Definition 0.1. [1] A σ^{+-} labeling of G is an injection $f : V(G) \to \{0, 1, \ldots, 2m-2\}$ which induces a bijective length function $\ell : E(G) \to \{1, 2, \ldots, m\}$ where $\ell(uv) = f(v) - f(u)$ for every edge $uv \in E(G)$ with $u \in A$ and $v \in B$, and f has the additional property that $f(u) - f(v) \neq m$ for all $u \in A$ and $v \in B$.

Theorem 0.2. *(Freyberg and Tran, [1]). Let G be a bipartite graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant edge e. Then there exists a G-decomposition of K_{2mr} and K_{2mr+1} for every positive integer r.*

Construction 0.3. *Let F be a forest on 7 edges and consider K_{21}. By **??**, any edge in K_{21} has a length in $L = \{1, \ldots, 10\}$ via ℓ as stated in Definition 0.1. Next, by Theorem **??**, if F_σ is a σ^{+-}−labeling of F, then $B_\sigma = \langle F_\sigma \rangle$ via $\phi_1 := v \mapsto v + 1$ contains all edges of K_{21} with lengths in $L_\sigma = \{1, \ldots, 7\}$ by the union of it's members. Additionally, all $G \in B_\sigma$ are isomorphic to F.*

Now, clearly each edge in K_{21} is uniquely determined by it's endpoints. So naturally observe that if we partition the vertices of K_{21} modulo 7, a singleton edge partition of K_{21} modulo 7×7 is induced. Let $\ell := ab \mapsto a + b \pmod 7$. We now partition the edges of K_{21} modulo $\sim$ where $ab \sim cd$ if $\ell(ab) = \ell(cd)$ and $\ell_7(ab) = \ell_7(cd)$, and specify each partite set $P_{i,j}$ via $ab \in P_{i,j}$ if $\ell \times \ell_7(ab) = (i, j)$. Each $P_{i,j}$ contains three edges in the same equivalence class modulo $\sim$; three edges of the same length i which are in the same endpoint equivalence class modulo 7×7. Lastly, let $L_i = \{uv \in E(K_{21}) \mid \ell(uv) = i\}, \forall i \in L$, then $L_i = \bigcup_{j \in \mathbb{Z}_7} P_{i,j}$.

Let $\phi_7 := v \mapsto v + 7$ and $H_{i,j} \subseteq K_{21}$ be a subgraph containing some edge ab such that $\ell \times \ell_7(ab) = (i, j)$. Then if $G_{i,j} = \bigcup_{G \in \langle H_{i,j} \rangle_{\phi_7}} G$, $P_{i,j} \subseteq E(G_{i,j})$; each $G_{i,j}$ contains the partition $P_{i,j}$. Then since for any $(i, j) \in L \times \mathbb{Z}_7$, such a subgraph $H_{i,j}$ exists, so does such a $G_{i,j}$. Therefore, for any $i \in L$, $L_i \subseteq E(\bigcup_{j \in \mathbb{Z}_7} G_{i,j})$ for such $G_{i,j}$. In other words, we can generate all edges of length $i \in L$ in K_{21} by finding subgraphs containing edges of length i from each distinct equivalence class modulo 7×7 and applying ϕ_7 to their vertices.

So finally, if there exist edge-disjoint subgraphs $F_1, F_2, F_3 \cong F$ of K_{21} with only edges of lengths in $L^* = \{8, 9, 10\}$, let $B_i = \langle F_i \rangle_{\phi_7}$ for each $i = 1, 2, 3$. Then $\bigcup_{i=1,2,3} B_i$ must contain all edges of K_{21} with lengths in L^* by the union of it's members. Let $B_{21} = B_\sigma \cup B_1 \cup B_2 \cup B_3$ and recall that B_σ contains all edges of lengths in L_σ by the union of it's members. So then since $L = L_\sigma \cup L^*$, $K_{21} = \bigcup_{G \in B_{21}} G$, and all $G \in B_{21}$ are isomorphic to F and edge disjoint by definition. Therefore,

the existence of such subgraphs F_σ, F_1, F_2, F_3 give $B_{21} = B_\sigma \cup B_1 \cup B_2 \cup B_3$ an F−decomposition of K_{21}.

