

# **Seven Edge Forest Designs**

**A THESIS**

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# Dedication

I dedicate this thesis to my advisor Professor Bryan Freyberg, to my family who has supported me throughout this process, and to Jers, Aaron, Mike B, Joe, Max, Jordi, Ian, TK, Parker, Jehan and Torta from Tuscarora and West 7th in Saint Paul. Thank you for believing in me when I couldn't and helping me realize what is possible when I apply myself.

## Abstract

Let  $G$  be a subgraph of  $K_n$  where  $n \in \mathbb{N}$ . A  $G$ -decomposition of  $K_n$ , or  $G$ -design of order  $n$ , is a finite collection  $\{G_1, \dots, G_k\}$  of pairwise edge-disjoint subgraphs of  $K_n$  that are all isomorphic to some graph  $G$  and whose union is  $K_n$ . We prove that an  $F$ -decomposition of  $K_n$  exists for every seven edge forest  $F$  if and only if  $n \equiv 0, 1, 7$ , or  $8 \pmod{14}$  and  $n \geq 14$ . We also share some additional results on edge mappings and galaxy decompositions of complete bipartites along with some `python` programs related to graphs and decompositions.

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# Chapter 1

## Background

### 1.1 Fundamentals of Graph Theory

Graph Theory is the study of objects called *vertices* or *nodes* and their relationships which we call *edges*. An edge between vertices  $u$  and  $v$  is typically denoted  $uv$  or  $(u, v)$ . A graph  $G$  is formally defined as an ordered pair  $G = (V, E)$  where  $V$  is the set of all vertices in  $G$  and  $E$  is the set of all edges between vertices in  $G$ . These sets are sometimes denoted  $V(G)$  and  $E(G)$ , respectively.

$G$  is called a *simple graph* if (1) there is at most one edge between any two vertices, (2) there are no edges from a vertex to itself and (3) all edges have no directionality to them, meaning  $uv = vu$  for any edge  $uv \in E(G)$ . For the rest of this paper all graphs we work with are finite simple graphs, but note that many of the proceeding definitions are defined in the same or very similar ways for infinite and directed graphs.

Graphs are more intuitive to work with through their visual representations instead of their formal definitions. Let  $G$  be a simple graph where  $V(G) = \{A, B, C, D, E, a, b, c, d, e\}$  and  $E(G) = \{Aa, Bb, Cc, Dd, Ee, AB, BC, CD, DE, EA, ac, ce, eb, bd, da\}$ .  $G$  is often called the *Petersen* graph. It's a bit unwieldy when described formally, yet its visual representation is very easy to understand.

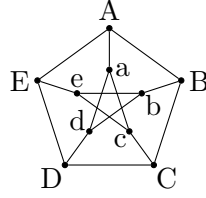
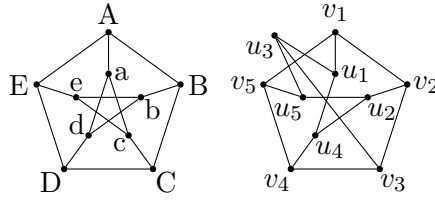


Figure 1.1: The Petersen graph

We say two graphs  $G$  and  $H$  are *isomorphic* if there exists a bijection from  $V(G)$  to  $V(H)$  called an *isomorphism* that induces a bijection from  $E(G)$  to  $E(H)$  and we denote this relationship via  $G \cong H$ . In other words, we consider two graphs  $G$  and  $H$  to be the ‘same’ if we can relabel and move vertices in some fashion (without adding/removing vertices and edges) in visual representations of  $G$  and  $H$  to go between the two.

Figure 1.2: (left)  $G \cong H$  (right)

Graph theorists generally consider two graphs to be the same if they are isomorphic because an isomorphism preserves all meaningful structural properties of a graph. We will wrap up the fundamentals with a few definitions and some important operations.

**Definition 1.1.1** (Subgraph). A subgraph  $G \subseteq K$  is a graph whose vertices and edges are subsets of the vertices and edges of  $K$ ;  $G \subseteq K$  if  $V(G) \subseteq V(K)$  and  $E(G) \subseteq E(K)$ .

**Definition 1.1.2** (Vertex-induced Subgraph). A *vertex-induced* subgraph  $G \subseteq K$  is one whose vertices are some subset of  $V(K)$  and whose edges are all edges between those vertices in  $K$ ;  $V(G) \subseteq V(K)$  and  $E(G) = \{uv \in E(K) \mid u, v \in V(G)\}$ . If  $G$  is such a subgraph we say that  $G$  is induced by  $S = V(G) \subseteq V(K)$ .

**Definition 1.1.3** (Edge-induced Subgraph). A *edge-induced* subgraph  $G \subseteq K$  is one whose edges are some subset of  $E(K)$  and whose vertices are all those who appear

as an endpoint in that subset of edges;  $E(G) \subseteq E(K)$  and  $V(G) = \{u \in V(K) \mid uv \in E(G) \text{ for some } v \in V(K)\}$ . If  $G$  is such a subgraph we say that  $G$  is induced by  $S = E(G) \subseteq E(K)$

Here is a visual example of these types of graphs: Let  $K$  be the Petersen graph from Figure 1.1.

**Subgraph:**  $G \subseteq K$  where  $V(G) = \{E, e, b\}$ ,  $E(G) = \{Ee\}$ .

**Vertex-induced Subgraph:**  $H \subseteq K$  is induced by  $\{a, A, B\} \subseteq V(K)$

**Edge-induced Subgraph:**  $M \subseteq K$  is induced by  $\{Dd, DC, Cc\} \subseteq E(K)$

The figure below shows  $K$  and it's color-coded subgraphs.

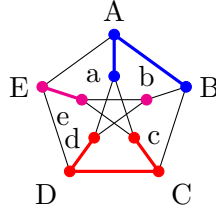


Figure 1.3:  $K$  and subgraphs  $G, H, M \subseteq K$

Next, we will talk about two important operations done on graphs.

**Definition 1.1.4** (Graph Union). The union of two graphs  $G$  and  $H$  is simply the graph resulting from the union of their vertices and the union of their edges and is denoted  $G \cup H$ ;  $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ . If  $G$  and  $H$  are vertex-disjoint, we may denote their union via  $G \sqcup H$  and call it a *disjoint union* of  $G$  and  $H$ . If they are only edge-disjoint, we may call it an *edge-disjoint union*. We also use  $\sqcup$  in general to denote a union between disjoint sets;  $A \sqcup B$  to denotes a union between sets  $A$  and  $B$  where  $A \cap B = \emptyset$ .

Here is an example of a union and a disjoint union of graphs. Let  $G = (\{a, b, c, d\}, \{ab, bc, cd, da\})$ ,  $H = (\{a, b, c\}, \{ab, bc, ca\})$ , and  $K = (\{A, B, C\}, \{AB, BC, CA\})$  Then:

$$G \cup H = (\{a, b, c, d\}, \{ab, bc, cd, da, ca\})$$

$$G \sqcup K = (\{a, b, c, d, A, B, C\}, \{ab, bc, cd, da, AB, BC, CA\})$$

These unions are depicted in the following figure.

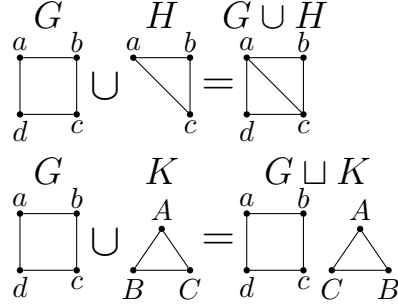


Figure 1.4: (above)  $G \cup H$  and (below)  $G \sqcup K$

Next, we define another very important operation that combines two vertex disjoint graphs in a different manner.

**Definition 1.1.5** (Join). Let  $G$  and  $H$  be vertex disjoint graphs. Their *join*, denoted  $G \vee H$ , is the graph obtained by taking the disjoint union of  $G$  and  $H$  and adding all possible edges between every vertex in  $G$  and every vertex in  $H$ . Formally:

$$G \vee H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}).$$

Here is an example. Let  $G = (\{a, b, c\}, \{ab, bc, ca\})$  and  $H = (\{A, B, C\}, \{AB, BC, CA\})$ , then  $G \vee H = (\{a, b, c, A, B, C\}, E(G) \sqcup E(H) \sqcup \{aA, aB, aC, bA, bB, bC, cA, cB, cC\})$ . This join is depicted in the figure below.

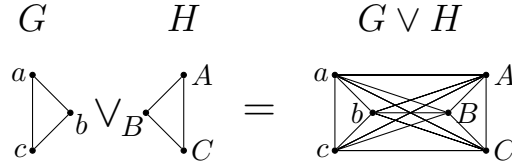


Figure 1.5:  $G \vee H$

Lastly, we define a few characteristics of graphs and their components. These may or may not be used frequently in this paper, but are important concepts to know in

order to be able to talk about graphs comfortably.

Let  $G$  be a simple graph. We say two vertices  $u, v \in V(G)$  are *adjacent* or *neighbors* if they share an edge  $uv \in E(G)$ . Similarly, we say a vertex is *incident* with an edge if it is one of its endpoints;  $u \in V(G)$  is incident with  $e \in E(G)$  if  $e = uv$  for some  $v \in V(G)$ . The set of all vertices adjacent to  $v$  in  $G$  is called the *neighborhood* of  $v$  denoted  $N_G(v)$  or simply  $N(v)$ . Sometimes this is referred to as the open neighborhood of  $v$  in  $G$  and then the closed neighborhood is defined via  $N_G[v] = N_G(v) \sqcup \{v\}$ . The *degree* of a vertex  $v \in V(G)$  is the number of vertices adjacent to it and is denoted  $\deg_G(v)$  or simply  $\deg(v)$ . Equivalently, the degree of a vertex  $v$  is the number of edges incident with it or the number of  $|N_G(v)|$  neighbors that  $v$  has. We call a vertex of degree 1 a *leaf* and an edge incident with a leaf a *pendant* edge.

The following are three similar types of objects found in graphs.

**Definition 1.1.6** (Walk). Let  $G$  be a graph on  $n$  vertices. A *walk* in  $G$  is a sequence  $(w_0, w_1, \dots, w_k)$  of vertices in  $G$  whose adjacent elements must be adjacent in  $G$ . Adjacent elements in a walk must be distinct vertices but a vertex may be repeated multiple times throughout the sequence.

**Definition 1.1.7** (Path). Let  $G$  be a graph on  $n$  vertices. A *path* in  $G$  is a sequence  $(v_0, v_1, \dots, v_k)$  of distinct vertices in  $G$  whose adjacent elements must be adjacent in  $G$ , and where no vertex is repeated throughout the sequence. This sequence gives the subgraph of  $G$  induced by  $\{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\}$ .

**Definition 1.1.8** (Cycle). Let  $G$  be a graph on  $n$  vertices. A *cycle* in  $G$  is a sequence  $(v_0, v_1, \dots, v_k, v_0)$  of internally distinct vertices (distinct except on the endpoints) that begins and terminates at the same vertex  $v_0$ . Often such a cycle is denoted via  $(v_0v_1 \cdots v_k)$  and it is understood that the sequence wraps back around to  $v_0$  after  $v_k$ . Additionally, the cycle  $(v_0v_1 \cdots v_k)$  is equivalent to  $(v_1 \cdots v_kv_0)$ ,  $(v_2 \cdots v_kv_0v_1)$ ,  $\dots$  and so on.

We call a simple graph  $G$  *acyclic* if it contains no cycles. If there exists a path from any vertex to every other vertex in  $G$ , then we call  $G$  *connected*. If not, we call  $G$  *disconnected*. We call the set of connected subgraphs of  $G$  whose disjoint union equals  $G$  the *connected components* of  $G$ .

This concludes the fundamental concepts needed to understand this project. The next and final section of this chapter will introduce all the fundamental families of graphs we refer to in subsequent chapters.

## 1.2 Fundamental Families of Graphs

In this section we introduce some fundamental families of graphs which we refer to throughout this paper. Often, instead of fully defining the graphs being worked with, we simply refer to it as a member of a larger family of graphs or as isomorphic to a family member. Some of these families overlap. It may be helpful to view a graph as a member of one family or another depending on the context.

Recall that a graph is acyclic if it contains no cycles. Similarly, we call a graph  $k$ -cyclic if it contains exactly  $k$  distinct cycles. If  $k = 2$  or  $3$  we call it *bicyclic* or *tricyclic*, respectively. In a similar vein, we call a graph  $k$ -partite if we can partition its vertices into  $k$  disjoint sets such that no two vertices in the same partite set share an edge. If  $k = 2$  or  $3$ , we call it *bipartite* or *tripartite*, respectively. These are broad families of graphs often used to characterize subsets of graphs within another family. The following are some more important families of graphs.

**Definition 1.2.1** (Complete Graph). The *complete graph* on  $n$  vertices, denoted  $K_n$ , is the graph on  $n$  vertices such that every pair of distinct vertices shares an edge.

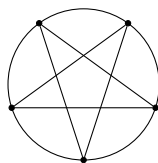


Figure 1.6: The Complete Graph  $K_5$

**Definition 1.2.2** (Complete Bipartite Graph). Let  $m, n \in \mathbb{N}$ . The *complete bipartite graph*  $K_{m,n}$  is the bipartite graph whose vertices can be partitioned into two disjoint sets of sizes  $m$  and  $n$ , respectively, such that every vertex in the one partite set is adjacent to **every** vertex in the other partite set and no two vertices in the same partite set are adjacent.



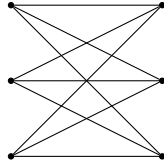


Figure 1.7: The Complete Bipartite Graph  $K_{3,3}$

**Definition 1.2.3** (Complete Multipartite Graph). The *complete  $k$ -partite graph* or *complete multipartite graph*  $K_{n_1, \dots, n_k}$  is the graph whose vertices can be partitioned into  $k$  disjoint sets of sizes  $n_1, n_2, \dots, n_k$ , respectively such that every vertex in the one partite set is adjacent to **every** vertex in the other  $k - 1$  partite sets and no two vertices in the same partite set are adjacent.

If all partite sets are the same size  $n$  we call this graph the *complete equipartite graph* and denote it via  $K_{n:k}$  or  $K_{n \times m}$ .

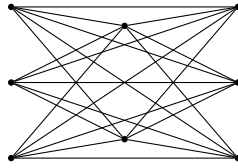


Figure 1.8: The Complete Multipartite Graph  $K_{3,2,3}$

**Definition 1.2.4** (Cycle Graph). The *cycle graph* on  $n$  vertices denoted  $C_n$  is a graph with exactly one distinct cycle containing all of its vertices.

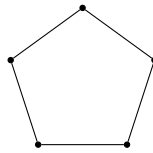


Figure 1.9: The Cycle Graph  $C_5$

**Definition 1.2.5** (Tree). A *tree* is any connected acyclic graph. Trees on  $n$  vertices have  $n - 1$  edges. All trees are bipartite and connected.

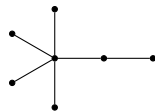
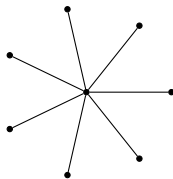


Figure 1.10: A tree on 6 vertices

**Definition 1.2.6** (Path Graph). The *path* graph on  $n$  vertices, denoted  $P_n$ , is an acyclic graph with exactly one path containing all of its vertices. All paths are trees.

Figure 1.11: The path  $P_4$ 

**Definition 1.2.7** (Star Graph). The *star graph* on  $n + 1$  vertices, denoted  $K_{1,n}$  (or  $S_{n+1}$  which we never use in this paper) consists of one central *hub* vertex adjacent to  $n$  *outer* vertices, with no other edges. All stars are trees. Sometimes this graph is referred to as an *n-star* or *n-edge star*.

Figure 1.12: The 7-star  $K_{1,7}$ 

**Definition 1.2.8** (Forest Graph). Any disjoint union of tree graphs is called a *forest* graph. These graphs are all bipartite and disconnected.

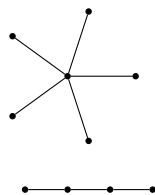


Figure 1.13: A forest on 9 vertices

**Definition 1.2.9** (Galaxy Graph). Any disjoint union of star graphs is called a *galaxy* graph. We refer to  $G = G_1 \sqcup \cdots \sqcup G_k$  as the  $\{G_1, \dots, G_k\}$ -*galaxy* graph if  $G_1, \dots, G_k$  are all stars. All galaxies are forests, disconnected, and bipartite.

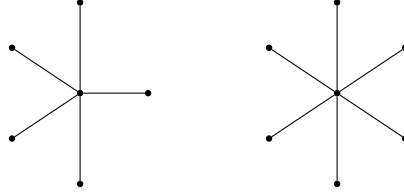


Figure 1.14: The  $\{K_{1,5}, K_{1,6}\}$ -Galaxy

We have now defined a few important families of graphs which we will refer to throughout the rest of this paper. Again, we generally don't explicitly define every graph by its vertex and edge sets and simply refer to it as some member of a family or say that it is isomorphic to a member of a family. This is much more efficient and concise than listing out all vertices and edges as we did in the beginning of this chapter.

We are now ready to move on and introduce graph decompositions, the objects which are the subject of this project.

## Chapter 2

# Introduction

### 2.1 Decompositions

Suppose you have  $n$  translucent sheets of tracing paper with some points drawn on all  $n$  sheets of paper in the same set arrangement. Now, draw lines connecting points on each sheet of paper, so that no line appears on two distinct sheets of paper.

A graph  $K$  is depicted when all  $n$  sheets of tracing paper are aligned and stacked on top of each other with some light source present. Call the graph depicted on the  $i$ th sheet of paper  $G_i$  for  $i = 1, \dots, n$ . The stacking of these sheets of paper depicts the edge-disjoint union  $G_1 \cup \dots \cup G_n = K$ , and this collection of papers depicts the set  $\{G_1, \dots, G_n\}$  which we call a *graph decomposition* of  $K$ . This is defined formally below.

**Definition 2.1.1** (Graph Decomposition). Let  $K$  be a simple graph. We call a collection  $\{G_1, \dots, G_n\}$  of pairwise edge-disjoint subgraphs  $G_1, \dots, G_n \subseteq K$  of  $K$  a *decomposition* of  $K$  if their union equals  $K$ .

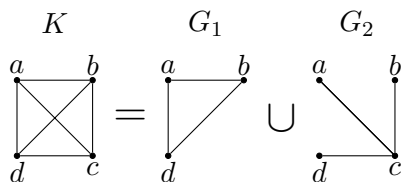


Figure 2.1:  $\{G_1, G_2\}$  is a decomposition of  $K_4$

Graph decompositions are an important topic in combinatorics, graph theory, and design theory, with origins dating back to the 1800s. Notably, in 1850, Reverend Thomas Kirkman, a full-time clergyman and legendary mathematician, posed an important problem in *The Lady's and Gentleman's Diary* [8] now known as *the school girl problem*. It goes

*Fifteen young ladies in a school walk out three abreast for  
seven days in succession: it is required to arrange them daily  
so that no two shall walk twice abreast.*

The problem asks if we can form five distinct rows of three school girls on each day of the week so that no two school girls walk in the same row more than once in the week. This is equivalent to finding a decomposition of  $K_{15}$  whose members are all triangles; whose members are isomorphic to  $C_3$  or  $K_3$ . Both Kirkman and Arthur Cayley independently solved the the schoolgirl problem and published their solutions in the 1851 edition of *The Lady's and Gentleman's Diary* [9]. Kirkman's solution is provided below.

