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Sagan

Problem 5.1. (**Sagan, Chapter 5, Exercise 1**)
This exercise refers to the list of examples just after the definition of a poset.

(a) Verify that they satisfy the definition of a poset.

(b) Show that the partial order in Π_n is equivalent to defining $\rho \leq \pi$ if every block of π is a union of blocks of ρ.

(c) Describe the cover relations in the list. For example, in C_n the covers are of the form $i \prec i+1$ for $0 \leq i < n$.

Proof. **(a) Two Posets** We prove that the set of all positive divisors of $n \in \mathbb{Z}$ with $|$:$(D_n, |)$ and the set of all subspaces of a vector space $\mathcal{V}$ over $\mathbb{F}_q$ with subspace containment $\leq$: $(L(\mathcal{V}), \leq)$ are posets. For any $a, b, c \in D_n$ and any $\mathcal{U}_1, \mathcal{U}_2, \mathcal{U}_3 \in L(\mathcal{V})$,

(reflexivity)	$a \mid a$	$\mathcal{U}_1 \leq \mathcal{U}_1$
(antisymmetry)	$a \neq b$ and $a \mid b \implies a < b \implies b \nmid a$	$\mathcal{U}_1 < \mathcal{U}_2 \implies \mathcal{U}_2 \not\leq \mathcal{U}_1$
(transitivity)	$a \mid b \mid c \implies \exists k, q \in \mathbb{Z}$ s.t. $\substack{ak=b \\ bq=c \\ \implies akq=c} \implies a \mid c$	$\mathcal{U}_1 \leq \mathcal{U}_2 \leq \mathcal{U}_3 \implies \mathcal{U}_1 \leq \mathcal{U}_3$

(c) In D_n, if $|n| = \prod_i p_i^{a_i}$ is a prime decomposition, then each positive divisor is of the form $d = \prod_i p_i^{b_i}$ where $0 \leq b_i \leq a_i$ is some 'subdecomposition'. So then each cover relation just looks like

$$\prod_i p_i^{b_i} \mid \prod_i p_i^{\beta_i} \text{ where each } \beta_i \geq b_i \text{ and } \sum_i (\beta_i - b_i) = 1$$

In $L(\mathcal{V})$, cover relations look like

$$\mathcal{W} \leq \mathcal{U} \text{ and } \mathcal{B}_{\mathcal{U}} = \mathcal{B}_{\mathcal{W}} \sqcup \{b\} \text{ is a basis for } \mathcal{U} \text{ and } b \in \mathcal{U}.$$

That is, $\mathcal{W} \leq \mathcal{U}$ and $\dim \mathcal{U} = \dim \mathcal{W} + 1$.

$\square$

- **Sagan—Chapter 5:** 1(ac), 4, 5abc, 6a, 7bc, 8c, 9 (2 parts), 10a, 15 (1 poset not C_n), 17ab, 18ab, 19a (oneway), 20ab
- **Stanley—Chapter 3:** 7, 14a, 25, 30a, 34, 38, 33, 52, 62abc
- **Fulton—Chapter 1:** Exercises 1, 2 (pp. 15–16), compute product both ways
- **Fulton—Chapter 2:** Exercises 1, 2 (pp. 24–26)

Problem 5.4. (Sagan, Chapter 5, Exercise 4)
Complete the proof of Proposition 5.1.3. To show that $K_n \cong B_{n-1}$ it may be simpler to show that $K_n \cong B^*_{n-1}$ using the map ϕ from Section 1.7.

Proof. Recall the following bijection from **Theorem 1.7.1**: $B_{n-1} \longleftrightarrow K_n$ via

$$\phi(S = \{s_1, \ldots, s_k\}) = (x_1, \ldots, x_{k+1}) = (s_1 - 0, s_2 - s_1, \ldots, s_k - s_{k-1}, n - s_k)$$
$$\phi^{-1}(X = (x_1, \ldots x_{k+1})) = \{\sum_{i=1}^{j} x_i \mid 1 \leq j \leq k\}$$

We represent ordered k-sums in K_n as a $k-$tuple of their parts.

We show that this is a poset isomorphism from B^*_{n-1} to K_n. Recall that *Any subset of* $[n-1]$ *is a chain*

$(\leq)$ Consider any $B = \{b_1 < \cdots < b_k\} \in B_{n-1}$. Obviously, $B = A \implies \phi(B) = \phi(A)$. So, we show that $\phi(B \sqcup \{a\}) <_{K_n} \phi(B)$.

$$A = B \sqcup \{a\} = \begin{cases} (i)\{a < b_1 < \cdots < b_k\} \text{ if } a < b_1, \\ (ii)\{b_1 < \cdots < b_j < a < b_{j+1} < \cdots < b_k\} \text{ if } b_j < a < b_{j+1} \text{ for some } j \in [k], \\ (iii)\{b_1 < \cdots < b_k < a\} \text{ if } b_k < a. \end{cases}$$

