

Seven Edge Forest Designs

A THESIS

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Dedication

I dedicate this Thesis to my advisor Professor Bryan Freyberg, to my family who has supported me throughout this process, and to Jordi, Ian, TK, and Torta from Tuscarora Ave.

Abstract

Let G be a subgraph of K_n where $n \in \mathbb{N}$. A G -decomposition of K_n , or G -design of order n , is a finite collection $\mathcal{G} = \{G_1, \dots, G_k\}$ of pairwise edge-disjoint subgraphs of K_n that are all isomorphic to some graph G . We prove that an F -decomposition of K_n exists for every seven-edge forest F if and only if $n \equiv 0, 1, 7$, or $8 \pmod{14}$.

Along the way, we introduce new methods, constraint programming algorithms in Python, and some bonus results for Galaxy graph decompositions of complete bipartite, and eventually multipartite graphs.

Contents

Acknowledgements	i
Dedication	ii
Abstract	iii
List of Tables	vi
List of Figures	vii
1 Background	1
1.1 Fundamentals of Graph Theory	1
1.2 Fundamental Families of Graphs	6
2 Introduction	10
2.1 Decompositions	10
2.2 Graph labeling	13
2.3 Seven edge forests	16
3 $n \equiv 0, 1 \pmod{14}$	18
3.1 Construction for $n \equiv 0, 1 \pmod{14}$	18
3.2 Results for $n \equiv 0, 1 \pmod{14}$	21
4 $n \equiv 7, 8 \pmod{14}$	25
4.1 Construction	25
4.2 Results	28

5	$T_7^{11} \sqcup T_2^1$	33
6	Additional Results	36
6.1	Wraparound edge mappings that preserve length	36
7	Conclusion and Discussion	37
8	Appendix	38
	References	55
	Appendix A. Glossary and Acronyms	56
A.1	Glossary	56
A.2	Acronyms	56

List of Tables

3.1	σ^{+-} -labelings for forests with seven edges	23
3.1	σ^{+-} -labelings for forests with seven edges	24
8.1	(1-2-3)-labelings	38
8.1	(1-2-3)-labelings	39
8.1	(1-2-3)-labelings	40
8.1	(1-2-3)-labelings	41
8.1	(1-2-3)-labelings	42
8.1	(1-2-3)-labelings	43
8.1	(1-2-3)-labelings	44
8.2	(1-2-3)-labelings	45
8.2	(1-2-3)-labelings	46
8.2	(1-2-3)-labelings	47
8.2	(1-2-3)-labelings	48
8.2	(1-2-3)-labelings	49
8.2	(1-2-3)-labelings	50
8.2	(1-2-3)-labelings	51
8.2	(1-2-3)-labelings	52
8.3	A $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of K_{21}	53
8.4	A $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of K_{22}	53
8.4	A $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of K_{22}	54
A.1	Acronyms	56

List of Figures

1.1	The Petersen graph	2
1.2	$G \cong H$	2
1.3	K and subgraphs $G, H, M \subseteq K$	3
1.4	(above) $G \cup H$ and (below) $G \sqcup K$	4
1.5	(above) $G \cup H$ and (below) $G \sqcup K$	4
1.6	The Complete Graph K_5	6
1.7	The Complete Bipartite Graph $K_{3,3}$	7
1.8	The Complete Multipartite Graph $K_{3,2,3}$	7
1.9	The Cycle Graph C_5	7
1.10	A Tree Graph on 6 vertices	8
1.11	The Path Graph P_4	8
1.12	The 7-star $(K_{1,7})$	8
1.13	A forest on 9 vertices	9
1.14	The $(K_{1,6}, K_{1,7})$ -Galaxy	9
2.1	$\{G_1, G_2\}$ is a decomposition of K_4	10
2.2	$\{G_1, G_2\}$ is a P_3 -decomposition of K_4 or a P_3 -design of order 4	12
2.3	K_5 with lengths colored	14
2.4	A cyclic P_3 -decomposition of K_5	15
2.5	Trees with less than seven edges	16
3.1	K_{14} (left) and K_{15} (right) with edges colored by length	18
3.2	$K_{13} \vee K_1$ is isomorphic to K_{14}	20

3.3	A labeling (left) that gives the $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of K_{14} (middle) and K_{15} (right), respectively, when developed by 1. The leaf 7 in the pendant edge $(0, 7)$ (of length 7 modulo 14 and 15) is relabeled as ∞ for K_{14}	20
3.4	σ^{+-} -labeling of $\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	22
4.1	K_{21} (left) and K_{22} (right) with edges colored by length.	25
4.2	$K_{21} \vee K_1$ is isomorphic to K_{22} (right)	26
4.3	Three labelings of $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ (left column) that generate all edges of lengths 1, 2, and 3 in K_{21} when developed by 7 repeatedly.	27
4.4	Labelings for $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	31
4.5	A σ^{+-} -labeling of $F \cong \mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ (left) and generating presentations for the F -decomposition of K_n where $n = 35$ (middle) and $n = 36$ (right)	32
5.1	G_0 in a generating presentation of the $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of $K_{n,7}$	34

Chapter 1

Background

1.1 Fundamentals of Graph Theory

Graph Theory is the study of objects called *vertices* or *nodes* and their relationships which we call *edges*. An edge between vertices u and v is typically denoted uv or (u, v) . A graph G is formally defined as an ordered pair $G = (V, E)$ where V is the set of all vertices in G and E is the set of all edges between vertices in G . These sets are sometimes referred to as $V(G)$ and $E(G)$, respectively.

G is called a *simple graph* if: (1) there is at most 1 edge between any two vertices, (2) there are no edges from a vertex to itself and (3) all edges have no directionality to them, meaning $uv = vu$ for any edge $uv \in E(G)$. For the rest of this paper all graphs are finite simple graphs, but note that unions and subgraphs are defined the same way for directed graphs and infinite graphs.

Graphs are more intuitive to work with through their visual representations instead of their formal definitions. Let the simple graph G where $V(G) = \{A, B, C, D, E, a, b, c, d, e\}$ and $E(G) = \{Aa, Bb, Cc, Dd, Ee, AB, BC, CD, DE, EA, ac, ce, eb, bd, da\}$. G is often called the *Petersen* graph. It's unwieldy when described formally, yet its visual representation is very easy to understand.

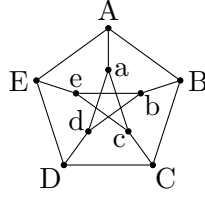
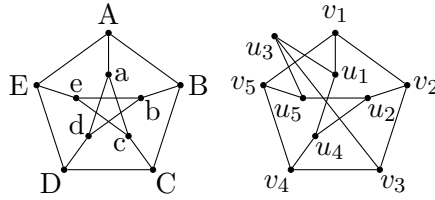


Figure 1.1: The Petersen graph

We say two graphs G and H are *isomorphic* if there exists a bijection from $V(G)$ to $V(H)$ that induces a bijection from $E(G)$ to $E(H)$ and we denote this relationship via $G \cong H$. In other words, we consider two graphs G, H to be the 'same' if we can relabel and/or move vertices in some fashion (without adding/removing vertices edges) in a visual representations of G and H to go back and forth between the two.

Figure 1.2: $G \cong H$

Graph theorists casually refer to two graphs as the 'same' graph if they are in the same isomorphism class. We will wrap up the fundamentals with a few mdefinitions and some algebraic tools.

Definition 1.1.1 (Subgraph). A subgraph $G \subseteq K$ is a graph whose vertices and edges are subsets of the vertices and edges of K ; $G \subseteq K$ if $V(G) \subseteq V(K)$ and $E(G) \subseteq E(K)$.

Definition 1.1.2 (Vertex-induced Subgraph). A *vertex-induced* subgraph $G \subseteq K$ is one whose vertices are some subset of $V(K)$ and whose edges are all edges between those vertices in K ; $V(G) \subseteq V(K)$ and $E(G) = \{uv \in E(K) \mid u, v \in E(G)\}$. If G is such a subgraph we say that G is induced by $S = V(G) \subseteq V(K)$.

Definition 1.1.3 (Edge-induced Subgraph). A *edge-induced* subgraph $G \subseteq K$ is one whose edges are some subset of $E(K)$ and whose vertices are all those who appear as

an endpoint in that subset of edges; $E(G) \subseteq E(K)$ and $V(G) = \{v \in V(K) \mid vu \in E(G) \text{ or } uv \in E(G) \text{ for some } u \in V(K)\}$. If G is such a subgraph we say that G is induced by $S = E(G) \subset E(K)$

Here is a visual example of these types of graphs: Let K be the Petersen graph from Figure 1.1. Now, let

Subgraph: $G \subseteq K$ where $V(G) = \{E, e, b\}$, $E(G) = \{Ee\}$.

Vertex-induced Subgraph: $H \subseteq K$ is induced by $\{a, A, B\} \subseteq V(K)$

Edge-induced Subgraph: $M \subseteq K$ is induced by $\{Dd, DC, Cc\} \subseteq E(K)$

The figure below shows K and its color-coded subgraphs.

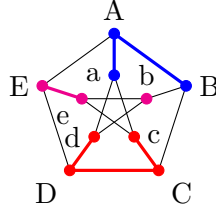


Figure 1.3: K and subgraphs $G, H, M \subseteq K$

Next, we will talk about two important operations done on graphs.

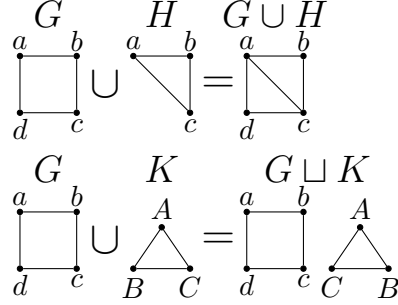
Definition 1.1.4 (Graph Union). The union of two graphs G and H is simply the graph resulting from the union of their vertices and the union of their edges and is denoted $G \cup H$; $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. If G and H are edge-disjoint, we denote their union via $G \sqcup H$ and call it a *disjoint union* of G and H .

Here is an example of a union and a disjoint union of graphs. Let $G = (\{a, b, c, d\}, \{ab, bc, cd, da\})$, $H = (\{a, b, c\}, \{ab, bc, ca\})$, and $K = (\{A, B, C\}, \{AB, BC, CA\})$ Then:

$$G \cup H = (\{a, b, c, d\}, \{ab, bc, cd, da, ca\})$$

$$G \sqcup K = V(G \sqcup K) = (\{a, b, c, d, A, B, C\}, \{ab, bc, cd, da, AB, BC, CA\})$$

These unions are depicted in the following figure.

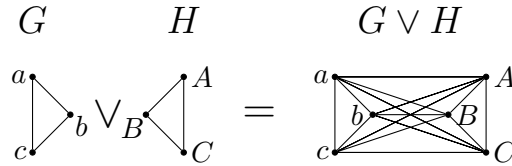
Figure 1.4: (above) $G \cup H$ and (below) $G \sqcup K$

Next, we define another very important operation that combines two graphs in a different manner.

Definition 1.1.5 (Join). Let G and H be vertex disjoint graphs. Their *join*, denoted via $G \vee H$, is the graph obtained by taking the disjoint union of G and H and adding all possible edges between every vertex in G and every vertex in H . Formally:

$$G \vee H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}).$$

Here is an example. Let $G = (\{a, b, c\}, \{ab, bc, ca\})$ and $H = (\{A, B, C\}, \{AB, BC, CA\})$, then $G \vee H = (\{a, b, c, A, B, C\}, E(G) \sqcup E(H) \sqcup \{aA, aB, aC, bA, bB, bC, cA, cB, cC\})$. This join is depicted in the figure below.

Figure 1.5: (above) $G \cup H$ and (below) $G \sqcup K$

Lastly, we define a few characteristics of graphs and their components. These may or may not be used frequently in this paper, but are important concepts to know in order to be able to talk about graphs comfortably.

Let G be a simple graph. We say two vertices $u, v \in V(G)$ are *adjacent* or *neighbors* if they share an edge $uv \in E(G)$. Similarly, we say a vertex is *incident* with an edge if it is one of its endpoints; $u \in V(G)$ is incident with $e \in E(G)$ if $e = uv$ for some $v \in V(G)$. The set of all vertices adjacent to u in G is called the *neighborhood* of u denoted $N_G(u)$ or simply $N(u)$. Sometimes this is referred to as the open neighborhood of u in G and then the closed neighborhood is defined via $N_G[u] = N_G(u) \cup \{u\}$. The *degree* of a vertex $u \in V(G)$ is the number of vertices adjacent to it and is denoted via $\deg_G(u) = |N_G(u)|$ or simply $\deg(u)$. Equivalently, the degree is the number of edges incident to it or the number of neighbors that G has.

The following are three similar types of objects we can form from graphs.

Definition 1.1.6 (Walk). Let G be a graph on n vertices. A *walk* in G is a sequence (w_0, w_1, \dots, w_k) of vertices in G whose adjacent elements must be adjacent in G . Adjacent elements in a walk must be distinct vertices but a vertex may be repeated multiple times.

Definition 1.1.7 (Path). Let G be a graph on n vertices. A *path* in G is a sequence (v_0, v_1, \dots, v_k) of distinct vertices in G whose adjacent elements must be adjacent in G , and where no vertex is repeated. This sequence gives the subgraph of G induced by $\{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\}$.

Definition 1.1.8 (Cycle). Let G be a graph on n vertices. A *cycle* in G is a sequence $(v_0, v_1, \dots, v_k, v_0)$ of internally distinct vertices (distinct except on the endpoints) that begins and terminates at the same vertex v_0 . Often such a cycle is denoted via $(v_0v_1 \cdots v_k)$ and it is understood that the sequence wraps back around to v_0 after v_k . Additionally, the cycle $(v_0v_1 \cdots v_k)$ is equivalent to $(v_1 \cdots v_kv_0)$, $(v_2 \cdots v_kv_0v_1)$, \dots and so on.