*Denoting the ladies by  $a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d_1, d_2, d_3; e_1, e_2, e_3$ , the following arrangement will be found to answer the question:*

$a_1 a_2 a_3$	$a_1 b_1 c_1$	$a_1 d_1 e_1$	$a_1 b_2 d_2$	$a_1 c_2 e_2$	$a_1 b_3 e_3$	$a_1 c_3 d_3$
$b_1 b_2 b_3$	$a_2 b_2 c_2$	$a_2 d_2 e_2$	$a_2 b_3 d_3$	$a_2 c_3 e_3$	$a_2 b_1 e_1$	$a_2 c_1 d_1$
$c_1 c_2 c_3$	$a_3 d_3 e_3$	$a_3 b_3 c_3$	$a_3 c_1 e_1$	$a_3 b_1 d_1$	$a_3 c_2 d_2$	$a_3 b_2 e_2$
$d_1 d_2 d_3$	$b_3 d_1 e_2$	$b_1 c_1 e_3$	$b_1 c_3 e_1$	$b_2 c_3 d_1$	$c_2 b_3 e_1$	$c_2 b_3 e_1$
$e_1 e_2 e_3$	$c_3 d_2 e_1$	$e_3 b_2 c_1$	$d_1 c_2 e_3$	$c_1 d_2 b_3$	$d_2 b_1 c_2$	$c_1 d_3 b_2$

*This is the symmetrical and only possible solution. All others differ from this only in disturbing the alphabetical order, or that of the three subindices in certain triplets of the first column, or in both these together.*

Each triple in the array above gives an edge-distinct triangle subgraph of  $K_{15}$  whose vertex set we take to be  $\{a_1, a_2, \dots, e_4, e_5\}$ . The set of all these subgraphs is a decomposition of  $K_{15}$ . Since all of these subgraphs are isomorphic to  $C_3$ , we call it a  $C_3$ -decomposition. This is a special type of decomposition which is defined formally on the following page.

**Definition 2.1.2** (*G*-decomposition). A *G*-decomposition of a graph  $K$  is a decomposition  $\{G_1, \dots, G_t\}$  of  $K$  whose members are all isomorphic to some graph  $G$ . If such a set exists we say that  $K$  allows a *G*-decomposition or equivalently, that  $G$  decomposes  $K$ . If  $K \cong K_n$  we sometimes call the decomposition a *G*-design of order  $n$ .

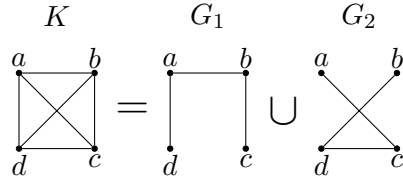


Figure 2.2:  $\{G_1, G_2\}$  is a  $P_3$ -decomposition of  $K_4$  or a  $P_3$ -design of order 4

We know that if a *G*-decomposition of some graph  $K$  exists, that all of its members have the same number of edges and vertices. This allows us to find constraints on graphs  $K$  that can be decomposed by some subgraph  $G \subseteq K$  on  $m$  edges solely based on divisibility properties.

**Lemma 2.1.3** (Necessary Condition (general)). *Let  $G$  be a simple graph on  $m$  edges. There exists a  $G$ -decomposition of a graph  $K$  only if  $|E(G)| = m$  divides  $|E(K)|$ .*

*Proof.* Suppose there exists a  $G$ -decomposition  $\{G_1, \dots, G_n\}$  of  $K$ . Then  $E(G_1) \sqcup \dots \sqcup E(G_t) = E(K)$  and so  $|E(K)| = |E(G_1) \sqcup \dots \sqcup E(G_t)| = |E(G_1)| + \dots + |E(G_t)| = tm$ . So  $|E(G)| = m$  divides  $|E(K)|$ .  $\square$

**Theorem 2.1.4** (Necessary Condition ( $K_n$ )). *Let  $G$  be a simple graph on  $m$  edges. There exists a  $G$ -decomposition of  $K_n$  only if  $n$  is idempotent modulo  $2m$ ; only if  $n^2 \equiv n \pmod{2m}$ .*

*Proof.* Suppose there exists a  $G$ -decomposition of  $K_n$ . Then  $|E(G)| = m$  divides  $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$  by Lemma 2.1.3. Therefore,  $\frac{n^2-n}{2} = mt$  for some  $t \in \mathbb{N}$ . Observe.

$$n^2 - n = 2mt \implies n^2 - n \equiv 0 \pmod{2m} \implies n^2 \equiv n \pmod{2m}.$$

$\square$

By the previous theorem, any graph on  $m$  edges decomposes  $K_n$  only if  $n$  is idempotent modulo  $2m$ . Note that the converse isn't necessarily true. However, for a graph  $G$  on  $m$  edges, this finite set of constraints allows us to ask:

For what  $n$  is  $K_n$   $G$ -decomposable?

This question is known as the *spectrum problem* for graph decompositions. Pioneering work by Rosa and Kotzig in the 1960s—especially in the development of graph labelings—helped shape the modern approach to  $G$ -decomposition problems. Since then, labeling-based techniques and tools from design theory have driven significant progress. In particular, graph labeling methods have played a central role in addressing the spectrum problem for small graphs. This thesis project directly builds upon work by Freyberg and Peters, who recently solved the spectrum problem for forests with six edges [4]. Their paper provides a comprehensive summary of known decompositions for graphs  $G$  with fewer than seven edges.

Using graph labelings to solve  $G$ -decomposition problems is basically about doing algebra on subgraphs in order to generate other edge-disjoint subgraphs while preserving isomorphism to  $G$ . If we take the vertices of a graph  $K$  to be elements of a group, we can use the structure of the group to our advantage. Specifically, when  $K \cong K_n$ , and we take its vertices to be  $\mathbb{Z}_n$ , and then we label the vertices of  $G$  with some subset of  $\mathbb{Z}_n$ . There are various labeling techniques of this kind stemming from Rosa's work in the 1960s that allow us to permute or act on the labels of the vertices of  $G$  with subgroups of  $\mathbb{Z}_n$  to generate new isomorphic copies of  $G$  which are pairwise edge-disjoint. In the next section, we provide an example which outlines in some detail how this machinery works for  $G$ -decompositions of complete graphs.

## 2.2 Graph labelings

Take the vertices of  $K_5$  to be  $\mathbb{Z}_5$  and arrange it in the same manner as in 1.6. Notice that every vertex shares an edge with two vertices directly adjacent to it and two vertices that are 'two adjacencies away' on the outer cycle (01234). We call this idea *length* denoted  $\ell$  where edges joining two vertices  $u, v$  have length  $\ell(uv) = l$  if they are ' $l$  adjacencies away' from each other on the outer cycle.

Formally, for  $K_n$  we define the edge length function  $\ell$  as follows:

$$\ell(uv) = \min\{|u - v|, n - |u - v|\}$$

Notice that for  $K_5$ , we only have lengths 1 or 2 as previously observed. Color the length 1 edges **blue**, and the length 2 edges **red**. This is depicted in the figure below.

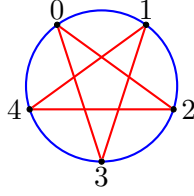


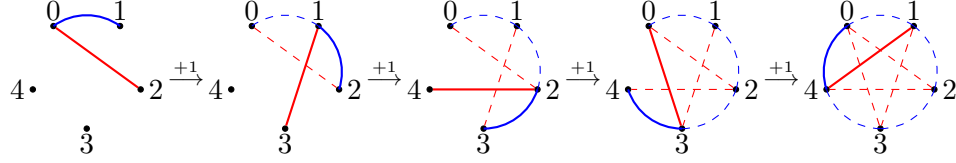
Figure 2.3:  $K_5$  with lengths colored

Now, consider  $P_3$ . It has 2 edges, and  $K_5$  has  $\binom{5}{2} = 10$  edges. Since  $2|10$ , by Lemma 2.1.3 it's *possible* that a  $P_3$ -decomposition of  $K_5$  exists. Now since  $P_3$  has 2 edges and there are 2 lengths in  $K_5$ , what if we can just make sure each copy of  $P_3$  has both a **blue** edge and a **red** edge? How can we do that while ensuring that no edge is repeated?

It turns out that if we take the vertices of  $K_n$  to be  $\mathbb{Z}_n$ , adding 1 (and therefore anything) modulo  $n$  to the endpoints of an edge preserves its length. We call the act of permuting vertices in this manner (permuting by an element of the group) *clicking* or *developing*. If we are permuting all vertices of a labeling by the same group element, we often will just say we are developing the labeling by that group element.

In the context of our problem with  $P_3$  and  $K_5$ , this means that if we label  $P_3$  with elements of  $\mathbb{Z}_5$  such that we have one **blue** edge of length 1, and one **red** edge of length 2, we can generate all edges of length 1 and 2 in  $K_n$  (which is just all edges of  $K_n$ ) while preserving the structure of the graph by developing our labeling by 1. Label the defining path of  $P_3$  via  $(2,0,1)$ . Developing the vertices by 1 (modulo 5) will give all members of a  $P_3$ -decomposition of  $K_5$ . A decomposition that can be generated by permuting all vertices of one labeling repeatedly in this fashion is called a *cyclic decomposition*. This is depicted in the following figure.



Figure 2.4: A cyclic  $P_3$ -decomposition of  $K_5$ 

Nice and easy right? But that's just one complete graph that  $P_3$  can decompose. Remember, that it is *possible* that  $P_3$  can decompose any  $K_n$  where  $n \equiv n^2 \pmod{4}$  by Theorem 2.1.4. This equivalent to saying  $n = 4t + r$  where  $r$  is an idempotent in the ring  $\mathbb{Z}_4$  and  $t \geq 1$ . The idempotents in  $\mathbb{Z}_4$  are 0, 1. So this means  $\mathbb{Z}_5$  is just a special case of  $n$  where  $n = 4t + 1$  where  $t = 1$ . Luckily, even though these are infinite families, it is known that for each step  $t \mapsto t + 1$ , new lengths come 2 at a time. This means if we can somehow transform our labeling at each step to include the new lengths, we can maybe take care of the entire family  $K_{4t+1}$ . We want to fine tune our labeling to be able to weather this process. This is what graph labeling is all about. Note that if  $r$  was not 0 or 1, we would need multiple labelings to take care of the whole family. This is explained later in this paper.

Lastly, some basic observations about a general subgraph  $G \subseteq K_n$  with  $m$  edges of  $K_n$ . The maximal length in  $K_n$  is  $\lfloor \frac{n}{2} \rfloor$ . This is intuitive, since when you travel halfway across the outer cycle from some vertex, the lengths start going back down again as you begin nearing that vertex again. Now,  $n$  must be of the form  $2mt + r$  where  $t \geq 1$  and  $r$  is an idempotent in the ring  $\mathbb{Z}_{2m}$ . This means that in  $K_{2mt+r}$  if  $\ell(uv) = |u - v| < \lfloor \frac{2m+r}{2} \rfloor < \lfloor \frac{2mt+r}{2} \rfloor$  for  $t > 1$ , then  $\ell(uv) = |u - v|$  in all  $K_{2mt+r}$  for  $t \geq 1$ . this is important, because at each step  $t \mapsto t + 1$ , new lengths come  $m$  at a time.

Now, for  $r = 0$  or  $1$ , if a certain labeling of a graph  $G$  on  $m$  edges exists, there exists a  $G$ -decomposition of  $K_{2mt+r}$  for  $t \geq 1$ . However, if  $r \neq 0, 1$ , one labeling will not suffice and other techniques are needed to prove that  $G$  decomposes  $K_{2mt+r}$  for  $t \geq 1$ . These labelings and techniques are defined as they are needed in the proceeding chapters. Finally, we are ready to introduce the focus of this project.

## 2.3 Seven edge forests

This project continues on Freyberg and Peters' work on six edge forests by asking the same question about seven edge forests:

Let  $F$  be a forest on seven edges. For which  $n$  does  $F$  decompose  $K_n$ ?

The spectrum problem for the matching  $7\mathbf{T}_2^{11}$  was solved by de Werra in 1970. Every component of a forest on seven edges is a tree on six or less edges which are cataloged in Figure 2.5. We use the naming convention  $\mathbf{T}_j^i$  to denote the  $i^{\text{th}}$  tree with  $j$  vertices and we index the vertices  $v_1$  through  $v_j$  for each tree as specified below.

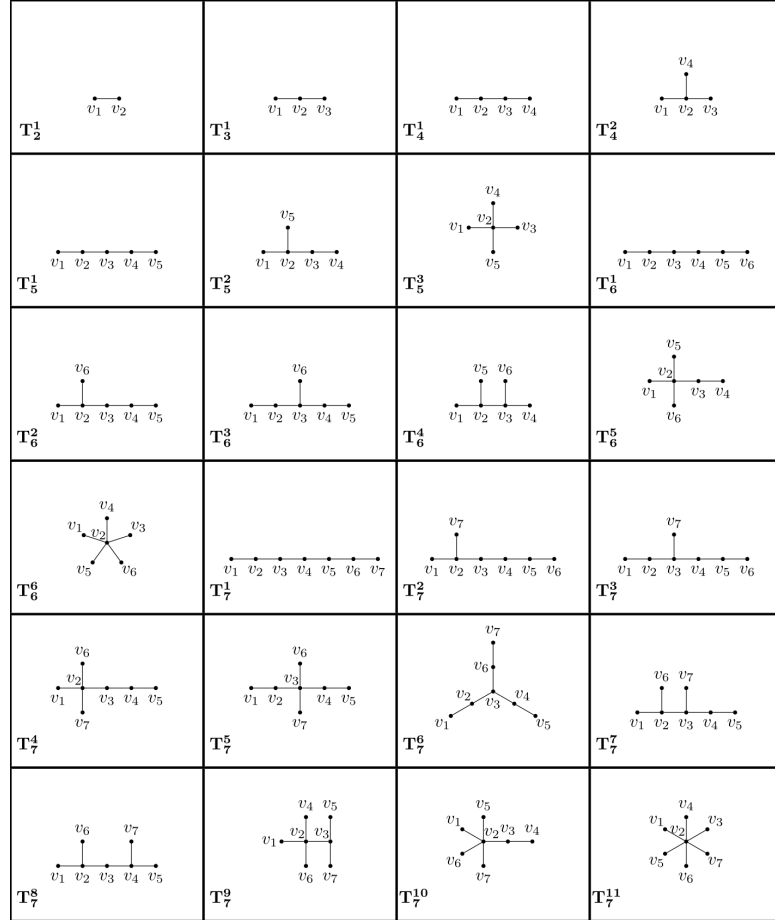


Figure 2.5: Trees with less than seven edges

The next theorem gives the necessary conditions for the existence of a  $G$ -decomposition of  $K_n$  when  $G$  is a graph with 7 edges.

**Theorem 2.3.1.** *If  $G$  is a graph with 7 edges and a  $G$ -decomposition of  $K_n$  exists, then  $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$ .*

*Proof.* If a  $G$ -decomposition of  $K_n$  exists, then  $n$  is idempotent modulo  $2(7) = 14$  by Theorem 2.1.4 which immediately implies that  $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$  since those are all the idempotents in  $\mathbb{Z}_{14}$ .  $\square$

For this project, we do not define the graph on one vertex to be a tree. This means that any connected component in a forest has at least one edge and we also require there to be at least two connected components. There are 47 such forests with 7 edges up to isomorphism. As stated previously, the matching on seven edges is solved, so only the remaining 46 forests need be considered in the subsequent chapters. Chapter 3 handles decomposing  $K_n$  into all 47 forests when  $n \equiv 0 \text{ or } 1 \pmod{14}$ . Chapter 4 applies to all the forests when  $n \equiv 7 \text{ or } 8 \pmod{14}$  with the lone exception of  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ , which is solved for those values of  $n$  in Chapter 5.

After proving the main result of this project, we provide a couple of additional results in Chapter 6: (1) An edge mapping that depends on  $t$  and preserves lengths for wraparound edges in  $K_{3m}$  and  $K_{3m+1}$  to  $K_{2mt+m}$  and  $K_{2mt+m+1}$ , respectively, for each step  $t \mapsto t + 1$ . (2) Galaxy decompositions of complete bipartite graphs. This chapter concludes the main results of this thesis project.

We then present some `python` programs in Chapter 7 that were created as a result of this thesis project. One is `tikzgrapher`: A new graph visualization program, which was built from scratch only using `pygame` and `NetworkX` and several other basic libraries. It allows one to visualize simple `NetworkX` graphs in an interactive `pygame` window that allows for colorings and custom labelings along with dragging and moving components of the graph. A main feature is that the user can save the graph layout depicted in the `pygame` window as a `TikZ` graph in a standalone `LATEX` file. `tikzgrapher` is paired with a graph labeling solver. This is a constraint programming project that can find several labelings on graphs if they exist. The Conclusion follows this chapter, then a list of References and an Appendix of labelings marks the end of the paper.

## Chapter 3

### $n \equiv 0, 1 \pmod{14}$

To begin this chapter, we extend intuition developed in the Introduction to present some machinery specific to  $K_n$  where  $n \equiv 0, 1 \pmod{14}$ . This informs the formal definitions and theorems we use for this case.

#### 3.1 Construction for $n \equiv 0, 1 \pmod{14}$

$K_0$  and  $K_1$  don't have enough vertices to contain a forest on seven edges. So  $K_{14}$  and  $K_{15}$  are the base graphs for  $K_n$  where  $n \equiv 0$  or  $1 \pmod{14}$ , respectively. We first show how to decompose  $K_{14}$  and  $K_{15}$  in Subsection 3.1.1 and then show how to generalize this to their respective families in Subsection 3.1.2

To be absolutely clear, **these are not original ideas as a result of this project**. Rosa and his colleagues, developed these labeling ideas decades ago. Other mathematicians since also have helped in developing labeling techniques since. This is just an explanation from our perspective, of the amazing work of Rosa and his successors in creating graph labelings techniques to solve decomposition problems.

### 3.1.1 $K_{14}$ and $K_{15}$

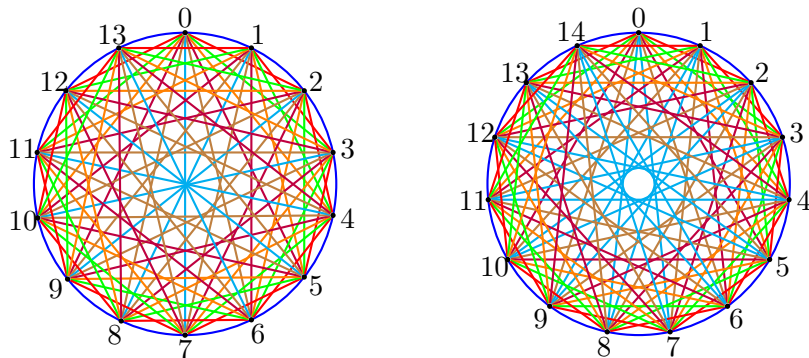


Figure 3.1:  $K_{14}$  (left) and  $K_{15}$  (right) with edges colored by length

Both  $K_{14}$  and  $K_{15}$  only have edges with lengths 1 through 7, however, there is important distinction in counting edges of one length in  $K_{14}$  versus counting them in  $K_{15}$ . The cyan edges have length 7, and conveniently because of the way these graphs are drawn in the figure above, we can easily count them as they form the innermost spoke of the graphs. Because these graphs are so similar, we can use one labeling to deal with both at the same time! However, we must be very careful in doing so.

In  $K_{14}$  there are only 7 edges of length 7 and 14 edges of all other lengths, yet in  $K_{15}$  there are 15 edges of length 7 along with 15 edges of all other lengths. It turns out this pattern generalizes. For  $K_n$ ,

**If  $n$  is odd:** There are  $n$  edges of lengths 1 through  $\frac{n-1}{2}$ .

**If  $n$  is even:** There are  $n$  edges of lengths 1 up to  $\frac{n}{2}$ , and then  $\frac{n}{2}$  edges of length  $\frac{n}{2}$ . This is why the labeling of  $P_3$  in Figure 2.4 used to decompose  $K_5$  in the Introduction worked so easily.  $K_5$  has odd order, so there are 5 edges of each length in  $\{1, 2\}$ . The same applies here for  $K_{15}$ , we just label each seven edge forest  $F$  so that they contain all seven lengths in  $K_{15}$ , and then develop the labelings by 1 to generate the entire  $F$ -decomposition of  $K_{15}$ .

