Designs for forests with seven edges

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Abstract

Read our paper!

1 Introduction

Write stuff here.

2 Tools

I am putting various theorems and constructions with way too much detail here so we can trim them down.

Definition 2.1. [8] A σ^{+-} labeling of G is an injection $f : V(G) \to \{0, 1, \ldots, 2m-2\}$ which induces a bijective length function $\ell : E(G) \to \{1, 2, \ldots, m\}$ where $\ell(uv) = f(b) - f(u)$ for every edge $uv \in E(G)$ with $u \in A$ and $v \in B$, $a < b$ and f has the additional property that $f(u) - f(v) \neq m$ for all $u \in A$ and $v \in B$.

Theorem 2.2. *(Freyberg and Tran, [8]). Let G be a bipartite graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant edge e. Then there exists a G-decomposition of K_{2mr} and K_{2mr+1} for every positive integer r.*

Observation 2.3. *Consider K_n for any $n \in \mathbb{N}$ and ℓ on it's edges as stated in Definition 2.1. For any edge $e \in E(K_{21})$, $\ell(e) \in \{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$ and there exist n edges with each distinct length in $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$. Equivalently, each vertex $v \in V(K_{21})$ is a member of exactly two distinct edges e_1, e_2 of each length in $\{1, \ldots, \lfloor \frac{n}{2} \rfloor\}$.*

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Construction 2.4. *Let F be a forest on 7 edges and consider K_{21}. By 2.3, any edge in K_{21} has a length in $L = \{1, \ldots, 10\}$ via ℓ as stated in Definition 2.1. Next, by Theorem 2.2, if F_σ is a $\sigma^{+-}-$labeling of F, then $B_\sigma = \langle F_\sigma \rangle$ via $\phi_1 := v \mapsto v+1$ contains all edges of K_{21} with lengths in $L_\sigma = \{1, \ldots, 7\}$ by the union of it's members. Additionally, all $G \in B_\sigma$ are isomorphic to F.*

Now, clearly each edge in K_{21} is uniquely determined by it's endpoints. So naturally observe that if we partition the vertices of K_{21} modulo 7, a singleton edge partition of K_{21} modulo 7×7 is induced. Let $\ell := ab \mapsto a + b \pmod 7$. We now partition the edges of K_{21} modulo $\sim$ where $ab \sim cd$ if $\ell(ab) = \ell(cd)$ and $\ell_7(ab) = \ell_7(cd)$, and specify each partite set $P_{i,j}$ via $ab \in P_{i,j}$ if $\ell \times \ell_7(ab) = (i, j)$. Each $P_{i,j}$ contains three edges in the same equivalence class modulo $\sim$; three edges of the same length i which are in the same endpoint equivalence class modulo 7×7. Lastly, let $L_i = \{uv \in E(K_{21}) \mid \ell(uv) = i\}, \forall i \in L$, then $L_i = \bigcup_{j \in \mathbb{Z}_7} P_{i,j}$.

Let $\phi_7 := v \mapsto v+7$ and $H_{i,j} \subseteq K_{21}$ be a subgraph containing some edge ab such that $\ell \times \ell_7(ab) = (i, j)$. Then if $G_{i,j} = \bigcup_{G \in \langle H_{i,j} \rangle_{\phi_7}} G$, $P_{i,j} \subseteq E(G_{i,j})$; each $G_{i,j}$ contains the partition $P_{i,j}$. Then since for any $(i, j) \in L \times \mathbb{Z}_7$, such a subgraph $H_{i,j}$ exists, so does such a $G_{i,j}$. Therefore, for any $i \in L$, $L_i \subseteq E(\bigcup_{j \in \mathbb{Z}_7} G_{i,j})$ for such $G_{i,j}$. In other words, we can generate all edges of length $i \in L$ in K_{21} by finding subgraphs containing edges of length i from each distinct equivalence class modulo 7×7 and applying ϕ_7 to their vertices.

So finally, if there exist $\ell \times \ell_7-$wise edge-disjoint subgraphs $F_1, F_2, F_3 \cong F$ of K_{21} with only edges of lengths in $L^* = \{8, 9, 10\}$, let $B_i = \langle F_i \rangle_{\phi_7}$ for each $i = 1, 2, 3$. Then $\bigcup_{i=1,2,3} B_i$ must contain all edges of K_{21} with lengths in L^* by the union of it's members. Let $B_{21} = B_\sigma \cup B_1 \cup B_2 \cup B_3$ and recall that B_σ contains all edges of lengths in L_σ by the union of it's members. So then since $L = L_\sigma \cup L^*$, $K_{21} = \bigcup_{G \in B_{21}} G$, and all $G \in B_{21}$ are isomorphic to F and edge disjoint by definition. Therefore,

the existence of such subgraphs F_σ, F_1, F_2, F_3 give $B_{21} = B_\sigma \cup B_1 \cup B_2 \cup B_3$, an $F-$decomposition of K_{21}.

