Seven Edge Forest Designs

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There are many people that have earned my gratitude for their contribution to my time in graduate school. However, without Professor Bryan Freyberg I would not have had the opportunity to become a graduate student, and Professor Joseph Gallian's courses and guidance has been very influential to the way I organize myself mathematically today.

Dedication

I dedicate this Thesis to my advisor Professor Bryan Freyberg, to my family who has supported me throughout this process, and to Jordi, Ian, TK, and Torta from Tuscarora Ave.

Abstract

Let G be a subgraph of K_n where $n \in \mathbb{N}$. A G-decomposition of K_n, or G-design of order n, is a finite collection $\mathcal{G} = \{G_1, \ldots, G_k\}$ of pairwise edge-disjoint subgraphs of K_n that are all isomorphic to some graph G. We prove that an F-decomposition of K_n exists for every seven-edge forest F if and only if $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$.

Along the way, we introduce new methods, constraint programming algorithms in Python, and some bonus results for Galaxy graph decompositions of complete bipartite, and eventually multipartite graphs.

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Chapter 1

Background

1.1 Fundamentals of Graph Theory

Graph Theory is the study of objects called *vertices* or *nodes* and their relationships which we call *edges*. An edge between vertices u and v is typically denoted via uv or (u, v). A graph G is completely defined by an ordered pair $G = (V, E)$ where V is the set of all vertices in G and E is the set of all edges between vertices in G. These sets are sometimes refered to as $V(G)$ and $E(G)$, respectively.

We call G a simple graph if (1) there is at most 1 edge between any two vertices, (2) there are no edges from a vertex to itself and (3) all edges have no directionality to them, meaning $uv = vu$ for any edge $uv \in E(G)$. For the rest of this paper all graphs are finite simple graphs, but note that unions and subgraphs are defined the same way for directed graphs and infinite graphs.

Graphs are more intuitive to work with through their visual representations instead of their formal definitions. Let the simple graph G where $V(G) = \{A, B, C, D, E, a, b, c, d, e\}$ and $E(G) = \{Aa, Bb, Cc, Dd, Ee, AB, BC, CD, DE, EA, ac, ce, eb, bd, da\}$. G is often called the *Petersen* graph. It's unwieldy when described formally, yet its visual representation is very easy to understand.

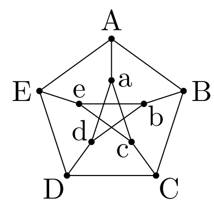

Figure 1.1: The Petersen graph

We say two graphs G and H are *isomorphic* if there exists a bijection from $V(G)$ to $V(H)$ that induces a bijection from $E(G)$ to $E(H)$ and we denote this relationship via $G \equiv H$. In other words, we consider two graphs G, H to be the 'same' if we can relabel and/or move vertices in some fashion (without adding/removing vertices edges) in a visual representations of G and H to go back and forth between the two.

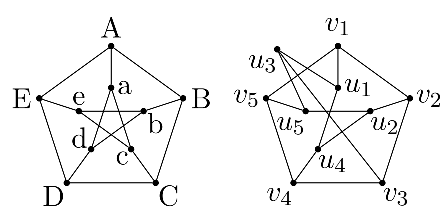

Figure 1.2: $G \cong H$

Graph theorists casually refer to two graphs as the 'same' graph if they are in the same isomorphism class. We will wrap up the fundamentals with a few mdefinitions and some algebraic tools.

Definition 1.1.1 (Subgraph)**.** A subgraph $G \subseteq K$ is a graph whose vertices and edges are subsets of the vertices and edges of K; $G \subseteq K$ if $V(G) \subseteq V(K)$ and $E(G) \subseteq E(K)$.

Definition 1.1.2 (Vertex-induced Subgraph)**.** A *vertex-induced* subgraph $G \subseteq K$ is one whose vertices are some subset of $V(K)$ and whose edges are all edges between those vertices in K; $V(G) \subseteq V(K)$ and $E(G) = \{uv \in E(K) \mid u, v \in E(G)\}$. If G is such a subgraph we say that G is induced by $S = V(G) \subseteq V(K)$.

Definition 1.1.3 (Edge-induced Subgraph)**.** A *edge-induced* subgraph $G \subseteq K$ is one whose edges are some subset of $E(K)$ and whose vertices are all those who appear as an endpoint in that subset of edges; $E(G) \subseteq E(K)$ and $V(G) = \{v \in V(K) \mid vu \in E(G) \text{or } uv \in E(G) \text{ for some } u \in V(K)\}$. If G is such a subgraph we say that G is induced by $S = E(G) \subset E(K)$

Here is a visual example of these types of graphs: Let K be the Petersen graph from Figure 1.1. Now, let

Subgraph: $G \subseteq K$ where $V(G) = \{E, e, b\}$, $E(G) = \{Ee\}$.

Vertex-induced Subgraph: $H \subseteq K$ is induced by $\{a, A, B\} \subseteq V(K)$

Edge-induced Subgraph: $M \subseteq K$ is induced by $\{Dd, DC, Cc\} \subseteq E(K)$

The figure below shows K and it's color-coded subgraphs.

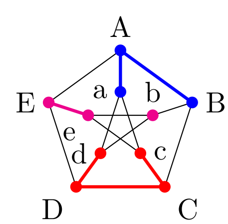

Figure 1.3: K and subgraphs $G, H, M \subseteq K$

Next, we will talk about two important operations done on graphs.

Definition 1.1.4 (Graph Union)**.** The union of two graphs G and H is simply the graph resulting from the union of their vertices and the union of their edges and is denoted $G \cup H$; $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. If G and H are vertex disjoint, we denote their union via $G \sqcup H$ and call it a *disjoint union* of G and H.

Here is an example of a union and a disjoint union of graphs. Let $G = (\{a, b, c, d\}, \{ab, bc, cd, da\}), H = (\{a, b, c\}, \{ab, bc, ca\}),$ and $K = (\{A, B, C\}, \{AB, BC, CA\})$ Then:

$$G \cup H = (\{a, b, c, d\}, \{ab, bc, cd, da, ca\})$$

$$G \sqcup K = V(G \sqcup K) = (\{a, b, c, d, A, B, C\}, \{ab, bc, cd, da, AB, BC, CA\})$$

These unions are depicted in the following figure.

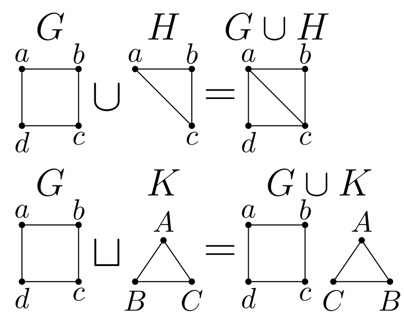

Figure 1.4: (above) $G \cup H$ and (below) $G \sqcup K$

Next, we define another very important operation that combines two graphs in a different manner.