But notice that if we develop the same labeling of  $F$  in  $K_{14}$  (assuming we don't use the vertex 14) we would overcount length 7 edges. There is a simple remedy for this, but it requires a shift in perspective.

Take  $V(K_{14})$  to be  $\mathbb{Z}_{13} \cup \{\infty\}$ , and label all edges via the length function modulo

13 except for edges incident to  $\infty$  which will refer to as length  $\infty$ . We do this so that intuitively, developing vertices by 1 fix the  $\infty$  vertex so that  $\infty \mapsto \infty + 1 = \infty$ . Formally:

$$\ell(uv) = \begin{cases} \min\{|u - v|, 13 - |u - v|\}, & u, v \neq \infty, \\ \infty, & u \text{ or } v = \infty \end{cases} \quad \text{and } v \mapsto \begin{cases} v + 1, & v \in \mathbb{Z}_{13}, \\ \infty, & v = \infty. \end{cases}$$

Another reason for doing this is that we will now have 13 edges of lengths 1 through 6 as well as of length  $\infty$ , since the  $\infty$  vertex will have all 13 edges of length  $\infty$  to vertices in  $\mathbb{Z}_{13}$  adjacent to it. Now if we develop the endpoints of an edge with any length by 1 repeatedly, we will get all 13 distinct edges of that length. So we can hopefully cyclically generate  $F$ -decompositions with this new construction for  $K_{14}$  for each forest.

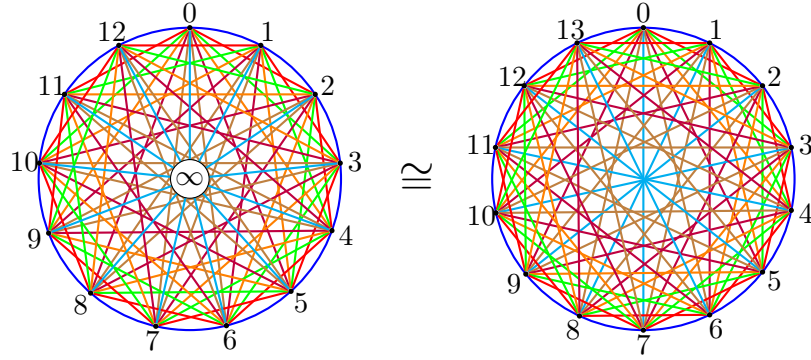


Figure 3.2:  $K_{13} \vee K_1$  is isomorphic to  $K_{14}$

It is important to note that we **must** use edges  $uv$  of length  $|u - v|$  in  $K_{13}$  so they will also have length  $|u - v|$  in  $K_{14}$  and  $K_{15}$  if we want to use one labeling for both cases. We also must ensure that the edge of length 7 is a pendant edge, so that we can relabel a leaf incident with that edge as  $\infty$  without disturbing other edge lengths.

Putting this all together: if we can just use vertex labels from  $\mathbb{Z}_{13}$  for a labeling of a forest  $F$  but only use edges  $uv$  of lengths  $|u - v| = l$  modulo 14 for each length  $l \in \{1, \dots, 7\}$ , ensuring that the edge of length 7 is a pendant edge: we can (1) simply develop the labeling by 1 to get the  $F$ -decomposition of  $K_{15}$ , then (2) just relabel a leaf incident with the length seven edge with  $\infty$ , and then once more develop by 1 to get the  $F$ -decomposition of  $K_{14}$ . We show an example on the next page.

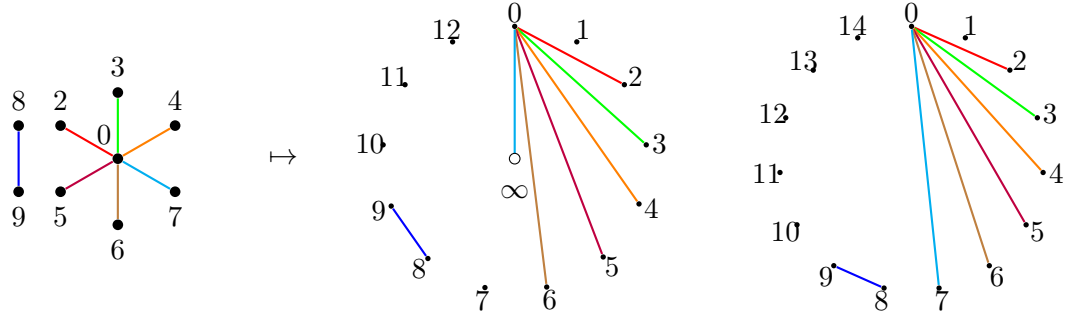


Figure 3.3: A labeling (left) that gives the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of  $K_{14}$  (middle) and  $K_{15}$  (right), respectively, when developed by 1. The leaf 7 in the pendant edge  $(0, 7)$  (of length 7 modulo 14) is relabeled as  $\infty$  for  $K_{14}$

### 3.1.2 Stretching a labeling and generalizing to $K_{14t}$ and $K_{14t+1}$ for $t > 1$

Now, we have learned how to decompose  $K_{14}$  and  $K_{15}$ , but what about the rest of the family? Well,  $K_{14t}$  and  $K_{14t+1}$  actually contain edges of lengths in

$$\bigsqcup_{0 \leq i < t} \{1 + 7i, \dots, 7 + 7i\} \text{ for } t \geq 1$$

This makes sense because the maximal length in  $K_{14t}$  and  $K_{14t+1}$  is  $\lfloor \frac{14t+1}{2} \rfloor = 7t$ . Luckily, it turns out that we can actually just replicate this process we did for lengths 1 through 7 for each step  $t \mapsto t + 1$  for  $K_{14t}$  and  $K_{14t+1}$  for the new lengths that come 7 at a time if we add some more restrictions.

There are two types of edges in complete graphs. In  $K_n$ , we call edges  $uv$  of length  $n - |u - v|$  *wraparound edges*, and edges  $ab$  of length  $|a - b|$  *short edges*. If we want to build a labeling that can generalize to an entire family of complete graphs, we need to understand how these types of edges would generalize as the order of the complete graph increases.

$$\text{In } K_{14(1)}: \ell((0, 8)) = \min\{|0 - 8|, 14 - |0 - 8|\} = \min\{8, 6\} = 6.$$

$$\text{In } K_{14(2)}: \ell((0, 8)) = \min\{|0 - 8|, 28 - |0 - 8|\} = \min\{8, 20\} = 8.$$

There is a way to map wraparound edges at each step to preserve length which is described in Chapter 6, but it can get ugly. Luckily, there is a way to avoid this

problem entirely. If we only use *short edges*  $uv$  which have length  $|u - v|$  in  $K_{14}$  for our labelings, then that edge will have length  $|u - v|$  in every complete graph  $K_n$  where  $n > 14$ . We show another example below for  $t > 2$ .

$$\text{In } K_{14(1)}: \ell((0, 6)) = \min\{|0 - 6|, 14 - |0 - 6|\} = \min\{6, 8\} = 6.$$

$$\text{In } K_{14(2)}: \ell((0, 6)) = \min\{|0 - 6|, 28 - |0 - 6|\} = \min\{6, 22\} = 6.$$

$\vdots$

$$\text{In } K_{14(t)}: \ell((0, 6)) = \min\{|0 - 6|, 14t - |0 - 6|\} = \min\{6, 22 + 14(t - 2)\} = 6.$$

So we see that if we only use short edges, the length of edges in our labeling will be preserved as we scale up. So for that reason, we simply **need** to use them. However, another important feature is that if we only use short edges  $uv$ , we know that WLOG  $v > u$  and so  $|u - v| = v - u$ . This introduces another extremely important mechanism we can exploit to generalize labelings.

If we bundle the short edge requirement and the requirement that the maximal edge is a pendant edge with a new ordered bipartition  $V(G) = A \sqcup B$  requirement on the vertices of a labeling  $G$  of  $F$  such that all vertices  $a \in A$  only have neighbors in  $B$  which are larger than  $a$ , then we get a really nice property. All edges are now of the form  $ab$  where  $a \in A$  and  $b \in B$  so that  $a < b$  and so all such edges  $ab$  have length  $b - a$  modulo 14. This means that if  $\ell(ab) = l$  in  $K_{14}$ , then  $b - a = l$  and so  $(b + c) - a = l + c$ . More importantly, if we then add any  $c \in \mathbb{Z}_{14t}$  to all vertices in the larger partite set  $B$  of our labeling, we will simply increase the lengths of all edges in our labeling by  $c$  in  $\mathbb{Z}_{14t}$  and  $\mathbb{Z}_{14t+1}$  for  $t \geq 1$  as a result of bundling these restrictions together! We call the act of adding some constant  $c$  to all vertices of the larger partite set  $B$  in a labeling of this type *stretching*.

Let  $t > 1$ . Then for a labeling  $G$  of  $F$  in  $K_{14t+1}$ , developing by 1 will generate lengths in  $\{1, \dots, 7\}$ , then if we stretch that labeling by  $7i$  and develop once more by 1 for each  $0 < i < t$ , we generate all edges of  $K_{14t+1}$  and get an  $F$ -design of order  $14t + 1$ .

Now, in the same labeling  $G$  of  $F$  in  $K_{14t}$ , recall that there are only  $7t$  edges of the maximal length  $7t$ . This means that we want to take it's vertices to be  $\mathbb{Z}_{14t-1} \cup \{\infty\}$  and so there are now  $14t - 1$  edges of length 7 along with all lengths less than  $7t$ . So then we simply develop  $G$  by 1 to generate lengths in  $\{1, \dots, 7\}$ , then stretch that labeling



by  $7i$  and develop by 1 for each  $0 < i < t$  **except in the last labeling** which was stretched by  $7(t-1)$  with lengths in  $\{7t-6, \dots, 7t\}$ . Since the pendant edge of length 7 was stretched to be one of maximal length  $7t$ , it is still a pendant edge. So now we relabel the leaf as  $\infty$  and so its length becomes  $\infty$ . We then develop it by 1 and have collected all edges in  $K_{14t}$  while generating an  $F$ -design of order  $14t$ .

We show how the labeling from Figure 3.3 gives the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of  $K_{14(2)} = K_{28}$  and  $K_{14(2)+1} = K_{29}$  below.

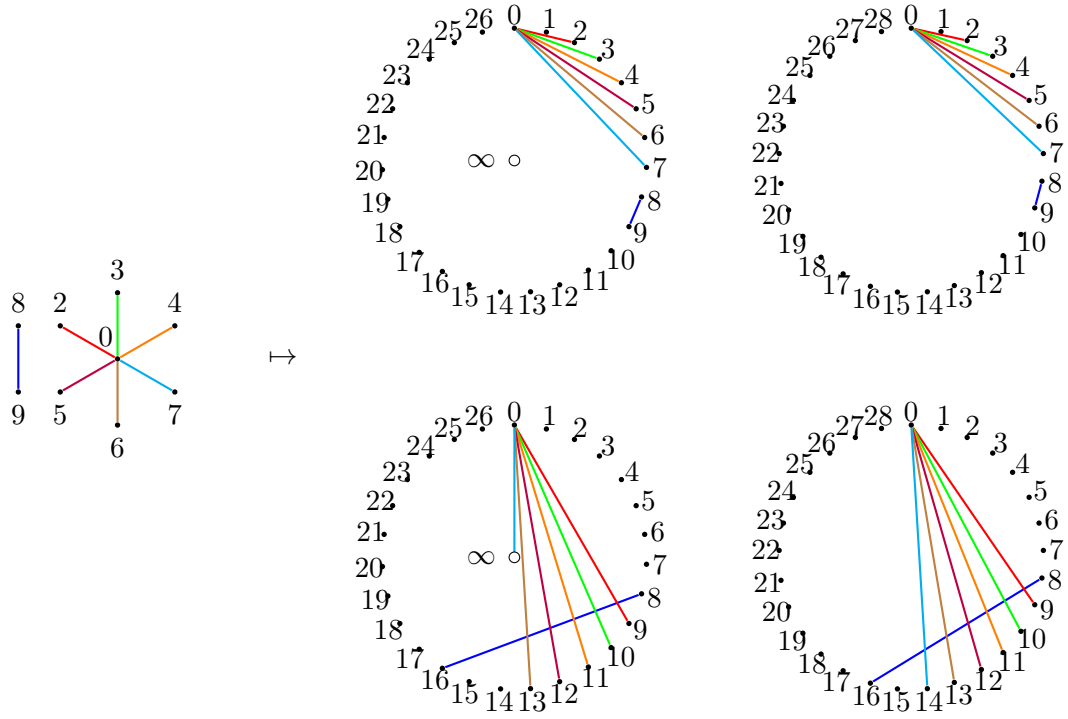


Figure 3.4: A single labeling (left) gives four labelings which when developed by 1 give the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of (middle)  $K_{28}$  and (right)  $K_{29}$ .

We will refer to this act of stretching a labeling for the next construction, but first the ideas in this section are now put together formally in the next section.

### 3.2 Results for $n \equiv 0, 1 \pmod{14}$

We begin with the original  $\rho$ -labeling of a bipartite graph  $G$  on  $m$  edges which uses lengths modulo  $2m$  (in  $K_{2m}$ ).

**Definition 3.2.1** ((Rosa [7])). Let  $G$  be a graph with  $m$  edges. A  $\rho$ -labeling of  $G$  is an injection  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2m\}$  that induces a bijective *length function*  $\ell : E(G) \rightarrow \{1, 2, \dots, m\}$  where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all  $uv \in E(G)$ .

Rosa showed that if a  $\rho$ -labeling of a graph  $G$  with  $m$  edges exists, then a cyclic  $G$ -decomposition of  $K_{2m+1}$  exists, which is the next theorem. Later, Rosa and his peers began studying more restrictive types of  $\rho$ -labelings to decompose more complete graphs. Next, we define some of these labelings and theorems associated with them.

**Theorem 3.2.2** ((Rosa [7])). *Let  $G$  be a graph with  $m$  edges. There exists a cyclic  $G$ -decomposition of  $K_{2m+1}$  if and only if  $G$  admits a  $\rho$ -labeling.*

**Definition 3.2.3** ((Rosa [7])). A  $\sigma$ -labeling of a graph  $G$  is a  $\rho$ -labeling such that  $\ell(uv) = |f(u) - f(v)|$  for all  $uv \in E(G)$ .

**Definition 3.2.4** ((El-Zanati, Vanden Eynden [6])). A  $\rho$ - or  $\sigma$ -labeling of a bipartite graph  $G$  with bipartition  $(A, B)$  is called an *ordered*  $\rho$ - or  $\sigma$ -labeling and denoted  $\rho^+, \sigma^+$ , respectively, if  $f(a) < f(b)$  for each edge  $ab$  with  $a \in A$  and  $b \in B$ .

**Theorem 3.2.5** ((El-Zanati, Vanden Eynden [6])). *Let  $G$  be a graph with  $m$  edges which has a  $\rho^+$ -labeling. Then  $G$  decomposes  $K_{2mk+1}$  for all positive integers  $k$ .*

**Definition 3.2.6** ((Freyberg, Tran [5])). A  $\sigma^{+-}$ -labeling of a bipartite graph  $G$  with  $m$  edges and bipartition  $(A, B)$  is a  $\sigma^+$ -labeling with the property that  $f(a) - f(b) \neq m$  for all  $a \in A$  and  $b \in B$ , and  $f(v) \notin \{2m, 2m - 1\}$  for any  $v \in V(G)$ .

**Theorem 3.2.7** ((Freyberg, Tran [5])). *Let  $G$  be a graph with  $m$  edges and a  $\sigma^{+-}$ -labeling such that the edge of length  $m$  is a pendant. Then there exists a  $G$ -decomposition of both  $K_{2mk}$  and  $K_{2mk+1}$  for every positive integer  $k$ .*

Table 3.1 gives  $\sigma^{+-}$ -labelings of all forests on 7 edges except the matching. The vertex labels of each connected component with  $k$  vertices are given as a  $k$ -tuple,  $(v_1, \dots, v_k)$  corresponding to the vertices  $v_1, \dots, v_k$  positioned as shown in Figure 2.5. We leave it to the reader to infer the bipartition  $(A, B)$ .

**Example 3.2.8.** A  $\sigma^{+-}$ -labeling of  $\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$  is shown in Figure 3.5. The vertices labeled 1, 2 and 9 belong to  $A$ , and the others belong to  $B$ . The lengths of each edge are indicated on the edge.

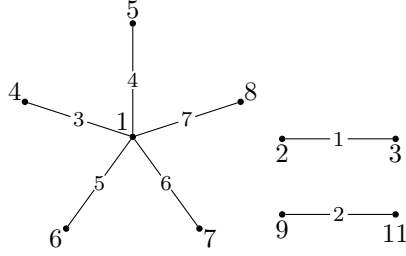


Figure 3.5:  $\sigma^{+-}$ -labeling of  $\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$

The labelings given in Table 3.1 along with Theorem 3.2.7 are enough to conclude this case.

Forest	Vertex Labels
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5, 2, 9, 7) \sqcup (3, 4)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(9, 2, 5, 1, 6, 0, 3) \sqcup (8, 7)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(9, 2, 5, 1, 6, 0, 4) \sqcup (8, 7)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(5, 1, 4, 2, 9, 6, 7) \sqcup (10, 11)$
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(3, 8, 1, 4, 2, 5, 7) \sqcup (9, 10)$
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(7, 8, 1, 6, 0, 4, 3) \sqcup (9, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(8, 1, 6, 3, 4, 5, 7) \sqcup (9, 10)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(6, 1, 5, 3, 8, 4, 7) \sqcup (9, 10)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(5, 11, 9, 10, 6, 12, 7) \sqcup (8, 1)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(4, 8, 1, 6, 0, 5, 3) \sqcup (9, 10)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(0, 6, 1, 5, 2, 9) \sqcup (11, 10, 12)$
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(3, 6, 1, 8, 4, 0) \sqcup (10, 9, 11)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(5, 11, 9, 12, 7, 10) \sqcup (1, 8, 4)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(3, 8, 4, 1, 6, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(5, 1, 8, 3, 4, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(4, 1, 8, 5, 6, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(0, 6, 1, 5, 2) \sqcup (9, 8, 10, 3)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(7, 1, 8, 5, 6) \sqcup (0, 4, 2, 3)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(7, 1, 8, 4, 6) \sqcup (10, 9, 11, 12)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(6, 0, 3, 4, 5) \sqcup (8, 7, 9, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(4, 8, 1, 7, 2) \sqcup (10, 9, 11, 12)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(6, 0, 3, 4, 5) \sqcup (8, 9, 2, 7)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(0, 6, 1, 5, 2, 9) \sqcup (8, 10) \sqcup (3, 4)$
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(3, 6, 1, 8, 4, 0) \sqcup (5, 7) \sqcup (9, 10)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(4, 1, 8, 3, 5, 7) \sqcup (0, 2) \sqcup (9, 10)$
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(5, 8, 4, 1, 6, 7) \sqcup (0, 2) \sqcup (9, 10)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(5, 11, 9, 12, 7, 10) \sqcup (8, 1) \sqcup (0, 4)$

Table 3.1:  $\sigma^{+-}$ -labelings for forests with seven edges

Forest	Vertex Labels
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(4, 1, 8, 5, 6, 7) \sqcup (2, 3) \sqcup (9, 11)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5, 2) \sqcup (8, 10, 9) \sqcup (11, 4)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(7, 1, 8, 5, 6) \sqcup (10, 9, 11) \sqcup (0, 4)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(6, 0, 3, 4, 5) \sqcup (1, 8, 7) \sqcup (9, 11)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (2, 9, 7, 10) \sqcup (3, 4)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, 10, 7) \sqcup (4, 0, 5, 6) \sqcup (8, 1)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (10, 9, 11, 12) \sqcup (8, 1)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(0, 6, 1, 5) \sqcup (8, 10, 9) \sqcup (11, 4, 7)$
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(4, 0, 5, 6) \sqcup (1, 8, 7) \sqcup (11, 9, 12)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (8, 10, 7) \sqcup (11, 4) \sqcup (2, 3)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (11, 9, 12) \sqcup (2, 3) \sqcup (8, 1)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(0, 6, 1, 5, 2) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(6, 1, 8, 4, 7) \sqcup (3, 5) \sqcup (9, 12) \sqcup (10, 11)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(3, 0, 4, 5, 6) \sqcup (8, 1) \sqcup (10, 11) \sqcup (9, 7)$
$3\mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1) \sqcup (4, 8, 5) \sqcup (2, 9, 7) \sqcup (10, 11)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (9, 2) \sqcup (8, 10) \sqcup (4, 7) \sqcup (11, 12)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (2, 3) \sqcup (9, 11) \sqcup (8, 1) \sqcup (10, 7)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(0, 6, 1) \sqcup (4, 8, 5) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(0, 6, 1) \sqcup (8, 4) \sqcup (2, 5) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$

Table 3.1:  $\sigma^{+-}$ -labelings for forests with seven edges

**Theorem 3.2.9.** *Let  $F$  be a forest with 7 edges. There exists an  $F$ -decomposition of  $K_n$  whenever  $n \equiv 0$  or  $1 \pmod{14}$ .*

*Proof.* The matching has been solved, and then the rest of the proof follows from Theorem 3.2.7 and the labelings given in Table 3.1.  $\square$

## Chapter 4

### $n \equiv 7, 8 \pmod{14}$

In this chapter, we will use our own constructions based on edge lengths in  $K_n$  where  $n \equiv 7$  or  $8 \pmod{14}$ . We first describe our construction in the context of  $K_{21}$  and  $K_{22}$  in Subsection 4.1.1, and generalize our construction in Subsection 4.1.2. We then formalize these ideas in Section 4.2.