Construction 2.5. *Let F be a forest graph on seven edges and consider K_{22}. For this construction we simply take the 22nd vertex to be ∞; we add ∞ to the neighborhood of each vertex in K_{21}, and let $\ell(x\infty) = \infty$ for all $x \in V(K_{22})$. Next, we let B_σ refer to the same block as previously described in Construction 2.4. Similarly to in the previous construction, if $F_1, F_2, F_3, F_4 \cong F$ are $\ell \times \ell_7-$wise edge-wise disjoint subgraphs of K_{21} with only edges of lengths in $L^* = \{8, 9, 10\} \cup \{\infty\}$, we let $B_i = \langle F_i \rangle_{\phi_7}$ via ϕ_7 as defined in Construction 2.4 with the new condition $\ell(x\infty) = x \pmod 7$. Then $B_{22} = B_\sigma \cup B_1 \cup B_2 \cup B_3 \cup B_4$ has the same properties as B_{21} from Construction 2.4 with L^* as defined in this construction. Therefore,*

the existence of such subgraphs $F_\sigma, F_1, F_2, F_3, F_4$ give $B_{22} = B_\sigma \cup B_1 \cup B_2 \cup B_3 \cup B_4$, an $F-$decomposition of K_{22}.

Construction 2.6. *Let F be a forest on seven edges and consider K_{14t+7} and K_{14t+8} for some $t > 1 \in \mathbb{N}$. By Observation 2.3, K_{14t+7} has edge-lengths in $L = [1, 7t{+}3]$. Let us define the following edge length intervals: $L_\sigma = [1, 7]$, $I_0 = [8, 10]$ and $I_j = (I_\sigma{+}10){+}7(j{-}1) = [11{+}7(j{-}1), 17{+}7(j{-}1)]$ for all $0 < j \leq t$. Lastly let $L^*_\sigma = \bigcup_{0<j\leq t} I_j$ and $L^* = [8, 10]$.*

Then if there exists a $\sigma^{+-}-$labeling F_σ of F, we let B_σ be defined as in Constructions 2.4 and 2.5, and notice that all edges from K_{14t+7} and K_{14t+8} with lengths in I_σ are contained in the union of the members of B_σ. Next, recall the vertex partition of F_σ where all for all $u \in A$ and $v \in B$ with $uv \in E(F_\sigma)$, $u < v$ and additionally $\ell(uv) = |u - v|$.

Let $\varphi_j := v \mapsto v + 10(j-1)$ for all $0 < j \leq t$ and let F^j_σ simply be F_σ with φ_j on B and subsequently each $ab \mapsto a\varphi_j(b)$. Also let $B^j_\sigma = \langle F^j_\sigma \rangle_{\phi_7}$, for each $0 < j \leq t$ Recall that since F_σ contains no wraparound edges in in K_{21}, for any edge ab with length $|a - b| \in I_\sigma$, $|a - \varphi_j(b)| \in I_j$. So then since uv is a wraparound edge of K_{14t+7} and K_{14t+8} if and only if $|a - b| > 7t + 3$, the maximal edge length, F^j_σ must not contain any wraparound edges for each $0 < j \leq t$. So then each F^j_σ must contain exactly one edge of each length in I_j. Therefore for any $1 < j \leq t$, since $|B^j_\sigma| = 14t + 7$, B^j_σ must contain all edges K_{14t+7} and K_{14t+8} with lengths in I_j. Therefore $B^*_\sigma = \bigcup_{0<j\leq t} B^j_\sigma$ must contain all edges of K_{14t+7} and K_{14t+8} lengths in L^*_σ in the union of it's members.

