\varepsilon, \alpha\beta\}$ is $Q(\sqrt{3}\sqrt{5})$. The lattice of subgroups of $\mathrm{Gal}(Q(\sqrt{3}, \sqrt{5})/Q)$ and the lattice of subfields of $Q(\sqrt{3}, \sqrt{5})$ are shown in Figure 30.4. ∎

Example 5 is a bit more complicated than our previous examples. In particular, the automorphism group is non-Abelian.

EXAMPLE 5 Direct calculations show that $\omega = -1/2 + i\sqrt{3}/2$ satisfies the equations $\omega^3 = 1$ and $\omega^2 + \omega + 1 = 0$. Now, consider

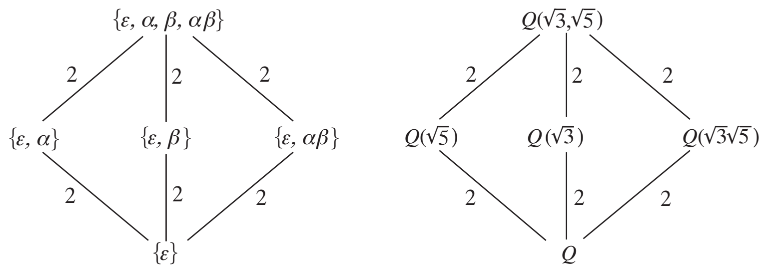


Figure 30.4 Lattice of subgroups of Gal$(Q(\sqrt{3}, \sqrt{5})/Q)$ and lattice of subfields of $Q(\sqrt{3}, \sqrt{5})$.

the extension $Q(\omega, \sqrt[3]{2})$ of Q. We may describe the automorphisms of $Q(\omega, \sqrt[3]{2})$ by specifying how they act on ω and $\sqrt[3]{2}$. There are six in all.

ε	α	β	β^2	$\alpha\beta$	$\alpha\beta^2$
$\omega \to \omega$ $\sqrt[3]{2} \to \sqrt[3]{2}$	$\omega \to \omega^2$ $\sqrt[3]{2} \to \sqrt[3]{2}$	$\omega \to \omega$ $\sqrt[3]{2} \to \omega\sqrt[3]{2}$	$\omega \to \omega$ $\sqrt[3]{2} \to \omega^2\sqrt[3]{2}$	$\omega \to \omega^2$ $\sqrt[3]{2} \to \omega^2\sqrt[3]{2}$	$\omega \to \omega^2$ $\sqrt[3]{2} \to \omega\sqrt[3]{2}$

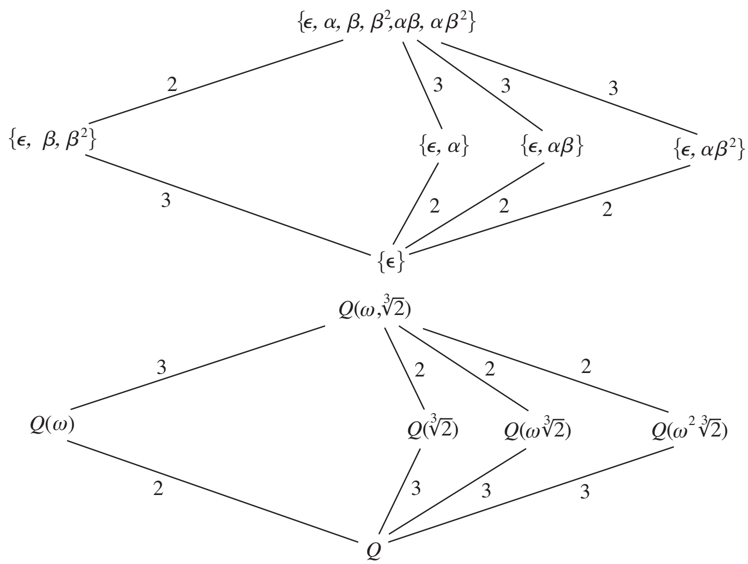


Figure 30.5 Lattice of subgroups of Gal$(Q(\omega, \sqrt[3]{2})/Q)$ and lattice of subfields of $Q(\omega, \sqrt[3]{2})$, where $\omega = -1/2 + i\sqrt{3}/2$.

Since $\alpha\beta \neq \beta\alpha$, we know that $\text{Gal}(Q(\omega, \sqrt[3]{2})/Q)$ is isomorphic to S. (See Theorem 7.2.) The lattices of subgroups and subfields are shown in Figure 30.5.

The lattices in Figure 30.5 have been arranged so that each nontrivial proper field occupying the same position as some group is the fixed field of that group. For instance, $Q(\omega\sqrt[3]{2})$ is the fixed field of $\{\varepsilon, \alpha\beta\}$. ∎

The preceding examples show that, in certain cases, there is an intimate connection between the lattice of subfields between E and F and the lattice of subgroups of $\text{Gal}(E/F)$. In general, if E is an extension of F, and we let $\mathcal{F}$ be the lattice of subfields of E containing F and let $\mathcal{G}$ be the lattice of subgroups of $\text{Gal}(E/F)$, then for each K in $\mathcal{F}$, the group $\text{Gal}(E/K)$ is in $\mathcal{G}$, and for each H in $\mathcal{G}$, the field E_H is in $\mathcal{F}$. Thus, we may define a mapping $g\colon \mathcal{F} \to \mathcal{G}$ by $g(K) = \text{Gal}(E/K)$ and a mapping $f\colon \mathcal{G} \to \mathcal{F}$ by $f(H) = E_H$. It is easy to show that if K and L belong to $\mathcal{F}$ and $K \subseteq L$, then $g(K) \supseteq g(L)$. Similarly, if G and H belong to $\mathcal{G}$ and $G \subseteq H$, then $f(G) \supseteq f(H)$. Thus, f and g are inclusion-reversing mappings between $\mathcal{F}$ and $\mathcal{G}$. We leave it to the reader to show that for any K in $\mathcal{F}$, we have $(fg)(K) \supseteq K$, and for any G in $\mathcal{G}$, we have $(gf)(G) \supseteq G$. When E is an arbitrary extension of F, these inclusions may be strict. However, when E is a suitably chosen extension of F, the Fundamental Theorem of Galois Theory, Theorem 30.1, says that f and g are inverses of each other, so that the inclusions are equalities. In particular, f and g are inclusion-reversing isomorphisms between the lattices $\mathcal{F}$ and $\mathcal{G}$. A stronger result than that given in Theorem 30.1 is true, but our theorem illustrates the fundamental principles involved.

Theorem 30.1 Fundamental Theorem of Galois Theory

Let F be a field of characteristic 0 or a finite field. If E is the splitting field over F for some polynomial in $F[x]$, then the mapping from the set of subfields of E containing F to the set of subgroups of $\text{Gal}(E/F)$ given by $K \to \text{Gal}(E/K)$ is a one-to-one correspondence. Furthermore, for any subfield K of E containing F,

1. $[E:K] = |\text{Gal}(E/K)|$ ***and*** $[K:F] = |\text{Gal}(E/F)|/|\text{Gal}(E/K)|$. [***The index of*** $\text{Gal}(E/K)$ ***in*** $\text{Gal}(E/F)$ ***equals the degree of K over F.***]

2. ***If K is the splitting field of some polynomial in $F[x]$, then $\mathrm{Gal}(E/K)$ is a normal subgroup of $\mathrm{Gal}(E/F)$ and $\mathrm{Gal}(K/F)$ is isomorphic to $\mathrm{Gal}(E/F)/\mathrm{Gal}(E/K)$.***
3. ***$K = E_{\mathrm{Gal}(E/K)}$. [The fixed field of $\mathrm{Gal}(E/K)$ is K.]***
4. ***If H is a subgroup of $\mathrm{Gal}(E/F)$, then $H = \mathrm{Gal}(E/E_H)$. [The automorphism group of E fixing E_H is H.]***

Generally speaking, it is much easier to determine a lattice of subgroups than a lattice of subfields. For example, it is usually quite difficult to determine, directly, how many subfields a given field has, and it is often difficult to decide whether or not two extensions are the same. The corresponding questions about groups are much more tractable. Hence, the Fundamental Theorem of Galois Theory can be a great labor-saving device. Here is an illustration. [Recall from Chapter 19 that if $f(x) \in F[x]$ and the zeros of $f(x)$ in some extension of F are $a_1, a_2, \ldots, a_n$, then $F(a_1, a_2, \ldots, a_n)$ is the splitting field of $f(x)$ over F.]

EXAMPLE 6 Let $\omega = \cos(2\pi/7) + i\sin(2\pi/7)$, so that $\omega^7 = 1$, and consider the field $Q(\omega)$. How many subfields does it have and what are they? First, observe that $Q(\omega)$ is the splitting field of $x^7 - 1$ over Q, so that we may apply the Fundamental Theorem of Galois Theory. A simple calculation shows that the automorphism ϕ that sends ω to ω^3 has order 6. Thus,

$$[Q(\omega) : Q] = |\,Gal(Q(\omega)/Q)\,| \geq 6.$$

Also, since

$$x^7 - 1 = (x-1)(x^6 + x^5 + x^4 + x^3 + x^2 + x + 1)$$

and ω is a zero of $x^7 - 1$, we see that

$$|\,\mathrm{Gal}(Q(\omega)/Q)\,| = [Q(\omega) : Q] \leq 6.$$

Thus, $\mathrm{Gal}(Q(\omega)/Q)$ is a cyclic group of order 6. So, the lattice of subgroups of $\mathrm{Gal}(Q(\omega)/Q)$ is trivial to compute. See Figure 30.6. This means that $Q(\omega)$ contains exactly two proper extensions of Q: one of degree 3 corresponding to the fixed field of $\langle\phi^3\rangle$ and one of degree 2 corresponding to the fixed field of $\langle\phi^2\rangle$. To find the fixed field of $\langle\phi^3\rangle$, we must find a member of $Q(\omega)$ that is

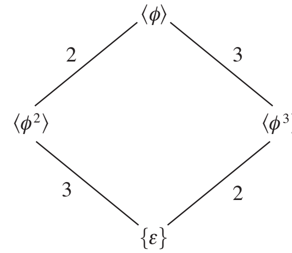


Figure 30.6 Lattice of subgroups of $\mathrm{Gal}(Q(\omega)/Q)$, where $\omega = \cos(2\pi/7) + i\sin(2\pi/7)$.

not in Q and that is fixed by ϕ^3. Experimenting with various possibilities leads us to discover that $\omega + \omega^{-1}$ is fixed by ϕ^3 (see Exercise 9), and it follows that $Q \subset Q(\omega+\omega^{-1}) \subseteq Q(\omega)_{\langle\phi^3\rangle}$. Since $[Q(\omega)_{\langle\phi^3\rangle} : Q] = 3$ and $[Q(\omega+\omega^{-1}) : Q]$ divides $[Q(\omega)_{\langle\phi^3\rangle} : Q]$, we see that $Q(\omega+\omega^{-1}) = Q(\omega)_{\langle\phi^3\rangle}$. A similar argument shows that $Q(\omega^3+\omega^5+\omega^6)$ is the fixed field of $\langle\phi^2\rangle$. Thus, we have found all subfields of $Q(\omega)$. ∎

EXAMPLE 7 Consider the extension $E = \mathrm{GF}(p^n)$ of $F = \mathrm{GF}(p)$. Let us determine $\mathrm{Gal}(E/F)$. By Corollary 2 of Theorem 21.2, E has the form $F(b)$ for some b where b is the zero of an irreducible polynomial $p(x)$ of the form $x^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$, where $a_{n-1}, a_{n-2}, \ldots, a_0$ belong to F. Since any field automorphism ϕ of E must take 1 to itself, it follows that ϕ acts as the identity on F. Thus, $p(b) = 0$ implies that $p(\phi(b)) = 0$. And because $p(x)$ has at most n zeros, we know that there are at most n possibilities for $\phi(b)$. On the other hand, by Exercise 49 in Chapter 13, we know that the mapping $\sigma(a) = a^p$ for all $a \in E$ is an automorphism of E, and it follows from the fact that E^* is cyclic (Theorem 21.2) that the group $\langle\sigma\rangle$ has order n (see Exercise 9 in Chapter 21). Thus, $\mathrm{Gal}(\mathrm{GF}(p^n)/\mathrm{GF}(p)) \approx Z_n$. ∎

Solvability of Polynomials by Radicals

For Galois, the elegant correspondence between groups and fields given by Theorem 30.1 was only a means to an end. Galois sought to solve a problem that had stymied mathematicians for centuries. Methods for solving linear and quadratic equations were known thousands of years ago (the quadratic formula). In the 16th century, Italian mathematicians developed formulas for solving any third- or fourth-degree equation. Their formulas involved only the operations of addition, subtraction, multiplication, division, and extraction of roots (radicals). For example, the equation

$$x^3 + bx + c = 0$$

has the three solutions

$$\begin{gathered} A + B, \\ -(A+B)/2 + (A-B)\sqrt{-3}/2, \\ -(A+B)/2 - (A-B)\sqrt{-3}/2, \end{gathered}$$

where

$$A = \sqrt[3]{\frac{-c}{2} + \sqrt{\frac{b^3}{27} + \frac{c^2}{4}}} \quad \text{and} \quad B = \sqrt[3]{\frac{-c}{2} - \sqrt{\frac{b^3}{27} + \frac{c^2}{4}}}.$$

The formulas for the general cubic $x^3 + ax^2 + bx + c = 0$ and the general quartic (fourth-degree polynomial) are even more complicated, but nevertheless can be given in terms of radicals of rational expressions of the coefficients.

Both Abel and Galois proved that there is no general solution of a fifth-degree equation by radicals. In particular, there is no "quintic formula." Before discussing Galois's method, which provided a group theoretic criterion for the solution of an equation by radicals and led to the modern-day Galois theory, we need a few definitions.

Definition Solvable by Radicals

Let F be a field, and let $f(x) \in F[x]$. We say that $f(x)$ is *solvable by radicals over F* if $f(x)$ splits in some extension $F(a_1, a_2, \ldots, a_n)$ of F and there exist positive integers $k_1, \ldots, k_n$ such that $a_1{}^{k_1} \in F$ and $a_i{}^{k_i} \in F(a_1, \ldots, a_{i-1}$ for $i = 2, \ldots, n$.

So, a polynomial in $F[x]$ is solvable by radicals if we can obtain all of its zeros by adjoining nth roots (for various n) to F. In other words, each zero of the polynomial can be written as an expression (usually a messy one) involving elements of F combined by the operations of addition, subtraction, multiplication, division, and extraction of roots.

EXAMPLE 8 Let $\omega = \cos(2\pi/8) + i\sin(2\pi/8) = \sqrt{2}/2 + i\sqrt{2}/2$. Then $x^8 - 3$ splits in $Q(\omega, \sqrt[8]{3}), \omega^8 \in Q$, and $(\sqrt[8]{3})^8 \in Q \subset Q(\omega)$. Thus, $x^8 - 3$ is solvable by radicals over Q. Although the zeros of $x^8 - 3$ are most conveniently written in the form $\sqrt[8]{3}, \sqrt[8]{3}\omega, \sqrt[8]{3}\omega^2, \ldots, \sqrt[8]{3}\omega^7$, the notion of solvable by radicals is best illustrated by writing them in the form

$$\pm\sqrt[8]{3}, \pm\sqrt{-1}\sqrt[8]{3}, \pm\sqrt[8]{3}\left(\frac{\sqrt{2}}{2} + \frac{\sqrt{-1}\sqrt{2}}{2}\right),$$
$$\pm\sqrt[8]{3}\left(\frac{\sqrt{2}}{2} - \frac{\sqrt{-1}\sqrt{2}}{2}\right).$$

∎

Thus, the problem of solving a polynomial equation for its zeros can be transformed into a problem about field extensions. At the same time, we can use the Fundamental Theorem of Galois Theory to transform a problem about field extensions into a problem about groups. This is exactly how Galois showed that there are fifth-degree polynomials that cannot be solved by radicals, and this is exactly how we will do it. Before giving an example of such a polynomial, we need some additional group theory.

Definition Solvable Group

We say that a group G is solvable if G has a series of subgroups

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k = G,$$

where, for each $0 \leq i < k$, H_i is normal in H_{i+1} and H_{i+1}/H_i is Abelian.

Obviously, Abelian groups are solvable. So are the dihedral groups and any group whose order has the form p^n, where p is a prime (see Exercises 28 and 29). The monumental Feit–Thompson Theorem (see Chapter 24) says that every group of odd order is solvable. In a certain sense, solvable groups are almost Abelian. On the other hand, it follows directly from the definitions that any non-Abelian simple group is not solvable. In particular, A_5 is not solvable. It follows from Exercise 21 in Chapter 24 that S_5 is not solvable. Our goal is to connect the notion of solvability of polynomials by radicals to that of solvable groups. The next theorem is a step in this direction.

Theorem 30.2 Condition for Gal(E/F) to be Solvable

Let F be a field of characteristic 0 and let $a \in F$. If E is the splitting field of $x^n - a$ over F, then the Galois group Gal(E/F) is solvable.

PROOF We first handle the case where F contains a primitive nth root of unity ω. Let b be a zero of $x^n - a$ in E. Then the zeros of $x^n - a$ are $b, \omega b, \omega^2 b, \ldots, \omega^{n-1} b$, and therefore $E = F(b)$. In this case, we claim that $\text{Gal}(E/F)$ is Abelian and hence solvable. To see this, observe that any automorphism in $\text{Gal}(E/F)$ is completely determined by its action on b. Also, since b is a zero of $x^n - a$, we know that any element of $\text{Gal}(E/F)$ sends b to another zero of $x^n - a$. That is, any element of $\text{Gal}(E/F)$ takes b to $\omega^i b$ for some i. Let ϕ and σ be two elements of $\text{Gal}(E/F)$. Then, since $\omega \in F$, ϕ and σ fix ω and $\phi(b) = \omega^j b$ and $\sigma(b) = \omega^k b$ for some j and k. Thus,

$$(\sigma\phi)(b) = \sigma(\phi(b)) = \sigma(\omega^j b) = \sigma(\omega^j)\sigma(b) = \omega^j \omega^k b = \omega^{j+k} b,$$

whereas

$$(\phi\sigma)(b) = \phi(\sigma(b)) = \phi(\omega^k b) = \phi(\omega^k)\phi(b) = \omega^k \omega^j b = \omega^{k+j} b,$$

so that $\sigma\phi$ and $\phi\sigma$ agree on b and fix the elements of F. This shows that $\sigma\phi = \phi\sigma$, and therefore $\text{Gal}(E/F)$ is Abelian.

Now suppose that F does not contain a primitive nth root of unity. Let ω be a primitive nth root of unity and let b be a zero of $x^n - a$ in E. The case where $a = 0$ is trivial, so we may assume that $b \neq 0$. Since ωb is also a zero of $x^n - a$, we know that both b and ωb belong to E, and therefore $\omega = \omega b / b$ is in E as well. Thus, $F(\omega)$ is contained in E, and $F(\omega)$ is the splitting field of $x^n - 1$ over F. Analogously to the case above, for any automorphisms ϕ and σ in $\text{Gal}(F(\omega)/F)$ we have $\phi(\omega) = \omega^j$ for some j and $\sigma(\omega) = \omega^k$ for some k. Then,

$$\begin{aligned}(\sigma\phi)(\omega) &= \sigma(\phi(\omega)) = \sigma(\omega^j) = (\sigma(\omega))^j = (\omega^k)^j \\ &= (\omega^j)^k = (\phi(\omega))^k = \phi(\omega^k) = \phi(\sigma(\omega)) = (\phi\sigma)(\omega).\end{aligned}$$

Since elements of $\text{Gal}(F(\omega)/F)$ are completely determined by their action on ω, this shows that $\text{Gal}(F(\omega)/F)$ is Abelian.

Because E is the splitting field of $x^n - a$ over $F(\omega)$ and $F(\omega)$ contains a primitive nth root of unity, we know from the case we have already done that $\mathrm{Gal}(E/F(\omega))$ is Abelian and, by Part 2 of Theorem 30.1, the series

$$\{e\} \subseteq \mathrm{Gal}(E/F(\omega)) \subseteq \mathrm{Gal}(E/F)$$

is a normal series. Finally, since both $\mathrm{Gal}(E/F(\omega))$ and

$$\mathrm{Gal}(E/F)/\mathrm{Gal}(E/F(\omega)) \approx \mathrm{Gal}(F(\omega)/F)$$

are Abelian, $\mathrm{Gal}(E/F)$ is solvable. ∎

To reach our main result about polynomials that are solvable by radicals, we need two important facts about solvable groups.

Theorem 30.3 Factor Group of a Solvable Group Is Solvable

A factor group of a solvable group is solvable.

PROOF Suppose that G has a series of subgroups

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_k = G,$$

where, for each $0 \leq i < k, H_i$ is normal in H_{i+1} and H_{i+1}/H_i is Abelian. If N is any normal subgroup of G, then

$$\{e\} = H_0N/N \subset H_1N/N \subset H_2N/N \subset \cdots \subset H_kN/N = G/N$$

is the requisite series of subgroups that guarantees that G/N is solvable. (See Exercise 31.) ∎

Theorem 30.4 N and G/N Solvable Implies G Is Solvable

Let N be a normal subgroup of a group G. If both N and G/N are solvable, then G is solvable.

PROOF Let a series of subgroups of N with Abelian factors be

$$N_0 \subset N_1 \subset \cdots \subset N_t = N$$

and let a series of subgroups of G/N with Abelian factors be

$$N/N = H_0/N \subset H_1/N \subset \cdots \subset H_s/N = G/N.$$

Then the series

$$N_0 \subset N_1 \subset \cdots \subset N_t = H_0 \subset H_1 \subset \cdots \subset H_s = G$$

has Abelian factors (see Exercise 33). ∎

We are now able to make the critical connection between solvability of polynomials by radicals and solvable groups.

Theorem 30.5 (Galois) Solvable by Radicals Implies Solvable Group

Let F be a field of characteristic 0 and let $f(x) \in F[x]$. Suppose that $f(x)$ splits in $F(a_1, a_2, \ldots, a_t)$, where $a_1{}^{n_1} \in F$ and $a_i{}^{n_i} \in F(a_1, \ldots, a_{i-1})$ for $i = 2, \ldots, t$. Let E be the splitting field for $f(x)$ over F in $F(a_1, a_2, \ldots, a_t)$. Then the Galois group $\mathrm{Gal}(E/F)$ is solvable.

PROOF We use induction on t. For the case $t = 1$, we have $F \subseteq E \subseteq F(a_1)$. Let $a = a_1{}^{n_1}$ and let L be a splitting field of $x^{n_1} - a$ over F. Then $F \subseteq E \subseteq L$, and both E and L are splitting fields of polynomials over F. By part 2 of Theorem 30.1, $\mathrm{Gal}(E/F) \approx \mathrm{Gal}(L/F)/\mathrm{Gal}(L/E)$. It follows from Theorem 30.2 that $\mathrm{Gal}(L/F)$ is solvable, and from Theorem 30.3 we know that $\mathrm{Gal}(L/F)/\mathrm{Gal}(L/E)$ is solvable. Thus, $\mathrm{Gal}(E/F)$ is solvable.

Now suppose $t > 1$. Let $a = a_1{}^{n_1} \in F$, let L be a splitting field of $x^{n_1} - a$ over E, and let $K \subseteq L$ be the splitting field of $x^{n_1} - a$ over F. Then L is a splitting field of $(x^{n_1} - a)f(x)$ over F, and L is a splitting field of $f(x)$ over K. Since $F(a_1) \subseteq K$, we know that $f(x)$ splits in $K(a_2, \ldots, a_t)$, so the induction hypothesis implies that $\mathrm{Gal}(L/K)$ is solvable. Also, Theorem 30.2 asserts that $\mathrm{Gal}(K/F)$ is solvable, which, from Theorem 30.1, tells us that $\mathrm{Gal}(L/F)/\mathrm{Gal}(L/K)$ is solvable. Hence, Theorem 30.4 implies that $\mathrm{Gal}(L/F)$ is solvable. So, by part 2 of Theorem 30.1 and Theorem 30.3, we know that the factor group $\mathrm{Gal}(L/F)/\mathrm{Gal}(L/E) \approx \mathrm{Gal}(E/F)$ is solvable. ∎

It is worth remarking that the converse of Theorem 30.3 is true also; that is, if E is the splitting field of a polynomial $f(x)$ over a field F of characteristic 0 and $\mathrm{Gal}(E/F)$ is solvable, then $f(x)$ is solvable by radicals over F.

It is known that every finite group is a Galois group over some field. However, one of the major unsolved problems in algebra, first posed by Emmy Noether, is determining which finite groups can occur as Galois groups over Q. Many people suspect that the answer is "all of them." It is known that every solvable group is a Galois group over Q. John Thompson proved that certain kinds of simple groups, including the Monster, are Galois groups over Q.

Insolvability of a Quintic

We will finish our introduction to Galois theory by explicitly exhibiting a polynomial that has integer coefficients and that is not solvable by radicals over Q.

Consider $g(x) = 3x^5 - 15x + 5$. By Eisenstein's Criterion (Theorem 17.4), $g(x)$ is irreducible over Q. Since $g(x)$ is continuous and $g(-2) = -61$ and $g(-1) = 17$, we know that $g(x)$ has a real zero between -2 and -1. A similar analysis shows that $g(x)$ also has real zeros between 0 and 1 and between 1 and 2.

Each of these real zeros has multiplicity 1, as can be verified by long division or by appealing to Theorem 19.6. Furthermore, $g(x)$ has no more than three real zeros, because Rolle's Theorem from calculus guarantees that between each pair of real zeros of $g(x)$ there must be a zero of $g'(x) = 15x^4 - 15$. So, for $g(x)$ to have four real zeros, $g'(x)$ would have to have three real zeros, and it does not. Thus, the other two zeros of $g(x)$ are nonreal complex numbers, say, $a + bi$ and $a + bi$. (See Exercise 73 in Chapter 15.)

Now, let's denote the five zeros of $g(x)$ by a_1, a_2, a_3, a_4, a_5. Since any automorphism of $K = Q(a_1, a_2, a_3, a_4, a_5)$ is completely determined by its action on the a's and must permute the a's, we know that $\text{Gal}(K/Q)$ is isomorphic to a subgroup of S_5, the symmetric group on five symbols. Since a_1 is a zero of an irreducible polynomial of degree 5 over Q, we know that $[Q(a_1) : Q] = 5$, and therefore 5 divides $[K : Q]$. Thus, the Fundamental Theorem of Galois Theory tells us that 5 also divides $|\text{Gal}(K/Q)|$. So, by Cauchy's Theorem (corollary to Theorem 24.3), we may conclude that $\text{Gal}(K/Q)$ has an element of order 5. Since the only elements in S_5 of order 5 are the 5-cycles, we know that $\text{Gal}(K/Q)$ contains a 5-cycle. The mapping from $\mathbf{C}$ to $\mathbf{C}$, sending $a+bi$ to $a-bi$, is also an element of $\text{Gal}(K/Q)$. Since this mapping fixes the three real zeros and interchanges the two complex zeros of $g(x)$, we know that $\text{Gal}(K/Q)$ contains a 2-cycle. But, the only subgroup of S_5 that contains both a 5-cycle and a 2-cycle is S_5. (See Exercise 25 in

Chapter 24.) So, $\mathrm{Gal}(K/Q)$ is isomorphic to S_5. Finally, since S_5 is not solvable (see Exercise 27), we have succeeded in exhibiting a fifth-degree polynomial that is not solvable by radicals.

Exercises

Seeing much, suffering much, and studying much are the three pillars of learning.

Benjamin Disraeli

1. Let E be an extension field of Q. Show that any automorphism of E acts as the identity on Q. (This exercise is referred to in this chapter.)
2. Determine the group of field automorphisms of $\mathrm{GF}(4)$.
3. Let E be an extension field of the field F. Show that the automorphism group of E fixing F is indeed a group. (This exercise is referred to in this chapter.)
4. Given that the automorphism group of $Q(\sqrt{2}, \sqrt{5}, \sqrt{7})$ is isomorphic to $Z_2 \oplus Z_2 \oplus Z_2$, determine the number of subfields of $Q(\sqrt{2}, \sqrt{5}, \sqrt{7})$ that have degree 4 over Q.
5. Let E be an extension field of a field F and let H be a subgroup of $\mathrm{Gal}(E/F)$. Show that the fixed field of H is indeed a field. (This exercise is referred to in this chapter.)
6. Let E be the splitting field of x^4+1 over Q. Find $\mathrm{Gal}(E/Q)$. Find all subfields of E. Find the automorphisms of E that have fixed fields $Q(\sqrt{2}), Q(\sqrt{-2})$, and $Q(i)$. Is there an automorphism of E whose fixed field is Q?
7. Let $f(x) \in F[x]$ and let the zeros of $f(x)$ be $a_1, a_2, \ldots, a_n$. If $K = F(a_1, a_2, \ldots, a_n)$, show that $\mathrm{Gal}(K/F)$ is isomorphic to a group of permutations of the a_i's. [When K is the splitting field of $f(x)$ over F, the group $\mathrm{Gal}(K/F)$ is called the *Galois group of* $f(x)$.]
8. Show that the Galois group of a polynomial of degree n has order dividing $n!$
9. Referring to Example 6, show that the automorphism ϕ has order 6. Show that $\omega + \omega^{-1}$ is fixed by ϕ^3 and $\omega^3 + \omega^5 + \omega^6$ is fixed by ϕ^2. (This exercise is referred to in this chapter.)
10. Let $E = Q(\sqrt{2}, \sqrt{5})$. What is the order of the group $\mathrm{Gal}(E/Q)$? What is the order of $\mathrm{Gal}(Q\sqrt{10}/Q)$?

11. Suppose that F is a field of characteristic 0 and E is the splitting field for some polynomial over F. If $\mathrm{Gal}(E/F)$ is isomorphic to $Z_{20} \oplus Z_2$, determine the number of subfields L of E there are such that L contains F and
a. $[L : F] = 4$.
b. $[L : F] = 25$.
c. $\mathrm{Gal}(E/L)$ is isomorphic to Z_5.

12. Determine the Galois group of $x^2 - 10x + 21$ over Q. (See Exercise 7 for the definition).

13. Determine the Galois group of $x^2 + 9$ over $\mathbf{R}$. (See Exercise 7 for the definition).

14. Suppose that F is a field of characteristic 0 and E is the splitting field for some polynomial over F. If $\mathrm{Gal}(E/F)$ is isomorphic to D_6, prove that there are exactly three fields L such that $E \supseteq L \supseteq F$ and $[E : L] = 6$.

15. Suppose that E is the splitting field for some polynomial over $\mathrm{GF}(p)$. If $\mathrm{Gal}(E/\mathrm{GF}(p)) = p^6$, how many fields are there strictly between E and $\mathrm{GF}(p)$?

16. Let p be a prime. Suppose that $|\,\mathrm{Gal}(E/F)\,| = p^2$. Draw all possible subfield lattices for fields between E and F.

17. Suppose that F is a field of characteristic 0 and E is the splitting field for some polynomial over F. If $\mathrm{Gal}(E/F)$ is isomorphic to A_4, show that there is no subfield K of E such that $[K : F] = 2$.

18. Determine the Galois group of $x^3 = 1$ over Q and $x^3 - 2$ over Q. (See Exercise 7 for the definition.)

19. Suppose that K is the splitting field of some polynomial over a field F of characteristic 0. If $[K : F] = p^2q$, where p and q are distinct primes, show that K has subfields L_1, L_2, and L_3 such that $[K : L_1] = p, [K : L_2] = p^2$, and $[K : L_3] = q$.

20. Suppose that E is the splitting field of some polynomial over a field F of characteristic 0. If $\mathrm{Gal}(E/F)$ is isomorphic to D_5, draw the subfield lattice for the fields between E and F.

21. Suppose that $F \subset K \subset E$ are fields and E is the splitting field of some polynomial in $F[x]$. Show, by means of an example, that K need not be the splitting field of some polynomial in $F[x]$.

22. Suppose that E is the splitting field of some polynomial over a field F of characteristic 0. If $[E : F]$ is finite, show that there is only a finite number of fields between E and F.

23. Suppose that E is the splitting field of some polynomial over a field F of characteristic 0. If $\mathrm{Gal}(E/F)$ is an Abelian group of order 10, draw the subfield lattice for the fields between E and F.

24. Let ω be a nonreal complex number such that $\omega^5 = 1$. If ϕ is the automorphism of $Q(\omega)$ that carries ω to ω^4, find the fixed field of $\langle \phi \rangle$.

25. Determine the isomorphism class of the group $\mathrm{Gal}(\mathrm{GF}(64)/\mathrm{GF}(2))$.

26. Determine the isomorphism class of the group $\mathrm{Gal}(\mathrm{GF}(729)/\mathrm{GF}(9))$.

Exercises 27, 28, and 29 are referred to in this chapter.

27. Show that S_5 is not solvable.

28. Show that the dihedral groups are solvable.

29. Show that a group of order p^n, where p is prime, is solvable.

30. Show that S_n is solvable when $n \leq 4$.

31. Complete the proof of Theorem 30.3 by showing that the given series of groups satisfies the definition for solvability.

32. Show that a subgroup of a solvable group is solvable.

33. Let N be a normal subgroup of G and let K/N be a normal subgroup of G/N. Prove that K is a normal subgroup of G. (This exercise is referred to in this chapter.)

