Decomposing Complete Graphs into Disconnected Graphs with Six Edges

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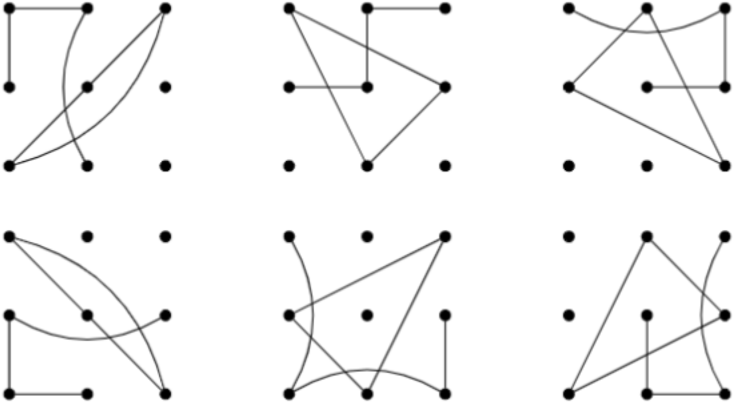
- If $K \cong K_n$, then we call the decomposition a G-*design* of order n

A G-design of Order 9 for $G \cong C_3 \cup P_4$

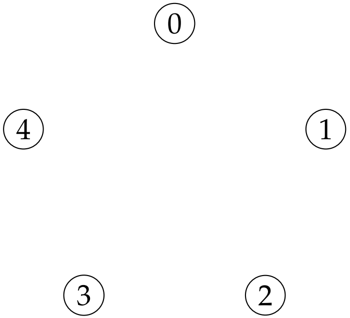


Cyclic Designs

- Let $V(K_n) = \mathbb{Z}_n$
- A G-design is *cyclic* if the permutation $v \mapsto v + 1$ on $V(K_n)$ is an automorphism of the design
- We call this *clicking*

Example of a Cyclic Design

Cyclic P_3-design of order 5



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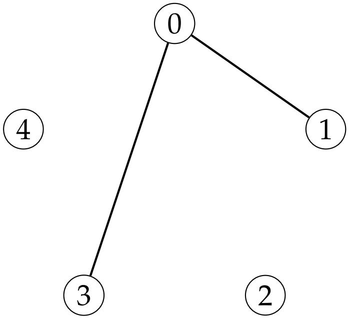

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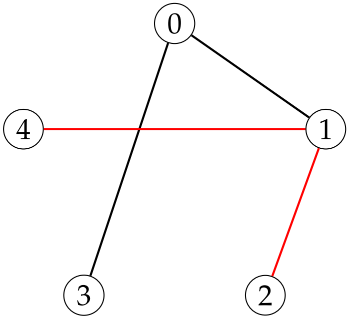

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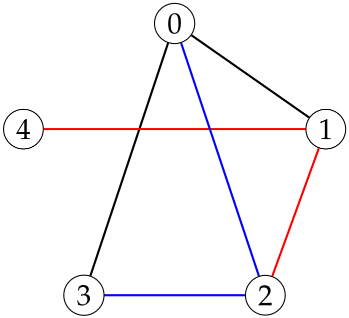

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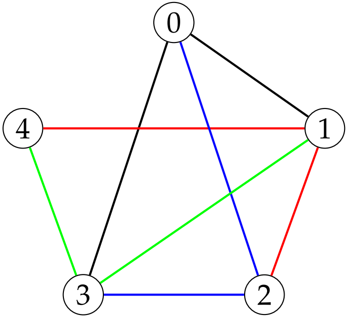

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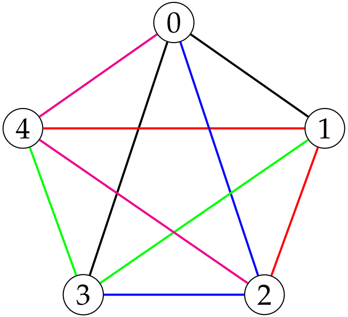

Small Graphs

- If $|E(G)| \leq 5$, then the spectrum of n such that a G-design of order n exists is known
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  - $|V(G)| = 7$ (Trees: Huang, Rosa, 1978)
  - $|V(G)| \geq 7$ (Disconnected graphs)
    - Forests (F, Peters 2023+)
    - Unicyclic graphs (Ahern, F, Froncek, Keranen, 2022+)

