

Here are some Lemmas which are referenced throughout the problems.

Lemma 1 (Sum of Ideals is an Ideal). *If I, J are ideals of a ring R , then $I + J = \{i + j \mid i \in I, j \in J\}$ is also an ideal of R .*

Proof. $\forall (i_1 + j_1), (i_2 + j_2) \in I + J$, $(i_1 + j_1) - (i_2 + j_2) = (i_1 - i_2) + (j_1 - j_2) \in I + J \implies I + J \leq_+ R$. Next for all $r \in R$ and all $(i + j) \in I + J$, $r(i + j) = ri + rj$, $(i + j)r = ir + jr \in I + J$ since $ir, ri \in I$ and $rj, jr \in J$. Thus,

If I, J are ideals of R , then $I + J$ is an ideal of R .

□

Lemma 2 (Inclusion Mapping is Ring Embedding). *If $S \subseteq R$ is a subring of a ring R , then the inclusion mapping $\iota : S \hookrightarrow R$ defined via $\iota(s) = s$ is a ring embedding (an injective ring homomorphism).*

Proof. For all $a, b \in S$,

$$\begin{aligned}\iota(a + b) &= a + b = \iota(a) + \iota(b) \\ \iota(ab) &= ab = \iota(a)\iota(b) \\ \iota(1_S) &= 1_S = 1_R \\ \iota(a) &= \iota(b) \implies a = b \text{ by definition.}\end{aligned}$$

□

Lemma 3 (Evaluation Mapping is a Surjective Ring Homomorphism). *The evaluation homomorphism $\varepsilon_a : R[x] \rightarrow R$ defined by $\varepsilon_a(f(x)) = f(a)$ is a surjective ring homomorphism for any $a \in R$.*

Proof. For all $f(x), g(x) \in R[x]$,

$$\begin{aligned}\varepsilon_a(f(x) + g(x)) &= f(a) + g(a) = \varepsilon_a(f(x)) + \varepsilon_a(g(x)) \\ \varepsilon_a(f(x)g(x)) &= f(a)g(a) = \varepsilon_a(f(x))\varepsilon_a(g(x)) \\ \varepsilon_a(1_{R[x]}) &= 1_R\end{aligned}$$

For any $r \in R$, $\varepsilon_a(r) = r$ so ε_a is surjective.

□

Lemma 4 (Telescoping Identity). *For any $k \in \mathbb{N}$ and any polynomial $x^k - y^k$ over a commutative ring R where x, y may either be indeterminates or elements of R , we have that:*

$$x^k - y^k = (x - y) \sum_{\substack{i+j=k-1 \\ i,j \in \mathbb{N}}} x^i y^j.$$

Proof. For $k > 0$:

$$\begin{aligned}
 x \sum_{\substack{i+j=k-1 \\ i,j \in \mathbb{N}}} x^i y^j &= x(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) = x^k + (x^{k-1}y + \cdots + x^2y^{k-2} + xy^{k-1}) \\
 y \sum_{\substack{i+j=k-1 \\ i,j \in \mathbb{N}}} x^i y^i &= y(x^{k-1} + x^{k-2}y + \cdots + xy^{k-2} + y^{k-1}) = (yx^{k-1} + x^{k-2}y^2 + \cdots + xy^{k-1}) + y^k \\
 &\quad = (x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1}) + y^k \\
 \implies (x-y) \sum_{\substack{i+j=k-1 \\ i,j \in \mathbb{N}}} x^i y^i &= x^k + (x^{k-1}y + \cdots + x^2y^{k-2} + xy^{k-1}) - (x^{k-1}y + x^{k-2}y^2 + \cdots + xy^{k-1}) - y^k \\
 &= x^k - y^k.
 \end{aligned}$$

□

Problem 1. In the following problems, we investigate the relationship between integral domains and fields.