34. Show that any automorphism of $\mathrm{GF}(p^n)$ acts as the identity on $\mathrm{GF}(p)$.

35. If G is a finite solvable group, show that there exist subgroups of G

$$\{e\} = H_0 \subset H_1 \subset H_2 \subset \cdots \subset H_n = G$$

such that H_{i+1}/H_i has prime order.

36. Show that the polynomial $x^5 - 6x + 3$ over Q is not solvable by radicals.

31 Cyclotomic Extensions

"... To regard old problems from a new angle requires creative imagination and marks real advances in science."

Albert Einstein

Innovation is taking two things that already exist and putting them together in a new way.

Tom Freston

Motivation

For the culminating chapter of this book, it is fitting to choose a topic that ties together results about groups, rings, fields, geometric constructions, and the history of mathematics. The so-called *cyclotomic extensions* is such a topic. We begin with the history.

The ancient Greeks knew how to construct regular polygons of 3, 4, 5, 6, 8, 10, 12, 15, and 16 sides with a straightedge and compass. And, given a construction of a regular n-gon, it is easy to construct a regular $2n$-gon. The Greeks attempted to fill in the gaps (7, 9, 11, 13, 14, 17, . . .) but failed. More than 2200 years passed before anyone was able to advance our knowledge of this problem beyond that of the Greeks. Incredibly, Gauss, at age 19, showed that a regular 17-gon is constructible, and shortly thereafter he completely solved the problem of exactly which n-gons are constructible. It was this discovery of the constructibility of the 17-sided regular polygon that induced Gauss to dedicate his life to the study of mathematics. Gauss was so proud of this accomplishment that he requested that a regular 17-sided polygon be engraved on his tombstone.

Gauss was led to his discovery of the constructible polygons through his investigation of the factorization of polynomials of the form $x^n - 1$ over Q. In this chapter, we examine the factors of $x^n - 1$ and show how Galois theory can be used to determine which regular n-gons are constructible with a straightedge and compass.

The irreducible factors of $x^n - 1$ are important in number theory and combinatorics.

Cyclotomic Polynomials

Recall from Example 2 in Chapter 16 that the complex zeros of $x^n - 1$ are $1, \omega = \cos(2\pi/n) + i\sin(2\pi/n), \omega^2, \omega^2, \ldots, \omega^{n-1}$. Thus, the splitting field of $x^n - 1$ over Q is $Q(\omega)$. This field is called the *nth cyclotomic extension* of Q, and the irreducible factors of $x^n - 1$ over Q are called the *cyclotomic polynomials.*

Since $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$ generates a cyclic group of order n under multiplication, we know from Corollary 3 of Theorem 4.2 that the generators of $\langle\omega\rangle$ are the elements of the form ω^k, where $1 \leq k \leq n$ and $\gcd(n, k) = 1$. These generators are called the *primitive nth roots of unity.* recalling that we use $\phi(n)$ to denote the number of positive integers less than or equal to n and relatively prime to n, we see that for each positive integer n there are precisely $\phi(n)$ primitive nth roots of unity. The polynomials whose zeros are the $\phi(n)$ primitive nth roots of unity have a special name.

Definition Cyclotomic Polynomial
For any positive integer n, let $\omega_1, \omega_2, \ldots, \omega_{\phi(n)}$ denote the primitive nth roots of unity. The *nth cyclotomic polynomial over Q* is the polynomial $\Phi_n(x) = (x - \omega_1)(x - \omega_2) \cdots (x - \omega_{\phi(n)})$.

In particular, note that $\Phi_n(x)$ is monic and has degree $\phi(n)$. In Theorem 31.2 we will prove that $\Phi_n(x)$ has integer coefficients, and in Theorem 31.3 we will prove that $\Phi_n(x)$ is irreducible over Z.

▮ **EXAMPLE 1** $\Phi_1(x) = x - 1$, since 1 is the only zero of $x - 1$. $\Phi_2(x) = x + 1$, since the zeros of $x^2 - 1$ are 1 and -1, and -1 is the only primitive root. $\Phi_3(x) = (x - \omega)(x - \omega^2)$, where $\omega = \cos(2\pi/3) + i\sin(2\pi/3) = (-1 + i\sqrt{3}/2$, and direct calculations show that $\Phi_3(x) = x^2 + x + 1$. Since the zeros of $x^4 - 1$ are ± 1 and $\pm i$ and only i and $-i$ are primitive, $\Phi_4(x) = (x - i)(x + i) = x^2 + 1$. ▮

In practice, one does not use the definition of $\Phi_n(x)$ to compute it. Instead, one uses the formulas given in the exercises and makes recursive use of the following result.

Theorem 31.1 $x^n - 1 = \prod_{d/n} \Phi_d(x)$

For every positive integer $n, x^n - 1 = \prod_{d|n} \Phi_d(x)$, where the product runs over all positive divisors d of n.

Before proving this theorem, let us be sure that the statement is clear. For $n = 6$, for instance, the theorem asserts that $x^6 - 1 = \Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_6(x)$, since 1, 2, 3, and 6 are the positive divisors of 6.

PROOF Since both polynomials in the statement are monic, it suffices to show that they have the same zeros and that all zeros have multiplicity 1. Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. Then $\langle\omega\rangle$ is a cyclic group of order n, and $\langle\omega\rangle$ contains all the nth roots of unity. From Theorem 4.3 we know that for each j, $|\omega^j|$ divides n so that $(x - \omega^j)$ appears as a factor in $\Phi_{|\omega^j|}(x)$. On the other hand, if $x - \alpha$ is a linear factor of $\Phi_d(x)$ for some divisor d of n, then $\alpha^d = 1$, and therefore $\alpha^n = 1$. Thus, $x - \alpha$ is a factor of $x^n - 1$. Finally, since no zero of $x^n - 1$ can be a zero of $\Phi_d(x)$ for two different d's, the result is proved. ▮

Before we illustrate how Theorem 31.1 can be used to calculate $\Phi_n(x)$ recursively, we state an important consequence of the theorem.

Theorem 31.2 $\Phi_d(x)$ has Integer Coefficients

For every positive integer n, $\Phi_n(x)$ has integer coefficients.

PROOF The case $n = 1$ is trivial. By induction, we may assume that $g(x) = \prod_{\substack{d|n \\ d<n}} \Phi_d(x)$ has integer coefficients. From Theorem 31.1 we know that $x^n - 1 = \Phi_n(x)g(x)$, and, because $g(x)$ is monic, we may carry out the division in $Z[x]$ (see Exercise 71 in Chapter 16). Thus, $\Phi_n(x) \in Z[x]$. ▮

Now let us do some calculations. If p is a prime, we have from Theorem 31.1 that $x^p - 1 = \Phi_1(x)\Phi_p(x) = (x - 1)\Phi_p(x)$, so that $\Phi_p(x) = (x^p - 1)/(x - 1) = x^{p-1} + x^{p-2} + \cdots + x + 1$. From Theorem

31.1 we have

$$x^6 - 1 = \Phi_1(x)\Phi_2(x)\Phi_3(x)\Phi_6(x),$$

so that $\Phi_6(x) = (x^6 - 1)/((x - 1)(x + 1)(x^2 + x + 1))$. So, by long division $\Phi_6(x) = x^2 - x + 1$. Similarly, $\Phi_{10}(x) = (x^{10} - 1)/((x - 1)(x + 1)(x^4 + x^3 + x^2 + x + 1)) = x^4 - x^3 + x^2 - x + 1$.

The exercises provide shortcuts that often make long division unnecessary. The values of $\Phi_n(x)$ for all n up to 15 are shown in Table 31.1. The software for the computer exercises provides the values for $\Phi_n(x)$ for all values of n up to 1000. Judging from Table 31.1, one might be led to conjecture that 1 and -1 are the only nonzero coefficients of the cyclotomic polynomials. However, it has been shown that every integer is a coefficient of some cyclotomic polynomial.

Table 31.1 The Cyclotomic Polynomials $\Phi_n(x)$ up to $n = 15$.

n	$\Phi_n(x)$
1	$x - 1$
2	$x + 1$
3	$x^2 + x + 1$
4	$x^2 + 1$
5	$x^4 + x^3 + x^2 + x + 1$
6	$x^2 - x + 1$
7	$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
8	$x^4 + 1$
9	$x^6 + x^3 + 1$
10	$x^4 - x^3 + x^2 - x + 1$
11	$x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
12	$x^4 - x^2 + 1$
13	$x^{12} + x^{11} + x^{10} + x^9 + x^8 + x^7 + x^6 + x^5 + x^4 + x^3 + x^2 + x + 1$
14	$x^6 - x^5 + x^4 - x^3 + x^2 - x + 1$
15	$x^8 - x^7 + x^5 - x^4 + x^3 - x + 1$

The next theorem reveals why the cyclotomic polynomials are important.

Theorem 31.3 (Gauss) $\Phi_d(x)$ is Irreducible Over Z

The cyclotomic polynomials $\Phi_n(x)$ are irreducible over Z.

PROOF Let $f(x) \in Z[x]$ be a monic irreducible factor of $\Phi_n(x)$. Because $\Phi_n(x)$ is monic and has no multiple zeros, it suffices to show that every zero of $\Phi_n(x)$ is a zero of $f(x)$.

Since $\Phi_n(x)$ divides $x^n - 1$ in $Z[x]$, we may write $x^n - 1 = f(x)g(x)$, where $g(x) \in Z[x]$. Let ω be a primitive nth root of unity that is a zero of $f(x)$. Then $f(x)$ is the minimal polynomial for ω over Q. Let p be any prime that does not divide n. Then, by Corollary 3 of Theorem 4.2, ω^p is also a primitive nth root of unity, and therefore $0 = (\omega^p)^n - 1 = f(\omega^p)g(\omega^p)$, and so $f(\omega^p) = 0$ or $g(\omega^p) = 0$. Suppose $f(\omega^p) \neq 0$. Then $g(\omega^p) = 0$, and so ω is a zero of $g(x^p)$. Thus, from Theorem 21.3, $f(x)$ divides $g(x^p)$ in $Q[x]$. Since $f(x)$ is monic, $f(x)$ actually divides $g(x^p)$ in $Z[x]$ (see Exercise 71 in Chapter 16). Say $g(x^p) = f(x)h(x)$, where $h(x) \in Z[x]$. Now let $\overline{g}(x), \overline{f}(x)$, and $\overline{h}(x)$ denote the polynomials in $Z_p[x]$ obtained from $g(x)$, $f(x)$, and $h(x)$, respectively, by reducing each coefficient modulo p. From Exercise 13 in Chapter 16 and Corollary 5 of Theorem 7.1, we then have $(\overline{g}(x))^p = \overline{g}(x^p) = \overline{f}(x)\overline{h}(x)$, and since $Z_p[x]$ is a unique factorization domain, it follows that $\overline{g}(x)$ and $\overline{f}(x)$ have an irreducible factor in $Z_p[x]$ in common; call it $m(x)$. Thus, we may write $\overline{f}(x) = k_1(x)m(x)$ and $\overline{g}(x) = k_2(x)m(x)$, where $k_1(x), k_2(x) \in Z_p[x]$. Then, viewing $x^n - 1$ as a member of $Z_p[x]$, we have $x^n - 1 = \overline{f}(x)\overline{g}(x) = k_1(x)k_2(x)(m(x))^2$. In particular, $x^n - 1$ has a multiple zero in some extension of Z_p. But because p does not divide n, the derivative nx^{n-1} or $x^n - 1$ is not 0, and so nx^{n-1} and $x^n - 1$ do not have a common factor of positive degree in $Z_p[x]$. Since this contradicts Theorem 19.5, we must have $f(\omega^p) = 0$.

We reformulate what we have thus far proved as follows: If β is any primitive nth root of unity that is a zero of $f(x)$ and p is any prime that does not divide n, then β^p is a zero of $f(x)$. Now let k be any integer between 1 and n that is relatively prime to n. Then we can write $k = p_1p_2 \cdots p_t$, where each p_i is a prime that does not divide n (repetitions are permitted). It follows then that each of $\omega, \omega^{p_1}, (\omega^{p_1})^{p_2}, \ldots, (\omega^{p_1p_2\cdots p_{t-1}})^{p_t} = \omega^k$ is a zero of $f(x)$. Since every zero of $\Phi_n(x)$ has the form ω^k, where k is between 1 and n and is relatively prime to n, we have proved that every zero of $\Phi_n(n)$ is a zero of $f(x)$. This completes the proof. ∎

Of course, Theorems 31.3 and 31.1 give us the factorization of $x^n - 1$ as a product of irreducible polynomials over Q. But Theorem 31.1 is also useful for finding the irreducible factorization of $x^n - 1$ over Z_p. The next example provides an illustration. Irreducible factors of $x^n - 1$ over Z_p are used to construct error-correcting codes.

EXAMPLE 2 We determine the irreducible factorization of x^6-1 over Z_2 and Z_3. From Table 31.1, we have $x^6-1=(x-1)(x+1)(x^2+x+1)(x^2-x+1)$. Taking all the coefficients on both sides mod 2, we obtain the same expression, but we must check that these factors are irreducible over Z_2. Since x^2+x+1 has no zeros in Z_2, it is irreducible over Z_2 (see Theorem 17.1). Finally, since $-1=1$ in Z_2, we have the irreducible factorization $x^6-1=(x+1)^2(x^2+x+1)^2$. Over Z_3, we again start with the factorization $x^6-1=(x-1)(x+1)(x^2+x+1)(x^2-x+1)$ over Z and view the coefficients mod 3. Then 1 is a zero of x^2+x+1 in Z_3, and by long division we obtain $x^2+x+1=(x-1)(x+2)=(x+2)^2$. Similarly, $x^2-x+1=(x-2)(x+1)=(x+1)^2$. So, the irreducible factorization of x^6-1 over Z_3 is $(x+1)^3(x+2)^3$. ∎

We next determine the Galois group of the cyclotomic extensions of Q.

Theorem 31.4 Gal$(Q(\omega)/Q \approx U(n)$

Let ω be a primitive nth root of unity. Then Gal$(Q(\omega)/Q \approx U(n)$.

PROOF Since $1, \omega, \omega^2, \ldots, \omega^{n-1}$ are all the nth roots of unity, $Q(\omega)$ is the splitting field of x^n-1 over Q. For each k in $U(n)$, ω^k is a primitive nth root of unity, and by the lemma preceding Theorem 19.4, there is a field automorphism of $Q(\omega)$, which we denote by ϕ_k, that carries ω to ω^k and acts as the identity on Q. Moreover, these are all the automorphisms of $Q(\omega)$, since any automorphism must map a primitive nth root of unity to a primitive nth root of unity. Next, observe that for every r, $s \in U(n)$,

$$(\phi_r\phi_s)(\omega)=\phi_r(\omega^s)=(\phi_r(\omega))^s=(\omega^r)^s=\omega^{rs}\phi_{rs}(\omega).$$

This shows that the mapping from $U(n)$ onto Gal$(Q(\omega)/Q)$ given by $k\to\phi_k$ is a group homomorphism. Clearly, the mapping is an isomorphism, since $\omega^r\neq\omega^s$ when $r,s\in U(n)$ and $r\neq s$. ∎

The next example uses Theorem 31.4 and the results of Chapter 8 to demonstrate how to determine the Galois group of cyclotomic extensions.

EXAMPLE 3 Let $\alpha = \cos(2\pi/9) + i\sin(2\pi/9)$ and let $\beta = \cos(2\pi/15) + i\sin(2\pi/15)$.Then

$$\mathrm{Gal}(Q(\alpha)/Q) \approx U(9) \approx Z_6$$

and

$$\mathrm{Gal}(Q(\beta)/Q) \approx U(15) \approx U(5) \oplus U(3) \approx Z_4 \oplus Z_2.$$

The Constructible Regular n-gons

As an application of the theory of cyclotomic extensions and Galois theory, we determine exactly which regular n-gons are constructible with a straightedge and compass. But first we prove a technical lemma.

Lemma $Q(\cos(2\pi n)) \subseteq Q(\omega)$

Let n be a positive integer and let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. Then $Q(\cos(2\pi/n)) \subseteq Q(\omega)$.

PROOF Observe that from $(\cos(2\pi/n) + i\sin(2\pi/n))(\cos(2\pi/n) - i\sin(2\pi/n)) = \cos^2(2\pi/n) + \sin^2(2\pi/n) = 1$, we have $\cos(2\pi/n) - i\sin(2\pi/n) = 1/\omega$. Moreover, $(\omega + 1/\omega)/2 = (2\cos(2\pi/n))/2 = \cos(2\pi/n)$. Thus, $\cos(2\pi/n) \in Q(\omega)$.

Theorem 31.5 (Gauss, 1796) Constructibility Criteria for a Regular n-gon

It is possible to construct the regular n-gon with a straightedge and compass if and only if n has the form $2^k p_1 p_2 \cdots p_t$, where $k \geq 0$ and the p_i's are distinct primes of the form $2^m + 1$.

PROOF If it is possible to construct a regular n-gon, then we can construct the angle $2\pi/n$ and therefore the number $\cos(2\pi/n)$. By the results of Chapter 22, we know that $\cos(2\pi/n)$ is constructible only if $[Q(\cos(2\pi/n)){:}Q]$ is a power of 2. To determine when this is so, we will use Galois theory.

Let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. Then $|\mathrm{Gal}(Q(\omega)/Q)| = [Q(\omega){:}Q] = \phi(n)$. By the lemma on the preceding page,

$Q(\cos(2\pi/n)) \subseteq Q(\omega)$, and by Theorem 30.1 we know that

$$\begin{aligned}[Q(\cos(2\pi/n)):Q] &= |\mathrm{Gal}(Q(\omega)/Q)|/|\mathrm{Gal}(Q(\omega)/Q(\cos(2\pi/n)))| \\ &= \phi(n)/|\mathrm{Gal}(Q(\omega)/Q(\cos(2\pi/n)))|.\end{aligned}$$

Recall that the elements σ of $\mathrm{Gal}(Q(\omega)/Q)$ have the property that $\sigma(\omega) = \omega^k$ for $1 \leq k \leq n$. That is, $\sigma(\cos(2\pi/n) + i\sin(2\pi/n)) = \cos(2\pi k/n) + i\sin(2\pi k/n)$. If such a σ belongs to $\mathrm{Gal}(Q(\omega)/Q(\cos(2\pi/n)))$, then we must have $\cos(2\pi k/n) = \cos(2\pi/n)$. Clearly, this holds only when $k = 1$ and $k = n - 1$. So, $|\mathrm{Gal}(Q(\omega)/Q(\cos(2\pi/n)))| = 2$, and therefore $[Q(\cos(2\pi/n)):Q] = \phi(n)/2$. Thus, if an n-gon is constructible, then $\phi(n)/2$ must be a power of 2. Of course, this implies that $\phi(n)$ is a power of 2.

Write $n = 2^k p_1^{n_1} p_2^{n_2} \cdots p_t^{n_t}$, where $k \geq 0$, the p_i's are distinct odd primes, and the n_i's are positive. Then

$$\begin{aligned}\phi(n) &= |U(n)| = |U(2^k)||U(p_1^{n_1})||U(p_2^{n_2})| \cdots |U(p_t^{n_t})| \\ &= 2^{k-1} p_1^{n_1-1}(p_1 - 1) p_2^{n_2-1}(p_2 - 1) \cdots p_t^{n_t-1}(p_t - 1)\end{aligned}$$

must be a power of 2. Clearly, this implies that each $n_i = 1$ and each $p_i - 1$ is a power of 2. This completes the proof that the condition in the statement is necessary.

To prove that the condition given in Theorem 31.5 is also sufficient, suppose that n has the form $2^k p_1 p_2 \cdots p_t$, where the p_i's are distinct odd primes of the form $2^m + 1$, and let $\omega = \cos(2\pi/n) + i\sin(2\pi/n)$. By Theorem 31.3, $Q(\omega)$ is a splitting field of an irreducible polynomial over Q, and therefore, by the Fundamental Theorem of Galois Theory, $\phi(n) = [Q(\omega):Q] = |\mathrm{Gal}(Q(\omega)/Q)|$. Since $\phi(n)$ is a power of 2 and $\mathrm{Gal}(Q(\omega)/Q)$ is an Abelian group, it follows by induction (see Exercise 15) that there is a series of subgroups

$$H_0 \subset H_1 \subset \cdots \subset H_t = \mathrm{Gal}(Q(\omega)/Q),$$

where H_0 is the identity, H_1 is the subgroup of $\mathrm{Gal}(Q(\omega)/Q)$ of order 2 that fixes $\cos(2\pi/n)$, and $|H_{i+1}:H_i| = 2$ for $i = 0, 1, 2, \ldots, t-1$. By the Fundamental Theorem of Galois Theory, we then have a series of subfields of the real numbers

$$Q = E_{H_t} \subset E_{H_{t-1}} \subset \cdots \subset E_{H_1} = Q(\cos(2\pi/n)),$$

where $[E_{H_{i-1}}:E_{H_i}] = 2$. So, for each i, we may choose $\beta_i \in E_{H_{i-1}}$ such that $E_{H_{i-1}} = E_{H_i}(\beta_i)$. Then β_i is a zero of a polynomial of the form $x^2 + b_i x + c_i \in E_{H_i}[x]$, and it follows that $E_{H_{i-1}} = E_{H_i}(\sqrt{b_i^2 - 4c_i})$. Thus, it follows from Exercise 3 in Chapter 22 that every element of $Q(\cos(2\pi/n))$ is constructible. ∎

It is interesting to note that Gauss did not use Galois theory in his proof. In fact, Gauss gave his proof 15 years before Galois was born.

Some authors write the expression $2^m + 1$ in the statement of Theorem 31.5 in the form $2^{2^k} + 1$. These expressions are equivalent since if a prime $p > 2$ can be written in the form $2^m + 1$ then m must be a power of 2 (see Exercise 21).

Exercises

Difficulties should act as a tonic. They should spur us to greater exertion.
B. C. Forbes

1. Determine the minimal polynomial for $\cos(\pi/3) + i\sin(\pi/3)$ over Q.
2. Factor $x^{12} - 1$ as a product of irreducible polynomials over Z.
3. Factor $x^8 - 1$ as a product of irreducible polynomials over Z_2, Z_3, and Z_5.
4. For any $n > 1$, prove that the sum of all the nth roots of unity is 0.
5. For any $n > 1$, prove that the product of the nth roots of unity is $(-1)^{n+1}$.
6. Let ω be a primitive 12th root of unity over Q. Find the minimal polynomial for ω^4 over Q.
7. Let F be a finite extension of Q. Prove that there are only a finite number of roots of unity in F.
8. For any $n > 1$, prove that the irreducible factorization over Z of $x^{n-1} + x^{n-2} + \cdots + x + 1$ is $\prod \Phi_d(x)$, where the product runs over all positive divisors d of n greater than 1.
9. If $2^n + 1$ is prime for some $n \geq 1$, prove that n is a power of 2. (Primes of the form $2^n + 1$ are called *Fermat primes*.)
10. Prove that $\Phi_n(0) = 1$ for all $n > 1$.

11. Prove that if a field contains the nth roots of unity for n odd, then it also contains the $2n$th roots of unity.

12. Let m and n be relatively prime positive integers. Prove that the splitting field of $x^{mn} - 1$ over Q is the same as the splitting field of $(x^m - 1)(x^n - 1)$ over Q.

13. Prove that $\Phi_{2n}(x) = \Phi_n(-x)$ for all odd integers $n > 1$.

14. Prove that if p is a prime and k is a positive integer, then $\Phi_{p^k}(x) = \Phi_p(x^{p^{k-1}})$. Use this to find $\Phi_8(x)$ and $\Phi_{27}(x)$.

15. Prove the assertion made in the proof of Theorem 31.5 that there exists a series of subgroups $H_0 \subset H_1 \subset \cdots \subset H_t$ with $|H_{i+1}{:}H_i| = 2$ for $i = 0, 1, 2, \ldots, t-1$. (This exercise is referred to in this chapter.)

16. Prove that $x^9 - 1$ and $x^7 - 1$ have isomorphic Galois groups over Q. (See Exercise 7 in Chapter 30 for the definition.)

17. Let p be a prime that does not divide n. Prove that $\Phi_{pn}(x) = \Phi_n(x^p)/\Phi_n(x)$.

18. Prove that the Galois groups of $x^{10} - 1$ and $x^8 - 1$ over Q are not isomorphic.

19. Let E be the splitting field of $x^5 - 1$ over Q. Show that there is a unique field K with the property that $Q \subset K \subset E$.

20. Let E be the splitting field of $x^6 - 1$ over Q. Show that there is no field K with the property that $Q \subset K \subset E$.

21. If $p > 2$ is a prime of the form $2^m + 1$, prove that m is a power of 2.

22. Let $\omega = \cos(2\pi/15) - i\sin(2\pi/15)$. Find the three elements of $\mathrm{Gal}(Q(\omega)/Q)$ of order 2.

Computer Exercises

A computer program returns the n-th cyclotomic polynomial, where n is a product of distinct primes is available at the website:

http://www.d.umd/~jgallian

Carl Friedrich Gauss

The Granger Collection, New York

He [Gauss] lives everywhere in mathematics.

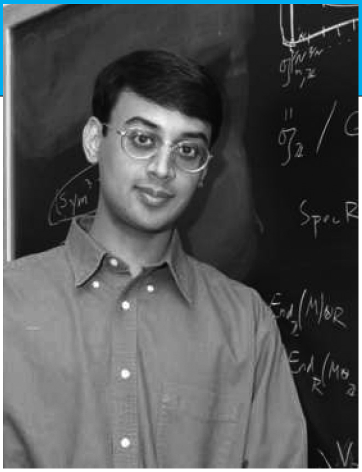
E. T. BELL, *Men of Mathematics*

CARL FRIEDRICH GAUSS, considered by many to be the greatest mathematician who has ever lived, was born in Brunswick, Germany, on April 30, 1777. While still a teenager, he made many fundamental discoveries. Among these were the method of "least squares" for handling statistical data, and a proof that a 17-sided regular polygon can be constructed with a straightedge and compass (this result was the first of its kind since discoveries by the Greeks 2000 years earlier). In his Ph.D. dissertation in 1799, he proved the Fundamental Theorem of Algebra.

Throughout his life, Gauss largely ignored the work of his contemporaries and, in fact, made enemies of many of them. Young mathematicians who sought encouragement from him were usually rebuffed. Despite this fact, Gauss had many outstanding students, including Eisenstein, Riemann, Kummer, Dirichlet, and Dedekind.

Gauss died in Göttingen at the age of 77 on February 23, 1855. At Brunswick, there is a statue of him. Appropriately, the base is in the shape of a 17-point star. In 1989, Germany issued a bank note (see page 117) depicting Gauss and the Gaussian distribution.

Manjul Bhargava



Office of Communications,
Princeton University

We are watching him [Bhargava] very closely. He is going to be a superstar. He's amazingly mature mathematically. He is changing the subject in a fundamental way.

PETER SARNAK

MANJUL BHARGAVA was born in Canada on August 8, 1974, and grew up in Long Island, New York. After graduating from Harvard in 1996, Bhargava went to Princeton to pursue his Ph.D. under the direction of Andrew Wiles (see biography after Chapter 18). Bhargava investigated a "composition law" first formulated by Gauss in 1801 for combining two quadratic equations (equations in a form such as $x^2 + 3xy + 6y^2 = 0$) in a way that was very different from normal addition and revealed a lot of information about number systems. Bhargava tackled an aspect of the problem in which no progress had been made in more than 200 years. He not only broke new ground in that area but also discovered 13 more composition laws and developed a coherent mathematical framework to explain them. He then applied his theory of composition to solve a number of fundamental problems concerning the distribution of extension fields of the rational numbers and of other, related algebraic objects. What made Bhargava's work especially remarkable is that he was able to explain all his revolutionary ideas using only elementary mathematics. In commenting on Bhargava's results, Wiles said, "He did it in a way that Gauss himself could have understood and appreciated."

Among Bhargava's many awards are the Blumenthal Award for the Advancement of Research in Pure Mathematics, the SASTRA Ramanujan Prize, the Cole Prize in number theory, the Fermat Prize, Infosys Prize, and the Fields Medal (see Chapter 24). He has been elected to the National Academy of Sciences, the American Academy of Arts and Sciences, and is a Fellow of the Royal Society.

In addition to doing mathematics, Bhargava is an accomplished tabla player who has studied with the world's most distinguished tabla masters. He performs extensively in the New York and Boston areas. To hear him play the tabla, visit

http://www.npr.org/templates/story/story.php?storyId=4111253

Selected Answers

Failures, repeated failures, are finger posts on the road to achievement. One fails forward toward success.

C. S. Lewis

For some exercises only partial answers are provided. Many of the proofs given below are merely sketches. In these cases, the student should supply the complete proof

Chapter 0

In short, if we adhere to the standard of perfection in all our endeavors, we are left with nothing but mathematics and the White Album.

DANIEL GILBERT, *Stumbling on Happiness*

1. $\{1, 2, 3, 4\}$; $\{1, 3, 5, 7\}$; $\{1, 5, 7, 11\}$; $\{1, 3, 7, 9, 11, 13, 17, 19\}$; $\{1, 2, 3, 4, 6, 7, 8, 9, 11, 12, 13, 14, 16, 17, 18, 19, 21, 22, 23, 24\}$
3. 12, 2, 2, 10, 1, 0, 4, 5
5. Let a be the least common multiple of every element of the set and b be any common multiple of every element of the set. Write $b = aq + r$ where $0 \leq r \leq a$. Then for any element c in the set we have that c divides $b - aq = r$. This means that r is common multiple of every element of the set and therefore is greater than or equal to a, which is a contradiction.
7. By using 0 as an exponent if necessary, we may write $a = p_1^{m_1} \cdots p_k^{m_k}$ and $b = p_1^{n_1} \cdots p_k^{n_k}$, where the p's are distinct primes and the m's and n's are nonnegative. Then $\text{lcm}(a, b) = p_1^{s_1} \cdots p_k^{s_k}$, where $s_i = \max(m_i, n_i)$, and $\gcd(a, b) = p_1^{t_1} \ldots p_k^{t_k}$, where $t_i = \min(m_i, n_i)$. Then $\text{lcm}(a, b) \cdot \gcd(a, b) = p_1^{m_1+n_1} \cdots p_k^{m_k+n_k} = ab$.
9. Write $a = nq_1 + r_1$ and $b = nq_2 + r_2$, where $0 \leq r_1, r_2 < n$. We may assume that $r_1 \geq r_2$. Then $a - b = n(q_1 - q_2) + (r_1 - r_2)$, where $r_1 - r_2 \geq 0$. If $a \bmod n = b \bmod n$, then $r_1 = r_2$ and n divides $a - b$. If n divides $a - b$, then by the uniqueness of the remainder, we have $r_1 - r_2 = 0$.
11. Use Exercise 9.
13. Use Theorem 0.2.
15. By Theorem 0.2 there are integers s and t such that $ms + nt = 1$. Then $m(sr) + n(tr) = r$.
17. Let p be a prime greater than 3. By the division algorithm, we can write p in the form $6n + r$, where r satisfies $0 \leq r < 6$. Now observe that $6n$, $6n + 2$, $6n + 3$, and $6n + 4$ are not prime.
19. Since st divides $a - b$, both s and t divide $a - b$. The converse is true when $\gcd(s, t) = 1$.
21. Use Euclid's Lemma and the Fundamental Theorem of Arithmetic.

23. Use proof by contradiction.
25. x NAND y is 1 if and only if both inputs are 0; x XNOR y is 1 if and only if both inputs are the same.
27. Let S be a set with $n+1$ elements and pick some a in S. By induction, S has 2^n subsets that do not contain a. But there is a one-to-one correspondence between the subsets of S that do not contain a and those that do. So, there are $2 \cdot 2^n = 2^{n+1}$ subsets in all.
29. Consider $n = 200! + 2$.
31. Say $p_1 p_2 \ldots p_r = q_1 q_2 \ldots q_s$, where the p's and q's are primes. By the Generalized Euclid's Lemma, p_1 divides some q_i, say q_1 (we may relabel the q's if necessary). Then $p_1 = q_1$ and $p_2 \ldots p_r = q_2 \ldots q_s$. Repeating this argument at each step, we obtain $p_2 = q_2, \ldots, p_r = q_r$ and $r = s$.
33. Suppose that S is a set that contains a and whenever $n \geq a$ belongs to S, then $n+1 \in S$. We must prove that S contains all integers greater than or equal to a. Let T be the set of all integers greater than a that are not in S and suppose that T is not empty. Let b be the smallest integer in T (if T has no negative integers, b exists because of the Well Ordering Principle; if T has negative integers, it can have only a finite number of them so that there is a smallest one). Then $b - 1 \in S$, and therefore $b = (b-1) + 1 \in S$.
35. For $n = 1$, observe that $1^3 + 2^3 + 3^3 = 36$. Assume that $n^3 + (n+1)^3 + (n+2)^3 = 9m$ for some integer m. We must prove that $(n+1)^3 + (n+2)^3 + (n+3)^3$ is a multiple of 9. Using the induction hypothesis we have that $(n+1)^3 + (n+2)^3 + (n+3)^3 = 9m - n^3 + (n+3)^3 = 9m - n^3 + n^3 + 3 \cdot n^2 \cdot 3 + 3 \cdot n \cdot 9 + 3^3 = 9m + 9n^2 + 27n + 27$.
37. The statement is true for any divisor of $8^3 - 4 = 508$.
39. 6 P.M.
41. Observe that the number with the decimal representation $a_9 a_8 \ldots a_1 a_0$ is $a_9 \cdot 10^9 + a_8 \cdot 10^8 + \cdots + a_1 \cdot 10 + a_0$. Then use Exercise 9 and the fact that $a_i 10^i \bmod 9 = a_i \bmod 9$ to deduce that the check digit is $(a_9 + a_8 + \cdots + a_1 + a_0) \bmod 9$.
43. For the case in which the check digit is not involved, see the answer to Exercise 41. If a transposition involving the check digit $c = (a_1 + a_2 + \cdots + a_{10}) \bmod 9$ goes undetected, then $a_{10} = (a_1 + a_2 + \cdots + a_9 + c) \bmod 9$. Substitution yields $2(a_1 + a_2 + \cdots + a_9) \bmod 9 = 0$. Therefore, modulo 9, we have $10(a_1 + a_2 + \cdots + a_9) = a_1 + a_2 + \cdots + a_9 = 0$. It follows that $c = a_{10}$. In this case the transposition does not yield an error.
47. Cases where $(2a - b - c) \bmod 11 = 0$ are undetected.
49. No. $(1, 0) \in R$ and $(0, -1) \in R$, but $(1, -1) \in R$.
51. a belongs to the same subset as a. If a and b belong to the subset A, then b and a also belong to A. If a and b belong to the subset A and b and c belong to the subset B, then $A = B$, since the distinct subsets of P are disjoint. So, a and c belong to A.