Definition 1.1.5 (Join)**.** Let G and H be vertex disjoint graphs. Their *join*, denoted via $G \vee H$, is the graph obtained by taking the disjoint union of G and H and adding all possible edges between every vertex in G and every vertex in H. Formally:

$$G \vee H \;=\; \big(V(G) \cup V(H), E(G) \;\cup\; E(H) \;\cup\; \{\, xy \mid x \in V(G),\, y \in V(H)\}\big).$$

Here is an example. Let $G = (\{a, b, c\}, \{ab, bc, ca\})$ and $H = (\{A, B, C\}, \{AB, BC, CA\})$, then $G \vee H = (\{a, b, c, A, B, C\}, E(G) \sqcup E(H) \sqcup \{aA, aB, aC, bA, bB, bC, cA, cB, cC\})$. This join is depicted in the figure below.

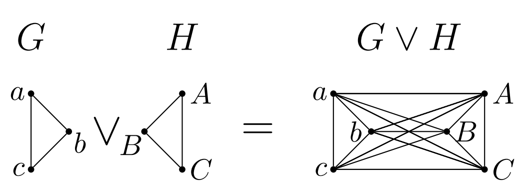

Figure 1.5: (above) $G \cup H$ and (below) $G \sqcup K$

Lastly, we define a few characteristics of graphs and their components. These may or may not be used frequently in this paper, but are important concepts to know in order to be able to talk about graphs comfortably.

Let G be a simple graph. We say two vertices $u, v \in V(G)$ are *adjacent* or *neighbors* if they share an edge $uv \in E(G)$. Similarly, we say a vertex is *incident* with an edge if it is one of it's endpoints; $u \in V(G)$ is incident with $e \in E(G)$ if $e = uv$ for some $v \in V(G)$. The set of all vertices adjacent to u in G is called the *neighborhood* of v denoted $N_G(v)$ or simply $N(v)$. Sometimes this is refered to as the open neighborhood of v in G and then the closed neighborhood is defined via $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is the number of vertices adjacent to it and is denoted via $deg_G(v) = |N_G(v)|$ or simply $deg(v)$. Equivalently, the degree is the number or edges incident to it or the number of neighbors that G has.

The following are three similar types of objects we can form from graphs.

Definition 1.1.6 (Walk)**.** Let G be a graph on n vertices. A *walk* in G is a sequence $(w_0, w_1, \ldots, w_k)$ of vertices in G whose adjacent elements must be adjacent in G. Adjacent elements in a walk much be distinct vertices but a vertex may be repeated multiple times.

Definition 1.1.7 (Path)**.** Let G be a graph on n vertices. A *path* in G is a sequence $(v_0, v_1, \ldots, v_k)$ of distinct vertices in G whose adjacent elements must be adjacent in G, and where no vertex is repeated. This sequence gives the subgraph of G induced by $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$.

Definition 1.1.8 (Cycle)**.** Let G be a graph on n vertices. A *cycle* in G is a sequence $(v_0, v_1, \ldots, v_k, v_0)$ of internally distinct vertices (distinct except on the endpoints) that begins and terminates at the same vertex v_0. Often such a cycle is denoted via $(v_0v_1 \cdots v_k)$ and it is understood that the sequence wraps back around to v_0 after v_k. Additionally, the cycle $(v_0v_1 \cdots v_k)$ is equivalent to $(v_1 \cdots v_kv_0)$, $(v_2 \cdots v_kv_0v_1), \ldots$ and so on.

let G be a simple graph. We call G *acyclic* if it contains no cycles. If there exists a path from any vertex to every other vertex in G, then we call G *connected.* If not, we call G *disconnected.* We call the set of connected subgraphs of G whose disjoint union equals G the *connected components* of G.

This concludes the fundamental concepts needed to understand this project. The next and final section of this chapter will introduce all the fundamental families of graphs we refer to in the proceeding chapters.

1.2 Fundamental Families of Graphs

In this section introduce some fundamental families of graphs which we refer to throughout this paper. Often instead of fully defining the graphs being worked with, we simply refer to it as a member of a larger family of graphs. These families are not completely distinct, but sometimes it is helpful to view graphs as a member of one family or another depending on the context.

Recall that a graph is acyclic if it contains no cycles. Similarly, we call a graph *k-cyclic* if it contains exactly k distinct cycles. If $k = 2$ or 3 we call it *bicyclic* or *tricyclic*, respectively. In a similar vein, we call a graph *k-partite* if we can partition it's vertices into k disjoint sets.If $k = 2$ or 3, we call it *bipartite* or *tripartite*, respectively. These are all very broad families of graphs often used to characterize graphs within another family. The following are more nuanced, and more popular families of graphs to work with.

Definition 1.2.1 (Complete Graph)**.** The *complete graph* on n vertices, denoted K_n, is the graph on n vertices such that every pair of distinct vertices shares an edge.

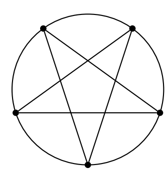

Figure 1.6: The Complete Graph K_5

Definition 1.2.2 (Complete Bipartite Graph)**.** Let $m, n \in \mathbb{N}$. The *complete bipartite graph* $K_{m,n}$ is the bipartite graph whose vertices can be partitioned into two disjoint sets of sizes m and n, respectively, such that every vertex in the one partite set is adjacent to every vertex in the other partite set and there no edges between vertices in the same partite set.

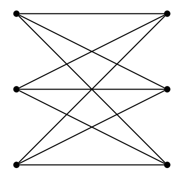

Figure 1.7: The Complete Bipartite Graph $K_{3,3}$

Definition 1.2.3 (Complete Multipartite Graph)**.** The *complete k-partite graph* or *complete multipartite graph* $K_{n_1,\ldots,n_k}$ is the graph whose vertices can be partitioned into k disjoint sets of sizes n_1, n_2, $\ldots$, n_k, respectively such that such that every vertex in the one partite set is adjacent to every vertex in the other $k-1$ partite sets and there no edges between vertices in the same partite set.

If all partite sets are the same size n we call this graph the *complete equipartite graph* $K_{n:k}$ or $K_{n\times m}$.

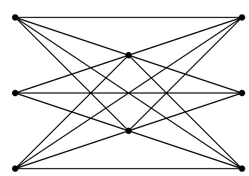

Figure 1.8: The Complete Multipartite Graph $K_{3,2,3}$

Definition 1.2.4 (Cycle Graph)**.** The *cycle graph* on n vertices denoted C_n is a graph with exactly one cycle containing all of it's edges.

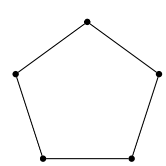

Figure 1.9: The Cycle Graph C_5

Definition 1.2.5 (Tree)**.** A *tree* is any connected acyclic graph. Trees on n vertices have $n-1$ edges. Equivalently, these graphs are any connected bipartite graphs.

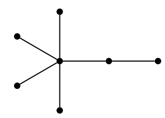

Figure 1.10: A Tree Graph on 6 vertices

Definition 1.2.6 (Path Graph)**.** The *path* graph on n vertices, denoted P_n, is an acyclic graph with exactly one path containing all of it's edges. All paths are trees.

Figure 1.11: The Path Graph P_4

Definition 1.2.7 (Star Graph)**.** The *star graph* on $n + 1$ vertices, denoted $K_{1,n}$ (or S_{n+1} which we never use in this paper) consisting of one central *hub* vertex adjacent to n *outer* vertices, with no other edges. All stars are trees. Sometimes this graph is refered to as an *n-star*.