Let G be a simple graph. We call G *acyclic* if it contains no cycles. If there exists a path from any vertex to every other vertex in G , then we call G *connected*. If not, we call G *disconnected*. We call the set of connected subgraphs of G whose disjoint union equals G the *connected components* of G .

This concludes the fundamental concepts needed to understand this project. The next and final section of this chapter will introduce all the fundamental families of graphs we refer to in the proceeding chapters.

1.2 Fundamental Families of Graphs

In this section introduce some fundamental families of graphs which we refer to throughout this paper. Often instead of fully defining the graphs being worked with, we simply refer to it as a member of a larger family of graphs. These families are not completely distinct, but sometimes it is helpful to view graphs as a member of one family or another depending on the context.

Recall that a graph is acyclic if it contains no cycles. Similarly, we call a graph k -cyclic if it contains exactly k distinct cycles. If $k = 2$ or 3 we call it *bicyclic* or *tricyclic*, respectively. In a similar vein, we call a graph k -partite if we can partition its vertices into k disjoint sets. If $k = 2$ or 3 , we call it *bipartite* or *tripartite*, respectively. These are all very broad families of graphs often used to characterize graphs within another family. The following are more nuanced, and more popular families of graphs to work with.

Definition 1.2.1 (Complete Graph). The *complete graph* on n vertices, denoted K_n , is the graph on n vertices such that every pair of distinct vertices shares an edge.

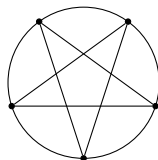


Figure 1.6: The Complete Graph K_5

Definition 1.2.2 (Complete Bipartite Graph). Let $m, n \in \mathbb{N}$. The *complete bipartite graph* $K_{m,n}$ is the bipartite graph whose vertices can be partitioned into two disjoint sets of sizes m and n , respectively, such that every vertex in the one partite set is adjacent to every vertex in the other partite set and there are no edges between vertices in the same partite set.

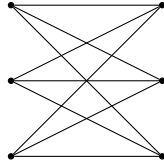


Figure 1.7: The Complete Bipartite Graph $K_{3,3}$

Definition 1.2.3 (Complete Multipartite Graph). The *complete k -partite graph* or *complete multipartite graph* K_{n_1, \dots, n_k} is the graph whose vertices can be partitioned into k disjoint sets of sizes n_1, n_2, \dots, n_k , respectively such that every vertex in the one partite set is adjacent to every vertex in the other $k - 1$ partite sets and there are no edges between vertices in the same partite set.

If all partite sets are the same size n we call this graph the *complete equipartite graph* $K_{n:k}$ or $K_{n \times m}$.

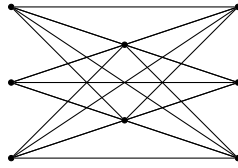


Figure 1.8: The Complete Multipartite Graph $K_{3,2,3}$

Definition 1.2.4 (Cycle Graph). The *cycle graph* on n vertices denoted C_n is a graph with exactly one cycle containing all of its edges.

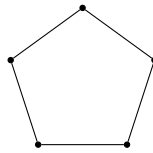


Figure 1.9: The Cycle Graph C_5

Definition 1.2.5 (Tree). A *tree* is any connected acyclic graph. Trees on n vertices have $n - 1$ edges. Equivalently, these graphs are any connected bipartite graphs.

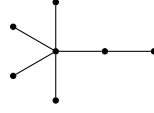
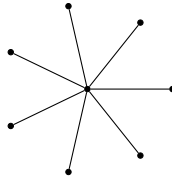


Figure 1.10: A Tree Graph on 6 vertices

Definition 1.2.6 (Path Graph). The *path* graph on n vertices, denoted P_n , is an acyclic graph with exactly one path containing all of its edges. All paths are trees.

Figure 1.11: The Path Graph P_4

Definition 1.2.7 (Star Graph). The *star graph* on $n + 1$ vertices, denoted $K_{1,n}$ (or S_{n+1} which we never use in this paper) consisting of one central *hub* vertex adjacent to n *outer* vertices, with no other edges. All stars are trees. Sometimes this graph is referred to as an *n-star*.

Figure 1.12: The 7-star ($K_{1,7}$)

Definition 1.2.8 (Forest Graph). Any disjoint union of tree graphs is called a *forest* graph. These graphs are all bipartite and can be equivalently defined as disconnected bipartite graphs.

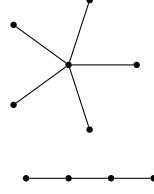
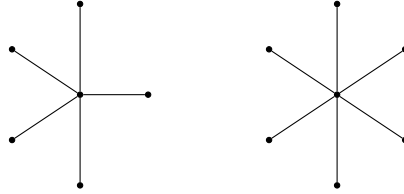


Figure 1.13: A forest on 9 vertices

Definition 1.2.9 (Galaxy Graph). Any disjoint union of star graphs is called a *galaxy* graph. We refer to $G = G_1 \sqcup \dots \sqcup G_k$ as a (G_1, \dots, G_k) -*galaxy* graph if G_1, \dots, G_k are all stars. This family is a subset of the forest family.

Figure 1.14: The $(K_{1,6}, K_{1,7})$ -Galaxy

We have now defined a few important families of graphs which we will refer to throughout the rest of this paper. We generally don't explicitly define every graph by its vertices and edges and simply refer to it as some member of one family or say this it is isomorphic to one. This is much more efficient and concise than listing out all vertices and edges as we did in the beginning of this chapter.

We are now ready to move on and introduce graph decompositions, the objects which are the subject of this project.

Chapter 2

Introduction

2.1 Decompositions

Suppose you have n translucent sheets of tracing paper with some points drawn on all n sheets of paper in the same set arrangement. Now, draw lines connecting points on each sheet of paper, so that no line appears on two distinct sheets of paper.

A graph is depicted when all n sheets of tracing paper are aligned and stacked on top of each other with some light source present, call this graph K . Call the graph depicted on the i th sheet of paper G_i for $i = 1, \dots, n$. The stacking of these sheets of paper depicts $G_1 \sqcup \dots \sqcup G_n = K$, and this collection of papers depicts the set $\{G_1, \dots, G_n\}$ which we call a *graph decomposition* of K . This is defined formally below.

Definition 2.1.1 (Graph Decomposition). Let K be a simple graph. We call a collection $\mathcal{G} = \{G_1, \dots, G_n\}$ of pairwise edge-disjoint subgraphs $G_1, \dots, G_n \subseteq K$ of K a *decomposition* of K if their disjoint union equals K ; $G_1 \sqcup \dots \sqcup G_n = K$ and $\{G_1, \dots, G_t\}$.

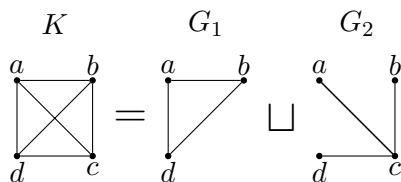


Figure 2.1: $\{G_1, G_2\}$ is a decomposition of K_4

Graph decompositions are an important topic combinatorics, graph theory, and design theory, with origins dating back to the 1800s. Notably, in 1850 Reverend Thomas Kirkman, a full-time clergyman and legendary mathematician, posed an important problem in *The Lady's and Gentleman's Diary* [8] now known as *the school girl problem*. It goes

*Fifteen young ladies in a school walk out three abreast for
seven days in succession: it is required to arrange them daily
so that no two shall walk twice abreast.*

The problem asks if we can form five distinct rows of three school girls on each day of the week so that no two school girls walk in the same row more than once in a week. This is equivalent to finding a decomposition of K_{15} whose members are all triangles; whose members are isomorphic to C_3 or K_3 . Both Kirkman and Arthur Cayley independently solved the the schoolgirl problem and published their solutions in the 1851 edition of *The Lady's and Gentleman's Diary* [9]. Kirkman's solution is provided below.

Denoting the ladies by $a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d_1, d_2, d_3; e_1, e_2, e_3$, the following arrangement will be found to answer the question:

$a_1 a_2 a_3$	$a_1 b_1 c_1$	$a_1 d_1 e_1$	$a_1 b_2 d_2$	$a_1 c_2 e_2$	$a_1 b_3 e_3$	$a_1 c_3 d_3$
$b_1 b_2 b_3$	$a_2 b_2 c_2$	$a_2 d_2 e_2$	$a_2 b_3 d_3$	$a_2 c_3 e_3$	$a_2 b_1 e_1$	$a_2 c_1 d_1$
$c_1 c_2 c_3$	$a_3 d_3 e_3$	$a_3 b_3 c_3$	$a_3 c_1 e_1$	$a_3 b_1 d_1$	$a_3 c_2 d_2$	$a_3 b_2 e_2$
$d_1 d_2 d_3$	$b_3 d_1 e_2$	$b_1 c_1 e_3$	$b_1 c_3 e_1$	$b_2 c_3 d_1$	$c_2 b_3 e_1$	$c_2 b_3 e_1$
$e_1 e_2 e_3$	$c_3 d_2 e_1$	$e_3 b_2 c_1$	$d_1 c_2 e_3$	$c_1 d_2 b_3$	$d_2 b_1 c_2$	$c_1 d_3 b_2$

This is the symmetrical and only possible solution. All others differ from this only in disturbing the alphabetical order, or that of the three subindices in certain triplets of the first column, or in both these together.

Each triple in this array above gives a edge-distinct triangle subgraph of K_{15} whose vertex set we take to be $\{a_1, a_2, \dots, e_4, e_5\}$. The set of all these subgraphs is a decomposition of K_{15} . Since all of these subgraphs are isomorphic to C_3 , we call it a C_3 -decomposition. This is a special type of decomposition which is defined formally on the following page.

Definition 2.1.2 (*G*-decomposition). A *G*-decomposition of a graph K is a decomposition $\mathcal{G} = \{G_1, \dots, G_t\}$ whose members are all isomorphic to some graph G ; $\mathcal{G} = \{G_1, \dots, G_t\}$ such that $K = G_1 \sqcup \dots \sqcup G_t$ and $G_i \cong G$ for $i = 1, \dots, t$. If such a set exists we say that K allows a *G*-decomposition or equivalently, that G decomposes K . If $K \cong K_n$ we sometimes call the decomposition a *G*-design of order n .

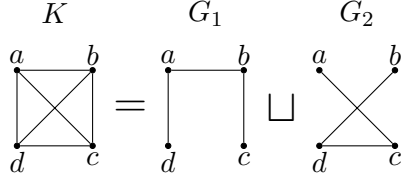


Figure 2.2: $\{G_1, G_2\}$ is a P_3 -decomposition of K_4 or a P_3 -design of order 4

We know that if a *G*-decomposition of some graph K exists, that all of its members have the same number of edges and vertices. This allows us to find constraints on the types of bigger graphs K that can be decomposed by some subgraph $G \subseteq K$ on m edges solely based on divisibility properties.

Lemma 2.1.3 (Necessary Condition (general)). *Let G be a simple graph on m edges. There exists a G -decomposition of a graph K only if $|E(G)| = m$ divides $|E(K)|$.*

Proof. Suppose there exists a G -decomposition $\mathcal{G} = \{G_1, \dots, G_n\}$ of K . Then $E(G_1) \sqcup \dots \sqcup E(G_t) = E(K)$ and so $|E(K)| = |E(G_1) \sqcup \dots \sqcup E(G_t)| = |E(G_1)| + \dots + |E(G_t)| = tm$. So $|E(G)| = m$ divides $|E(K)|$. \square

Theorem 2.1.4 (Necessary Condition (K_n)). *Let G be a simple graph on m edges. There exists a G -decomposition of K_n only if n is idempotent modulo $2m$; only if $n^2 \equiv n \pmod{2m}$.*

Proof. Suppose there exists a G -decomposition of K_n . Then $|E(G)| = m$ divides $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$ by Lemma 2.1.3. Therefore, $\frac{n^2-n}{2} = mt$ for some $t \in \mathbb{N}$. Observe.

$$n^2 - n = 2mt \implies n^2 - n \equiv 0 \pmod{2m} \implies n^2 \equiv n \pmod{2m}.$$

\square

By the previous theorem, any graph on m edges decomposes K_n only if n is idempotent modulo $2m$. Note that the converse isn't necessarily true. However, for a graph G on m edges, this finite set of constraints allows us to ask:

For what n is K_n G -decomposable?

This question is known as the *spectrum problem* for decompositions. Pioneering work by Rosa and Kotzig in the 1960s—especially in the development of graph labeling—helped shape the modern treatment of G -decomposition problems. Since then, labeling-based techniques and tools from design theory have driven significant progress. In particular, graph labeling methods have played a central role in addressing the spectrum problem for small graphs. This directly continues contributions by Freyberg and Peters, who recently solved the spectrum problem for forests with six edges [4]. Their paper provides a comprehensive summary of known decompositions for graphs G with fewer than seven edges.

Using graph labelings to solve G -decomposition problems is basically about doing algebra on graphs in order to generate edge-disjoint subgraphs. If we take the vertices of a graph K to be elements of a group, we can use the structure of the group to our advantage. Specifically, when $K \cong K_n$, and we take its vertices to be \mathbb{Z}_n , and then we label the vertices of G with some subset of \mathbb{Z}_n . There are various labeling techniques of this kind stemming from Rosa's work in the 1960s that allow us to permute or act on the labels of the vertices of G with subgroups of \mathbb{Z}_n to generate new isomorphic copies of G which are pairwise edge-disjoint. In the next section, we provide an example which outlines in some detail how this machinery works for G -decompositions of Complete Graphs.