Chapter 1

If you are ready to learn from them, mistakes can be your friend.

FRANK WILCZEK

1. Three rotations $0°$, $120°$, $240°$—and three reflections across lines from vertices to midpoints of opposite sides. See the back inside cover for a picture.
3. **a.** V **b.** R_{270} **c.** R_0 **d.** $R_0, R_{180}, H, V, D, D'$ **e.** none
5. D_n has n rotations of the form $k(360°/n)$, where $k = 0, \ldots, n-1$. In addition, D_n has n reflections. When n is odd, the axes of reflection are the lines from the vertices to the midpoints of the opposite sides. When n is even, half of the axes of reflection

are obtained by joining opposite vertices; the other half, by joining midpoints of opposite sides.

7. A rotation followed by a rotation either fixes every point (and so is the identity) or fixes only the center of rotation. However, a reflection fixes a line.
9. Observe that $1 \cdot 1 = 1; 1(-1) = -1; (-1)1 = -1; (-1)(-1) = 1$. These relationships also hold when 1 is replaced by "rotation" and -1 is replaced by "reflection."
11. Thinking geometrically and observing that even powers of elements of a dihedral group do not change orientation, we note that each of a, b and c appears an even number of times in the expression. So, there is no change in orientation. Thus, the expression is a rotation.
13. In $D_4, HD = DV$ but $H \neq V$.
15. R_0, R_{180}, H, V. Use the table in Chapter 1 to construct the Cayley table. Yes. Each has two rotations and two reflections.
17. See answer for Exercise 15.
19. In each case, the group is D_6.
21. First observe that squaring R_0, R_{180} or any reflection gives R_0 and squaring R_{90} or R_{270} gives R_{180}. Thus $X^2Y = Y$ or $X^2Y = R_{180}Y$. Since $Y \neq R_{90}$ we have $X^2Y = R_{180}Y$ and $X^2Y = R_{90}$. Thus $R_{180}Y = R_{90}$. Solving for Y gives $Y = R_{270}$.
23. By Exercise 8 the elements listed are reflections. No two of these are equal for if so then multiplying by F on the right would yield equal rotations, which is not true.

Chapter 2

There are no secrets to success. It is the result of preparation, hard work, and learning from failure.

COLIN POWELL

1. **c, d**
3. none
5. 7; 13; $n-1$; $\frac{3}{13} + \frac{2}{13}i$
7. G_3.
9. If $5x = 3$ then multiply both sides by 4 we get $0 = 12$. If $3x = 5$ then multiply both sides by 7 we get $x = 15$. Checking we see that $3 \cdot 15 = 5 \bmod 20$.
11. One is Socks-Shoes-Boots.
13. Under multiplication modulo 4, 2 does not have an inverse. Under multiplication modulo 5, each element has an inverse.
15. a^{11}, a^6, a^4, a.
17. **a.** $2a + 3b$ **b.** $-2a + 2(-b + c)$ **c.** $-3(a + 2b) + 2c = 0$
19. Observe that $a^5 = e$ and $b^7 = e$ imply that $a^{-2} = a^3$ and $b^{-4} = b^3$. Thus, $a^{-2}b^{-4} = a^3b^3$. Moreover, $(a^2b^4)^{-2} = ((a^2b^4)^{-1})^2 = (b^{-4}a^{-2})^2 = (b^3a^3)^2 = b^3a^3b^3a^3$.
21. Use det $(AB) =$ (det A)(det B).
23. 29
25. For $n \geq 0$, use induction. For $n < 0$, note that $e = (ab)^0 = (ab)^n(ab)^{-n} = (ab)^n a^{-n}b^{-n}$ so that $a^nb^n = (ab)^n$. In a non-Abelian group $(ab)^n$ need not equal a^nb^n.
27. Use the Socks–Shoes Property.
29. For the case $n > 0$, use induction. For $n < 0$, note that $e = (a^{-1}ba)^n(a^{-1}ba)^{-n} = (a^{-1}ba)^n(a^{-1}b^{-n}a)$ and solve for $(a^{-1}ba)^n$.
31. {1, 3, 5, 9, 13, 15, 19, 23, 25, 27, 39, 45}
33. Suppose x appears in a row labeled with a twice; say, $x = ab$ and $x = ac$. Then cancellation yields $b = c$. But we use distinct elements to label the columns.
35. Closure and associativity follow from the definition of multiplication; $a = b = c = 0$ gives the identity; we may find inverses by solving the equations $a + a' = 0$, $b' + ac' + b = 0$, $c' + c = 0$ for a', b', c'.
37. Since e is one solution it suffices to show that nonidentity solutions come in distinct pairs. To this end note that if $x^n = e$ and $x \neq e$, then $(x^{-1})^n = e$

and $x \neq x^{-1}$. So if we can find one nonidentity solution we can find a second one. Now suppose that a and a^{-1} are nonidentity elements that satisfy $x^n = e$ and b is a nonidentity element such that $b \neq a$ and $b \neq a^{-1}$ and $b^n = e$. Then, as before, $(b^{-1})^n = e$ and $b \neq b^{-1}$. Moreover, $b^{-1} \neq a$ and $b^{-1} \neq a^{-1}$. Thus, finding a third nonidentity solution gives a fourth one. Continuing in this fashion we see that we always have an even number of nonidentity solutions to the equation $x^3 = e$.
To prove the second statement note that if $x^2 \neq e$, then $x^{-1} \neq x$ and $(x^{-1})^2 \neq e$. So, arguing as in the preceding case we see that solutions to $x^2 \neq e$ come in distinct pairs.

39. If $F_1F_2 = R_0$, then $F_1F_1 = F_1F_2$ and by cancellation $F_1 = F_2$.
41. Since FR^k is a reflection we know that $(FR^k)\,(FR^k) = R_0$. So $R^kFR^k = F^{-1} = F$.
43. R, $R^{-1}F$
45. Since $a^2 = b^2 = (ab)^2 = e$, we have $aabb = abab$. Now cancel on the left and right.
47. The matrix $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ is in GL(2, Z_2) if and only if $ad \neq bc$. This happens when a and d are 1 and at least 1 of b and c is 0, and when b and c are 1 and at least 1 of a and d is 0. $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$ and $\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}$ do not commute.
49. Use Exercise 33.
51. Let a be any element in G. Then for each b in G a appears exactly once in the row headed by b in the Cayley table for G.
53. Z_{16}, D_8, $U(17)$ and $U(32)$.

Chapter 3

Success is the ability to go from one failure to another with no loss of enthusiasm.

SIR WINSTON CHURCHILL

1. $|Z_{12}| = 12$; $|U(10)| = 4$; $|U(12)| = 4$; $|U(20)| = 8$; $|D_4| = 8$
In $Z_{12}, |0| = 1; |1| = |5| = |7| = |11| = 12; |2| = |10| = 6; |3| = |9| = 4; |4| = |8| = 3; |6| = 2$.
In $U(10), |1| = 1; |3| = |7| = 4; |9| = 2$.
In $U(12), |1| = 1; |5| = 2; |7| = 2; |11| = 2$. In $U(20), |1| = 1; |3| = |7| = |13| = |17| = 4; |9| = |11| = |19| = 2$.
In $D_4, |R_0| = 1; |R_{90}| = |R_{270}| = 4; |R_{180}| = |H| = |V| = |D| = |D'| = 2$.
In each case, notice that the order of the element divides the order of the group.
3. In $Q, |0| = 1$ and all other elements have infinite order. In $Q^*, |1| = 1, |-1| = 2$, and all other elements have infinite order.
5. Use Theorem 0.2.
7. Each is the inverse of the other.
9. $(a^4c^{-2}b^4)^{-1} = b^{-4}c^2a^{-4} = b^3c^2a^2$
11. $\{R_0, R_{120}, R_{240}, F, R_{120}F, R_{240}F\}$ where F is any reflection in D_6.
13. D_4; D_4; it contains $\{R_0, R_{180}, H, V\}$
15. If n is a positive integer, the real solutions of $x^n = 1$ are 1 when n is odd and ± 1 when n is even. So, the only elements of finite order in R^* are ± 1.
17. By Exercise 29 of Chapter 2 we have $e = (xax^{-1})^n = xa^nx^{-1}$ if and only if $a^n = e$.
19. Suppose $G = H \cup K$. Pick $h \in H$ with $h \notin k$. Pick $k \in K$ with $k \notin H$. Then, $hk \in G$ but $hk \notin H$ and $hk \notin K$. $U(8) = \{1, 3\} \cup \{1, 5\} \cup \{1, 7\}$.
21. $U_4(20) = \{1, 9, 13, 17\}; U_5(20) = \{1, 11\}; U_5(30) = \{1, 11\}; U_{10}(30) = \{1, 11\}$. To prove that $U_k(n)$ is a subgroup, it suffices to show that it is closed. Suppose that a and b belong to $U_k(n)$. We must show that in $U(n)$, ab mod $k = 1$. That is, $(ab \bmod n)$ mod $k = 1$. Let $n = kt$ and $ab = qn + r$ where $0 \leq r < n$. Then $(ab \bmod n)$ mod $k = r$ mod $k = (ab - qn)$ mod $k = (ab - qkt)$ mod $k = ab$ mod $k = (a \bmod k)(b \bmod k) = 1 \cdot 1 = 1$. H is not a subgroup because $7 \in H$ but $7 \cdot 7 = 9$ is not 1 mod 3.

23. Suppose that $m < n$ and $a^m = a^n$. Then $e = a^n a^{-m} = a^{n-m}$. This contradicts the assumption that a has infinite order.
25. det $A = \pm 1$
27. $\langle 3 \rangle = \{3, 3^2, 3^3, 3^4, 3^5, 3^6\} = \{3, 9, 13, 11, 5, 1\} = U(14)$. $\langle 5 \rangle = \{5, 5^2, 5^3, 5^4, 5^5, 5^6\} = \{5, 11, 13, 9, 3, 1\} = U(14)$. $\langle 11 \rangle = \{11, 9, 1\} \neq U(14)$. Since $|U(20)| = 8$, for $U(20) = \langle k \rangle$ for some k it must the case that $|k| = 8$. But $1^1 = 1$, $3^4 = 1$, $7^4 = 1$, $9^2 = 1$, $11^2 = 1$, $13^4 = 1$, $17^4 = 1$, and $19^2 = 1$. So, the maximum order of any element is 4.
29. By Exercise 28, either every element of H is even or exactly half are even. Since H has odd order the latter cannot occur.
31. By Exercise 30, either every element of H is a rotation or exactly half are rotations. Since H has odd order the latter cannot occur.
33. Since n is even, D_n contains R_{180}. Let F be any reflection in D_n. Then the set $\{R_0, R_{180}, F, R_{180}F\}$ is closed and therefore is a subgroup of D_n.
35. $\langle 2 \rangle, \langle 3 \rangle, \langle 6 \rangle$
37. Suppose that H is a subgroup of D_3 of order 4. Since D_3 has only two elements of order 2, H must contain R_{120} or R_{240}. By closure, it follows that H must contain R_0, R_{120}, and R_{240} as well as some reflection F. But then H must also contain the reflection $R_{120}F$.
39. $\{R_0, R_{180}\}$
41. If $x \in Z(G)$, then $x \in C(a)$ for all a, so $x \in \cap_{a \in G} C(a)$. If $x \in \cap_{a \in G} C(a)$, then $xa = ax$ for all a in G, so $x \in Z(G)$.
43. The case that $k = 0$ is trivial. Let $x \in C(a)$. If k is positive, then by induction on k, $xa^{k+1} = xaa^k = axa^k = aa^k x = a^{k+1}x$. The case where k is negative now follows from Exercise 34. In a group, if x commutes with a, then x commutes with all powers of a. If x commutes with a^k for some k, then x need not commute with a.
45. **a**. First observe that because $\langle S \rangle$ is a subgroup of G containing S, it is a member of the intersection. So, $H \subseteq \langle S \rangle$. On the other hand, since H is a subgroup of G and H contains S, by definition $\langle S \rangle \subseteq H$.
    b. Let $K = \{s_1^{n_1} s_2^{n_2} \ldots s_m^{n_m} \mid m \geq 1, s_i \in S, n_i \in Z\}$. Then because K satisfies the subgroup test and contains S, we have $\langle S \rangle \subseteq K$. On the other hand, if L is any subgroup of G that contains S, then L also contains K by closure. Thus, by part **a**, $H = \langle S \rangle$ contains K.
47. Mimic the proof of Theorem 3.5.
49. No. In $D_4, C(R_{180}) = D_4$. Yes. Elements in the center commute with all elements.
51. For the first part, see Example 4. For the second part, use D_4.
53. Note that $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}^n = \begin{bmatrix} 1 & n \\ 0 & 1 \end{bmatrix}$.
55. First observe that $(a^d)^{n/d} = a^n = e$, so $|a^d|$ is at most n/d. Moreover, there is no positive integer $t < n/d$ such that $(a^d)^t = a^{dt} = e$, for otherwise $|a| \neq n$.
57. Let G be a group of even order. Observe that for each element x of order greater than 2 x and x^{-1} are distinct elements of the same order. So, because elements of order greater than 2 come in pairs, there is an even number of elements of order greater than 2 (possibly 0). This means that the number of elements of order 1 or 2 is even. Since the identity is the unique element of order 1, it follows that the number of order 2 is odd.
59. For any positive integer n, a rotation of $360°/n$ has order n. A rotation of $\sqrt{2}$ degrees has infinite order.
61. Inscribe a regular n-gon in a circle. Then every element of D_n is a symmetry of the circle.
63. Let $|g| = m$ and write $m = nq + r$, where $0 \leq r < n$. Then $g^r = g^{m-nq} = g^m(g^n)^{-q} = (g^n)^{-q}$ belongs to H. So, $r = 0$.
65. $1 \in H$. Let $a, b \in H$. Then $(ab^{-1})^2 = a^2(b^2)^{-1}$, which is the product of two rationals. 2 can be

replaced by any positive integer.

67. Let $|a| = n$ and $m = nq = r$ where $0 \le r < n$. Then $a^r = a^{m-nq} = a^m(a^{-q})^n = ee = e$. So, $r = 0$.
69. In Z_6, $H = \{0, 1, 3, 5\}$ is not closed.
71. **a.** Let xh_1x^{-1} and xh_2x^{-1} belong to xHx^{-1}. Then $(xh_1x^{-1})(xh_2x^{-1})^{-1} = xh_1h_2^{-1}x^{-1} \in xHx^{-1}$ also.
b. Let $\langle h \rangle = H$. Then $\langle xhx^{-1} \rangle = xHx^{-1}$.
c. $(xh_1x^{-1})(xh_2x^{-1}) = xh_1h_2x^{-1} = xh_2h_1x^{-1} = (xh_2x^{-1})(xh_1x^{-1})$.
73. Let a/b and c/d belong to the set. By observation ac/bd and b/a have odd numerators and denominators. If ac/bd reduces lowest terms to x/y then x divides ac and y divides bd. So they are odd.
75. If 2^a and $2^b \in K$, then $2^a(2^b)^{-1} = 2^{a-b} \in K$, since $a - b \in H$.
77. $\begin{bmatrix} 2 & 0 \\ 0 & 2 \end{bmatrix}^{-1} = \begin{bmatrix} \frac{1}{2} & n \\ 0 & \frac{1}{2} \end{bmatrix}$ is not in H.
79. If $a + bi$ and $c + di \in H$, then $(a+bi)(c+di)^{-1} = (ac+bd)+(bc-ad)i$ and $(ac + bd)^2 + (bc - ad)^2 = 1$, so that H is a subgroup. H is the unit circle in the complex plane.
81. $\{1, 2n - 1, 2n + 1, 4n - 1\}$
83. In D_{10} let a be any reflection and $b = R_{36}$.
85. First observe that $2^n - 1$ and $2^{n-2} \pm 1$ are in $U(2^n)$ and satisfy $x^2 = 1$. Now suppose that $x \in U(2^n)$, $x \ne 1$, and $x^2 = 1 \bmod 2^n$. From $x^2 = 1 \bmod 2^n$ we have that $x^2 - 1 = (x - 1)(x + 1)$ is divisible by 2^n. Since $x - 1$ and $x + 1$ are even and $n \ge 3$, we know that at least one of $x - 1$ and $x + 1$ is divisible by 4. Moreover, it cannot be the case that both $x - 1$ and $x + 1$ are divisible by 4 for then so would $(x + 1) - (x - 1) = 2$. If $x - 1$ is not divisible by 4, then $x + 1$ is divisible by 2^{n-1}. Thus $x + 1 = k2^{n-1}$ for some integer k and $k2^{n-1} = x + 1 \le 2^n$. So, $k = 1$ or $k = 2$. For $k = 1$, we have $x = 2^{n-1} - 1$. For $k = 2$, we have $x = 2^n - 1$. If $x + 1$ is not divisible by 4, then $x - 1$ is divisible by 2^{n-1}. Thus $x - 1 = k2^{n-1}$ for some integer k and $k2^{n-1} = x - 1 < 2^n$. So, $k = 1$ and $x = 2^{n-1} + 1$.
87. Since $ee = e$ is in $HZ(G)$, it is nonempty. Let h_1z_1 and h_2z_2 belong to $HZ(G)$. Then $h_1z_1(h_2z_2)^{-1} = h_1z_1z_2^{-1}h_2^{-1} = h_1h_2^{-1}z_1z_2^{-1} \in HZ(G)$.
89. Use Exercise 88.
91. In a finite group G, $|C(x)|/|G|$ is the probability that x commutes with every element of G. Let x be any element in D_4. If $x = R_0$ or R_{180} the probability that x commutes with every element is 1. If $x = R_{90}$ or R_{270} the probability that x commutes with every element is .5. If x is a reflection the probability that x commutes with every element is .5 (x commutes with R_0, R_{180}, x, and $R_{180}x$). So, the probability that any two elements commute exceeds .5 (the exact probability is 5/8). For D_3, the probability is .5.

Chapter 4

A mistake is to commit a misunderstanding.

BOB DYLAN

1. For Z_6, generators are 1 and 5; for Z_8, generators are 1, 3, 5, and 7; for Z_{20}, generators are 1, 3, 7, 9, 11, 13, 17, and 19.
3. $\langle 20 \rangle = \{20, 10, 0\}$; $\langle 10 \rangle = \{10, 20, 0\}$; $\langle a^{20} \rangle = \{a^{20}, a^{10}, a^0\}$; $\langle a^{10} \rangle = \{a^{10}, a^{20}, a^0\}$
5. $\langle 3 \rangle = \{3, 9, 7, 1\}$; $\langle 7 \rangle = \{7, 9, 3, 1\}$
7. $U(8)$ or D_3
9. Six subgroups; generators are the divisors of 20. Six subgroups; generators are a^k, where k is a divisor of 20.
11. By definition, $a^{-1} \in \langle a \rangle$. So, $\langle a^{-1} \rangle \subseteq \langle a \rangle$. By definition, $a = (a^{-1})^{-1} \in \langle a^{-1} \rangle$. So, $\langle a \rangle \subseteq \langle a^{-1} \rangle$.
13. In Z, $\langle m \rangle \cap \langle n \rangle = \operatorname{lcm}(m, n)$. In Z_k, $\langle m \rangle \cap \langle n \rangle = \operatorname{lcm}(m, n) \bmod k$. If $|a| = \infty$, $\langle a^m \rangle \cap \langle a^n \rangle = a^{\operatorname{lcm}(m,n)}$. If

$|a| = k$, $\langle a^m \rangle \cap \langle a^n \rangle = a^{\text{lcm}(m,n)}$ mod k.

15. $|g|$ divides 12 is equivalent to $g^{12} = e$. So, if $a^{12} = e$ and $b^{12} = e$, then $(ab^{-1})^{12} = a^{12}(b^{12})^{-1} = ee^{-1} = e$. The same argument works when 12 is replaced by any integer (see Exercise 51 of Chapter 3).
17. By Theorem 4.2 we have $|\langle a^6 \rangle| = n/\gcd(n, 6)$. Since n is odd and $\langle a^6 \rangle$ is a proper subgroup we have $\gcd(n, 6) = 3$. So, $|\langle a^6 \rangle| = n/3$.
19. If $|a^2| = 3$, $|a|$ is 3 or 6. If $|a^2| = 4$, $|a| = 8$.
21. For every a and b we have $ab = (ab)^{-1} = b^{-1}a^{-1} = ba$.
23. Let $|a| = m$, $|b| = n$, $|ab| = k$ and $\gcd(m, n) = d$. Then $\text{lcm}(m, n) = mn/d$ and $(ab)^{mn/d} = (a^m)^{n/d}(b^{n/d})^m = ee = e$ so k divides $\text{lcm}(m, n)$. So, if $d > 1$, then $k < mn$. If $d = 1$, then $\langle a \rangle \cap \langle b \rangle = \{e\}$ because $|\langle a \rangle \cap \langle b \rangle|$ divides both $|\langle a \rangle|$ and $|\langle b \rangle|$. We also have $e = (ab)^k = a^k b^k$ and therefore $a^k = b^{-k} \in \langle a \rangle \cap \langle b \rangle = \{e\}$.
25. Exercise 31 in Chapter 3 tells us that H is a subgroup of the cyclic group of n rotations in D_n. So, by Theorem 4.3, H is cyclic.
27. one
29. **a.** $|a|$ divides 12. **b.** $|a|$ divides m. **c.** By Theorem 4.3, $|a| = 1, 2, 3, 4, 6, 8, 12$, or 24. If $|a| = 2$, then $a^8 = (a^2)^4 = e^4 = e$. A similar argument eliminates all other possibilities except 24.
31. Yes, by Theorem 4.3. The subgroups of Z are of the form $\langle n \rangle = \{0, \pm n, \pm 2n, \pm 3n, \dots\}$, $n = 0, 1, 2, 3, \dots$. The subgroups of $\langle a \rangle$ are of the form $\langle a^n \rangle$ for $n = 0, 1, 2, 3, \dots$.
33. For the first part, apply Theorem 4.3 to the subgroup of rotations; D_n has n elements of order 2 when n is odd and $n + 1$ elements of order 2 when n is even.
35. See Example 16 of Chapter 2.
37. 1000000, 3000000, 5000000, 7000000; by Theorem 4.3, $\langle 1000000 \rangle$ is the unique subgroup of order 8, and only those on the list are generators. $a^{1000000}, a^{3000000}, a^{5000000}, a^{7000000}$; by Theorem 4.3, $\langle a^{1000000} \rangle$ is the unique subgroup of order 8, and only those on the list are generators.
39. Let $G = \{a_1, a_2, \dots, a_k\}$. Now let $|a_i| = n_i$. Consider $n = n_1 n_2 \cdots n_k$.
41. The lattice is a vertical line with successive terms from top to bottom $\langle p^0 \rangle, \langle p^1 \rangle, \langle p^3 \rangle, \dots, \langle p^{n-1} \rangle, \langle 0 \rangle$.
43. Suppose that a/b generates are positive rationals under multiplication. Because $\langle a/b \rangle = \langle (a/b)^{-1} \rangle = \langle b/a \rangle$ we may assume that $1 < a/b$. Then from $1 < a/b < (a/b)^2 < (a/b)^3 < \dots$, we see that $\langle a/b \rangle$ does not contain any rational number k strictly between 1 and ab.
45. For 7, use Z_{2^6}. For n, use $Z_{2^{n-1}}$.
47. Suppose that $|ab| = n$. Then $(ab)^n = e$ implies that $b^n = a^{-n} \in \langle a \rangle$, which is finite. Thus $b^n = e$.
49. 50; 10
51. All divisors of 60
53. The argument given in the proof of the corollary to Theorem 4.4 shows that in an infinite group, the number of elements of finite order n is a multiple of $\phi(n)$ or there is an infinite number of elements of order n.
55. It follows from Example 16 in Chapter 2 and Example 15 in Chapter 0 that the group $H = \langle \cos(360°/n) + i \sin(360°/n) \rangle$ is a cyclic group of order n and every member of this group satisfies $x^n - 1 = 0$. Moreover, since every element of order n satisfies $x^n - 1 = 0$ and there can be at most n such elements, all complex numbers of order n are in H. Thus, by Theorem 4.4, $\mathbf{C}^*$ has exactly $\phi(n)$ elements of order n.
57. Let $x \in Z(G)$ and $|x| = p$ where p is prime. Say $y \in G$ with $|y| = q$ where q is prime. Then $(xy)^{pq} = e$ and therefore $|xy| = 1, p$, or q. If $|xy| = 1$, then $p = q$. If $|xy| = p$, then $e = (xy)^p = y^p$ and q divides p. Thus,

$q = p$. A similar argument applies if $|xy| = q$.

59. An infinite cyclic group does not have an element of prime order. A finite cyclic group can have only one subgroup for each divisor of its order. A subgroup of order p has exactly $p-1$ elements of order p. Another element of order p would give another subgroup of order p.
61. $1 \cdot 4, 3 \cdot 4, 7 \cdot 4, 9 \cdot 4; x^4, (x^4)^3, (x^4)^7, (x^4)^9$
63. 1 of order 1; 33 of order 2; 2 of order 3; 10 of order 11; 20 of order 33
65. 1, 2, 10, 20. In general, if an Abelian group contains cyclic subgroups of order m and n where m and n are relatively prime, then it contains subgroups of order d for each divisor d of mn.
67. Say a and b are distinct elements of order 2. If a and b commute, then ab is a third element of order 2. If a and b do not commute, then aba is a third element of order 2.
69. Use Exercise 38 of Chapter 3 and Theorem 4.3.
71. 1 and 2.
73. Suppose that G has 14 elements of order 3. Let $a \in G$ and $a \neq e$. Let $b \in G$ and $b \notin \langle a \rangle$. Then, by cancellation, $H = \{a^i b^j \mid i, j \text{ are } 0, 1, 2\}$ has exactly nine elements and is closed and therefore is a subgroup of G. Let $c \in G$ and $c \notin H$. Then, by cancellation, the nine expressions of the form $a^i b^j c$ where i, j are $0, 1, 2$ are distinct and have no overlap with the nine elements of H. But that gives 18 elements in G.
75. 12 or 60; 48
77. 3; 2; 6
79. Since $|e| = 1$, H is nonempty. Assume $a, b \in H$ and let $|a| = m$ and $|b| = n$. Then $(ab)^{mn} = (a^m)^n (b^n)^m = e^n e^m = ee = e$. So, $|ab|$ divides mn. Since mn is odd, so is $|ab|$.
81. See Example 7 in Chapter 3.
83. Observe that $n^2 - 1 = -1$ and n are in $U(n^2-1)$ and have order 2. (It follows from the Corollary of Theorem 0.2 in Chapter 0 that n is relatively prime to both $n-1$ and $n+1$) Thus $\{\pm 1, \pm n\}$ is closed and therefore is a subgroup.
85. Observe that among the integers from 1 to p^n, the p^{n-1} integers $p, 2p, 3p, \ldots, p^{n-1}p$ are exactly the ones that are not relatively prime to p.
87. Let $d_1, d_2, \ldots, d_k$ be the distinct positive divisors of n and for each i from 1 to k let $D_i = \{x \in Z_n \mid |x| = d_i\}$. By Theorem 4.3 and the definition of order each element of Z_n is in exactly one D_i. So, $n = |D_1| + |D_2| + \cdots + |D_k|$ and by Theorem 4.4 $D_i = \phi(d_i)$.
89. First note that $x \neq e$. If $x^3 = x^5$, then $x^2 = e$. By Corollary 2 Theorem 4.1 and Theorem 4.3 we then have $|x|$ divides both 2 and 15. Thus $|x| = 1$ and $x = e$. If $x^3 = x^9$, then $x^6 = e$ and therefore $|x|$ divides 6 and 15. This implies that $|x| = 3$. Then $|x^{13}| = |x(x^3)^4| = |x| = 3$. If $x^5 = x^9$, then $x^4 = e$ and $|x|$ divides both 4 and 15, and therefore $x = e$.

Chapter 5

Mistakes are often the best teachers.

JAMES A. FROUDE

1. **a.** $\alpha^{-1} = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 2 & 1 & 3 & 5 & 4 & 6 \end{bmatrix}$
   b. $\beta\alpha = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 6 & 2 & 3 & 4 & 5 \end{bmatrix}$
   c. $\alpha\beta = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 & 6 \\ 6 & 2 & 1 & 5 & 3 & 4 \end{bmatrix}$
3. **a.** (15)(234) **b.** (124)(35)(6) **c.** (1423)
5. **a.** 3 **b.**12 **c.** 6 **d.** 6 **e.** 12 **f.** 2
7. 12
9. $((14562)(2345)(136)(235))^{10} = ((153)(46))^{10} = (135)^{10}(46)^{10} = (153)$.
11. even; odd
13. Say $S = \{s_1, \ldots, s_n\}$ and ϕ is one-to-one from S to S. Then $\phi(s_1), \ldots, \phi(s_n)$ are all distinct and all in S so $\phi(S) = S$. On the other hand, if $\phi(s_i) = \phi(s_j)$ for some $i \neq j$, then $\phi(S)$ has at most $n-1$ members. The

mapping from Z to Z that takes x to $2x$ is one-to-one but not onto.

15. Suppose that α can be written as a product of m 2-cycles and β can be written as a product of n 2-cycles. Then $\alpha\beta$ can be written as a product of $m+n$ 2-cycles. Now observe that $m+n$ is even if and only if m and n are both even or both odd.
17. n is odd.
19. If α is the product of m 2-cycles and β is the product of n 2-cycles then $\alpha^{-5}\beta\alpha^3$ is the product of $8m+n$ and $8m+n$ is odd if and only if n is odd.
21. even.
23. For S_6, the possible orders are $1, 2, 3, 4, 5, 6$; for A_6, $1, 2, 3, 4, 5$; for A_7, $1, 2, 3, 4, 5, 6, 7$.
25. Since $|\beta| = 21$, we have $n = 16$.
27. Suppose H contains at least one odd permutation, σ. Imitate the proof of Theorem 5.7 with σ in place of (12).
29. The identity is even; the set is not closed.
31. $(8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/(2 \cdot 2 \cdot 2 \cdot 2 \cdot 4!)$
33. In A_6 elements of order 2 in disjoint cycle form must be the product of two 2-cycles. So the number of elements of order 2 is $6 \cdot 5 \cdot 4 \cdot 3/(2 \cdot 2 \cdot 2)$.
35. Since $|x^5| = 5$, we know $|x| = 25$. One solution is $(1, 6, 7, 8, 9, 2, 10, 11, 13, 3, 14, 15, 16, 17, 4, 18, 19, 20, 21, 5, 22, 23, 24, 25)$. The number of solutions is 20!.
37. 180; 75
39. In $S_7, \beta = (2457136)$. In S_9, $\beta = (2457136)$ or $\beta = (2457136)(89)$.
41. Since $|(a_1a_2a_3a_4)(a_5a_6)| = 4$ such an x would have order 8. But the elements in S_{10} of order 8 are 8-cycles or the disjoint product of 8-cycle and a 2-cycle. In both cases the square of such an element is the product of two 4-cycles.
43. Let $\alpha, \beta \in \text{stab}(a)$. Then $\alpha\beta(a) = \alpha(\beta(a)) = \alpha(a) = a$. Also, $\alpha(a) = a$ implies $\alpha^{-1}(\alpha(a)) = \alpha^{-1}(a)$ or $a = \alpha^{-1}(a)$.
45. The Finite Subgroup Test shows that H is a subgroup. $|H| = 2(n-2)!$.
47. The first part follows directly from Corollary 3 of Theorem 4.2. For the second part look at (123456).
49. Let $\alpha = (123)$ and $\beta = (145)$.
51. $(123)(12) \neq (12)(123)$ in $S_n (n \geq 3)$.
53. An even number of 2-cycles followed by an even number of 2-cycles gives an even number of 2-cycles in all. So the Finite Subgroup Test is verified.
55. $\langle(1234)\rangle$; $\{(1),(12),(34),(12)(34)\}$.
57. R_0, R_{180}, H, V
59. The permutation corresponding to the rotation of $360/n$ degrees, $(1, 2, \ldots, n)$, is an even permutation so all rotations are even. Labeling consecutive vertices of a regular 5-gon $1, 2, 3, 4, 5$ the even permutation $(14)(23)(5)$ is the reflection that fixes 5 and switches vertices 1 and 4 and 2 and 3. Multiplying the n rotations by this reflection yields all n reflections. There is no reflection in D_7 since their disjoint cycle form is a 1-cycle and three 2-cycles, which is an odd permutation.
61. Since $(1234)^2$ is in B_n it is non-empty. If $\alpha = \alpha_1\alpha_2 \cdots \alpha_i$ and $\beta = \beta_1\beta_2 \cdots \beta_j$ where i and j are even and all the α's and β's are 4-cycles, then $\alpha\beta = \alpha_1\alpha_2 \cdots \alpha_i\beta_1\beta_2 \cdots \beta_j$ is the product of $i+j$ 4-cycles and $i+j$ is even. So, by the Finite Subgroup Test B_n is a subgroup. To show that B_n is a subgroup of A_n note that 4-cycles are odd permutations and the product of any two odd permutations is even. So, for the product of any even number of 4-cycles the product of the first two 4-cycles is even, then the product of the next two 4-cycles is even, and so on. This proves that B_n is a subgroup of A_n.
63. By the previous exercise B_n contains all 3-cycles in A_n. By Exercise 60 every element of A_n is a 3-cycle or a product of 3-cycles. Since 3-cycles are even permutations any product of them is an even permutation.
65. Cycle decomposition shows that any nonidentity element of A_5 is a 5-cycle,

a 3-cycle, or a product of a pair of disjoint 2-cycles. Then, observe that there are $(5 \cdot 4 \cdot 3 \cdot 2 \cdot 1)/5 = 24$ group elements of the form $(abcde)$, $(5 \cdot 4 \cdot 3)/3 = 20$ group elements of the form (abc), and $(5 \cdot 4 \cdot 3 \cdot 2)/(2 \cdot 2 \cdot 2) = 15$ group elements of the form $(ab)(cd)$.