Therefore,

$$\phi(A) = \phi(B \sqcup \{a\}) = \begin{cases} (i)(\overbrace{a - 0, b_1 - a}^{b_1 - 0}, b_2 - b_1, \ldots, n - b_k) \text{ if } a < b_1, \\ (ii)(b_1 - 0, b_2 - b_1, \ldots, \overbrace{a - b_j, b_{j+1} - a}^{b_j,\, b_{j+1}}) \text{ if } b_j < a < b_{j+1} \text{ for some } j \in [k], \\ (iii)(b_1 - 0, \ldots, \overbrace{a - b_k, n - a}^{n - b_k}) \text{ if } b_k < a. \end{cases}$$

Each case gives a refinement of $\phi(B) = (b_1 - 0, b_2 - b_1, \ldots, n - b_k)$, so $\phi(B \sqcup \{a\}) <_{K_n} \phi(B)$. Finally, for any $A \supset B$ with $A \setminus B = \{a_1, \ldots, a_m\}$ in B_{n-1}

$$\phi(A) = \phi(B \sqcup \{a_1\} \sqcup \cdots \sqcup \{a_m\}) <_{K_n} \phi(B \sqcup \{a_1\} \sqcup \cdots \sqcup \{a_{m-1}\}) <_{K_n} \cdots <_{K_n} \phi(B \sqcup \{a_1\}) <_{K_n} \phi(B) \; (*)$$

So we see that $A \supseteq B \implies \phi(A) \leq_{K_n} \phi(B)$.

$(\geq)$ On the otherhand, if $\phi(A) \leq_{K_n} \phi(B)$ suppose $A \subset B$. But then by $(*)$, $B = A \sqcup \{b_1\} \sqcup \cdots \sqcup \{b_m\}$ where $B \setminus A = \{b_1, \ldots, b_m\} \implies \phi(B) <_{K_n} \phi(A)$, a contradiction. So $\phi(A) \leq_{K_n} \phi(B) \implies A \supseteq B$ and ϕ is a poset isomorphism from B^*_{n-1} to K_n.

Lastly, we show that B_N is *self-dual* for $N \in \mathbb{Z}^+$. Let $\psi : B_N \longrightarrow B_N$ be defined by $\psi(A) = A^c$. $\psi(\psi(A)) = (A^c)^c = A$ and so ψ is it's own bijective inverse. For any $A \subseteq B$ in B_n, we have $\psi(A) = A^c \supseteq B^c = \psi(B)$. Then if $\psi(A) = A^c \supseteq B^c = \psi(B)$, we could apply the previous logic, but also obviously $A \subseteq B$. so ψ is a poset isomorphism from B_n to B^*_n and thus $B_n \cong B^*_n$. Thus,

$$B_{n-1} \cong B^*_{n-1} \cong K_n.$$

$\square$

Problem 5.5. (**Sagan, Chapter 5, Exercise 5**)
Let $f : P \to Q$ be an isomorphism of posets.

(a) Show that f is also an isomorphism of P^* with Q^*.

(b) Show that if P has a $\hat{0}$, then so does Q.

(c) Show in two ways that if P has a $\hat{1}$, then so does Q: by mimicking the proof of part (b) and by using the result of (b) together with part (a).

Proof. (**a**) For any $a, b \in P^*$, $(a \leq_P b \iff f(a) \leq_Q f(b)) \implies (b \geq_P a \iff f(b) \geq_Q f(a))$ is immediate. Then since f is a bijection, it must be a poset isomorphism from P^* to Q^*.

(b) $\hat{0} \leq_P p, \forall p \in P \implies f(\hat{0}) \leq_Q f(p), \forall p \in P$. Now, since f is surjective, $\forall q \in Q$, $\exists p' \in P$ such that $f(p') = q$. Thus, $f(\hat{0}) \leq_Q f(p') = q, \forall q \in Q \implies f(\hat{0}) = \hat{0}_Q \in Q$.

(c) $p \leq_P \hat{1}, \forall p \in P \implies f(p) \leq_Q f(\hat{1}), \forall p \in P$. Now, since f is surjective, $\forall q \in Q$, $\exists p' \in P$ such that $f(p') = q$. Thus, $q = f(p') \leq_Q f(\hat{1}), \forall q \in Q \implies f(\hat{1}) = \hat{1}_Q \in Q$. Alternatively, $\hat{1} \in P$ is $\hat{0}^* \in P^*$. So then by (b), Q^* has a $\hat{0}^*_Q$, which must be $\hat{1}_Q$ in Q.

□

Problem 5.6. (**Sagan, Chapter 5, Exercise 6**)
Show that the axioms for a partially ordered set are satisfied by $P \sqcup Q$, $P + Q$, and $P \times Q$.