- (a) Prove that every field is an integral domain.
- (b) Prove that if R is a finite integral domain, then R is a field. **Hint:** Consider the set function $a : R \rightarrow R$ given by multiplication by an element a (it will not be a ring homomorphism). If a is a non-zero divisor, prove this is an injective function, and so must also be surjective since R has finitely many elements.
- (c) Find an example of an integral domain which is not a field.

Proof. (a) Let \mathbb{K} be a field, and suppose it contains some non-zero zero divisor $b \in \mathbb{K} \setminus \{0\}$. Then there exists some $a \in \mathbb{K} \setminus \{0\}$ such that $ab = ba = 0$. Since $a, b \in \mathbb{K} \setminus \{0\}$, they are both units and so there exist $a^{-1}, b^{-1} \in \mathbb{K}$ such that $a^{-1}a = bb^{-1} = 1 \implies a^{-1}(ab)b^{-1} = 1 = a^{-1}(0)b^{-1} = 0 \implies 1 = 0$. But then $\mathbb{K} = \{0\}$ which is not a field, a contradiction. Therefore, all non-zero elements are not zero divisors and so \mathbb{K} is a commutative ring with unity and no zero divisors, which is the definition of an integral domain.

(b) Let R be a finite integral domain, and let a be some non-zero zero divisor $a \in R \setminus \{0\}$. We show the mapping $a : R \rightarrow R$ via $a(x) = ax, \forall x \in R$ is a bijection. For all $x, y \in R$,

$$[1-1] \quad a(x) = a(y) \implies ax = ay \implies ax - ay = a(x - y) = 0 \implies x = y$$

otherwise a is a zero divisor, and we get a contradiction.

[Onto] $a : R \hookrightarrow R \implies$ the domain is in bijection with its image $\implies |R| = |a(R)|$

and since $a(R) \subseteq R$ (the domain is equal to the codomain), we must have that $a(R) = R$.

(This image is just the orbit of a in the canonical multiplicative monoid action of R on itself.) So then for each non-zero zero divisor $a \in R \setminus \{0\}$, $\exists x \in R$ such that $a(x) = ax = 1$. That is, each non-zero zero divisor, which is just all of $R \setminus \{0\}$ is a unit. Therefore, R is a commutative ring with unity and inverses for all non-zero elements, which is the definition of a field.

(c) \mathbb{Z} is an integral domain but not a field.

□

Problem 2. Let I, J be ideals in a commutative ring R such that $I + J = (1)$. Prove that $IJ = I \cap J$.

Proof. (\subseteq) Recall that since I, J are ideals of R , $ir \in I$ and $rz \in J, \forall i \in I, \forall j \in J, \forall r \in R$. So then for all $ij \in IJ$, we have that $(i)j \in I$ and $i(j) \in J \implies ij \in I \cap J \implies IJ \subseteq I \cap J$.

(\supseteq) $I + J = \langle 1 \rangle \implies \hat{i} + \hat{j} = 1$ for some $\hat{i} \in I, \hat{j} \in J$. Then for all $a \in I \cap J$, $a = a(1) = (1)a = a(\hat{i} + \hat{j}) = a\hat{i} + a\hat{j} \in IJ + JI = IJ$, since commutativity in R implies $IJ = JI$. Therefore, $I \cap J \subseteq IJ$.

Thus,

$$IJ = I \cap J$$

□

Problem 3. Let $\varphi : R \rightarrow S$ be a ring homomorphism, and let J be an ideal of S . Prove that $I = \varphi^{-1}(J) = \{i \in R \mid \varphi(i) \in J\}$ is an ideal of R .

Proof. $\forall a, b \in \varphi^{-1}(J)$, $\varphi(a), \varphi(b) \in J$ and so $\varphi(a - b) = \varphi(a) - \varphi(b) \in J$. So then $a - b \in \varphi^{-1}(J)$ and $\varphi^{-1}(J) \leq_+ R$.