67. If α has odd order k and α is an odd permutation, then $\epsilon = \alpha^k$ would be odd.
69. The product is the n-cycle $(1, n, 2, n-1, 3, n-2, \ldots, (n-1)/2, (n+3)/2, (n+1)/2)$. Labeling the vertices of a regular n-gon in consecutive order 1 through n counterclockwise we can think of $(12 \cdots n)$ as a $360/n$ degree rotation and $(2, n)(3, n-1) \cdots ((n+1)/2), (n+3)/2)$ as reflection through the vertex labeled 1 to the midpoint of the opposite edge.
71. Verifying that $a * \sigma(b) \neq b * \sigma(a)$ is done by examining all cases. To prove the general case, observe that $\sigma^i(a) * \sigma^{i+1}(b) \neq \sigma^i(b) * \sigma^{i+1}(a)$ can be written in the form $\sigma^i(a) * \sigma(\sigma^i(b)) \neq \sigma^i(b) * \sigma(\sigma^1(a))$, which is the case already done. If a transposition were not detected, then $\sigma(a_1) * \cdots * \sigma^i(a_i) * \sigma^{i+1}(a_{i+1}) * \cdots * \sigma^n(a_n) = \sigma(a_1) * \cdots * \sigma^i(a_{i+1}) * \sigma^{i+1}(a_i) * \cdots * \sigma^n(a_n)$, which implies $\sigma^i(a_i) * \sigma^{i+1}(a_{i+1}) = \sigma^i(a_{i+1}) * \sigma^{i+1}(a_i)$.
73. By Theorem 5.4 it is enough to prove that every 2-cycle can be expressed as a product of elements of the form $(1k)$. To this end, observe that if $a \neq 1, b \neq 1$, then $(ab) = (1a)(1b)(1a)$.
75. By case-by-case analysis, H is a subgroup for $n = 1, 2, 3$, and 4. For $n \geq 5$, observe that $(12)(34)$ and $(12)(35)$ belong to H but their product does not.
77. The product of an element of $Z(A_4)$ of order 2 and an element of A_4 of order 3 would have order 6. But A_4 has no element of order 6.
79. TAAKTPKSTOOPEDN

Chapter 6

Think and you won't sink.

B. C. FORBES, *Epigrams*

1. Try $n \to 2n$.
3. $\phi(xy) = \sqrt{xy} = \sqrt{x}\sqrt{y} = \phi(x)\phi(y)$.
5. The mapping $\phi(x) = (3/2)x$ is an isomorphism from G onto H. Multiplication is not preserved since $\phi(4) = 6$ but $\phi(2)\phi(2) = 3 \cdot 3 = 9$. When $G = \langle m \rangle$ and $H = \langle n \rangle$ the mapping $\phi(x) = (n/m)x$ is an isomorphism from G onto H.
7. D_{12} has an element of order 12 and S_4 does not; D_{12} has an element of order 6 and S_4 does not; D_{12} has exactly 2 elements of order 4 and S_4 has more than 2; $|Z(D_6)| = 2$ whereas $|Z(S_4)| = 1$.
9. Since $T_e(x) = ex = x$ for all x, T_e is the identity. For the second part, observe that $T_g \circ (T_g)^{-1} = T_e = T_{gg^{-1}} = T_g \circ T_{g^{-1}}$ and cancel.
11. $3\overline{a} - 2\overline{b}$.
13. For any x in the group, we have $(\phi_g \phi_h)(x) = \phi_g(\phi_h(x)) = \phi_g(hxh^{-1}) = ghxh^{-1}g^{-1} = (gh)x(gh)^{-1} = \phi_{gh}(x)$.
15. $\phi_{R_{90}}$ and ϕ_{R_0} disagree on H; $\phi_{R_{90}}$ and ϕ_H disagree on R_{90}; $\phi_{R_{90}}$ and ϕ_D disagree on R_{90}. The remaining cases are similar.
17. Let $\alpha \in \mathrm{Aut}(G)$. We show that α^{-1} is operation-preserving: $\alpha^{-1}(xy) = \alpha^{-1}(x)\,\alpha^{-1}(y)$ if and only if $\alpha(\alpha^{-1}(xy)) = \alpha(\alpha^{-1}(x)\alpha^{-1}(y))$, that is, if and only if $xy = \alpha(\alpha^{-1}(x))\alpha(\alpha^{-1}(y)) = xy$. So α^{-1} is operation-preserving. That $\mathrm{Inn}(G)$ is a group follows from the equation $\phi_g \phi_h = \phi_{gh}$.
19. Note that for $n > 1$, $(\phi_a)^n = (\phi_a)^{n-1}\phi_a$ so an induction argument gives $(\phi_a)^n = (\phi_a)^{n-1})\phi_a = \phi_{a^{n-1}}\phi_a$. Thus $(\phi_{a^{n-1}}\phi_a)(x) = \phi_{a^{n-1}}(\phi_a(x)) = \phi_{a^{n-1}}(\phi_a)(x)) = \phi_{a^{n-1}}(axa^{-1}) = a^{n-1}(axa^{-1})(a^{n-1})^{-1} =$

$a^{n-1}(axa^{-1})(a^{-n+1}) = a^n x a^{-n} = \phi_{a^n}(x)$. To handle the case where n is negative we note that $\phi_e = \phi_{a^n a^{-n}} = \phi_{a^n}\phi_{a^{-n}} = \phi_{a^n}(\phi_a)^{-n}$ (because $-n$ is positive). Solving for ϕ_{a^n} we obtain $\phi_{a^n} = (\phi_a)^n$.

21. Since $b = \phi(a) = a\phi(1)$, it follows that $\phi(1) = a^{-1}b$ and therefore $\phi(x) = a^{-1}bx$. [Here a^{-1} is the multiplicative inverse of a mod n, which exists because $a \in U(n)$.]
23. Note that both H and K are isomorphic to the group of all permutations on four symbols, which is isomorphic to S_4. The same is true when 5 is replaced by n, since both H and K are isomorphic to S_{n-1}.
25. Recall that, when n is even, $Z(D_n) = \{R_0, R_{180}\}$. Since R_{180} and $\phi(R_{180})$ are not the identity and belong to $Z(D_n)$, they must be equal.
27. Z_{60}; D_{60}
29. See Example 16 of Chapter 2.
31. That α is one-to-one follows from the fact that r^{-1} exists modulo n. The operation-preserving condition is Exercise 11 in Chapter 0.
33. By Part 1 of Theorem 6.2, we have $\phi(a^n) = \phi(a)^n = \gamma(a)^n = \gamma(a^n)$ thus ϕ and γ agree on all elements of $\langle a\rangle$.
35. First observe that because $2^5 = 10 = -1$ we have $|2| = 10$. So, by parts 4 and 2 of Theorem 6.1, the mapping that takes $\phi(x) = 2^x$ is an isomorphism.
37. $T_g(x) = T_g(y)$ if and only if $gx = gy$ or $x = y$. This shows that T_g is a one-to-one function. Let $y \in G$. Then $T_g(g^{-1}y) = y$, so that T_g is onto.
39. Apply the appropriate definitions.
41. See Exercise 43 in Chapter 4.
43. Try $a + bi \to \begin{bmatrix} a & -b \\ b & a \end{bmatrix}$.
45. Yes, by Cayley's Theorem.
47. Observe that $\phi_g(y) = gyg^{-1}$ and $\phi_{zg}(y) = zgy(zg)^{-1} = zgyg^{-1}z^{-1} = gyg^{-1}$ since $z \in Z(G)$. So, $\phi_g = \phi_{zg}$.
49. $\phi_g = \phi_h$ implies $gxg^{-1} = hxh^{-1}$ for all x. This implies $h^{-1}gx(h^{-1}g)^{-1} = x$, and therefore $h^{-1}g \in Z(G)$.
51. By Exercise 49 $\phi_\alpha = \phi_\beta$ implies $\beta^{-1}\ \alpha$ is in $Z(S_n)$ and by Exercise 70 in Chapter 5, $Z(S_n) = \{\epsilon\}$.
53. Since both ϕ and γ take e to itself, H is not empty. Assume a and b belong to H. Then $\phi(ab^{-1}) = \phi(a)\phi(b^{-1}) = \phi(a)\phi(b)^{-1} = \gamma(a)\gamma(b)^{-1} = \gamma(a)\gamma(b^{-1}) = \gamma(ab^{-1})$. Thus, ab^{-1} is in H.
55. Two of many are $(12)H(12) = \{(1), (235), (253), (16)(35), (16)(23), (16)(25)\}$; $(123)H(123)^{-1} = \{(1), (152), (125), (15)(36), (25)(36), (12)(36)\}$.
57. Since -1 is the unique element of $\mathbf{C}^*$ of order 2, $\phi(-1) = -1$. Since i and $-i$ are the only elements of $\mathbf{C}^*$ of order 4, $\phi(i) = i$ or $-i$.
59. Z_{120}, D_{60}, S_5. Z_{120} is Abelian, the other two are not. D_{60} has an element of order 60 and S_5 does not.
61. For the first part use $D = R_{90}V$ and $H = R_{90}D$. For the second part use $H = R_{180}V$.
63. $T_{R_{90}} = (R_0R_{90}R_{180}R_{270})(HD'\ VD)$.
65. The first statement follows from the fact that every element of D_n has the form $(R_{360/n})^i$ or $(R_{360/n})^iF$. Because α must map an element of order n to an element of order n $R_{360/n}$ must map to $(R_{360/n})^i$ where $i \in U(n)$. Moreover, F must map to a reflection (see Exercise 20). Thus we have at most $n|U(n)|$ choices.
67. In both cases H is isomorphic to the set of all even permutations of the set of four integers so it is isomorphic to A_4.
69. Consider the mapping $\phi(x) = x^2$ and note that 2 is not in the image.
71. Use the fact that if $a > 0$, then $a = \sqrt{a}\sqrt{a}$. For the second part, use the first part together with the fact that the inverse of an automorphism is an automorphism.
73. Say ϕ is an isomorphism from Q to R^+ and ϕ takes 1 to a. It follows that the integer r maps to a^r. Then $a = \phi(1) = \phi(s\frac{1}{s}) = \phi(\frac{1}{s} + \cdots + \frac{1}{s}) = \phi(\frac{1}{s})^s$

and therefore $a^{\frac{1}{s}} = \phi(\frac{1}{s})$. Thus the rational r/s maps to $a^{r/s}$. But $a^{r/s} \neq a^{\pi}$ for any rational number r/s.

75. Send each even permutation in S_n to itself. Send each odd permutation α in S_n to $(n+1, n+2)\alpha$. This does not contradict Theorem 5.5 because the subgroup is merely isomorphic to A_{n+2}, not the same as A_{n+2}. In particular, this example shows that an isomorphism from one permutation group to another permutation group need not preserve oddness.

Chapter 7

Use missteps as stepping stones to deeper understanding and greater achievement.

SUSAN TAYLOR

1. For the general case the distinct left cosets are $H, 1+H, 2+H$, an eclipses $n-1+H$.
3. **a.** yes **b.** yes **c.** no
5. For $\langle a^5\rangle$ there are 5: $\langle a^5\rangle, a\langle a^5\rangle, a^2\langle a^5\rangle, a^3\langle a^5\rangle, a^4\langle a^5\rangle$. By Collorary 2 of Theorem 4.2, for $\langle a^4\rangle = \langle a^2\rangle$ there are two left cosets: $\langle a^2\rangle, a\langle a^2\rangle$.
7. Suppose that $H \neq \langle 3\rangle$. Let $a \in H$ but a not in $\langle 3\rangle$. Then $a + \langle 3\rangle = 1 + \langle 3\rangle$ or $a + \langle 3\rangle = 2 + \langle 3\rangle$. If $a + \langle 3\rangle = 1 + \langle 3\rangle$ then $1 \in H$ and $H = Z$. If $a + \langle 3\rangle = 2 + \langle 3\rangle$ then both 2 and 3 belong to H and therefore $3 - 2 = 1$ belongs to H.
9. Let ga belong to $g(H \cap K)$, where a is in $H \cap K$. Then by definition ga is in $gH \cap gK$. Now let $x \in gH \cap gK$. Then $x = gh$ for some $h \in H$, and $x = gk$ for some $k \in K$. Cancellation then gives $h = k$. Thus, $x \in g(H \cap K)$.
11. Suppose that $h \in H$ and $h < 0$. Then $h\mathbf{R}^+ \subseteq hH = H$. But $h\mathbf{R}^+$ is the set of all negative real numbers. Thus, $H = \mathbf{R}^*$.
13. 1, 2, 3, 4, 5, 6, 10, 12, 15, 20, 30, 60
15. Use Lagrange's Theorem (Theorem 7.1) and Corollary 3.
17. By Exercise 16, we have $5^6 \bmod 7 = 1$. So, using mod 7, we have $5^{15} = 5^6 \cdot 5^6 \cdot 5^2 \cdot 5 = 1 \cdot 1 \cdot 4 \cdot 5 = 6; 7^{13} \bmod 11 = 2$.
19. Use Corollary 4 of Lagrange's Theorem (Theorem 7.1) together with Theorem 0.2.
21. First observe that for all $n \geq 3$ the subgroup of rotations of D_n is isomorphic to Z_n. If n is even let F be any reflection in D_n. Then the set $\{R_0, R_{180}, F, FR_{180}\}$ is closed and therefore a subgroup of order 4. Now suppose that D_n has a subgroup K of order 4. By Lagrange, $|D_n| = 2n = 4k$ and therefore $n = 2k$.
23. Since G has odd order, no element can have order 2. Thus, for each $x \neq e$, we know that $x \neq x^{-1}$. So, we can write the product of all the elements in the form $ea_1a_1^{-1}a_2a_2^{-1} \cdots a_na_n^{-1} = e$.
25. Let $e \neq g \in G$. Then $|g| = 5$ or 25. If $|g| = 25$ for some g, then G is cyclic. If there is no such g, then $|g| = 1$ or 5 for all g.
27. 1, 3, 11, 33. If some x has order 33, then $|x^{11}| = 3$. Otherwise, use the Corollary to Theorem 4.4.
29. If the group is cyclic Theorem 4.3 says that it has exactly one subgroup of order 5. So, assume the group is not cyclic. Not all of the 54 nonidentity elements can have order 5 because the number of elements of order 5 is a multiple of $\phi(5) = 4$. So the group has an element of order 11. Also since $\phi(11) = 10$ the number of elements of order 11 is a multiple of 10. If there were more than 10 the group would have distinct subgroups H and K of order 11. But then $|HK| = |H||K|/|H \cap K| = 121$. So, excluding the subgroup of order 11, there are 44 elements remaining and each has order 5. That gives us exactly 11 subgroups of order 5.
31. By Lagrange's Theorem every element in G has an order that is a divisor of n. So, we can partition the n elements of G according to their orders. For

each divisor d of n let m_d be the number elements in G of order d. By our assumption m_d is $\phi(d)$ where ϕ is the Euler phi function. (If there were more than $\phi(d)$ elements of order d in G then G would have at least 2 subgroups of order d.) So $n = \Sigma m_d$ where d ranges over all divisors d of n. We also have from Exercise 87 of Chapter 4 that $n = \Sigma\phi(d)$ where d ranges over all divisors d of n. This proves that each $n_d = \phi(d)$. In particular, $m_n \neq 0$.

33. Suppose that H and K are distinct subgroups of order m. Then $|HK| = |H||K| = \frac{m \cdot m}{|H \cap K|} \leq 2m$ and therefore $\frac{m}{2} \leq |H \cap K|$. Since m is odd and H are K are distinct we know that $\frac{m}{2} < |H \cap K| < m$ and that $|H \cap K|$ divides m. This is impossible.
35. Observe that $|G : H| = |G|/|H|$, $|G : K| = |G|/|K|$, and $|K : H| = |K|/|H|$.
37. Since the reflections in a dihedral group have order 2, the generators of the subgroups of orders 12 and 20 must be rotations. The smallest rotation subgroup of a dihedral group that contains rotations of orders 12 and 20 must have order divisible by 12 and 20 and therefore must be a multiple of 60. So, D_{60} is the smallest such dihedral group.
39. Let a have order 3 and b be an element of order 3 not in $\langle a\rangle$. Then $\langle a\rangle\langle b\rangle$ is a subgroup of G of order 9. Now use Lagrange's Theorem.
41. By Corollary 5 of Theorem 7.1, the statement is true for $n = 1$. For the sake of induction assume that $a^{p^k} = a$. Then $a^{p^{k+1}} = a^{p^k} a^p = a^p = a$.
43. Let $a \in G$ and $|a| = 5$. Then the set $\langle a\rangle H$ has exactly $5 \cdot |H|/\langle a\rangle \cap H|$ elements and $|\langle a\rangle \cap H|$ divides $|\langle a\rangle| = 5$. It follows that $|\langle a\rangle \cap H| = 5$ and therefore $\langle a\rangle \cap H = \langle a\rangle$.
45. First observe that by Corollary 2 of Lagrange's Theorem every positive integer k with the property that $x^k = e$ for all x in G is a common multiple of orders of all the elements in G. So, d is the least common multiple of the orders of the elements of G. Since $|G|$ is a common multiple of the orders of all the elements of G it follows directly from the division algorithm (Theorem 0.1) that $|G|$ is divisible by d.
47. Let G be a finite Abelian group. The case G has 0 or 1 element of order 2 corresponds to the cases $n = 0$ and $n = 1$. Let $H = \{x \in G \mid x^2 = e\}$. Then H is a subgroup of G that consists of the identity and all elements of order 2. It suffices to prove $|H| = 2^n$. If G has at least two elements of order 2, say a_1 and b. Then $H_1 = \{e, a_1, b, a_1 b\}$ is a subgroup of order 2^2. If $H_1 = H$, we are done. If not, let $a_2 \in H$ but not in H_1. Then $H_2 = H_1 \cup \langle a_2\rangle H_1$ is a subgroup of H of order $2|H_1| = 2^3$. If $H_2 = H$ we are done. If not, let $a_3 \in H$ but not in H_2 and let $H_3 = H_2 \cup \langle a_3\rangle H_2$. We can continue this argument until we reach H.
49. Consider the mapping from G to G defined by $\phi(x) = x^2$ and let $|G| = 2k + 1$. Use the observation that $x = xe = xx^{2k+1} = x^{2k+2} = (x^2)^{k+1}$ to prove that ϕ is one-to-one and Exercise 13 of Chapter 5 to show that ϕ is onto.
51. If H and K are distinct subgroups of order p^m then $np^m = |G| \geq |HK| = |H||K|/|H \cap K| \geq p^m p^m / p^{m-1} = pp^m$, which is obviously false.
53. Let G be the group and H the unique subgroup of order q. We must show that G has an element of order pq. Let a belong to G but not in H. By Lagrange, $|a| = p$ or pq. If $|a| = pq$ we are done. So, we may assume that a has order p and we let $K = \langle a\rangle$. Then $H \cup K$ accounts for $q + p - 1$ elements (the identity appears twice). Pick $b \in G$ but b not one of the elements in $H \cup K$. Then $L = \langle b\rangle$ is a subgroup of G of order p different than K. Then $K \cap L = \{e\}$ because $|K \cap L|$ must

divide p and is not p. By Theorem 7.2 $|KL| = |K||L|/|K \cap L| = (p \cdot p)/1 = p^2$. But a group of order pq with $q < p$ cannot have p^2 elements. This shows that b cannot have order p. So $|b| = pq$.

55. Use Theorem 7.2.
57. 50
59. Let K be the set of all even permutations in H. Since K is closed it is a subgroup of H. If $K = H$, we are done. If not, let α be an element in H that is odd. Then αK must be the set of all odd permutations in H, for if β is any odd permutation in H, we have $\alpha^{-1}\beta \in H$, which means $\beta \in H$. Thus $|H| = |K \cup \alpha K| = 2|K|$.
61. Suppose that H is a subgroup of A_5 of order 30. We claim that H contains all 20 elements of A_5 that have order 3. To verify this, assume that there is some α in A_5 of order 3 that is not in H. Then $A_5 = H \cup \alpha H$. It follows that $\alpha^2 H = H$ or $\alpha^2 = \alpha H$. Since the latter implies that $\alpha \in H$, we have that $\alpha^2 H = H$, which implies that $\alpha^2 \in H$. But then $\langle \alpha \rangle = \langle \alpha^2 \rangle \subseteq H$, which is a contradiction of our assumption that α is not in H. The same argument, shows that H must contain all 24 elements of order 5. Since $|H| = 30$, we have a contradiction.
63. Say H is a subgroup of order 30. By Exercise 61 H is not a subgroup of A_5 and by Exercise 59 $H \cap A_5$ is a subgroup of A_5 of order 15. But this contradicts Exercise 62.
65. If H is a subgroup of S_5 of order 60 other than A_5, then it follows from Theorem 7.2 that $|A_5 \cap H| = 30$, which contradicts Exercise 61.
67. Certainly, $a \in \mathrm{orb}_G(a)$. Now suppose that $c \in \mathrm{orb}_G(a) \cap \mathrm{orb}_G(b)$. Then $c = \alpha(a)$ and $c = \beta(b)$ for some α and β, and therefore $(\beta^{-1}\alpha)(a) = b$. So, if $x \in \mathrm{orb}_G(b)$, then $x = \gamma(b) = (\gamma\beta^{-1}\alpha)(a)$ for some γ. This proves that $\mathrm{orb}_G(b) \subseteq \mathrm{orb}_G(a)$. By symmetry, $\mathrm{orb}_G(a) \subseteq \mathrm{orb}_G(b)$.
69. **a.** $\mathrm{stab}_G(1) = \{(1), (24)(56)\}; \mathrm{orb}_G(1) = \{1, 2, 3, 4\}$
b. $\mathrm{stab}_G(3) = \{(1), (24)(56)\}; \mathrm{orb}_G(3) = \{3, 4, 1, 2\}$
c. $\mathrm{stab}_G(5) = \{(1), (12)(34), (13)(24), (14)(23)\}; \mathrm{orb}_G(5) = \{5, 6\}$
71. Suppose that $B \in G$ and det $(B) = 2$. Then det $(A^{-1}B) = 1$, so that $A^{-1}B \in H$ and therefore $B \in AH$. Conversely, for any $Ah \in AH$ we have det $(Ah) = (\det(A))(\det(B)) = 2 \cdot 1 = 2$.
73. It is the set of all permutations that carry face 2 to face 1.
75. $aH = bH$ if and only if det $(a) = \pm$det (b).
77. Closure of the set follows from using $\alpha\beta^2 = \beta^2\beta^3$.

Chapter 8

Practice isn't the thing you do when you're good. It's the thing you do that makes you good.

MALCOLM GLADWELL

1. Closure and associativity in the product follow from the closure and associativity in each component. The identity in the product is the n-tuple with the identity in each component. The inverse of $(g_1, g_2, \ldots, g_n)$ is $(g_1^{-1}, g_2^{-1}, \ldots, g_n^{-1})$.
3. Use $g \to (g, e_H)$ and $h \to (e_G, h)$.
5. To show that $Z \oplus Z$ is not cyclic, note that $(a, b+1) \notin \langle (a, b) \rangle$.
7. Use $(g_1, g_2) \to (g_2, g_1)$. In general, $G_1 \oplus G_2 \cdots \oplus G_n$ is isomorphic to the external direct product of any rearrangement of $G_1, G_2, \ldots, G_n$.
9. In $Z_6 \oplus Z_2$ take $\langle (1, 0) \rangle$ and $\langle (2, 1) \rangle$. In $Z_{pm} \oplus Z_p$ take $\langle (1, 0) \rangle$ and $\langle (p, 1) \rangle$.
11. There are 12 elements of order 4. Observe by Theorem 4.4 that as long as d divides n, the number of elements of order d in a cyclic group depends only on d. So, in both $Z_{8000000}$ and Z_4 there are $\phi(4) = 2$ elements of order 4 and $\phi(2) = 1$ element of order 2. Similarly for $Z_m \oplus Z_n$.
13. Z_{n^2} and $Z_n \oplus Z_n$.
15. Try $a + bi \to (a, b)$.

17. Use Exercise 3 and Theorem 4.3.
19. $\langle m/r\rangle \oplus \langle n/s\rangle$
21. Since $\langle (g,h)\rangle \subseteq \langle g\rangle \oplus \langle h\rangle$, a necessary and sufficient condition for equality is that lcm $(|g|,|h|) = |(g,h)| = |\langle g\rangle \oplus \langle h\rangle)| = |g||h|$. This is equivalent to gcd $(|g|,|h|) = 1$.
23. $(m+2)(n+1)-1$ of order 2; $2(n+1)$ of order 4.
25. Map $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ to (a,b,c,d). Let $\mathbf{R}^k$ denote $\mathbf{R} \oplus \mathbf{R} \oplus \cdots \oplus \mathbf{R}$ (k factors). Then the group of $m \times n$ matrices under addition is isomorphic to $\mathbf{R}^{mn}$.
27. $(g,g)(h,h)^{-1} = (gh^{-1}, gh^{-1})$. When $G = \mathbf{R}$, $G \oplus G$ is the plane and H is the line $y = x$.
29. $\langle (3,0)\rangle, \langle (3,1)\rangle, \langle (3,2)\rangle, \langle (0,1)\rangle$
31. $\mathrm{lcm}(6,10,15) = 30; \mathrm{lcm}(n_1, n_2, \ldots, n_k)$.
33. Observe that $Z_4 \oplus Z_3 \oplus Z_2 \approx Z_4 \oplus Z_6 \approx \langle 25\rangle \oplus \langle 10\rangle$. Also $Z_4 \oplus Z_3 \oplus Z_2 \approx Z_2 \oplus Z_{12} \approx \langle 50\rangle \oplus \langle 5\rangle$.
35. Let F be a reflection in D_3. $\{R_0, F\} \oplus \{R_0, R_{180}, H, V\}$.
37. Compare the number of elements of order 2 in each group.
39. The mapping $\phi(3^m 6^n) = (m,n)$ is an isomorphism. The mapping $\phi(3^m 9^n) = (m,n)$ is not well-defined, since $\phi(3^2 9^0) \neq \phi(3^0 9^1)$.
41. In both cases they are the same.
43. $\mathbf{C}^*$ has only one element of order 2 whereas $Z_m \oplus Z_n$ has exactly one element of order 2 if and only if it is cyclic, which is true if and only if $\gcd(m,n) = 1$.
45. 12
47. $\mathrm{Aut}(U(25)) \approx \mathrm{Aut}(Z_{20}) \approx U(20) \approx U(4) \oplus U(5) \approx Z_2 \oplus Z_4$
49. $2^k - 1; 2^t - 1$, where t is the number of the integers $n_1, n_2, \ldots, n_k$ that are even.
51. For part **a**, there are $\phi(18) = 6$. Two are $\phi(x) = (x,x)$ and $\phi(x) = (x, 2x)$. For part **b**, there are none because $Z_2 \oplus Z_3 \oplus Z_3$ is not cyclic.
53. Since (2, 0) has order 2, it must map to an element in Z_{12} of order 2. The only such element in Z_{12} is 6. The isomorphism defined by (1, 1) $x \to 5x$ with $x = 6$ takes (2, 0) to 6. Since (1, 0) has order 4, it must map to an element in Z_{12} of order 4. The only such elements in Z_{12} are 3 and 9. The first case occurs for the isomorphism defined by (1, 1) $x \to 7x$ with $x = 9$ [recall that (1, 1) is a generator of $Z_4 \oplus Z_3$]; the second case occurs for the isomorphism defined by (1, 1) $x \to 5x$ with $x = 9$.
55. Since $a \in Z_m$ and $b \in Z_n$, we know that $|a|$ divides m and $|b|$ divides n. So, $|(a,b)| = \mathrm{lcm}(|a|,|b|)$ divides $\mathrm{lcm}(m,n)$.
57. Z, Z_3, Z_4, Z_6
59. Observe that every nonidentity element of $Z_p \oplus Z_p$ has order p and each subgroup of order p contains $p-1$ of them. So, there are exactly $(p^2-1)/(p-1) = p+1$ subgroups of order p.
61. Look at $Z \oplus Z_2$.
63. $U(165) \approx U(11) \oplus U(15) \approx U(5) \oplus U(33) \approx U(3) \oplus U(55) \approx U(3) \oplus U(5) \oplus U(11)$
65. $U(2n) \approx U(2) \oplus U(n) = \{1\} \oplus U(n) \approx U(n)$.
67. Mimic the analysis for elements of order 12 in $U(105)$ in this chapter. The number is 14.
69. $A_4 \oplus Z_4$ has exactly 7 elements of order 2 wheres as $D_{12} \oplus Z_2$ has more.
71. $Z \oplus D_4$.
73. Observe that $U(55)$ and $U(75)$ are both isomorphic to $Z_4 \oplus Z_{10}$ and $U(144)$ and $U(140)$ are both isomorphic to $Z_2 \oplus Z_4 \oplus Z_6$.
75. From Theorem 6.5 we know $\mathrm{Aut}(Z_n) \approx U(n)$. So, $n = 8$ and 12 are the two smallest.
77. Since $U(pq) \approx U(p) \oplus U(q) \approx Z_{p-1} \oplus Z_{q-1}$ if follows that $k = \mathrm{lcm}\,(p-1, q-1)$.
79. $Z_{p^{n-1}}$
81. $U_8(40) \approx U(5) \approx Z_4$.
83. $U_5(140) \approx U(28)$; $U_4(140) \approx U(35) \approx Z_4 \oplus Z_6$.
85. If $x \in U_{st}(n)$ then $x \in U(n)$ and $x - 1 = stm$ for some m. So,

$x-1=s(tm)$ and $x-1=t(sm)$. If $x \in U_s(n) \cap U_t(n)$ then $x \in U(n)$ and both s and t divide $x-1$. So, by Exercise 6 in Chapter 0, st divides $x-1$.

87. $Z_2 \oplus Z_2$.
89. Since $5 \cdot 29 = 1$ mod 36, we have that $s = 29$. So, we need to compute 34^{29} mod 2701. The result is 1415, which converts to NO.
91. Because the block 2505 exceeds the modulus 2263, sending 2505^e mod 2263 is the same as sending 242^e mod 2263 which decodes as 242 instead of 2505.

Chapter 9

There's a mighty big difference between good, sound reasons and reasons that sound good.