#### 4.1 Construction for $n \equiv 7, 8 \pmod{14}$

The number of vertices in  $K_7$  and  $K_8$  is less than 9, the minimum number of vertices of a seven edge forest. So neither are decomposable by seven edge forests and our base graphs are  $K_{21}$  and  $K_{22}$  for  $n \equiv 7$  and  $8 \pmod{14}$ , respectively. We show these base graphs in the figure on the following page.

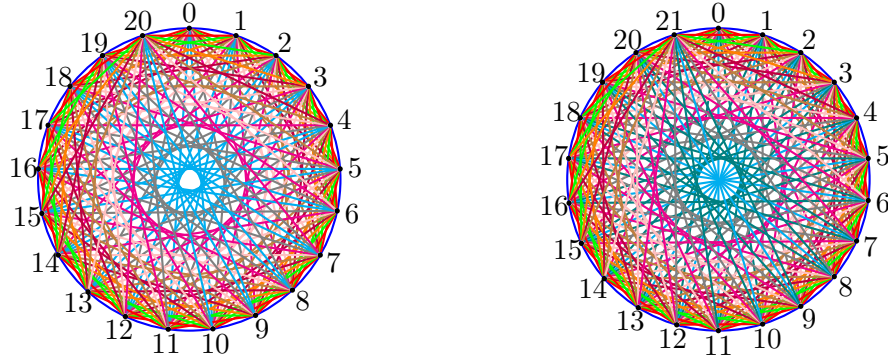


Figure 4.1:  $K_{21}$  (left) and  $K_{22}$  (right) with edges colored by length.

#### 4.1.1 $K_{21}$ and $K_{22}$

We have lost some luxuries present in the  $n \equiv 0, 1 \pmod{14}$  case. There are two main problems in this case that we need to solve:

- (1) In  $K_{22}$  there are 11 total maximal length 11 edges but 22 of every other length.
- (2)  $K_{21}$  and  $K_{22}$  have lengths  $\{1, \dots, 10\}$  and  $\{1, \dots, 11\}$ , respectively.

(1) is a problem because we want to be able to develop the vertices of our labelings to collect all edges of  $K_{22}$  in the same number of steps. (2) is a problem because this means we cannot just fit 7 distinct lengths on a single labeling and collect all edges when we develop the vertices by 1. Furthermore, since  $K_{22}$  has one more length than  $K_{21}$ , we don't have a single labeling strategy in this project that takes care of both cases where  $n \equiv 7$  and  $8 \pmod{14}$  at once like  $\sigma^{+-}$  did for  $n \equiv 0$  and  $1 \pmod{14}$ . We address these problems in order.

We have a remedy for length 11 edges in  $K_{22}$  which is similar to what we did for  $K_{14}$ . We take the vertices of  $K_{22}$  to be  $\mathbb{Z}_{21} \cup \{\infty\}$  and redefine length of edges and development for vertices in  $K_{22}$ :

$$\ell(uv) = \begin{cases} \min\{|u - v|, 21 - |u - v|\}, & u, v \neq \infty, \\ \infty, & u \text{ or } v = \infty \end{cases} \quad \text{and } v \mapsto \begin{cases} v + 1, & v \in \mathbb{Z}_{n-1}, \\ \infty, & v = \infty. \end{cases}$$

Now, we have 21 edges of lengths 1 through 10 as well as  $\infty$ , since the  $\infty$  vertex will have all 21 edges of length  $\infty$  to vertices in  $\mathbb{Z}_{21}$  adjacent to it. Now we can theoretically

cyclically generate edges of every length in  $K_{21}$  in the same number of steps, as well as all edges of every length in  $K_{22}$  in the same number of steps, but we just can't generate them all on one labeling for either case since the number of edges on our forests doesn't divide the number of distinct lengths in  $K_{21}$  or  $K_{22}$ .

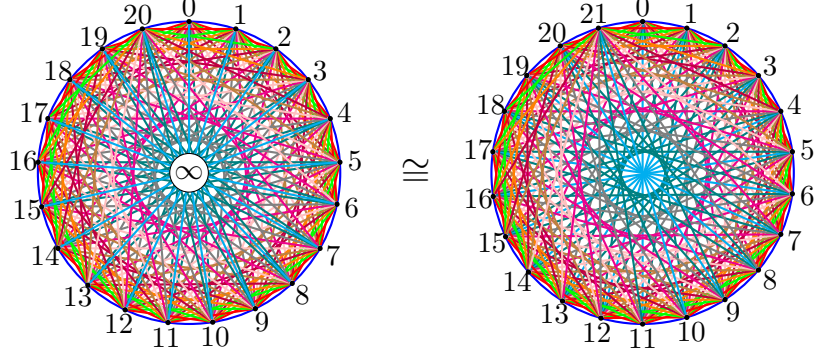


Figure 4.2:  $K_{21} \vee K_1$  is isomorphic to  $K_{22}$  (right)

Let's begin solving (2) by just looking at  $K_{21}$ . If we can generate all 21 distinct edges of lengths 1, 2 and 3 separately, then we can generate the remaining 7 lengths with one labeling. But how can we collect 21 edges of length 1, 2, and 3 in isomorphic copies? By imposing a new edge length on top of the standard length  $\ell$ , we can achieve this. We define  $\ell_7^+$  from  $\mathbb{Z}_{21} \cup \{\infty\}$  to  $\mathbb{Z}_7$  as follows

$$\ell_7^+(uv) = \begin{cases} u + v \bmod 14, & u, v \neq \infty \\ v, & u = \infty \end{cases}$$

Now, every edge has a standard length  $\ell$  and an additive length  $\ell_7^+$  modulo 7. Previously, we partitioned the edges into sets  $E_i$  of edges of length  $i$  for each length  $i \in \{1, \dots, 10\}$  via the standard length function  $\ell$ . Now, within each partite set  $E_i$ , we have further partitioned the edges into sets  $E_{i,j}$  of standard length  $i$  and additive length  $j$  modulo 7. For example: the edge  $(1, 8)$  has length  $\ell((1, 8)) = 7$  and  $\ell_7^+((1, 8)) = 8 + 1 \bmod 7 = 2$ , so  $(1, 8) \in E_{7,2}$ . We show a bundle of labelings of  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$  that utilizes this new partition on the next page.



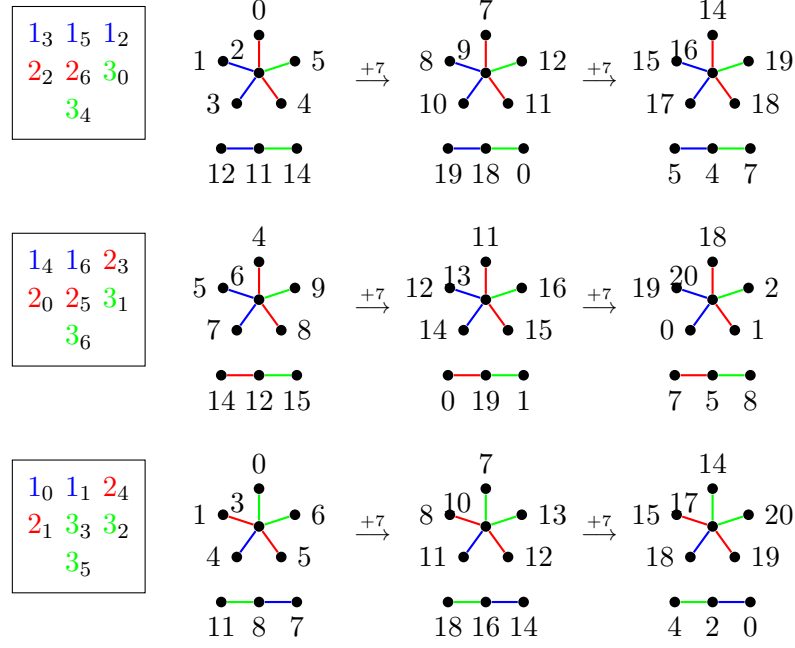


Figure 4.3: Three labelings of  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$  (left column) that generate all edges of lengths 1, 2, and 3 in  $K_{21}$  when developed by 7

Each square in the figure above contains the edge length pairs of the edges in the labelings to its right. Standard lengths  $\ell$  are colored in the same manner as in Figure 4.3 and the new length  $\ell_7^+$  appears in black as the subscript for each of these lengths. Across all three labelings, there is exactly one representative for all 7 distinct equivalence classes modulo  $\ell_7^+$  for each each length in  $\{1, 2, 3\}$ . Now, we already know developing the vertices by 7 preserves  $\ell$ , but it also preserves  $\ell_7^+$  since  $u+7+v+7 \bmod 7 = u+v \bmod 7$ . So then developing all labelings by 7 will give us all edges of lengths in  $\{1, 2, 3\}$  across 9 isomorphic copies of  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ .

An equivalent way of understanding development is through *group actions*. Let  $S$  be the set of all labelings in Figure 4.3. The cyclic subgroup  $\langle 7 \rangle = \{0, 7, 14\} \subseteq \mathbb{Z}_{21}$  acts on  $S$ ;  $\langle 7 \rangle \curvearrowright S$  via developing the vertices by 7. Then if we call the labelings in column 1:  $F_1, F_2, F_3$  from top to bottom, the set of labelings in Row 1 is simply  $\text{Orb}_{\langle 7 \rangle}(F_1)$ , Row 2 is  $\text{Orb}_{\langle 7 \rangle}(F_2)$  and Row 3 is  $\text{Orb}_{\langle 7 \rangle}(F_3)$ . We employ the same technique for  $K_{22}$ , we just need four labelings  $F_1, F_2, F_3, F_4$  with lengths 1, 2, 3,  $\infty$ .

Now have a technique to collect all edges of lengths  $\{1, 2, 3\}$  and  $\{1, 2, 3, \infty\}$  in  $K_{21}$  and  $K_{22}$ , respectively, in isomorphic copies of graphs. So we just need to collect edges of lengths in  $\{4, \dots, 10\}$ . Well, if we just stretch our  $\sigma^{+-}$ -labelings by 3, we will now have lengths  $\{4, \dots, 10\}$  instead of  $\{1, \dots, 7\}$ . So developing those stretched labelings by 1 will give us the remaining edges of both  $K_{21}$  and  $K_{22}$  in isomorphic copies of our forests.

#### 4.1.2 Generalizing to $K_{14t+7}$ and $K_{14t+8}$ for $t > 1$

Recall our labeling constructions for collecting edges of lengths  $\{1, 2, 3\}$  and  $\{1, 2, 3, \infty\}$  in  $K_{21}$  and  $K_{22}$ . As long as we only use short edges, we can actually just generate all edges of these lengths for the entire families with the same set of labelings we used for  $K_{14t+7}$  and  $K_{14t+8}$  where  $t > 1$ . The only thing that changes at each step  $t \mapsto t + 1$  is that more steps of development is needed to collect all the edges, and therefore the number of isomorphic copies.

Equivalently, the orbits of the labelings will simply grow at each step  $t \mapsto t + 1$ . Let  $\langle 7 \rangle_n$  denote the subgroup  $\langle 7 \rangle \subset K_n$ . Then for each forest and its set of labelings  $F_1, F_2, F_3$  used for  $K_{21}$ ,  $\text{Orb}_{\langle 7 \rangle_{21}}(F_i) \subset \text{Orb}_{\langle 7 \rangle_{14t+7}}(F_i)$  for each  $i = 1, 2, 3$  and  $t > 1$ .

Next, recall how we generalized labelings to  $K_{14t}$  and  $K_{14t+1}$  in Chapter 3 for edges of subsequent lengths  $\bigcup_{0 < i < t} \{1 + 7i, \dots, 7 + 7i\}$  where  $t > 1$ . We do the same thing here for  $K_{14t+7}$  and  $K_{14t+8}$  except for all subsequent edges of lengths in  $\bigcup_{0 < i < t} \{4 + 7i, \dots, 10 + 7i\}$ . We simply stretch our labelings (previously  $\sigma^{+-}$ -labelings which were stretched by 3) for both  $K_{21}$  and  $K_{22}$  that contain edge lengths in  $\{4, \dots, 10\}$  by 7 at each step  $t \mapsto t + 1$  to collect each set of edges with new lengths that come 7 at a time. The nice thing is, we don't need to do anything but this stretching on our one labeling at each step, since we took care of  $\infty$  in  $K_{22}$ . Developing all of these by 1 will give us the remaining edges in isomorphic copies of our forests and complete the forest decompositions of  $K_{14t+7}$  and  $K_{14t+8}$  where  $t \geq 1$ .

We formalize these constructions in the next section.

## 4.2 Results for $n \equiv 7, 8 \pmod{14}$

We begin with a formal definition for the set of labelings used to generate edges in  $K_{14t+7}$  of lengths in  $\{1, 2, 3\}$  in isomorphic copies of a graph  $G$  on seven edges for  $t \geq 1$ . Then we prove that if a graph  $G$  admits such a labeling and a  $\rho^+$ -labeling, then it decomposes  $K_{14t+7}$  for  $t \geq 1$ .

**Definition 4.2.1.** Let  $G$  be a graph with 7 edges. A (1-2-3)-labeling of  $3G$  is an assignment  $f$  of the integers  $\{0, \dots, 20\}$  to the vertices of  $3G$  such that

- (1)  $f(u) \neq f(v)$  whenever  $u$  and  $v$  belong to the same connected component,

and

- (2)

$$\bigcup_{uv \in E(3G)} \{(f(u) \bmod 7, f(v) \bmod 7)\} = \bigcup_{i=0}^6 \bigcup_{j=1}^3 \{(i, i+j \bmod 7)\}.$$

Notice that the second condition of a (1-2-3)-labeling **demands** that  $3G$  contains exactly 7 edges of each length in  $\{1, 2, 3\}$ . Additionally, the second condition requires that no two edges of the same length have the same end labels modulo 7. A (1-2-3)-labeling of every forest with 7 edges except  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  is given in Table A.1. This exceptional graph does not admit such a labeling, and we deal with it in Chapter 5.

**Theorem 4.2.2.** *Let  $G$  be a bipartite graph with 7 edges. If  $3G$  admits a (1-2-3)-labeling and  $G$  admits a  $\rho^+$ -labeling, then  $G$  decomposes  $K_{14k+7}$  for every  $k \geq 1$ .*

*Proof.* Let  $n = 14k + 7$  and notice that  $K_n$  has  $|E(K_n)| = (7k+3)(14k+7)$  edges, which can be partitioned into  $14k+7$  edges of each of the lengths in  $\{1, 2, \dots, 7k+3\}$ . We will construct the  $G$ -decomposition in two steps. First, we use the (1-2-3)-labeling to identify all the edges of lengths 1, 2, and 3 accounting for  $3(2k+1)$  copies of  $G$ . Then, we use the  $\rho^+$ -labeling to identify edges of the remaining lengths in  $7k(2k+1)$  copies of  $G$ . In total, the decomposition consists of  $|E(K_n)|/7 = (7k+3)(2k+1)$  copies of  $G$ .

Let  $f_1$  be a (1-2-3)-labeling of  $3G$  and identify this graph as a block  $B_0$ . Then develop  $B_0$  by 7 modulo  $n$ . Since the order of the development is  $\frac{n}{7} = 2k+1$  and there are 7 edges of each of the lengths 1, 2, and 3 in  $B_0$ , we have identified  $3(2k+1)$  copies

of  $G$  containing all  $14k + 7 = n$  edges of each length 1, 2, and 3. Notice (2) of Definition 4.2.1 ensures no edge has been counted more than once in the development.

Let  $f_2 : V(G) \rightarrow \{0, \dots, 14\}$  be a  $\rho^+$ -labeling of  $G$  with associated vertex partition  $(A, B)$ . For  $i = 1, 2, \dots, k$ , identify blocks  $B_i \cong G$  with vertex labels  $\ell$  such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i - 1), & \text{if } v \in B \end{cases}$$

Notice that the  $i^{\text{th}}$  block contains exactly one edge of each length  $7i - 3, 7i - 2, \dots$ , and  $7i + 3$ . This is because every edge  $ab$  has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i - 1)$$

and  $f_2(b) - f_2(a)$  is a length in  $\{1, \dots, 7\}$ . Developing each block  $B_i$  by 1 yields  $14k + 7$  copies of  $G$  per block and accounts for  $14k + 7$  edges of each of the lengths 4, 5,  $\dots$ , and  $7k + 3$ .

Since we have identified

$$3(2k + 1) + k(14k + 7) = (7k + 3)(2k + 1)$$

edge-disjoint copies of  $G$ , the proof is complete. □

Next we formalize the set of labelings used to generate edges in  $K_{14t+8}$  of lengths in  $\{1, 2, 3, \infty\}$  in isomorphic copies of a graph  $G$  on seven edges for  $t \geq 1$ . Then we prove that if a graph  $G$  admits such a labeling and a  $\rho^+$ -labeling, then it decomposes  $K_{14t+8}$  for  $t \geq 1$ .

**Definition 4.2.3.** Let  $G$  be a graph with 7 edges. A *1-rotational (1-2-3)-labeling* of  $4G$  is an assignment  $f$  of  $\{0, \dots, 20\} \cup \infty$  to the vertices of  $4G$  such that

- (1)  $f(u) \neq f(v)$  whenever  $u$  and  $v$  belong to the same connected component,

and

- (2)

$$\bigcup_{uv \in E(4G)} \{(f(u) \bmod 7, f(v) \bmod 7)\} = \bigcup_{i=0}^6 \bigcup_{j=1}^3 \{(i, i + j \bmod 7), (i, \infty)\}.$$

Notice that the second condition of a 1-rotational (1-2-3)-labeling demands that  $4G$  contains exactly 7 edges of each length in  $\{1, 2, 3, \infty\}$ . Additionally, the second condition requires that no two edges of the same length have the same end labels modulo 7. A 1-rotational (1-2-3)-labeling of every forest with 7 edges except  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  is given in Table A.2. This exceptional graph does not admit this labeling either, and we deal with it for this case as well in Chapter 5.

