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DECOMPOSITION OF COMPLETE GRAPHS INTO FORESTS WITH SEVEN EDGES

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Abstract

Let K be a graph and G a subgraph of K. If $E(K)$ can be partitioned into edge-disjoint copies of G, we call the partition a G-decomposition of K and say that G decomposes K. There are 47 forests with exactly 7 edges. We prove that every one decomposes the complete graph K_n if and only if $n \equiv 0, 1, 7 \text{ or } 8 \pmod{14}$.

Keywords: Graph decomposition, forests, ρ-labeling.

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1. Introduction

A *G-decomposition* of a graph K is a set of mutually edge-disjoint subgraphs of K which are isomorphic to a graph G. If such a set exists we say that K *allows* a G-decomposition, and if $K \cong K_n$ we sometimes call the decomposition a *G-design of order n.*

G-decompositions are a longstanding topic in combinatorics, graph theory, and design theory, with roots tracing back to at least the 19th century. The work of Rosa and Kotzig in the 1960s on what are now known as graph labelings laid the foundation for the modern treatment of such problems. Using adaptations of these labelings alongside techniques from design theory, numerous papers have since been published on G-decompositions. This work is a natural continuation of Freyberg and Peters' recent paper on decomposing complete graphs into forests with six edges [4]. Their paper also includes a summary of G-decompositions for graphs G with less than 7 edges.

Every connected component of a forest with 7 edges is a tree with 6 or less edges. All such trees are cataloged in Figure 1. We use the naming convention $\mathbf{T_j^i}$ to denote the i^{th} tree with j vertices. For each tree $\mathbf{T_j^i}$, the names of the vertices, v_t for $1 \leq t \leq j$, will be used in the decompositions described in Section 3.