BURTON HILLIS

1. No.
3. $HR_{90} = R_{270}H; DR_{270} = R_{90}D; R_{90}V = VR_{270}$
5. Say $i<j$ and let $h \in H_i \cap H_j$. Then $h \in H_1H_2 \cdots H_i \cdots H_{j-1} \cap H_j = \{e\}$.
7. Recall that if A and B are matrices, then det $(ABA^{-1}) = (\det A)(\det B)(\det A)^{-1}$.
9. Let $x \in G$. If $x \in H$, then $xH = H = Hx$. If $x \notin H$, then xH is the set of elements in G, not in H. But Hx is also the set of elements in G, not in H.
11. Let $G = \langle a \rangle$. Then $G/H = \langle aH \rangle$
13. in H.
15. $|9H| = 2$; $|13H| = 4$.
17. $1 + \langle 3.5 \rangle$.
19. Observe that in a group G if $|a| = 2$ and $\{e, a\}$ is a normal subgroup then $xax^{-1} = a$ for all x in G. Thus $a \in Z(G)$. So, the only normal subgroup of order 2 in D_n is $\{R_0, R_{180}\}$ when n is even.
21. By Theorem 9.5, the group has an element a of order 3 and an element b of order 11. Then $|ab| = 33$. For the general case use induction.
23. 4; no
25. Z_8.
27. Yes; no
29. The subgroups would have orders 2 or 4 and therefore are Abelian. But the internal direct products of Abelian groups are Abelian.
31. Certainly, every nonzero real number is of the form $\pm r$, where r is a positive real number. Real numbers commute, and $\mathbf{R}^+ \cap \{1, -1\} = \{1\}$.
33. No. If $G = H \times K$, then $|g| = \text{lcm}(|h|, |k|)$, provided that $|h|$ and $|k|$ are finite. If $|h|$ or $|k|$ is infinite, so is $|g|$.
35. For the first question, note that $\langle 3 \rangle \cap \langle 6 \rangle = \{1\}$ and $\langle 3 \rangle \langle 6 \rangle \cap \langle 10 \rangle = \{1\}$. For the second question, observe that $12 = 3^{-1}6^2$. So $\langle 3 \rangle \langle 6 \rangle \cap \langle 12 \rangle \neq \{1\}$.
37. Say $|g| = n$. Then $(gH)^n = g^nH = eH = H$. Now use Corollary 2 to Theorem 4.1.
39. Let x be in G and h be in H. Then $xhx^{-1}H = (xh)x^{-1}H = (xh)Hx^{-1}H = xhHx^{-1}H = xHx^{-1}H = xx^{-1}H = H$, so xhx^{-1} belongs to H.
41. Suppose that H is a proper subgroup of Q of index n. Then Q/H is a finite group of order n. By Corollary 4 of Theorem 7.1, we know that for every x in Q we have nx is in H. Now observe that the function $f(x) = nx$ maps Q onto Q. So, $Q \subseteq H$.
43. Take $G = Z_6$, $H = \{0, 3\}, a = 1$, and $b = 9$.
45. Normality follows directly from Theorem 4.3 and Example 7.
47. In general, if p is a prime and $|H| = p^2m$ and $|K| = p^2n$ where $\gcd(m, n) = 1$, Then $H \cap K| = 1, p$, or p^2. So, by Corollary 3 of Theorem 7.1 and Theorem 9.7, $H \cap K$ is Abelian.
49. Use the G/Z Theorem.
51. Suppose that K is a normal subgroup of G and let $gH \in G/H$ and $kH \in K/H$. Then $gHkH(gH)^{-1} = gHkHg^{-1}H = gkg^{-1}H \in K/H$. Now suppose that K/H is a normal subgroup of G/H and let $g \in G$ and $k \in K$. Then

$gkg^{-1}H = gHkHg^{-1}H = gHkH(gH)^{-1} \in K/H$ so $gkg^{-1} \in K$.

53. Say H has index n. Then $(\mathbf{R}^*)^n = \{x^n \mid x \in \mathbf{R}^*\} \subseteq H$. If n is odd, then $(\mathbf{R}^*)^n = \mathbf{R}^*$; if n is even, then $(\mathbf{R}^*)^n = \mathbf{R}^+$. So, $H = \mathbf{R}^*$ or $H = \mathbf{R}^+$.
55. Use Exercise 9 and observe that $VK \neq KV$.
57. G has elements of orders 1, 2, 3, and 6.
59. Let $N = \langle a \rangle$, $H = \langle a^k \rangle$, and x be in G. Then, $x(a^k)^m x^{-1} = (xa^m x^{-1})^k = (a^r)^k = (a^k)^r \in H$.
61. $\gcd(|x|, |G/H|) = 1$ implies $\gcd(|xH|, |G/H|) = 1$. But $|xH|$ divides $|G/H|$. Thus $|xH| = 1$ and therefore $xH = H$.
63. Use Example 5 and Lagrange's Theorem.
65. Observe that for every positive integer $n, (1+i)^n$ is not a real number. So, $(1+i)$ R* has infinite order.
67. Use Theorems 9.4 and 9.3.
69. Because $|g| = 16$ implies that $|gH|$ divides 16 it suffices to show that $(gH)^4$ is not H. Suppose that $(gH)^4 = g^4H = H$. Then g^4 is in H. But then e, g^4, and g^8 are in H, which is a contradiction. In the general case, say $|gH| = k$. Then $(gH)^k = g^kH = H$. So, g^k is in H and therefore $|g^k| = 1$ or 2. It follows that $k = 2n$ or n.
71. Use Theorem 9.3 and Theorem 7.3.
73. If A_5 had a normal subgroup of order 2 then, by Exercise 72, it would have an element of the form $(ab)(cd)$ that commutes with every element of A_5. Try (abc).
75. Note that $U(72) \approx U(8) \oplus U(9) \approx U_9(72) \oplus U_8(72) \approx \oplus Z_2 \oplus Z_2 \oplus Z_6$. $|U_9(72)| = 4$, $|U_8(72)| = 6$, and $|\langle 19 \rangle U_8(72)| = 12$.
77. First recall that A_5 contains $5 \cdot 4 \cdot 3/3 = 20$ elements of order 3. Let α be any of these 20. Then, since $|A_5/H| = 5$, we have that $H = (\alpha^2 H)^5 = \alpha^{10}H = \alpha H$. But this means that all 20 elements of order 5 in A_5 are in H, which has order 12.
79. Because $51H = H$ we have $153H = (3 \cdot 51)H = 3(51H) = 3H$.
81. Let $|gH| = d$ and $|g| = m$. We know by Exercise 37 that $|gH|$ divides $|g|$ and, because $g^d \in H$, we also know that $|g^d| = m/d$ divides $|H|$. This means that $m/d = 1$.
83. By definition, every element of G can be written in the form $a_{j_1}a_{j_2}\ldots a_{j_k}$ where $a_{j_1}, a_{j_2}, \ldots, a_{j_k} \in \langle a_1, a_2, \ldots, a_n \rangle$. Then $gH = a_{j_1}Ha_{j_2}H \ldots a_{j_k}H$.
85. Since G is Abelian the subgroups $H_1, H_2, \ldots H_k$ are normal in G. By assumption, $G = H_1H_2 \cdots H_k$, So, all that remains to prove is that for all $i = 2, 3, \ldots, k-1$ we have $H_1H_2 \cdots H_i \cap H_{i+1} = \{e\}$. But if $x \in H_1H_2 \cdots H_i \cap H_{i+1}$ and $x \neq e$ then x can be written in the two distinct forms $h_1h_2 \cdots h_i e \cdots e$ $(k-i\ \ e$ terms) and $e \cdots eh_{i+1}e \cdots e$ $(i\ \ e$ terms on the left and $k-(i+1)$ e terms on the right) and each $h_j \in H_j$. This contradicts our assumption about G.
87. We know from Theorem 9.7 that $G/Z(G) \approx Z_{p^2}$ or $Z_p \oplus Z_p$ and from the G/Z Theorem (Theorem 9.3) Z_{p^2} is ruled out.
89. By Theorem 7.2 and Example 5 in Chapter 9, if H and K were distinct subgroups of order p^2 then HK would be a subgroup of order p^3 or p^4, which contradicts Lagrange.
91. If G is cyclic then Theorem 4.4 says that G has exactly one element of order 2. If G is not cyclic, let a be any non-identity element of G and b be any element of G not in $\langle a \rangle$. Then $\langle a \rangle \times \langle b \rangle$ is isomorphic to a group of the form $Z_{2^s} \oplus Z_{2^t}$ where s and t are positive. But then G has at least three elements of order 2. The appropriate generalization is: "An Abelian group of order p^n for a prime p and some positive integer n is cyclic if and only if it has exactly $p-1$ elements of order p."
93. Observe that for every two distinct primes p and q we have $pH \neq qH$.

(For if there are integers a, b, c, d such that $pa^2/b^2 = qc^2d^2$ then p occurs an odd number of times on the left side of $pa^2d^2 = qb^2d^2$ but an even number of times on the right side). Every nonidentity element in Q/H has order 2.

Chapter 10

It's always helpful to learn from your mistakes, because then your mistakes seem worthwhile.

GARRY MARSHALL

1. Note that $\det(AB) = (\det A)(\det B)$.
3. Note that $(f+g)' = f' + g'$.
5. Observe that $(xy)^r = x^r y^r$. Odd values of r yield an isomorphism. For even values of r the kernel is $\{1, -1\}$.
7. $(\sigma\phi)(g_1g_2) = \sigma(\phi(g_1g_2)) = \sigma(\phi(g_1)\phi(g_2)) = \sigma(\phi(g_1))\sigma(\phi(g_2)) = (\sigma\phi)(g_1)(\sigma\phi)(g_2)$. Ker ϕ is a normal subgroup of Ker $\sigma\phi$. $|H|/|K| = [\text{Ker } \sigma\phi\text{:Ker } \phi]$.
9. $\phi((g,h)(g',h')) = \phi((gg',hh')) = gg' = \phi((g,h))\phi((g',h'))$. The kernel is $\{(e,h)|h \in H\}$.
11. Consider ϕ: $Z \oplus Z \to Z_a \oplus Z_b$ given by $\phi((x,y)) = (x \bmod a, y \bmod b)$ and use Theorem 10.3.
13. $(a,b) \to b$ is a homomorphism from $A \oplus B$ onto B with kernel $A \oplus \{e\}$.
15. 3, 13, 23
17. Suppose ϕ is such a homomorphism. By Theorem 10.3, Ker $\phi = \langle(8,1)\rangle, \langle(0,1)\rangle$, or $\langle(8,0)\rangle$. In these cases, (1, 0) + Ker ϕ has order either 16 or 8. So, $(Z_{16} \oplus Z_2)/\text{Ker } \phi$ is not isomorphic to $Z_4 \oplus Z_4$.
19. Since |Ker ϕ| is not 1 and divides 17, ϕ is the trivial map.
21. $\langle 5 \rangle$
23. $|\phi^{-1}(H)| = |H||\text{Ker } \phi|$.
25. 4 onto; 10 to
27. For each k with $0 \le k \le n-1$, the mapping $1 \to k$ determines a homomorphism.
29. Use Theorem 10.3 and properties 5, 7, and 9 of Theorem 10.2.
31. $\phi^{-1}(7) = 7$ Ker $\phi = \{7, 17\}$
33. 11Ker ϕ
35. $\phi((a,b)+(c,d)) = \phi((a+c,b+d)) = (a+c)-(b+d) = a-b+c-d = \phi((a,b)) + \phi((c,d))$. Ker $\phi = \{(a,a) \mid a \in Z\}$. $\phi^{-1}(3) = \{(a+3,a) \mid a \in Z\}$.
37. Use the property of complex numbers that $|xy| = |x||y|$ and the First Isomorphism Theorem.
39. $\phi(xy) = (xy)^6 = x^6y^6 = \phi(x)\phi(y)$. Ker $\phi = \langle 60^\circ + i \sin 60^\circ \rangle$.
41. Since $\phi(e) = e = e^{-2}$, $e \in H$. If $a \in H$, then $\phi(ab) = \phi(a)\phi(b) = a^{-2}b^{-2} = (ab)^{-2} \in H$. Also, $\phi(a^{-1}) = \phi(a)^{-1} = (a^{-2})^{-1} = (a^{-1})^{-2} \in H$. If $\phi(x) = x^3$ and $a \in H$, then $\phi(a) = a^3 = a^{-2}$ implies that $a^5 = e$. Thus, $|a| = 5$ or 1.
43. Property 2 of Theorem 10.2 handles the 2^m case. Suppose that there is a homomorphism from $G = Z_{2^m} \oplus Z_{2^n}$ onto $Z_2 \oplus Z_2 \oplus Z_2$ where m and n are at least 1 and let H be the kernel. We may assume that $m + n \ge 3$. Then $|H| = 2^{m+n-3}$. Because every nonidentity element of G/H has order 2 we know that $((1,0)H)^2 = (2,0)H = H$ and $((0,1)H)^2 = (0,2)H = H$. This means that $H_1 = \langle(2,0)\rangle$ and $H_2 = \langle(0,2)\rangle$ are subgroups of H. Then H_1H_2 is also a subgroup of H. But $|H_1H_2| = 2^{m-1} \cdot 2^{n-1} = 2^{m+n-2}$ exceeds $|H| = 2^{m+n-3}$. The argument works for any prime p.
45. Since S_4/H is a group of order 6 but has no element of order 6 (because S_4 does not have one) it follows from Theorems 7.3 and 10.3 that S_4/H is isomorphic S_3.
47. It follows from Exercise 11 in Chapter 0 that the mapping ϕ from $U(st)$ to $U(s)$ given by $\phi(x) = x \bmod s$ is a homomorphism. Since Ker $\phi = U_s(st)$ we have by Theorem 10.3 that $U(st)/U_s(st)$ is isomorphic to a subgroup of $U(s)$. To see that ϕ is onto note that it follows from Theorem 8.3 that $|U(st)/U_s(st)| = |U(st)|/|U_s(st)| =$

$|U(s) \oplus U(t)|/|U(t)| =$
$|U(s)||U(t)|/|U(t)| = |U(s)|$.

49. Show that the mapping from K to KN/N given by $k \to kN$ is an onto homomorphism with kernel $K \cap N$.
51. Since the eight elements of A_4 of order 3 must map to an element of order that divides 3, by Lagrange's Theorem, each of them must map to the identity. But then the kernel has at least 8 elements and its order and divides 12. So, the kernel has order 12.
53. $D_4, \{e\}, Z_2, Z_2 \oplus Z_2$
55. It is divisible by 10. 10 can be replaced by any positive integer.
57. It is infinite. Look at Z.
59. Let γ be the natural homomorphism from G onto G/N. Let $\overline{H}$ be a subgroup of G/N and let $\gamma^{-1}(\overline{H}) = H$. Then H is a subgroup of G and $H/N = \gamma(H) = \gamma(\gamma^{-1}(\overline{H})) = \overline{H}$.
61. The mapping $g \to \phi_g$ is a homomorphism with kernel $Z(G)$.
63. $(f+g)(3) = f(3) + g(3)$. The kernel is the set of elements in $Z[x]$ whose graphs pass through the point (3, 0). 3 can be replaced by any integer.
65. Let g belong to G. Since $\phi(g)$ belongs to $Z_2 \oplus Z_2 = \langle(1,0)\rangle \cup \langle(0,1)\rangle \cup \langle(1,1)\rangle$, it follows that $G = \phi^{-1}(\langle(1,0)\rangle) \cup \phi^{-1}(\langle(0,1)\rangle) \cup \phi^{-1}(\langle(1,1)\rangle)$. Moreover, each of these three subgroups is proper and by property 9 of Theorem 10.2 normal.
67. $\phi(a,b) = (2a, b)$.
69. It fails because 5 does divide $|\mathrm{Aut}(Z_{11})| = 10$.
71. Mimic Example 19.
73. Let ϕ be a homomorphism from S_3 to G. Since $|\phi(S_3)|$ must divide 6, we have that $|\phi(S_3)| = 1, 2, 3,$ or 6. In the first case, ϕ maps every element to 0. If $|\phi(S_3)| = 2$, then n is even and ϕ maps the even permutations to 0 and the odd permutations to an element of order 2. The case that $|\phi(S_3)| = 3$ cannot occur, because it implies that Ker ϕ is a normal subgroup of order 2, whereas S_3 has no normal subgroup of order 2. The case that $|\phi(S_3)| = 6$ cannot occur, because it implies that ϕ is an isomorphism from a non-Abelian group to an Abelian group.
75. $\phi(zw) = z^2w^2 = \phi(z)\phi(w)$. Ker $\phi = \{1, -1\}$ and, because ϕ is onto $\mathbf{C}^*$ we have by Theorem 10.3, $\mathbf{C}^*/\{1,-1\}$ is isomorphic to $\mathbf{C}^*$. When $\mathbf{C}^*$ is replaced by $\mathbf{R}^*$ we have that ϕ is onto $\mathbf{R}^+$ and by Theorem 10.3, $\mathbf{R}^*/\{1,-1\}$ is is isomorphic to $\mathbf{R}^+$.

Chapter 11

Ever tried. Ever failed. No matter. Try again. Fail again. Fail better.

SAMUEL BECKETT

1. $n = 4; Z_4, Z_2 \oplus Z_2$
3. $n = 36; Z_9 \oplus Z_4, Z_3 \oplus Z_3 \oplus Z_4, Z_9 \oplus Z_2 \oplus Z_2, Z_3 \oplus Z_3 \oplus Z_2 \oplus Z_2$
5. The only Abelian groups of order 45 are Z_{45} and $Z_3 \oplus Z_3 \oplus Z_5$. In the first group, $|3| = 15$; in the second one, $|(1,1,1)| = 15$. $Z_3 \oplus Z_3 \oplus Z_5$ does not have an element of order 9.
7. $Z_9 \oplus Z_3 \oplus Z_4; Z_9 \oplus Z_3 \oplus Z_2 \oplus Z_2$
9. $Z_4 \oplus Z_2 \oplus Z_3 \oplus Z_5$
11. By the Fundamental Theorem, any finite Abelian group G is isomorphic to some direct product of cyclic groups of prime-power order. Now go across the direct product and, for each distinct prime you have, pick off the largest factor of the prime power. Next, combine all of these into one factor (you can do this, since the subscripts are relatively prime). Let us call the order of this new factor n_1. Now repeat this process with the remaining original factors and call the order of the resulting factor n_2. Then n_2 divides n_1, since each prime-power divisor of n_2 is also a prime-power divisor of n_1. Continue in this fashion. Example: If $G \approx Z_{27} \oplus Z_3 \oplus Z_{125} \oplus Z_{25} \oplus Z_4 \oplus Z_2 \oplus Z_2$, then $G \approx Z_{27 \cdot 125 \cdot 4} \oplus Z_{3 \cdot 25 \cdot 2} \oplus Z_2$. Now note that 2 divides $3 \cdot 25 \cdot 2$ and $3 \cdot 25 \cdot 2$ divides $27 \cdot 125 \cdot 4$.

13. $Z_2 \oplus Z_2$
15. **a.** 1 **b.** 1 **c.** 1 **d.** 1 **e.** 1 **f.** There is a unique Abelian group of order n if and only if n is not divisible by the square of any prime.
17. This is equivalent to asking how many Abelian groups of order 16 have no elements of order 8. From the Fundamental Theorem of Finite Abelian Groups the only choices are $Z_4 \oplus Z_4, Z_4 \oplus Z_2 \oplus Z_2$, and $Z_2 \oplus Z_2 \oplus Z_2 \oplus Z_2$.
19. $Z_2 \oplus Z_2$
21. $Z_3 \oplus Z_3$
23. n is square-free (no prime factor of n occurs more than once).
25. Among the first 11 elements in the table, there are nine elements of order 4. None of the other isomorphism classes has this many.
27. $Z_4 \oplus Z_2 \oplus Z_2$; one internal direct product is $\langle 7\rangle \times \langle 101\rangle \times \langle 199\rangle$.
29. Since $|\langle(2,2)\rangle| = 8$ we know $|(Z_{16} \oplus Z_{16})/\langle(2,2)\rangle = 32$. Then observing that $|(1,0) + \langle(2,2)\rangle| = 16$ and $|(0,1) + \langle(2,2)\rangle| = 16$, we know that the maximum order of any element in the factor group is 16. So, the isomorphism class is $Z_{16} \oplus Z_2$.
31. 3; 6; 12
33. $Z_4 \oplus Z_4$
35. Use Theorems 11.1, 8.1, and 4.3.
37. $|\langle a\rangle K| = |a||K|/|\langle a\rangle \cap K| = |a||K| = |\overline{a}||\overline{K}|p = |\overline{G}|p = |G|$
39. By the Fundamental Theorem of Finite Abelian Groups, it suffices to show that every group of the form $Z_{{p_1}^{n_1}} \oplus Z_{{p_2}^{n_2}} \oplus \ldots \oplus Z_{{p_k}^{n_k}}$ is a subgroup of a U-group. Consider first a group of the form $Z_{{p_1}^{n_1}} \oplus Z_{{p_2}^{n_2}}$ (p_1 and p_2 need not be distinct). By Dirichlet's Theorem, for some s and t there are distinct primes q and r such that $q = t{p_1}^{n_1} + 1$ and $r = s{p_2}^{n_2} + 1$. Then $U(qr) = U(q) \oplus U(r) \approx Z_{t{p_1}^{n_1}} \oplus Z_{s{p_2}^{n_2}}$, and this latter group contains a subgroup isomorphic to $Z_{{p_1}^{n_1}} \oplus Z_{{p_2}^{n_2}}$. The general case follows in the same way.
41. Look at D_4.
43. If G has an element of order greater than 2, then $\phi(x) = x^{-1}$ is a nontrivial automorphism of G (see Exercise 12 of Chapter 6).
45. By Theorem 11.1 and Corollary 1 of Theorem 8.2 it suffices to do the case where $|G| = p^m$ and p is prime. Then, if G is not cyclic, it follows from Theorem 11.1 together with Theorem 4.3 that $x^p = e$ has more than p solutions.
47. First observe by direct calculations we have $|8| = |12| = |18| = |21| = |27| = 4$.

Chapter 12

Mistakes are the portals of discovery.

JAMES JOYCE

1. For any $n > 1$, the ring $M_2(Z_n)$ of 2×2 matrices with entries from Z_n is a finite noncommutative ring. The set $M_2(2Z)$ of 2×2 matrices with even integer entries is an infinite noncommutative ring that does not have a unity.
3. In $\mathbf{R}$, consider $\{n\sqrt{2} \mid n \in Z\}$.
5. **a**, **b**
7. In Z_p, nonzero elements have multiplicative inverses. Use them.
9. If a and b belong to the intersection, then they belong to each member of the intersection. Thus, $a - b$ and ab belong to each member of the intersection. So, $a - b$ and ab belong to the intersection.
11. Rule 3: $0 = 0(-b) = (a + (-a))(-b) = a(-b) + (-a)(-b) = -(ab) + (-a)(-b)$. So, $ab = (-a)(-b)$.
    Rule 4: $a(b - c) = a(b + (-c)) = ab + a(-c) = ab + (-(ac)) = ab - ac$.
    Rule 5: Use rule 2.
    Rule 6: Use rule 3.
13. *Hint:* Z is a cyclic group under addition, and every subgroup of a cyclic group is cyclic.
15. For positive m and n, observe that

$(m \cdot a)(n \cdot b) = (a + a + \cdots + a)(b + b + \cdots + b) = (ab + ab + \cdots + ab)$, where the last term has mn summands. Similar arguments apply in the remaining cases.

17. From Exercise 15, we have $(n \cdot a)(m \cdot a) = (nm) \cdot a^2 = (mn) \cdot a^2 = (m \cdot a)(n \cdot a)$.
19. Let a, b belong to the center. Then $(a-b)x = ax-bx = xa-xb = x(a-b)$. Also, $(ab)x = a(bx) = a(xb) = (ax)b = (xa)b = x(ab)$.
21. $(x_1, \ldots, x_n)(a_1, \ldots, a_n) = (x_1, \ldots, x_n)$ for all x_i in R_i if and only if $x_i a_i = x_i$ for all x_i in R_i and $i = 1, \ldots, n$.
23. $\{1, -1, i, -i\}$
25. $f(x) = 1$ and $g(x) = -1$.
27. If a is a unit, then $b = a(a^{-1}b)$.
29. Consider $a^{-1} - a^{-2}b$.
31. $0^1 = 0$ so the set is nonempty. Let $a^m = 0$ and $b^n = 0$. We may assume that $m \geq n$. Then in the expansion of $(a-b)^{2m}$ each term has an expression of the form $a^{2m-i}b^i$. So when $i = 0, 1, \ldots, m$ we have $a^{2m-i} = 0$ and when $i = m+1, m+2, m+m$ we have $b^i = 0$. So, all terms in the expansion are 0. (This argument also works when the exponent of $(a-b)$ is $m+n-1$.) Finally, if r is any element in the ring then$(ab)^m = a^m b^m = 0$.
33. Try the ring $M_2(Z)$.
35. By inspection, R is closed under addition and multiplication. The elements $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}$ do not commute,

    For the general case use $m \times m$ matrices with the first $m-1$ columns all 0 and the last column any elements from Z_n.
37. Observe that $-x = (-x)^4 = x^4 = x$.
39. For Z_6, use $n = 3$. For Z_{10}, use $n = 5$. Say $m = p^2 t$, where p is a prime. Then $(pt)^n = 0$ in Z_m, since m divides $(pt)^n$.
41. Every subgroup of Z_n is closed under multiplication.
43. $ara - asa = a(r-s)a$. $(ara)(asa) = ara^2sa = arsa$. $a1a = a^2 = 1$, so $1 \in S$.
45. The Subring Test is satisfied.
47. They satisfy the subring test but the multiplication is trivial. This is, the product of any two elements is zero.
49. Look at (1, 0, 1) and (0, 1, 1).
51. Observe that $n \cdot 1 - m \cdot 1 = (n-m) \cdot 1$. Also, $(n \cdot 1)(m \cdot 1) = (nm) \cdot ((1)(1)) = (nm) \cdot 1$.
53. $\{2n/3^m \mid n \in Z, m$ is a positive integer$\}$. This set is a ring that contains $2/3$ and is contained in every ring that contains $2/3$.
55. $(a+b)(a-b) = a^2 + ba - ab - b^2 = a^2 - b^2$ if and only if $ba - ab = 0$.
57. $Z_2 \oplus Z_2; Z_2 \oplus Z_2 \oplus \cdots$ (infinitely many copies)
59. If (a, b) is a zero-divisor in $R \oplus S$ then there is a $(c, d) \neq (0, 0)$ such that $(a, b)(c, d) = (0, 0)$. Thus $ac = 0$ and $bd = 0$. So, a or b is a zero-divisor or exactly one of a or b is 0. Conversely, if a is a zero-divisor in R then there is a $c \neq 0$ in R such that $ac = 0$. In this case $(a, b)(c, 0) = (0, 0)$. A similar argument applies if b is a zero-divisor. If $a = 0$ and $b \neq 0$ then $(a, b)(x, 0) = (0, 0)$ where x is any nonzero element in A . A similar argument applies if $a \neq 0$ and $b = 0$.
61. Fix some a in $R, a \neq 0$. Then there is a b in R such that $ab = a$. Now if x in R and $x \neq 0$ then there is an element c in R such that $ac = x$. Then $xb = acb = c(ab) = ca = x$. Thus b is the unity. To show that every nonzero element r of R has an inverse note that since $rR = R$ there is an element s in R such that $rs = b$.
63. If det $A = \pm 1$ then by Exercise 9 in Chapter 2 A^{-1} is in $M_2(Z)$ and by definition A is a unit. On the other hand, if A is a unit in $M_2(Z)$ then $1 = \det(AA^{-1}) = (\det A)(\det A^{-1})$ where both det A and det A^{-1} are integers and units in Z. But the only units in Z are ± 1.

Chapter 13

Mathematics is not a careful march down a well-cleared highway, but a journey into a strange wilderness, where the explorers get lost.

W. S. Anglin

1. The verifications for Example 16 follow from elementary properties of real and complex numbers. For Example 7, note that

$$\begin{bmatrix}1 & 0\\0 & 0\end{bmatrix}\begin{bmatrix}0 & 0\\0 & 1\end{bmatrix}=\begin{bmatrix}0 & 0\\0 & 0\end{bmatrix}.$$

For Example 8, note that (1, 0)(0, 1) = (0, 0).

3. Let $ab = 0$ and $a \neq 0$. Then $ab = a \cdot 0$, so $b = 0$.
5. Let $k \in Z_n$. If $\gcd(k, n) = 1$, then k is a unit. If $\gcd(k, n) = d > 1$, write $k = sd$. Then $k(n/d) = sd(n/d) = sn = 0$.
7. Let $s \in R, s \neq 0$. Consider the set $S = \{sr \mid r \in R\}$. If $S = R$, then $sr = 1$ (the unity) for some r. If $S \neq R$, then there are distinct r_1 and r_2 such that $sr_1 = sr_2$. In this case, $s(r_1 - r_2) = 0$. To see what happens when the "finite" condition is dropped, consider Z.
9. Take $a = (1, 1, 0), b = (1, 0, 1)$, and $c = (0, 1, 1)$.
11. $(a_1+b_1\sqrt{d})-(a_2+b_2\sqrt{d}) = (a_1-a_2)+(b_1-b_2)\sqrt{d}$; $(a_1+b_1\sqrt{d})(a_2+b_2\sqrt{d}) = (a_1a_2+b_1b_2d)+(a_1b_2+a_2b_1)\sqrt{d}$. Thus, the set is a ring. Since $Z[\sqrt{d}]$ is a subring of the ring of complex numbers, it has no zero-divisors.
13. The even integers.
15. $(1-a)(1+a+a^2+\cdots+a^{n-1}) = 1+a+a^2+\cdots+a^{n-1}-a-a^2-\cdots-a^n = 1-a^n = 1-0 = 1$.
17. Suppose $a \neq 0$ and $a^n = 0$ (where we take n to be as small as possible). Then $a \cdot 0 = 0 = a^n = a \cdot a^{n-1}$, so by cancellation, $a^{n-1} = 0$.
19. If $a^2 = a$ and $b^2 = b$, then $(ab)^2 = a^2b^2 = ab$. The other cases are similar.
21. Let $f(x) = x$ on $[-1, 0]$, $f(x) = 0$ on $(0, 1]$, $g(x) = 0$ on $[-1, 0]$, and $g(x) = x$ on $(0, 1]$. Then $f(x)$ and $g(x)$ are in R and $f(x)g(x) = 0$ on $[-1, 1]$.
23. Suppose that a is an idempotent and $a^n = 0$. By the previous exercise, $a = 0$.
25. The generators are: $2+i$, $1+2i$, $1+i$, $2+2i$.
27. $a^2 = a$ implies $a(a-1) = 0$. So if a is a unit, $a-1 = 0$ and $a = 1$.
29. See Theorems 3.1 and 12.3.
31. Note that $ab = 1$ implies $aba = a$. Thus $0 = aba - a = a(ba-1)$. So, $ba - 1 = 0$.
33. A subdomain of an integral domain D is a subset of D that is an integral domain under the operations of D. To show that P is a subdomain, show that it is a subring and contains 1. Every subdomain contains 1 and is closed under addition and subtraction, so every subdomain contains P. $|P| =$ char D when char D is prime and $|P|$ is infinite when char D is 0.
35. Use Theorems 13.3, Lagrange's Theorem, and Theorem 13.4.
37. By Exercise 36, 1 is the only element of an integral domain that is its own multiplicative inverse if and only if $1 = -1$. This is true only for integral domains of characteristic 2.
39. **a.** Since $a^3 = b^3, a^6 = b^6$. Then $a = b$ because we can cancel a^5 from both sides (since $a^5 = b^5$).
b. Use the fact that there exist integers s and t such that $1 = sn + tm$, but remember that you cannot use negative exponents in a ring.
41. Observe that F^* (the nonzero elements of F) form a group of order 31. So, for any a in F other than 0 or 1, we know by Lagrange's Theorem that $|a| = 31$.
43. In $Z_p[k]$ note that $(a+b\sqrt{k})^{-1} = \frac{1}{(a+b\sqrt{k})}\frac{(a-b\sqrt{k})}{(a-b\sqrt{k})} = \frac{a-b\sqrt{k}}{a^2-b^2k}$ exists if and only if $a^2 - b^2k \neq 0$ where $a \neq 0$ and $b \neq 0$.

45. Let $S = \{a_1, a_2, \ldots, a_n\}$ be the nonzero elements of the ring. First show that $S = \{a_1a_1, a_1a_2, \ldots, a_1a_n\}$. Thus, $a_1 = a_1a_i$ for some i. Then a_i is the unity, for if a_k is any element of S, we have $a_1a_k = a_1a_ia_k$, so that $a_1(a_k - a_ia_k) = 0$.
47. Say $|x| = n$ and $|y| = m$ with $n < m$. Consider $(nx)y = x(ny)$.
49. **a.** For $n = 2$ the Binomial Theorem gives us $(x_1 + x_2)^p = x_1^p + px_1^{p-1}x_2 + \cdots + px_1x_2^{p-1} + x_2^p$, where each coefficient $p!/k!(p-k)!$ of every term between x_1^p and x_2^p is divisible by p. Thus, $(x_1 + x_2)^p = x_1^p + x_2^p$. The general case follows by induction on n.
b. This case follows from Part a and induction on m.
c. Note Z_4 is a ring of characteristic 4 and $(1+1)^4 = 2^4 = 0$, but $1^4 + 1^4 = 1 + 1 = 2$.
51. Use Theorems 13.4 and 9.5 and Exercise 47.
53. $n\begin{bmatrix} a & b \\ c & d \end{bmatrix} = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ for all members of $M_2(R)$ if and only if $na = 0$ for all a in R.
55. Use Exercise 54.
57. **a.** 2 **b.** 2, 3 **c.** 2, 3, 6, 11 **d.** 2, 3, 9, 10
59. 2
61. If $a \in Z_p$ and $a^2 + 1 = 0$, then $a^2 + 1 = (a+i)(a-i) = 0$.
63. Suppose that F is a field of order 16 and K is a subfield of F of order 8. Then K^* is a subgroup of F^* and $|K^*| = 7$ and $|F^*| = 15$, which contradicts LaGrange's Theorem.
65. Use Exercise 29 and part **a** of Exercise 49.
67. char $R = 0$. For positive integer n the subring $\{0\} \oplus \{0\} \cdots \oplus \{0\} \oplus \cdots \oplus Z_n \oplus \{0\} \cdots$ is subring of characteristic n.
69. Observe that because $u_Su_R = u_S = u_Su_S$ we have $u_S(u_R - u_S) = 0$.
71. Since a field of order 27 has characteristic 3, we have $3a = 0$ for all a. From this, we have $6a = 0$ and $5a = -a$.
73. Let $F^* = \langle a \rangle$. Then $-1 = a^n$ for some n. Thus $1 = a^{2n}$ and $|a|$ divides $2n$.