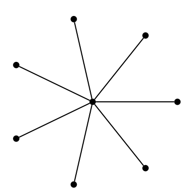

Figure 1.12: The 7-star ($K_{1,7}$)

Definition 1.2.8 (Forest Graph)**.** Any disjoint union of tree graphs is called a *forest* graph. These graphs are all bipartite and can be equivalently defined as disconnected bipartite graphs.

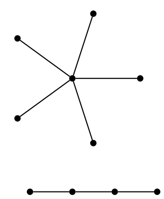

Figure 1.13: A forest on 9 vertices

Definition 1.2.9 (Galaxy Graph)**.** Any disjoint union of star graphs is called a *galaxy* graph. We refer to $G = G_1 \sqcup \cdot \sqcup G_k$ as a $(G_1, \ldots, G_k)$-*galaxy* graph if $G_1, \ldots, G_k$ are all stars. This family is a subset of the forest family.

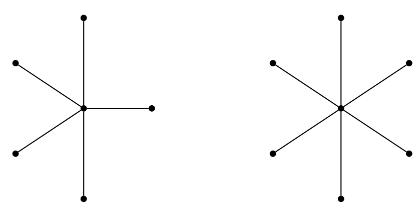

Figure 1.14: The $(K_{1,6}, K_{1,7})$-Galaxy

We have now defined a few very important families of graphs which we will refer to throughout the rest of this paper. We generally don't explicitly define every graph by its vertices and edges and simply refer to it as some member of one family or say this it is isomorphic to one. This is much more efficient and concise than listing out all vertices and edges as we did in the beginning of this chapter.

We are now ready to move on and introduce graph decompositions, the objects which are the subject of this project.

Chapter 2

Introduction

A *G-decomposition* of a graph K is a set of mutually edge-disjoint subgraphs of K which are isomorphic to a given graph G. If such a set exists we say that K *allows* a G-decomposition, and if $K \cong K_n$ we sometimes call the decomposition a *G-design of order n*.

G-decompositions are a longstanding topic in combinatorics, graph theory, and design theory, with roots tracing back to at least the 19th century. The work of Rosa and Kotzig in the 1960s on what are now known as graph labelings laid the foundation for the modern treatment of such problems. Using adaptations of these labelings alongside techniques from design theory, numerous papers have since been published on G-decompositions. This work is a natural continuation of Freyberg and Peters' recent paper on decomposing complete graphs into forests with six edges [4]. Their paper also includes a summary of G-decompositions for graphs G with less than 7 edges.

Every connected component of a forest with 7 edges is a tree with 6 or less edges. All such trees are cataloged in Figure 2.1. We use the naming convention $\mathbf{T_j^i}$ to denote the i^{th} tree with j vertices. For each tree $\mathbf{T_j^i}$, the names of the vertices, v_t for $1 \leq t \leq j$, will be referred to in the decompositions presented in the main results of this project.

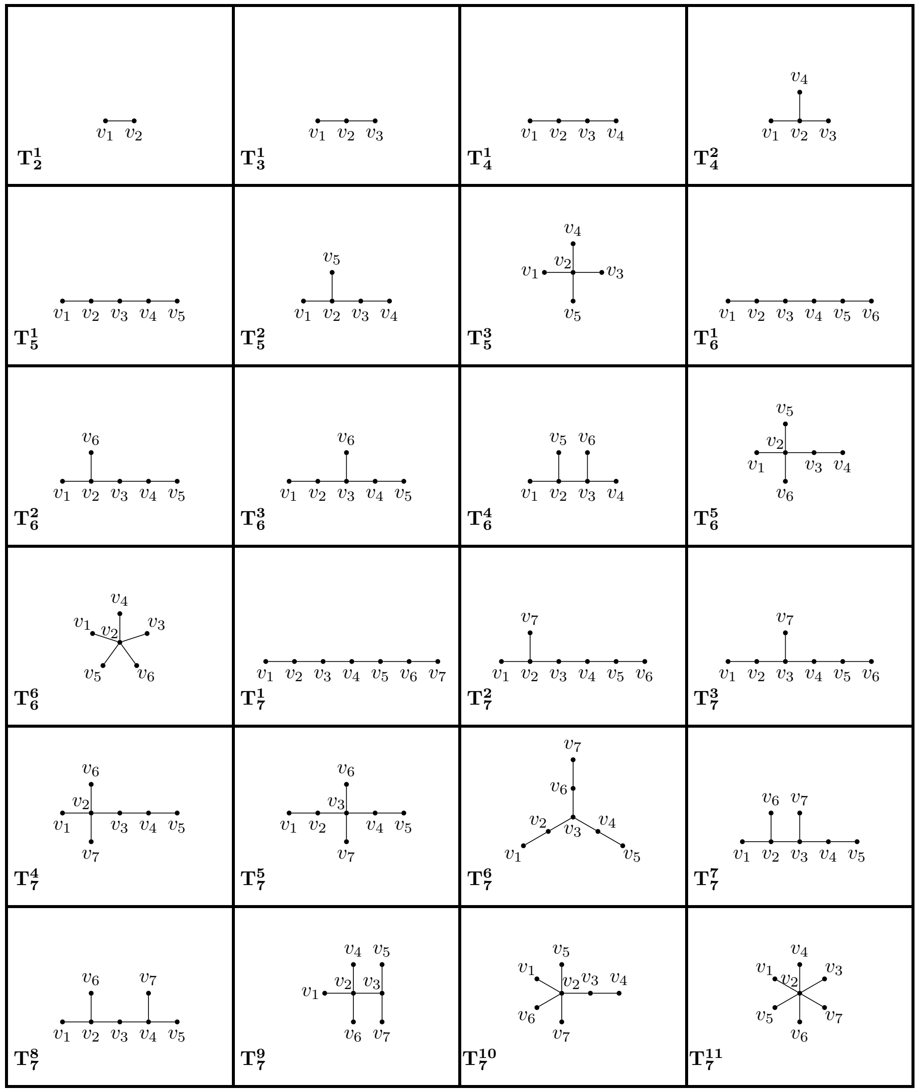

Figure 2.1: trees with less than seven edges

The next theorem gives the necessary conditions for the existence of a G-decomposition of K_n when G is a graph with 7 edges.

Theorem 2.0.1. *If G is a graph with 7 edges and a G-decomposition of K_n exists, then $n \equiv 0, 1, 7,$ or $8 \pmod{14}$.*

Proof. If a G-decomposition exists, then $7 | \binom{n}{2}$ which immediately implies $n \equiv 0, 1, 7,$ or 8 (mod 14).

□

In this article, we only consider simple graphs without isolated vertices. There are 47 non-isomorphic forests with 7 edges. Chapter 3 treats all 47 forests when $n \equiv 0$ or $1 \pmod{14}$. Chapter 4 applies to all the forests when $n \equiv 7$ or $8 \pmod{14}$ with the lone exception of $F \cong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$, which is solved for those values of n in Chapter 5.

Chapter 3

$\mathbf{n \equiv 0, 1 (mod\ 14)}$

In this section, we use established graph labeling techniques to construct the G-decompositions of K_n when $n \equiv 0$ or $1 \pmod{14}$.

