Seven Edge Forest Designs

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Dedication

I dedicate this Thesis to my advisor Professor Bryan Freyberg, to my family who has supported me throughout this process, and to Jordi, Ian, TK, and Torta from Tuscarora Ave.

Abstract

Let G be a subgraph of K_n where $n \in \mathbb{N}$. A G-decomposition of K_n, or G-design of order n, is a finite collection $\mathcal{G} = \{G_1, \ldots, G_k\}$ of pairwise edge-disjoint subgraphs of K_n that are all isomorphic to some graph G. We prove that an F-decomposition of K_n exists for every seven-edge forest F if and only if $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$.

Along the way, we introduce new methods, constraint programming algorithms in Python, and some bonus results for Galaxy graph decompositions of complete bipartite, and eventually multipartite graphs.

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Chapter 1

Introduction

A *G-decomposition* of a graph K is a set of mutually edge-disjoint subgraphs of K which are isomorphic to a given graph G. If such a set exists we say that K *allows* a G-decomposition, and if $K \cong K_n$ we sometimes call the decomposition a *G-design of order n*.

G-decompositions are a longstanding topic in combinatorics, graph theory, and design theory, with roots tracing back to at least the 19th century. The work of Rosa and Kotzig in the 1960s on what are now known as graph labelings laid the foundation for the modern treatment of such problems. Using adaptations of these labelings alongside techniques from design theory, numerous papers have since been published on G-decompositions. This work is a natural continuation of Freyberg and Peters' recent paper on decomposing complete graphs into forests with six edges [**?**]. Their paper also includes a summary of G-decompositions for graphs G with less than 7 edges.

Every connected component of a forest with 7 edges is a tree with 6 or less edges. All such trees are cataloged in Figure **??**. We use the naming convention $\mathbf{T^i_j}$ to denote the i^{th} tree with j vertices. For each tree $\mathbf{T^i_j}$, the names of the vertices, v_t for $1 \le t \le j$, will be referred to in the decompositions described in Section **??**.