**Theorem 4.2.4.** *Let  $G$  be a bipartite graph with 7 edges. If  $4G$  admits a 1-rotational (1-2-3)-labeling and  $G$  admits a  $\rho^+$ -labeling, then  $G$  decomposes  $K_{14k+8}$  for every  $k \geq 1$ .*

*Proof.* Let  $n = 14k + 8$  and notice that  $K_n$  has  $|E(K_n)| = (7k + 4)(14k + 7)$  edges, which can be partitioned into  $14k + 7$  edges of each of the lengths in  $\{1, 2, \dots, 7k + 3, \infty\}$ . We will construct the  $G$ -decomposition in two steps. First, we use the 1-rotational (1-2-3)-labeling to identify all the edges of lengths 1, 2, 3, and  $\infty$  accounting for  $4(2k + 1)$  copies of  $G$ . Then, we use the  $\rho^+$ -labeling to identify edges of the remaining lengths in  $7k(2k + 1)$  copies of  $G$ . In total, the decomposition consists of  $|E(K_n)|/7 = (7k + 4)(2k + 1)$  copies of  $G$ . Let  $f_1$  be a 1-rotational (1-2-3)-labeling of  $4G$  and identify this graph as a block  $B_0$ . Then develop  $B_0$  by 7 modulo  $n - 1$ . Since the order of the development is  $\frac{n-1}{7} = 2k + 1$  and there are 7 edges of each of the lengths 1, 2, 3 and  $\infty$  in  $B_0$ , we have identified  $4(2k + 1)$  copies of  $G$  containing all  $14k + 7 = n - 1$  edges of each length 1, 2, 3 and  $\infty$ . Notice (2) of Definition 4.2.3 ensures no edge has been counted more than once in the development.

Let  $f_2 : V(G) \rightarrow \{0, \dots, 14\}$  be a  $\rho^+$ -labeling of  $G$  with associated vertex partition  $(A, B)$ . For  $i = 1, 2, \dots, k$ , identify blocks  $B_i \cong G$  with vertex labels  $\ell$  such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i - 1), & \text{if } v \in B \end{cases}$$

Notice that the  $i^{\text{th}}$  block contains exactly one edge of each length  $7i - 3, 7i - 2, \dots$ , and  $7i + 3$ . This is because every edge  $ab$  has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i - 1)$$

and  $f_2(b) - f_2(a)$  is a length in  $\{1, \dots, 7\}$ . Developing each block  $B_i$  by 1 yields  $14k + 7$  copies of  $G$  per block and accounts for  $14k + 7$  edges of each of the lengths 4, 5,  $\dots$ , and  $7k + 3$ .

Since we have identified

$$4(2k + 1) + k(14k + 7) = (7k + 4)(2k + 1)$$

edge-disjoint copies of  $G$ , the proof is complete.  $\square$

We are now able to state the main theorem of this chapter.

**Theorem 4.2.5.** *Let  $F$  be a forest with 7 edges and  $F \not\cong \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ . There exists an  $F$ -decomposition of  $K_n$  whenever  $n \equiv 7$  or  $8 \pmod{14}$  and  $n \geq 21$ .*

*Proof.* If  $n \equiv 7 \pmod{14}$ , a (1-2-3)-labeling of  $3F$  can be found in Table A.1. On the other hand, if  $n \equiv 8 \pmod{14}$ , then a 1-rotational (1-2-3)-labeling of  $4F$  can be found in Table A.2. In either case, a  $\rho^+$ -labeling of  $F$  can be found in Table 3.1 (recall that a  $\sigma^{+-}$ -labeling is a  $\rho^+$ -labeling). The result now follows from Theorems 4.2.2 and 4.2.4.  $\square$

We illustrate how to interpret the tables of labelings and realize the constructions from the last two chapters by building an  $F$ -decomposition of  $K_{35}$  and  $K_{36}$  for the forest graph  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ .

**Example 4.2.6.** *Here are excerpts from Tables 3.1, A.1, and A.2 for  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$*

Labeling Type	Labelings
$\sigma^{+-}$	$(4, 1, 8, 5, 6, 7) \sqcup (10, 9, 11)$
(1-2-3)	$(0, 2, 1, 3, 4, 5) \sqcup (12, 11, 14)$ $(4, 6, 8, 9, 5, 7) \sqcup (14, 12, 15)$ $(0, 3, 1, 4, 5, 6) \sqcup (11, 8, 7)$
1-rotational (1-2-3)	$(1, 2, 0, 3, 4, 5) \sqcup (11, 8, \infty)$ $(2, \infty, 3, 4, 5, 6) \sqcup (12, 13, 15)$ $(6, 7, 8, 4, 5, \infty) \sqcup (11, 12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (9, 6, 4)$

Figure 4.4: Labelings for  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$

The  $\rho^+$  labelings obtained by stretching the  $\sigma^{+-}$  labeling are bottommost labelings in the following generating presentations and are developed by 1.

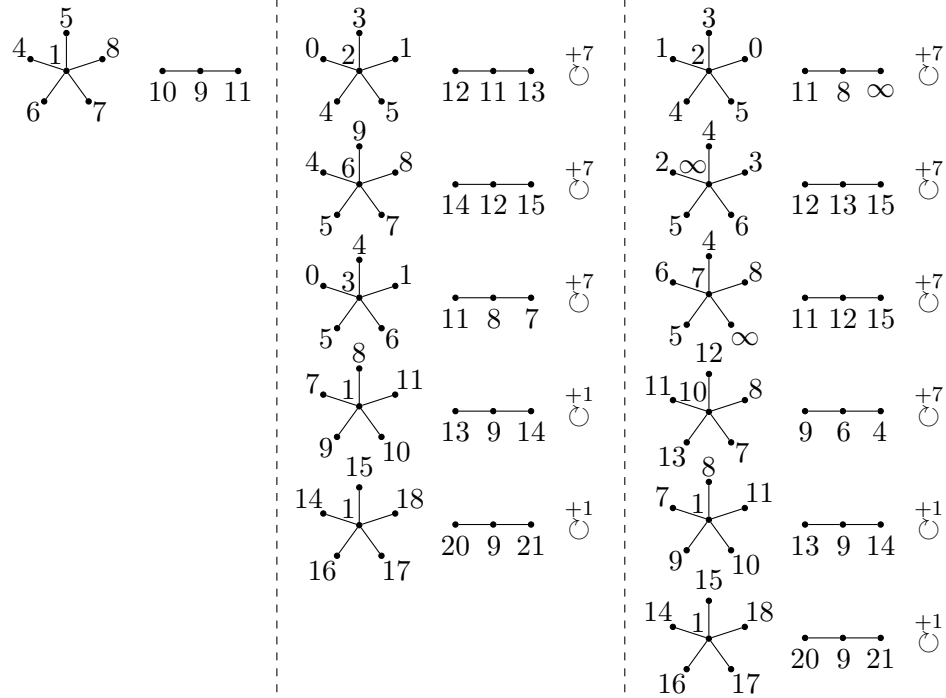


Figure 4.5: A  $\sigma^{+-}$ -labeling of  $F \cong \mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$  (left) and generating presentations for the  $F$ -decomposition of  $K_n$  where  $n = 35$  (middle) and  $n = 36$  (right)

We have now proven that every seven edge forest except  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_n$  if and only if  $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$ . As stated earlier, we deal with the this exceptional forest  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  in the next chapter.

## Chapter 5

### $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$

To begin we construct  $K_{14t+7}$  and  $K_{14t+8}$  using joined copies of  $K_{22}$ ,  $K_{21}$ , and  $K_{14}$ . Let  $t$  be a positive integer. Now join  $t - 1$  copies of  $K_{14}$  with each other and a lone copy of  $K_{21}$ . The resulting graph is  $K_{14(t-1)+21} \cong K_{14t+7}$ . Similarly,  $K_{14t+8}$  can be constructed by joining  $t - 1$  copies of  $K_{14}$  with each other and 1 copy of  $K_{22}$ .

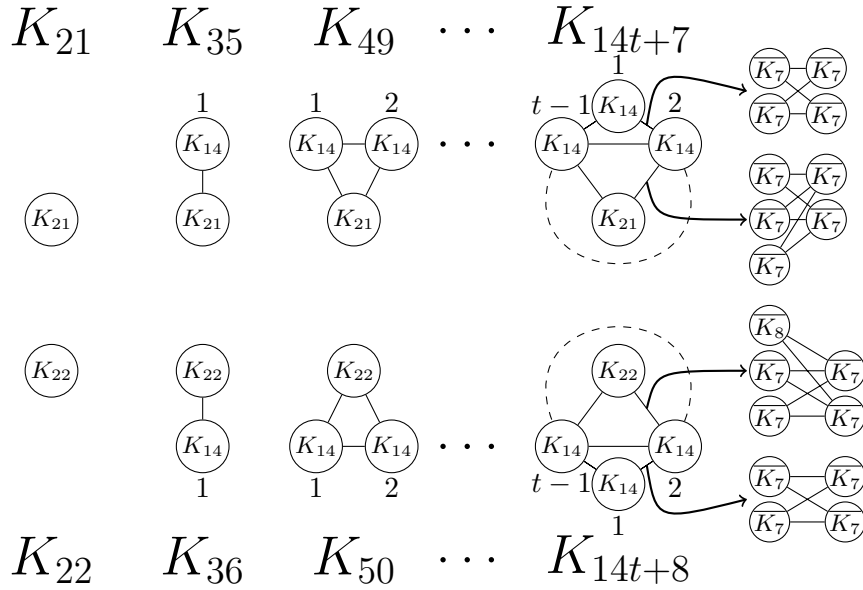


Figure 5.1: A different construction for  $K_{14t+7}$  and  $K_{14t+8}$



Equivalently, We can view  $K_{14t+7}$  as  $K_t$  whose ‘vertices’ are  $K_{14}$  except one which is  $K_{21}$ , and whose edges are the join between them. We will refer to these ‘vertices’ as nodes. Similarly, we can view  $K_{14t+8}$  as  $K_t$  whose nodes are  $K_{14}$  except one which is  $K_{22}$ , and whose edges are the join between them. Notice in Figure 5.1 that all edges in the  $K_t$  constructions of these families are then the edges of  $K_{14,14}$ ,  $K_{21,14}$ , and  $K_{22,14}$ .

We show that  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_n$  for  $n \equiv 7$  or  $8 \pmod{14}$  where  $n \geq 21$  by proving that  $K_{22}$ ,  $K_{21}$ ,  $K_{14}$ ,  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$  can each be decomposed by  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  since these six graphs make up the nodes and edges of the  $K_t$  representations of  $K_{14t+7}$  and  $K_{14t+8}$  where  $t \geq 1$ .

We begin with  $K_{21}$  and  $K_{22}$ . The proof of the next theorem was obtained by manipulating a  $K_{1,7}$ -decomposition of  $K_{21}$  by Cain in [1]. We ‘plucked edges off’ of every 7-edge star in the decomposition, put them to the side, and then sent them to 6-edge stars that they were vertex disjoint from. This gave us a  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{21}$  which can be found in Table A.3. Then, we just added three 6-edge stars centered at  $\infty$ , and three distinct single edge paths from  $\infty$  to the remaining three neighbors of  $\infty$  in  $K_{22}$  not covered in the stars. We once again put all the lone paths aside, and sent them to 6-edge stars that they were vertex disjoint from them. This gave us a  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{22}$  which can be found in Table A.4.

**Theorem 5.0.1.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{21}$  and  $K_{22}$ .

*Proof.* Figures 8 and 9 give  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decompositions of  $K_{21}$  and  $K_{22}$ , respectively. □

We didn’t need to develop anything formally to do this as we only performed this process once for the given 7-edge star decomposition of  $K_{21}$ . Initially, we were interested in proving a stronger statement resulting from Cain’s work, but found it difficult. Investigating implications of Cain’s work [1] from 1974 is something we are interested in for future work.

Next, we address  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$ .

**Theorem 5.0.2.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{n,7}$  for all  $n \geq 2$ .

*Proof.* Consider  $K_{n,7}$  where  $n \geq 2$ . Take the partite set of  $n$  vertices to be  $\mathbb{Z}_n$  and color them white. Similarly, take the partite set of 7 vertices to be  $K_7$  and color them black. Naturally we refer to *white-black* vertices  $uv$  in  $K_{n,7}$  via  $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_7$  and vice versa. Finally, let  $E_i = \{(i, 0)\} \sqcup (\{i+1\} \times \{1, \dots, 6\})$  and  $G_i \subset K_{n,7}$  be the subgraph induced by  $E_i$  for each  $i \in \mathbb{Z}_n$ . Note that  $G_i \cong \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  for all  $i \in \mathbb{Z}_n$ .

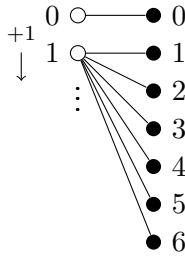


Figure 5.2:  $G_0$  in a generating presentation of the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{n,7}$ .

Notice that  $E_i \cap E_j = \emptyset$  if  $i \neq j$ , so by definition all distinct  $G_i$ 's are pairwise edge disjoint. Lastly,

$$\bigcup_{i \in \mathbb{Z}_n} E_i = [\bigcup_{i \in \mathbb{Z}_n} \{(i, 0)\}] \sqcup [\bigcup_{i \in \mathbb{Z}_n} (\{i+1\} \times \{1, \dots, 6\})] = [\mathbb{Z}_n \times \{0\}] \sqcup [\mathbb{Z}_n \times \{1, \dots, 6\}] = \mathbb{Z}_n \times \mathbb{Z}_7$$

So  $G_0 \cup \dots \cup G_{n-1} = K_{n,7}$  and  $\{G_i \mid i \in \mathbb{Z}_n\}$  is a  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{n,7}$ . Furthermore, it is generated by developing the white vertices of  $G_0$  by 1.  $\square$

**Corollary 5.0.3.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$ .

*Proof.*  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{7,7}$  and  $K_{8,7}$  by Theorem 5.0.2.  $K_{14,14}$  can be expressed as the edge-disjoint union of four copies of  $K_{7,7}$ ,  $K_{21,14}$  can be expressed as the edge-disjoint union of six copies of  $K_{7,7}$ , and  $K_{22,14}$  can be expressed as the edge-disjoint union of two copies of  $K_{8,7}$  and four copies of  $K_{7,7}$ . Therefore,  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes them all.  $\square$

Recall that we proved  $K_{14}$  and  $K_{15}$  are  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposable in Chapter 3. So we are now ready to put everything together to state the main theorem of this Chapter, completing the main result of this thesis.

**Theorem 5.0.4.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{14t+7}$  and  $K_{14t+8}$  where  $t$  is a positive integer.

*Proof.*  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{14}$  by Theorem 3.2.7,  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$  by Corollary 5.0.3, and lastly  $K_{22}, K_{21}$  by Theorem 5.0.1.

Therefore,  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes the join of  $(t - 1)$  copies of  $K_{14}$  with each other and 1 copy of  $K_{21}$ , which is isomorphic to  $K_{14t+7}$ . Similarly  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes the join of  $(t - 1)$  copies of  $K_{14}$  with each other and 1 copy of  $K_{22}$  which is isomorphic to  $K_{14t+8}$ .

□

We now present some additional results.

## Chapter 6

# Additional Results

We present two main additional results produced by work done on this project. (1) Wraparound edge mappings that preserve lengths and (2) Galaxy graph decompositions of bipartite graphs.

### 6.1 Wraparound edge mappings that preserve lengths

These mappings only apply to vertices in  $K_{2mt+m}$  and  $K_{2mt+(m+1)}$  where  $m > 1$  is odd. Also, these are meant to be used in labelings which will be developed by  $m$ , such as our (1-2-3)-labelings and 1-rotational (1-2-3)-labelings, otherwise you could just map them anywhere really. Anyways, this is a somewhat invasive way to deal with wraparound edges, and it should be noted that this is a barebones sort of framework to begin dealing with wraparound edges in labelings. Note that ‘new’ short edge mappings will be fixed after the first step from the base graphs. However the ‘new’ wraparound edges will change at every step.

If  $uv$  is a wraparound edge of length  $l$  in  $K_{3m}$ , it’s length eventually won’t be preserved at some step  $t \mapsto t + c$  for  $c \geq 1$  in the infinite family of graphs of the form  $K_{2mt+m}$ . So the idea is, maybe we can map the wraparound edge to a (1) a short edge of length  $l$  at each step  $t \mapsto t + 1$  or (2) a wraparound edge of length  $l$  at each step  $t \mapsto t + 1$  so that lengths will be preserved. The next theorem is something alluded to as an observation to previously, but we prove it anyways.

**Theorem 6.1.1.** *In  $K_n$  whose vertices we take to be  $\mathbb{Z}_n$ ,  $uv$  is a wraparound edge if and only if the absolute difference of its endpoints is greater than the maximal length in  $K_n$ ;  $\lfloor \frac{n}{2} \rfloor < |u - v|$ .*

*Proof.* Let  $uv$  be an edge in  $K_n$  via  $\ell$  defined previously. If  $uv$  is a wraparound edge, then  $n - |u - v| < |u - v|$ . So then  $\frac{n}{2} - \frac{|u-v|}{2} < \frac{|u-v|}{2}$ , and therefore  $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < |u - v|$ . If  $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < |u - v|$ , note that without loss of generality  $u < v$ . Then necessarily  $n - |u - v| = n - (v - u) < v - u = |u - v|$ , so then  $n < 2(v - u)$  and  $\lfloor \frac{n}{2} \rfloor \leq \frac{n}{2} < (v - u) < |u - v|$ .

Thus,

$$\lfloor \frac{n}{2} \rfloor < |u - v| \iff uv \text{ is a wraparound edge.}$$

□

There is likely a simpler proof for the following theorem. However, time was spent on more important tasks and this is what we have to offer in this paper. Also we use the notation  $uv$  and  $(u, v)$  for edges interchangeably here to clean things up a bit.

**Theorem 6.1.2.** *Let us define the following edge length functions*

$$\ell_n(u, v) = \begin{cases} \min\{|u - v|, n - |u - v|\}, & u, v \in \mathbb{Z}_n, \\ \infty, & u = \infty \text{ or } v = \infty, \end{cases}$$

$$\ell_n^+(u, v) = \begin{cases} u + v \bmod n, & u, v \in \mathbb{N}, \\ u \bmod n, & u = \infty, \\ v \bmod n, & v = \infty. \end{cases}$$

Now, let  $t > 1$ , and let  $m > 1$  be odd. Lastly, let  $h = 2m(t - 1)$ . For any wraparound edge  $(a, b)$  in  $K_{21}$  such that  $a < b$ , we have

$$\ell_{3m}(a, b) = \ell_{2mt+m}(a - h, b - h) = \ell_{2mt+m}(a + h, b + h) \quad (\text{short})$$

$$\ell_{3m}(a, b) = \ell_m^+(a - h, b - h) = \ell_m^+(a + h, b + h) \quad (\text{wraparound})$$

That is, these mappings preserve the standard length  $\ell$  and additive length  $\ell_m^+$  modulo  $m$  of  $ab \in E(K_{3m})$  in  $K_{2mt+m}$  and in  $K_{2mt+(m+1)}$  where we take  $K_{2mt+(m+1)}$  to be  $\mathbb{Z}_{2mt+m} \cup \{\infty\}$ .