2.2 Graph labeling

Take the vertices of K_5 to be \mathbb{Z}_5 and arrange it in the same manner as in 1.6. Notice that every vertex shares an edge with two vertices directly adjacent to it and two vertices that are 'two adjacencies away' on the outer cycle (01234). We call this idea *length* denoted ℓ where edges joining two vertices u, v have length $\ell(uv) = l$ if they are ' l adjacencies' away from each other on the outer cycle.

Formally, for K_n we define the edge length function ℓ as follows:

$$\ell(uv) = \min\{|u - v|, n - |u - v|\}$$

Notice that for K_5 , we only have lengths 1 or 2 as previously observed. Color the length 1 edges **blue**, and the length 2 edges **red**. This is depicted in the figure below.

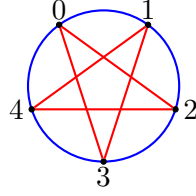
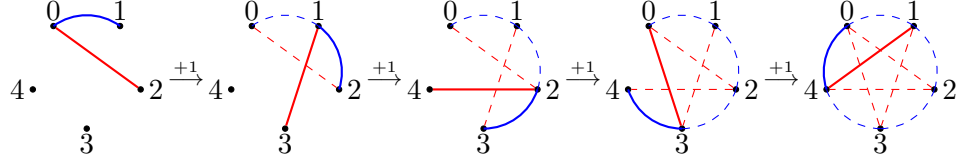


Figure 2.3: K_5 with lengths colored

Now, consider P_3 . It has 2 edges, and K_5 has $\binom{5}{2} = 10$ edges. Since $2|10$, by Lemma 2.1.3 it's *possible* that a P_3 -decomposition of K_5 exists. Now since P_3 it has 2 edges and there are 2 lengths in K_5 , what if we can just make sure each copy of P_3 has both a **blue** edge and a **red** edge? How can we do that while ensuring that no edge is repeated?

It turns out that if we take the vertices of K_n to be \mathbb{Z}_n , adding 1 (and therefore anything) modulo n to the endpoints of an edge preserves its length. We call the act of permuting vertices in this manner *clicking* or *developing*.

In the context of our problem with P_3 and K_5 , this means that if we label P_3 with elements of \mathbb{Z}_5 such that just have one **blue** edge of length 1, and one **red** edge of length 2, that we can simply generate all edges of length 1 and 2 in K_n (which is just all edges of K_n) while preserving the structure of the graph by developing all vertices of our labeling by 1 at the same time repeatedly. Label the defining path of P_3 via $(2,0,1)$. Developing the vertices 1 modulo 5 will give all members of a P_3 -decomposition of K_5 . A decomposition that can be generated by permuting all vertices of one labeling repeatedly in this fashion is called a *cyclic decomposition*. This is depicted in the following figure.

Figure 2.4: A cyclic P_3 -decomposition of K_5

Nice and easy right? But that's just one complete graph that P_3 can decompose. Remember, that it is *possible* that P_3 can decompose any K_n where $n \equiv n^2 \pmod{4}$ by Theorem 2.1.4. This equivalent to saying $n = 4t + r$ where r is an idempotent in the ring \mathbb{Z}_4 and $t \geq 1$. The idempotents in \mathbb{Z}_4 are 0, 1. So this means \mathbb{Z}_5 is just a special case of n where $n = 4t + 1$ where $t = 1$. Luckily, even though these are infinite families, it is known that for each step $t \mapsto t + 1$, new lengths come 2 at a time. This means if we can somehow transform our labeling at each step to include the new lengths, we can maybe take care of the entire family K_{4t+1} . We want to fine tune our labeling to be able to weather this process. This is what graph labeling is all about. Note that if r was not 0 or 1, we would need multiple labelings to take care of the whole family. This is explained later in this paper.

Lastly, some basic observations about a general G with m edges and K_n . The maximal length in K_n is $\lfloor \frac{n}{2} \rfloor$. This is intuitive, since when you travel halfway across the outer cycle from some vertex, the lengths start going back down again and you are nearing that vertex. Now, n must be of the form $2mt + r$ where $t \geq 1$ and r is an idempotent in the ring \mathbb{Z}_{2m} . This means that in K_{2m+r} if $\ell(uv) = |u - v| < \lfloor \frac{2m+r}{2} \rfloor < \lfloor \frac{2mt+r}{2} \rfloor$ for $t > 1$, then $\ell(uv) = |u - v|$ in all K_{2mt+r} for $t \geq 1$. this is important, because at each step $t \mapsto t + 1$, new lengths come m at a time. This means that some wraparound edges xy in K_{2m+r} are short edges of length $|x - y|$ in K_{2mt+r} for $t > 1$.

Now, for $r = 0$ or 1 , if a certain labeling of a graph G on m edges exists, there exists a G -decomposition of K_{2mt+r} for $t \geq 1$. However, if $r \neq 0, 1$, one labeling will not suffice and other techniques are needed to prove that G decomposes K_{2mt+r} for $t \geq 1$. These labelings and techniques are defined as they are needed in the proceeding chapters. Finally, we are ready to introduce the focus of this project.

2.3 Seven edge forests

This project continues on Freyberg and Peters' work on six edge forests by asking the same question about seven edge forests:

Let F be a forest on seven edges. For which n does F decompose K_n ?

The spectrum problem for the matching $7\mathbf{T}_2^{11}$ was solved by de Werra in 1970. Every component of a forest on seven edges is a tree on six or less edges which are cataloged in Figure 2.5. We use the naming convention \mathbf{T}_j^i to denote the i^{th} tree with j vertices and we index the vertices v_1 through v_j for each tree as specified below.

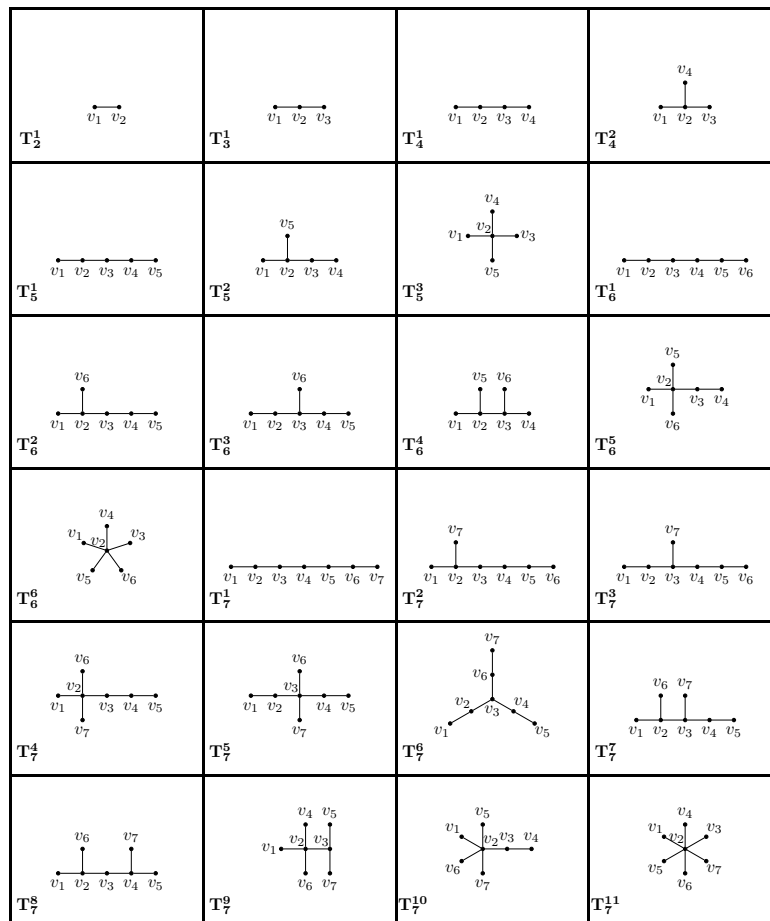


Figure 2.5: Trees with less than seven edges

The next theorem gives the necessary conditions for the existence of a G -decomposition of K_n when G is a graph with 7 edges.

Theorem 2.3.1. *If G is a graph with 7 edges and a G -decomposition of K_n exists, then $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$.*

Proof. If a G -decomposition of K_n exists, then n is idempotent modulo $2(7) = 14$ by Theorem 2.1.4 which immediately implies that $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$ since those are all the idempotents in \mathbb{Z}_{14} . \square

For this project, we do not define the graph on one vertex to be a tree. This means that any connected component in a forest has at least one edge and we also require there to be at least two connected components. There are 47 such forests with 7 edges up to isomorphism. As stated previously, the matching on seven edges is solved, so only the remaining 46 trees need be considered in the subsequent chapters. Chapter 3 handles all decomposing K_n into all 47 forests when $n \equiv 0 \text{ or } 1 \pmod{14}$. Chapter 4 applies to all the forests when $n \equiv 7 \text{ or } 8 \pmod{14}$ with the lone exception of $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$, which is solved for those values of n in Chapter 5.

After proving the main result of this project, we provide a couple of additional results in Chapter 6: (1) An edge mapping depending on t that preserves length for wraparound edges in $2m+r$ to $2mt+r$ at each step $t \mapsto t+1$. (2) Galaxy decompositions of complete multipartite graphs. These two results conclude this thesis project, and produce some open questions.

Lastly, we present some *Python* programs that were created as a result of this thesis project. One is *tikzgrapher*: my graph visualization software, which was built from scratch only using *Pygame* and *NetworkX*. It allows one to visualize simple *NetworkX* graphs in an interactive *Pygame* window that allows for colorings and custom labelings along with dragging and moving components of the graph. The main feature however, is that the user can save the graph layout depicted in the *Pygame* window as a *tikZ* graph in a standalone *L^AT_EX* file. *tikzgrapher* is paired with a graph labeling solver: a constraint programming project that can find several labelings on graphs if they exist. Chapter 6 concludes the results of this project.

Chapter 3

$n \equiv 0, 1 \pmod{14}$

To begin this chapter, we extend intuition developed in the Introduction to present some machinery specific to K_n where $n \equiv 0, 1 \pmod{14}$. This informs the formal definitions and theorems to take care of this case.

3.1 Construction for $n \equiv 0, 1 \pmod{14}$

K_1 and K_2 don't have enough vertices to contain a forest on seven edges. So K_{14} and K_{15} are the base graphs for K_n where $n \equiv 0$ or $1 \pmod{14}$, respectively. We will only show how to decompose these graphs, and then later explain how we can extend these strategies to the rest of their families.

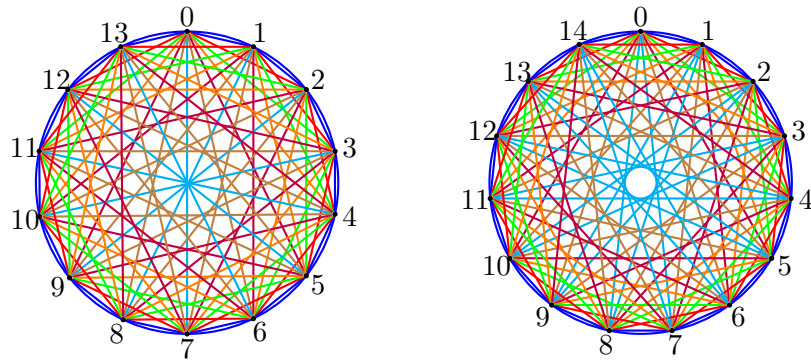


Figure 3.1: K_{14} (left) and K_{15} (right) with edges colored by length

Both K_{14} and K_{15} both only have edges with lengths 1 through 7, however, there is important distinction in counting edges of one length K_{14} versus counting them in K_{15} . The cyan edges have length 7, and conveniently because of the way these graphs are drawn in the figure above, we can easily count them as they form the innermost spoke of the graphs. Because these graphs are so similar we can use one labeling to deal with both at the same time! However, we must be very careful.

In K_{14} there are only 7 edges of length 7 and 14 edges of all other lengths, yet in K_{15} there are 15 edges of length 7 along with 15 edges of all other lengths. it turns out this pattern generalizes. For K_n where

n is odd: There are n edges of lengths 1 through $\frac{n-1}{2}$

n is even: There are n edges of lengths 1 up to $\frac{n}{2}$, and then $\frac{n}{2}$ edges of length $\frac{n}{2}$

This is why our labeling of P_3 to decompose K_5 in the Introduction worked so easily. Because there are the same number of edges of every length (in that case $n = 5$ edges of lengths 1 and 2). The same applies here for K_{15} , we just label our seven edge forests so that we get all seven lengths and we can generate the entire decomposition through developing the nodes by 1 repeatedly.

But notice that if we repeatedly develop nodes of the same labeling in K_{14} (assuming we don't use the vertex 14) we would *overcount* length 7 edges. There is a simple remedy for this where we redefine some aspects of our labeling when using it for the K_{14} .

Take $V(K_{14})$ to be $\mathbb{Z}_{13} \cup \{\infty\}$, and label all edges via the length function modulo 13 except for edges incident to ∞ which will refer to as length ∞ . Now, in the context of our labelings, we define developing nodes by 1 in a slightly different fashion effectively fixing the ∞ vertex. We redefine length and development of vertices in K_{14} as follows:

$$\ell(uv) = \begin{cases} \min\{|u-v|, 13-|u-v|\}, & u, v \neq \infty, \\ \infty, & u \text{ or } v = \infty \end{cases} \quad \text{and } v \mapsto \begin{cases} v+1, & v \in \mathbb{Z}_{13}, \\ \infty, & v = \infty. \end{cases}$$

If we do this, we will have $n-1$ (13 in our case) edges of lengths 1 through 6 as well as ∞ , since the infinity node will have all $n=13$ edges of length ∞ to nodes in \mathbb{Z}_{13} incident to it. So if we partition the edges by length, we can now cyclically generate all

edges of K_{15} and K_{14} ! In the context of our labeling of K_{14} and K_{15} we will (1) simply generate the develop by 1 to generate the decomposition for K_{15} and then (2) relabel the leaf on the pendant edge of length 7 ∞ and use the new machinery defined above to develop the nodes by 1 repeatedly and get $K_{13} \vee K_1$ which is isomorphic to K_{14} .