Proof. (**a**) Recall that $a \leq b$ if and only if $a, b \in P$ and $a \leq_P b$ or $a, b \in Q$ and $a \leq_Q b$. So for any $a, b, c \in P \sqcup Q$,

(reflexivity) $a \in P$ or $a \in Q \implies a \leq a$
(antisymmetry) $a \leq b$ and $b \leq a \implies$ either $a, b \in P$ and $a = b$ or $a, b \in Q$ and $a = b \implies a = b$
(transitivity) $a \leq b \leq c \implies$ either $a, b, c \in P$ and $a \leq_P b \leq_P c$ or $a, b, c \in Q$ and $a \leq_Q b \leq_Q c \implies a \leq c$

For $P \oplus Q$, recall that $a \leq b$ if and only if $a, b \in P$ and $a \leq_P b$ or $a, b \in Q$ and $a \leq_Q b$ or $a \in P$ and $b \in Q$. So for any $a, b, c \in P + Q$,

(reflexivity) $a \in P$ or $a \in Q \implies a \leq a$
(antisymmetry) $a \leq b$ and $b \leq a \implies$ either $a, b \in P$ and $a = b$ or $a, b \in Q$ and $a = b \implies a = b$
(transitivity) $a \leq b \leq c \implies$ either $a, b, c \in P$ and $a \leq_P b \leq_P c$ or $a, b, c \in Q$ and $a \leq_Q b \leq_Q c$
or $a, b \in P$ and $c \in Q$ or $a \in P$ and $b, c \in Q \implies a \leq c$

Lastly, for $P \times Q$, recall that $(a_1, b_1) \leq (a_2, b_2)$ if and only if $a_1 \leq_P a_2$ and $b_1 \leq_Q b_2$. So for any (a_1, b_1) , $(a_2, b_2), (a_3, b_3) \in P \times Q$,

(reflexivity) $a_1 \leq_P a_1$ & $b_1 \leq_Q b_1 \implies (a_1, b_1) \leq (a_1, b_1)$
(antisymmetry) $(a_1, b_1) \leq (a_2, b_2)$ & $(a_2, b_2) \leq (a_1, b_1) \implies a_1 = a_2$ & $b_1 = b_2 \implies (a_1, b_1) = (a_2, b_2)$
(transitivity) $(a_1, b_1) \leq (a_2, b_2) \leq (a_3, b_3) \implies a_1 \leq a_2 \leq a_3$ & $b_1 \leq b_2 \leq b_3$
$\implies a_1 \leq a_3$ & $b_1 \leq b_3 \implies (a_1, b_1) \leq (a_3, b_3)$

□

Problem 5.7. (**Sagan, Chapter 5, Exercise 7**(b),(c))
Complete the proof of Proposition 5.2.1.

(b) If $n = \prod_{i=1}^k p_i^{n_i}$ is a prime decomposition of n, then

$$D_n \cong C_{n_1} \times \cdots \times C_{n_k}$$

Proof. **(b)** Every divisor $d \in D_n$ has a unique prime factorization $d = \prod_i^k p_i^{a_i}$ where $0 \leq a_i \leq n_i$ for each $i \in [k]$. So let $\phi : D_n \to C_{n_1} \times \cdots \times C_{n_k}$ be defined via $d = \prod_i^k p_i^{a_i} \mapsto (a_1, \ldots, a_k)$. Then $\phi^{-1} : C_{n_1} \times \cdots C_{n_k} \to D_n$ defined by $(a_1, \ldots, a_k) \mapsto d = \prod_{i=1}^k p_i^{a_i}$ is a bijective inverse of ϕ :

$$\phi(\phi^{-1}(a_1, \ldots, a_k)) = \phi(\prod_{i=1}^k p_i^{a_i}) = (a_1, \ldots, a_k)$$
$$\phi^{-1}(\phi(d)) = \phi^{-1}(a_1, \ldots, a_k) = \prod_{i=1}^k p_i^{a_i} = d$$

We show ϕ is order preserving.

$$d \leq d' \implies d' = \prod_{i=1}^k p_i^{a'_i} \text{ where } a'_i \geq a_i \text{ for each } i \in [k] \implies \phi(d) = (a_1, \ldots, a_k) \leq (a'_1, \ldots, a'_k) = \phi(d')$$

$$\phi(d) \leq \phi(d') \implies (a_1, \ldots, a_k) \leq (a'_1, \ldots, a'_k) \;\&\; a_i \leq a'_i,\ \forall i \in [k] \implies d = \prod_{i=1}^k p_i^{a_i} \leq \prod_{i=1}^k p_i^{a'_i} = d'$$

Thus,

$$\phi \text{ is a poset isomorphism from } D_n \text{ to } C_{n_1} \times \cdots \times C_{n_k}.$$

$\square$

(c) If $\rho \leq \tau$ in Π_n, then

$$[\rho, \tau] \cong \Pi_{n_1} \times \cdots \times \Pi_{n_k}$$

where $\tau = T_1 \sqcup \cdots \sqcup T_k$ and n_i is the number of blocks of ρ contained in T_i for each $i \in [k]$.