Now, if $\ell \times \ell_7-$wise edge-disjoint subgraphs isomorphic to F with the properties defined at end of Constructions 2.4 and 2.5 exist in K_{14t+7} and K_{14t+8}, respectively, let B_1, B_2, B_3 be defined as in Construction 2.4 and $\beta_i = B_i$ from Construction 2.5 for $i = 1, 2, 3, 4$. Finally, let $B_{14t+7} = B_\sigma \cup B^*_\sigma \cup B_1 \cup B_2 \cup B_3$ and let $B_{14t+8} = B_\sigma \cup B^*_\sigma \cup \beta_1 \cup \beta_2 \cup \beta_3 \cup \beta_4$. Notice that B_{14t+7} contains all edges in K_{14t+7} of lengths in $L = L_\sigma \cup L^*_\sigma \cup L^*$ and B_{14t+8} contains all edges in K_{14t+8} of lengths in $L \cup \{\infty\} = L_\sigma \cup L^*_\sigma \cup L^* \cup \{\infty\}$. Therefore since all $G \in B_{14t+7} \cup B_{14t+8}$ are isomorphic to F and edge-disjoint,

the existence of such subgraphs give B_{14t+7} and $B_{14t+8}, F-$decompositions of K_{14t+7} and K_{14t+8}, respectively.

Theorem 2.7. *In K_n, uv is a wraparound edge if and only if the absolute difference of its endpoints is greater than the maximal length in K_n; $\lfloor \frac{n}{2} \rfloor < |u{-}v|$.*

Proof. Let uv be an edge in K_n via ℓ defined previously. If uv is a wraparound edge, then $n - |u - v| < |u - v|$. So then $\frac{n}{2} - \frac{|u-v|}{2} < \frac{|u-v|}{2}$, and therefore $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < |u - v|$. If $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < |u - v|$, note that without loss of generality $u < v$. Then necessarily $n - |u - v| = n - (v - u) < v - u = |u - v|$, so then $n < 2(v - u)$ and $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < (v - u) < |u - v|$.

Thus,

$$\lfloor \frac{n}{2} \rfloor < |u - v| \iff \text{ uv is a wraparound edge.}$$

□

Theorem 2.8. *Let us refer to K_n whose vertices we take to be $\mathbb{Z}_n$ and $\mathbb{Z}_n \cup \{\infty\}$ when $n \equiv 7$ or $8 \pmod{14}$, respectively, as simply K_n. Next let us define the edge operation $\ell_n := \ell_n(uv) = \min\{|u-v|, n-|u-v|\}$, and let $t > 1,\ m > 0$ with $h = 14(t-1)$. For wraparound edge ab in K_{21}, without loss of generality $a < b$ and*

$$\ell_{14t+7}[(a-mh)\ (b-(m-1)h)] = \ell_{14t+7}[(a+(m-1)h\ (b+mh))] = \ell_{21}[ab],$$
$$(a-mh)+(b-(m-1)h) \equiv (a+(m-1)h+(b+mh)) \equiv a+b \pmod{7}.$$

Note that K_{14t+7} and K_{14t+8} share vertex labels with the exception of ∞ in K_{14t+8} and both come equipped with the ℓ_{14t+7} for all non-∞ edges.

Proof. Consider some wraparound edge $ab \in K_{21}$ with $\ell_{21}(ab) = \ell_{ab}$, without loss of generality $a < b$ and $|a-b| = b-a$. Then by definition, $\ell_{ab} = 21-(b-a)$ and $\ell_{ab} = (21+a)-b$. Note that $a < b < 21 \Rightarrow b < 21+a$. Suppose $a > 13$, then $|a-b| = b-a < 21-13 = 8 < 10$, the maximal length in K_{21}, so by Theorem 2.7, ab is not a wraparound edge, a contradiction. Therefore, $a \leq 13$.

Let $t > 1,\ h = 14(t-1)$ and let $\alpha = a-h, \beta = b \in V(K_{14t+7})$. Well, $(14t+7)-(14(t-1)) = 14(t-(t-1))+7 = 21$, so $21+a = (14t+7)-(14(t-1))+a = a-h \in V(K_{14t+7})$. So then since $a \leq 13$, $21+a \leq 34 < 14t+7$. So, $|\alpha-\beta| = |(21+a)-b| = |\ell_{ab}| = \ell_{ab}$. Now recall that $\ell_{ab} \leq \lfloor \frac{21}{2} \rfloor = 10$, the maximal length in K_{21}. Therefore, $|\alpha-\beta| \leq 10 = \lfloor \frac{14(1)+7}{2} \rfloor < \lfloor \frac{14t+7}{2} \rfloor$ and so by Theorem 2.7, $\lfloor \frac{14t+7}{2} \rfloor \not< |\alpha-\beta| \Rightarrow \alpha\beta$ is not a wraparound edge and therefore by definition $\ell_{14t+7}(\alpha\beta) = |\alpha-\beta| = \ell_{ab} = \ell_{14t+7}[(a-h)\ b] = \ell_{21}(ab)$.