Definition 3.0.1 ((Rosa [7]))**.** Let G be a graph with m edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, 2, \ldots, 2m\}$ that induces a bijective *length function* $\ell : E(G) \to \{1, 2, \ldots, m\}$ where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all $uv \in E(G)$.

Rosa showed that a ρ-labeling of a graph G with m edges and a cyclic G-decomposition of K_{2m+1} are equivalent, which the next thm shows. Later, Rosa, his students, and colleagues began considering more restrictive types of ρ-labeling to address decomposing complete graphs of more orders. Definitions of these labelings and related results follow.

Theorem 3.0.2 ((Rosa [7]))**.** *Let G be a graph with m edges. There exists a cyclic G-decomposition of K_{2m+1} if and only if G admits a ρ-labeling.*

Definition 3.0.3 ((Rosa [7]))**.** A σ-labeling of a graph G is a ρ-labeling such that $\ell(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$.

Definition 3.0.4 ((El-Zanati, Vanden Eynden [2]))**.** A ρ- or σ-labeling of a bipartite graph G with bipartition (A, B) is called an *ordered* ρ- or σ-labeling and denoted ρ^+, σ^+, respectively, if $f(a) < f(b)$ for each edge ab with $a \in A$ and $b \in B$.

Theorem 3.0.5 ((El-Zanati, Vanden Eynden [2])). *Let G be a graph with m edges which has a ρ^+-labeling. Then G decomposes K_{2mk+1} for all positive integers k.*

Definition 3.0.6 ((Freyberg, Tran [5])). A σ^{+-}-*labeling* of a bipartite graph G with m edges and bipartition (A, B) is a σ^+-labeling with the property that $f(a) - f(b) \neq m$ for all $a \in A$ and $b \in B$, and $f(v) \notin \{2m, 2m-1\}$ for any $v \in V(G)$.

Theorem 3.0.7 ((Freyberg, Tran [5])). *Let G be a graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant. Then there exists a G-decomposition of both K_{2mk} and K_{2mk+1} for every positive integer k.*

Figure 3.1 gives a σ^{+-}-labeling of every forest with 7 edges. The vertex labels of each connected component with k vertices are given as a k-tuple, $(v_1, \ldots, v_k)$ corresponding to the vertices $v_1, \ldots, v_k$ given in Figure 2.1. We leave it to the reader to infer the bipartition (A, B).

Example 3.0.8. *A σ^{+-}-labeling of $\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$ is shown in Figure 3.1. The vertices labeled 1, 2 and 9 belong to A, and the others belong to B. The lengths of each edge are indicated on the edge.*

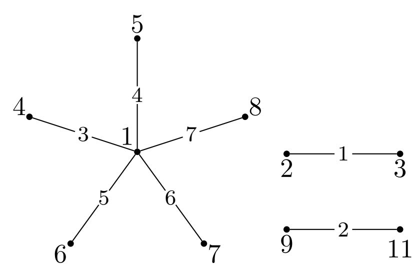

Figure 3.1: σ^{+-}-labeling of $\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$

The labelings given in Figure 3.1 along with thm 3.0.7 are enough to prove the following thm.

Forest	Vertex Labels
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2,9,7) \sqcup (3,4)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,3) \sqcup (8,7)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,4) \sqcup (8,7)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(5,1,4,2,9,6,7) \sqcup (10,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(3,8,1,4,2,5,7) \sqcup (9,10)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(7,8,1,6,0,4,3) \sqcup (9,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,1,6,3,4,5,7) \sqcup (9,10)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(6,1,5,3,8,4,7) \sqcup (9,10)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(5,11,9,10,6,12,7) \sqcup (8,1)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(4,8,1,6,0,5,3) \sqcup (9,10)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(0,6,1,5,2,9) \sqcup (11,10,12)$
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(3,6,1,8,4,0) \sqcup (10,9,11)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(5,11,9,12,7,10) \sqcup (1,8,4)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(3,8,4,1,6,7) \sqcup (10,9,11)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(5,1,8,3,4,7) \sqcup (10,9,11)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(4,1,8,5,6,7) \sqcup (10,9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(0,6,1,5,2) \sqcup (9,8,10,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(7,1,8,5,6) \sqcup (0,4,2,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(7,1,8,4,6) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(6,0,3,4,5) \sqcup (8,7,9,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(4,8,1,7,2) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(6,0,3,4,5) \sqcup (8,9,2,7)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5,2,9) \sqcup (8,10) \sqcup (3,4)$
$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(3,6,1,8,4,0) \sqcup (5,7) \sqcup (9,10)$
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,3,5,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(5,8,4,1,6,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(5,11,9,12,7,10) \sqcup (8,1) \sqcup (0,4)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,5,6,7) \sqcup (2,3) \sqcup (9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (8,10,9) \sqcup (11,4)$

Table 3.1: σ^{+-}-labelings for forests with seven edges

Forest	Vertex Labels
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(7,1,8,5,6) \sqcup (10,9,11) \sqcup (0,4)$
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(6,0,3,4,5) \sqcup (1,8,7) \sqcup (9,11)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5) \sqcup (2,9,7,10) \sqcup (3,4)$
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11,9,10,7) \sqcup (4,0,5,6) \sqcup (8,1)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(4,0,5,6) \sqcup (10,9,11,12) \sqcup (8,1)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(0,6,1,5) \sqcup (8,10,9) \sqcup (11,4,7)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(4,0,5,6) \sqcup (1,8,7) \sqcup (11,9,12)$
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (8,10,7) \sqcup (11,4) \sqcup (2,3)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (11,9,12) \sqcup (2,3) \sqcup (8,1)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(6,1,8,4,7) \sqcup (3,5) \sqcup (9,12) \sqcup (10,11)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(3,0,4,5,6) \sqcup (8,1) \sqcup (10,11) \sqcup (9,7)$
$3\mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (2,9,7) \sqcup (10,11)$
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (9,2) \sqcup (8,10) \sqcup (4,7) \sqcup (11,12)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (2,3) \sqcup (9,11) \sqcup (8,1) \sqcup (10,7)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(0,6,1) \sqcup (8,4) \sqcup (2,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$

Table 3.1: σ^{+-}-labelings for forests with seven edges

Theorem 3.0.9. *Let F be a forest with 7 edges. There exists an F-decomposition of K_n whenever $n \equiv 0$ or $1 \pmod{14}$.*

Proof. The proof follows from thm 3.0.7 and the labelings given in Figure 3.1. $\square$

Chapter 4

$\mathbf{n \equiv 7, 8 (mod\ 14)}$

In this section, we use our own constructions based on the same edge length definition as in the previous section. The first one addresses the $n \equiv 7 \pmod{14}$ case.

Definition 4.0.1. Let G be a graph with 7 edges. A (1-2-3)-*labeling* of $3G$ is an assignment f of the integers $\{0, \ldots, 20\}$ to the vertices of $3G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

   and

2. $$\bigcup_{uv \in E(3G)} \{(f(u) \textbf{ mod } 7, f(v) \textbf{ mod } 7)\} = \bigcup_{i=0}^{6} \bigcup_{j=1}^{3} \{(i, i+j \textbf{ mod } 7)\}.$$

Notice that the second condition of a (1-2-3)-labeling says that $3G$ contains exactly 7 edges of each of the lengths $1, 2,$ and 3. Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A (1-2-3) labeling of every forest with 7 edges with the exception of $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ is given in Figure 8.1. This exceptional forest does not admit such a labeling and is dealt with in Section 5.