Chapter 14

The paradox of excellence is that it is built upon the foundations of necessary failure.

MATTHEW SYED

1. Let r_1a and r_2a belong to $\langle a \rangle$. Then $r_1a - r_2a = (r_1 - r_2)a \in \langle a \rangle$. If $r \in R$ and $r_1a \in \langle a \rangle$, then $r(r_1a) = (rr_1)a \in \langle a \rangle$.
3. Clearly, I is not empty. Now observe that $(r_1a_1 + \cdots + r_na_n) - (s_1a_1 + \cdots + s_na_n) = (r_1 - s_1)a_1 + \cdots + (r_n - s_n)a_n \in I$. Also, if $r \in R$, then $r(r_1a_1 + \cdots + r_na_n) = (rr_1)a_1 + \cdots + (rr_n)a_n \in I$. That $I \subseteq J$ follows from closure under addition and multiplication by elements from R.
5. Let $a + bi$, $c + di \in S$. Then $(a+bi) - (c+di) = a - c + (b-d)i$ and $b - d$ is even. Also, $(a+bi)(c+di) = ac - bd + (ad+cb)i$ and $ad + cb$ is even. Finally, $(1+2i)(1+i) = -1 + 3i \notin S$.
7. Suppose that s is not prime. Then we can write $s = pm$ where p is prime and $m > 1$. Then $\langle s \rangle$ is properly contained in $\langle p \rangle$ and $\langle p \rangle$ is properly contained in Z_n. So $\langle s \rangle$ is not maximal. Now suppose that s is prime and there is a divisor $t > 1$ of n such that $\langle t \rangle$ properly contains $\langle s \rangle$ (recall every subgroup of Z_n has the form $\langle k \rangle$ where k is a divisor of n.) Then $s = rt$ for some r. So we have $t = s$.
9. If aR is a non-zero ideal of R we know that $aR = R$. So a belongs to R.
11. Since $ar_1 - ar_2 = a(r_1 - r_2)$ and $(ar_1)r = a(r_1r), 4R = \{\cdots, -16, -8, 0, 8, 16, \cdots\}$.
13. If n is prime, use Eucid's Lemma (Chapter 0). If n is not prime, say $n = st$ where $s < n$ and $t < n$; then st belongs to nZ but s and t do not.
15. **a.** $a = 1$ **b.** $a = 2$ **c.** $a = \gcd(m, n)$

17. **a.** $a = 12$
b. $a = 48$. To see this, note that every element of $\langle 6\rangle\langle 8\rangle$ has the form $6t_1 8k_1 + 6t_2 8k_2 + \cdots + 6t_n 8k_n = 48s \in \langle 48\rangle$. So, $\langle 6\rangle\langle 8\rangle \subseteq \langle 48\rangle$. Also, since $48 \in \langle 6\rangle\langle 8\rangle$, we have $\langle 48\rangle \subseteq \langle 6\rangle\langle 8\rangle$.
c. $a = mn$
19. Let $r \in R$. Then $r = 1r \in A$.
21. Let $u \in I$ be a unit and let $r \in R$. Then $r = r(u^{-1}u) = (ru^{-1})u \in I$.
23. Observe that $\langle 2\rangle$ and $\langle 3\rangle$ are the only nontrivial ideals of Z_6, so both are maximal. More generally, Z_{pq}, where p and q are distinct primes, has exactly two maximal ideals.
25. Clearly, I is closed under subtraction. Also, if b_1, b_2, b_3, and b_4 are even, then every entry of $\begin{bmatrix} a_1 & a_2 \\ a_3 & a_4 \end{bmatrix}\begin{bmatrix} b_1 & b_2 \\ b_3 & b_4 \end{bmatrix}$ is even.
27. The proof that I is an ideal is the same as the case that $n = 2$ in Exercise 25. The number of elements in I is n^4.
29. That I satisfies the ideal test follows directly from the definitions of matrix addition and multiplication. To see that R/I is field first observe that $\begin{bmatrix} a & b \\ 0 & c \end{bmatrix} + I = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & b \\ 0 & c \end{bmatrix} + I = \begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + I.$ Thus we need only show that $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + I$ has an inverse in R/I when $a \neq 0$. To this end note that $\left(\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix} + I\right)\left(\begin{bmatrix} a^{-1} & 0 \\ 0 & 0 \end{bmatrix} + I\right) = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I = \left(\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + I\right) + \left(\begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + I\right) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} + I.$ $\left(\begin{bmatrix} 2 & 0 \\ 0 & 0 \end{bmatrix} + I\right)^{-1} = \begin{bmatrix} 1/2 & 0 \\ 0 & 0 \end{bmatrix} + I.$
31. $R = \{0 + \langle 2i\rangle, 1 + \langle 2i\rangle, i + \langle 2i\rangle, 1 + i + \langle 2i\rangle\}$. R is not an integral domain because $(1 + i + \langle 2i\rangle)^2 = (1+i)^2 + \langle 2i\rangle = 1 + 1 + \langle 2i\rangle = 0 + \langle 2i\rangle$.
33. First note that every element of R has the form $ax + b + I$ where $a,\ b \in Z_5$. Since 1 and -1 are zeros of $x^2 - 1$ we know that $0 + I = x^2 - 1 + I = (x - 1 + I)(x + 1 + I)$ and that $x - 1 + I = x + 4 + I$ and $x + 1 + I$ are zero-divisors in R. Then for every nonzero c in Z_5, $c(x+1) + I$ and $c(x+4) + I$ are distinct zero-divisors in R. By Exercise 7 in Chapter 13 every nonzero element element of R that is not a zero-divisor is a unit. So, $|U(R)| = 16$. Because $|4| = 2$ and $|x| = 2$, we know that $U(R)$ is not cyclic. Next we look for the unit of maximum order. Trying various possibilities we find that $(3x+1)^4 = 4$ and $(3x+1)^8 = 1$. So $U(R) \approx Z_8 \oplus Z_2$.
35. Use the observation that every member of R can be written in the form $\begin{bmatrix} 2q_1 + r_1 & 2q_2 + r_2 \\ 2q_3 + r_3 & 2q_4 + r_4 \end{bmatrix}$. Then note that $\begin{bmatrix} 2q_1 + r_1 & 2q_2 + r_2 \\ 2q_3 + r_3 & 2q_4 + r_4 \end{bmatrix} + I = \begin{bmatrix} r_1 & r_2 \\ r_3 & r_4 \end{bmatrix} + I.$
37. $(br_1 + a_1) - (br_2 + a_2) = b(r_1 - r_2) + (a_1 - a_2) \in B$; $r'(br + a) = b(r'r) + r'a \in B$ since $r'a$ is in A.
39. Use Exercise 21.
41. Let a be an idempotent other that 0 or 1. Then $a^2 = a$ implies that $a(a-1) = 0$.
43. Since every element of $\langle x\rangle$ has the form $xg(x)$, we have $\langle x\rangle \subseteq I$. If $f(x) \in I$, then $f(x) = a_n x^n + \cdots + a_1 x = x(a_n x^{n-1} + \cdots + a_1) \in \langle x\rangle$.
45. Suppose $f(x) + I \neq I$. Then $f(x) + I = f(0) + I$ and $f(0) \neq 0$. Thus,
$$(f(x) + I)^{-1} = \frac{1}{f(0)} + I.$$
This shows that R/I is a field. Now use Theorem 14.4.

47. Since $(3+i)(3-i) = 10, 10 + \langle 3+i\rangle = 0 + \langle 3+i\rangle$. Also, $i + \langle 3+i\rangle = -3 + \langle 3+i\rangle = 7 + \langle 3+i\rangle$. So, $Z[i]/\langle 3+i\rangle = \{k + \langle 3+i\rangle \mid k = 0, 1, \ldots, 9\}$, since $1 + \langle 3+i\rangle$ has additive order 10.
49. Use Theorems 14.3 and 14.4.
51. Since every $f(x)$ in $\langle x, 2\rangle$ has the form $f(x) = xg(x) + 2h(x)$, we have $f(0) = 2h(0)$, so that $f(x) \in I$. If $f(x) \in I$, then $f(x) = a_n x^n + \cdots + a_1 x + 2k = x(a_n x^{n-1} + \cdots + a_1) + 2k \in \langle x, 2\rangle$. I is prime and maximal. $Z[x]/I$ has two elements.
53. One example is $J = \langle x^2+1, 2\rangle$. To see that 1 is not in J note that if there were $f(x), g(x) \in Z[x]$ such that $(x^2+1)f(x) + 2g(x) = 1$ then evaluating the left side at 1 yields an even integer.
55. $3x + 1 + I$
57. Every ideal is a subgroup. Every subgroup of a cyclic group is cyclic.
59. Let I be any ideal of $R \oplus S$ and let $I_R = \{r \in R \mid (r, s) \in I$ for some $s \in S\}$ and $I_S = \{s \in S \mid (r, s) \in I$ for some $r \in R\}$. Then I_R is an ideal of R and I_S is an ideal of S. Let $I_R = \langle r\rangle$ and $I_S = \langle s\rangle$. Since, for any $(a, b) \in I$ there are elements $a' \in R$ and $b' \in S$ such that $(a, b) = (a'r, b's) = (a', b')(r, s)$, we have that $I = \langle (r, s)\rangle$.
61. Say $b, c \in \text{Ann}(A)$. Then $(b-c)a = ba - ca = 0 - 0 = 0$. Also, $(rb)a = r(ba) = r \cdot 0 = 0$.
63. **a.** $\langle 3\rangle$ **b.** $\langle 3\rangle$ **c.** $\langle 3\rangle$
65. Suppose $(x + N(\langle 0\rangle))^n = 0 + N(\langle 0\rangle)$. We must show that $x \in N(\langle 0\rangle)$. We know that $x^n + N(\langle 0\rangle) = 0 + N(\langle 0\rangle)$, so that $x^n \in N(\langle 0\rangle)$. Then, for some m, $(x^n)^m = 0$, and therefore $x \in N(\langle 0\rangle)$.
67. The set $Z_2[x]/\langle x^2+x+1\rangle$ has only four elements and each of the nonzero ones has a multiplicative inverse. For example, $(x + \langle x^2+x+1\rangle)(x + 1 + \langle x^2+x+1\rangle) = 1 + \langle x^2+x+1\rangle$.
69. $x + 2 + \langle x^2+x+1\rangle$ is not zero, but its square is.
71. If f and $g \in A$, then $(f-g)(0) = f(0) - g(0)$ is even and $(f \cdot g)(0) = f(0) \cdot g(0)$ is even. $f(x) = \frac{1}{2} \in R$ and $g(x) = 2 \in A$, but $f(x)g(x) \notin A$.
73. *Hint:* Any ideal of R/I has the form A/I, where A is an ideal of R.
75. In Z, $\langle 2\rangle \cap \langle 3\rangle = \langle 6\rangle$ is not prime.
77. According to Theorem 13.3 we need only determine the additive order of $1 + \langle 2+i\rangle$. Since $5(1 + \langle 2+i\rangle) = 5 + \langle 2+i\rangle = (2+i)(2-i) + \langle 2+i\rangle = 0 + \langle 2+i\rangle$, we know that $1 + \langle 2+i\rangle$ has order 5.
79. The set K of all polynomials whose coefficients are even is closed under subtraction and multiplication by elements from $Z[x]$ and therefore K is an ideal. By Theorem 14.3 to show that K is prime it suffices to show that $Z[x]/K$ has no zero-divisors. Suppose that $f(x) + K$ and $g(x) + K$ are nonzero elements of $Z[x]/K$. Since K absorbs all terms that have even coefficients we may assume that $f(x) = a_m x^m + \cdots + a_0$ and $g(x) = b_n x^n + \cdots + b_0$ are in $Z[x]$ and a_m and b_n are odd integers. Then $(f(x) + K)(g(x) + K) = a_m b_n x^{m+n} + \cdots + a_0 b_0 + K$ and $a_m e_n$ is odd. So, $f(x)g(x) + K$ in nonzero.
81. Use the fact that R/I is an integral domain to show that $R/I = \{I, 1 + I\}$.
83. $\langle x\rangle \subset \langle x, 2^n\rangle \subset \langle x, 2^{n-1}\rangle \subset \cdots \subset \langle x, 2\rangle$

Chapter 15

For every problem there is a solution which is simple, clean and wrong.

H. L. MENCKEN

1. Property 3 $\phi(A)$ is a subgroup because ϕ is a group homomorphism. Let $s \in S$ and $\phi(r) = s$. Then $s\phi(a) = \phi(r)\phi(a) = \phi(ra)$ and $\phi(a)s = \phi(a)\phi(r) = \phi(ar)$. Property 4: Let a and b belong to $\phi^{-1}(B)$ and r belong to R. Then $\phi(a)$ and $\phi(b)$ are

in B. So, $\phi(a) = \phi(b) = \phi(a) + \phi(-b) = \phi(a - b) \in B$. Thus, $a - b \in B$. Also, $\phi(ra) = \phi(r)\phi(a) \in B$ and $\phi(ar) = \phi(a)\phi(r) \in B$. So, ra and $ar \in \phi^{-1}(B)$.

3. We already know the mapping is an isomorphism of groups. Let $\Phi(x + \text{Ker }\phi) = \phi(x)$. Note that $\Phi((r + \text{Ker }\phi)(s + \text{Ker }\phi)) = \Phi(rs + \text{Ker }\phi) = \phi(rs) = \phi(r)\phi(s) = \Phi(r + \text{Ker }\phi)\Phi(s + \text{Ker }\phi)$.
5. $\phi(2 + 4) = \phi(1) = 5$, whereas $\phi(2) + \phi(4) = 0 + 0 = 0$.
7. Observe that $(x + y)/1 = x/1 + y/1$ and $(xy)/1 = (x/1)(y/1)$.
9. $a = \phi(1) = \phi(1 \cdot 1) = \phi(1)\phi(1) = aa = a^2$. For the example look at Z_6.
11. For groups, $\phi(x) = ax$ for $a = 2, 4, 6, 8$ since each of these has additive order 5. For rings, only $\phi(x) = 6x$ since 6 is the only non-zero idempotent in R.
13. If a and $b(b \neq 0)$ belong to every member of the collection, then so do $a - b$ and ab^{-1}. Thus, by Exercise 29 in Chapter 13, the intersection is a subfield.
15. Apply the definition.
17. Multiplication is not preserved.
19. Yes.
21. The set of all polynomials passing through the point (1, 0).
23. $a = a^2$ implies that $\phi(a) = \phi(a^2) = \phi(a)\phi(a) = (\phi(a))^2$.
25. For Z_6 to Z_6, $1 \to 0$, $1 \to 1$, $1 \to 3$, and $1 \to 4$ each define a homomorphism. For Z_{20} to Z_{30}, $1 \to 0$, $1 \to 6$, $1 \to 15$, and $1 \to 21$ each define a homomorphism.
27. The zero map and the identity map.
29. Use Exercise 28.
31. First note that every element of $R[x]/\langle x^2 \rangle$ can be written uniquely in the form $a_1x + a_0 + \langle x^2 \rangle$. Then mapping that takes $a_1x + a_0 + \langle x^2 \rangle$ to $\left\{\begin{bmatrix} a_0 & a_1 \\ 0 & a_0 \end{bmatrix}\right\}$ is a ring isomorphism.
33. First observe that $\phi((0, 1)) = \phi((1, 1)) - \phi((1, 0)) = (1, 1) - (0, 1) = (1, 0)$. Then $\phi((a, b)) = a\phi((1, 0)) + b\phi((0, 1)) = a(0, 1) + b(1, 0) = (b, a)$.
35. Observe that an idempotent must map to an idempotent. So, (1, 0) and (0, 1) must map to 0 or 1. It follows that $(a, b) \to a$, $(a, b) \to b$, and $(a, b) \to 0$ are the only ring homomorphisms.
37. Say $m = a_k a_{k-1} \cdots a_1 a_0$ and $n = b_k b_{k-1} \cdots b_1 b_0$. Then $m - n = (a_k - b_k)10^k + (a_{k-1} - b_{k-1})10^{k-1} + \cdots + (a_1 - b_1)10 + (a_0 - b_0)$. Now use the test for divisibility by 9.
39. Use the appropriate divisibility tests.
41. Mimic Example 8.
43. Observe that the mapping ϕ from $Z_n[x]$ is isomorphic to Z_n given by $\phi(f(x)) = f(0)$ is a ring-homomorphism onto Z_n with kernel $\langle x \rangle$ and use Theorem 15.3.
45. The ring homomorphism from $Z \oplus Z$ to Z given by $\phi(a, b) = a$ takes (1, 0) to 1. Or define ϕ from Z_6 to Z_6 by $\phi(x) = 3x$ and let $R = Z_6$ and $S = \phi(Z_6)$. Then 3 is a zero-divisor in R and $\phi(3) = 3$ is the unity of S.
47. Observe that $(2 \cdot 10^{75} + 2) \bmod 3 = 1$ and $(10^{100} + 1) \bmod 3 = 2 = -1 \bmod 3$.
49. This follows directly from Theorem 13.3 and Theorem 10.1, part 3.
51. No. The kernel must be an ideal.
53. **a.** Suppose $ab \in \phi^{-1}(A)$. Then $\phi(a)\phi(b) \in A$, so that $a \in \phi^{-1}(A)$ or $b \in \phi^{-1}(A)$.
    b. Consider the natural homomorphism from R to S/A. Then use Theorems 15.3 and 14.4.
55. **a.** $\phi((a, b) + (a', b')) = \phi((a + a', b + b')) = a + a' = \phi((a, b)) + \phi((a', b'))$, so ϕ preserves addition. Also, $\phi((a, b)(a', b')) = \phi((aa', bb')) = aa' = \phi((a, b))\phi((a', b'))$.
    b. $\phi(a) = \phi(b)$ implies that $(a, 0) = (b, 0)$, which implies that $a = b$. $\phi(a + b) = (a + b, 0) = (a, 0) + (b, 0) = \phi(a) + \phi(b)$. Also, $\phi(ab) = (ab, 0) = (a, 0)(b, 0) = \phi(a)\phi(b)$.
    c. Use $(r, s) \to (s, r)$.

57. The mapping $\phi(x) = (x \bmod m, x, \bmod n)$ from Z_{mn} to $Z_m \oplus Z_n$ is a ring isomorphism.
59. First note that $\phi(1) = 1$ implies that $\phi(m) = \phi(m \cdot 1) = m\phi(1) = m$. Now let $\phi(\sqrt[3]{2}) = a$. Then $2 = \phi(2) = \phi(\sqrt[3]{2}^3) = (\phi(\sqrt[3]{2}))^3$ and therefore $\phi(\sqrt[3]{2}) = \sqrt[3]{2}$.
61. Use Exercises 52 and 60.
63. If $a/b = a'/b'$ and $c/d = c'/d'$, then $ab' = ba'$ and $cd' = dc'$. So, $acb'\,d' = (ab')(cd') = (ba')(dc') = bda'\,c'$. Thus, $ac/bd = a'c'/b'd'$ and therefore $(a/b)(c/d) = (a'/b')(c'/d')$.
65. First note that any field containing Z and i must contain $Q[i]$. Then prove $(a + bi)/(c + di) \in Q[i]$.
67. The subfield of E is $\{ab^{-1} | a, b \in D, b \neq 0\}$.
69. Reflexive and symmetric properties follow from the commutativity of D. For transitivity, assume $a/b \equiv c/d$ and $c/d \equiv e/f$. Then $adf = (bc)f = b(cf) = bde$, and cancellation yields $af = be$.
71. Try $ab^{-1} \to a/b$.
73. The mapping $a + bi \to a \to bi$ is a ring isomorphism of $\mathbf{C}$.
75. Certainly the unity 1 is contained in every subfield. So, if a field has characteristic p, the subfield $\{0, 1, \ldots, p - 1\}$ is contained in every subfield. If a field has characteristic 0, then $\{(m \cdot 1)(n \cdot 1)^{-1} \,|\, m, n \in Z, n \neq 0\}$ is a subfield contained in every subfield. This subfield is isomorphic to Q [map$(m \cdot 1)(n \cdot 1)^{-1}$ to m/n].

Chapter 16

You know my methods. Apply them!

SHERLOCK HOMES
The Hound of the Baskervilles[1]

1. $f + g = 3x^4 + 2x^3 + 2x + 2$; $f \cdot g = 2x^7 + 3x^6 + x^5 + 2x^4 + 3x^2 + 2x + 2$.
3. 1, 2, 4, 5
5. The only place in the proof of Theorem 16.2 and its corollaries that uses the fact the coefficients were from a field is where we used the multiplicative inverse of lead coefficient b_m of $g(x)$.
7. Note the functions defined by $f(x) = x^3, x^5, x^7, \ldots$, are the same one defined by $f(x) = x$ and the ones defined by $f(x) = x^4, x^6, x^8, \ldots$, are the same one defined by $f(x) = x^2$. So all such terms may be replaced by x and x^2. In the general case note that by Fermat's Little Theorem (Corollary 5 to Theorem 7.1) the function from Z_p to Z_p defined by $g(x) = x^p$ is the same as the function $f(x) = x$ from Z_p to Z_p. So, every polynomial function with coefficients from Z_p can be written in the form $a_{p-1}x^{p-1} + \cdots + a_0$ where $a_{p-1}, \ldots, a_0 \in Z_p$.
9. $(x - 1)^2(x - 2)$
11. $4x^2 + 3x + 6$ is the quotient and $6x + 2$ is the remainder.
13. Let $f(x), g(x) \in R[x]$. By inserting terms with the coefficient 0, we may write $f(x) = a_nx^n + \cdots + a_0$ and $g(x) = b_nx^n + \cdots + b_0$. Then

$$\begin{aligned}\overline{\phi}(f(x) + g(x)) &= \phi(a_n + b_n)x^n + \cdots + \\ &\quad \phi(a_0 + b_0) \\ &= (\phi(a_n) + \phi(b_n))x^n + \\ &\quad \cdots + \phi(a_0) + \phi(b_0) \\ &= (\phi(a_n)x^n + \cdots + \phi(a_0)) + \\ &\quad (\phi(b_n)x^n + \cdots + \phi(b_0)) \\ &= \overline{\phi}(f(x)) + \overline{\phi}(g(x)).\end{aligned}$$

Multiplication is done similarly.
15. Note that $(2x^n + 1)^2 = 1$ and $(2x^n)^2 = 0$ for all n.
17. It is its own inverse.
19. If $f(x) = a_nx^n + \cdots + a_0$ and $g(x) = b_mx^m + \cdots + b_0$, then $f(x) \cdot g(x) = a_nb_mx^{m+n} + \cdots + a_0b_0$.

21. Let m be the multiplicity of b in $q(x)$. Then we may write $f(x) = (x-a)^n(x-b)^m q'(x)$, where $q'(x)$ is in $F[x]$ and $q'(b) \neq 0$. This means that b is a zero of $f(x)$ of multiplicity at least m. If b is a zero of $f(x)$ greater than m, then b is a zero of $g(x) = f(x)/(x-b)^m = (x-a)^n q'(x)$. But then $0 = g(b) = (b-a)^n q'(b)$, and therefore $q'(b) = 0$.
23. Apply the definitions of function addition and multiplication and ring homomorphism.
25. Since $f(2) = 16 - 4 - 2 = 10,\ p = 2$ or 5.
27. By observation, $U(2)$ and $U(3)$ are cyclic. If $U(p)$ is not cyclic and $p > 3$, then by the Fundamental Theorem of Finite Abelian groups there is some prime q such that $U(p)$ has a subgroup isomorphic to $Z_q \oplus Z_q$. But the polynomial $x^q - 1$ in $Z_q[x]$ has $q^2 - 1$ zeros, which contradicts Theorem 16.3.
29. In Z_{10}, let $f(x) = 5x$. Then $0, 2, 4, 6, 8$ are zeros.
31. If $(f(x)/g(x))^2 = x$ then $x^2(k(x))^2 = x(g(x))^2$. But the right side has even degree whereas the left side has odd degree.
33. Suppose that $f(x) \in D[x]$ has degree at least 1 and there is a $g(x) \in D[x]$ such that $f(x)g(x) = 1$. Then by Exercise 19 we have $0 = \deg f(x)g(x) =$ $\deg\ (f(x) + \deg g(x) \geq 1$.
35. For char 2, 1 is a zero; char 3, 2 is a zero; char $p > 3$, 3 is a zero.
37. By Theorem 16.5, $g(x) = (x-1)(x-2)$.
39. First note that $-1 = 16$ is zero. Since $x^9 + 1 = 0$ implies $x^{18} = 1$ in the group $U(17)$, for any solution a of $x^9 + 1 = 0$ in the group $U(17)$ we know that $|a|$ must divide 18 and $|a|$ must divide $|U(17)| = 16$. This gives us $|a| = 2$ and $a^9 + 1 = a + 1$ so that $a = -1$. Because $U(17)$ is cyclic, 16 is the unique element of order 2. $Q[x]/\langle x^2+1\rangle$ is isomorphic to $Q(i)$.
41. Since -1 is a zero of $x^{25} + 1,\ x + 1$ is a factor. Suppose that $x^{25} + 1 = (x+1)^2 g(x)$ for some $g(x) \in Z_{37}[x]$. Then the derivative $f'(x) = 25x^{24} =$ $(x+1)^2 g'(x) + g(x)2(x+1)$. This gives $f'(-1) = 25 = 0$, which is false.
43. *Hint:* $F[x]$ is a PID. So $\langle f(x), g(x)\rangle = \langle a(x)\rangle$ for some $a(x) \in F[x]$. Thus, $a(x)$ divides both $f(x)$ and $g(x)$. This means that $a(x)$ is a constant.
45. Suppose that $I = \langle f(x)\rangle$. Then there is some $g(x) \in Z[x]$ such that $2 = f(x)g(x)$. This implies that $f(x) = \pm 2$. But $x + 2 \in I$ and is not in $\langle 2\rangle$.
47. If $f(x) \neq g(x)$, then deg $[f(x) - g(x)] <$ deg $p(x)$. But the minimum degree of any member of $\langle p(x)\rangle$ is deg $p(x)$.
49. For any positive integer k that are at most k zeros of $x^k - 1$. So, there are at most k elements in the field that are solutions to $x^k = 1$.
51. "Long divide" $x - a$ into $f(x)$ and induct on deg $f(x)$.
53. By Theorem 16.5, $I = \langle x - 1\rangle$.
55. Observe that every term of $f(a)$ has the form $c_i a^i$ and $c_i a^i$ mod $m = c_i b^i$mod m. To prove the second statement, assume that there is some integer k such that $f(k) = 0$. If k is even, then because k mod $2 = 0$, we have by the first statement $0 = f(k)$ mod $2 = f(0)$ mod 2 so that $f(0)$ is even. This shows that k is not even. If k is odd, then $k \bmod 2 = 1$, so by the first statement $f(k) = 0$ is odd. This contradiction completes the proof .
57. For any a in $U(p), a^{p-1} = 1$, so every member of $U(p)$ is a zero of $x^{p-1} - 1$. Now use the Factor Theorem and a degree argument.
59. Use Exercise 58.
61. Let $x^{48} + x^{21} + a$. Since $x + 4 = x - 1 \in Z_5[x]$ from the Factor Theorem we need only find an a in Z_5 such that $f(1) = 1 + 1 + a = 0$.

63. $\mathbf{C}(x)$ (field of quotients of $\mathbf{C}[x]$).
65. Take $R = Z$ and $I = \langle 2 \rangle$.
67. A solution to $x^{25} - 1 = 0$ in Z_{37} is a solution to $x^{25} = 1$ in $U(37)$. So, by Corollary 2 of Theorem 4.1, $|x|$ divides 25. Moreover, we must also have that $|x|$ divides $|U(37)| = 36$. So, $|x| = 1$ and therefore $x = 1$.
69. Write $f(x) = (x - a)g(x)$. Use the product rule to compute $f'(x)$.
71. Say deg $g(x) = m$, deg $h(x) = n$, and $g(x)$ has leading coefficient a. Let $k(x) = g(x) - ax^{m-n}h(x)$. Then deg $k(x) <$ deg $g(x)$ and $h(x)$ divides $k(x)$ in $Z[x]$ by induction. So, $h(x)$ divides $k(x) + ax^{m-n}h(x) = g(x)$ in $Z[x]$.
73. If $f(x)$ takes on only finitely many values, then there is at least one a in Z with the property that $f(x) = a$ for infinitely many x in Z. But then $g(x) = f(x) - a$ has infinitely many zeros. This contradicts Corollary 3 of Theorem 16.2.
75. Use Theorem 15.3, Theorem 14.4, and Example 14 in Chapter 14.
77. Let $f(x) = a_nx^n + a_{n-1}x^{n-1} + \cdots + a_1x + a_0$ and assume that p/q is a zero of $f(x)$, where p and q are integers and n is even. We may assume that p and q are relatively prime. Substituting p/q for x and clearing fractions, we have $a_np^n + a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} = -a_0q^n$. If p is even, then the left side is even. If p is odd, then each summand on the left side is odd and since there is an even number of summands, the left side is still even. Because a_0 is odd, we then have that q is even. It follows that $a_np^n = -(a_{n-1}p^{n-1}q + \cdots + a_1pq^{n-1} + a_0q^n)$ is even, since the right side is divisible by q. This implies that p is even. This contradicts the assumption that p and q are relatively prime.
79. Consider the remainder when x^{43} is divided by $x^2 + x + 1$.

Chapter 17

Experience enables you to recognize a mistake when you make it again.

FRANKLIN P. JONES[2]

1. Use Theorem 17.1.
3. If $f(x)$ is not primitive, then $f(x) = ag(x)$, where a is an integer greater than 1. Then a is not a unit in $Z[x]$ and $f(x)$ is reducible.
5. **a.** If $f(x) = g(x)h(x)$, then $af(x) = ag(x)h(x)$.
   b. If $f(x) = g(x)h(x)$, then $f(ax) = g(ax)h(ax)$.
   c. If $f(x) = g(x)h(x)$, then $f(x + a) = g(x + a)h(x + a)$.
   d. Try $a = 1$.
7. Suppose that $r + 1/r = 2k + 1$ where k is an integer. Then $r^2 - 2kr - r + 1 = 0$. It follows from Exercise 4 of this chapter that r is an integer. But the mod 2 irreducibility test shows that the polynomial $x^2 - (2k + 1)\, x + 1$ is irreducible over Q and an irreducible quadratic polynomial cannot have a zero in Q.
9. Use part a Exercise 5 and clear fractions.
11. Find an irreducible polynomial $p(x)$ of degree 2 over Z_5. Then $Z_5[x]/\langle p(x)\rangle$ is a field of order 25.
13. Note that -1 is a zero. No, since 4 is not a prime.
15. x; $x + 1$ What can you conclude about a polynomial $f(x)$ in $Z[x]$ if $f(x)$ is reducible over $Z_p[x]$ for every prime p (i.e., when the coefficients of $f(x)$ are reduced modulo p)?
17. $f(x)$ is irreducible over Q. Nothing.
19. $|x + I| = 12$; $|x + 1 + I| = 48$; $(x + I)^{-1} = 3x + I$.
21. Let $f(x) = x^4 + 1$ and $g(x) = f(x+1) = x^4+4x^3+6x^2+4x+2$. Then $f(x)$ is irreducible over Q if $g(x)$ is. Eisenstein's Criterion shows that $g(x)$ is irreducible over Q. To see that

x^4+1 is reducible over $\mathbf{R}$, observe that $x^8-1=(x^4+1)(x^4-1)$, so any complex zero of x^4+1 is a complex zero of x^8-1. Also note that the complex zeros of x^4+1 must have order 8 (when considered as an element of $\mathbf{C}$). Let $\omega=\sqrt{2}/2+i\sqrt{2/2}$. Then Example 2 in Chapter 16 tells us that the complex zeros of x^4+1 are $\omega, \omega^3, \omega^5$, and ω^7, so $x^4+1=(x-\omega)(x-\omega^3)(x-\omega^5)(x-\omega^7)$. But we may pair these factors up as $((x-\omega)(x-\omega^7))((x-\omega^3)(x-\omega^5))=(x^2-\sqrt{2}\,x+1)\cdot(x^2+\sqrt{2}\,x+1)$ to factor using reals (see DeMoivre's Theorem, Example 15 in Chapter 0).