Proof. **(c)** Let $P_i = \{p \in \rho \mid p \in T_i\}$ and arbitrarily index the members $P_i = \{p_{i,1}, \dots, p_{i,n_1}\}$ for each $i \in [k]$. Now let $\sigma \in [\rho, \tau]$. The blocks of σ are collapsed blocks of ρ which form refinements of the blocks of τ. So $\sigma = \sigma_1 \| \cdots \| \sigma_k$ where each σ_i is some union of collapsed blocks in P_i which refines T_i.

Each σ_i is not a block, but a 'superblock' which refines T_i. We use $\sqcup$ to denote a union of blocks forming a partition, $\|$ to denote 'pasting' together superblocks, $\cup$ to denote the collapsing of blocks. We just use superblocks here to specify/group together collapsed blocks of ρ which realize refined blocks of τ

Now, for each $i \in [k]$, let $\lambda_i(\rho) = \{\pi \in \Pi_{n_i} \mid \bigcup_{j \in \pi} p_{i,j}$ is a block in $\sigma_i\}$. Each $\lambda_i(\rho)$ indexes σ_i, and the union of its parts forms a partition of $[n_i]$. Let $\lambda_i(\sigma) = \bigsqcup_{\pi \in \lambda_i(\rho)} \pi$ be this partition. Zooming out, $\forall i \in [k]: \ \sigma_i = \bigsqcup_{\pi \in \lambda_i(\rho)} (\bigcup_{j \in \pi} p_{i,j})$ This construction gives us our bijection $\varphi : [\rho, \tau] \to \Pi_{n_1} \times \cdots \times \Pi_{n_k}$ via

$$\varphi(\sigma = \sigma_1 \| \cdots \| \sigma_k) = (\lambda_1, \dots, \lambda_k), \ \forall \sigma \in [\rho, \tau]$$

where $\varphi^{-1} \coloneqq (\lambda_1, \dots, \lambda_k) \mapsto \sigma_1 \| \cdots \| \sigma_k$ and each $\sigma_i = \bigsqcup_{\pi \in \lambda_i} (\bigcup_{j \in \pi} p_{i,j})$. **For the sake of hygiene, we skip making new notation and just clarify that the disjoint union here just iterates over the parts of λ_i.** Immediately, by definition,

$$\varphi^{-1}(\varphi(\sigma)) = \varphi^{-1}(\lambda_1(\sigma), \dots, \lambda_k(\sigma)) = \sigma_1 \| \cdots \| \sigma_k = \sigma$$
$$\varphi(\varphi^{-1}(\lambda_1, \dots, \lambda_k)) = \varphi(\sigma_1 \| \cdots \| \sigma_k) = (\lambda_1, \dots, \lambda_k)$$

We show φ is order preserving.

($\leq$) Let $\sigma = \sigma_1 \| \cdots \| \sigma_k, \varsigma = \varsigma_1 \| \cdots \| \varsigma_k$ in $[\rho, \tau]$ with $\sigma \leq \varsigma$. Each σ_i refines ς_i since they both refine T_i. So then of course $\sigma_i = \bigsqcup_{\pi \in \lambda_i(\sigma)} (\bigcup_{j \in \pi} p_{i,j}) \leq \bigsqcup_{\pi \in \lambda_i(\varsigma)} (\bigcup_{j \in \pi} p_{i,j}) \implies \lambda_i(\sigma) \leq \lambda_i(\varsigma)$ in Π_{n_i} for each $i \in [k]$ and then $\varphi(\sigma) = (\lambda_1(\sigma), \dots, \lambda_k(\sigma)) \leq (\lambda_1(\varsigma), \dots, \lambda_k(\varsigma)) = \varphi(\varsigma)$.

($\geq$) On the other hand, if $\varphi(\sigma) = (\lambda_1(\sigma), \dots, \lambda_k(\sigma)) \leq (\lambda_1(\varsigma), \dots, \lambda_k(\varsigma)) = \varphi(\varsigma)$, then $\lambda_i(\sigma) \leq \lambda_i(\varsigma)$ in Π_{n_i} for each $i \in [k]$. So then $\sigma_i = \bigsqcup_{\pi \in \lambda_i(\sigma)} (\bigcup_{j \in \pi} p_{i,j}) \leq \bigsqcup_{\pi \in \lambda_i(\varsigma)} (\bigcup_{j \in \pi} p_{i,j}) = \varsigma_i$ for each $i \in [k]$ and so $\sigma = \sigma_1 \| \cdots \| \sigma_k \leq \varsigma = \varsigma_1 \| \cdots \| \varsigma_k$.