Now let $\alpha' = a, \beta' = b+h \in V(K_{14t+7})$. Well, $\alpha' < \beta'$ and so $|\alpha'-\beta'| = \beta'-\alpha' = b+h-a = (b-a)+h$. Since, ab is a wraparound edge in K_{21}, $b-a > 10$, the maximal length. So $10+14(t-1) = 14t-4 < |\alpha'-\beta'|$. $(\lfloor \frac{14n+7}{2} \rfloor)$ and $(\lfloor \frac{28n-8}{2} \rfloor)$ are arithmetic sequences with increments 7 and 14, respectively, which are equal at $n = 1$. So for $t > 1$, $\lfloor \frac{14t+7}{2} \rfloor < \lfloor \frac{28t-8}{2} \rfloor = 14t-4 < |\alpha'-\beta'|$ and $\alpha'\beta'$ is a wraparound edge. Therefore, $\ell_{14t+7}(\alpha'\beta') = 14t+7-|\alpha'-\beta'| = 14t+7-((b-a)+h) = (14t+7)-(14(t-1))-(b-a) = 21-(b-a) = \ell_{ab} = \ell_{21}(ab)$ as well. Now by [8], for any edge $uv \in E(K_n)$, edge length via ℓ_n is preserved under the operation $uv \mapsto (u+1)\ (v+1)$ modulo n. So then the edge lengths of $\alpha\beta$ and $\alpha'\beta'$ are also preserved under $\theta := uv \mapsto (u-(m-1)h)\ (v-(m-1)h)$ and $\theta' := uv \mapsto (u+(m-1)h)\ (v+(m-1)h)$, respectively.

Finally, $(a-mh)+(b-(m-1)h) = a+b+((m-1)-m)(t-1)14 \equiv a+b+((m-1)+m)(t-1)14 \equiv (a+(m-1)h+(b+mh)) \equiv a+b \pmod{7}$. So then,

$$\alpha\beta \overset{\theta}{\mapsto} (\alpha-(m-1)h)\ (\beta-(m-1)h) = (a-mh)\ (b-(m-1)h),$$
$$\alpha'\beta' \overset{\theta'}{\mapsto} (\alpha'+(m-1)h)\ (\beta'+(m-1)h) = (a+(m-1)h)\ (b+mh)$$

preserves edge length and sum modulo 7 and therefore the statement is proven. $\square$

Theorem 2.9. *Let us refer to K_n whose vertices we take to be $\mathbb{Z}_n$ and $\mathbb{Z}_n \cup \{\infty\}$ when $n \equiv 7$ or $8 \pmod{14}$, respectively, as simply K_n. Next, let a, b be distinct vertices in K_{21} with $a < b$.*

If $t > 1,\ h = 14(t-1)$ and $\alpha = a - h,\ \beta = b + h \in V(K_{14t+7})$, then

$$b - a \neq 7 \textit{ or } b \not\equiv a \pmod 7 \Rightarrow \alpha \neq \beta.$$

Note that since we let $V(K_{14t+8}) = V(K_{14t+7}) \cup \{\infty\}$ this statement also holds for all non-∞ vertices in K_{14t+8}.

Proof. Recall that since $a, b \in \mathbb{Z}_{21}, a < b$, and they are distinct, $1 \leq b - a \leq 20$.

If $b-a \neq 7$, suppose $\alpha = \beta$. Then in $\mathbb{Z}_{14t+7}$: $a - 14(t-1) = b + 14(t-1)$ and so $b - a \equiv -28(t-1) \pmod{14t+7}$. Note that in $\mathbb{Z}_{14t+7}$: $-28 = 14t + 7 - (28) = 14t - 21 = 7(2t-3) \Rightarrow b - a \equiv 7(2t-3)(t-1) \pmod{14t+7}$.

So if $t = 2$, then $b - a \equiv 7 \pmod{14t+7}$ and so $b - a = 7$ or $b - a \geq 35 + 7 = 42$, both contradictions. If $t > 2$, since $-28 \equiv 21 \pmod{14(3)+7}$ and $(7(2n-3))_{n>2}$ is strictly increasing, $b - a \geq 21$, a contradiction. So $b - a \neq 7 \Rightarrow \alpha \neq \beta$.