Construction 0.4. *Let F be a forest graph on seven edges and consider K_{22}. For this construction we simply take the 22nd vertex to be ∞; we add ∞ to the neighborhood of each vertex in K_{21}, and let $\ell(x\infty) = \infty$ for all $x \in V(K_{22})$. Next, we let B_σ refer to the same block as previously described in Construction 0.3. Similarly to in the previous construction, if $F_1, F_2, F_3, F_4 \cong F$ are edge-wise disjoint subgraphs of K_{21} with only edges of lengths in $L^* = \{8, 9, 10\} \cup \{\infty\}$, we let $B_i = \langle F_i \rangle_{\phi_7}$ via ϕ_7 as defined in Construction 0.3 with the new condition $\ell(x\infty) = x \pmod 7$. Then $B_{22} = B_\sigma \cup B_1 \cup B_2 \cup B_3 \cup B_4$ has the same properties as B_{21} from Construction 0.3 with L^* as defined in this construction. Therefore,*

the existence of such subgraphs $F_\sigma, F_1, F_2, F_3, F_4$ give $B_{22} = B_\sigma \cup B_1 \cup B_2 \cup B_3 \cup B_4$, an F−decomposition of K_{22}.

Definition 0.5. Consider any edge $uv \in E(K_n)$. We say:

$$
\begin{aligned}
uv \text{ is a wraparound edge if } \ell(uv) &= n - |u-v| && (1)\\
uv \text{ is a short edge if } \ell(uv) &= |u-v| && (2)\\
& && (3)
\end{aligned}
$$

.

Now for any edge $uv \in E(K_n)$, without loss of generality $v < u$ and we say:

$$
\begin{aligned}
v \text{ is a wraparound vertex if } uv \text{ is a wraparound edge} && (4)\\
u \text{ is a short vertex if } \forall ux \in E(K_n), ux \text{ is a short edge} && (5)
\end{aligned}
$$

.

Algorithm 0.6. *Let F be a forest on seven edges, $\mathcal{B}_{21}, \mathcal{B}_{22}$ be the $F-$decompositions of K_{21}, K_{22}, respectively, given by Constructions* **??** *and* **??***, and consider K_{14t+7} along with K_{14+8} where we take the vertices to be members of $\mathbb{Z}_{21} \cup \{\infty\}$ in the natural way (according to C_{14t+7} and C_{14+8} where $22 \mapsto \infty$ along with all it's incident edge lengths).*

For each block in

Construction 0.7. *Let F be a forest on seven edges and consider K_{14t+7} and K_{14t+8} for $t > 1 \in \mathbb{N}$. By definition, K_{14t+7} has edge-lengths $[1, 7t+3]$. Let us define the following edge length intervals: $I_\sigma = [1, 7]$, $I_0 = [8, 10]$ and $I_j = (I_\sigma + 10) + 7(j-1) = [11 + 7(j-1), 17 + 7(j-1)]$ for all $j > 0$. So $L = \bigcup_{0 \leq j \leq t} I_j \cup I_\sigma$ is the set of all distinct lengths in K_{14t+7} and $L \cup \{\infty\}$ is the set of all distinct lengths in K_{14t+8} for $t > 1$.*

References

[1] B. Freyberg and N. Tran, Decomposition of complete graphs into bipartite unicyclic graphs with eight edges, J. Combin. Math. Combin. Comput., *114* (2020), 133-142.

[2] R. C. Bunge, A. Chantasartrassmee, S.I. El-Zanati, and C. Vanden Eyn-den, On cyclic decompositions of complete graphs into tripartite graphs, *J. Graph Theory* **72** (2013), 90–111.