Theorem 4.0.2. *Let G be a bipartite graph with 7 edges. If $3G$ admits a (1-2-3)-labeling and G admits a ρ^+-labeling, then G decomposes K_{14k+7} for every $k \geq 1$.*

Proof. Let $n = 14k + 7$ and notice that K_n has $|E(K_n)| = (7k + 3)(14k + 7)$ edges, which can be partitioned into $14k + 7$ edges of each of the lengths in $\{1, 2, \ldots, 7k + 3\}$.

We will construct the G-decomposition in two steps. First, we use the 1-2-3-labeling to identify all the edges of lengths $1, 2$, and 3 accounting for $3(2k+1)$ copies of G. Then, we use the ρ^+-labeling to identify edges of the remaining lengths in $7k(2k+1)$ copies of G. In total, the decomposition consists of $|E(K_n)|/7 = (7k+3)(2k+1)$ copies of G.

Let f_1 be a (1-2-3)-labeling of $3G$ and identify this graph as a block B_0. Then develop B_0 by 7 modulo n. Since the order of the development is $\frac{n}{7} = 2k+1$ and there are 7 edges of each of the lengths $1, 2$, and 3 in B_0, we have identified $3(2k+1)$ copies of G containing all $14k+7 = n$ edges of each length $1, 2$, and 3. Notice (2) of Definition 4.0.1 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \to \{0, \ldots, 14\}$ be a ρ^+-labeling of G with associated vertex partition (A, B). For $i = 1, 2, \ldots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i-1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i-3, 7i-2, \ldots,$ and $7i+3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i-1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \ldots, 7\}$. Developing each block B_i by 1 yields $14k+7$ copies of G per block and accounts for $14k+7$ edges of each of the lengths $4, 5, \ldots,$ and $7k+3$.

Since we have identified

$$3(2k+1) + k(14k+7) = (7k+3)(2k+1)$$

edge-disjoint copies of G, the proof is complete. $\square$

To address the $n \equiv 8 \pmod{14}$ case, we define the following labeling.

Definition 4.0.3. Let G be a graph with 7 edges. A *1-rotational* (1-2-3)*-labeling* of $4G$ is an assignment f of $\{0, \ldots, 20\} \cup \infty$ to the vertices of $4G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

   and

2.

$$\bigcup_{uv\in E(4G)}\{(f(u)\ \mathbf{mod}\ 7, f(v)\ \mathbf{mod}\ 7)\} = \bigcup_{i=0}^{6}\bigcup_{j=1}^{3}\{(i, i+j\ \mathbf{mod}\ 7), (i,\infty)\}.$$

Notice that the second condition of a 1-rotational (1-2-3)-labeling says that $4G$ contains exactly 7 edges of each of the lengths $1, 2, 3$, and ∞. Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A 1-rotational (1-2-3)-labeling of every forest with 7 edges with the exception of $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ is given in Figure 8.2.

Theorem 4.0.4. *Let G be a bipartite graph with 7 edges. If $4G$ admits a 1-rotational (1-2-3)-labeling and G admits a ρ^+-labeling, then G decomposes K_{14k+8} for every $k \geq 1$.*

Proof. Let $n = 14k+8$ and notice that K_n has $|E(K_n)| = (7k+4)(14k+7)$ edges, which can be partitioned into $14k+7$ edges of each of the lengths in $\{1, 2, \ldots, 7k+3, \infty\}$. We will construct the G-decomposition in two steps. First, we use the 1-rotational (1-2-3)-labeling to identify all the edges of lengths $1, 2, 3$, and ∞ accounting for $4(2k+1)$ copies of G. Then, we use the ρ^+-labeling to identify edges of the remaining lengths in $7k(2k+1)$ copies of G. In total, the decomposition consists of $|E(K_n)|/7 = (7k+4)(2k+1)$ copies of G. Let f_1 be a 1-rotational (1-2-3)-labeling of $4G$ and identify this graph as a block B_0. Then develop B_0 by 7 modulo $n-1$. Since the order of the development is $\frac{n-1}{7} = 2k+1$ and there are 7 edges of each of the lengths $1, 2, 3$ and ∞ in B_0, we have identified $4(2k+1)$ copies of G containing all $14k+7 = n-1$ edges of each length $1, 2, 3$ and ∞. Notice (2) of Definition 4.0.3 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \to \{0, \ldots, 14\}$ be a ρ^+-labeling of G with associated vertex partition (A, B). For $i = 1, 2, \ldots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i-1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i-3, 7i-2, \ldots,$ and $7i+3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i-1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \ldots, 7\}$. Developing each block B_i by 1 yields $14k+7$ copies of G per block and accounts for $14k+7$ edges of each of the lengths $4, 5, \ldots,$ and $7k+3$.

Since we have identified

$$4(2k+1) + k(14k+7) = (7k+4)(2k+1)$$

edge-disjoint copies of G, the proof is complete. $\square$

We are now able to state the main thm of this section.

Theorem 4.0.5. *Let F be a forest with 7 edges and $F \ncong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$. There exists an F-decomposition of K_n whenever $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$.*

Proof. If $n \equiv 7 \pmod{14}$, a (1-2-3)-labeling of $3F$ can be found in Figure 8.1. On the other hand, if $n \equiv 8 \pmod{14}$, then a 1-rotational (1-2-3)-labeling of $4F$ can be found in Figure 8.2. In either case, a ρ^+-labeling of F can be found in Figure 3.1 (recall that a σ^{+-}-labeling is a ρ^+-labeling). The result now follows from Theorems 4.0.2 and 4.0.4. $\square$

Example 4.0.6. *We illustrate the constructions in the previous two thms by finding an F-decomposition of K_{35} and K_{36} for the forest graph $F \cong \mathbf{T_6^6} \sqcup \mathbf{T_3^1}$.*

Here are excerpts from the preceding tables for $\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$

Labeling Type	Labelings
σ^{+-}	$(4,1,8,5,6,7) \sqcup (10,9,11)$
(1-2-3)	$(0,2,1,3,4,5) \sqcup (12,11,14)$
	$(4,6,8,9,5,7) \sqcup (14,12,15)$
	$(0,3,1,4,5,6) \sqcup (11,8,7)$
1-rotational (1-2-3)	$(1,2,0,3,4,5) \sqcup (11,8,\infty)$
	$(2,\infty,3,4,5,6) \sqcup (12,13,15)$
	$(6,7,8,4,5,\infty) \sqcup (11,12,15)$
	$(11,10,8,12,13,7) \sqcup (9,6,4)$

Figure 4.1: Labelings for $\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$

The ρ^+ *labelings obtained by stretching the* σ^{+-} *labeling are bottommost labelings in the following generating presentations and are developed by* 1.