Thus,

$$[\rho, \tau] \cong \Pi_{n_1} \times \cdots \times \Pi_{n_k}$$

$\square$

Problem 5.8. (Sagan, Chapter 5, Exercise 8)
(c) Show that if P, Q are ranked posets, then so is $P \times Q$ with rank function

$$\mathrm{rk}_{P\times Q}(x, y) = \mathrm{rk}_P x + \mathrm{rk}_Q y.$$

Proof. **(c)** Let R_P and R_Q be rank functions for P and Q, respectively, and let $R_{P\times Q} : P \times Q \to \mathbb{N}$ be defined by $R_{P\times Q}(x, y) = R_P x + R_Q y$. Obviously, $R_{P\times Q}$ is well-defined since R_P and R_Q are. We show $R_{P\times Q}$ is a rank function for $P \times Q$.

$(\hat{0} \mapsto 0)$ If P has $\hat{0}_P$ then Q has $\hat{0}$ and so $R_P(\hat{0}) = 0 = R_Q(\hat{0}_Q) \implies R_{P\times Q}(\hat{0}_P, \hat{0}_Q) = 0 + 0 = 0$

$(a \lessdot b \implies R(a) \lessdot R(b))$ If $R_P(x) = m$ and $R_Q(y) = n$, and $(x, y) \lessdot (x', y')$, suppose $x \not\lessdot x'$ and $y \not\lessdot y'$. But then there exist $x < x* < x'$ and $y < y* < y'$ such that $(x, y) < (x*, y*) < (x', y')$, a contradiction. So $x \lessdot x'$ or $y \lessdot y'$. Now, $(x, y) < (x', y), (x, y') < (x', y')$ so either $x \lessdot x'$ and $y = y'$ or $x = x'$ and $y \lessdot y'$. Therefore,

$$R(x', y') = R_P(x) + (R_Q(y) + 1) = (R_P(x) + 1) + R_Q(y) = R_P(x) + R_Q(y) + 1$$

Thus,

$$R_{P\times Q}(x, y) = R_P(x) + R_Q(y) \text{ is a valid rank function for } P \times Q.$$

□

Problem 5.9. (Sagan, Chapter 5, Exercise 9)
Prove Proposition 5.2.2. **Two Parts**

(a) If $k \in C_n$, then $R(k) = k, R^{-1}(k) = \{k\}$, and $\mathrm{Rank}\ C_n = n$.

(b) If $S \in B_n$, then $R(S) = |S|, R^{-1}(k) = \binom{[n]}{k} = \{k\text{-subsets of } [n]\}$, and $\mathrm{Rank}\ B_n = n$.

Proof. **(a)** Let $R : C_n \to \mathbb{N}$ be defined by $R(k) = k$. Obviously R is well-defined. We show R is a valid rank function for C_n.

$(\hat{0} \mapsto 0)$ $R(0) = 0$

$(a \lessdot b \implies R(a) \lessdot R(b))$ All covers in C_n are of the form $k \lessdot k+1$, and so $R(k) = k \lessdot k+1 = R(k+1)$.

Thus, $R(k) = k$ is a valid rank function for C_n and $R(\hat{1}_{C_n}) = R(n) = \mathrm{Rank}\ C_n = n$.

(b) Let $R : B_n \to \mathbb{N}$ be defined by $R(S) = |S|$. Obviously, R is well-defined. We show R is a rank function for B_n.

$(\hat{0} \mapsto 0)$ $R(\emptyset) = 0$

$(a \lessdot b \implies R(a) \lessdot R(b))$ Covers in B_n are of the form $S \lessdot S \sqcup \{a\}$, since any superset $T = S \sqcup (T \setminus S)$ of S must contain all supersets of the form $S \sqcup \{a\}$ where $a \in T \setminus S$. So then any cover $S \lessdot S \sqcup \{a\}$ is such that $R(S \sqcup \{a\}) = |S| + 1$.

Thus, $R(S) = |S|$ is a valid rank function for B_n and $R(\hat{1}_{B_n}) = R([n]) = \mathrm{Rank}\ B_n = n$.

□

Problem 5.10. (**Sagan, Chapter 5, Exercise 10**)
Prove Proposition 5.2.3. **Two Parts**:

(a) If $i, j \in C_n$, $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$.

(b) If $S, T \in B_n$, $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.

Proof. **(a)** Let $u = \max\{i, j\}$ and $l = \min\{i, j\}$. Obviously $l \leq i, j \leq u$, so u and l are upper and lower bounds, respectively, for i as well as j. Well, u and l are either i or j by definition, so no other upper bound μ can be less than u, and no other lower bound ℓ can be greater than l otherwise

$$i \text{ or } j = l < \ell \leq i, j \leq \mu < u = i \text{ or } j$$

Which is nonsense.

Thus, $i \wedge j = \min\{i, j\}$ and $i \vee j = \max\{i, j\}$.

