Definition 0.1. [1] A σ^{+-} labeling of G is an injection $f : V(G) \to \{0, 1, \ldots, 2m-2\}$ which induces a bijective length function $\ell : E(G) \to \{1, 2, \ldots, m\}$ where $\ell(uv) = f(v) - f(u)$ for every edge $uv \in E(G)$ with $u \in A$ and $v \in B$, and f has the additional property that $f(u) - f(v) \neq m$ for all $u \in A$ and $v \in B$.

Theorem 0.2. *(Freyberg and Tran, [1]). Let G be a bipartite graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant edge e. Then there exists a G-decomposition of K_{2mr} and K_{2mr+1} for every positive integer r.*

Construction 0.3. *Let F be a forest graph on seven edges, and consider K_{21}. Recall that any K_n via ℓ has edge lengths in $L = \{1, 2, \ldots, \lfloor \frac{n}{2} \rfloor\}$. If B_σ is the σ^{+-} labeling of F, then by Theorem 0.2 every edge of length in $L_\sigma = \{1, 2, \ldots, 7\} \subset L$ is generated by B_σ via $\phi_1 := v \mapsto v+1$ on $V(B_\sigma)$. Let $\mathcal{B}_\sigma = \langle B_\sigma \rangle_{\phi_1}$. So then $F \setminus \bigcup_{G \in \mathcal{B}_\sigma} G$ contains all edges with lengths in L_σ.*

Recall that in any K_n there exist n edges of each distinct length, but every edge is also uniquely determined by it's endpoints. If we partition the vertices of K_{21} modulo 7, naturally this induces a singleton edge partition modulo 7×7. We can consolidate this partition via $\ell_7 :=\ ab \mapsto a+b \pmod 7$ on the edges. So now we see that in partitioning the edges of K_{21} via ℓ_7 and ℓ, each partite set $P_{i,j}$ will have three edges of the same length which also belong to the same equivalence class with respect to ℓ_7 where $(i, j) \in L \times \mathbb{Z}_7$. The result is that $L_i = \bigcup_{j \in \mathbb{Z}_7} P_{i,j} = \{uv \in E(K_{21}) \mid \ell(uv) = i\}$.

Let $\phi_7 := v \mapsto v+7$, $B_{i,j} \subset K_{21}$ be a subgraph which contains some edge ab with $(\ell \times \ell_7\ (ab)) = (i, j)$, and $\mathcal{B}_{i,j} = \langle B_{i,j} \rangle_{\phi_7}$. Then if $\mathcal{H}_{i,j} = \bigcup_{G \in \mathcal{B}_{i,j}} G$ we must have that $P_{i,j} \subset E(\mathcal{H}_{i,j})$. Therefore, we must have that $L_i \subset E(\bigcup_{j \in \mathbb{Z}_7} \mathcal{H}_{i,j}) = \{uv \in E(K_{21}) \mid \ell(uv) = i\}$.

So if there exist edge-disjoint blocks $B_1, B_2, B_3 \cong F$ with only edges of lengths in L^*, let $\mathcal{B}_i = \langle B_i \rangle_{\phi_7}$ for $i \in \{1, 2, 3\}$. So $\bigcup_{i \in \{1,2,3\}} \mathcal{B}_i$ must contain all edges of lengths in L^* in K_{21} and therefore,

the existence of such blocks give $\mathcal{B}_{21} = \mathcal{B}_\sigma \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3$, an F-design of order 21.

Construction 0.4. *Let F be a forest graph on seven edges and consider K_{22}. We let B_σ refer to the same block as previously described in the construction for $F-$designs of order 21. Similarly to in the previous construction, if B_1, B_2, B_3, B_4 are edge-wise disjoint blocks which are isomorphic to F with only edges of lengths $\{8, 9, 10\} \cup \{\infty\}$ via ϕ_7 where we take $\infty = 0$ with respect to computing ℓ_7. Then using the previous conventions for $\mathcal{B}_i$ where $i \in \{1, 2, 3, 4, \sigma\}$,*

the existence of such blocks give $\mathcal{B}_{22} = \mathcal{B}_\sigma \cup \mathcal{B}_1 \cup \mathcal{B}_2 \cup \mathcal{B}_3 \cup \mathcal{B}_4$, an F-design of order 22.

Definition 0.5. Consider any edge $uv \in E(K_n)$. We say:

$$\begin{aligned} uv \text{ is a wraparound edge if } \ell(uv) &= n - |u-v| && (1)\\ uv \text{ is a short edge if } \ell(uv) &= |u-v| && (2)\\ & && (3) \end{aligned}$$

.

Now for any edge $uv \in E(K_n)$, without loss of generality $v < u$ and we say:

$$\begin{aligned} v \text{ is a wraparound vertex if } uv \text{ is a wraparound edge} && (4)\\ u \text{ is a short vertex if } \forall ux \in E(K_n), ux \text{ is a short edge} && (5) \end{aligned}$$

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Algorithm 0.6. *Let F be a forest on seven edges, $\mathcal{B}_{21}, \mathcal{B}_{22}$ be the $F-$decompositions of K_{21}, K_{22}, respectively, given by Constructions 0.3 and 0.4, and consider K_{14t+7} along with K_{14+8} where we take the vertices to be members of $\mathbb{Z}_{21} \cup \{\infty\}$ in the natural way (according to C_{14t+7} and C_{14+8} where $22 \mapsto \infty$ along with all it's incident edge lengths).*

For each block in

Construction 0.7. *Let F be a forest on seven edges and consider K_{14t+7} and K_{14t+8} for $t > 1 \in \mathbb{N}$. By definition, K_{14t+7} has edge-lengths $[1, 7t+3]$. Let us define the following edge length intervals: $I_\sigma = [1, 7]$, $I_0 = [8, 10]$ and $I_j = (I_\sigma + 10) + 7(j-1) = [11 + 7(j-1), 17 + 7(j-1)]$ for all $j > 0$. So $L = \bigcup_{0 \le j \le t} I_j \cup I_\sigma$ is the set of all distinct lengths in K_{14t+7} and $L \cup \{\infty\}$ is the set of all distinct lengths in K_{14t+8} for $t > 1$.*

References

[1] B. Freyberg and N. Tran, Decomposition of complete graphs into bipartite unicyclic graphs with eight edges, J. Combin. Math. Combin. Comput., *114* (2020), 133-142.

[2] R. C. Bunge, A. Chantasartrassmee, S.I. El-Zanati, and C. Vanden Eyn-den, On cyclic decompositions of complete graphs into tripartite graphs, *J. Graph Theory* **72** (2013), 90–111.