Next, for any $r \in R$ and any $i \in \varphi^{-1}(J)$, $\varphi(r i) = \varphi(r)\varphi(i) \in J$ and $\varphi(i r) = \varphi(i)\varphi(r) \in J$, since J is an ideal of S . So then $ri, ir \in \varphi^{-1}(J)$.

Thus,

If $\varphi : R \rightarrow S$ is a ring homomorphism and J is an ideal of S , then $\varphi^{-1}(J)$ is an ideal of R .

□

Problem 4. Let $\varphi : R \rightarrow S$ be a ring homomorphism, and let J be an ideal of R .

- Prove that $\varphi(J) = \{\varphi(j) \mid j \in J\}$ need not be an ideal of S .
- Prove that if φ is surjective, then $\varphi(J)$ is an ideal of S .
- Prove that if φ is surjective, and $I = \ker \varphi$, then $S \cong R/I$ and if we let $\bar{J} \subseteq R/I$ be the image of $\varphi(J)$ under this isomorphism, then

$$(R/I)/\bar{J} \cong R/(I+J).$$

Proof. (a) Consider the inclusion map $\iota : \mathbb{Z} \hookrightarrow \mathbb{R}$ defined by $\iota(n) = n \in R, \forall n \in \mathbb{Z}$. This is a ring embedding by **Lemma 2**. Now look at $2 \in 2\mathbb{Z}$, which is a well-known ideal of \mathbb{Z} . Well, $\pi \in \mathbb{R}$ but $2\pi = \pi 2 \notin \iota(2\mathbb{Z}) = 2\mathbb{Z}$. So then $\iota(2\mathbb{Z})$ is not an ideal and we see that the ring homomorphic image of an ideal need not be an ideal of the codomain.

(b) For all $\varphi(a), \varphi(b) \in \varphi(J)$ with preimages $a, b \in J$, $\varphi(a) - \varphi(b) = \varphi(a - b) \in \varphi(J)$ since $a - b \in J$. Therefore $\varphi(J) \leq_+ S$. Next, since φ is surjective, for any $s \in S$, $\exists r \in R$ such that $s = \varphi(r)$. So then $s\varphi(a) = \varphi(r)\varphi(a) = \varphi(ra)$ and $\varphi(a)s = \varphi(a)\varphi(r) = \varphi(ar)$ which must both belong to $\varphi(J)$ since $J \subseteq R$ is an ideal $\implies ar, rs \in J$. Therefore, $\varphi(J)$ is an ideal of S . That is, a surjective ring homomorphic image of an ideal is in fact an ideal of the codomain.

(c) φ is surjective, so by the First Isomorphism Theorem $R/\ker \varphi = R/I \cong \varphi(R) = S$ and so $S \cong R/\ker \varphi = R/I$. Let $\psi : S \rightarrow R/I$ be this pullback isomorphism. That is,

$$\overbrace{r}^R \xrightarrow{\varphi} \overbrace{\varphi(r)}^S = s \xleftarrow{\psi} \overbrace{[r]_I}^{R/I=R/\ker \varphi} = [r]_{\ker \varphi} \quad (1)$$

$$\text{So: } \psi(s) = \psi(\varphi(r)) = [r]_I \in R/I, \text{ for each } s = \varphi(r) \in \varphi(R) = S \quad (2)$$

$$\implies \psi(\varphi(j)) = [j]_I \in R/I \text{ for each } j \in J \quad (3)$$

$$\text{Now let: } \bar{J} = \psi(\varphi(J)) = \{\psi(\varphi(j)) = [j]_I \mid j \in J\} \subseteq R/I \quad (4)$$

Recall the Third Isomorphism Theorem. For ideals A, B of a ring R where $A \subseteq B$ is a subset:

$$\textcircled{1} B/A \text{ is an ideal of } R/A \text{ and } \textcircled{2} \frac{(R/A)}{(B/A)} \cong \frac{R}{B}.$$