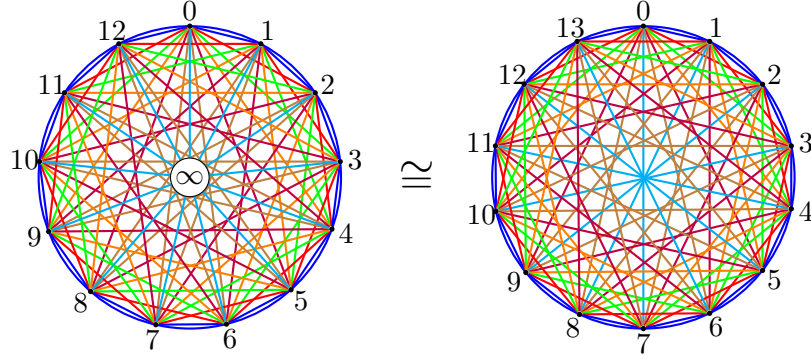


Figure 3.2: $K_{13} \vee K_1$ is isomorphic to K_{14}

Now, notice that if we use this strategy, we must ensure that we only use nodes in \mathbb{Z}_{2m-2} which is \mathbb{Z}_{13} in our case. We must also have a bipartition and another restriction which is given in the formal definition of the labeling in the results section of this chapter. However, we will give a small example here before defining things formally.

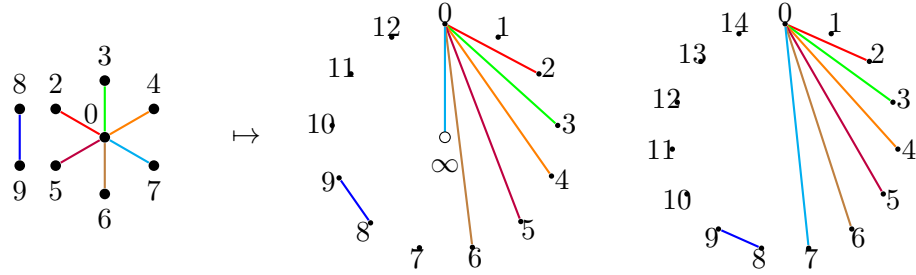


Figure 3.3: A labeling (left) that gives the $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of K_{14} (middle) and K_{15} (right), respectively, when developed by 1. The leaf 7 in the pendant edge $(0, 7)$ (of length 7 modulo 14 and 15) is relabeled as ∞ for K_{14}

It turns out that we can actually just replicate this process at each step $t \mapsto t + 1$ for K_{14t} and K_{14t+1} for new lengths that come 7 at a time if we add some restrictions.

We formally define sufficiently restrictive labelings in the next section that allow us to develop vertices and get certain decompositions.

3.2 Results for $n \equiv 0, 1 \pmod{14}$

We begin with the original ρ -labeling of a bipartite graph G on m edges which uses lengths modulo $2m$ (in K_{2m}).

Definition 3.2.1 ((Rosa [7])). Let G be a graph with m edges. A ρ -labeling of G is an injection $f : V(G) \rightarrow \{0, 1, 2, \dots, 2m\}$ that induces a bijective *length function* $\ell : E(G) \rightarrow \{1, 2, \dots, m\}$ where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all $uv \in E(G)$.

Rosa showed that if a ρ -labeling of a graph G with m edges exists, then a cyclic G -decomposition of K_{2m+1} exists, which is presented formally later. Later, Rosa and his peers began studying more restrictive types of ρ -labelings to decompose more complete graphs. Next, we define some of these labelings and theorems associated with them.

Theorem 3.2.2 ((Rosa [7])). *Let G be a graph with m edges. There exists a cyclic G -decomposition of K_{2m+1} if and only if G admits a ρ -labeling.*

Definition 3.2.3 ((Rosa [7])). A σ -labeling of a graph G is a ρ -labeling such that $\ell(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$.

Definition 3.2.4 ((El-Zanati, Vanden Eynden [2])). A ρ - or σ -labeling of a bipartite graph G with bipartition (A, B) is called an *ordered* ρ - or σ -labeling and denoted ρ^+, σ^+ , respectively, if $f(a) < f(b)$ for each edge ab with $a \in A$ and $b \in B$.

Theorem 3.2.5 ((El-Zanati, Vanden Eynden [2])). *Let G be a graph with m edges which has a ρ^+ -labeling. Then G decomposes K_{2mk+1} for all positive integers k .*

Definition 3.2.6 ((Freyberg, Tran [5])). A σ^{+-} -labeling of a bipartite graph G with m edges and bipartition (A, B) is a σ^+ -labeling with the property that $f(a) - f(b) \neq m$ for all $a \in A$ and $b \in B$, and $f(v) \notin \{2m, 2m - 1\}$ for any $v \in V(G)$.

Theorem 3.2.7 ((Freyberg, Tran [5])). *Let G be a graph with m edges and a σ^{+-} -labeling such that the edge of length m is a pendant. Then there exists a G -decomposition of both K_{2mk} and K_{2mk+1} for every positive integer k .*

Table 3.1 gives σ^{+-} -labelings of all forests on 7 edges except the matching. The vertex labels of each connected component with k vertices are given as a k -tuple, (v_1, \dots, v_k) corresponding to the vertices v_1, \dots, v_k positioned as shown in Figure 2.5. We leave it to the reader to infer the bipartition (A, B) .

Example 3.2.8. *A σ^{+-} -labeling of $\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$ is shown in Figure 3.4. The vertices labeled 1, 2 and 9 belong to A , and the others belong to B . The lengths of each edge are indicated on the edge.*

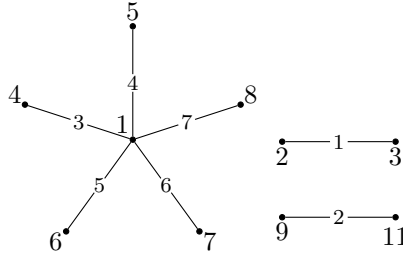


Figure 3.4: σ^{+-} -labeling of $\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$

The labelings given in Figure 3.1 along with thm 3.2.7 are enough to conclude this case.

Forest	Vertex Labels
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5, 2, 9, 7) \sqcup (3, 4)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(9, 2, 5, 1, 6, 0, 3) \sqcup (8, 7)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(9, 2, 5, 1, 6, 0, 4) \sqcup (8, 7)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(5, 1, 4, 2, 9, 6, 7) \sqcup (10, 11)$
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(3, 8, 1, 4, 2, 5, 7) \sqcup (9, 10)$
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(7, 8, 1, 6, 0, 4, 3) \sqcup (9, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(8, 1, 6, 3, 4, 5, 7) \sqcup (9, 10)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(6, 1, 5, 3, 8, 4, 7) \sqcup (9, 10)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(5, 11, 9, 10, 6, 12, 7) \sqcup (8, 1)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(4, 8, 1, 6, 0, 5, 3) \sqcup (9, 10)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(0, 6, 1, 5, 2, 9) \sqcup (11, 10, 12)$
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(3, 6, 1, 8, 4, 0) \sqcup (10, 9, 11)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(5, 11, 9, 12, 7, 10) \sqcup (1, 8, 4)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(3, 8, 4, 1, 6, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(5, 1, 8, 3, 4, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(4, 1, 8, 5, 6, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(0, 6, 1, 5, 2) \sqcup (9, 8, 10, 3)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(7, 1, 8, 5, 6) \sqcup (0, 4, 2, 3)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(7, 1, 8, 4, 6) \sqcup (10, 9, 11, 12)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(6, 0, 3, 4, 5) \sqcup (8, 7, 9, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(4, 8, 1, 7, 2) \sqcup (10, 9, 11, 12)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(6, 0, 3, 4, 5) \sqcup (8, 9, 2, 7)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(0, 6, 1, 5, 2, 9) \sqcup (8, 10) \sqcup (3, 4)$
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(3, 6, 1, 8, 4, 0) \sqcup (5, 7) \sqcup (9, 10)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(4, 1, 8, 3, 5, 7) \sqcup (0, 2) \sqcup (9, 10)$
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(5, 8, 4, 1, 6, 7) \sqcup (0, 2) \sqcup (9, 10)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(5, 11, 9, 12, 7, 10) \sqcup (8, 1) \sqcup (0, 4)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(4, 1, 8, 5, 6, 7) \sqcup (2, 3) \sqcup (9, 11)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5, 2) \sqcup (8, 10, 9) \sqcup (11, 4)$

Table 3.1: σ^{+-} -labelings for forests with seven edges

Forest	Vertex Labels
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(7, 1, 8, 5, 6) \sqcup (10, 9, 11) \sqcup (0, 4)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(6, 0, 3, 4, 5) \sqcup (1, 8, 7) \sqcup (9, 11)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (2, 9, 7, 10) \sqcup (3, 4)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, 10, 7) \sqcup (4, 0, 5, 6) \sqcup (8, 1)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (10, 9, 11, 12) \sqcup (8, 1)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(0, 6, 1, 5) \sqcup (8, 10, 9) \sqcup (11, 4, 7)$
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(4, 0, 5, 6) \sqcup (1, 8, 7) \sqcup (11, 9, 12)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (8, 10, 7) \sqcup (11, 4) \sqcup (2, 3)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (11, 9, 12) \sqcup (2, 3) \sqcup (8, 1)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(0, 6, 1, 5, 2) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(6, 1, 8, 4, 7) \sqcup (3, 5) \sqcup (9, 12) \sqcup (10, 11)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(3, 0, 4, 5, 6) \sqcup (8, 1) \sqcup (10, 11) \sqcup (9, 7)$
$3\mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1) \sqcup (4, 8, 5) \sqcup (2, 9, 7) \sqcup (10, 11)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (9, 2) \sqcup (8, 10) \sqcup (4, 7) \sqcup (11, 12)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (2, 3) \sqcup (9, 11) \sqcup (8, 1) \sqcup (10, 7)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(0, 6, 1) \sqcup (4, 8, 5) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(0, 6, 1) \sqcup (8, 4) \sqcup (2, 5) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$

Table 3.1: σ^{+-} -labelings for forests with seven edges

Theorem 3.2.9. *Let F be a forest with 7 edges. There exists an F -decomposition of K_n whenever $n \equiv 0$ or $1 \pmod{14}$.*

Proof. The proof follows from thm 3.2.7 and the labelings given in Figure 3.1. \square

Chapter 4

$n \equiv 7, 8 \pmod{14}$

In this chapter, we will use our own constructions based on edge lengths in K_n where $n \equiv 7$ or $8 \pmod{14}$. We begin by describing our construction and then later formalize these ideas in the results section.

4.1 Construction

The minimum number of vertices in a forest on seven edges is 9 (like in $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$) but K_7 and K_8 each have less vertices than that, so none of our forests could be a subgraph of either one. For this reason, K_{21} and K_{22} are the base graphs for decomposing K_n where $n \equiv 7$ and 8 , respectively, into seven edge forests.

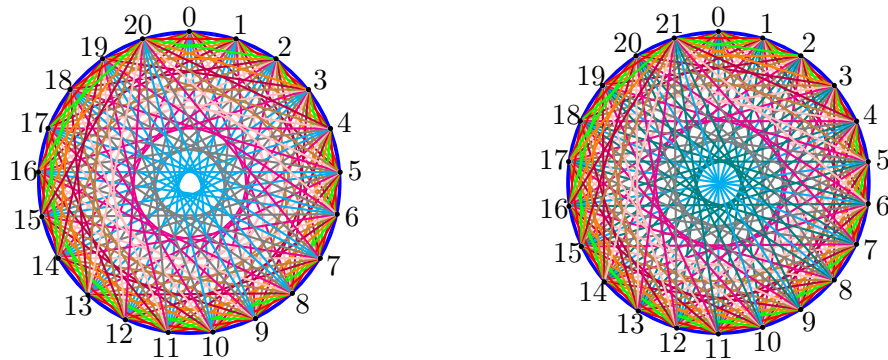


Figure 4.1: K_{21} (left) and K_{22} (right) with edges colored by length.

Here is problem with this case relative to our forests on seven edges. K_{21} has lengths 1 through 10 and K_{21} has lengths 1 through 11. So the number of distinct lengths in the base graphs is no longer 7. Furthermore, K_{22} has one more length than K_{21} . This means we cannot just fit 7 distinct lengths on a single labeling and collect all edges when we develop the vertices by 1. However, new lengths still come 7 at a time at each step $t \mapsto t + 1$: K_{14t+7} and K_{14t+8} have lengths in $\{1, \dots, 10\} \cup \dots \cup \{7t - 3, \dots, 3 + 7t\}$ and $\{1, \dots, 11\} \cup \dots \cup \{7t - 2, \dots, 4 + 7t\}$, respectively.

If we could just take care of lengths $\{1, 2, 3\}$ and $\{1, 2, 3, \infty\}$, respectively, we could just use some rosa-type labeling to collect each of the remaining lengths using some rosa-type labeling, since 7 lengths will remain. Before we think about that, recall that in K_{22} there will only be 11 edges of length 11. We want to make sure we are wary of that as well. We want the number of each length to be the same to generate them with a labeling.