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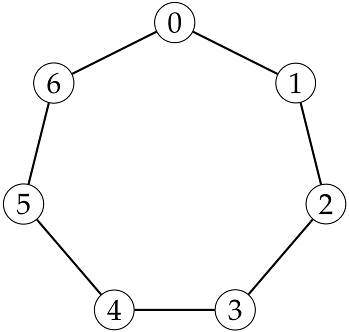

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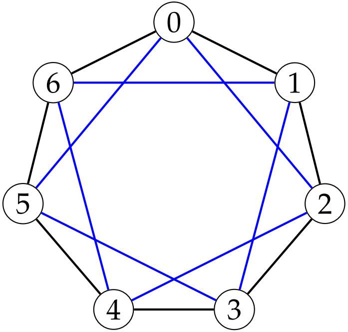

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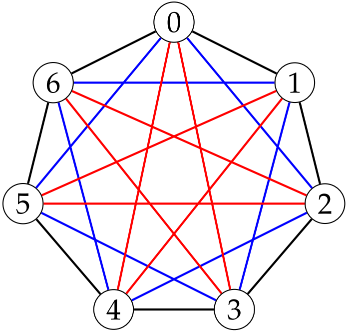

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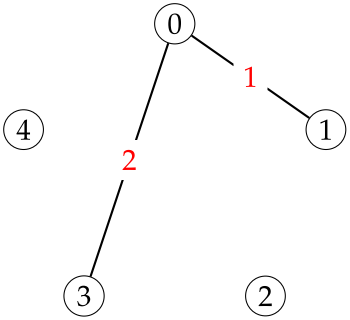

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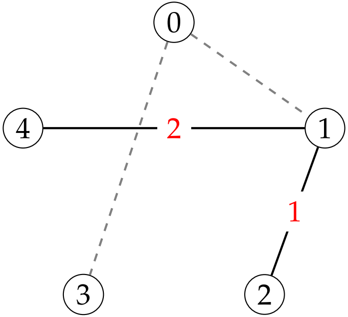

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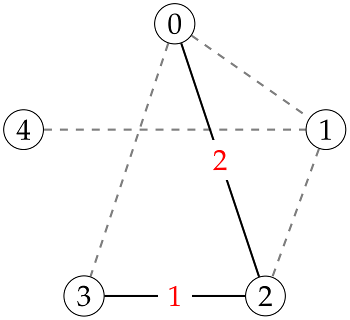

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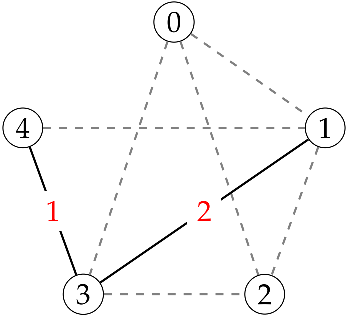

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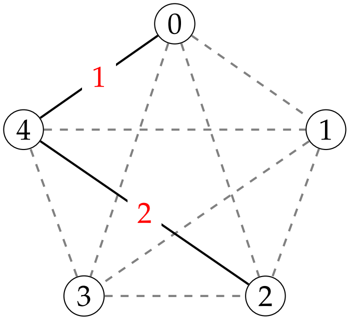

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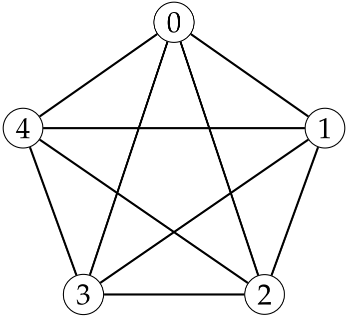
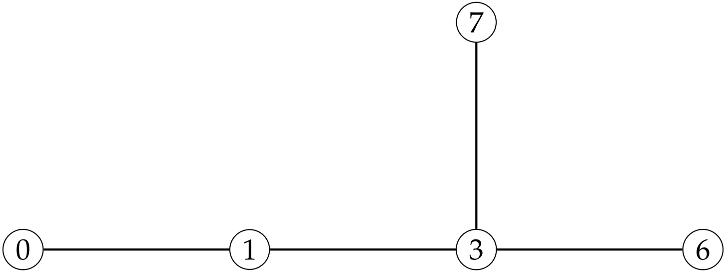


Rosa-type Labelings

ρ-labeling

Let G be a simple graph with n edges. A ρ-*labeling* of G is a one-to-one function $f : V(G) \to \{0, 1, ..., 2n\}$ such that the set of induced edge lengths is $\{1, 2, ..., n\}$.