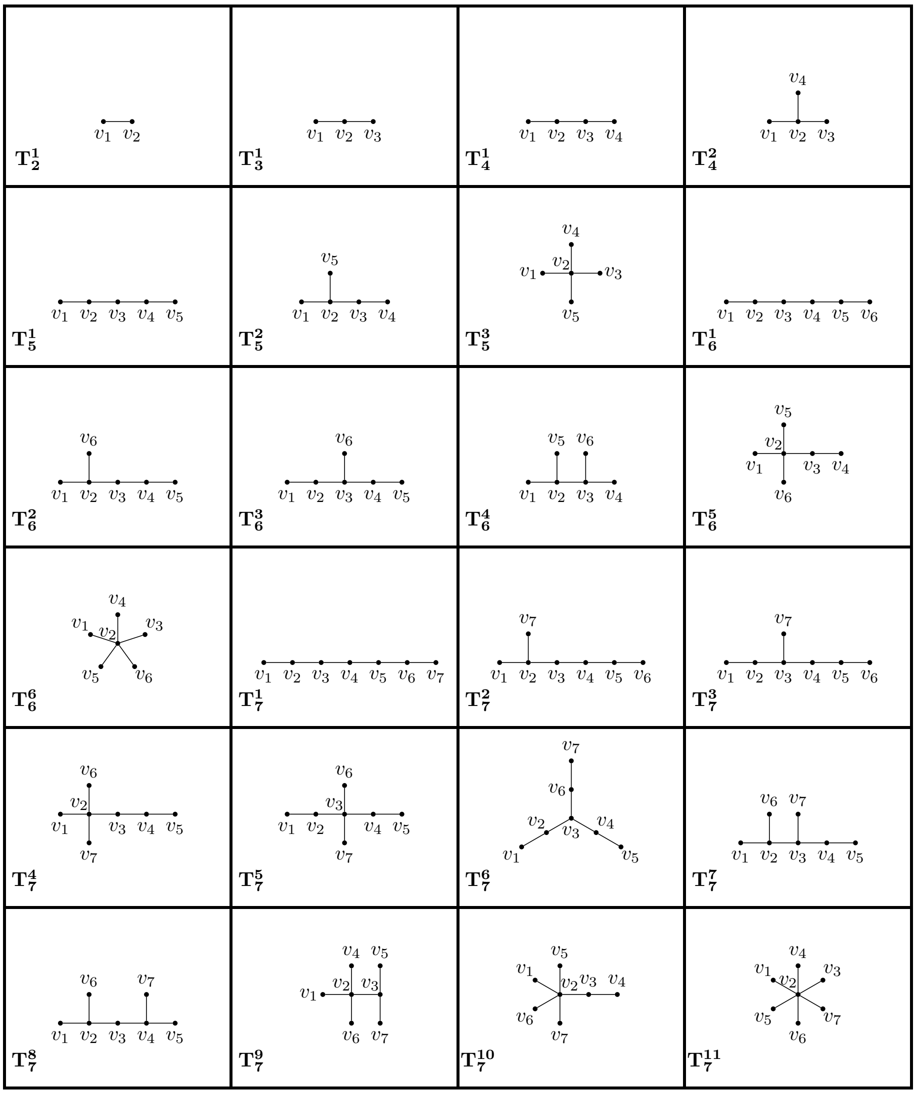

Figure 1.: Trees with less than 7 edges

2. $n \equiv 0$ or 1 (mod 14)

Definition (Rosa [1])**.** Let G be a graph with m edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, 2, \dots, 2m\}$ that induces a bijective *length function* $\ell : E(G) \to \{1, 2, \dots, m\}$ where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all $uv \in E(G)$.

Rosa showed that a ρ-labeling of a graph G with m edges and a cyclic G-decomposition of K_{2m+1} are equivalent, which the next theorem shows. Later, Rosa, his students, and colleagues began considering more restrictive types of ρ-labeling to address decomposing complete graphs of more orders. Definitions of these labelings and related results follow.

Theorem 1 (Rosa [1]). *Let G be a graph with m edges. There exists a cyclic G-decomposition of K_{2m+1} if and only if G admits a ρ-labeling.*

Definition (Rosa [1]). A σ-labeling of a graph G is a ρ-labeling such that $\ell(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$.

Definition (El-Zanati, Vanden Eynden [7]). A ρ- or σ-labeling of a bipartite graph G with bipartition (A, B) is called an *ordered* ρ- or σ-labeling and denoted ρ^+, σ^+, respectively, if $f(a) < f(b)$ for each edge ab with $a \in A$ and $b \in B$.

Theorem 2 (El-Zanati, Vanden Eynden [7]). *Let G be a graph with m edges which has a ρ^+-labeling. Then G decomposes K_{2mk+1} for all positive integers k.*

Definition (Freyberg, Tran [3]). A σ^{+-}-*labeling* of a bipartite graph G with m edges and bipartition (A, B) is a σ^+-labeling with the property that $f(a)-f(b) \neq m$ for all $a \in A$ and $b \in B$, and $f(v) \notin \{2m, 2m-1\}$ for any $v \in V(G)$.

Theorem 3 (Freyberg, Tran [3]). *Let G be a graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant. Then there exists a G-decomposition of both K_{2mk} and K_{2mk+1} for every positive integer k.*

Figure **??** shows a σ^{+-}-labeling of every forest with 7 edges. These labelings along with Theorem 3 are enough to prove the following theorem.

Theorem 4. *Let F be a forest with 7 edges. There exists an F-decomposition of K_n whenever $n \equiv 0$ or $1 \pmod{14}$.*

Proof. The proof follows from Theorem 3 and the labelings given in Figure **??**.
■