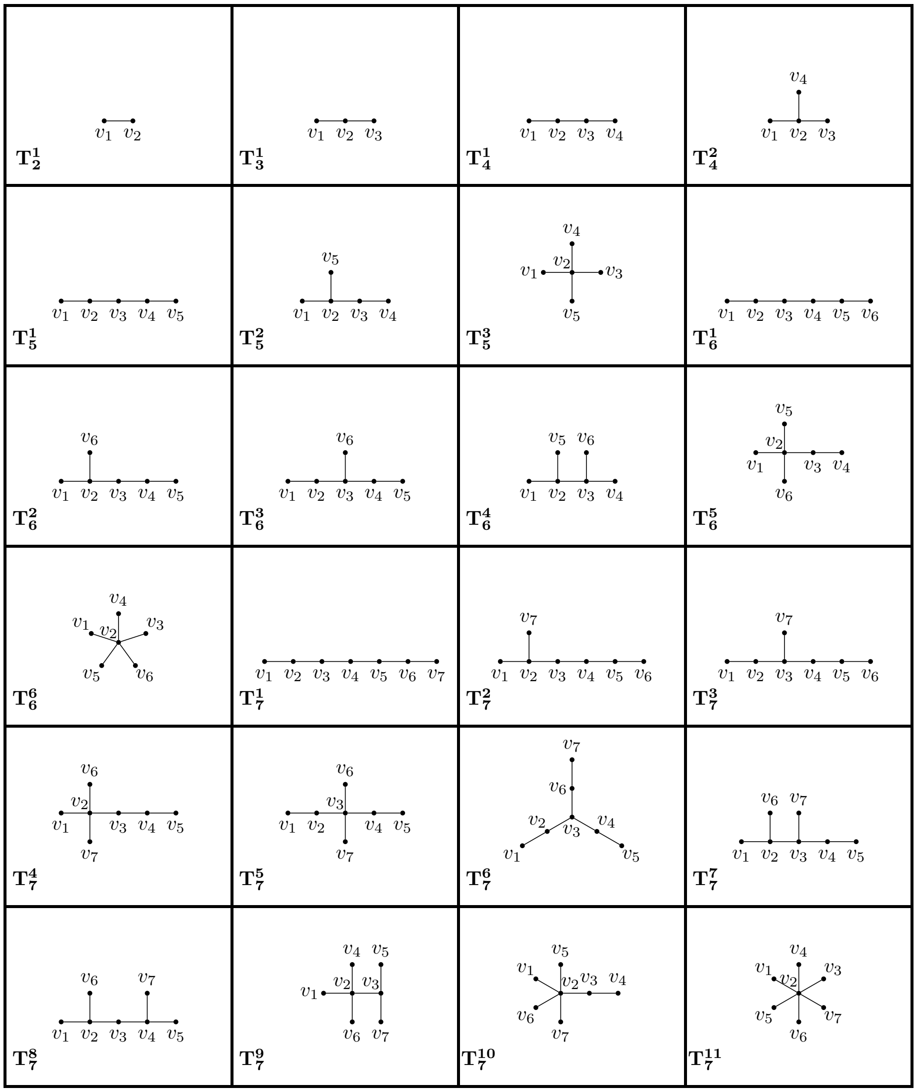

Figure 1.1: trees with less than seven edges

The next theorem gives the necessary conditions for the existence of a G-decomposition of K_n when G is a graph with 7 edges.

Theorem 1.0.1. *If G is a graph with 7 edges and a G-decomposition of K_n exists, then $n \equiv 0, 1, 7,$ or $8 \pmod{14}$.*

Proof. If a G-decomposition exists, then $7 | \binom{n}{2}$ which immediately implies $n \equiv 0, 1, 7,$ or 8 (mod 14).

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$\square$

In this article, we only consider simple graphs without isolated vertices. There are 47 non-isomorphic forests with 7 edges. Section **??** treats all 47 forests when $n \equiv 0$ or $1 \pmod{14}$. Section **??** applies to all the forests when $n \equiv 7$ or $8 \pmod{14}$ with the lone exception of $F \cong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$, which is solved for those values of n in Section **??**.

Chapter 2

$n \equiv 0, 1 \pmod{14}$

In this section, we use established graph labeling techniques to construct the G-decompositions of K_n when $n \equiv 0$ or $1 \pmod{14}$.

Definition 2.0.1 ((Rosa [**?**])). Let G be a graph with m edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, 2, \ldots, 2m\}$ that induces a bijective *length function* $\ell : E(G) \to \{1, 2, \ldots, m\}$ where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all $uv \in E(G)$.

Rosa showed that a ρ-labeling of a graph G with m edges and a cyclic G-decomposition of K_{2m+1} are equivalent, which the next thm shows. Later, Rosa, his students, and colleagues began considering more restrictive types of ρ-labeling to address decomposing complete graphs of more orders. Definitions of these labelings and related results follow.

Theorem 2.0.2 ((Rosa [**?**])). *Let G be a graph with m edges. There exists a cyclic G-decomposition of K_{2m+1} if and only if G admits a ρ-labeling.*

Definition 2.0.3 ((Rosa [**?**])). A σ-labeling of a graph G is a ρ-labeling such that $\ell(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$.

Definition 2.0.4 ((El-Zanati, Vanden Eynden [**?**])). A ρ- or σ-labeling of a bipartite graph G with bipartition (A, B) is called an *ordered* ρ- or σ-labeling and denoted ρ^+, σ^+, respectively, if $f(a) < f(b)$ for each edge ab with $a \in A$ and $b \in B$.

Theorem 2.0.5 ((El-Zanati, Vanden Eynden [**?**])). *Let G be a graph with m edges which has a ρ^{+}-labeling. Then G decomposes K_{2mk+1} for all positive integers k.*

Definition 2.0.6 ((Freyberg, Tran [**?**])). A σ^{+-}-*labeling* of a bipartite graph G with m edges and bipartition (A, B) is a σ^{+}-labeling with the property that $f(a) - f(b) \neq m$ for all $a \in A$ and $b \in B$, and $f(v) \notin \{2m, 2m-1\}$ for any $v \in V(G)$.

Theorem 2.0.7 ((Freyberg, Tran [**?**])). *Let G be a graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant. Then there exists a G-decomposition of both K_{2mk} and K_{2mk+1} for every positive integer k.*

Figure **??** gives a σ^{+-}-labeling of every forest with 7 edges. The vertex labels of each connected component with k vertices are given as a k-tuple, $(v_1, \ldots, v_k)$ corresponding to the vertices $v_1, \ldots, v_k$ given in Figure **??**. We leave it to the reader to infer the bipartition (A, B).