23. $(x+3)(x+5)(x+6)$
25. By the Mod 2 Irreducibility Test (Theorem 17.3 with $p=2$) it is enough to show that x^4+x^3+1 is irreducible over Z_2. By inspection, x^4+x^3+1 has no zeros in Z_2 and so it has no linear factors over Z_2. The only quadratic irreducible in $Z_2[x]$ is x^2+x+1 and it is ruled out as a factor by long division.
27. **a.** Consider the number of distinct expressions of the form $(x-c)(x-d)$. **b.** Reduce the problem to the case considered in part a.
29. Use Exercise 28, and imitate Example 10.
31. Map $Z_3[x]$ onto $Z_3[i]$ by $f(x)\to f(i)$. This is a ring homomorphism with kernel $\langle x^2+1\rangle$.
33. x^2+1, x^2+x+2, x^2+2x+2
35. We know that $a_n(r/s)^n+a_{n-1}(r/s)^{n-1}+\cdots+a_0=0$. So $a_nr^n+sa_{n-1}r^{n-1}+\cdots+s^na_0=0$. This shows that $s\,|\,a_nr^n$ and $r\,|\,s^na_0$. Now use Euclid's Lemma and the fact that r and s are relatively prime.
37. Suppose that $p(x)$ can be written in the form $g(x)h(x)$ where deg $g(x)<$ deg $p(x)$ and deg $h(x)<$ deg $p(x)$ with $g(x), h(x)\in F[x]$. By Theorem 14.4 $F[x]/\langle p(x)\rangle$ is a field with $0+\langle p(x)\rangle=p(x)+\langle p(x)\rangle=g(x)h(x)+\langle p(x)\rangle=(g(x)+\langle p(x)\rangle)(h(x)+\langle p(x)\rangle)$. Thus $g(x)+\langle p(x)\rangle=0+\langle p(x)\rangle$ or $h(x)+\langle p(x)\rangle=0+\langle p(x)\rangle$. This implies that $g(x)\in\langle p(x)\rangle$ or $h(x)\in\langle p(x)\rangle$. In either case we have contradicted Theorem 16.4.
39. Since $(f+g)(a)=f(a)+g(a)$ and $(f\cdot g)\ (a)=f(a)g(a)$, the mapping is a homomorphism. Clearly, $p(x)$ belongs to the kernel. By Theorem 17.5, $\langle p(x)\rangle$ is a maximal ideal, so the kernel is $\langle p(x)\rangle$.
41. The mapping $a\to a+\langle p(x)\rangle$ is an isomorphism.
43. $f(x)$ is primitive.

Chapter 18

If you are going through hell, keep going.

Winston Churchill

1. 1. $|a^2-db^2|=0$ implies $a^2=db^2$. Thus, $a=0=b$, since otherwise $d=1$ or d is divisible by the square of a prime.
   2. $N((a+b\sqrt{d})(a'+b'\sqrt{d}))=N(aa'+dbb'+(ab'+a'b)\sqrt{d})=|(aa'+dbb')^2-d(ab'+a'b)^2|=|a^2a'^2+d^2b^2b'^2-da^2b'^2-da'^2b^2|=|a^2-db^2||a'^2-db'^2|=N(a+b\sqrt{d})N(a'+b'\sqrt{d})$.
   3. If $xy=1$, then $1=N(1)=N(xy)=N(x)N(y)$ and $N(x)=1=N(y)$. If $N(a+b\sqrt{d})=1$, then $\pm1=a^2-db^2=(a+b\sqrt{d})(a-b\sqrt{d})$ and $a+b\sqrt{d}$ is a unit.
   4. This property follows directly from properties 2 and 3.
3. Let $I=\cup I_i$. Let $a, b\in I$ and $r\in R$. Then $a\in I_i$ for some i and $b\in I_j$ for some j. Thus, $a, b\in I_k$, where $k=\max\{i, j\}$. So, $a-b\in I_k\subseteq I$ and $ra, ar\in I_k\subseteq I$.
5. Clearly, $\langle ab\rangle\subseteq\langle b\rangle$. If $\langle ab\rangle=\langle b\rangle$, then $b=rab$, so that $1=ra$ and a is a unit. If a is a unit then $b=a^{-1}(ab)\in\langle ab\rangle$. Thus $\langle b\rangle\subseteq\langle ab\rangle$
7. Say $x=a+bi$ and $y=c+di$. Then

$$xy=(ac-bd)+(bc+ad)i.$$

So

$$d(xy) = (ac - bd)^2 + (bc + ad)^2 = (ac)^2 + (bd)^2 + (bc)^2 + (ad)^2.$$

On the other hand,

$$d(x)d(y) = (a^2 + b^2)(c^2 + d^2) = a^2c^2 + b^2d^2 + b^2c^2 + a^2d^2.$$

9. Suppose $a = bu$, where u is a unit. Then $d(b) \leq d(bu) = d(a)$. Also, $d(a) \leq d(au^{-1}) = d(b)$.
11. Use the fact that x is a unit if and only if $N(x) = 1$.
13. $3 \cdot 7$ and $(1 + 2\sqrt{-5})(1 - 2\sqrt{-5})$. Mimic Example 8 to show that these are irreducible.
15. Observe that $10 = 2 \cdot 5$ and $10 = (2 - \sqrt{-6})(2 + \sqrt{-6})$ and mimic Example 8. A PID is a UFD.
17. Suppose $3 = \alpha\beta$, where $\alpha, \beta \in Z[i]$ and neither is a unit. Then $9 = d(3) = d(\alpha)d(\beta)$, so that $d(\alpha) = 3$. But there are no integers such that $a^2 + b^2 = 3$. Observe that $2 = -i(1 + i)^2$ and $5 = (1 + 2i)(1 - 2i)$.
19. Use Exercise 1 with $d = -1.5$ and $1 + 2i$; 13 and $3 + 2i$; 17 and $4 + i$.
21. Mimic Example 1.
23. $(-1 + \sqrt{5})(1 + \sqrt{5}) = 4 = 2 \cdot 2$. Now use Exercise 22.
25. $m = 0$ and $n = -1$ give $q = -i$, $r = -2 - 2i$.
27. $1 = N(ab) = N(a)N(b)$, so that $N(a) = 1 = N(b)$.
29. Suppose that $bc = pt$ in Z_n. Then there exists an integer k such that $bc = pt + kn$. This implies that p divides bc in Z, and by Euclid's Lemma we know that p divides b or p divides c.
31. See Example 3.
33. $p|(a_1a_2 \cdots a_{n-1})a_n$ implies that $p|a_1a_2 \cdots a_{n-1}$ or $p|a_n$. Thus, by induction, p divides some a_i.
35. Use Exercise 10 and Theorem 14.4.
37. Suppose R satisfies the ascending chain condition and there is an ideal I of R that is not finitely generated. Then pick $a_1 \in I$. Since I is not finitely generated, $\langle a_1 \rangle$ is a proper subset of I, so we may choose $a_2 \in I$ but $a_2 \notin \langle a_1 \rangle$. As before, $\langle a_1, a_2 \rangle$ is proper, so we may choose $a_3 \in I$ but $a_3 \notin \langle a_1, a_2 \rangle$. Continuing in this fashion, we obtain a chain of infinite length $\langle a_1 \rangle \subset \langle a_1, a_2 \rangle \subset \langle a_1, a_2, a_3 \rangle \subset \cdots$. Now suppose every ideal of R is finitely generated and there is a chain $I_1 \subset I_2 \subset I_3 \subset \cdots$. Let $I = \cup I_i$. Then $I = \langle a_1, a_2, \ldots, a_n \rangle$. Since $I = \cup I_i$, each a_i belongs to some member of the union, say $I_{i'}$. Letting $k = \max\{i' | i = 1, \ldots, n\}$, we see that all $a_i \in I_k$. Thus, $I \subseteq I_k$ and the chain has length at most k.
39. Say $I = \langle a + bi \rangle$. Then $a^2 + b^2 + I = (a + bi)(a - bi) + I = I$ and $a^2 + b^2 \in I$. For any $c, d \in Z$, let $c = q_1(a^2 + b^2) + r_1$ and $d = q_2(a^2 + b^2) + r_2$, where $0 \leq r_1, r_2 < a^2 + b^2$. Then $c + di + I = r_1 + r_2i + I$.
41. $N(6 + 2\sqrt{-7}) = 64 = N(1 + 3\sqrt{-7})$. For the other part, use Exercise 25.
43. Theorem 18.1 shows that primes are irreducible. So, assume that a is an irreducible in a UFD R and that $a|bc$ in R. We must show that $a|b$ or $a|c$. Since $a|bc$, there is an element d in R such that $bc = ad$. Now replace b, c, and d by their factorizations as a product of irreducibles and use uniqueness.
45. See Exercise 21 in Chapter 0.
47. $13 = (2 + 3i)(2 - 3i)$; $5 + i = (1 + i)((3 - 2i)$
49. The case that $I = R$ is trivial. So we can write $I = \langle a \rangle$ where a is not a zero or a unit. Let J/I be any non-trivial ideal in R/I and let $J = \langle b \rangle$. Since I properly contains I we have that $a = br$ where r is not a unit. Then from Theorem 18.3 we know that a can be written uniquely (up to associates) as a product of irreducibles in R. So there are only finite possibilities for b.

Chapter 19

All things are difficult before they are easy.

THOMAS FULLER

1. $\{a5^{2/3} + b5^{1/3} + c \mid a, b, c \varepsilon Q\}$.
3. $Q(\sqrt{-3})$
5. $Q(\sqrt{-3})$
7. Since $ac + b \in F(c)$ we have $F(ac + b) \subseteq F(c)$. But $c = a^{-1}(ac + b) - a^{-1}$b, so $F(c) \subseteq F(ac + b)$.
9. $a^5 = a^2 + a + 1; a^{-2} = a^2 + a + 1; a^{100} = a^2$
11. The set of all expressions of the form
$$(a_n\pi^n + a_{n-1}\pi^{n-1} + \cdots + a_0)/(b_m\pi^m + b_{m-1}\pi^{m-1} + \cdots + b_0),$$
where $b_m \neq 0$.
13. $x^7 - x = x(x^6 - 1) = x(x^3 + 1)(x^3 - 1) = x(x - 1)^3(x + 1)^3; x^{10} - x = x(x^9 - 1) = x(x - 1)^9$ (see Exercise 49 in Chapter 13).
15. *Hint:* Use Exercise 49 in Chapter 13.
17. $a = 4/3,\ b = 2/3,\ c = 5/6$
19. Use the fact that $1 + i = -(4 - i) + 5$ and $4 - i = 5 - (1 + i)$.
21. If the zeros of $f(x)$ are $a_1, a_2, \ldots, a_n$, then the zeros of $f(x + a)$ are $a_1 - a, a_2 - a, \cdots, a_n - a$. Now use Exercise 20.
23. Q and $Q(\sqrt{2})$
25. 64
27. Let $F = Z_3[x]/\langle x^3 + 2x + 1\rangle$ and denote the cosets $x + \langle x^3 + 2x + 1\rangle$ by β and $2 + \langle x^3 + 2x + 1\rangle$ by 2. Then $x^3 + 2x + 1 = (x - \beta)(x - \beta - 1)(x + 2\beta + 1)$.
29. Suppose that $\phi: Q(\sqrt{-3}) \to Q(\sqrt{3})$ is an isomorphism. Since $\phi(1) = 1$, we have $\phi(-3) = -3$. Then $-3 = \phi(-3) = \phi(\sqrt{-3}\sqrt{-3}) = (\phi(\sqrt{-3})^2$. This is impossible, since $\phi(\sqrt{-3})$ is a real number.
31. Use long division.
33. Use Theorem 19.5.
35. Use Theorem 19.5.
37. Let K be the intersection of all subfields of E that contain F and the set $\{a_1, a_2, \ldots, a_n\}$. It follows from the subfield test given in Exercise 29 Chapter 13 that K is a subfield of E and, by the definition, that K contains F and the set $\{a_1, a_2, \ldots, a_n\}$. Since $F(a_1, a_2, \ldots, a_n)$ is the smallest such field we have $F(a_1, a_2, \ldots, a_n) \subseteq K$. Moreover, since the field $F(a_1, a_2, \ldots, a_n)$ is one member of the intersection we have $K \subseteq F(a_1, a_2, \ldots, a_n\}$. This proves that $K = F(a_1, a_2, \ldots, a_n\}$.
39. Since $|(Z_2[x]/\langle f(x)\rangle)^*| = 31$, every nonidentity is a generator.
41. Use the Fundamental Theorem of Field Theory (Theorem 19.1) and the Factor Theorem (Corollary 2 of Theorem 16.2).
43. Mimic the argument given in Example 9 of this chapter.
45. Since $-1 = 1$, $x^n - x$ would have 1 as a multiple zero. But then, by Theorem 19.5, $x^n - x$ and its derivative, which is $-1 = 1$, must have a common factor of positive degree. This is impossible.
47. $Q(\sqrt[3]{2})(\omega)$ where $\omega = -1/2 + \sqrt{3}i/2$; $Q(\sqrt{3}i)(\sqrt[3]{2})$.
49. Observe that the polynomial $x^2 - 2x - 1$ is irreducible over Z_5. So Theorems 19.1 and 19.3 shows that such a field exists.
51. If $\alpha \in F(\beta)$, then we have $\alpha = a\beta + b$ for some $a, b \in F$. Squaring both sides, replacing β^2 with $-\beta - 1$, and solving for β, we find that $\beta \in F$. For the second part, if $\beta \in F(\alpha)$ we have $\beta = a\alpha + b$ for some $a, b \in F$. Solving for α in terms of β and proceding as before we get that β is in F.

Chapter 20

A good proof is one which makes us wiser.

YU. MANIN

1. It follows from Theorem 20.1 that if $p(x)$ and $q(x)$ are both monic

irreducible polynomials in $F[x]$ with $p(a) = q(a) = 0$, then deg $p(x) =$ deg $q(x)$. If $p(x) \neq q(x)$, then $(p - q)(a) = p(a) - q(a) = 0$ and deg $(p(x) - q(x)) <$ deg $p(x)$, contradicting Theorem 20.1. To prove Theorem 20.3, use the Division Algorithm for $F[x]$ (Theorem 16.2).

3. Note that $[Q(\sqrt[n]{2}):Q] = n$ and use Theorem 20.5.
5. Use Exercise 4.
7. Suppose $Q(\sqrt{a}) = Q(\sqrt{b})$. If $\sqrt{b} \in Q$, then $\sqrt{a} \in Q$ and we may take $c = \sqrt{a}/\sqrt{b}$. If $\sqrt{b} \notin Q$, then $\sqrt{a} \notin Q$. Write $\sqrt{a} = r + s\sqrt{b}$. It follows that $r = 0$ and $a = bs^2$. The other direction follows from Exercise 7 in Chapter 19.
9. Since $[F(a) : F] = 5$, $\{1, a, a^2, a^3, a^4\}$ is a basis for $F(a)$ over F. Also, from $5 = [F(a) : F] = [F(a) : F(a^3)][F(a^3) : F]$ we know that $[F(a^3) : F] = 1$ or 5. However, $[F(a^3) : F] = 1$ implies that $a^3 \in F$ and therefore the elements $1, a, a^2, a^3, a^4$ are not linearly independent over F. So, $[F(a^3) : F] = 5$.
11. If a is a zero of $f(x)$ in E then $n = [E : F] = [E : F(a)][F(a) : F] = [E : F](\deg f(x))$.
13. $g(x) = f(x/b - c/b)$
15. By the Primitive Element Theorem there is an element b in $F(a_1, a_2)$ such that $F(a_1, a_2) = F(b)$. Then, by induction on n, there is an element c in $F(b, a_3, \ldots, a_n)$ such that $F(c) = F(b, a_3, \ldots, a_n) = F(a_1, a_2, \ldots, a_n)$.
17. 6; 3; 2.
19. They are the same.
21. If $b = 0$ for then $x - c$ is minimal. If $b \neq 0$ for $g(x) = f((x - c)/b)$ we have $g(ab+c) = f((ab+c-c)/b) = f(a) = 0$.
23. If an irreducible polynomial $p(x)$ in $\mathbf{R}[x]$ has degree n and a is a zero in $\mathbf{C}$ of $p(x)$ then $2 = [\mathbf{C}:\mathbf{R}] = [\mathbf{C}: \mathbf{R}(a)][\mathbf{R}(a):\mathbf{R}] = [\mathbf{C}: \mathbf{R}(a)]n$. So, $n = 1$ or 2.
25. $Q(\sqrt[4]{2})$.
27. Suppose that $[E : F] = 1$. Because $\{1\}$ is a linearly independent set over F it is a basis for E over F. So every element of E has the form $a \cdot 1 = a$ for some a in F. If $E = F$, then $\{1\}$ is a basis of E over F and therefore $[E : F] = 1$.
29. Pick a in K but not in F. Now use Theorem 20.5.
31. Mimic Example 5.
33. Mimic Example 6.
35. Suppose $E_1 \cap E_2 \neq F$. Then $[E_1{:}E_1 \cap E_2]\ [E_1 \cap E_2{:}F] = [E_1{:}F]$ implies $[E_1{:}E_1 \cap E_2] = 1$, so that $E_1 = E_1 \cap E_2$. Similarly, $E_2 = E_1 \cap E_2$.
37. Observe that $F(a) = F(1 + a^{-1})$.
39. We need only show that if $a \in R$, then $a^{-1} \in R$. But $a^{-1} \in F(a) \subseteq R$ (see Theorem 19.3).
41. Every element of $F(a)$ can be written in the form $f(a)/g(a)$, where $f(x), g(x) \in F[x]$. If $f(a)/g(a)$ is algebraic and not a member of F, then there is some $h(x) \in F[x]$ such that $h(f(a)/g(a)) = 0$. By clearing fractions and collecting like powers of a, we obtain a polynomial in a with coefficients from F equal to 0. But then a would be algebraic over F.
43. Note that a is a zero of $x^3 - a^3$ over $F(a^3)$. For the second part, take $F = Q, a = 1$; $F = Q, a = (-1 + i\sqrt{3})/2$; $F = Q, a = 3\sqrt{2}$.
45. E must be an algebraic extension of R, so that $E \subseteq$ C. But then $[\mathrm{C}{:}E][E{:}\mathrm{R}] = [\mathrm{C}{:}\mathrm{R}] = 2$.
47. Let a be a zero of $p(x)$ in some extension of F. First note $[E(a){:}E] \leq [F(a){:}F] = deg\ p(x)$. Then observe that $[E(a){:}F(a)][F(a){:}F] = [E(a){:}F] = [E(a){:}E][E{:}F]$. This implies that deg $p(x)$ divides $[E(a){:}E]$, so that deg $p(x) = [E(a){:}E]$. It now follows from Theorem 19.3 that $p(x)$ is irreducible over E.
49. *Hint:* If $\alpha + \beta$ and $\alpha\beta$ are algebraic, then so is $\sqrt{(\alpha + \beta)^2 - 4\alpha\beta}$.
51. $\sqrt{b^2 - 4ac}$
53. Because $a \in Q(\sqrt{a}\,)$ it suffices to show that $\sqrt{a} \in Q(a)$. Since $a^3 = 1$ we have $a^4 = a$. Then $a^2 = \sqrt{a} \in Q(a)$.
55. Say a is a generator of F^*. If char F

$= 0$, then the prime subfield of F is isomorphic to Q. Since Q^* is not cyclic, we have that $F = Z_p(a)$, and it suffices to show that a is algebraic over Z_p. If $a \in Z_p$, we are done. Otherwise, $1 + a = a^k$ for some $k \neq 0$. If $k > 0$, we are done. If $k < 0$, then $a^{-k} + a^{1-k} = 1$ and we are done.

57. If $[K{:}F] = n$, then there are elements $v_1, v_2, \ldots, v_n$ in K that constitute a basis for K over F. The mapping $a_1v_1 + \cdots + a_nv_n \to (a_1, \ldots, a_n)$ is a vector space isomorphism from K to F^n. If K is isomorphic to F^n, then the n elements in K corresponding to $(1, 0, \ldots, 0), (0, 1, \ldots, 0), \ldots, (0, 0, \ldots, 1)$ in F^n constitute a basis for K over F.
59. Observe that $[F(a, b){:}F(a)] = [F(a)(b){:}F(a)] \leq [F(b){:}F] \leq [F(a)(b){:}F(b)][F(b){:}F] = [F(a)(b){:}F] = [F(a, b){:}F]$.
61. Observe that $K = F(a_1, a_2, \ldots, a_n)$, where $a_1, a_2, \ldots, a_n$ are the zeros of the polynomial. Now use Theorem 20.5.
63. Elements of $Q(\pi)$ have the form $(a_m\pi^m + a_{m-1}\pi^{m-1} + \cdots + a_0)/(b_n\pi^n + b_{n-1}\pi^{n-1} + \cdots + b_0)$, where the a's and b's are rational numbers. So, if $\sqrt{2} \in Q(\pi)$, we have an expression of the form $2(b_n\pi^n + b_{n-1}\pi^{n-1} + \cdots + b_0)^2 = (a_m\pi^m + a_{m-1}\pi^{m-1} + \cdots + a_0)^2$. Equating the lead terms of both sides, we have $2b_n^2\pi^{2n} = a_m^2\pi^{2m}$. But then we have $m = n$, and $\sqrt{2}$ is equal to the rational number a_m/b_n.
65. If $f(a^m) = 0$ for some polynomial $f(x)$ in $F[X]$, then a is a zero of $g(x) = f(x^m)$ which is in $F[x]$.

Chapter 21

Difficulties strengthen the mind, as labor does the body.

SENECA

1. $[\mathrm{GF}(729){:}\mathrm{GF}(9)] = 3$; $[\mathrm{GF}(64){:}\mathrm{GF}(8)] = 2$
3. The lattice of subfields of GF(64) looks like Figure 21.3 with GF(2) at the bottom, GF(64) at the top, and GF(4) and GF(8) on the sides.
5. $\mathrm{GF}(2^6)$
7. $2a + 1$
9. Since each binomial coefficient $\binom{p^i}{j}$ other than $j = 1$ and $j = p^i$ is divisible by p we have $(a + b)^p = a^{p^i} + b^{p^i}$. Clearly $(ab)^{p^i} = a^{p^i}b^{p^i}$. Since $\mathrm{GF}(p^{p^i})$ is a field and $a^{p^i} = 0$ only when $a = 0$ so the Ker $\phi = \{0\}$.
11. Observe that $x^2 + 1 = (x + 1)^2$
13. The only possibilities for $f(x)$ are $x^3 + x + 1$ and $x^3 + x^2 + 1$. See Exercise 8 in Chapter 19 for the first case. See Example 2 in this chapter for the second case.
15. By Theorem 21.4 an element of GF(64) has the desired property if and only if it is not in GF(4) or GF(8). Since GF(4) is not a subgroup of GF(8) their intersection is $\{0, 1\}$. This means there are 54 elements with the property. Given that a^7 in GF(16) is a zero of the irreducible polynomial of degree 4 over Z_2, find the other three zeros.
17. Let $|F| = p^n$. Then n must be divisible by both 4 and 6. So, by Theorem 21.4, F must also have subfields of order p^{12}, p^3, p^2 and p.
19. Because $|\mathrm{GF}(8)^*| = 7$, $\mathrm{GF}(2)(a) = \mathrm{GF}(8)$.
21. The statement is trivially true for 0 and 1. Since $a^{16} = a^4$ implies that $a^{12} = 1$ we know that $|a|$ divides 12. But $|a|$ also divides $|\mathrm{GF}(2^n)^*| = 2^n - 1$, which is odd. So, $|a| = 3$ and the statement follows.
23. First note that $x, x - 1$ and $x - 2$ are factors and that are the only linear factors. If $p(x)$ is an irreducible factor of $x^{27} - x$ then and a is a zero of an irreducible factor $x^{27} - x$ of degree d then $Z_3(a)$ is a subfield of GF(27) of order 3^d and by Theorem 21.4 we have that $d = 1$ or 3. So, the irreducible factorization of $x^{27} - x$ consists of

three linear factors and eight cubic factors.

25. $\mathrm{GF}(p^{2n})$.
27. To show that F is field let $a, b \in F$. Then $a \in F_i$ for some i and $b \in F_j$ for some j and $a, b \in F_k$ where k is the maximum of i and j. It follows that $a - b \in F_k, ab \in F_k$ and $a^{-1} \in F_k$ when $a \neq 0$.
29. Since 0, 1, and 2 are zeros we know $x, x-1$ and $x-2$ are factors and by Theorem 19.8 these each have multiplicity 1. Theorem 21.5 tells us that all other irreducible factors have degree 2 and, since their degrees must sum to six, there are three of them. Finally, from the proof of Theorem 21.1 we know that each of the nine elements of GF(9) are zeros of $x^9 - x$. So no two quadratic reducibles can be the same. The factorization $x^9 - x = x(x-1)(x-2)(x^2+1)(x^2+x+2)(x^2+2x+2)$.
31. Direct calculations show that given $x^3 + 2x + 1 = 0$, we have $x^2 \neq 1$ and $x^{13} \neq 1$.
33. Direct calculations show that $x^{13} = 1$, whereas $(2x)^2 \neq 1$ and $(2x)^{13} \neq 1$. Thus, $2x$ is a generator.
35. First observe that for any field F, the set F^* is a group under multiplication. Now use Theorem 21.2 and Theorem 4.3.
37. Let $a, b \in K$. Then, by Exercise 49b in Chapter 13, $(a-b)^{p^m} = a^{p^m} - b^{p^m} = a - b$. Also, $(ab)^{p^m} = a^{p^m} b^{p^m} = ab$. So, K is a subfield.
39. Consider $x^{p^n - 1} - 1$ and use Corollary 4 of Lagrange's Theorem (Theorem 7.1).
41. Structurally identical
43. Consider $g(x) = x^2 - a$. Note that $|GF(p)[x]/\langle g(x)\rangle| = p^2$, so that $g(x)$ has a zero in $\mathrm{GF}(p^2)$. Now use Theorem 21.3.
45. F^*; $\mathrm{GF}(2^5)^* = \langle a^{33}\rangle$; $\mathrm{GF}(2^2)^* = \langle a^{341}\rangle$; $\mathrm{GF}(2)^* = \langle a^{1023}\rangle = \{1\}$.
47. Since F^* is a cyclic group of order 124, it has a unique element of order 2.
49. Use Corollary 2 of Theorem 21.2.
51. Observe that $p - 1 = -1$ has multiplicative order 2 and $a^{(p^n-1)/2}$ is the unique element in $\langle a\rangle$ of order 2.
53. Since $0 = 0^2$ it suffices to show that in F^* there are exactly $(p^n - 1)/2$ elements that can be written as a square of an element of F^*. Observe that the mapping from the cyclic group F^* to itself that takes x to x^2 is a group homomorphism with the kernel $\{\pm 1\}$. So, the mapping is 2-1.
55. Since $p = 1 \pmod 4$ we have $p^n = 1$ mod 4 and $\mathrm{GF}(p^n)^*$ is a cyclic of order $p^n - 1$. So, by Theorem 4.4 there is exactly two elements of order 4.
57. First note that is a not in Z_5 for then $x - a$ would be a factor of the irreducible. Then taking $b = 0$ and we solve for $c = (4a+1)(3a+2)^{-1}$.
59. It is a field of order 4^5.
61. For the case $\mathrm{GF}(p^n)$ observe that if $1 + a + a^2 + a^3 + \cdots + a^i = 1 + a + a^2 + a^3 + \cdots + a^j$ for some $i < j$ then $0 = a^{i+1} + \cdots + a^j = a^{i+1}(1 + a + a^2 + \cdots + a^{j-i-1})$ and therefore a^{i+1} is a zero-divisor.
63. Let $F_1 = \mathrm{GF}(p^n), F_2 = \mathrm{GF}(p^{2n}), F_3 = \mathrm{GF}(p^{4n}), F_4 = \mathrm{GF}(p^{8n}), \ldots$.
65. The algebraic closure of Z_2.
67. By Theorem 21.4, for each prime q the only proper subfield of $\mathrm{GF}(p^q)$ is $\mathrm{GF}(p)$.

Chapter 22

Why, sometimes I've believed as many as six impossible things before breakfast.

LEWIS CARROLL

1. To construct $a + b$, first construct a. Then use a straightedge and compass to extend a to the right by marking off the length of b. To construct $a - b$, use the compass to mark off a length of b

from the right endpoint of a line of length a.

3. Let y denote the length of the hypotenuse of the right triangle with base 1, and let x denote the length of the hypotenuse of the right triangle with base $|c|$. Then $y^2 = 1 + d^2, y^2 + x^2 = (1 + |c|)^2$, and $|c|^2 + d^2 = x^2$. So, $1 + 2|c| + |c|^2 = 1 + d^2 + |c|^2 + d^2$, which simplifies to $|c| = d^2$.
5. Use $\sin^2\theta + \cos^2\theta = 1$.
7. Use $\cos 2\theta = 2\cos^2\theta - 1$.
9. Use $\sin(\alpha - \beta) = \sin\alpha\cos\beta - \cos\alpha\sin\beta$ and Exercise 8.
11. Solving two linear equations with coefficients from F involves only the operations of F.
13. Use Theorem 17.1.
15. If so, then an angle of $40°$ is constructible. Now use Exercise 10.
17. This amounts to showing that $\sqrt{\pi}$ is not constructible. But if $\sqrt{\pi}$ is constructible, so is π. However, $[Q(\pi){:}Q]$ is infinite.
19. No, since $[Q(\sqrt[3]{3}){:}Q] = 3$.
21. No, since $[Q(\sqrt[3]{\pi}){:}Q]$ is infinite.

Chapter 23

Difficulty, my brethren, is the nurse of greatness.

WILLIAM CULLEN BRYANT

1. $a = eae^{-1}; cac^{-1} = b$ implies $a = c^{-1}bc = c^{-1}b(c^{-1})^{-1}; a = xbx^{-1}$ and $b = ycy^{-1}$ imply $a = xycy^{-1}x^{-1} = xyc(xy)^{-1}$.
3. Note that $|a^2| = |a|/2$ and appeal to Exercise 2.
5. Observe that $T(xC(a)) = xax^{-1} = yay^{-1} = T(yC(a))$ if and only if $y^{-1}xa = ay^{-1}x$, which is true if and only if $y^{-1}x \in C(a)$, which in turn is true if and only if $yC(a) = xC(a)$. This proves that T is well-defined and one-to-one. T is onto by definition.
7. Say $\text{cl}(e)$ and $\text{cl}(a)$ are the only two conjugacy classes of a group G of order n. Then $\text{cl}(a)$ has $n - 1$ elements all of the same order, say m. If $m = 2$, then it follows from Exercise 45 in Chapter 2 that G is Abelian. But then $\text{cl}(a) = \{a\}$ and so $n = 2$. If $m > 2$, then $\text{cl}(a)$ has at most $n - 2$ elements, since conjugation of a by e, a, and a^2 each yields a.
9. Consider the correspondence T from the left cosets of $N(H)$ in G to the conjugates of H in G given by $T(xN(H)) = xHx^{-1}$.
11. Say $\text{cl}(x) = \{x, g_1xg_1^{-1}, g_2xg_2^{-1}, \ldots, g_kxg_k^{-1}\}$. If $x^{-1} = g_ixg_i^{-1}$, then for each $g_jxg_j^{-1}$ in $\text{cl}(x)$, we have $(g_jxg_j^{-1})^{-1} = g_jx^{-1}g_j^{-1} = g_j(g_ixg_i^{-1})g_j^{-1} \in \text{cl}(x)$. Because $|G|$ has odd order, $g_jxg_j^{-1} \neq (g_jxg_j^{-1})^{-1}$. It follows that $|\text{cl}(x)|$ is even. But $|\text{cl}(x)|$ divides $|G|$.
13. Part **a** is not possible by the corollary of Theorem 23.2. Part **b** is not possible because it implies that the center would have order 2, and 2 does not divide 21. Part **c** is the class equation for D_5. Part **d** is not possible because of Corollary 1 of Theorem 23.1.
15. Use Theorem 7.2.
17. Use Example 5 of Chapter 9 and Theorem 7.2.
19. Use Theorem 23.5 and its corollary.
21. 8
23. 15
25. The number of Sylow q-subgroups has the form $1 + qk$ and divides p. So, $k = 0$.
27. A group of order 100 has 1, 5, or 25 subgroups of order 4; exactly one subgroup of order 25 (which is normal); at least one subgroup of order 5; and at least one subgroup of order 2.
29. Let H be a Sylow 5-subgroup. Since the number of Sylow 5-subgroups is 1 mod 5 and divides $7 \cdot 17$, the only possibility is 1. So, H is normal in G. Then by the N/C Theorem (Example 17 of Chapter 10), $|G/C(H)|$ divides both 4 and $|G|$. Thus $C(H) = G$.