If $b \not\equiv a \pmod 7$, then suppose $\alpha = \beta$. By the sequence $(7(2n-3))_{n>1}$, $b - a \equiv 7(2t-3)(t-1) \pmod{14t+7}$ and so $a - b \equiv 0 \pmod 7$, a contradiction. So $b \not\equiv a \pmod 7 \Rightarrow \alpha \neq \beta$. Therefore the statement is proven. □

Corollary 2.10. *Let ab be a wraparound edge in K_{21}, $t > 1$ and $h = 14(t-1)$ such that $a < b$. Then in K_{14t+7} : $a - h \geq 21$ and $b + h > 21$. Next, let $u, v \in K_{21}$ with $v \in [u]_7$. If $t = 2$, then $|u - v| \neq 14 \Rightarrow u \pm h \neq v$ and $v \pm h \neq u$. If $t > 2$, $u \pm h \neq v$ and $v \pm h \neq u$ in K_{14t+7}.*

Proof. In the proof of Theorem 2.9 it is shown that $a - h = 21 + a$, so then since $0 \leq a \leq 13$, $21 \leq 21 + a = a - h$ in K_{14t+7}. Now, suppose $b \leq 7$. Then since $a < b \leq 7$ and $|a - b| = b - a$, $1 \leq |a - b| \leq 7 < 10$, the maximal length in K_{21}. But then by Theorem 2.7, ab is not a wraparound edge, a contradiction. So $b > 7$. Therefore $b + h > 7 + h \geq 21$.

Next if $t = 2$, then $h = 14$. So if $u + h = v$ or $u - h = v$, $|u - v| = h$. On the other hand if $|u - v| = h$, then $u - v = h$ or $u - v = -h$ so $u + h = v$ or $u - h = v$.

Lastly if $t > 2$, then recall that $14t + 7 = 21 + 14(t-1) = 21 + h$ so then since $0 < u, v < 21$, $u + h, v + h < 21 + h = 14t + 7$. So then $u + h, v + h \notin \mathbb{Z}_{21}$ and so necessarily $u + h \neq v$ and $v + h \neq u$. Now, since $14t + 7 - h = 21$, $14t + 7 + (v - h) \equiv 21 + v \pmod{14t+7}$ and similarly $14t + 7 + (u - h) \equiv 21 + u \pmod{14t+7}$, $u - h, v - h$ are simply $21 + u$ and $21 + v$ in K_{14t+7}, respectively. Lastly, Since $u, v < 20$ and $49 \leq 14t + 7$ we must have that $21 \geq 21 + u, 21 + v < 20 + 21 = 41 < 49 \leq 14t + 7$. So then $u - h, v - h \notin K_{21}$ and therefore necessarily $u - h \neq v$ and $v - h \neq u$. So the statement is proven.

□

Summary of 2.8 through 2.10

I do everything on an edge to edge basis. Similarly, in the last theorems I am basically creating pairwise vertex criteria to list for our blocks to say that they work.

Theorem 2.8 tells us that the vertex mappings in our general constructions preserve edge length and sums modulo 7 from K_{21} to larger members of the family. I prove that for any wraparound edge in K_{21}, my mapping does this. Therefore we can extend this to our wraparound edges from K_{22} as well.

Theorem 2.9 tells us that even if we have mapped vertices in the same equivalence class modulo 7, as long as their absolute difference is not 7 they will not map to the same vertex. I did not explicitly state that if we add or subtract our fixed increment h uniformly to members in the same equivalence class modulo 7 this won't happen because this is obvious and even should be given via the $v \mapsto v + 1$ being an automorphism.

Corollary 2.10 states that these mappings take wraparound edge incident vertices from K_{21} to 'new' vertices; the mapped vertices will never be elements in $\mathbb{Z}_{21}$. So for example if we have not touched 0 but then 14 is a member of a wraparound edge and we send 14 somewhere, it can't be anywhere in $\mathbb{Z}_{21}$. Additionally it has a weaker statement: if we map a vertex in the same equivalence class modulo 7 as one we don't map anywhere, they will not map to the same vertex as long as their difference is greater than 14. This is important for the case where we are forced to send a vertex somewhere and it must also take it's non-wraparound neighbors on the journey as well. This entire corollary is pretty much to deal with the small size of K_{35} relative to our increment.