*Proof.* Since  $uv$  is a wraparound edge where  $u < v$ ,  $\ell_{3m}(u, v) = 3m - |u - v| = 3m - (v - u) = 3m + u - v$ . Let us simply denote this via  $\ell_{ab} = 3m + a - b$ . Now, let  $k = \lfloor \frac{m}{2} \rfloor = \frac{m-1}{2}$  so the maximal length in  $K_{3m}$  is  $\lfloor \frac{3m}{2} \rfloor = \lfloor \frac{2m+m}{2} \rfloor = \lfloor \frac{2m}{2} + \frac{m}{2} \rfloor = m + \lfloor \frac{m}{2} \rfloor = m + k$  since  $k, m \geq 1$ . Suppose  $a \geq 2m$ . Then,  $(2m \geq a < b < 3m) \implies (1 \leq b - a < m < m + k)$ . But then  $|a - b| < m + k$ , the maximal length and so  $ab$  is not a wraparound edge, a contradiction. So  $a < b < 2m$ .

(short): Let  $\alpha = a - h, \beta = b \in \mathbb{Z}_{2mt+m}$ . Note:  $2mt + m - h \equiv 2mt + m - 2m(t - 1) = 2m + m = 3m$ . Therefore,  $3m + a \equiv (2mt + m - h) + a \equiv a - h \pmod{2mt + m}$ . So then since  $1 < t$ , we have that  $3m + a < 3m + 2m = 2m(2) + m \leq 2mt + m$ , and so in fact  $\alpha = 3m + a$ . Recall that  $\beta = b < 3m < 3m + a = \alpha$ . So then we have that  $|\alpha - \beta| = \alpha - \beta = (3m + a) - b$ . Well,  $3m < 2mt + m$  for  $t > 1$ . So then  $\ell_{2mt+m}(\alpha\beta) = \min\{|\alpha - \beta|, 2mt + m - |\alpha - \beta|\} = \min\{3m + a - b, 2mt + m + a - b\} = 3m + a - b = \ell_{ab}$ .

(wraparound): Instead, let  $\alpha = a, \beta = b + h \in \mathbb{Z}_{2mt+m}$ . Clearly,  $\alpha = a < b + h = \beta$ . So  $|\alpha - \beta| = \beta - \alpha = b + h - a$ . Recall that  $ab$  is a wraparound edge in  $K_{3m}$  with  $a < b$ . So then  $|a - b| = b - a > m + k$ , the maximal length in  $K_{3m}$ . So then  $|\alpha - \beta| = b - a + h > m + k + h = m + k + 2m(t - 1)$ . Now, the maximal length in  $K_{2mt+m}$  is  $\lfloor \frac{2mt+m}{2} \rfloor = mt + k = m + k + m(t - 1)$ . Well, (i)  $(2m(t^* - 1))_{t^* \geq 2}$  and (ii)  $(m(t^* - 1))_{t^* \geq 2}$  are both arithmetic sequences with increments  $2m$  and  $m$ , respectively. Both expressions are only equal at  $t^* = 1$ , and then afterwards (i) increases faster than (ii). So then we see that  $2m(t - 1) > m(t - 1)$  for all  $t > 1$ , and thus,  $|\alpha - \beta| = b - a + h > m + k + h = m + k + 2m(t - 1) > m + k + m(t - 1)$ , the maximal length in  $K_{2mt+m}$  for  $t > 1$ . So then  $\alpha\beta$  is a wraparound edge in  $K_{2mt+m}$  of length  $2mt - m - |\alpha - \beta| = 2mt + m - \beta - \alpha = 2mt + m - (b - a + h) = 2mt + m - 2m(t - 1) + a - b = 3m + a - b = \ell_{ab}$ .

□

Note that if we develop the mapped endpoints in (Short) and (Wraparound) by  $c \in \mathbb{N}$ , edge lengths  $\ell_{2mt+m}, \ell_m^+$  are still preserved. Now, we don't provide any kind of labeling to extend this theorem, but we do provide some 'guardrails' that one can implement in their labeling to avoid certain problems. Besides changing lengths of edges incident to vertices mapped this way in higher order family members, the only thing

to worry about is the possibility that an endpoint mapped in a higher order complete graph actually collides with another vertex incident to a short edge (since most labels will presumably not be mapped). This can happen. However the next theorem gives way to a corollary that guarantees this won't happen given certain conditions as met.

**Theorem 6.1.3.** *Let  $m > 1$  be odd,  $t > 1$ , and  $h = 2m(t-1)$ . Now take the vertices of  $K_{3m}$  and  $K_{2mt+m}$  to be  $\mathbb{Z}_{3m}$  and  $\mathbb{Z}_{2mt+m}$ , respectively, and let  $a, b$  be distinct vertices in  $K_{3m}$  with  $a < b$ . Then,*

$$b - a \neq m \text{ or } b \not\equiv a \pmod{m} \Rightarrow a - h \neq b + h \text{ in } \mathbb{Z}_{2mt+m}$$

*Proof.* Recall that since  $a, b \in \mathbb{Z}_{2mt+m}$  and are distinct,  $1 \leq b - a < 3m$ .

If  $b - a \neq m$ , suppose  $\alpha = \beta = a - h = b + h$ . Then  $b - a \equiv -2h \equiv -4m(t-1) \pmod{2mt+m}$ . Well,  $-4m \equiv 2mt + m - 4m \equiv 2mt - 3m \equiv m(2t - 3) \pmod{2mt + m}$ . So we have that

$$b - a \equiv [m(2t - 3)][(t - 1)] \pmod{2mt + m}.$$

If  $t = 2$ ,  $b - a \equiv m(2(2 - 3)(2 - 1)) \equiv m \pmod{2m(2) + m}$  and so for  $k \geq 1$ , we have  $b - a = m$  or  $b - a = k(m + 2m(2)) + m = 5mk > 3m$ , both contradictions. So  $\alpha \neq \beta$ .

If  $t > 2$ ,  $b - a \equiv [m(2t - 3)][(t - 1)] \equiv m(2t^2 - 5t + 3) \pmod{2mt + m}$ . So then  $b - a = m(2t^2 - 5t + 3)$  or  $b - a = m(2t^2 - 5t + 3) + k(2mt + m)$  for  $t \geq 3$  and  $k \geq 1$ . Well, since  $t \geq 3$ , we have  $m(2t^2 - 5t + 3) \geq m(2(3)^2 - 5(3) + 3) = 6m$  and  $2mt + m \geq 2m(3) + m = 7m$  since both are strictly increasing for  $t > 2$ . So then  $b - a \geq 6m$  or  $b - a \geq 6m + 7mk$  for  $t > 2$ , both contradictions since  $1 < b - a < 3m$ .

Finally, on the other hand if  $b \equiv a \pmod{m}$ , suppose  $\alpha = \beta$ . Then  $a - h \equiv b - h \pmod{2mt + m}$  and  $b - a \equiv -2h \equiv -4m(t-1) \pmod{2mt + m}$ . So then  $b - a \equiv 0 \pmod{m}$  since  $m \mid -4m(t-1)$  and  $m \mid 2mt+m$ . But then  $b \equiv a \pmod{m}$ , a contradiction. So  $\alpha \neq \beta$ , and the statement is proven.

□

**Corollary 6.1.4.** *Let  $m > 1$  be odd,  $t > 1$  and  $h = 2m(t - 1)$  and take the vertices of  $K_{3m}$  and  $K_{2mt+m}$  to be  $\mathbb{Z}_{3m}$  and  $\mathbb{Z}_{2mt+m}$ , respectively. Next, let  $a, b$  be distinct vertices in  $K_{3m}$  such that  $a < b$  and  $a \equiv b \pmod{m}$ . Then,*

$$t = 2 \text{ and } |a - b| \neq 2m \iff a \pm h \neq b \text{ and } b \pm h \neq a \text{ in } \mathbb{Z}_{2mt+m} \quad (6.1)$$

$$t > 2 \implies a \pm h \neq b \text{ and } b \pm h \neq a \text{ in } \mathbb{Z}_{2mt+m} \quad (6.2)$$

*Proof.* Let  $k = \lfloor \frac{m}{2} \rfloor$ . In the proof of Theorem 6.1.2 it is shown that  $a - h = 3m + a$ , so then since  $0 \leq a < 3m$ ,  $3m \leq 3m + a = a - h$  in  $K_{2mt+m}$ . Now, suppose  $b \leq m$ . Then since  $a < b \leq m$  and  $|a - b| = b - a$ , necessarily  $1 \leq |a - b| \leq m < m + k$ , the maximal length in  $K_{3m}$ . But then by Theorem 6.1.1,  $ab$  is not a wraparound edge, a contradiction. So  $b > m$ . Therefore  $b + h > m + h \geq 3m$ .

We prove (6.1) by contrapositive. For  $t = 2$ ,  $h = 2m$  and if  $a + h = b$  or  $a - h = b$ , then  $|a - b| = h = 2m$ . On the other hand if  $|a - b| = h$ , then  $a - b = h$  or  $a - b = -h$  and then  $a + h = b$  or  $a - h = b$ . Thus, the contrapositive holds and (6.1) is proven.

If  $t > 2$ , then recall that  $2mt + m = 3m + 2m(t - 1) = 3m + h$ . So then since  $0 \leq u < v < 3m$ , we have that  $h \leq u + h < v + h < 3m + h = 2mt + m$ . Now,  $h = 2m(t - 1) \geq 2m(3 - 1) = 4m$  for  $t > 2$ . So,

$$0 \leq u < v < 3m < 4m \leq u + h < v + h < 3m + h = 2mt + m \text{ for } t > 2.$$

Therefore,  $u + h \neq v$  and  $v + h \neq u$ . Now, since  $2mt + m - h = 3m$ ,  $2mt + m + (v - h) \equiv 3m + v \pmod{2mt + m}$  and similarly  $2mt + m + (u - h) \equiv 3m + u \pmod{2mt + m}$ , we have that  $u - h$  and  $v - h$  are simply  $3m + u$  and  $3m + v$ , respectively, in  $\mathbb{Z}_{2mt+m}$ . Well,

$$0 \leq u < v < 3m < 4m \leq 3m + u < 3m + v < 5m \leq 2mt + m \text{ for } t > 2.$$

Therefore,  $u - h \neq v$  and  $v - h \neq u$  in  $\mathbb{Z}_{2mt+m}$ , and (6.2) is proven. □

This concludes the results of this section. The next section is a fun result that came as a result of dealing with the exceptional graph in Chapter 5.



## 6.2 Galaxy Decompositions of Complete Bipartite Graphs

In Chapter 5, we address  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  in the context of our forest decompositions but it is also a galaxy. After finding the labeling in 5.2, it became clear we could generalize this idea to all galaxies. We begin with galaxy decompositions of complete bipartites.

**Theorem 6.2.1.** *Let  $N$  and  $m$  be positive integers and consider the complete bipartite  $K_{N,m}$  with a finite collection  $C = \{G_i \mid i \in \mathbb{Z}_n\}$  of vertex disjoint stars. If  $0 < n \leq N$  and  $m = \sum_{i \in \mathbb{Z}_n} |E(G_i)|$ , then there exists a  $C$ -galaxy decomposition of  $K_{N,m}$ .*

*Proof.* Take the partite set of  $N$  vertices to be  $\mathbb{Z}_N$  and color them white. Similarly, take the partite set of  $m$  vertices to be  $\mathbb{Z}_m$  and color them black. Note that we will be using vertices as group elements. Naturally, we refer to *white-black* vertices  $uv$  in  $K_{N,m}$  via  $(u, v) \in \mathbb{Z}_N \times \mathbb{Z}_m$  and vice versa. Next, let  $\mathcal{G} = \sqcup_{i \in \mathbb{Z}_n} G_i$ , a  $C$ -galaxy. Lastly, let  $L_0 = \{0, \dots, |E(G_0)|\}$ ,  $L_i = \{|E(G_{i-1})|, \dots, |E(G_i)| - 1\}$  for  $i \in \mathbb{Z}_n^*$  and  $\mathcal{G}_j \subseteq K_{N,m}$  be the subgraph induced by  $E_j = \sqcup_{i \in \mathbb{Z}_n} (\{i + j\} \times L_i)$  for each  $j \in \mathbb{Z}_N$  and note that these subgraphs are all isomorphic to  $\mathcal{G}$ .

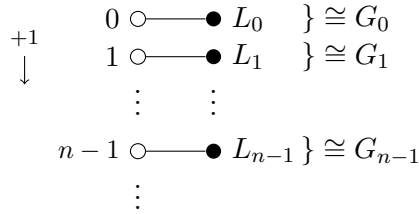


Figure 6.1:  $\mathcal{G}_0$  in a generating presentation of the  $C$ -galaxy decomposition of  $K_{N,m}$ .

Consider any edge  $(u, v) \in K_{N,m}$ . Well,  $\mathbb{Z}_m = L_0 \sqcup \dots \sqcup L_{n-1}$ , so  $v \in L_a$  for exactly one  $a \in \mathbb{Z}_n$ . So then  $(u, v) \in \{u\} \times L_a = \{a + b\} \times L_i \subseteq \sqcup_{i \in \mathbb{Z}_n} (\{i + b\} \times L_a) = E_b$  where  $b = u - a \in \mathbb{Z}_N$ . There is only one such  $b \in \mathbb{Z}_N$ . Therefore, any edge  $(u, v) \in K_{N,m}$  belongs to exactly one isomorphic copy  $\mathcal{G}_b$  of  $\mathcal{G}$  in  $\{\mathcal{G}_i \mid i \in \mathbb{Z}_N\}$ .

Thus,

$\{\mathcal{G}_i \mid i \in \mathbb{Z}_N\}$  is a  $C$ -galaxy decomposition of  $K_{N,m}$ .

□

Next, we extend this idea to complete multipartite graphs.

**Corollary 6.2.2.** *Let  $m$  be a positive integer and  $N_0, \dots, N_{k-1}$  be positive integers which are divisible by  $m$ . If  $C = \{G_i \mid i \in \mathbb{Z}_n\}$  is a collection of vertex disjoint stars where  $0 < n \leq \min\{N_0, \dots, N_{k-1}\}$  and  $m = \sum_{i \in \mathbb{Z}_n} |E(G_i)|$ , then there exists a  $C$ -galaxy decomposition of the complete  $k$ -partite graph  $K_{N_0, \dots, N_{k-1}}$ .*

*Proof.* Let  $d_i = \frac{n_i}{m}$  for each  $i \in \mathbb{Z}_k$ .  $K_{N_1, \dots, N_k}$  can be viewed as the union of  $\sum_{0 \leq i < j < k} d_i d_j$  copies of  $K_{m, m}$ . Well,  $m = \sum_{i \in \mathbb{Z}_n} |E(G_i)|$  so the maximal number of stars  $n$  in  $C$  is  $m$ , so  $n \leq m$  and by Theorem 6.2.1 the  $C$ -galaxy decomposes  $K_{m, m}$ . Therefore the  $C$ -galaxy decomposes any number of copies of  $K_{m, m}$  and so it decomposes  $K_{N_1, \dots, N_k}$ .  $\square$

This corollary concludes this chapter and the results of this thesis project. We hope to extend this galaxy idea in future work. The next section contains `python` programs made for this project.

## Chapter 7

# Programming

This project required over 450 labelings of various forests, all of which were found by hand. At around 200 labelings it became clear that due to the sheer number of labelings being done, just probabilistically some labelings would (1) have typos (2) have incorrect computations and (3) violate some constraint of the labeling.

All this programming began because we wanted some sort of local program that could display the labelings, so we could check with some level of certainty that they were isomorphic to the forest being worked with. Everything we found that displayed graphs didn't allow for dragging vertices and/or interacting with graphs at a high level. So we decided to make our own programs.

There are three groups of programs in a dedicated github repository that we provide links to, for the sake of space. Note: the dependencies required to run these are:

- (1) A `python 3.13` installation
- (2) The `NetworkX` library for `python`
- (3) the `pygame` or `pygame-ce` library for `python`
- (4) the `itertools` library for `python`
- (5) the `z3` library for `python`

It should be clear when looking at the code what the dependencies are. If you use `pip` in `vscode`, you may simply input the following into your workspace terminal:

```

py -m pip install #enter library name here
# OR
python -m pip install #enter library name here

```

depending on how you install and work with `python`, installing packages may not work this way. `Anaconda` is a popular `python` bundle that likely comes with some of these.

## 7.1 Tikzgrapher

The file shared here is an earlier version of a program named `tikzgrapher` that was specifically built to display (1-2-3)-labelings and 1-rotational (1-2-3)-labelings. There is a less restrictive version where arguments are optional and customizable (custom edge and vertex labelings, colorings of vertices, no side tab for squares containing lengths). Here is a link to the newer version of `tikzgrapher`:

<https://github.com/tucxy/Programming/tree/main/Python/tikzgrapher>

These are the features ordered earliest to latest in this older build of `tikzgrapher`:

- (1) Displays a list of `NetworkX` graphs together on one `pygame` window, starting from top to bottom.
- (2) Reduces vertices modulo  $n$  and computes the standard edge length for each edge modulo  $n$ , and has the subscript as the additive edge length  $\ell_7^+$ .
- (3) For each graph in the list given as input: uses a longest path search algorithm and by default displays the longest path of a graph in the center row of a grid of coordinates, then displays vertices coming off of that row.
- (4) Has a tab on the left that displays all standard edge lengths  $\ell$  and a chart for the subscript labels of the labelings in order. The window with the tab open looks similar to Figure 4.3 except it doesn't have colored edges.
- (5) Allows user to save displayed graphs as a `Tikz` graph in a standalone `LATEX` file to a specified path.

We have every single forest labeling in files that import this version `tikzgrapher`. If you simply uncomment below a forest, a `pygame` window will pop out and display the labeling. Here are links to those files.

(1)  $\sigma^{+-}$ -labelings:

<https://github.com/tucxy/Thesis-Programs/blob/main/sigma.py>

(2) (1-2-3)-labelings:

<https://github.com/tucxy/Thesis-Programs/blob/main/7mod14.py>

(3) 1-rotational (1-2-3)-labelings:

<https://github.com/tucxy/Thesis-Programs/blob/main/8mod14.py>

(4)  $T_7^{11} \sqcup T_2^1$ -decomposition of  $K_{21}$  and  $K_{22}$ :

<https://github.com/tucxy/Thesis-Programs/blob/main/starpath.py>

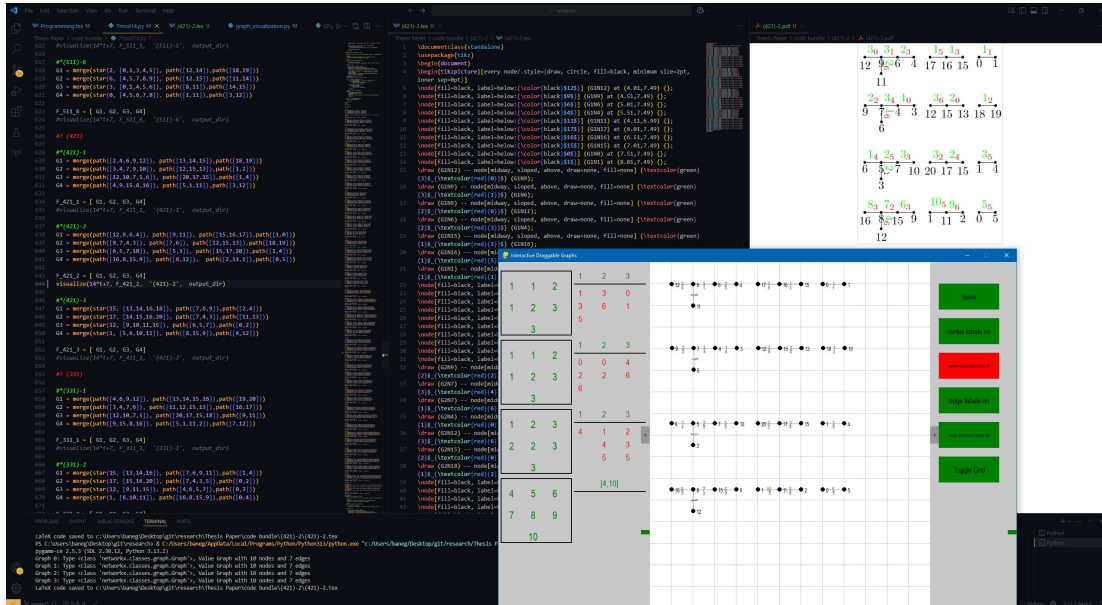


Figure 7.1: A snippet of *tikzgrapher*

Once again, this version is no longer being updated, the following link will take you to the latest version:

<https://github.com/tucxy/Programming/tree/main/Python/tikzgrapher>

## 7.2 Labeling Solvers

This next program is a bit more ambitious. After all labelings were found (of course) we thought, "Hey what if we didn't have to find these by hand?" Initially, we tried to use a genetic learning algorithm but found that the fitness function was too rigid, and realized that reinforcement learning was not the way to go. We found much more success using constraint programming. Using the **z3** SAT solver, we created (1) a solver that outputs a  $\sigma^{+-}$ -labeling of a graph (2) a graceful labeling of a graph (3) a solver that outputs a more generalized version of the (1-2-3)-labeling for a graph on  $m$  edges in  $K_{2mt+r}$  where  $r$  is an odd idempotent modulo  $2m$ .