So, we redefine length of edges and development for vertices in K_{22} :

$$\ell(uv) = \begin{cases} \min\{|u - v|, 21 - |u - v|\}, & u, v \neq \infty, \\ \infty, & u \text{ or } v = \infty \end{cases} \quad \text{and } v \mapsto \begin{cases} v + 1, & v \in \mathbb{Z}_{n-1}, \\ \infty, & v = \infty. \end{cases}$$

If we do this, we will have 21 edges of lengths 1 through 6 as well as ∞ , since the infinity node will have all 21 edges of length ∞ to nodes in \mathbb{Z}_{21} incident to it. So if we partition the edges by these notions of length, we can now cyclically generate all edges of K_{21} and K_{22} with labelings. Now the only problem is that there are still 10 and 11 distinct lengths in K_{21} and K_{21} , respectively.

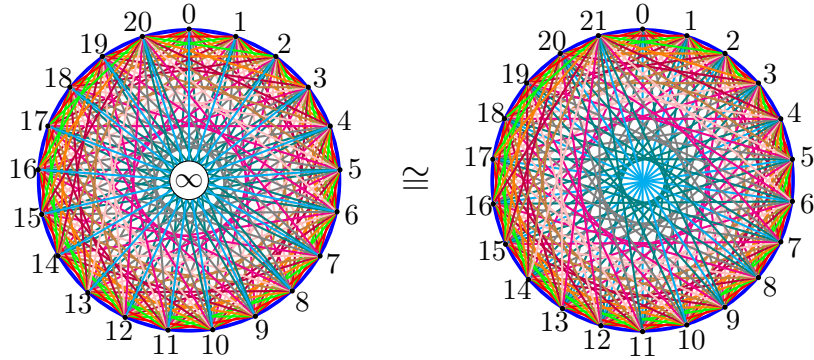


Figure 4.2: $K_{21} \vee K_1$ is isomorphic to K_{22} (right)

Now, let's begin by looking at K_{21} . How can we collect 21 edges of length 1, 2, and 3? Well, what if we have some labelings that generate 21 distinct edges of length 8, 9 and 10 between them? What if we can do something similar for K_{22} with edges of lengths 1, 2, 3, and ∞ ? By imposing a new edge label on top of the standard length ℓ , we can achieve this. We define ℓ_7^+ from $\mathbb{Z}_{21} \cup \{\infty\}$ to \mathbb{Z}_7 as follows

$$\ell_7^+(uv) = \begin{cases} u + v \bmod 14, & u, v \neq \infty \\ v, & u = \infty \end{cases}$$

Now, every edge has a standard length ℓ and an additive length ℓ_7^+ modulo 7. Previously, we partitioned the edges into sets E_i of edges of length i for each length $i \in \{1, \dots, 10\}$ via the standard length function ℓ . Now, within each set E_i of length i , we have further partitioned the edges into sets $E_{i,j}$ of standard length i and additive length j modulo 7. For example: the edge $(1, 8)$ has length $\ell((1, 8)) = 7$ and $\ell_7^+((1, 8)) = 8 + 1 \bmod 7 = 2$, so $(1, 8) \in E_{7,2}$. Here is a labeling of $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ that utilizes this new partition.

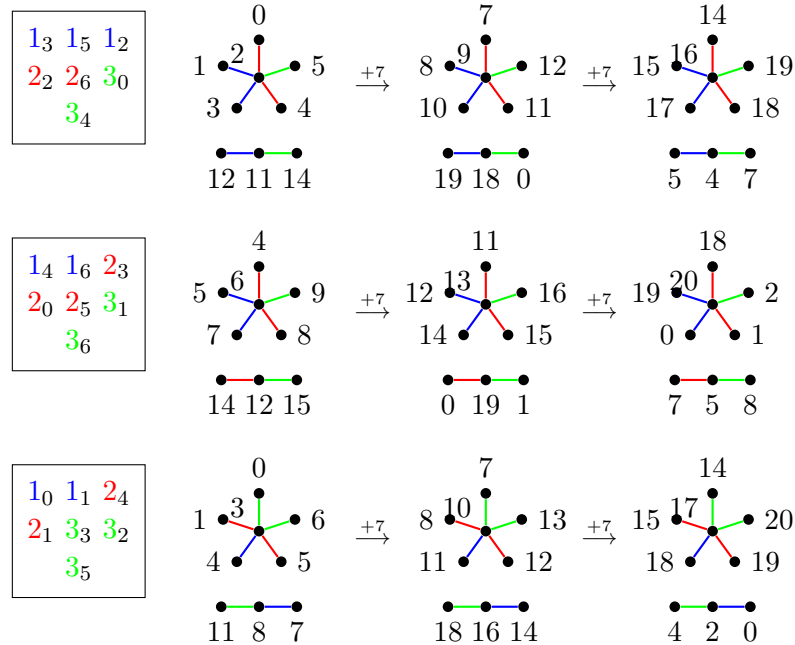


Figure 4.3: Three labelings of $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ (left column) that generate all edges of lengths 1, 2, and 3 in K_{21} when developed by 7 repeatedly.

The squares to the left of the labelings contain the standard lengths ℓ with the same colors we have been using, and then the new length ℓ_7^+ in black as the subscript, for each of the seven edges in each labeling. If you look at each edge length, you will find that there is a representative for each equivalence class via ℓ_7^+ somewhere exactly once across the three labelings.

Such K_n are of the form K_{14t+7} and K_{14t+8} where $t \geq 1$. K_7 and K_8 only have 7 and 8 distinct vertices, respectively, and since the minimum number of distinct vertices a forest on seven edges (such as $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$) has is 9 vertices, it is impossible for any seven edge forest to even be a subgraph of K_7 and K_8 . So K_{21} and K_{22} are the base graphs for K_n where $n \equiv 7$ or $8 \pmod{14}$.

4.2 Results

Definition 4.2.1. Let G be a graph with 7 edges. A (1-2-3)-labeling of $3G$ is an assignment f of the integers $\{0, \dots, 20\}$ to the vertices of $3G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

and

- 2.

$$\bigcup_{uv \in E(3G)} \{(f(u) \bmod 7, f(v) \bmod 7)\} = \bigcup_{i=0}^6 \bigcup_{j=1}^3 \{(i, i+j \bmod 7)\}.$$

Notice that the second condition of a (1-2-3)-labeling says that $3G$ contains exactly 7 edges of each of the lengths 1, 2, and 3. Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A (1-2-3) labeling of every forest with 7 edges with the exception of $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ is given in Figure 8.1. This exceptional forest does not admit such a labeling and is dealt with in Section 5.

Theorem 4.2.2. *Let G be a bipartite graph with 7 edges. If $3G$ admits a (1-2-3)-labeling and G admits a ρ^+ -labeling, then G decomposes K_{14k+7} for every $k \geq 1$.*

Proof. Let $n = 14k + 7$ and notice that K_n has $|E(K_n)| = (7k + 3)(14k + 7)$ edges, which can be partitioned into $14k + 7$ edges of each of the lengths in $\{1, 2, \dots, 7k + 3\}$. We will construct the G -decomposition in two steps. First, we use the 1-2-3-labeling to

identify all the edges of lengths 1, 2, and 3 accounting for $3(2k+1)$ copies of G . Then, we use the ρ^+ -labeling to identify edges of the remaining lengths in $7k(2k+1)$ copies of G . In total, the decomposition consists of $|E(K_n)|/7 = (7k+3)(2k+1)$ copies of G .

Let f_1 be a (1-2-3)-labeling of $3G$ and identify this graph as a block B_0 . Then develop B_0 by 7 modulo n . Since the order of the development is $\frac{n}{7} = 2k+1$ and there are 7 edges of each of the lengths 1, 2, and 3 in B_0 , we have identified $3(2k+1)$ copies of G containing all $14k+7 = n$ edges of each length 1, 2, and 3. Notice (2) of Definition 4.2.1 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \rightarrow \{0, \dots, 14\}$ be a ρ^+ -labeling of G with associated vertex partition (A, B) . For $i = 1, 2, \dots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i-1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i-3, 7i-2, \dots$, and $7i+3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i-1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \dots, 7\}$. Developing each block B_i by 1 yields $14k+7$ copies of G per block and accounts for $14k+7$ edges of each of the lengths $4, 5, \dots$, and $7k+3$.

Since we have identified

$$3(2k+1) + k(14k+7) = (7k+3)(2k+1)$$

edge-disjoint copies of G , the proof is complete. \square

To address the $n \equiv 8 \pmod{14}$ case, we define the following labeling.

Definition 4.2.3. Let G be a graph with 7 edges. A *1-rotational (1-2-3)-labeling* of $4G$ is an assignment f of $\{0, \dots, 20\} \cup \infty$ to the vertices of $4G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

and

2.

$$\bigcup_{uv \in E(4G)} \{(f(u) \bmod 7, f(v) \bmod 7)\} = \bigcup_{i=0}^6 \bigcup_{j=1}^3 \{(i, i+j \bmod 7), (i, \infty)\}.$$

Notice that the second condition of a 1-rotational (1-2-3)-labeling says that $4G$ contains exactly 7 edges of each of the lengths 1, 2, 3, and ∞ . Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A 1-rotational (1-2-3)-labeling of every forest with 7 edges with the exception of $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ is given in Figure 8.2.

Theorem 4.2.4. *Let G be a bipartite graph with 7 edges. If $4G$ admits a 1-rotational (1-2-3)-labeling and G admits a ρ^+ -labeling, then G decomposes K_{14k+8} for every $k \geq 1$.*

Proof. Let $n = 14k + 8$ and notice that K_n has $|E(K_n)| = (7k + 4)(14k + 7)$ edges, which can be partitioned into $14k + 7$ edges of each of the lengths in $\{1, 2, \dots, 7k + 3, \infty\}$. We will construct the G -decomposition in two steps. First, we use the 1-rotational (1-2-3)-labeling to identify all the edges of lengths 1, 2, 3, and ∞ accounting for $4(2k + 1)$ copies of G . Then, we use the ρ^+ -labeling to identify edges of the remaining lengths in $7k(2k + 1)$ copies of G . In total, the decomposition consists of $|E(K_n)|/7 = (7k + 4)(2k + 1)$ copies of G . Let f_1 be a 1-rotational (1-2-3)-labeling of $4G$ and identify this graph as a block B_0 . Then develop B_0 by 7 modulo $n - 1$. Since the order of the development is $\frac{n-1}{7} = 2k + 1$ and there are 7 edges of each of the lengths 1, 2, 3 and ∞ in B_0 , we have identified $4(2k + 1)$ copies of G containing all $14k + 7 = n - 1$ edges of each length 1, 2, 3 and ∞ . Notice (2) of Definition 4.2.3 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \rightarrow \{0, \dots, 14\}$ be a ρ^+ -labeling of G with associated vertex partition (A, B) . For $i = 1, 2, \dots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i - 1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i - 3, 7i - 2, \dots$, and $7i + 3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i - 1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \dots, 7\}$. Developing each block B_i by 1 yields $14k + 7$ copies of G per block and accounts for $14k + 7$ edges of each of the lengths $4, 5, \dots$, and $7k + 3$.

Since we have identified

$$4(2k + 1) + k(14k + 7) = (7k + 4)(2k + 1)$$

edge-disjoint copies of G , the proof is complete. \square

We are now able to state the main thm of this section.

Theorem 4.2.5. *Let F be a forest with 7 edges and $F \not\cong \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$. There exists an F -decomposition of K_n whenever $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$.*

Proof. If $n \equiv 7 \pmod{14}$, a (1-2-3)-labeling of $3F$ can be found in Figure 8.1. On the other hand, if $n \equiv 8 \pmod{14}$, then a 1-rotational (1-2-3)-labeling of $4F$ can be found in Figure 8.2. In either case, a ρ^+ -labeling of F can be found in Figure 3.1 (recall that a σ^{+-} -labeling is a ρ^+ -labeling). The result now follows from Theorems 4.2.2 and 4.2.4. \square

Example 4.2.6. *We illustrate the constructions in the previous two theorems by finding an F -decomposition of K_{35} and K_{36} for the forest graph $F \cong \mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$.*

Here are excerpts from the preceding tables for $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$

Labeling Type	Labelings
σ^{+-}	$(4, 1, 8, 5, 6, 7) \sqcup (10, 9, 11)$
(1-2-3)	$(0, 2, 1, 3, 4, 5) \sqcup (12, 11, 14)$ $(4, 6, 8, 9, 5, 7) \sqcup (14, 12, 15)$ $(0, 3, 1, 4, 5, 6) \sqcup (11, 8, 7)$
1-rotational (1-2-3)	$(1, 2, 0, 3, 4, 5) \sqcup (11, 8, \infty)$ $(2, \infty, 3, 4, 5, 6) \sqcup (12, 13, 15)$ $(6, 7, 8, 4, 5, \infty) \sqcup (11, 12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (9, 6, 4)$

Figure 4.4: Labelings for $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$

The ρ^+ labelings obtained by stretching the σ^{+-} labeling are bottommost labelings in the following generating presentations and are developed by 1.