* Note that in normal rings without unity required, B/A is actually a quotient ring since ideals are always rings. That is: A is an ideal of B if every ideal is a ring. That need not be true if you require rings to have unity and then B/A need not be a quotient ring, but just a set of cosets. However, the resulting quotient is always a quotient ring since the set of cosets (B/A) is an ideal of the quotient ring (R/A) . Yet another reason why (a) this coset notation sucks, and (b) this ring definition sucks.*

By **Lemma 1**, $I+J$ is an ideal of R since I, J are ideals of R . Also, $I \subseteq I+J$ is a subset. Well, we can simply compute that $(I+J)/I = \{(i+j)+I \mid (i+j) \in I+J\} = \{j+I = [j]_I \mid j \in J\} = \bar{J}$ by (4).

Therefore, since $I, (I+J)$ are ideals of R with $I \subseteq (I+J)$, by the Third Isomorphism Theorem we have that

$$\textcircled{1} \bar{J} = (I+J)/I \text{ is an ideal of } (R/I) \text{ and } \textcircled{2} \frac{(R/I)}{\bar{J}} = \frac{(R/I)}{(I+J)/I} \cong \frac{R}{(I+J)}.$$

□

Problem 5. Let R be a commutative ring, $a \in R$, and let $f_1(x), \dots, f_r(x) \in R[x]$.

- (a) Prove that $R[x]/(x-a) \cong R$.
- (b) Prove the equality of ideals

$$(f_1(x), \dots, f_n(x), x-a) = (f_1(a), \dots, f_n(a), x-a).$$

- (c) Prove the useful substitution trick

$$R[x]/(f_1(x), \dots, f_n(x), x-a) \cong R/(f_1(a), \dots, f_n(a)).$$

Hint: Use part (c) of the previous problem.

Proof. (a) Consider $\varepsilon_a : R[x] \rightarrow R$ defined by $\varepsilon_a(f(x)) = f(a)$, an evaluation which was proven to be a surjective homomorphism in **Lemma 3**. We show that $\ker \varepsilon_a = \langle x-a \rangle = \{f(x)(x-a) \mid f(x) \in R[x]\}$.

(\subseteq) For any $B(x) = \sum_{k=0}^n b_k x^k \in \ker \varepsilon_a$, $B(a) = 0$. Well, $B(x) - B(a) = (\sum_{k=0}^n b_k x^k) - (\sum_{k=0}^n b_k a^k) = \sum_{k=0}^n b_k (x^k - a^k) = \sum_{k=1}^n b_k (x^k - a^k)$. Then by **Lemma 4**, $(x^k - a^k) = (x-a) \sum_{i+j=k-1} x^i y^j$ for each $1 \leq k \leq n$. Therefore,

$$B(x) - B(a) = \sum_{k=1}^n b_k (x^k - a^k) = \sum_{k=1}^n b_k ((x-a) \sum_{i+j=k-1} x^i y^j) = (x-a) \sum_{k=1}^n b_k (\sum_{i+j=k-1} x^i y^j)$$

and since in fact $B(a) = 0$, we have that $B(x) = (x-a) \sum_{k=1}^n b_k (\sum_{i+j=k-1} x^i y^j) \in \langle x-a \rangle$ since $\sum_{k=1}^n b_k (\sum_{i+j=k-1} x^i y^j)$ belongs to $R[x]$. So then $\ker \varepsilon_a \subseteq \langle x-a \rangle$..