The  $\sigma^{+-}$ -labeling is quite fast, but the other labelings can take up to five minutes or so. In the future, we hope to translate this code to C++, to hopefully speed up the process. As of now, it does work and the beefier the processor the better. Here are the links to this project:

- (1) Labeling Solvers:

<https://github.com/tucxy/Thesis-Programs/blob/main/CP.py>

- (2) Notebook to test the solvers:

<https://github.com/tucxy/Thesis-Programs/blob/main/main.py>

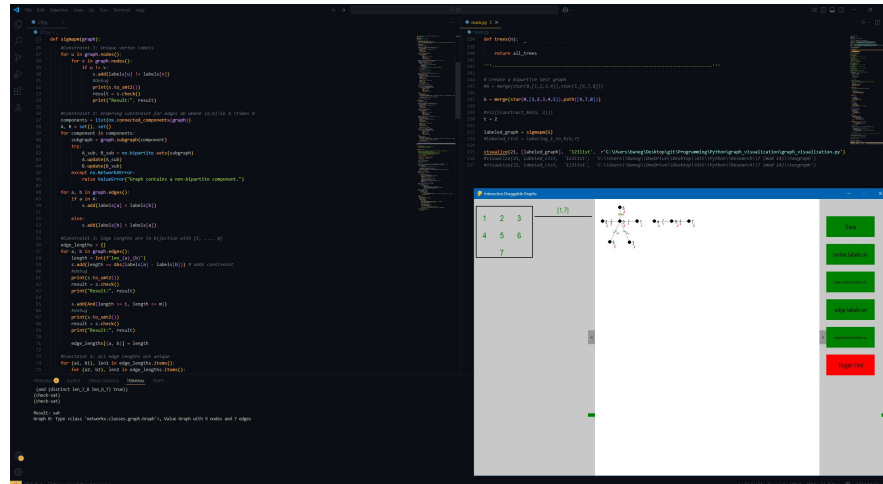


Figure 7.2: A snippet of the  $\sigma^{+-}$ -labeling solver

These are set up so that if you visit this link:  
<https://github.com/tucxy/Thesis-Programs/tree/main> and click the code button:  
the .zip file installed will give a folder when extracted. Make that folder your working  
directory, and everything should just work.

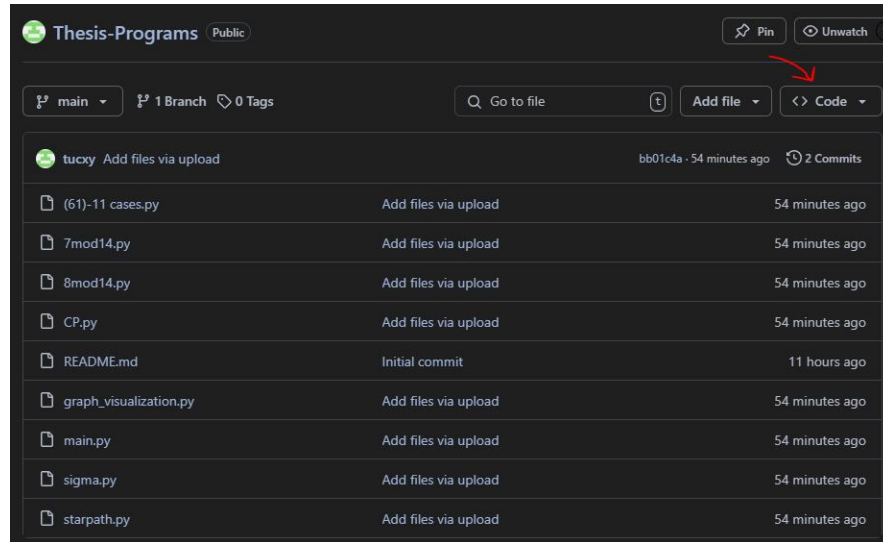


Figure 7.3: click on the code button to download the zip file. Then extract the folder and set it as your working directory.

Feel free to email me with questions: [baneg003@outlook.com](mailto:baneg003@outlook.com)

## Chapter 8

# Conclusion

Using established graph labeling techniques, namely  $\sigma^{+-}$ -labelings and  $\rho^+$ -labelings, along with our own original constructions and techniques we have proven that every seven edge forest decomposes  $K_n$  if and only if  $n \equiv 0, 1, 7$ , or  $8 \pmod{14}$  and  $n \geq 14$ . We also proved some results on wraparound edge mappings that preserve lengths across higher order complete graphs in the same families and on galaxy decompositions of complete bipartite graphs. Both of these came out of our work on seven edge forest decompositions of complete graphs. There was also a short chapter on programming which included a new graph visualization software and some labeling solvers.

Of course, a natural continuation of this work would be investigating eight edge forests designs. Additionally, the results on wraparound edge mappings and galaxy graph decompositions are very preliminary and there is a lot open in those areas as well. Specifically, developing a labeling that allows for wraparound edges, and investigating galaxy decompositions of complete graphs. With respect to programming, `tikzgrapher` will continue to be improved but perhaps more exciting and useful are the labeling solvers. Very basic constraint programming algorithms were used for the solvers shared in this paper, so creating more efficient labeling solvers is another area that could be explored and could have a big impact in future research on graph decompositions by eliminating the need to find labelings by hand.



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# Appendix A

## Labelings

### A.1 (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 1, 2, 4, 6, 9, 12) \sqcup (13, 14)$ $(3, 4, 7, 9, 10, 13, 15) \sqcup (8, 5)$ $(8, 11, 12, 10, 7, 5, 6) \sqcup (1, 3)$ $(0, 4, 9, 15, 8, 16, 7) \sqcup (1, 11)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(12, 9, 6, 4, 2, 1, 7) \sqcup (14, 15)$ $(15, 13, 10, 9, 7, 4, 11) \sqcup (8, 5)$ $(8, 11, 12, 10, 7, 5, 13) \sqcup (1, 3)$ $(16, 8, 15, 9, 4, 0, 6) \sqcup (1, 11)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(0, 1, 2, 4, 6, 9, 3) \sqcup (16, 19)$ $(15, 13, 10, 9, 7, 4, 14) \sqcup (17, 18)$ $(6, 5, 7, 10, 12, 11, 8) \sqcup (18, 15)$ $(7, 16, 8, 15, 9, 4, 12) \sqcup (1, 11)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(8, 6, 4, 2, 1, 9, 7) \sqcup (14, 15)$ $(8, 10, 9, 7, 4, 11, 13) \sqcup (12, 15)$ $(9, 12, 10, 7, 5, 11, 13) \sqcup (1, 4)$ $(7, 15, 9, 4, 0, 8, 6) \sqcup (1, 11)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12, 8, 7) \sqcup (11, 14)$ $(0, 2, 3, 6, 5, 1, 4) \sqcup (8, 7)$ $(0, 3, 5, 4, 1, 8, 7) \sqcup (16, 15)$ $(4, 9, 15, 8, 12, 6, 7) \sqcup (1, 11)$
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 6, 8, 5, 9) \sqcup (12, 15)$ $(4, 7, 9, 10, 11, 8, 13) \sqcup (1, 3)$ $(5, 7, 10, 12, 11, 6, 13) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12, 6) \sqcup (1, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(8, 6, 4, 2, 5, 9, 7) \sqcup (12, 14)$ $(1, 3, 2, 0, 5, 4, 6) \sqcup (10, 12)$ $(9, 8, 7, 10, 4, 11, 5) \sqcup (12, 13)$ $(7, 15, 9, 4, 13, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(7, 6, 4, 2, 8, 9, 5) \sqcup (12, 14)$ $(2, 3, 4, 7, 0, 5, 6) \sqcup (9, 12)$ $(7, 8, 5, 4, 9, 10, 11) \sqcup (0, 2)$ $(6, 15, 9, 4, 8, 11, 7) \sqcup (2, 12)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 8, 7, 9, 12) \sqcup (13, 14)$ $(0, 2, 3, 4, 7, 6, 5) \sqcup (8, 10)$ $(0, 3, 5, 8, 9, 4, 1) \sqcup (12, 14)$ $(4, 9, 15, 8, 12, 7, 16) \sqcup (1, 11)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12, 1, 8) \sqcup (14, 15)$ $(5, 6, 3, 2, 0, 7, 4) \sqcup (8, 9)$ $(0, 3, 5, 4, 7, 1, 8) \sqcup (12, 14)$ $(4, 9, 15, 8, 12, 18, 7) \sqcup (1, 11)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 6, 9, 12) \sqcup (13, 14, 15)$ $(3, 4, 7, 9, 10, 13) \sqcup (5, 8, 6)$ $(11, 12, 10, 7, 5, 6) \sqcup (3, 1, 4)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11, 2)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14, 15)$ $(13, 10, 9, 7, 4, 11) \sqcup (5, 8, 6)$ $(11, 12, 10, 7, 5, 13) \sqcup (3, 1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(0, 1, 2, 4, 6, 5) \sqcup (16, 13, 14)$ $(8, 6, 3, 2, 0, 4) \sqcup (14, 12, 15)$ $(7, 4, 5, 3, 0, 6) \sqcup (10, 8, 11)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(1, 2, 5, 4, 6, 7) \sqcup (16, 14, 13)$ $(8, 6, 9, 3, 2, 4) \sqcup (14, 12, 15)$ $(4, 5, 6, 3, 0, 1) \sqcup (11, 8, 7)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(0, 2, 4, 7, 1, 5) \sqcup (12, 11, 13)$ $(7, 6, 3, 2, 8, 9) \sqcup (14, 12, 15)$ $(4, 3, 5, 6, 0, 1) \sqcup (11, 8, 7)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(0, 2, 1, 3, 4, 5) \sqcup (12, 11, 14)$ $(4, 6, 8, 9, 5, 7) \sqcup (14, 12, 15)$ $(0, 3, 1, 4, 5, 6) \sqcup (11, 8, 7)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(2, 4, 6, 9, 12) \sqcup (16, 15, 14, 13)$ $(3, 4, 7, 9, 10) \sqcup (11, 12, 15, 13)$ $(12, 10, 7, 5, 6) \sqcup (18, 15, 17, 20)$ $(4, 9, 15, 8, 16) \sqcup (2, 11, 1, 5)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(12, 9, 6, 4, 11) \sqcup (17, 16, 15, 14)$ $(9, 7, 4, 3, 6) \sqcup (11, 12, 15, 13)$ $(6, 5, 7, 10, 3) \sqcup (18, 15, 17, 20)$ $(16, 8, 15, 9, 12) \sqcup (2, 11, 1, 6)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(4, 6, 9, 11, 8) \sqcup (16, 15, 18, 14)$ $(9, 7, 4, 3, 6) \sqcup (16, 17, 20, 15)$ $(6, 5, 7, 10, 3) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 6, 7)$ $(14, 17, 16, 20, 15) \sqcup (9, 7, 4, 3)$ $(9, 12, 10, 11, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(7, 6, 9, 11, 8) \sqcup (16, 15, 13, 14)$ $(9, 7, 4, 3, 5) \sqcup (16, 17, 20, 15)$ $(4, 6, 5, 7, 10) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 12, 6)$ $(18, 17, 16, 20, 15) \sqcup (9, 7, 10, 4)$ $(10, 12, 11, 14, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 4, 6, 9, 12) \sqcup (13, 14) \sqcup (8, 7)$ $(3, 4, 7, 9, 10, 13) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 6) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14) \sqcup (8, 7)$ $(13, 10, 9, 7, 4, 11) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 13) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(0, 1, 2, 4, 7, 5) \sqcup (9, 6) \sqcup (8, 10)$ $(8, 6, 3, 2, 0, 4) \sqcup (5, 7) \sqcup (12, 13)$ $(6, 4, 5, 3, 0, 8) \sqcup (13, 14) \sqcup (18, 15)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11) \sqcup (5, 14)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 5, 4, 6, 7) \sqcup (13, 14) \sqcup (12, 15)$ $(8, 6, 9, 3, 2, 4) \sqcup (12, 14) \sqcup (18, 15)$ $(4, 5, 6, 3, 0, 1) \sqcup (8, 7) \sqcup (16, 14)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(0, 2, 4, 7, 1, 5) \sqcup (11, 13) \sqcup (12, 15)$ $(7, 6, 3, 2, 8, 9) \sqcup (11, 12) \sqcup (1, 4)$ $(4, 3, 5, 6, 0, 1) \sqcup (8, 7) \sqcup (12, 14)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(0, 2, 1, 3, 4, 5) \sqcup (12, 14) \sqcup (18, 19)$ $(4, 6, 8, 9, 5, 7) \sqcup (12, 15) \sqcup (11, 14)$ $(0, 3, 1, 4, 5, 6) \sqcup (8, 11) \sqcup (14, 15)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12) \sqcup (13, 14, 15) \sqcup (18, 19)$ $(3, 4, 7, 9, 10) \sqcup (12, 15, 13) \sqcup (1, 2)$ $(12, 10, 7, 5, 6) \sqcup (20, 17, 15) \sqcup (1, 4)$ $(4, 9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(12, 9, 6, 4, 11) \sqcup (17, 16, 15) \sqcup (0, 1)$ $(9, 7, 4, 3, 6) \sqcup (12, 15, 13) \sqcup (18, 19)$ $(6, 5, 7, 10, 3) \sqcup (20, 17, 15) \sqcup (1, 4)$ $(16, 8, 15, 9, 12) \sqcup (1, 11, 2) \sqcup (0, 5)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(13, 15, 16, 18, 14) \sqcup (9, 6, 7) \sqcup (2, 4)$ $(14, 17, 16, 20, 15) \sqcup (3, 4, 7) \sqcup (11, 13)$ $(9, 12, 10, 11, 15) \sqcup (6, 5, 7) \sqcup (0, 2)$ $(5, 1, 10, 11, 6) \sqcup (8, 15, 9) \sqcup (4, 12)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(4, 6, 9, 12) \sqcup (16, 15, 14, 13) \sqcup (19, 20)$ $(9, 7, 4, 3) \sqcup (11, 12, 15, 13) \sqcup (16, 17)$ $(12, 10, 7, 5) \sqcup (18, 15, 17, 20) \sqcup (9, 11)$ $(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, 6, 7) \sqcup (16, 15, 13, 14) \sqcup (1, 4)$ $(5, 3, 4, 7) \sqcup (16, 17, 20, 15) \sqcup (0, 2)$ $(4, 6, 5, 7) \sqcup (9, 12, 11, 15) \sqcup (0, 3)$ $(16, 8, 15, 9) \sqcup (10, 1, 11, 6) \sqcup (0, 4)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(18, 15, 13, 14) \sqcup (11, 9, 12, 6) \sqcup (1, 2)$ $(18, 17, 20, 15) \sqcup (9, 7, 10, 4) \sqcup (2, 3)$ $(11, 12, 14, 15) \sqcup (4, 6, 5, 7) \sqcup (17, 19)$ $(11, 1, 5, 6) \sqcup (16, 8, 14, 15) \sqcup (0, 9)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(16, 15, 14, 13) \sqcup (0, 3, 5) \sqcup (12, 9, 6)$ $(11, 12, 15, 13) \sqcup (10, 9, 7) \sqcup (16, 18, 20)$ $(18, 15, 17, 20) \sqcup (10, 11, 14) \sqcup (6, 5, 7)$ $(2, 12, 3, 11) \sqcup (8, 1, 7) \sqcup (4, 0, 5)$
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(11, 9, 12, 6) \sqcup (18, 15, 13) \sqcup (0, 1, 2)$ $(9, 7, 10, 4) \sqcup (18, 17, 20) \sqcup (1, 3, 2)$ $(11, 12, 14, 15) \sqcup (4, 6, 7) \sqcup (17, 19, 20)$ $(16, 8, 14, 15) \sqcup (11, 1, 6) \sqcup (9, 0, 4)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(8, 6, 9, 11) \sqcup (0, 1, 2) \sqcup (16, 19) \sqcup (18, 15)$ $(8, 10, 7, 9) \sqcup (18, 17, 20) \sqcup (11, 14) \sqcup (2, 3)$ $(13, 11, 12, 14) \sqcup (17, 19, 20) \sqcup (6, 7) \sqcup (8, 5)$ $(0, 5, 1, 7) \sqcup (3, 10, 2) \sqcup (4, 13) \sqcup (16, 6)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(11, 9, 12, 6) \sqcup (0, 1, 2) \sqcup (18, 15) \sqcup (13, 14)$ $(9, 7, 10, 4) \sqcup (18, 17, 20) \sqcup (11, 13) \sqcup (2, 3)$ $(11, 12, 14, 15) \sqcup (17, 19, 20) \sqcup (8, 6) \sqcup (1, 3)$ $(4, 0, 5, 6) \sqcup (8, 1, 9) \sqcup (3, 12) \sqcup (17, 7)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(2, 4, 6, 9, 12) \sqcup (13, 14) \sqcup (18, 19) \sqcup (0, 1)$ $(3, 4, 7, 9, 10) \sqcup (13, 15) \sqcup (1, 2) \sqcup (8, 5)$ $(6, 5, 7, 10, 12) \sqcup (17, 20) \sqcup (8, 11) \sqcup (1, 3)$ $(4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12) \sqcup (2, 6)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(11, 9, 6, 4, 12) \sqcup (16, 15) \sqcup (8, 10) \sqcup (2, 3)$ $(6, 7, 4, 3, 9) \sqcup (13, 15) \sqcup (18, 19) \sqcup (8, 5)$ $(3, 5, 7, 10, 6) \sqcup (17, 20) \sqcup (8, 11) \sqcup (0, 1)$ $(12, 8, 15, 9, 16) \sqcup (2, 11) \sqcup (0, 5) \sqcup (3, 13)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(13, 15, 16, 18, 14) \sqcup (9, 6) \sqcup (2, 4) \sqcup (5, 7)$ $(14, 17, 16, 20, 15) \sqcup (4, 7) \sqcup (11, 13) \sqcup (5, 6)$ $(9, 12, 10, 11, 15) \sqcup (6, 7) \sqcup (0, 2) \sqcup (3, 4)$ $(5, 1, 10, 11, 6) \sqcup (9, 15) \sqcup (4, 12) \sqcup (0, 7)$
$3\mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(18, 15, 13) \sqcup (11, 9, 6) \sqcup (0, 1, 2) \sqcup (16, 19)$ $(18, 17, 20) \sqcup (9, 7, 10) \sqcup (1, 3, 2) \sqcup (11, 14)$ $(11, 12, 14) \sqcup (4, 6, 7) \sqcup (17, 19, 20) \sqcup (8, 5)$ $(11, 1, 6) \sqcup (16, 8, 14) \sqcup (9, 0, 4) \sqcup (10, 3)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(9, 6, 4, 2) \sqcup (13, 14) \sqcup (18, 19) \sqcup (0, 1) \sqcup (10, 12)$ $(9, 7, 4, 3) \sqcup (13, 15) \sqcup (1, 2) \sqcup (8, 5) \sqcup (16, 17)$ $(10, 7, 5, 6) \sqcup (17, 20) \sqcup (8, 11) \sqcup (1, 3) \sqcup (9, 12)$ $(9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12) \sqcup (2, 6) \sqcup (0, 5)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(16, 15, 18, 13) \sqcup (9, 6) \sqcup (2, 4) \sqcup (5, 7) \sqcup (0, 1)$ $(16, 17, 20, 14) \sqcup (4, 7) \sqcup (11, 13) \sqcup (5, 6) \sqcup (1, 3)$ $(9, 12, 10, 11) \sqcup (6, 7) \sqcup (0, 2) \sqcup (3, 4) \sqcup (8, 5)$ $(10, 1, 11, 5) \sqcup (9, 15) \sqcup (4, 12) \sqcup (0, 7) \sqcup (8, 3)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(11, 9, 6) \sqcup (0, 1, 2) \sqcup (18, 15) \sqcup (16, 19) \sqcup (17, 20)$ $(9, 7, 10) \sqcup (1, 3, 2) \sqcup (17, 18) \sqcup (11, 14) \sqcup (8, 5)$ $(11, 12, 14) \sqcup (4, 6, 7) \sqcup (19, 20) \sqcup (13, 15) \sqcup (3, 5)$ $(11, 1, 6) \sqcup (16, 8, 14) \sqcup (0, 9) \sqcup (10, 3) \sqcup (17, 13)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(0, 1, 2) \sqcup (18, 15) \sqcup (9, 11) \sqcup (16, 19) \sqcup (5, 6) \sqcup (10, 7)$ $(1, 3, 2) \sqcup (17, 18) \sqcup (9, 7) \sqcup (11, 14) \sqcup (8, 5) \sqcup (16, 13)$ $(4, 6, 7) \sqcup (12, 14) \sqcup (3, 5) \sqcup (13, 15) \sqcup (17, 20) \sqcup (18, 19)$ $(16, 8, 14) \sqcup (1, 11) \sqcup (0, 9) \sqcup (10, 3) \sqcup (17, 13) \sqcup (2, 7)$