Example 2.0.8. *A σ^{+-}-labeling of $\mathbf{T^6_6} \sqcup 2\mathbf{T^1_2}$ is shown in Figure* **??**. *The vertices labeled $1, 2$ and 9 belong to A, and the others belong to B. The lengths of each edge are indicated on the edge.*

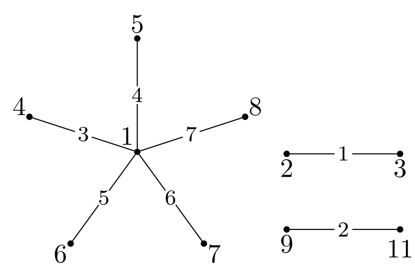

Figure 2.1: σ^{+-}-labeling of $\mathbf{T^6_6} \sqcup 2\mathbf{T^1_2}$

The labelings given in Figure **??** along with thm **??** are enough to prove the following thm.

Forest	Vertex Labels
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2,9,7) \sqcup (3,4)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,3) \sqcup (8,7)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,4) \sqcup (8,7)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(5,1,4,2,9,6,7) \sqcup (10,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(3,8,1,4,2,5,7) \sqcup (9,10)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(7,8,1,6,0,4,3) \sqcup (9,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,1,6,3,4,5,7) \sqcup (9,10)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(6,1,5,3,8,4,7) \sqcup (9,10)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(5,11,9,10,6,12,7) \sqcup (8,1)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(4,8,1,6,0,5,3) \sqcup (9,10)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(0,6,1,5,2,9) \sqcup (11,10,12)$
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(3,6,1,8,4,0) \sqcup (10,9,11)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(5,11,9,12,7,10) \sqcup (1,8,4)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(3,8,4,1,6,7) \sqcup (10,9,11)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(5,1,8,3,4,7) \sqcup (10,9,11)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(4,1,8,5,6,7) \sqcup (10,9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(0,6,1,5,2) \sqcup (9,8,10,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(7,1,8,5,6) \sqcup (0,4,2,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(7,1,8,4,6) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(6,0,3,4,5) \sqcup (8,7,9,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(4,8,1,7,2) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(6,0,3,4,5) \sqcup (8,9,2,7)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5,2,9) \sqcup (8,10) \sqcup (3,4)$

$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(3,6,1,8,4,0) \sqcup (5,7) \sqcup (9,10)$
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,3,5,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(5,8,4,1,6,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(5,11,9,12,7,10) \sqcup (8,1) \sqcup (0,4)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,5,6,7) \sqcup (2,3) \sqcup (9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (8,10,9) \sqcup (11,4)$
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(7,1,8,5,6) \sqcup (10,9,11) \sqcup (0,4)$
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(6,0,3,4,5) \sqcup (1,8,7) \sqcup (9,11)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5) \sqcup (2,9,7,10) \sqcup (3,4)$
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11,9,10,7) \sqcup (4,0,5,6) \sqcup (8,1)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(4,0,5,6) \sqcup (10,9,11,12) \sqcup (8,1)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(0,6,1,5) \sqcup (8,10,9) \sqcup (11,4,7)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(4,0,5,6) \sqcup (1,8,7) \sqcup (11,9,12)$
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (8,10,7) \sqcup (11,4) \sqcup (2,3)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (11,9,12) \sqcup (2,3) \sqcup (8,1)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(6,1,8,4,7) \sqcup (3,5) \sqcup (9,12) \sqcup (10,11)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(3,0,4,5,6) \sqcup (8,1) \sqcup (10,11) \sqcup (9,7)$
$3\mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (2,9,7) \sqcup (10,11)$
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (9,2) \sqcup (8,10) \sqcup (4,7) \sqcup (11,12)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (2,3) \sqcup (9,11) \sqcup (8,1) \sqcup (10,7)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(0,6,1) \sqcup (8,4) \sqcup (2,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$

Figure 2.2: σ^{+-}-labelings for forests with 7 edges

Theorem 2.0.9. *Let F be a forest with* 7 *edges. There exists an F-decomposition of K_n whenever $n \equiv 0$ or $1 \pmod{14}$.*

Proof. The proof follows from thm **??** and the labelings given in Figure **??**. $\square$

Chapter 3

Conclusion and Discussion

Appendix A

Glossary and Acronyms

Care has been taken in this thesis to minimize the use of jargon and acronyms, but this cannot always be achieved. This appendix defines jargon terms in a glossary, and contains a table of acronyms and their meaning.

A.1 Glossary

- **Cosmic-Ray Muon** (**CR** μ) – A muon coming from the abundant energetic particles originating outside of the Earth's atmosphere.

A.2 Acronyms

Table A.1: Acronyms

Acronym	Meaning
CRμ	Cosmic-Ray Muon