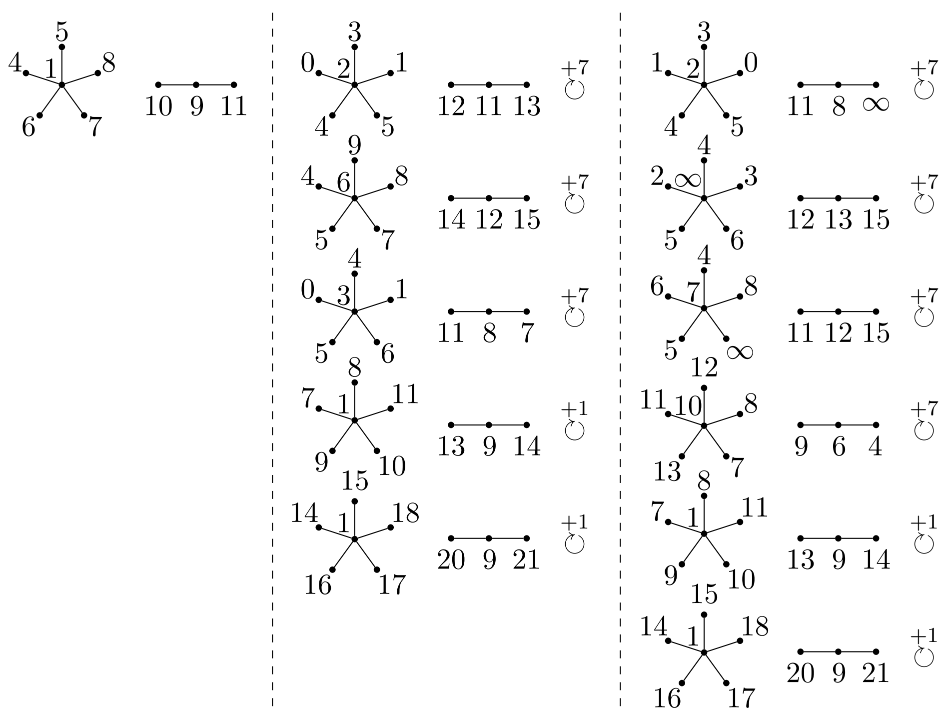

Figure 4.2: A σ^{+-}-labeling of $F \cong \mathbf{T_6^6} \sqcup \mathbf{T_3^1}$ (left) and generating presentations for the F-decomposition of K_n where $n = 35$ (middle) and $n = 36$ (right)

Chapter 5

$\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$

We begin this case by constructing K_n for $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$ using *joined* copies of K_{22}, K_{21}, and K_{14}. Recall, the *join* of two graphs G_1 and G_2 is the graph obtained by adding an edge $\{g_1, g_2\}$ for every vertex $g_1 \in V(G_1)$ and every vertex of $g_2 \in V(G_2)$.

Let t be a positive integer and join $t-1$ copies of K_{14} with each other and a lone copy of K_{21}. The resulting graph is $K_{14(t-1)+21} \cong K_{14t+7}$. So we can think of K_{14t+7} as K_t whose t "vertices" consist of $t-1$ copies of K_{14} and 1 copy of K_{21} and whose edges are the join between them. From now on, we will refer to these "vertices" as nodes. Similarly, K_{14t+8} can be constructed as K_t whose nodes are $t-1$ copies of K_{14} and 1 copy of K_{22} and whose edges are the join between them.

We show that $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes K_n for $n \equiv 7$ or $8 \pmod{14}$ by proving that K_{22}, K_{21}, K_{14}, $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ are each $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposable. Notice that these 6 graphs make up the nodes and edges of the K_t representations of K_{14t+7} and K_{14t+8} stated in the constructions above.
The proof of the next theorem was obtained by manipulating a $K_{1,7}$-decomposition of K_{22} by Cain in [1].

Theorem 5.0.1. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* K_{21} *and* K_{22}.

Proof. Figures 8 and 9 give $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decompositions of K_{21} and K_{22}, respectively. $\square$

Theorem 5.0.2. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* $K_{n,7}$ *for all* $n \geq 2$.

Proof. Consider $K_{n,7}$ where $n \geq 2$. Take the partite set of n vertices to be $\mathbb{Z}_n$ and color them white. Similarly, take the partite set of 7 vertices to be K_7 and color them black. Naturally we refer to *white-black* vertices uv in $K_{n,7}$ via $(u,v) \in \mathbb{Z}_n \times \mathbb{Z}_7$ and vice versa. Finally, let $E_i = \{(i,0)\} \sqcup (\{i+1\} \times \{1,\ldots,6\})$ and $G_i \subset K_{n,7}$ be the subgraph induced by E_i for each $i \in \mathbb{Z}_n$. Note that $G_i \cong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ for all $i \in \mathbb{Z}_n$.

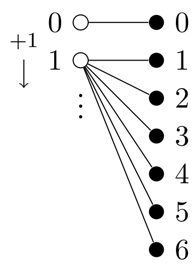

Figure 5.1: G_0 in a generating presentation of the $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}-$decomposition of $K_{n,7}$.

Notice that $E_i \cap E_j = \emptyset$ if $i \neq j$, so by definition all distinct G_i's are pairwise edge disjoint. Lastly,

$$\bigcup_{i\in\mathbb{Z}_n} E_i = [\bigcup_{i\in\mathbb{Z}_n}\{(i,0)\}] \sqcup [\bigcup_{i\in\mathbb{Z}_n}(\{i+1\}\times\{1,\ldots,6\})] = [\mathbb{Z}_n\times\{0\}]\sqcup[\mathbb{Z}_n\times\{1,\ldots,6\}] = \mathbb{Z}_n\times\mathbb{Z}_7$$

.

So $G_0 \sqcup \cdots \sqcup G_{n-1} = K_{n,7}$ and $\{G_i \mid i \in \mathbb{Z}_n\}$ is a $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}-$decomposition of $K_{n,7}$. Furthermore, it is generated by developing the white nodes of G_0 by 1. □

Corollary 5.0.3. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* $K_{22,14}$, $K_{21,14}$, *and* $K_{14,14}$.

Proof. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes $K_{7,7}$ and $K_{8,7}$ by Theorem 5.0.2. $K_{14,14}$ can be expressed as the edge-disjoint union of four copies of $K_{7,7}$, $K_{21,14}$ can be expressed as the edge-disjoint union of six copies of $K_{7,7}$, and $K_{22,14}$ can be expressed as the edge-disjoint union of two copies of $K_{8,7}$ and four copies of $K_{7,7}$. Therefore, $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes them all. □

Theorem 5.0.4. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* K_{14t+7} *and* K_{14t+8} *where* t *is a positive integer.*

Proof. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes K_{14} by Theorem 3.0.7, $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ by Corollary 5.0.3, and lastly K_{22}, K_{21} by Theorem 5.0.1.

Therefore, $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes the join of $(t-1)$ copies of K_{14} with each other and 1 copy of K_{21}, which is isomorphic to K_{14t+7}. Similarly $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes the join of $(t-1)$ copies of K_{14} with each other and 1 copy of K_{22} which is isomorphic to K_{14t+8}. $\square$

Chapter 6

Additional Results

Recall the definition of a (1-2-3)-labeling of $3G$ and a 1-rotational (1-2-3)-labeling of $4G$. When first solving the $7, 8 \pmod{14}$ cases, we used (8-9-10)-labelings allowing for wraparound edges for all forests except $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$. We did this so that we could simply use the σ^{+-} labeling of each forest to complete the decompositions.