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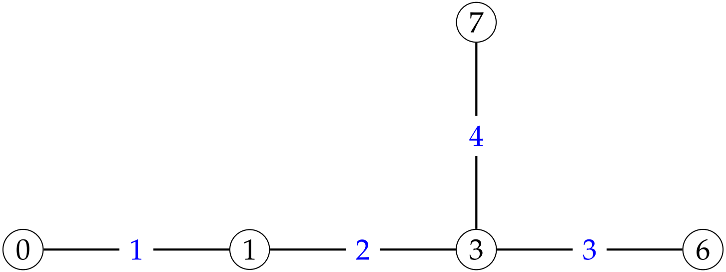

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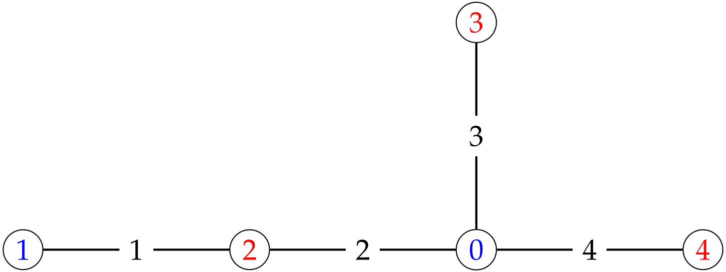
Theorem (Rosa, 1967)

Let G be a graph with n edges. A cyclic decomposition of K_{2n+1} exists if and only if G admits a ρ-labeling.

Rosa-type Labelings

Ordered ρ-labeling

A ρ-labeling of a bipartite graph G with bipartition (X, Y) is called an *ordered* ρ-labeling and denoted ρ^+, if $f(x) < f(y)$ for each edge xy with $x \in X$ and $y \in Y$.



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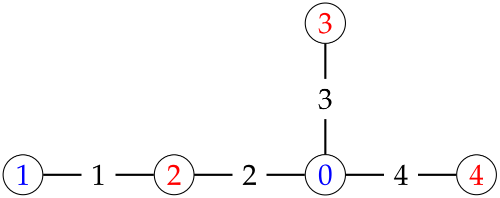

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  2. For every edge bc with $b \in B$ and $c \in C$, there exists a complementary edge $b'c'$ with $b' \in B$ and $c' \in C$ such that
  $$|f(b) - f(c)| + |f(b') - f(c')| = 2n.$$
  3. For all $b \in B$ and $c \in C$, we have
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Example

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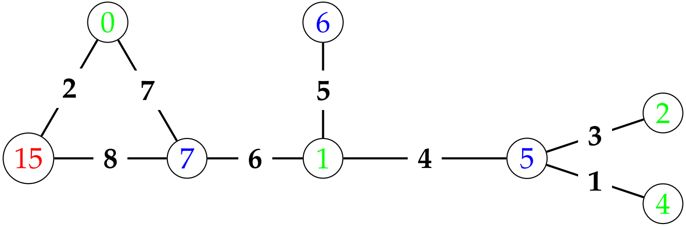

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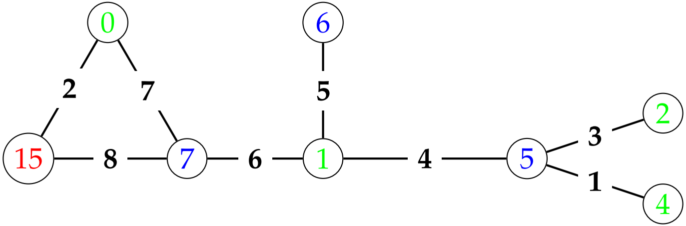

1-Rotational Designs

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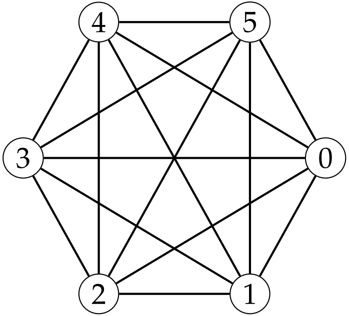