The first part of Corollary 2.10 is to deal with the situation where we have a wraparound vertex mapped somewhere, and then another vertex which is untouched and both are equivalent modulo 7. The second part ('Next,let...') is to deal with the situation where we have an untouched vertex, and a vertex which is mapped somewhere but which isnt a member of a wraparound edge in K_{21}. It's important to distinguish this case, because wraparound edge vertices are necessarily bounded more strictly than just any vertex.

So to summarize the summary, basically if we have representative labelings that generate the blocks we can put together as outlined in Constructions 2.4 and 2.5 to get a decomposition for K_{21} and K_{22}, respectively, if we can apply my algorithm (vertex mappings) such that the above conditions hold: length, sum modulo 7, and isomorphism to the specified graph are preserved from our K_{21} and K_{22} designs and therefore we have the decompositions for the whole family.

Generalized Versions

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3 Main Results

Theorem 3.1. *For any* $7-$*edge forest* F*, there exists an* $F-$*decomposition of* K_{21} *and* K_{22} *as well as* K_{14t} *and* K_{14t+1} *where* t *is a positive integer.*

Proof. We direct the reader to Section A of the appendix. Set $t = 1$. For each $7-$edge forest F, we present three $\ell \times \ell_7$-wise edge- disjoint subgraphs F_1, F_2, F_3 with only edges in $\{8, 9, 10\}$ and a σ^{+-} labeling F_σ at the bottom. So then by Theorem 2.2 there exists an $F-$decomposition of $K_{2(7)r}$ and $K_{2(7)r+1}$ for every positive integer r. Additionally, by Construction 2.4, the existence of these subgraphs gives $B_{21} = B_1 \cup B_2 \cup B_3 \cup B_\sigma$, an $F-$decomposition of K_{21} with blocks $B_i = \langle F_i \rangle_{\phi_7}$ for $i = 1, 2, 3$ and $B_\sigma = \langle F_\sigma \rangle_{\phi_1}$. The same holds for $t = 1$ in section B of the appendix where we have four $\ell \times \ell_7$-wise edge-disjoint subgraphs F_1, F_2, F_3, F_4 with only edges in $\{8, 9, 10\} \cup \{\infty\}$ and a σ^{+-} labeling F_σ at the bottom. By Construction 2.5, $B_{22} = B_1 \cup B_2 \cup B_3 \cup B_4 \cup B_\sigma$ is the $F-$decomposition of K_{22} with blocks $B_i = \langle F_i \rangle_{\phi_7}$ for $i = 1, 2, 3$ and $B_\sigma = \langle F_\sigma \rangle_{\phi_1}$. So the statement is proven.

□

Theorem 3.2. *For any* $7-$*edge forest* F*, there exists an* $F-$*decomposition of* K_{14t+7} *and* K_{14t+8} *where* t *is a positive integer.*

Proof. Notice the following properties of any labeling F with only lengths in $\{8, 9, 10\} \cup \{\infty\}$:

(i) Any wraparound edge for $t = 1$ is of the form $a(b + h)$ or $(a - h)b$.

(ii) For any vertices of the form $a - h, b + h$ where $a \in [b]_7$ and $a < b : a - b \neq 7$.

(iii) For any vertices $u, v \pm h$, $|u - v| \neq 14$.

(iv) Any wraparound edge for $t = 1$ is of the form $(a - h)b, a(b + h)$if and only if $a < b$.

By $(i), (iv)$ and Theorem 2.8, edge lengths and sums modulo 7 are preserved by the vertex mappings on our blocks from the K_{21} and K_{22} construction which is simply our blocks where $t = 1$. By (ii) and Theorem 2.9, any two vertices of forms $a - h, b + h$ are not equal in K_{14t+7} or K_{14t+8}. Lastly, by (iii) and Corollary 2.10, no two vertices of the form $a, b \pm h$ are equal in K_{14t+7} or K_{14t+8}. So then we see that edge length and sums modulo 7 from our K_{21} and K_{22} constructions are preserved in the blocks for K_{14t+7} and K_{14t+8}, respectively. Most importantly, since no two vertices map to the same place in $14t + 7$ and $14t + 8$ for $t > 1$, the isomorphism between F and our blocks is also preserved. Note that no infinity edge is a wraparound edge.

Therefore, the labelings in the appendix indeed satisfy the properties described in Construction 2.6. Therefore B_{14t+7} and B_{14t+8} generated by these labelings in section A and B are $F-$decompositions of K_{14t+7} and K_{14t+8}, respectively. So the statement is proven. □

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