31. By Theorem 23.6 $G/Z(G)$ would be cyclic and therefore by Theorem 9.3 G would be Abelian. But then $G = Z(G)$.
33. If p does not divide $q-1$, and q does not divide p^2-1, then a group of order p^2q is Abelian.
35. Sylow's Third Theorem implies that the Sylow 3- and Sylow 5-subgroups are unique. Pick any x not in the union of these. Then $|x| = 15$.
37. By Sylow's Third Theorem, $n_{17} = 1$ or 35. Assume $n_{17} = 35$. Then the union of the Sylow 17-subgroups has 561 elements. By Sylow's Third Theorem, $n_5 = 1$. Thus, we may form a cyclic subgroup of order 85 (Theorem 23.6). But then there are 64 elements of order 85. This gives too many elements.
39. Use the G/Z Theorem (Theorem 9.3) and Theorem 23.6.
41. Let H be the Sylow 3-subgroup and suppose that the Sylow 5-subgroups are not normal. By Sylow, there must be six Sylow 5-subgroups, call them $K_1, \ldots, K_6$. These subgroups have 24 elements of order 5. Also, each of the cyclic subgroups $HK_1, \ldots, HK_6$ has eight generators. Thus, there are 48 elements of order 15, which results in more than 60 elements in the group.
43. By Theorem 23.2 and Theorem 9.5, $Z(G)$ has an element x of order p. By induction, the group $G/\langle x\rangle$ has normal subgroups of order p^k for every k between 1 and $n-1$, inclusively. Now use Exercise 59 in Chapter 9 and Exercise 59 of Chapter 10.
45. Pick $x \in Z(G)$ such that $|x| = p$. If $x \in H$, by induction, $N(H/\langle x\rangle) > H/\langle x\rangle$, say $y\langle x\rangle \in N(H/\langle x\rangle)$ but not $H/\langle x\rangle$. Now show $y \in N(H)$ but not H. If $x \notin H$, then $x \in N(H)$, so that $N(H) > H$.
47. Since 7 does not divide $11^2 - 1 = 120$ and 11 does not divide $7^2 - 1 = 48$, we know that the Sylow 7-subgroup of G and the Sylow 11-subgroup of G are nornmal in G. Call them H and K, respectively. Then, by Example 5 of Chapter 9, Theorem 7.2, Theorem 9.6, and the Corollary to Theorem 23.2, HK is subgroup of order $7^2 \cdot 11^2$ and $HK = H \times K \approx H \oplus K$, which is Abelian by Exercise 4 of Chapter 8.
49. Sylow's Third Theorem shows that all the Sylow subgroups are normal. Then Theorem 7.2 and Example 5 of Chapter 9 ensure that G is the internal direct product of it Sylow subgroups. G is cyclic because of Theorem 9.6 and Corollary 1 of Theorem 8.2. G is Abelian because of Theorem 9.6 and Exercise 4 in Chapter 8.
51. Automorphisms preserve order.
53. That $|N(H)| = |N(K)|$ follows directly from the last part of Sylow's Third Theorem and Exercise 9.
55. Normality of H implies $\mathrm{cl}(h) \subseteq H$ for h in H. Now observe that $h \in \mathrm{cl}(h)$. This is true only when H is normal.
57. Suppose that G is a group of order 12 that has nine elements of order 2. By the Sylow theorems, G has three Sylow 2-subgroups whose union contains the identity and the nine elements of order 2. If H and K are both Sylow 2-subgroups, then by Theorem 7.2, $|H \cap K| = 2$. Thus, the union of the three Sylow 2-subgroups has at most seven elements of order 2, since there are three in H, two more in K that are not in H, and at most two more that are in the third but not in H or K.
59. By Lagrange's Theorem any nontrivial proper subgroup of G has order p or q. It follows from Theorem 23.5 and its corollary that there is exactly one subgroup of order q which is normal (for otherwise there would be $(q+1)(q-1) = q^2 - 1$ elements of order q). On the other hand, there cannot be a normal subgroup of order p for then G would be an internal direct product of a cyclic group of order q and a cyclic group of order p, which is Abelian. So, by Theorem 23.5 there must be exactly q subgroups of order p.
61. Note that any subgroup of order 4 in a group of order $4m$ where m is odd is a

Sylow 2-subgroup. By Sylow's Third Theorem, the Sylow 2-subgroups are conjugate and therefore isomorphic. S_4 contains both the subgroups $\langle(1234)\rangle$ and $\{(1),(12),(34),(12)(34)\}$.

63. By Sylow's Third Theorem, the number of Sylow 13-subgroups is equal to 1 mod 13 and divides 55. This means that there is only one Sylow 13-subgroup, so it is normal in G. Thus $|N(H)/C(H)| = 715/|C(H)|$ divides both 55 and 12. This forces $715/|C(H)| = 1$ and therefore $C(H) = G$. This proves that H is contained in $Z(G)$. Applying the same argument to K, we get that K is normal in G and $|N(K)/C(K)| = 715/|C(K)|$ divides both 65 and 10. This forces $715/|C(K)| = 1$ or 5. In the latter case, K is not contained in $Z(G)$.

65. First observe that because $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}^k = \begin{bmatrix} 1 & ka \\ 0 & 1 \end{bmatrix}$ the element $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ has order p. By Sylow's Second Theorem every element of order p is contained in a Sylow p-subgroup. So, if A is any element in $\mathrm{GL}(2, Z_p)$ of order p then A and $\begin{bmatrix} 1 & a \\ 0 & 1 \end{bmatrix}$ belong to Sylow p-subgroups and, by Sylow's Third Theorem, every two Sylow p-subgroups are conjugate.

Chapter 24

He who learns must suffer.

Aeschylus

1. Use the $2 \cdot \text{Odd}$ Test.
3. Use the Index Theorem.
5. Suppose G is a simple group of order 525. Let L_7 be a Sylow 7-subgroup of G. It follows from Sylow's theorems that $|N(L_7)| = 35$. Let L be a subgroup of $N(L_7)$ of order 5. Since $N(L_7)$ is cyclic (Theorem 23.6), $N(L) \geq N(L_7)$, so that 35 divides $|N(L)|$. But L is contained in a Sylow 5-subgroup (Theorem 23.4), which is Abelian (see the corollary to Theorem 23.2). Thus, 25 divides $|N(L)|$ as well. It follows that 175 divides $|N(L)|$. The Index Theorem now yields a contradiction.
7. $n_{11} = 12$. Use the N/C Theorem (Example 17 in Chapter 10) to show that there is an element of order 22; then use the Embedding Theorem and observe that A_{12} has no element of order 22.
9. Suppose that there is a simple group of order 396 and L_{11} is a Sylow 11-subgroup. Use the N/C Theorem given in Example 17 of Chapter 10 to show that $C(L_{11})$ has an element of order 33, whereas A_{12} does not.
11. If we can find a pair of distinct Sylow 2-subgroups A and B such that $|A \cap B| = 8$, then $N(A \cap B) \geq AB$, so that $N(A \cap B) = G$. Now let H and K be any distinct pair of Sylow 2-subgroups. Then $16 \cdot 16/|H \cap K| = |HK| \leq 112$ (Theorem 7.2), so that $|H \cap K|$ is at least 4. If $|H \cap K| = 8$, we are done. So, assume $|H \cap K| = 4$. Then $N(H \cap K)$ picks up at least 8 elements from H and at least 8 from K (see Exercise 45 in Chapter 23). Thus, $|N(H \cap K)| \geq 16$ and is divisible by 8. So, $|N(H \cap K)| = 16, 56,$ or 112. Since the latter two cases yield normal subgroups, we may assume $|N(H \cap K)| = 16$. If $N(H \cap K) = H$, then $|H \cap K| = 8$, since $N(H \cap K)$ contains at least 8 elements from K. So, we may assume that $N(H \cap K) \neq H$. Then, we may take $A = N(H \cap K)$ and $B = H$.
13. If H is a proper subgroup of A_{n+1} of order greater than $n!/2$, then $[A_{n+1} : H] < [A_{n+1} : A_n] = n + 1$ and it follows from the Embedding Theorem that A_{n+1} is isomorphic to a subgroup of A_n, which is impossible.
15. Use the Index Theorem.
17. (Gurmeet Singh) By Sylow's third theorem we know that number of Sylow 5-subgroups is 6. This means

that 6 is the index of the normalizer of a Sylow 5-subgroup. But then, by embedding theorem, G is isomorphic to a subgroup of A_6 of order 120. This contradicts Exercise 16.

19. Let α be as in the proof of the Generalized Cayley Theorem. Then $\text{Ker }\alpha \leq H$ and $|G/\text{ Ker }\alpha|$ divides $|G{:}H|!$. Now show $|\text{Ker }\alpha| = |H|$. A subgroup of index 2 is normal.
21. Since A_5 is simple, if H is a proper normal subgroup of S_5, then $H \cap A_5 = A_5$ or $\{\varepsilon\}$. But $H \cap A_5 = A_5$ implies $H = A_5$, whereas $H \cap A_5 = \{\varepsilon\}$ implies $H = \{\varepsilon\}$ or $|H| = 2$. (See Exercise 27 in Chapter 5.) Now use Exercise 72 in Chapter 9 and Exercise 70 in Chapter 5.
23. See Example 9 in Chapter 9.
25. Mimic Exercise 24.
27. Suppose there is a simple group of order 60 that is not isomorphic to A_5. The Index Theorem implies $n_2 \neq 1$ or 3, and the Embedding Theorem implies $n_2 \neq 5$. Thus, $n_2 = 15$. Counting shows that there must be two Sylow 2-subgroups whose intersection has order 2. Now mimic the argument used in showing that there is no simple group of order 144 to show that the normalizer of this intersection has index 5, 3, or 1, but the Embedding Theorem and the Index Theorem rule these out.
29. Suppose there is such a simple group G. Since the number of Sylow q-subgroups is 1 modulo q and divides p^2, it must be p^2. Thus there are $p^2(q-1)$ elements of order q in G. These elements, together with the p^2 elements in one Sylow p-subgroup, account for all p^2q elements in G. Thus, there cannot be another Sylow p-subgroup. But then the Sylow p-subgroup is normal in G.
31. Consider the right regular representation of G. Let g be a generator of the Sylow 2-subgroup and suppose that $|G| = 2^k n$ where n is odd. Then every cycle of the permutation T_g in the right regular representation of G has length 2^k. This means that there are exactly n such cycles. Since each cycle is odd and there is an odd number of them, T_g is odd. This means that the set of even permutations in the regular representations has index 2 and is therefore normal. (See Exercise 27 in Chapter 5 and Exercise 9 in Chapter 9).
33. By direct computation, show that $PSL(2, Z_7)$ has more than four Sylow 3-subgroups, more than one Sylow 7-subgroup, and more than one Sylow 2-subgroup. *Hint*: Observe that $\begin{bmatrix} 1 & 4 \\ 1 & 5 \end{bmatrix}$ has order 3. Now use conjugation to find four other subgroups of order 3; observe that $\left|\begin{bmatrix} 5 & 5 \\ 1 & 4 \end{bmatrix}\right| = 7$ and use conjugation to find another subgroup of order 7; observe that $\left|\begin{bmatrix} 5 & 1 \\ 3 & 5 \end{bmatrix}\right| = 4$ and use conjugation to find six more elements of order 4 (which guarantees that more than one Sylow 2-subgroup exists). Now argue as we did to show that A_5 is simple. In the cases that the supposed normal subgroup N has order 2 or 4, show that in G/N, the Sylow 7-subgroup is normal. But then, G has a normal subgroup of order 14 or 28, which were already ruled out.
35. By Sylow if the group has only one subgroup of order p^n it is normal. So suppose L_1 and L_2 are distinct subgroups of order p^n. Observe that if $|L_1 \cap L_2| \leq p^{n-2}$ then $|L_1 L_2| = p^n p^n / |L_1 \cap L_2| \geq p^n p^n / p^{n-2} = p^2 p^n > 4p^n$. So, $|L_1 \cap L_2| = p^{n-1}$. Then, by Exercise 45 of Chapter 23 $N(L_1 \cap L_2)$ contains L_1 and L_2. So, $|N(L_1 \cap L_2)| > p^{n+1} > 2p^n$ and is divisible by p^n. So, $N(L_1 \cap L_2) = G$ and therefore $L_1 \cap L_2$ is normal in G.
37. By Theorem 24.3 we know that there is a homomorphism ϕ from G into the

symmetric group S_p such that Ker ϕ is a subgroup of H. Then, because $G/\text{Ker }\phi$ is isomorphic to a subgroup of S_p, we have that $|G/\text{Ker }\phi|$ divides $p!$ and that $|G/\text{Ker }\phi|$ divides $|G|$. So, if $|G/\text{Ker }\phi|$ were anything other than p, it would have a prime divisor less than p, which would mean that G would have a prime divisor less than p. So, $p = |G : H| = |G/\text{Ker }\phi|$, which implies that $|H| = |\text{Ker }\phi|$. Since Ker ϕ is a subgroup of H they are equal.

Chapter 25

If you make a mistake, make amends.

Lou Holtz

1. u is related to u because u is obtained from itself by no insertions; if v can be obtained from u by inserting or deleting words of the form xx^{-1} or $x^{-1}x$, then u can be obtained from v by reversing the procedure; if u can be obtained from v and v can be obtained from w, then u can be obtained from w by obtaining first v from w and then u from v.
3. $$\begin{aligned} b(a^2N) &= b(aN)a = a^3bNa = a^3b(aN)\\ &= a^3a^3bN\\ &= a^6bN = a^6Nb = a^2Nb = a^2bN\\ b(a^3N) &= b(a^2N)a = a^2bNa = a^2b(aN)\\ &= a^2a^3bN\\ &= a^5bN = a^5Nb = aNb = abN\\ b(bN) &= b^2N = N\\ b(abN) &= baNb = a^3bNb = a^3b^2N = a^3N\\ b(a^2bN) &= ba^2Nb = a^2bNb = a^2b^2N\\ &= a^2N\\ b(a^3bN) &= ba^3Nb = abNb = ab^2N = aN \end{aligned}$$
5. Let F be the free group on $\{a_1, a_2, \ldots, a_n\}$. Let N be the smallest normal group containing $\{w_1, w_2, \ldots, w_t\}$ and let M be the smallest normal subgroup containing $\{w_1, w_2, \ldots, w_t, w_{t+1}, \ldots, w_{t+k}\}$. Then $F/N \approx G$ and $F/M \approx \overline{G}$. The homomorphism from F/N to F/M given by $aN \to aM$ induces a homomorphism from G onto $\overline{G}$. To prove the corollary, observe that the theorem shows that K is a homomorphic image of G, so $|K| \leq |G|$.
7. Clearly, a and ab belong to $\langle a, b\rangle$, so $\langle a, ab\rangle \subseteq \langle a, b\rangle$. Now show that a and b belong to $\langle a, ab\rangle$.
9. Show that $|G| \leq 2n$ and that D_n satisfies the relations that define G.
11. Since $x^2 = y^2 = e$, we have $(xy)^{-1} = y^{-1}x^{-1} = yx$. Also, $xy = z^{-1}yz$, so $(xy)^{-1} = (z^{-1}yz)^{-1} = z^{-1}y^{-1}z = z^{-1}yz = xy$.
13. **a.** b^6 **b.** b^7a
15. First observe that since $xy = (xy)^3(xy)^4 = (xy)^7 = (xy)^4(xy)^3 = yx$, x and y commute. Also, since $y = (xy)^4 = (xy)^3xy = x(xy) = x^2y$ we know that $x^2 = e$. Then $y = (xy)^4 = x^4y^4 = y^4$ and therefore, $y^3 = e$. This shows that $|G| \leq 6$. But Z_6 satisfies the defining relations with $x = 3$ and $y = 2$. So, $G \approx Z_6$.
17. Note that $yxyx^3 = e$ implies that $yxy^{-1} = x^5$ and therefore $\langle x\rangle$ is normal. So, $G = \langle x\rangle \cup y\langle x\rangle$ and $|G| \leq 16$. Use $y^2 = e$ and $yxyx^3 = e$, to prove that $x^2 \in Z(G)$. Then prove G is not Abelian and use Theorem 9.3 to show that $|Z(G)| \neq 8$. Thus, $Z(G) = \langle x^2\rangle$. Finally, prove that $(xy)^2 = x^{-2}$, so that $|xy| = 8$.
19. Use the fact that the mapping from G onto G/N given by $x \to xN$ is a homomorphism.
21. For H to be a normal subgroup we must have $yxy^{-1} \in H = \{e, y^3, y^6, y^9, x, xy^3, xy^6, xy^9\}$. But $yxy^{-1} = yxy^{11} = (yxy)y^{10} = xy^{10}$.
23. 6; the given relations imply that $a^2 = e$. G is isomorphic to Z_6.
25. 1, 2, and ∞
27. $ab = c \Rightarrow abc^{-1} = e$
$cd = a \Rightarrow (abc^{-1})cd = ae \Rightarrow bd = e \Rightarrow$

$d = b^{-1}$
$da = b \Rightarrow bda = b^2 \Rightarrow ea = b^2 \Rightarrow a = b^2$
$ab = c \Rightarrow b^3 = c$
So $G = \langle b \rangle$.
$bc = d \Rightarrow bb^3 = b^{-1} \Rightarrow b^5 = e$. So $|G| = 1$ or 5.
But Z_5 satisfies the defining relations with
$a = 1, b = 3, c = 4$, and $d = 2$.

29. Z_6

Chapter 26

If at first you don't succeed—that makes you about average.

BRADENTON, [*Florida*] *Herald*

1. If T is a distance-preserving function and the distance between points a and b is positive, then the distance between $T(a)$ and $T(b)$ is positive.
3. See Figure 1.5.
5. 12
7. $4n$
9. a. Z_2 b. $Z_2 \oplus Z_2$ c. $G \oplus Z_2$, where G is the plane symmetry group of a circle (see Exercise 55 of Chapter 3).
11. 6
13. An inversion in $\mathbf{R}^3$ leaves only a single point fixed, whereas a rotation leaves a line fixed.
15. In $\mathbf{R}^4$, a plane is fixed. In $\mathbf{R}^n$, a hyperplane of dimension $n - 2$ is fixed.
17. Create a coordinate system for the plane. Let T be an isometry; p, q, and r the three noncollinear points; and s any other point in the plane. Then the quadrilateral determined by $T(p), T(q), T(r)$, and $T(s)$ is congruent to the one formed by p, q, r, and s. Thus, $T(s)$ is uniquely determined by $T(p), T(q)$, and $T(r)$.
19. a rotation

Chapter 27

Failure is the key to success; each mistake teaches us something.

MORIHEI UESHIBA

1. 6
3. 30
5. 13
7. 45
9. 126
11. $\frac{1}{6}(n^6 + 2 \cdot n + 2 \cdot n^2 + n^3)$
13. For the first part, see Exercise 13 in Chapter 6. For the second part, try D_4.
15. R_0, R_{180}, H, V act as the identity and R_{90}, R_{270}, D, D' interchange L_1 and L_2.

Chapter 28

I am not bound to please thee with my answers.

SHAKESPEARE, *The Merchant of Venice*

1. $4 * (b, a)$
3. $(m/2) * \{3 * [(a, 0), (b, 0)], (a, 0), (e, 1)$ $3 * (a, 0), (b, 0), 3 * (a, 0), (e, 1)\}$
5. a^3b
7. Both yield paths from e to a^3b.
11. Say we start at x. Then we know the vertices $x, xs_1, xs_1s_2, \ldots, xs_1s_2 \cdots s_{n-1}$ are distinct and $x = xs_1s_2 \cdots s_n$. So if we apply the same sequence beginning at y, then cancellation shows that $y, ys_1, ys_1s_2, \ldots, ys_1s_2 \cdots s_{n-1}$ are distinct and $y = ys_1s_2 \cdots s_n$.
13. If there were a Hamiltonian path from $(0, 0)$ to $(2, 0)$, there would be a Hamiltonian circuit in the digraph, since $(2, 0) + (1, 0) = (0, 0)$. This contradicts Theorem 28.1.
15. **a.** If $s_1, s_2, \ldots, s_{n-1}$ traces a Hamiltonian path and $s_is_{i+1} \cdots s_j = e$, then the vertex $s_1s_2 \cdots s_{i-1}$ appears twice. Conversely, if $s_is_{i+1} \cdots s_j \neq e$, then the sequence $e, s_1, s_1s_2, \ldots, s_1s_2 \cdots s_{n-1}$ yields the n vertices (otherwise, cancellation gives a contradiction).
    b. This follows directly from part **a**.
17. The sequence traces the digraph in a clockwise fashion.
19. Abbreviate $(a, 0), (b, 0)$, and $(e, 1)$ by a, b, and 1, respectively. A circuit is

$4*(4*1,a),3*a,b,7*a,1,b,3*a,b,6*a,1,a,b,3*a,b,5*a,1,a,a,b,3*a,b,4*a,1,3*a,b,3*a,b,3*a,b.$

21. Abbreviate $(R_{90},0),(H,0)$, and $(R_0,1)$ by R,H, and 1, respectively. A circuit is $3*(R,1,1),H,2*(1,R,R),R,1,R,R,1,H,1,1.$
23. Abbreviate $(a,0),(b,0)$, and $(e,1)$ by a,b, and 1, respectively. A circuit is $2*(1,1,a),a,b,3*a,1,b,b,a,b,b,1,3*a,b,a,a.$
25. Abbreviate $(r,0),(f,0)$, and $(e,1)$ by r,f, and 1, respectively. Then the sequence is $r,r,f,r,r,1,f,r,r,f,r,1,r,f,r,r,f,1,r,r,f,r,r,1,f,r,r,f,r,1,r,f,r,r,f,1.$
27. $m^*[(n-1)^*(0,1),(1,1)]$
29. Abbreviate $(r,0),(f,0)$, and $(e,1)$ by r,f, and 1, respectively. A circuit is $1,r,1,1,f,r,1,r,1,r,f,1.$
31. $5*[3*(1,0),(0,1)],(1,0)$
33. $12*[(1,0),(0,1)]$
35. Letting V denote a vertical move and H a horizontal move and starting at $(1,0)$ a circuit is $V,V,H,6*(V,V,V,H)$.
37. In the proof of Theorem 28.3, we used the hypothesis that G is Abelian in two places: We needed H to satisfy the induction hypothesis, and we needed to form the factor group G/H. Now, if we assume only that G is Hamiltonian, then H also is Hamiltonian and G/H exists.

Chapter 29

We must view with profound respect the infinite capacity of the human mind to resist the introduction of useful knowledge.

THOMAS R. LOUNSBURY

1. wt(0001011) = 3; wt(0010111) = 4; wt(0100101) = 3; etc.
3. 1000110; 1110100
5. 000000, 100011, 010101, 001110, 110110, 101101, 011011, 111000
7. By using $t = 1/2$ in the proof of Theorem 29.2 we have that all single errors can be detected.
9. Observe that a vector has even weight if and only if it can be written as a sum of an even number of vectors of weight 1.
11. No, by Theorem 29.3.
13. 0000000, 1000111, 0100101, 0010110, 0001011, 1100010, 1010001, 1001100, 0110011, 0101110, 0011101, 1110100, 1101001, 1011010, 0111000, 1111111;
$$H=\begin{bmatrix}1&1&1\\1&0&1\\1&1&0\\0&1&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix};$$
yes.
15. Suppose that u is decoded as v and that x is the coset leader of the row containing u. Coset decoding means v is at the head of the column containing u. So, $x+v=u$ and $x=u-v$. Now suppose $u-v$ is a coset leader and u is decoded as y. Then y is at the head of the column containing u. Since v is a code word, $u=u-v+v$ is in the row containing $u-v$. Thus, $u-v+y=u$ and $y=v$.
17. 000000, 100110, 010011, 001101, 110101, 101011, 011110, 111000;
$$H=\begin{bmatrix}1&1&0\\0&1&1\\1&0&1\\1&0&0\\0&1&0\\0&0&1\end{bmatrix}.$$
001001 is decoded as 001101 by all four methods.
011000 is decoded as 111000 by all four methods.
000110 is decoded as 100110 by all four methods.

Since there are no code words whose distance from 100001 is 1 and three whose distance is 2, the nearest-neighbor method will not decode or will arbitrarily choose a code word; parity-check matrix decoding does not decode 100001; the standard-array and syndrome methods decode 100001 as 000000, 110101, or 101011, depending on which of 100001, 010100, or 001010 is a coset leader.

19. For any received word w, there are only eight possibilities for wH. But each of these eight possibilities satisfies condition 2 or the first portion of condition $3'$ of the decoding procedure, so decoding assumes that no error was made or one error was made.
21. There are 3^4 code words and 3^6 possible received words.
23. No; row 3 is twice row 1.
25. No. For if so, nonzero code words would be all words with weight at least 5. But this set is not closed under addition.
27. Use Exercise 24, together with the fact that the set of code words is closed under addition.
29. Abbreviate the coset $a + \langle x^2 + x + 1\rangle$ with a. The following generating matrix will produce the desired code:

$$\begin{bmatrix} 1 & 0 & 1 & 1 & x \\ 0 & 1 & x & x+1 & x+1 \end{bmatrix}.$$

31. Use Exercise 14.
33. Let $c, c' \in C$. Then, $c + (v + c') = v + c + c' \in v + C$ and $(v + c) + (v + c') = c + c' \in C$, so the set $C \cup (v + C)$ is closed under addition.
35. If the ith component of both u and v is 0, then so is the ith component of $u - v$ and au, where a is a scalar.

Chapter 30

Wisdom rises upon the ruins of folly.

THOMAS FULLER, *Gnomologia*

1. Note that $\phi(1) = 1$. Thus $\phi(n) = n$. Also, $1 = \phi(1) = \phi(nn^{-1}) = \phi(n)\phi(n^{-1}) = n\phi(n^{-1})$, so that $1/n = \phi(1/n)$.
3. If α and β are automorphisms of E fixing F, so are α^{-1} and $\alpha\beta$.
5. If a and b are fixed by elements of H, so are $a + b, a - b, a \cdot b$, and a/b.
7. It suffices to show that each member of $\mathrm{Gal}(K/F)$ defines a permutation on the a_i's. Let $a \in \mathrm{Gal}(K/F)$ and write

$$\begin{aligned} f(x) &= c_n x^n + c_{n-1}x^{n-1} + \cdots + c_0 \\ &= c_n(x - a_1)(x - a_2)\cdots(x - a_n). \end{aligned}$$

Then $f(x) = \alpha(f(x)) = c_n(x-\alpha(a_1))(x-\alpha(a_2))\cdots(x-\alpha(a_n))$. Thus, $f(a_i) = 0$ implies $a_i = \alpha(a_j)$ for some j, so that α permutes the a_i's.

9. Observe that $\phi^6(\omega) = \omega^{729} = \omega$ whereas $\phi^3(\omega) = \omega^{27} = \omega^{-1}$ and $\phi^2(\omega) = \omega^9 = \omega^2$. $\phi^3(\omega + \omega^{-1}) = \omega^{27} + \omega^{-27} = \omega^{-1} + \omega$; $\phi^2(\omega^3 + \omega^5 + \omega^6) = \omega^{27} + \omega^{45} + \omega^{54} = \omega^6 + \omega^3 + \omega^5$.
11. **a.** $Z_{20} \oplus Z_2$ has three subgroups of order 10.
    b. 25 does not divide 40, so there are none.
    c. $Z_{20} \oplus Z_2$ has one subgroup of order 5.
13. The splitting field over $\mathbf{R}$ is $\mathbf{R}(\sqrt{-3})$. The Galois group is the identity and the mapping $a + b\sqrt{-3} \to a - b\sqrt{-3}$.
15. Use Theorem 22.3.
17. Recall that A_4 has no subgroup of order 6. (See Example 5 in Chapter 7.)
19. Use Sylow's First Theorem.
21. Let ω be a primitive cube root of 1. Then $Q \subset Q(\sqrt[3]{2}) \subset Q(\omega, \sqrt[3]{2}$ and $Q(\sqrt[3]{2})$ is not the splitting field of a polynomial in $Q[x]$.
23. Use the lattice of Z_{10}.
25. Z_6 (Be sure you know why the group is cyclic.)
27. See Exercise 21 in Chapter 24.
29. Use Exercise 43 in Chapter 23.
31. Use Exercise 50 in Chapter 10.
33. Since $K/N \triangleleft G/N$, for any $x \in G$ and $k \in K$, there is a $k' \in K$ such that $k'N = (xN)(kN)(xN)^{-1} =$

$xNkNx^{-1}N = xkx^{-1}N$. So, $xkx^{-1} = k'n$ for some $n \in N$. And since $N \subseteq K$, we have $k'n \in K$.

35. Since G is solvable there is a series $\{e\} = K_0 \subset K_1 \subset \cdots \subset K_m = G$ such that K_{i+1}/K_i is Abelian. Now there is a series

$$\frac{K_i}{K_i} = \frac{L_0}{K_i} \subset \frac{L_1}{K_i} \subset \cdots \subset \frac{L_t}{K_i} = \frac{K_{i+1}}{K_i},$$

where $|(L_{j+1}/K_i)/(L_j/K_i)|$ is prime. Then $K_i = L_0 \subset L_1 \subset L_2 \subset \cdots \subset L_t = K_{i+1}$ and each $|L_{j+1}/L_j|$ is prime (see Exercise 42 of Chapter 10). We may repeat this process for each i.

Chapter 31

All wish to possess knowledge, but few, comparatively speaking, are willing to pay the price.

JUVENAL

1. $x^2 - x + 1$
3. Over $Z, x^8 - 1 = (x-1)(x+1)(x^2+1)(x^4+1)$. Over $Z_2, x^2+1 = (x+1)^2$ and $x^4 + 1 = (x+1)^4$. So, over Z_2, $x^8 - 1 = (x+1)^8$. Over Z_3, x^2+1 is irreducible, but x^4+1 factors into irreducibles as $(x^2+x+2)(x^2-x-1)$. So, $x^8-1 = (x-1)(x+1)(x^2+1)(x^2+x+2)(x^2-x-1)$. Over Z_5, $x^2+1 = (x-2)(x+2), x^4+1 = (x^2+2)(x^2-2)$, and these last two factors are irreducible. So, $x^8 - 1 = (x-1)(x+1)(x-2)(x+2)(x^2+2)(x^2-2)$.
5. Let ω be a primitive nth root of unity. We must prove $\omega\omega^2\cdots\omega^n = (-1)^{n+1}$. Observe that $\omega\omega^2\cdots\omega^n = \omega^{n(n+1)/2}$. When n is odd, $\omega^{n(n+1)/2} = (\omega^n)^{(n+1)/2} = 1^{(n+1)/2} = 1$. When n is even, $(\omega^{n/2})^{n+1} = (-1)^{n+1} = -1$.
7. If $[F{:}Q] = n$ and F has infinitely many roots of unity, then there is no finite bound on their multiplicative orders. Let ω be a primitive mth root of unity in F such that $\phi(m) > n$. Then $[Q(\omega){:}Q] = \phi(m)$. But $F \supseteq Q(\omega) \supseteq Q$ implies $[Q(\omega){:}Q] \leq n$.
9. Let $2^n + 1 = q$. Then $2 \in U(q)$ and $2^n = q - 1 = -1$ in $U(q)$ implies that $|2| = 2n$. So, by Lagrange's Theorem, $2n$ divides $|U(q)| = q - 1 = 2^n$.
11. Let ω be a primitive nth root of unity. Then $2n$th roots of unity are $\pm 1, \pm\omega, \ldots, \pm\omega^{n-1}$. These are distinct, since $-1 = (-\omega^i)^n$, whereas $1 = (\omega^i)^n$.
13. First observe that deg $\Phi_{2n}(x) = \phi(2n) = \phi(n)$ and deg $\Phi_n(-x) =$ deg $\Phi_n(x) = \phi(n)$. Thus, it suffices to show that every zero of $\Phi_n(-x)$ is a zero of $\Phi_{2n}(x)$. But the fact that ω is a zero of $\Phi_n(-x)$ means that $|-\omega| = n$, and because n is odd, this implies that $|\omega| = 2n$.
15. Let $G = \mathrm{Gal}(Q(\omega)/Q)$ and H_1 be the subgroup of G of order 2 that fixes $\cos(\frac{2\pi}{n})$. Then, by induction, G/H_1 has a series of subgroups $H_1/H_1 \subset H_2/H_1 \subset \cdots \subset H_t/H_1 = G/H_1$, so that $|H_{i+1}/H_1{:}H_i/H_1| = 2$. Now observe that $|H_{i+1}/H_1{:}H_i/H_1| = |H_{i+1}/H_i|$.
17. Instead, prove that $\Phi_n(x)\Phi_{pn}(x) = \Phi_n(x^p)$. Since both sides are monic and have degree $p\phi(n)$, it suffices to show that every zero of $\Phi_n(x)\Phi_{pn}(x)$ is a zero of $\Phi_n(x^p)$. If ω is a zero of $\Phi_n(x)$, then $|\omega| = n$. By Theorem 4.2, $|\omega^p| = n$ also. Thus, ω is a zero of $\Phi_n(x^p)$. If ω is a zero of $\Phi_{pn}(x)$, then $|\omega| = pn$ and therefore $|\omega^p| = n$.
19. Use Theorem 31.4 and Theorem 30.1.
21. Suppose that a prime $p = 2^m + 1$ and m is not a power of 2. Then $m = st$ where s is an odd integer greater than 1 (the case where $m = 1$ is trivial). Let $n = 2^t + 1$. Then $1 < n < p$ and $2^t \bmod n = -1$. Now looking at $p \bmod n$ and replacing 2^t with -1, we have $(2^t)^s + 1 = (-1)^s + 1 = 0$. This means that n divides the prime p, which is a contradiction.

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