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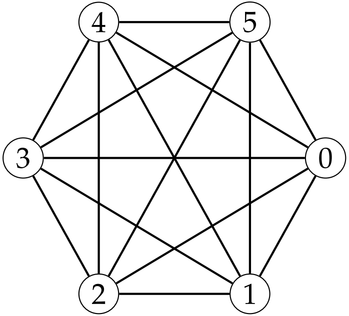

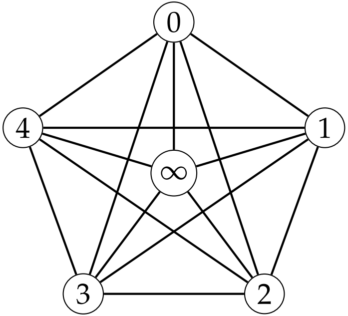

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- Let $V(K_{2n}) = \{0, 1, 2, ..., 2n-2, \infty\}$ and G be a graph with n edges and at least one pendant edge.

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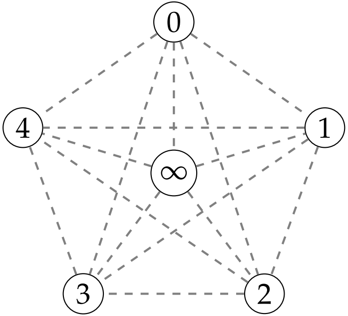
A 1-rotational P_4-design of order 6



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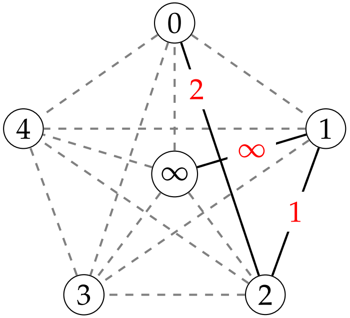

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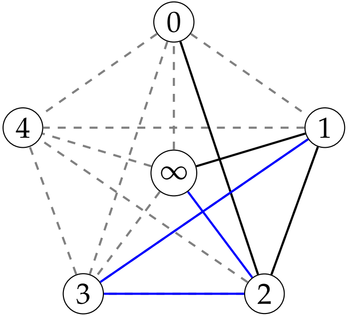

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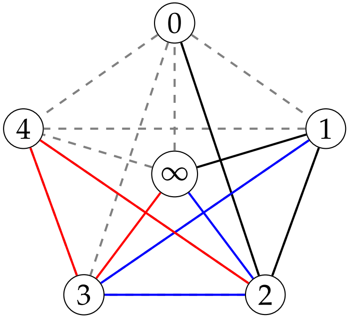

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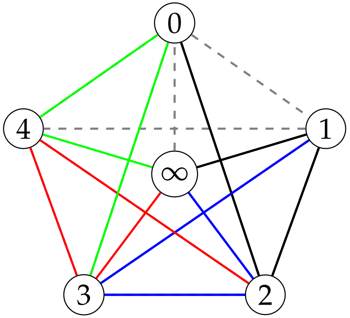

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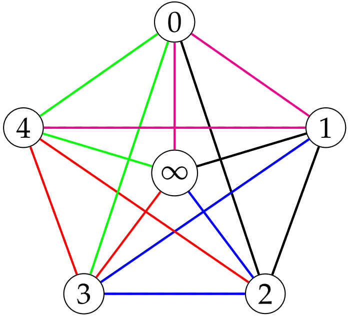

Bipartite Graphs

- Recall the length of $xy \in E(K_n)$ is $\min(|x-y|, n-|x-y|)$.
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- Recall the length of $xy \in E(K_n)$ is $\min(|x-y|, n-|x-y|)$.
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- F and Tran introduced the following restricted σ-labeling in 2020.