(\subseteq) $\forall C(x) = f(x)(x-a) \in \langle x-a \rangle$ obviously $C(a) = f(a)(a-a) = 0 \implies C(x) \in \ker \varepsilon_a$. So then $\langle x-a \rangle \subseteq \ker \varepsilon_a$. Thus, by the **First Isomorphism Theorem**,

$$R[x]/\ker \varepsilon_a = R[x]/\langle \langle x-a \rangle \cong \varepsilon_a(R[x]) = R.$$

(b) For each $1 \leq m \leq n$, there exist $b_{m,0}, \dots, b_{m,d_m} \in R$ and $d_m \in \mathbb{N}$ such that $f_m(x) = \sum_{k=0}^{d_m} b_{m,k} x^k$. Now we use a result obtained in (a):

$$f_k(x) - f_k(a) = (x-a) \sum_{i=1}^{d_k} b_{k,i} (\sum_{a+b=i-1} x^a y^a) \implies f_k(x) = f_k(a) + (x-a) \sum_{i=1}^{d_k} b_{k,i} (\sum_{a+b=i-1} x^a y^a)$$

So then

$$f_k(x) \in \langle f_k(a), x-a \rangle \subseteq \langle f_1(a), \dots, f_n(a), x-a \rangle, \forall 1 \leq k \leq n$$

and

$$f_k(a) \in \langle f_k(x), x-a \rangle \subseteq \langle f_1(x), \dots, f_n(x), x-a \rangle, \forall 1 \leq k \leq n$$

Therefore, all generators of $\langle f_1(x), \dots, f_n(x), x-a \rangle$ belong to $\langle f_1(a), \dots, f_n(a), x-a \rangle$ and vice versa. So the ideals are equal.

□

(c)

$$R[x]/(f_1(x), \dots, f_n(x), x-a) \cong R/(f_1(a), \dots, f_n(a)).$$

Proof. (c) In (a) we used the surjective homomorphism $\varepsilon_a : R[x] \rightarrow R$ defined by $\varepsilon_a(h(x)) = h(a)$ to show that $R[x]/\langle x-a \rangle = R[x]/I \cong R$ by the **First Isomorphism Theorem** where $I = \ker \varepsilon_a = \langle x-a \rangle$. Much like in **Problem 4** let $\psi : R \rightarrow R[x]/I$ be the pullback isomorphism and let $J = \langle f_1(x), \dots, f_n(x), x-a \rangle \subseteq R[x]$. Then:

$$\overbrace{h(x)}^{R[x]} \xrightarrow{\varepsilon_a} \overbrace{\varepsilon_a(h(x)) = h(a)}^R \xleftarrow{\psi} \overbrace{[h(x)]_I = [h(x)]_{\ker \varepsilon_a}}^{R[x]/I = R/\ker \varepsilon_a} \quad (1)$$

$$\text{So: } \psi(r) = \psi(\varepsilon_a(h(x))) = [h(x)]_I \in R[x]/I, \text{ for each } r = \varepsilon_a(h(x)) \in \varepsilon_a(R[x]) = R \quad (2)$$

$$\implies \psi(\varepsilon_a(j(x))) = [j(x)]_I \in R[x]/I \text{ for each } j(x) \in J \quad (3)$$

$$\text{Now let: } \bar{J} = \psi(\varepsilon_a(J)) = \{\psi(\varepsilon_a(j)) = [j(x)]_I \mid j(x) \in J\} = J/I \subseteq R[x]/I \text{ *an ideal of } R[x]/I* \quad (4)$$

Well, $(I+J)/I = (\langle x-a \rangle + \langle f_1(x), \dots, f_n(x), x-a \rangle)/\langle x-a \rangle = \langle f_1(x), \dots, f_n(x), x-a \rangle/\langle x-a \rangle = J/I = \bar{J}$. Since $I, (I+J)$ are ideals of $R[x]$ with $I \subseteq (I+J)$, by the Third Isomorphism Theorem we have that

$$\textcircled{1} \quad \bar{J} = (I+J)/I \text{ is an ideal of } (R[x]/I) \text{ and } \textcircled{2} \quad \frac{(R[x]/I)}{\bar{J}} = \frac{(R[x]/I)}{(I+J)/I} \cong \frac{R[x]}{I+J} = R[x]/J.$$