Chapter 7

Conclusion and Discussion

Chapter 8

Appendix

Forest	Labeling
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,1,2,4,6,9,12) \sqcup (13,14)$ $(3,4,7,9,10,13,15) \sqcup (8,5)$ $(8,11,12,10,7,5,6) \sqcup (1,3)$ $(0,4,9,15,8,16,7) \sqcup (1,11)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(12,9,6,4,2,1,7) \sqcup (14,15)$ $(15,13,10,9,7,4,11) \sqcup (8,5)$ $(8,11,12,10,7,5,13) \sqcup (1,3)$ $(16,8,15,9,4,0,6) \sqcup (1,11)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(0,1,2,4,6,9,3) \sqcup (16,19)$ $(15,13,10,9,7,4,14) \sqcup (17,18)$ $(6,5,7,10,12,11,8) \sqcup (18,15)$ $(7,16,8,15,9,4,12) \sqcup (1,11)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(8,6,4,2,1,9,7) \sqcup (14,15)$ $(8,10,9,7,4,11,13) \sqcup (12,15)$ $(9,12,10,7,5,11,13) \sqcup (1,4)$ $(7,15,9,4,0,8,6) \sqcup (1,11)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12,8,7) \sqcup (11,14)$
	$(0,2,3,6,5,1,4) \sqcup (8,7)$
	$(0,3,5,4,1,8,7) \sqcup (16,15)$
	$(4,9,15,8,12,6,7) \sqcup (1,11)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(1,2,4,6,8,5,9) \sqcup (12,15)$
	$(4,7,9,10,11,8,13) \sqcup (1,3)$
	$(5,7,10,12,11,6,13) \sqcup (1,4)$
	$(0,4,9,15,8,12,6) \sqcup (1,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,6,4,2,5,9,7) \sqcup (12,14)$
	$(1,3,2,0,5,4,6) \sqcup (10,12)$
	$(9,8,7,10,4,11,5) \sqcup (12,13)$
	$(7,15,9,4,13,8,6) \sqcup (1,11)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(7,6,4,2,8,9,5) \sqcup (12,14)$
	$(2,3,4,7,0,5,6) \sqcup (9,12)$
	$(7,8,5,4,9,10,11) \sqcup (0,2)$
	$(6,15,9,4,8,11,7) \sqcup (2,12)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(2,4,6,8,7,9,12) \sqcup (13,14)$
	$(0,2,3,4,7,6,5) \sqcup (8,10)$
	$(0,3,5,8,9,4,1) \sqcup (12,14)$
	$(4,9,15,8,12,7,16) \sqcup (1,11)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12,1,8) \sqcup (14,15)$
	$(5,6,3,2,0,7,4) \sqcup (8,9)$
	$(0,3,5,4,7,1,8) \sqcup (12,14)$
	$(4,9,15,8,12,18,7) \sqcup (1,11)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(1,2,4,6,9,12) \sqcup (13,14,15)$
	$(3,4,7,9,10,13) \sqcup (5,8,6)$
	$(11,12,10,7,5,6) \sqcup (3,1,4)$
	$(0,4,9,15,8,16) \sqcup (1,11,2)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(1,2,4,6,9,5) \sqcup (13,14,15)$
	$(13,10,9,7,4,11) \sqcup (5,8,6)$
	$(11,12,10,7,5,13) \sqcup (3,1,4)$
	$(0,4,9,15,8,12) \sqcup (1,11,2)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(0,1,2,4,6,5) \sqcup (16,13,14)$
	$(8,6,3,2,0,4) \sqcup (14,12,15)$
	$(7,4,5,3,0,6) \sqcup (10,8,11)$
	$(7,0,4,9,15,12) \sqcup (1,11,2)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(1,2,5,4,6,7) \sqcup (16,14,13)$
	$(8,6,9,3,2,4) \sqcup (14,12,15)$
	$(4,5,6,3,0,1) \sqcup (11,8,7)$
	$(7,0,6,4,9,12) \sqcup (1,11,2)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(0,2,4,7,1,5) \sqcup (12,11,13)$
	$(7,6,3,2,8,9) \sqcup (14,12,15)$
	$(4,3,5,6,0,1) \sqcup (11,8,7)$
	$(8,0,4,9,6,7) \sqcup (1,11,2)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(0,2,1,3,4,5) \sqcup (12,11,14)$
	$(4,6,8,9,5,7) \sqcup (14,12,15)$
	$(0,3,1,4,5,6) \sqcup (11,8,7)$
	$(4,0,8,5,6,7) \sqcup (1,11,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(2,4,6,9,12) \sqcup (16,15,14,13)$
	$(3,4,7,9,10) \sqcup (11,12,15,13)$
	$(12,10,7,5,6) \sqcup (18,15,17,20)$
	$(4,9,15,8,16) \sqcup (2,11,1,5)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(12,9,6,4,11) \sqcup (17,16,15,14)$
	$(9,7,4,3,6) \sqcup (11,12,15,13)$
	$(6,5,7,10,3) \sqcup (18,15,17,20)$
	$(16,8,15,9,12) \sqcup (2,11,1,6)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(4,6,9,11,8) \sqcup (16,15,18,14)$ $(9,7,4,3,6) \sqcup (16,17,20,15)$ $(6,5,7,10,3) \sqcup (9,12,11,15)$ $(16,8,15,9,12) \sqcup (10,1,11,6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(13,15,16,18,14) \sqcup (11,9,6,7)$ $(14,17,16,20,15) \sqcup (9,7,4,3)$ $(9,12,10,11,15) \sqcup (4,6,5,7)$ $(5,1,10,11,6) \sqcup (16,8,15,9)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(7,6,9,11,8) \sqcup (16,15,13,14)$ $(9,7,4,3,5) \sqcup (16,17,20,15)$ $(4,6,5,7,10) \sqcup (9,12,11,15)$ $(16,8,15,9,5) \sqcup (10,1,11,6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(13,15,16,18,14) \sqcup (11,9,12,6)$ $(18,17,16,20,15) \sqcup (9,7,10,4)$ $(10,12,11,14,15) \sqcup (4,6,5,7)$ $(5,1,10,11,6) \sqcup (16,8,14,15)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(1,2,4,6,9,12) \sqcup (13,14) \sqcup (8,7)$ $(3,4,7,9,10,13) \sqcup (8,6) \sqcup (12,15)$ $(11,12,10,7,5,6) \sqcup (1,4) \sqcup (17,15)$ $(0,4,9,15,8,16) \sqcup (1,11) \sqcup (3,12)$