Definition

Let G be a bipartite graph with n edges and bipartition $V(G) = A \cup B$. A σ^{+-}-*labeling* of G is a σ-labeling with:

1. $f(a) < f(b)$ for every edge $ab \in E(G)$ with $a \in A$ and $b \in B$
2. $f(a) - f(b) \neq n$ for all $a, b \in V(G)$
3. $f(v) \notin \{2n-1, 2n\}$ for all $v \in V(G)$

Bipartite Graphs

Theorem (F, Tran, 2020)

Let G *be a graph with* n *edges and a* σ^{+-}*-labeling such that the edge of length* n *is a pendant edge. Then there exists cyclic* G*-decompositions of* K_{2nt} *and* K_{2nt+1} *for every positive integer* t*.*

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- A *1-rotational* ρ*-tripartite labeling* of G is a 1-rotational ρ-labeling f that:
  1. $f(y) = \infty$.
  2. $f(a) < f(v)$ for every edge av with $a \in A$.
  3. For every edge bc with $b \in B$ and $c \in C$, there exists a complementary edge $b'c'$ with $b' \in B$ and $y' \in Y$ such that

$$|f(b) - f(c)| + |f(b') - f(c')| = 2n.$$

1-Rotational ρ-tripartite Labeling

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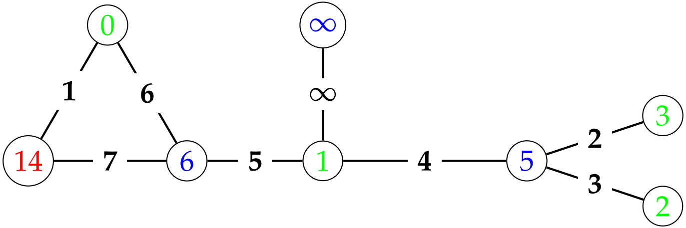
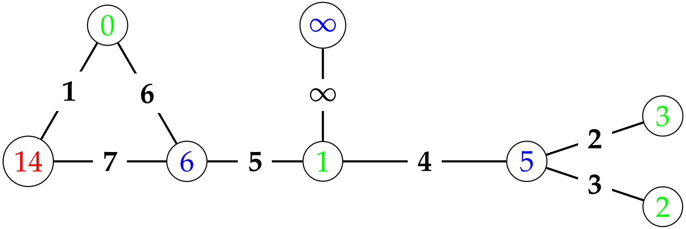


1-Rotational Designs

Theorem (Bunge, 2019)

Let G be a tripartite graph on n edges with at least one pendant edge. If G admits a 1-rotational ρ-tripartite labeling, then there exists a 1-rotational decomposition of K_{2nt} for any positive integer t.



Necessary Conditions

Observation

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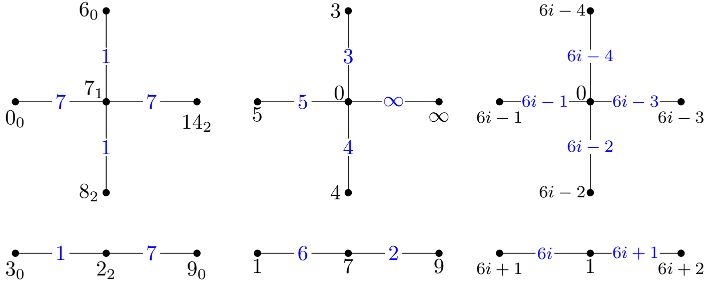
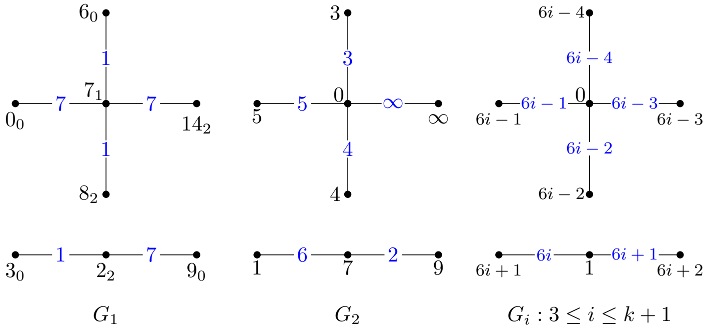
- We'll adapt the techniques; click multiple blocks

A Forest and $n = 12k + 4$



G_1 G_2 $G_i : 3 \leq i \leq k+1$

A Forest and $n = 12k + 4$



- Click G_1 by 3 and G_i by 1 for $2 \leq i \leq k+1$ (we're working in Z_{12k+3})
- Number of edges of each length $= 3 \times \frac{n-1}{3} = 1 \times (n-1) = n-1$
- Total number of edges $= (n-1) \times \frac{n}{2} = \binom{n}{2}$