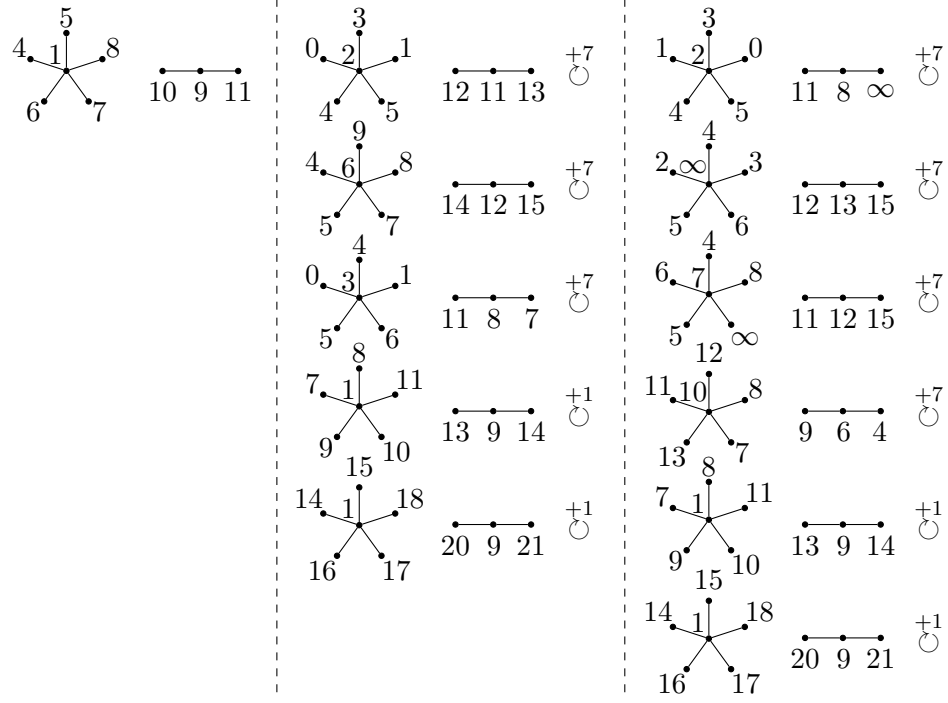


Figure 4.5: A σ^{+-} -labeling of $F \cong \mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ (left) and generating presentations for the F -decomposition of K_n where $n = 35$ (middle) and $n = 36$ (right)

Chapter 5

$\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$

To begin this case, we construct K_n for $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$ using *joined* copies of K_{22} , K_{21} , and K_{14} . Recall, the *join* of two graphs G and H is the graph obtained by adding an edge between every pair of vertices $(u, v) \in V(G) \times V(H)$.

Let t be a positive integer. Now join $t - 1$ copies of K_{14} with each other and a lone copy of K_{21} . The resulting graph is $K_{14(t-1)+21} \cong K_{14t+7}$. So we can think of K_{14t+7} as simply K_t whose t “vertices” consist of $t - 1$ copies of K_{14} and 1 copy of K_{21} and whose edges are the join between all of them. From now on, we will refer to these “vertices” as nodes. Similarly, K_{14t+8} can be constructed as K_t whose nodes are $t - 1$ copies of K_{14} and 1 copy of K_{22} and whose edges are the join between all of them.

We show that $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes K_n for $n \equiv 7$ or $8 \pmod{14}$ by proving that K_{22} , K_{21} , K_{14} , $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ can each be decomposed by $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$. Notice that these six graphs make up the nodes and edges of the K_t representations of K_{14t+7} and K_{14t+8} stated in the constructions above where t is a positive integer.

The proof of the next theorem was obtained by manipulating a $K_{1,7}$ -decomposition of K_{22} by Cain in [1].

Theorem 5.0.1. $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes K_{21} and K_{22} .

Proof. Figures 8 and 9 give $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decompositions of K_{21} and K_{22} , respectively. \square

Theorem 5.0.2. $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes $K_{n,7}$ for all $n \geq 2$.

Proof. Consider $K_{n,7}$ where $n \geq 2$. Take the partite set of n vertices to be \mathbb{Z}_n and color them white. Similarly, take the partite set of 7 vertices to be K_7 and color them black. Naturally we refer to *white-black* vertices uv in $K_{n,7}$ via $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_7$ and vice versa. Finally, let $E_i = \{(i, 0)\} \sqcup (\{i+1\} \times \{1, \dots, 6\})$ and $G_i \subset K_{n,7}$ be the subgraph induced by E_i for each $i \in \mathbb{Z}_n$. Note that $G_i \cong \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ for all $i \in \mathbb{Z}_n$.

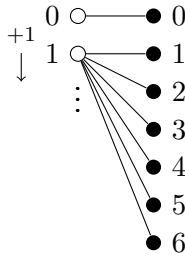


Figure 5.1: G_0 in a generating presentation of the $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of $K_{n,7}$.

Notice that $E_i \cap E_j = \emptyset$ if $i \neq j$, so by definition all distinct G_i 's are pairwise edge disjoint. Lastly,

$$\bigcup_{i \in \mathbb{Z}_n} E_i = [\bigcup_{i \in \mathbb{Z}_n} \{(i, 0)\}] \sqcup [\bigcup_{i \in \mathbb{Z}_n} (\{i+1\} \times \{1, \dots, 6\})] = [\mathbb{Z}_n \times \{0\}] \sqcup [\mathbb{Z}_n \times \{1, \dots, 6\}] = \mathbb{Z}_n \times \mathbb{Z}_7$$

So $G_0 \sqcup \dots \sqcup G_{n-1} = K_{n,7}$ and $\{G_i \mid i \in \mathbb{Z}_n\}$ is a $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of $K_{n,7}$. Furthermore, it is generated by developing the white nodes of G_0 by 1. \square

Corollary 5.0.3. $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$.

Proof. $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes $K_{7,7}$ and $K_{8,7}$ by Theorem 5.0.2. $K_{14,14}$ can be expressed as the edge-disjoint union of four copies of $K_{7,7}$, $K_{21,14}$ can be expressed as the edge-disjoint union of six copies of $K_{7,7}$, and $K_{22,14}$ can be expressed as the edge-disjoint union of two copies of $K_{8,7}$ and four copies of $K_{7,7}$. Therefore, $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes them all. \square

Theorem 5.0.4. $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes K_{14t+7} and K_{14t+8} where t is a positive integer.

Proof. $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes K_{14} by Theorem 3.2.7, $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ by Corollary 5.0.3, and lastly K_{22} , K_{21} by Theorem 5.0.1.

Therefore, $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes the join of $(t - 1)$ copies of K_{14} with each other and 1 copy of K_{21} , which is isomorphic to K_{14t+7} . Similarly $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ decomposes the join of $(t - 1)$ copies of K_{14} with each other and 1 copy of K_{22} which is isomorphic to K_{14t+8} . \square

Chapter 6

Additional Results

We present two main additional results produced by work done on this project. (1) Wraparound edge mappings that preserve length and (2) Galaxy graph decompositions of multipartite graphs.

6.1 Wraparound edge mappings that preserve length

Initially when I found labelings to take care of the $n \equiv 7, 8 \pmod{14}$ cases, we used lengths 8, 9, 10, and I mistakenly did not have all edges be short edges (non-wraparound edges). This meant they didn't generalize via the permutation $v \mapsto v + 1$ for K_{14t+7} and K_{14t+8} where $t > 1$. In an effort to not have to find 90 completely new labelings (I did later have to anyways), I tried to find a new way to map wraparound edges in my labelings. I was successful but it was very messy, and this caused us to pivot to using lengths 1, 2, 3 since a larger percent of edges with those lengths are short edges.

Note that this is very preliminary. In order to implement these mappings successfully, one has to ensure that new vertices won't collide with currently existing vertices elsewhere in the labelings. Because there was a pivot, this was not investigated and perhaps remains an open problem as a result of this project.