(b) Let $U = S \cup T$ and $L = S \cap T$. Obviously, $L \subseteq S, T \subseteq U$, so U and L are upper and lower bounds, respectively, for S as well as T. Any lowerbound $W \subseteq S, T$ can only contain elements that are shared by both sets, and so by definition $W \subseteq S \cap T = \{i \in [n] \mid i \in S \text{ and } i \in T\}$ which contains all of their common elements. Similarly, any upperbound $S, T \subseteq M$ must contain all elements of either set, and so by definition it must contain their union $S \cup T = \{i \in [n] \mid i \in S \text{ or } i \in T\} \subseteq M$.

Thus, $S \wedge T = S \cap T$ and $S \vee T = S \cup T$.

□

Problem 5.15. (**Sagan, Chapter 5, Exercise 15**)
Prove Proposition 5.3.4. **One Poset not** C_n:

The posets C_n, B_n, D_n, Y, K_n are distributive lattices for all n.

Proof. We show B_n is a distributive lattice. Recall that for all S, T in B_n, $S \vee T = S \cup T$ and $S \wedge T = S \cap T$. Now for any $A, B, C \in B_n$, since De Morgan laws hold for $\cap$ and $\cup$,

$$A \vee (B \wedge C) = A \cup (B \cap C) = (A \cup B) \cap (A \cup C) = (A \cup B) \wedge (A \cup C) = (A \vee B) \wedge (A \vee C)$$
$$A \wedge (B \vee C) = A \cap (B \cup C) = (A \cap B) \cup (A \cap C) = (A \cap B) \vee (A \cap C) = (A \wedge B) \vee (A \wedge C)$$

So B_n is a distributive lattice.

□

Problem 5.17. **(Sagan, Chapter 5, Exercise 17)**
Let P be a finite poset and let $L = J(P)$ be the corresponding distributive lattice. If $X \subseteq P$ is a lower-order ideal, then use the corresponding lowercase letter x to denote the associated element of L.

(a) Show that x covers y in L if and only if $Y = X - \{m\}$ where m is a maximal element of X.

(b) Show that x is join irreducible in L if and only if X is a principal ideal of P.

Proof. **(a)** $(\Rightarrow)$ Suppose $x, y \in L$ with $y \lessdot x$. Since P is finite and the order $\subseteq$ is set inclusion, just like in B_n : $x = y \sqcup \{m\}$ for some $m \in x - y$. That is, $y = x - \{m\}$. Suppose m is not a maximal element of x. But then there exists some element $m' < m$ such that $m' \in y - \{m\}$ and y is not a lower-order ideal, a contradiction. So $y = x - \{m\}$ where m is a maximal element of x.

$(\Longleftarrow)$ If $y = x - \{m\}$ where m is a maximal element of x, then $y \subset x$ and simply by subset containment the only $z \in L$ such that $y = x - \{m\} \subseteq z \subset x$ is $z = y$. Therefore, $\nexists z \in L$ such that $y \subset z \subset x$ and $y = x - \{m\} \lessdot x$.

Thus, $y = x - \{m\}$ for some maximal element m of x if and only if $y \lessdot x$.

(b) $(\implies)$ If x is join irreducible in L, then $\exists! y \in L$ such that $y \lessdot x$. That is, by (a), there is exactly one maximal element m of x such that $y = x - \{m\} \lessdot x$. By definition every element of a lower-order ideal x is bounded above by some maximal element of x. Well, m is the only maximal element of x and so $p \leq m, \ \forall p \in x$. Therefore, x is a principal ideal of P generated by m.

$(\Longleftarrow)$ If x is a principal ideal of P generated by m, then m is the only maximal element of x. By (a), $y \lessdot x \implies y = x - \{\mu\}$ for some maximal element μ of x, and since the only the only maximal element of x is m, $y = x - \{m\} \lessdot x$ is the only element covered by x. So x is join irreducible in L.

$\square$

Lemma 5.1. *$\lambda \lessdot \mu$ in $\mathcal{Y}$ if and only if λ and μ differ by exactly 1 in exactly 1 part.*

Proof. Recall that $\lambda = (\lambda_1, \dots \lambda_m) \leq \mu = (\mu_1, \dots, \mu_n)$ in $\mathcal{Y}$ if and only if $Y(\lambda) \subseteq Y(\mu)$ where $Y(\tau)$ denotes the Young diagram of a partition τ in $\mathcal{Y}$. Equivalently, $\lambda_i \leq \mu_i, \forall i \in [n]$. Also recall that if $m < n$ we treat λ as having an $n-m$-tail of 0s;

$$\lambda = (\lambda_1, \dots, \lambda_m, \overbrace{0, \dots, 0}^{m-n}) = \left(\substack{\lambda_i, \forall i \in [m] \\ 0, \text{otherwise}}\right)_{i \in [n]}$$