3. $n \equiv 7$ OR $8 \pmod{14}$

Forest	Design Generators	Forest	Design Generators
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2,9,7) \sqcup (3,4)$	$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,3) \sqcup (8,7)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,4) \sqcup (8,7)$	$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(5,1,4,2,9,6,7) \sqcup (10,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(3,8,1,4,2,5,7) \sqcup (9,10)$	$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(7,8,1,6,0,4,3) \sqcup (9,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,1,6,3,4,5,7) \sqcup (9,10)$	$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(6,1,5,3,8,4,7) \sqcup (9,10)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(5,11,9,10,6,12,7) \sqcup (8,1)$	$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(4,8,1,6,0,5,3) \sqcup (9,10)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(0,6,1,5,2,9) \sqcup (11,10,12)$	$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(3,6,1,8,4,0) \sqcup (10,9,11)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(5,11,9,12,7,10) \sqcup (1,8,4)$	$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(3,8,4,1,6,7) \sqcup (10,9,11)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(5,1,8,3,4,7) \sqcup (10,9,11)$	$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(4,1,8,5,6,7) \sqcup (10,9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(0,6,1,5,2) \sqcup (9,8,10,3)$	$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(7,1,8,5,6) \sqcup (0,4,2,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(7,1,8,4,6) \sqcup (10,9,11,12)$	$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(6,0,3,4,5) \sqcup (8,7,9,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(4,8,1,7,2) \sqcup (10,9,11,12)$	$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(6,0,3,4,5) \sqcup (8,9,2,7)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5,2,9) \sqcup (8,10) \sqcup (3,4)$	$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(3,6,1,8,4,0) \sqcup (5,7) \sqcup (9,10)$
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,3,5,7) \sqcup (0,2) \sqcup (9,10)$	$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(5,8,4,1,6,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(5,11,9,12,7,10) \sqcup (8,1) \sqcup (0,4)$	$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,5,6,7) \sqcup (2,3) \sqcup (9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (8,10,9) \sqcup (11,4)$	$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(7,1,8,5,6) \sqcup (10,9,11) \sqcup (0,4)$
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(6,0,3,4,5) \sqcup (1,8,7) \sqcup (9,11)$	$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5) \sqcup (2,9,7,10) \sqcup (3,4)$
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11,9,10,7) \sqcup (4,0,5,6) \sqcup (8,1)$	$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(4,0,5,6) \sqcup (10,9,11,12) \sqcup (8,1)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(0,6,1,5) \sqcup (8,10,9) \sqcup (11,4,7)$	$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(4,0,5,6) \sqcup (1,8,7) \sqcup (11,9,12)$
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (8,10,7) \sqcup (11,4) \sqcup (2,3)$	$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (11,9,12) \sqcup (2,3) \sqcup (8,1)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$	$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(6,1,8,4,7) \sqcup (3,5) \sqcup (9,12) \sqcup (10,11)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(3,0,4,5,6) \sqcup (8,1) \sqcup (10,11) \sqcup (9,7)$	$3\mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (2,9,7) \sqcup (10,11)$
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (9,2) \sqcup (8,10) \sqcup (4,7) \sqcup (11,12)$	$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (2,3) \sqcup (9,11) \sqcup (8,1) \sqcup (10,7)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$	$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(0,6,1) \sqcup (8,4) \sqcup (2,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$

Figure 2.: σ^{+-}-labelings for the forests with 7 edges

Example 5. Danny's example

4. $G \cong K_{1,6} \cup K_2$

Observation 6. *Consider $\mathcal{K} = K_{t-1}$ for $t \geq 1$ whose nodes are $K_{14's}$. Clearly, $\mathcal{K} = K_{14(t-1)}$. So then $\mathcal{K} \vee K_{21} = K_{14(t-1)} \vee K_{21} = K_{14(t-1)+21} = K_{14t+7}$. Similarly $\mathcal{K} \vee K_{22} = K_{14t+8}$.*

Observation 7. *Consider $\mathcal{K}_b = K_{i:j}$ for $i, j \in \mathbb{N}$ whose nodes are $\overline{K}_7's$. Clearly $\mathcal{K} = K_{7i:7j}$. Suppose we replace $0 \leq a \leq i$ nodes of partite set I in $\mathcal{K}$ with $\overline{K}_8's$. Then the resulting graph will be $K_{7(i-a)+8a:7j}$. The same holds if performed in partite set J.*

Theorem 8. *Let $F = \mathbf{T_6^{11}} \sqcup \mathbf{T_2^1}$. If F admits a σ^{+-} labeling of F and there exists an $F-$decomposition of $K_{21}, K_{22}, K_{7:7}, K_{8:7}$, then there exists an $F-$decomposition of K_{14t+7} and K_{14t+8} for all positive integers t.*

Proof. Suppose F admits a σ^{+-} labeling of K_{14} and that there exists an $F-$decomposition of $K_{21}, K_{22}, K_{7:7}, K_{8:7}$. Let $t \geq 1$ and $\mathcal{K} = K_{t-1}$ whose nodes are K_{14}'s as outlined in observation 6.