Chapter 7

Conclusion and Discussion

Chapter 8

Appendix

Forest	Labeling
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 1, 2, 4, 6, 9, 12) \sqcup (13, 14)$ $(3, 4, 7, 9, 10, 13, 15) \sqcup (8, 5)$ $(8, 11, 12, 10, 7, 5, 6) \sqcup (1, 3)$ $(0, 4, 9, 15, 8, 16, 7) \sqcup (1, 11)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(12, 9, 6, 4, 2, 1, 7) \sqcup (14, 15)$ $(15, 13, 10, 9, 7, 4, 11) \sqcup (8, 5)$ $(8, 11, 12, 10, 7, 5, 13) \sqcup (1, 3)$ $(16, 8, 15, 9, 4, 0, 6) \sqcup (1, 11)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(0, 1, 2, 4, 6, 9, 3) \sqcup (16, 19)$ $(15, 13, 10, 9, 7, 4, 14) \sqcup (17, 18)$ $(6, 5, 7, 10, 12, 11, 8) \sqcup (18, 15)$ $(7, 16, 8, 15, 9, 4, 12) \sqcup (1, 11)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(8, 6, 4, 2, 1, 9, 7) \sqcup (14, 15)$ $(8, 10, 9, 7, 4, 11, 13) \sqcup (12, 15)$ $(9, 12, 10, 7, 5, 11, 13) \sqcup (1, 4)$ $(7, 15, 9, 4, 0, 8, 6) \sqcup (1, 11)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12, 8, 7) \sqcup (11, 14)$ $(0, 2, 3, 6, 5, 1, 4) \sqcup (8, 7)$ $(0, 3, 5, 4, 1, 8, 7) \sqcup (16, 15)$ $(4, 9, 15, 8, 12, 6, 7) \sqcup (1, 11)$
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 6, 8, 5, 9) \sqcup (12, 15)$ $(4, 7, 9, 10, 11, 8, 13) \sqcup (1, 3)$ $(5, 7, 10, 12, 11, 6, 13) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12, 6) \sqcup (1, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(8, 6, 4, 2, 5, 9, 7) \sqcup (12, 14)$ $(1, 3, 2, 0, 5, 4, 6) \sqcup (10, 12)$ $(9, 8, 7, 10, 4, 11, 5) \sqcup (12, 13)$ $(7, 15, 9, 4, 13, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(7, 6, 4, 2, 8, 9, 5) \sqcup (12, 14)$ $(2, 3, 4, 7, 0, 5, 6) \sqcup (9, 12)$ $(7, 8, 5, 4, 9, 10, 11) \sqcup (0, 2)$ $(6, 15, 9, 4, 8, 11, 7) \sqcup (2, 12)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 8, 7, 9, 12) \sqcup (13, 14)$ $(0, 2, 3, 4, 7, 6, 5) \sqcup (8, 10)$ $(0, 3, 5, 8, 9, 4, 1) \sqcup (12, 14)$ $(4, 9, 15, 8, 12, 7, 16) \sqcup (1, 11)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12, 1, 8) \sqcup (14, 15)$ $(5, 6, 3, 2, 0, 7, 4) \sqcup (8, 9)$ $(0, 3, 5, 4, 7, 1, 8) \sqcup (12, 14)$ $(4, 9, 15, 8, 12, 18, 7) \sqcup (1, 11)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 6, 9, 12) \sqcup (13, 14, 15)$ $(3, 4, 7, 9, 10, 13) \sqcup (5, 8, 6)$ $(11, 12, 10, 7, 5, 6) \sqcup (3, 1, 4)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11, 2)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14, 15)$ $(13, 10, 9, 7, 4, 11) \sqcup (5, 8, 6)$ $(11, 12, 10, 7, 5, 13) \sqcup (3, 1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(0, 1, 2, 4, 6, 5) \sqcup (16, 13, 14)$ $(8, 6, 3, 2, 0, 4) \sqcup (14, 12, 15)$ $(7, 4, 5, 3, 0, 6) \sqcup (10, 8, 11)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(1, 2, 5, 4, 6, 7) \sqcup (16, 14, 13)$ $(8, 6, 9, 3, 2, 4) \sqcup (14, 12, 15)$ $(4, 5, 6, 3, 0, 1) \sqcup (11, 8, 7)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(0, 2, 4, 7, 1, 5) \sqcup (12, 11, 13)$ $(7, 6, 3, 2, 8, 9) \sqcup (14, 12, 15)$ $(4, 3, 5, 6, 0, 1) \sqcup (11, 8, 7)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(0, 2, 1, 3, 4, 5) \sqcup (12, 11, 14)$ $(4, 6, 8, 9, 5, 7) \sqcup (14, 12, 15)$ $(0, 3, 1, 4, 5, 6) \sqcup (11, 8, 7)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(2, 4, 6, 9, 12) \sqcup (16, 15, 14, 13)$ $(3, 4, 7, 9, 10) \sqcup (11, 12, 15, 13)$ $(12, 10, 7, 5, 6) \sqcup (18, 15, 17, 20)$ $(4, 9, 15, 8, 16) \sqcup (2, 11, 1, 5)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(12, 9, 6, 4, 11) \sqcup (17, 16, 15, 14)$ $(9, 7, 4, 3, 6) \sqcup (11, 12, 15, 13)$ $(6, 5, 7, 10, 3) \sqcup (18, 15, 17, 20)$ $(16, 8, 15, 9, 12) \sqcup (2, 11, 1, 6)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(4, 6, 9, 11, 8) \sqcup (16, 15, 18, 14)$ $(9, 7, 4, 3, 6) \sqcup (16, 17, 20, 15)$ $(6, 5, 7, 10, 3) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 6, 7)$ $(14, 17, 16, 20, 15) \sqcup (9, 7, 4, 3)$ $(9, 12, 10, 11, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(7, 6, 9, 11, 8) \sqcup (16, 15, 13, 14)$ $(9, 7, 4, 3, 5) \sqcup (16, 17, 20, 15)$ $(4, 6, 5, 7, 10) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 12, 6)$ $(18, 17, 16, 20, 15) \sqcup (9, 7, 10, 4)$ $(10, 12, 11, 14, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 4, 6, 9, 12) \sqcup (13, 14) \sqcup (8, 7)$ $(3, 4, 7, 9, 10, 13) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 6) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14) \sqcup (8, 7)$ $(13, 10, 9, 7, 4, 11) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 13) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(0, 1, 2, 4, 7, 5) \sqcup (9, 6) \sqcup (8, 10)$ $(8, 6, 3, 2, 0, 4) \sqcup (5, 7) \sqcup (12, 13)$ $(6, 4, 5, 3, 0, 8) \sqcup (13, 14) \sqcup (18, 15)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11) \sqcup (5, 14)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 5, 4, 6, 7) \sqcup (13, 14) \sqcup (12, 15)$ $(8, 6, 9, 3, 2, 4) \sqcup (12, 14) \sqcup (18, 15)$ $(4, 5, 6, 3, 0, 1) \sqcup (8, 7) \sqcup (16, 14)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(0, 2, 4, 7, 1, 5) \sqcup (11, 13) \sqcup (12, 15)$ $(7, 6, 3, 2, 8, 9) \sqcup (11, 12) \sqcup (1, 4)$ $(4, 3, 5, 6, 0, 1) \sqcup (8, 7) \sqcup (12, 14)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(0, 2, 1, 3, 4, 5) \sqcup (12, 14) \sqcup (18, 19)$ $(4, 6, 8, 9, 5, 7) \sqcup (12, 15) \sqcup (11, 14)$ $(0, 3, 1, 4, 5, 6) \sqcup (8, 11) \sqcup (14, 15)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12) \sqcup (13, 14, 15) \sqcup (18, 19)$ $(3, 4, 7, 9, 10) \sqcup (12, 15, 13) \sqcup (1, 2)$ $(12, 10, 7, 5, 6) \sqcup (20, 17, 15) \sqcup (1, 4)$ $(4, 9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(12, 9, 6, 4, 11) \sqcup (17, 16, 15) \sqcup (0, 1)$ $(9, 7, 4, 3, 6) \sqcup (12, 15, 13) \sqcup (18, 19)$ $(6, 5, 7, 10, 3) \sqcup (20, 17, 15) \sqcup (1, 4)$ $(16, 8, 15, 9, 12) \sqcup (1, 11, 2) \sqcup (0, 5)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(13, 15, 16, 18, 14) \sqcup (9, 6, 7) \sqcup (2, 4)$ $(14, 17, 16, 20, 15) \sqcup (3, 4, 7) \sqcup (11, 13)$ $(9, 12, 10, 11, 15) \sqcup (6, 5, 7) \sqcup (0, 2)$ $(5, 1, 10, 11, 6) \sqcup (8, 15, 9) \sqcup (4, 12)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(4, 6, 9, 12) \sqcup (16, 15, 14, 13) \sqcup (19, 20)$ $(9, 7, 4, 3) \sqcup (11, 12, 15, 13) \sqcup (16, 17)$ $(12, 10, 7, 5) \sqcup (18, 15, 17, 20) \sqcup (9, 11)$ $(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, 6, 7) \sqcup (16, 15, 13, 14) \sqcup (1, 4)$ $(5, 3, 4, 7) \sqcup (16, 17, 20, 15) \sqcup (0, 2)$ $(4, 6, 5, 7) \sqcup (9, 12, 11, 15) \sqcup (0, 3)$ $(16, 8, 15, 9) \sqcup (10, 1, 11, 6) \sqcup (0, 4)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(18, 15, 13, 14) \sqcup (11, 9, 12, 6) \sqcup (1, 2)$ $(18, 17, 20, 15) \sqcup (9, 7, 10, 4) \sqcup (2, 3)$ $(11, 12, 14, 15) \sqcup (4, 6, 5, 7) \sqcup (17, 19)$ $(11, 1, 5, 6) \sqcup (16, 8, 14, 15) \sqcup (0, 9)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(16, 15, 14, 13) \sqcup (0, 3, 5) \sqcup (12, 9, 6)$ $(11, 12, 15, 13) \sqcup (10, 9, 7) \sqcup (16, 18, 20)$ $(18, 15, 17, 20) \sqcup (10, 11, 14) \sqcup (6, 5, 7)$ $(2, 12, 3, 11) \sqcup (8, 1, 7) \sqcup (4, 0, 5)$
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(11, 9, 12, 6) \sqcup (18, 15, 13) \sqcup (0, 1, 2)$ $(9, 7, 10, 4) \sqcup (18, 17, 20) \sqcup (1, 3, 2)$ $(11, 12, 14, 15) \sqcup (4, 6, 7) \sqcup (17, 19, 20)$ $(16, 8, 14, 15) \sqcup (11, 1, 6) \sqcup (9, 0, 4)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(8, 6, 9, 11) \sqcup (0, 1, 2) \sqcup (16, 19) \sqcup (18, 15)$ $(8, 10, 7, 9) \sqcup (18, 17, 20) \sqcup (11, 14) \sqcup (2, 3)$ $(13, 11, 12, 14) \sqcup (17, 19, 20) \sqcup (6, 7) \sqcup (8, 5)$ $(0, 5, 1, 7) \sqcup (3, 10, 2) \sqcup (4, 13) \sqcup (16, 6)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(11, 9, 12, 6) \sqcup (0, 1, 2) \sqcup (18, 15) \sqcup (13, 14)$ $(9, 7, 10, 4) \sqcup (18, 17, 20) \sqcup (11, 13) \sqcup (2, 3)$ $(11, 12, 14, 15) \sqcup (17, 19, 20) \sqcup (8, 6) \sqcup (1, 3)$ $(4, 0, 5, 6) \sqcup (8, 1, 9) \sqcup (3, 12) \sqcup (17, 7)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(2, 4, 6, 9, 12) \sqcup (13, 14) \sqcup (18, 19) \sqcup (0, 1)$ $(3, 4, 7, 9, 10) \sqcup (13, 15) \sqcup (1, 2) \sqcup (8, 5)$ $(6, 5, 7, 10, 12) \sqcup (17, 20) \sqcup (8, 11) \sqcup (1, 3)$ $(4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12) \sqcup (2, 6)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(11, 9, 6, 4, 12) \sqcup (16, 15) \sqcup (8, 10) \sqcup (2, 3)$ $(6, 7, 4, 3, 9) \sqcup (13, 15) \sqcup (18, 19) \sqcup (8, 5)$ $(3, 5, 7, 10, 6) \sqcup (17, 20) \sqcup (8, 11) \sqcup (0, 1)$ $(12, 8, 15, 9, 16) \sqcup (2, 11) \sqcup (0, 5) \sqcup (3, 13)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(13, 15, 16, 18, 14) \sqcup (9, 6) \sqcup (2, 4) \sqcup (5, 7)$ $(14, 17, 16, 20, 15) \sqcup (4, 7) \sqcup (11, 13) \sqcup (5, 6)$ $(9, 12, 10, 11, 15) \sqcup (6, 7) \sqcup (0, 2) \sqcup (3, 4)$ $(5, 1, 10, 11, 6) \sqcup (9, 15) \sqcup (4, 12) \sqcup (0, 7)$
$3\mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(18, 15, 13) \sqcup (11, 9, 6) \sqcup (0, 1, 2) \sqcup (16, 19)$ $(18, 17, 20) \sqcup (9, 7, 10) \sqcup (1, 3, 2) \sqcup (11, 14)$ $(11, 12, 14) \sqcup (4, 6, 7) \sqcup (17, 19, 20) \sqcup (8, 5)$ $(11, 1, 6) \sqcup (16, 8, 14) \sqcup (9, 0, 4) \sqcup (10, 3)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(9, 6, 4, 2) \sqcup (13, 14) \sqcup (18, 19) \sqcup (0, 1) \sqcup (10, 12)$ $(9, 7, 4, 3) \sqcup (13, 15) \sqcup (1, 2) \sqcup (8, 5) \sqcup (16, 17)$ $(10, 7, 5, 6) \sqcup (17, 20) \sqcup (8, 11) \sqcup (1, 3) \sqcup (9, 12)$ $(9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12) \sqcup (2, 6) \sqcup (0, 5)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(16, 15, 18, 13) \sqcup (9, 6) \sqcup (2, 4) \sqcup (5, 7) \sqcup (0, 1)$ $(16, 17, 20, 14) \sqcup (4, 7) \sqcup (11, 13) \sqcup (5, 6) \sqcup (1, 3)$ $(9, 12, 10, 11) \sqcup (6, 7) \sqcup (0, 2) \sqcup (3, 4) \sqcup (8, 5)$ $(10, 1, 11, 5) \sqcup (9, 15) \sqcup (4, 12) \sqcup (0, 7) \sqcup (8, 3)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(11, 9, 6) \sqcup (0, 1, 2) \sqcup (18, 15) \sqcup (16, 19) \sqcup (17, 20)$ $(9, 7, 10) \sqcup (1, 3, 2) \sqcup (17, 18) \sqcup (11, 14) \sqcup (8, 5)$ $(11, 12, 14) \sqcup (4, 6, 7) \sqcup (19, 20) \sqcup (13, 15) \sqcup (3, 5)$ $(11, 1, 6) \sqcup (16, 8, 14) \sqcup (0, 9) \sqcup (10, 3) \sqcup (17, 13)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(0, 1, 2) \sqcup (18, 15) \sqcup (9, 11) \sqcup (16, 19) \sqcup (5, 6) \sqcup (10, 7)$ $(1, 3, 2) \sqcup (17, 18) \sqcup (9, 7) \sqcup (11, 14) \sqcup (8, 5) \sqcup (16, 13)$ $(4, 6, 7) \sqcup (12, 14) \sqcup (3, 5) \sqcup (13, 15) \sqcup (17, 20) \sqcup (18, 19)$ $(16, 8, 14) \sqcup (1, 11) \sqcup (0, 9) \sqcup (10, 3) \sqcup (17, 13) \sqcup (2, 7)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 1, \infty, 2, 4, 5, 3) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4, 1) \sqcup (10, 11)$ $(5, 7, \infty, 3, 6, 9, 10) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 6, 5) \sqcup (16, 15)$ $(0, 4, 9, 15, 8, 16, 7) \sqcup (1, 11)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 8, 1) \sqcup (12, 15)$ $(4, 6, \infty, 5, 2, 0, 18) \sqcup (10, 11)$ $(10, 9, 6, 3, \infty, 0, 7) \sqcup (12, 14)$ $(5, 6, 8, 10, 7, 4, 9) \sqcup (0, 1)$ $(16, 8, 15, 9, 4, 0, 6) \sqcup (1, 11)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 1, 6) \sqcup (9, 10)$ $(0, 2, 5, \infty, 6, 4, 1) \sqcup (10, 11)$ $(5, 7, \infty, 3, 6, 9, 8) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 6, 1) \sqcup (12, 15)$ $(7, 16, 8, 15, 9, 4, 12) \sqcup (1, 11)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 5, 8, 0, \infty) \sqcup (11, 13)$ $(4, \infty, 5, 2, 3, 8, 6) \sqcup (16, 13)$ $(6, 7, \infty, 10, 13, 8, 5) \sqcup (19, 20)$ $(11, 10, 7, 4, 1, 8, 12) \sqcup (13, 15)$ $(7, 15, 9, 4, 0, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 3, 6, 0, 1) \sqcup (9, \infty)$ $(2, 5, \infty, 6, 4, 8, 11) \sqcup (16, 13)$ $(10, \infty, 7, 8, 11, 5, 6) \sqcup (12, 13)$ $(4, 7, 10, 8, 5, 11, 12) \sqcup (13, 15)$ $(4, 9, 15, 8, 12, 6, 7) \sqcup (1, 11)$
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(8, 5, 4, 2, 0, 6, \infty) \sqcup (11, 13)$ $(3, 2, 5, \infty, 8, 1, 6) \sqcup (16, 13)$ $(5, 7, \infty, 3, 4, 8, 6) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 1, 12) \sqcup (13, 15)$ $(0, 4, 9, 15, 8, 12, 6) \sqcup (1, 11)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 5, 7, 0, 3) \sqcup (8, 11)$ $(11, \infty, 6, 4, 5, 8, 12) \sqcup (10, 13)$ $(6, 7, \infty, 10, 2, 8, 5) \sqcup (9, 12)$ $(11, 10, 8, 5, 6, 12, 7) \sqcup (16, 13)$ $(7, 15, 9, 4, 13, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 6, 0, 3, 5) \sqcup (8, 11)$ $(11, \infty, 6, 5, 8, 2, 12) \sqcup (13, 15)$ $(6, 7, \infty, 10, 8, 4, 5) \sqcup (11, 12)$ $(11, 10, 8, 5, 12, 13, 7) \sqcup (9, 6)$ $(6, 15, 9, 4, 8, 11, 7) \sqcup (2, 12)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 0, 1, 3, 6) \sqcup (9, \infty)$ $(4, 6, \infty, 1, 2, 12, 13) \sqcup (8, 11)$ $(10, \infty, 7, 5, 3, 6, 9) \sqcup (13, 15)$ $(5, 8, 10, 11, \infty, 7, 4) \sqcup (9, 12)$ $(4, 9, 15, 8, 12, 7, 16) \sqcup (1, 11)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 3, 6, \infty, 0) \sqcup (8, 7)$ $(13, 12, \infty, 6, 4, 10, 1) \sqcup (8, 11)$ $(10, \infty, 7, 6, 9, 2, 5) \sqcup (13, 15)$ $(5, 8, 10, 7, 4, 9, 11) \sqcup (16, 19)$ $(4, 9, 15, 8, 12, 18, 7) \sqcup (1, 11)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(3, 5, 4, 2, \infty, 1) \sqcup (13, 12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (8, 11, 10)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14, 15)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (17, 16, 15)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11, 2)$
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(\infty, 2, 4, 5, 8, 0) \sqcup (11, 13, 12)$ $(6, \infty, 5, 2, 3, 8) \sqcup (13, 16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14, 15)$ $(8, 10, 7, 4, \infty, 12) \sqcup (18, 15, 13)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11, 2)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(5, 4, 2, 3, 6, 0) \sqcup (9, \infty, 11)$ $(4, 6, \infty, 12, 13, 1) \sqcup (11, 8, 7)$ $(10, \infty, 7, 6, 9, 5) \sqcup (16, 15, 13)$ $(5, 8, 10, 7, 4, 11) \sqcup (16, 19, 17)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(5, 4, 7, 2, 1, 3) \sqcup (8, 11, \infty)$ $(12, \infty, 8, 6, 4, 5) \sqcup (13, 10, 7)$ $(10, \infty, 2, 7, 8, 5) \sqcup (19, 16, 14)$ $(11, 10, 12, 8, 5, 6) \sqcup (16, 13, 14)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 5, 0, 3) \sqcup (8, 11, 14)$ $(11, \infty, 6, 4, 8, 5) \sqcup (10, 13, 12)$ $(6, 7, \infty, 3, 8, 5) \sqcup (9, 12, 15)$ $(11, 10, 8, 6, 12, 7) \sqcup (13, 16, \infty)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(1, 2, 0, 3, 4, 5) \sqcup (11, 8, \infty)$ $(2, \infty, 3, 4, 5, 6) \sqcup (12, 13, 15)$ $(6, 7, 8, 4, 5, \infty) \sqcup (11, 12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (9, 6, 4)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(5, 4, 2, \infty, 1) \sqcup (11, 13, 12, 15)$ $(0, 2, 5, \infty, 6) \sqcup (8, 11, 10, 12)$ $(5, 7, \infty, 3, 6) \sqcup (16, 13, 14, 15)$ $(\infty, 4, 7, 10, 8) \sqcup (17, 16, 15, 13)$ $(4, 9, 15, 8, 16) \sqcup (2, 11, 1, 5)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(\infty, 2, 4, 5, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (8, 11, 10, 12)$ $(6, 3, \infty, 7, 1) \sqcup (16, 13, 14, 15)$ $(10, 7, 4, \infty, 5) \sqcup (17, 16, 15, 13)$ $(16, 8, 15, 9, 12) \sqcup (2, 11, 1, 6)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(\infty, 2, 4, 3, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (10, 12, 11, 15)$ $(6, 3, \infty, 7, 1) \sqcup (12, 14, 13, 15)$ $(\infty, 4, 7, 10, 1) \sqcup (17, 16, 13, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(0, 2, 1, 3, 4) \sqcup (11, 8, \infty, 6)$ $(2, \infty, 3, 4, 5) \sqcup (9, 12, 13, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15, 14)$ $(0, 3, 1, 5, 6) \sqcup (16, 13, 11, 10)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(10, 13, \infty, 8, 11) \sqcup (1, 2, 3, 4)$ $(15, 13, 12, 9, 7) \sqcup (3, \infty, 4, 5)$ $(11, 12, 15, 14, 13) \sqcup (4, 7, 5, \infty)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12, 7)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(0, 2, 3, 4, 5) \sqcup (9, 8, 11, \infty)$ $(2, \infty, 3, 4, 5) \sqcup (12, 13, 14, 15)$ $(4, 7, 8, 5, \infty) \sqcup (10, 12, 11, 15)$ $(0, 3, 1, 4, 6) \sqcup (16, 13, 11, \infty)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 1) \sqcup (19, 20) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (17, 18) \sqcup (8, 11)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14) \sqcup (0, 1)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (16, 15) \sqcup (2, 3)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(\infty, 2, 4, 5, 8, 0) \sqcup (18, 20) \sqcup (12, 13)$ $(13, \infty, 5, 2, 3, 8) \sqcup (9, 6) \sqcup (16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14) \sqcup (0, 1)$ $(15, 17, 14, 11, \infty, 19) \sqcup (8, 6) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(3, 2, 4, 5, 0, 1) \sqcup (18, 15) \sqcup (11, 14)$ $(5, \infty, 6, 4, 8, 11) \sqcup (10, 13) \sqcup (19, 20)$ $(8, 7, \infty, 3, 5, 6) \sqcup (16, 19) \sqcup (12, 15)$ $(7, 10, 8, 6, 11, 12) \sqcup (16, 13) \sqcup (9, \infty)$ $(6, 0, 8, 4, 5, 7) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(5, 4, 7, 2, 1, 3) \sqcup (8, 11) \sqcup (18, \infty)$ $(12, \infty, 8, 6, 4, 5) \sqcup (0, 3) \sqcup (10, 13)$ $(10, \infty, 2, 7, 8, 5) \sqcup (9, 6) \sqcup (16, 19)$ $(11, 10, 12, 8, 5, 6) \sqcup (13, 14) \sqcup (0, 2)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(5, 4, 2, 3, 6, 0) \sqcup (9, 12) \sqcup (11, \infty)$ $(4, 6, \infty, 12, 13, 15) \sqcup (0, 1) \sqcup (8, 11)$ $(10, \infty, 7, 6, 9, 5) \sqcup (13, 15) \sqcup (1, 2)$ $(5, 8, 10, 7, 4, 11) \sqcup (17, 19) \sqcup (9, \infty)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 0, 3, 4, 5) \sqcup (\infty, 15) \sqcup (8, 11)$ $(11, \infty, 2, 3, 5, 6) \sqcup (13, 15) \sqcup (19, 20)$ $(6, 7, 8, 4, 5, \infty) \sqcup (18, 19) \sqcup (12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (18, 20) \sqcup (9, 6)$ $(11, 1, 8, 9, 10, 7) \sqcup (0, 5) \sqcup (2, 6)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(10, 13, \infty, 8, 11) \sqcup (3, 2, 4) \sqcup (16, 15)$ $(15, 13, 12, 9, 7) \sqcup (10, \infty, 5) \sqcup (11, 14)$ $(11, 12, 15, 14, 13) \sqcup (4, \infty, 7) \sqcup (0, 3)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12) \sqcup (5, 7)$ $(0, 9, 1, 8, 2) \sqcup (5, 10, 6) \sqcup (3, 13)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(8, \infty, 13, 10, 9) \sqcup (3, 2, 4) \sqcup (14, 15)$ $(7, 9, 12, 13, 8) \sqcup (10, \infty, 5) \sqcup (11, 14)$ $(11, 12, 15, 18, 14) \sqcup (4, \infty, 7) \sqcup (0, 3)$ $(9, 6, 4, 3, 8) \sqcup (19, 17, 15) \sqcup (13, 14)$ $(1, 8, 0, 9, 2) \sqcup (5, 10, 6) \sqcup (3, 13)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(2, \infty, 3, 4, 5) \sqcup (12, 13, 15) \sqcup (16, 19)$ $(0, 2, 1, 3, 4) \sqcup (8, \infty, 6) \sqcup (18, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15) \sqcup (0, 1)$ $(8, 10, 12, 13, 7) \sqcup (9, 6, 4) \sqcup (17, 18)$ $(9, 0, 8, 6, 7) \sqcup (11, 1, 5) \sqcup (10, 15)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(1, \infty, 16, 18) \sqcup (11, 13, 12, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 12) \sqcup (9, 7)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14, 15) \sqcup (5, 7)$ $(10, 7, 4, \infty) \sqcup (17, 16, 15, 13) \sqcup (1, 3)$ $(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, \infty, 1) \sqcup (10, 12, 13, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 13) \sqcup (9, 7)$ $(0, \infty, 17, 20) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(10, 7, 4, \infty) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(2, 12, 6, 15) \sqcup (8, 0, 5, 7) \sqcup (9, 13)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(18, 16, 19, \infty) \sqcup (10, 12, 13, 15) \sqcup (3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 10, 13) \sqcup (4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(9, 7, 10, 4) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(9, 0, 8, 7) \sqcup (11, 1, 5, 6) \sqcup (10, 4)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$ $(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$ $(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (5, 3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (3, 4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13) \sqcup (6, 8, 7)$ $(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (2, 1, 3)$ $(9, 0, 8, 7) \sqcup (5, 1, 6) \sqcup (10, 4, 14)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (4, 5) \sqcup (16, 18)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (2, 5) \sqcup (16, 14)$ $(8, 10, 7, 4) \sqcup (0, \infty, 11) \sqcup (16, 17) \sqcup (9, 6)$ $(5, 7, 8, 6) \sqcup (20, 17, \infty) \sqcup (13, 14) \sqcup (1, 2)$ $(3, 10, 5, 11) \sqcup (0, 9, 1) \sqcup (2, 12) \sqcup (17, 13)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (3, 5) \sqcup (17, 20)$ $(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (4, 5) \sqcup (17, 18)$ $(3, \infty, 4, 7) \sqcup (12, 14, 13) \sqcup (8, 6) \sqcup (1, 2)$ $(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (1, 3) \sqcup (14, 15)$ $(9, 0, 8, 7) \sqcup (11, 1, 6) \sqcup (18, 12) \sqcup (10, 14)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(4, 1, \infty, 13, 10) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(5, \infty, 10, 11, 13) \sqcup (4, 7) \sqcup (0, 2) \sqcup (9, 12)$ $(7, \infty, 4, 5, 8) \sqcup (17, 19) \sqcup (0, 3) \sqcup (12, 14)$ $(7, 8, 6, 9, \infty) \sqcup (13, 14) \sqcup (1, 3) \sqcup (19, 20)$ $(1, 11, 2, 10, 3) \sqcup (0, 6) \sqcup (9, 4) \sqcup (8, 12)$
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(1, \infty, 13, 10, 7) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(5, \infty, 10, 11, 16) \sqcup (4, 7) \sqcup (0, 2) \sqcup (9, 12)$ $(6, 4, 5, 8, \infty) \sqcup (17, 19) \sqcup (0, 3) \sqcup (12, 14)$ $(7, 8, 6, 9, 11) \sqcup (13, 14) \sqcup (1, 3) \sqcup (19, 20)$ $(3, 10, 2, 11, 5) \sqcup (0, 6) \sqcup (4, 8) \sqcup (17, 7)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(1, \infty, 13, 5, 7) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(0, 3, 1, 4, \infty) \sqcup (2, 5) \sqcup (9, 7) \sqcup (10, 13)$ $(12, 11, 13, 14, \infty) \sqcup (17, 19) \sqcup (5, 7) \sqcup (9, 6)$ $(5, 8, 11, 6, 7) \sqcup (13, 14) \sqcup (2, \infty) \sqcup (19, 20)$ $(6, 0, 8, 9, 7) \sqcup (1, 11) \sqcup (10, 5) \sqcup (16, 12)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$ $(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$ $(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(9, \infty, 8, 6) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13, 14) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 4, 3) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(18, 20, 17, \infty) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(0, 9, 1, 11) \sqcup (10, 3) \sqcup (12, 6) \sqcup (19, 14) \sqcup (17, 13)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(8, \infty, 9, 5) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (3, 4)$ $(15, 13, 14, \infty) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 3, 4) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(17, 20, 18, 19) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(9, 0, 8, 7) \sqcup (1, 11) \sqcup (12, 6) \sqcup (10, 5) \sqcup (16, 20)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(8, \infty, 9) \sqcup (13, 12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2)$ $(19, \infty, 6) \sqcup (11, 10, 12) \sqcup (2, 5) \sqcup (18, 20) \sqcup (1, 4)$ $(11, \infty, 14) \sqcup (10, 7, 4) \sqcup (16, 17) \sqcup (0, 2) \sqcup (1, 3)$ $(20, 17, \infty) \sqcup (14, 13, 15) \sqcup (5, 7) \sqcup (9, 6) \sqcup (0, 1)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(8, \infty, 9) \sqcup (12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13) \sqcup (9, 6) \sqcup (0, 2) \sqcup (18, 20) \sqcup (1, 4) \sqcup (17, 19)$ $(11, \infty, 14) \sqcup (4, 7) \sqcup (16, 17) \sqcup (2, 5) \sqcup (8, 10) \sqcup (0, 3)$ $(20, 17, \infty) \sqcup (13, 14) \sqcup (5, 7) \sqcup (10, 11) \sqcup (0, 1) \sqcup (8, 6)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$