($\implies$) If $\lambda \lessdot \mu$, suppose that (i) λ and μ differ by more than 1 in some part or that (ii) λ and μ differ in more than 1 part. note that they must differ *in some part*, $\lambda_i \lessdot \mu_i, \forall i \in [n]$ and also:

$$\begin{aligned} &(i)\ \exists k \in [n] \text{ s.t. } \mu_k - \lambda_k = \delta > 1 \implies \lambda_k < \lambda_k + 1 < \lambda + \delta = \mu_k \\ &(ii)\ \exists a, b \in [n] \text{ s.t. } \substack{\lambda_a < \mu_a \\ \lambda_b < \mu_b} \end{aligned}$$

But then

$$(i)\ \lambda < \left(\substack{\lambda+1, \text{ for } i=k \\ \lambda, \text{ otherwise.}}\right) < \left(\substack{\lambda+\delta, \text{ for } i=k \\ \mu_i, \text{ otherwise.}}\right) = \mu \text{ or } (ii)\ \lambda < \left(\substack{\mu_i, \text{ for } i=a, \\ \lambda_i, \text{ otherwise.}}\right) < \left(\substack{\mu_i, \text{ for } i=a,b, \\ \lambda_i, \text{ otherwise.}}\right) = \mu$$

and then $\lambda \not\lessdot \mu$, a contradiction. So λ and μ differ by exactly 1 in exactly 1 part.

($\impliedby$) If λ and μ differ by exactly 1 in exactly 1 part, then $\exists! k \in [n]$ such that $\lambda_k < \mu_k$ and since $\mu_k - \lambda_k = 1$, obviously:$\lambda_k \lessdot \mu_k$ in $\mathbb{Z}^+ \implies \lambda \lessdot \mu$.

□

Problem 5.18. **(Sagan, Chapter 5, Exercise 18)**
(a) Given a poset P, let $\mathcal{A}(P)$ be the set of antichains of P. Show that the map

$$f : \mathcal{A}(P) \to J(P)$$

given by $f(A) = I(A)$ (where $I(A)$ is the order ideal generated by A) is a bijection.

Proof. **(a)** Let $f^{-1} : J(P) \to \mathcal{A}(P)$ be defined by $f^{-1}(I) = \max(I) = \{i \in I \mid i \text{ is maximal in } I\}$. This mapping is well-defined, since distinct maximal elements of an order ideal cannot be comparable, otherwise they wouldn't be maximal. Also, it is obvious you can't generate two distinct order ideals from the same antichain. Behold. $\forall A \in \mathcal{A}, \forall I \in J(P)$:

$$\begin{aligned} &f(f^{-1}(I)) = f(\max(I)) = I(\max(I)) = I \\ &f^{-1}(f(A)) = f^{-1}(I(A)) = \max\{i \in I(A) \mid i \text{ is maximal in } I(A)\} = A \end{aligned}$$

So f^{-1} is the bijective inverse of f and $\mathcal{A}(P)$ and $J(P)$ are in bijection via f, f^{-1}. □

We do (b) on the next page.

(b) Show that $\mu \in Y$ is join irreducible if and only if $\mu = (k^l)$ for some $k, l \in \mathbb{P}$

Proof. **(b)** Let $\mu = (\mu_1, \ldots, \mu_n)$.

($\implies$) If μ is join irreducible in $\mathcal{Y}$, then there exists exactly 1 partition λ in $\mathcal{Y}$ such that $\lambda = (\lambda_1, \ldots, \lambda_m) \lessdot \mu$. By **Lemma 5.1** any such λ must differ by 1 in exactly 1 part. Well, if λ is the only such partition, then removing 1 from any single part of μ results in the same partition λ and so μ must have all parts of equal size. That is, μ must be a rectangle of the form $\mu = (k^l)$ for some $k, l \in \mathbb{Z}^+$.

($\impliedby$) If μ is a rectangle, then removing 1 from any single part of μ results in the same partition, and since any covered element $\lambda \lessdot \mu$ is just the result of removing 1 from a single part of μ, there is only one such λ. $\exists! \lambda \in \mathcal{Y}$ such that $\lambda \lessdot \mu \implies \mu$ is join irreducible in $\mathcal{Y}$.

□

Problem 5.19. (**Sagan, Chapter 5, Exercise 19**)
(a) Rederive the formula for μ in B_n, equation (5.6), in two ways: by mimicking the proof of (5.7) and by constructing an $m \in P$ such that $D_m = B_n$ and then applying (5.7).