Approach for Tripartite Unicyclics and $n = 12k + 9$

- Partition $V(K_n)$ into three *groups* of cardinality $4k + 3$:

$$\begin{aligned} A &= \{(0,0),(1,0),\ldots,(4k+2,0)\} \\ B &= \{(0,1),(1,1),\ldots,(4k+2,1)\} \\ C &= \{(0,2),(1,2),\ldots,(4k+2,2)\} \end{aligned}$$

- The edge set of K_n can be expressed as

$$E(K_n) = \{(i,j)(i',j') : j = j', i \neq i'\} \cup \{(i,j)(i',j') : j \neq j'\}$$

- Intragroup edge length defined as usual for K_{4k+3}
- Intergroup edge length defined as $\ell((i,j)(i',j+1)) = i' - i$ where $i' - i$ is reduced modulo $4k + 3$ and $j + 1$ is taken modulo 3
- Exactly one edge of each length can be obtained by clicking $3k + 2$ blocks

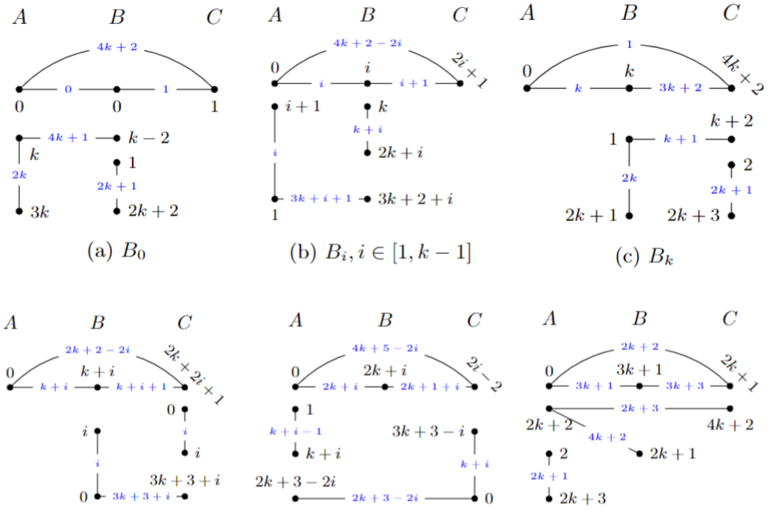
Approach for Tripartite Unicyclics and $n = 12k + 4$

- Partition $V(K_n)$ into $\{\infty\}$ and three *groups* of cardinality $4k + 1$:

$$\begin{aligned} A &= \{(0,0),(1,0),\ldots,(4k,0)\} \\ B &= \{(0,1),(1,1),\ldots,(4k,1)\} \\ C &= \{(0,2),(1,2),\ldots,(4k,2)\} \end{aligned}$$

- The edge set of K_n can be expressed as before with the addition of $n - 1$ edges of length ∞.
- Intragroup edges take on lengths $\{1, 2, \ldots, 2k\}$
- Intergroup edges take on lengths $\{0, 1, \ldots, 4k\}$ between each pair of groups

A $(C_3 \cup P_3 \cup P_2)$-design of order $12k+9$



(a) B_0 (b) $B_i, i \in [1, k-1]$ (c) B_k

(d) $B_{k+i}, i \in [1, k]$ (e) $B_{2k+i}, i \in [1, k]$ (f) B_{3k+1}

Main Result

Theorem (Ahern, F, Froncek, Keranen, Peters, 2022+)

If G *is a disconnected graph with six edges, then a* G*-design of order* n *exists if and only if* $n \equiv 0, 1, 4,$ *or* $9 \pmod{12}$ *unless either* $n = 4$ *or* $n = 9$ *and* G *is isomorphic to one of the graphs listed below.*

- ☐ $K_{1,5} \cup K_2$
- ☐ $K_{1,4} \cup 2K_2$
- ☐ $K_{1,3} \cup 3K_2$
- ☐ $P_4 \cup 3K_2$
- ☐ $2P_3 \cup 2K_2$
- ☐ $P_3 \cup 4K_2$
- ☐ $6K_2$
- ☐ $K_3 \cup K_{1,3}$

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- ☐ $6K_2$
- ☐ $K_3 \cup K_{1,3}$

THANK YOU!