By observation 7, if F decomposes $K_{7:7}$, then it decomposes $K_{14:14}$ since the edges of $K_{14:14}$ can be expressed as four copies of the edges of $K_{7:7}$. F also decomposes $K_{21:14}$ via six copies of the edges of $K_{7:7}$. Similarly, F then decomposes $K_{22:14}$ via two copies of the edges of $K_{8:7}$ and four copies of the edges of $K_{7:7}$. Therefore, F decomposes the edges between all nodes of $\mathcal{K}$. Furthermore, F also decomposes $\overline{\mathcal{K}} \vee K_{21}$ as well as $\overline{\mathcal{K}} \vee K_{22}$ since their edges are just many copies of $K_{21:14}$ and $K_{22:14}$.

Lastly, since F admits a σ^{+-}−labeling, by theorem 3 it decomposes K_{14t}. So then F decomposes the internal edges of the nodes of $\mathcal{K}$ which are just K_{14}'s. So then since F internally decomposes the nodes of $\mathcal{K}$ as well as the edges between them along with all edges of $\overline{\mathcal{K}} \vee K_{21}$ and $\overline{\mathcal{K}} \vee K_{22}$, F in fact decomposes $\mathcal{K} \vee K_{21}$ and $\mathcal{K} \vee K_{22}$ which we know to be K_{14t+7} and K_{14t+8} by observation 6.

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Theorem 9. $\mathbf{T_6^{11}} \sqcup \mathbf{T_2^{1}}$ *decomposes* K_{21} *and* K_{22}.

Proof. See table 4. ■

Theorem 10. $\mathbf{T_6^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* $K_{n,7}$ *for all* $n \geq 2$.

Proof. Take the partite set of n nodes to be $\mathbb{Z}_n$ and color them blue. Then, take the other partite set of 7 nodes to be $\mathbb{Z}_7$ and color them red. Notice that $|E(K_{n,7})| = |\mathbb{Z}_j \oplus \mathbb{Z}_7| = 7n$. So let us refer to edges of $K_{n,7}$ as elements of $\mathbb{Z}_n \oplus \mathbb{Z}_7$ and vice versa. Note that since $n \geq 2, (1,0) \neq (0,0)$.

Now, let $E_i = (i,0) + \{(0,0),(1,1),(1,2),\ldots,(1,6)\}$ for each $i \in \mathbb{Z}_n$ and F_i be the subgraph induced by E_i. Since each F_i contains a path $(i,0)$ which is vertex disjoint from the star centered at the blue $i+1$, it must be isomorphic to $\mathbf{T_6^{11}} \sqcup \mathbf{T_2^1}$.

Suppose that there exist distinct $i,j \in \mathbb{Z}_n$ such that $E_i \cap E_j \neq \emptyset$. But then we have that $(i,0) = (j,0)$ or $(i+1,a) = (j+1,b)$ for some $a,b \in \mathbb{Z}_7$, which is impossible. So all distinct E_i's are pairwise disjoint, and therefore all distinct F_i's are pairwise edge-disjoint. Lastly, $\bigcup_{i \in \mathbb{Z}_n} E_i = \langle(1,0)\rangle + [\{(0,0)\} \cup [(1,0) + \langle(0,1)\rangle] \setminus \{(1,0)\}] = \langle(1,0)\rangle + \langle(0,1)\rangle = \langle(1,0),(0,1)\rangle = \mathbb{Z}_n \oplus \mathbb{Z}_7$. Therefore, $\bigcup_{i \in \mathbb{Z}_n} F_i = K_{n,7}$.

Thus, $\{F_i \mid i \in \mathbb{Z}_n\}$ is a $\mathbf{T_6^{11}} \sqcup \mathbf{T_2^1}$−decomposition of $K_{n,7}$. Furthermore, This decomposition is generated by clicking the blue nodes of F_0 by 1.

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