Proof. **(I)** Recall the following:

$$B_n \cong C_1^n \text{ via } \phi(S) = \left(\substack{1, \text{ if } i \in S, \\ 0, \text{ otherwise.}}\right)_{i \in [n]} \text{ and } \mu_{C_1} := i \mapsto \begin{cases} 1 & \text{if } i = 0, \\ -1 & \text{if } i = 1 \end{cases}$$

Behold. By **Theorem 5.4.3** and **Theorem 5.4.4**

$$\mu_{C_1^n}(a_1, \ldots, a_n) = \prod_{i=1}^{n} \mu_{C_1}(a_i) = (-1)^k \text{ where } k = \sum_{i=1}^{n} a_i.$$

$$\text{If } \phi_i(S) \text{ is the } i\text{th component of } \phi(S), \forall S \in B_n, \text{ then } \sum_{i=1}^{n} \phi_i(S) = \sum_{i=1}^{n} \left(\substack{1, \text{ if } i \in S, \\ 0, \text{ otherwise.}}\right) = |S|.$$

$$\implies \mu_{B_n}(S) = \mu_{C_1^n}(\phi(S)) = \mu_{C_1^n}\left(\left(\substack{1, \text{ if } i \in S, \\ 0, \text{ otherwise.}}\right)_{i \in [n]}\right) = \prod_{i=1}^{n} \mu_{C_1}(\phi_i(S)) = (-1)^{|S|}.$$

(II) Let $m = \prod_{i=1}^n p_i$ be a product of n distinct primes. Then any $d \in D_m$ is of the form $d = \prod_{i=1}^n p_i^{a_i}$ where $a_i \in \{0, 1\}$ and then by **Problem 5.7** we get that $D_m \cong C_1^n \cong B_n$ via

$$d_S = \prod_{i=1}^{n} p_i^{a_i} \longleftrightarrow (a_i)_{i \in [n]} \longleftrightarrow \{i \in [n] \mid a_i = 1\} = S$$

$$\implies \sum_{i=1}^{n} a_i = |S| \text{ if } d_S \longleftrightarrow S$$

$$\text{and therefore } \mu_{D_m}(d_S) = (-1)^{\sum_{i=1}^n a_i} \implies \mu_{D_n}(d_S) = (-1)^{|S|} = \mu_{B_n}(S)$$

□

Problem 5.20. (Sagan, Chapter 5, Exercise 20)

(a) Let P be a locally finite poset with a $\hat{0}$. Show that if x covers exactly one element of P, then

$$\mu(x) = \begin{cases} -1 & \text{if } x \text{ covers } \hat{0}, \\ 0 & \text{otherwise.} \end{cases}$$

(b) Given any $n \in \mathbb{Z}$, construct a poset containing an element x with $\mu(x) = n$.

Proof. **(a)** Recall that for a locally finite poset P with $\hat{0}$,

$$\mu(x) = \begin{cases} 1 & \text{if } x = \hat{0} \\ -\sum_{y<x} \mu(y) & \text{if } x > \hat{0} \end{cases}, \ \forall x \in P.$$

If an element $x \in P$ covers exactly one element we have two cases. (i) $0 \lessdot x \implies \mu(x) = -\sum_{y<x} \mu(y) = -\mu(\hat{0}) = -1$. (ii) Otherwise $\exists! q \in P$ such that $\hat{0} < q \lessdot x$, and then since P is locally finite, $[\hat{0}, q] \cong C_n$ for some $n \in \mathbb{Z}^+$ and then

$$P \ni y \longleftrightarrow |[\hat{0}, y]| \in C_n \text{ and } \mu_{C_n}(i) = \begin{cases} 1 & \text{if } i = 0 \\ -1 & \text{if } i = 1 \\ 0 & \text{otherwise} \end{cases}$$

$$\implies \mu(x) = -\sum_{y<x} \mu(y) = -\sum_{i=1}^{n} \mu_{C_n}(i) = 1 + (-1) + 0 + \ldots + 0 = 0.$$

Thus,

$$\exists! q \in P \text{ such that } q \lessdot x, \text{ then } \mu(x) = \begin{cases} -1 & \text{if } \hat{0} \lessdot x \\ 0 & \text{otherwise.} \end{cases}$$

(b) $\mu_{C_3}(2) = 0$ as shown in the first example of **Chapter 5.5**. Next, For any $n \in \mathbb{Z}^+$, let $P = \{\hat{0}, x_1, \ldots, x_{n+1}, x\}$ be a poset whose only relations are $0 \lessdot x_i \lessdot x, \ \forall i \in [n+1]$. Then

$$\textbf{(a)} \text{ and } \hat{0} \lessdot x_i \implies \mu(x_i) = -1, \ \forall i \in [n+1]$$
$$\implies \mu(x) = -\sum_{y<x} \mu(y) = -(1 + \sum_{i \in [n+1]} \mu(x_i)) = -(1 + \sum_{i \in [n+1]} (-1)) = n$$

and then once again by **Theorem 5.4.4** $\mu_{C_1 \times P}(1, x) = \mu_{C_1}(1)\mu_P(x) = -1 \cdot n = -n$. □