Now, we pass elements of $\frac{R[x]/I}{\bar{J}}$ through $\Phi : \frac{R[x]/I}{\bar{J}} \rightarrow \frac{R[x]}{\langle f_1(a), \dots, f_n(a) \rangle}$ defined via

$$\Phi([h(x)]_I + \bar{J}) = \psi^{-1}([h(x)]_I) + \psi^{-1}(\bar{J}) = \psi^{-1}([h(x)]_I) + \varepsilon_a(J)$$

Obviously this is well defined since ψ^{-1} is an isomorphism, and then $\forall [[A(x)]_I]_{\bar{J}}, [[B(x)]_I]_{\bar{J}} \in \frac{R[x]/I}{\bar{J}} :$

$$\Phi([[f(x)]_I]_{\bar{J}} + [[g(x)]_I]_{\bar{J}}) = \psi^{-1}([[f(x)]_I]_{\bar{J}}) + \varepsilon_a(J) + \psi^{-1}([[g(x)]_I]_{\bar{J}}) + \varepsilon_a(J) = \phi^{-1}(f(x) + g(x)) + \varepsilon_a(J) =$$

□

Problem 6. If \mathbb{k} is an algebraic closed field, then the only maximal ideals of $\mathbb{k}[x]$ are of the form $(x-a)$ where $a \in \mathbb{k}$. In this problem, we'll see that this is not true when \mathbb{k} is not algebraically closed.

- (a) Use the first isomorphism theorem to show that $\mathbb{R}[x]/(x^2 + 1) \cong \mathbb{C}$.
- (b) Prove that $(x^2 + 1)$ is a maximal ideal of $\mathbb{R}[x]$.

Problem 7. Let R be a commutative ring with $0 \neq 1$. In this problem, we will prove that every proper ideal of R is contained in some maximal ideal.

- (a) Look up Zorn's Lemma and record it here.
- (b) Define $S = \{J \mid J \text{ is a proper ideal of } R \text{ and } J \supseteq I\}$. Explain why S is a partially ordered set (what is the ordering?).

- (c) Given a chain C in S , prove that $\bigcap_{J \in C} J$ is an ideal of R (you will use that C is totally ordered), and further that this ideal is in the set S .
- (d) Conclude using Zorn's Lemma that S has a maximal element.

Problem 8. Let \mathbb{k} be a field. In this problem, we will prove that the only maximal ideal of $\mathbb{k}[[x]]$ is (x) , which makes $\mathbb{k}[[x]]$ a local ring.

- (a) Explain why $(x) = \{f \in \mathbb{k}[[x]] \mid f \text{ has no constant term}\}$.
- (b) Compute $\mathbb{k}[[x]]/(x)$, and then explain why (x) is a maximal ideal.
- (c) You may freely use the following result from the optional hint last week: $f \in \mathbb{k}[[x]]$ is a unit if and only if f has a nonzero constant term. Use Proposition 1.41 to show that the only maximal ideal of $\mathbb{k}[[x]]$ is (x) .

Problem 9. Let d be an integer which is not the square of an integer, and consider

$$\mathbb{Q}(\sqrt{d}) = \{a + b\sqrt{d} \mid a, b \in \mathbb{Q}\}.$$

- (a) Prove that $\mathbb{Q}(\sqrt{d})$ is a subring of \mathbb{C} .
- (b) Define a function $N : \mathbb{Q}(\sqrt{d}) \rightarrow \mathbb{Q}$ by $N(a + b\sqrt{d}) = a^2 + b^2d$. Prove that $N(zw) = N(z)N(w)$ and that $N(z) \neq 0$ if $z \neq 0$. This function is often called the norm.
- (c) Prove that $\mathbb{Q}(\sqrt{d})$ is a field and is the smallest subfield of \mathbb{C} containing both \mathbb{Q} and \sqrt{d} (use N).
- (d) Prove that $\mathbb{Q}(\sqrt{d}) \cong \mathbb{Q}[t]/(t^2 - d)$.