Table A.1: (1-2-3)-labelings



## A.2 1-rotational (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 1, \infty, 2, 4, 5, 3) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4, 1) \sqcup (10, 11)$ $(5, 7, \infty, 3, 6, 9, 10) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 6, 5) \sqcup (16, 15)$ $(0, 4, 9, 15, 8, 16, 7) \sqcup (1, 11)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 8, 1) \sqcup (12, 15)$ $(4, 6, \infty, 5, 2, 0, 18) \sqcup (10, 11)$ $(10, 9, 6, 3, \infty, 0, 7) \sqcup (12, 14)$ $(5, 6, 8, 10, 7, 4, 9) \sqcup (0, 1)$ $(16, 8, 15, 9, 4, 0, 6) \sqcup (1, 11)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 1, 6) \sqcup (9, 10)$ $(0, 2, 5, \infty, 6, 4, 1) \sqcup (10, 11)$ $(5, 7, \infty, 3, 6, 9, 8) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 6, 1) \sqcup (12, 15)$ $(7, 16, 8, 15, 9, 4, 12) \sqcup (1, 11)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 5, 8, 0, \infty) \sqcup (11, 13)$ $(4, \infty, 5, 2, 3, 8, 6) \sqcup (16, 13)$ $(6, 7, \infty, 10, 13, 8, 5) \sqcup (19, 20)$ $(11, 10, 7, 4, 1, 8, 12) \sqcup (13, 15)$ $(7, 15, 9, 4, 0, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 3, 6, 0, 1) \sqcup (9, \infty)$ $(2, 5, \infty, 6, 4, 8, 11) \sqcup (16, 13)$ $(10, \infty, 7, 8, 11, 5, 6) \sqcup (12, 13)$ $(4, 7, 10, 8, 5, 11, 12) \sqcup (13, 15)$ $(4, 9, 15, 8, 12, 6, 7) \sqcup (1, 11)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(8, 5, 4, 2, 0, 6, \infty) \sqcup (11, 13)$ $(3, 2, 5, \infty, 8, 1, 6) \sqcup (16, 13)$ $(5, 7, \infty, 3, 4, 8, 6) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 1, 12) \sqcup (13, 15)$ $(0, 4, 9, 15, 8, 12, 6) \sqcup (1, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 5, 7, 0, 3) \sqcup (8, 11)$ $(11, \infty, 6, 4, 5, 8, 12) \sqcup (10, 13)$ $(6, 7, \infty, 10, 2, 8, 5) \sqcup (9, 12)$ $(11, 10, 8, 5, 6, 12, 7) \sqcup (16, 13)$ $(7, 15, 9, 4, 13, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 6, 0, 3, 5) \sqcup (8, 11)$ $(11, \infty, 6, 5, 8, 2, 12) \sqcup (13, 15)$ $(6, 7, \infty, 10, 8, 4, 5) \sqcup (11, 12)$ $(11, 10, 8, 5, 12, 13, 7) \sqcup (9, 6)$ $(6, 15, 9, 4, 8, 11, 7) \sqcup (2, 12)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 0, 1, 3, 6) \sqcup (9, \infty)$ $(4, 6, \infty, 1, 2, 12, 13) \sqcup (8, 11)$ $(10, \infty, 7, 5, 3, 6, 9) \sqcup (13, 15)$ $(5, 8, 10, 11, \infty, 7, 4) \sqcup (9, 12)$ $(4, 9, 15, 8, 12, 7, 16) \sqcup (1, 11)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 3, 6, \infty, 0) \sqcup (8, 7)$ $(13, 12, \infty, 6, 4, 10, 1) \sqcup (8, 11)$ $(10, \infty, 7, 6, 9, 2, 5) \sqcup (13, 15)$ $(5, 8, 10, 7, 4, 9, 11) \sqcup (16, 19)$ $(4, 9, 15, 8, 12, 18, 7) \sqcup (1, 11)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(3, 5, 4, 2, \infty, 1) \sqcup (13, 12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (8, 11, 10)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14, 15)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (17, 16, 15)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11, 2)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(\infty, 2, 4, 5, 8, 0) \sqcup (11, 13, 12)$ $(6, \infty, 5, 2, 3, 8) \sqcup (13, 16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14, 15)$ $(8, 10, 7, 4, \infty, 12) \sqcup (18, 15, 13)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(5, 4, 2, 3, 6, 0) \sqcup (9, \infty, 11)$ $(4, 6, \infty, 12, 13, 1) \sqcup (11, 8, 7)$ $(10, \infty, 7, 6, 9, 5) \sqcup (16, 15, 13)$ $(5, 8, 10, 7, 4, 11) \sqcup (16, 19, 17)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(5, 4, 7, 2, 1, 3) \sqcup (8, 11, \infty)$ $(12, \infty, 8, 6, 4, 5) \sqcup (13, 10, 7)$ $(10, \infty, 2, 7, 8, 5) \sqcup (19, 16, 14)$ $(11, 10, 12, 8, 5, 6) \sqcup (16, 13, 14)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 5, 0, 3) \sqcup (8, 11, 14)$ $(11, \infty, 6, 4, 8, 5) \sqcup (10, 13, 12)$ $(6, 7, \infty, 3, 8, 5) \sqcup (9, 12, 15)$ $(11, 10, 8, 6, 12, 7) \sqcup (13, 16, \infty)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(1, 2, 0, 3, 4, 5) \sqcup (11, 8, \infty)$ $(2, \infty, 3, 4, 5, 6) \sqcup (12, 13, 15)$ $(6, 7, 8, 4, 5, \infty) \sqcup (11, 12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (9, 6, 4)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(5, 4, 2, \infty, 1) \sqcup (11, 13, 12, 15)$ $(0, 2, 5, \infty, 6) \sqcup (8, 11, 10, 12)$ $(5, 7, \infty, 3, 6) \sqcup (16, 13, 14, 15)$ $(\infty, 4, 7, 10, 8) \sqcup (17, 16, 15, 13)$ $(4, 9, 15, 8, 16) \sqcup (2, 11, 1, 5)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(\infty, 2, 4, 5, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (8, 11, 10, 12)$ $(6, 3, \infty, 7, 1) \sqcup (16, 13, 14, 15)$ $(10, 7, 4, \infty, 5) \sqcup (17, 16, 15, 13)$ $(16, 8, 15, 9, 12) \sqcup (2, 11, 1, 6)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(\infty, 2, 4, 3, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (10, 12, 11, 15)$ $(6, 3, \infty, 7, 1) \sqcup (12, 14, 13, 15)$ $(\infty, 4, 7, 10, 1) \sqcup (17, 16, 13, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(0, 2, 1, 3, 4) \sqcup (11, 8, \infty, 6)$ $(2, \infty, 3, 4, 5) \sqcup (9, 12, 13, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15, 14)$ $(0, 3, 1, 5, 6) \sqcup (16, 13, 11, 10)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(10, 13, \infty, 8, 11) \sqcup (1, 2, 3, 4)$ $(15, 13, 12, 9, 7) \sqcup (3, \infty, 4, 5)$ $(11, 12, 15, 14, 13) \sqcup (4, 7, 5, \infty)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12, 7)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(0, 2, 3, 4, 5) \sqcup (9, 8, 11, \infty)$ $(2, \infty, 3, 4, 5) \sqcup (12, 13, 14, 15)$ $(4, 7, 8, 5, \infty) \sqcup (10, 12, 11, 15)$ $(0, 3, 1, 4, 6) \sqcup (16, 13, 11, \infty)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 1) \sqcup (19, 20) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (17, 18) \sqcup (8, 11)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14) \sqcup (0, 1)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (16, 15) \sqcup (2, 3)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(\infty, 2, 4, 5, 8, 0) \sqcup (18, 20) \sqcup (12, 13)$ $(13, \infty, 5, 2, 3, 8) \sqcup (9, 6) \sqcup (16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14) \sqcup (0, 1)$ $(15, 17, 14, 11, \infty, 19) \sqcup (8, 6) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(3, 2, 4, 5, 0, 1) \sqcup (18, 15) \sqcup (11, 14)$ $(5, \infty, 6, 4, 8, 11) \sqcup (10, 13) \sqcup (19, 20)$ $(8, 7, \infty, 3, 5, 6) \sqcup (16, 19) \sqcup (12, 15)$ $(7, 10, 8, 6, 11, 12) \sqcup (16, 13) \sqcup (9, \infty)$ $(6, 0, 8, 4, 5, 7) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(5, 4, 7, 2, 1, 3) \sqcup (8, 11) \sqcup (18, \infty)$ $(12, \infty, 8, 6, 4, 5) \sqcup (0, 3) \sqcup (10, 13)$ $(10, \infty, 2, 7, 8, 5) \sqcup (9, 6) \sqcup (16, 19)$ $(11, 10, 12, 8, 5, 6) \sqcup (13, 14) \sqcup (0, 2)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(5, 4, 2, 3, 6, 0) \sqcup (9, 12) \sqcup (11, \infty)$ $(4, 6, \infty, 12, 13, 15) \sqcup (0, 1) \sqcup (8, 11)$ $(10, \infty, 7, 6, 9, 5) \sqcup (13, 15) \sqcup (1, 2)$ $(5, 8, 10, 7, 4, 11) \sqcup (17, 19) \sqcup (9, \infty)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 0, 3, 4, 5) \sqcup (\infty, 15) \sqcup (8, 11)$ $(11, \infty, 2, 3, 5, 6) \sqcup (13, 15) \sqcup (19, 20)$ $(6, 7, 8, 4, 5, \infty) \sqcup (18, 19) \sqcup (12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (18, 20) \sqcup (9, 6)$ $(11, 1, 8, 9, 10, 7) \sqcup (0, 5) \sqcup (2, 6)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(10, 13, \infty, 8, 11) \sqcup (3, 2, 4) \sqcup (16, 15)$ $(15, 13, 12, 9, 7) \sqcup (10, \infty, 5) \sqcup (11, 14)$ $(11, 12, 15, 14, 13) \sqcup (4, \infty, 7) \sqcup (0, 3)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12) \sqcup (5, 7)$ $(0, 9, 1, 8, 2) \sqcup (5, 10, 6) \sqcup (3, 13)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(8, \infty, 13, 10, 9) \sqcup (3, 2, 4) \sqcup (14, 15)$ $(7, 9, 12, 13, 8) \sqcup (10, \infty, 5) \sqcup (11, 14)$ $(11, 12, 15, 18, 14) \sqcup (4, \infty, 7) \sqcup (0, 3)$ $(9, 6, 4, 3, 8) \sqcup (19, 17, 15) \sqcup (13, 14)$ $(1, 8, 0, 9, 2) \sqcup (5, 10, 6) \sqcup (3, 13)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(2, \infty, 3, 4, 5) \sqcup (12, 13, 15) \sqcup (16, 19)$ $(0, 2, 1, 3, 4) \sqcup (8, \infty, 6) \sqcup (18, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15) \sqcup (0, 1)$ $(8, 10, 12, 13, 7) \sqcup (9, 6, 4) \sqcup (17, 18)$ $(9, 0, 8, 6, 7) \sqcup (11, 1, 5) \sqcup (10, 15)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(1, \infty, 16, 18) \sqcup (11, 13, 12, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 12) \sqcup (9, 7)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14, 15) \sqcup (5, 7)$ $(10, 7, 4, \infty) \sqcup (17, 16, 15, 13) \sqcup (1, 3)$ $(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, \infty, 1) \sqcup (10, 12, 13, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 13) \sqcup (9, 7)$ $(0, \infty, 17, 20) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(10, 7, 4, \infty) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(2, 12, 6, 15) \sqcup (8, 0, 5, 7) \sqcup (9, 13)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(18, 16, 19, \infty) \sqcup (10, 12, 13, 15) \sqcup (3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 10, 13) \sqcup (4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(9, 7, 10, 4) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(9, 0, 8, 7) \sqcup (11, 1, 5, 6) \sqcup (10, 4)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$ $(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$ $(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (5, 3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (3, 4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13) \sqcup (6, 8, 7)$ $(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (2, 1, 3)$ $(9, 0, 8, 7) \sqcup (5, 1, 6) \sqcup (10, 4, 14)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (4, 5) \sqcup (16, 18)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (2, 5) \sqcup (16, 14)$ $(8, 10, 7, 4) \sqcup (0, \infty, 11) \sqcup (16, 17) \sqcup (9, 6)$ $(5, 7, 8, 6) \sqcup (20, 17, \infty) \sqcup (13, 14) \sqcup (1, 2)$ $(3, 10, 5, 11) \sqcup (0, 9, 1) \sqcup (2, 12) \sqcup (17, 13)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (3, 5) \sqcup (17, 20)$ $(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (4, 5) \sqcup (17, 18)$ $(3, \infty, 4, 7) \sqcup (12, 14, 13) \sqcup (8, 6) \sqcup (1, 2)$ $(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (1, 3) \sqcup (14, 15)$ $(9, 0, 8, 7) \sqcup (11, 1, 6) \sqcup (18, 12) \sqcup (10, 14)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(4, 1, \infty, 13, 10) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(5, \infty, 10, 11, 13) \sqcup (4, 7) \sqcup (0, 2) \sqcup (9, 12)$ $(7, \infty, 4, 5, 8) \sqcup (17, 19) \sqcup (0, 3) \sqcup (12, 14)$ $(7, 8, 6, 9, \infty) \sqcup (13, 14) \sqcup (1, 3) \sqcup (19, 20)$ $(1, 11, 2, 10, 3) \sqcup (0, 6) \sqcup (9, 4) \sqcup (8, 12)$
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(1, \infty, 13, 10, 7) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(5, \infty, 10, 11, 16) \sqcup (4, 7) \sqcup (0, 2) \sqcup (9, 12)$ $(6, 4, 5, 8, \infty) \sqcup (17, 19) \sqcup (0, 3) \sqcup (12, 14)$ $(7, 8, 6, 9, 11) \sqcup (13, 14) \sqcup (1, 3) \sqcup (19, 20)$ $(3, 10, 2, 11, 5) \sqcup (0, 6) \sqcup (4, 8) \sqcup (17, 7)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(1, \infty, 13, 5, 7) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(0, 3, 1, 4, \infty) \sqcup (2, 5) \sqcup (9, 7) \sqcup (10, 13)$ $(12, 11, 13, 14, \infty) \sqcup (17, 19) \sqcup (5, 7) \sqcup (9, 6)$ $(5, 8, 11, 6, 7) \sqcup (13, 14) \sqcup (2, \infty) \sqcup (19, 20)$ $(6, 0, 8, 9, 7) \sqcup (1, 11) \sqcup (10, 5) \sqcup (16, 12)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$ $(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$ $(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(9, \infty, 8, 6) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13, 14) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 4, 3) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(18, 20, 17, \infty) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(0, 9, 1, 11) \sqcup (10, 3) \sqcup (12, 6) \sqcup (19, 14) \sqcup (17, 13)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(8, \infty, 9, 5) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (3, 4)$ $(15, 13, 14, \infty) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 3, 4) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(17, 20, 18, 19) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(9, 0, 8, 7) \sqcup (1, 11) \sqcup (12, 6) \sqcup (10, 5) \sqcup (16, 20)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(8, \infty, 9) \sqcup (13, 12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2)$ $(19, \infty, 6) \sqcup (11, 10, 12) \sqcup (2, 5) \sqcup (18, 20) \sqcup (1, 4)$ $(11, \infty, 14) \sqcup (10, 7, 4) \sqcup (16, 17) \sqcup (0, 2) \sqcup (1, 3)$ $(20, 17, \infty) \sqcup (14, 13, 15) \sqcup (5, 7) \sqcup (9, 6) \sqcup (0, 1)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(8, \infty, 9) \sqcup (12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13) \sqcup (9, 6) \sqcup (0, 2) \sqcup (18, 20) \sqcup (1, 4) \sqcup (17, 19)$ $(11, \infty, 14) \sqcup (4, 7) \sqcup (16, 17) \sqcup (2, 5) \sqcup (8, 10) \sqcup (0, 3)$ $(20, 17, \infty) \sqcup (13, 14) \sqcup (5, 7) \sqcup (10, 11) \sqcup (0, 1) \sqcup (8, 6)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$

Table A.2: (1-2-3)-labelings



### A.3 $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decompositions of $K_{21}$ and $K_{22}$

No.	Block	No.	Block
1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (18, 7)$	8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	10	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
11	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	12	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
13	$(1, 7, 2, 3, 4, 5, 6) \sqcup (0, 14)$	14	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
15	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	16	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$
17	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	18	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
19	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	20	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (1, 3)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$

Table A.3: A  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{21}$ 

No.	Block	No.	Block
1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (6, \infty)$	8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	10	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
11	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	12	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
13	$(1, 7, 2, 3, 4, 5, 6) \sqcup (13, \infty)$	14	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
15	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	16	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$

Table A.4: A  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{22}$

No.	Block	No.	Block
17	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	18	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
19	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	20	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (20, \infty)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$
31	$(0, \infty, 1, 2, 3, 4, 5) \sqcup (18, 7)$	32	$(14, \infty, 15, 16, 17, 18, 19) \sqcup (1, 3)$
33	$(7, \infty, 8, 9, 10, 11, 12) \sqcup (0, 14)$		

Table A.4: A  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{22}$