Table 8.2: (1-2-3)-labelings

No.	Block	No.	Block
1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (18, 7)$	8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	10	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
11	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	12	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
13	$(1, 7, 2, 3, 4, 5, 6) \sqcup (0, 14)$	14	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
15	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	16	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$
17	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	18	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
19	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	20	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (1, 3)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$

Table 8.3: $A \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of K_{21}

No.	Block	No.	Block
1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (6, \infty)$	8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	10	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
11	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	12	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
13	$(1, 7, 2, 3, 4, 5, 6) \sqcup (13, \infty)$	14	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
15	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	16	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$
17	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	18	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
19	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	20	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$

Table 8.4: $A \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of K_{22}

No.	Block	No.	Block
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (20, \infty)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$
31	$(0, \infty, 1, 2, 3, 4, 5) \sqcup (18, 7)$	32	$(14, \infty, 15, 16, 17, 18, 19) \sqcup (1, 3)$
33	$(7, \infty, 8, 9, 10, 11, 12) \sqcup (0, 14)$		

Table 8.4: A $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of K_{22}

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Appendix A

Glossary and Acronyms

Care has been taken in this thesis to minimize the use of jargon and acronyms, but this cannot always be achieved. This appendix defines jargon terms in a glossary, and contains a table of acronyms and their meaning.

A.1 Glossary

- **Cosmic-Ray Muon (CR μ)** – A muon coming from the abundant energetic particles originating outside of the Earth’s atmosphere.

A.2 Acronyms

Table A.1: Acronyms

Acronym	Meaning
CR μ	Cosmic-Ray Muon