$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(1,2,4,6,9,5) \sqcup (13,14) \sqcup (8,7)$ $(13,10,9,7,4,11) \sqcup (8,6) \sqcup (12,15)$ $(11,12,10,7,5,13) \sqcup (1,4) \sqcup (17,15)$ $(0,4,9,15,8,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(0,1,2,4,7,5) \sqcup (9,6) \sqcup (8,10)$ $(8,6,3,2,0,4) \sqcup (5,7) \sqcup (12,13)$ $(6,4,5,3,0,8) \sqcup (13,14) \sqcup (18,15)$ $(7,0,4,9,15,12) \sqcup (1,11) \sqcup (5,14)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(1,2,5,4,6,7) \sqcup (13,14) \sqcup (12,15)$ $(8,6,9,3,2,4) \sqcup (12,14) \sqcup (18,15)$ $(4,5,6,3,0,1) \sqcup (8,7) \sqcup (16,14)$ $(7,0,6,4,9,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(0,2,4,7,1,5) \sqcup (11,13) \sqcup (12,15)$ $(7,6,3,2,8,9) \sqcup (11,12) \sqcup (1,4)$ $(4,3,5,6,0,1) \sqcup (8,7) \sqcup (12,14)$ $(8,0,4,9,6,7) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(0,2,1,3,4,5) \sqcup (12,14) \sqcup (18,19)$ $(4,6,8,9,5,7) \sqcup (12,15) \sqcup (11,14)$ $(0,3,1,4,5,6) \sqcup (8,11) \sqcup (14,15)$ $(4,0,8,5,6,7) \sqcup (1,11) \sqcup (3,12)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12) \sqcup (13,14,15) \sqcup (18,19)$ $(3,4,7,9,10) \sqcup (12,15,13) \sqcup (1,2)$ $(12,10,7,5,6) \sqcup (20,17,15) \sqcup (1,4)$ $(4,9,15,8,16) \sqcup (11,1,5) \sqcup (3,12)$
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(12,9,6,4,11) \sqcup (17,16,15) \sqcup (0,1)$ $(9,7,4,3,6) \sqcup (12,15,13) \sqcup (18,19)$ $(6,5,7,10,3) \sqcup (20,17,15) \sqcup (1,4)$ $(16,8,15,9,12) \sqcup (1,11,2) \sqcup (0,5)$
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(13,15,16,18,14) \sqcup (9,6,7) \sqcup (2,4)$ $(14,17,16,20,15) \sqcup (3,4,7) \sqcup (11,13)$ $(9,12,10,11,15) \sqcup (6,5,7) \sqcup (0,2)$ $(5,1,10,11,6) \sqcup (8,15,9) \sqcup (4,12)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(4,6,9,12) \sqcup (16,15,14,13) \sqcup (19,20)$ $(9,7,4,3) \sqcup (11,12,15,13) \sqcup (16,17)$ $(12,10,7,5) \sqcup (18,15,17,20) \sqcup (9,11)$ $(9,15,8,16) \sqcup (2,11,1,5) \sqcup (12,7)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11,9,6,7) \sqcup (16,15,13,14) \sqcup (1,4)$
	$(5,3,4,7) \sqcup (16,17,20,15) \sqcup (0,2)$
	$(4,6,5,7) \sqcup (9,12,11,15) \sqcup (0,3)$
	$(16,8,15,9) \sqcup (10,1,11,6) \sqcup (0,4)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(18,15,13,14) \sqcup (11,9,12,6) \sqcup (1,2)$
	$(18,17,20,15) \sqcup (9,7,10,4) \sqcup (2,3)$
	$(11,12,14,15) \sqcup (4,6,5,7) \sqcup (17,19)$
	$(11,1,5,6) \sqcup (16,8,14,15) \sqcup (0,9)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(16,15,14,13) \sqcup (0,3,5) \sqcup (12,9,6)$
	$(11,12,15,13) \sqcup (10,9,7) \sqcup (16,18,20)$
	$(18,15,17,20) \sqcup (10,11,14) \sqcup (6,5,7)$
	$(2,12,3,11) \sqcup (8,1,7) \sqcup (4,0,5)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(11,9,12,6) \sqcup (18,15,13) \sqcup (0,1,2)$
	$(9,7,10,4) \sqcup (18,17,20) \sqcup (1,3,2)$
	$(11,12,14,15) \sqcup (4,6,7) \sqcup (17,19,20)$
	$(16,8,14,15) \sqcup (11,1,6) \sqcup (9,0,4)$
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(8,6,9,11) \sqcup (0,1,2) \sqcup (16,19) \sqcup (18,15)$
	$(8,10,7,9) \sqcup (18,17,20) \sqcup (11,14) \sqcup (2,3)$
	$(13,11,12,14) \sqcup (17,19,20) \sqcup (6,7) \sqcup (8,5)$
	$(0,5,1,7) \sqcup (3,10,2) \sqcup (4,13) \sqcup (16,6)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(11,9,12,6) \sqcup (0,1,2) \sqcup (18,15) \sqcup (13,14)$
	$(9,7,10,4) \sqcup (18,17,20) \sqcup (11,13) \sqcup (2,3)$
	$(11,12,14,15) \sqcup (17,19,20) \sqcup (8,6) \sqcup (1,3)$
	$(4,0,5,6) \sqcup (8,1,9) \sqcup (3,12) \sqcup (17,7)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(2,4,6,9,12) \sqcup (13,14) \sqcup (18,19) \sqcup (0,1)$
	$(3,4,7,9,10) \sqcup (13,15) \sqcup (1,2) \sqcup (8,5)$
	$(6,5,7,10,12) \sqcup (17,20) \sqcup (8,11) \sqcup (1,3)$
	$(4,9,15,8,16) \sqcup (1,11) \sqcup (3,12) \sqcup (2,6)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(11,9,6,4,12) \sqcup (16,15) \sqcup (8,10) \sqcup (2,3)$ $(6,7,4,3,9) \sqcup (13,15) \sqcup (18,19) \sqcup (8,5)$ $(3,5,7,10,6) \sqcup (17,20) \sqcup (8,11) \sqcup (0,1)$ $(12,8,15,9,16) \sqcup (2,11) \sqcup (0,5) \sqcup (3,13)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(13,15,16,18,14) \sqcup (9,6) \sqcup (2,4) \sqcup (5,7)$ $(14,17,16,20,15) \sqcup (4,7) \sqcup (11,13) \sqcup (5,6)$ $(9,12,10,11,15) \sqcup (6,7) \sqcup (0,2) \sqcup (3,4)$ $(5,1,10,11,6) \sqcup (9,15) \sqcup (4,12) \sqcup (0,7)$
$3\mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(18,15,13) \sqcup (11,9,6) \sqcup (0,1,2) \sqcup (16,19)$ $(18,17,20) \sqcup (9,7,10) \sqcup (1,3,2) \sqcup (11,14)$ $(11,12,14) \sqcup (4,6,7) \sqcup (17,19,20) \sqcup (8,5)$ $(11,1,6) \sqcup (16,8,14) \sqcup (9,0,4) \sqcup (10,3)$
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(9,6,4,2) \sqcup (13,14) \sqcup (18,19) \sqcup (0,1) \sqcup (10,12)$ $(9,7,4,3) \sqcup (13,15) \sqcup (1,2) \sqcup (8,5) \sqcup (16,17)$ $(10,7,5,6) \sqcup (17,20) \sqcup (8,11) \sqcup (1,3) \sqcup (9,12)$ $(9,15,8,16) \sqcup (1,11) \sqcup (3,12) \sqcup (2,6) \sqcup (0,5)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(16,15,18,13) \sqcup (9,6) \sqcup (2,4) \sqcup (5,7) \sqcup (0,1)$ $(16,17,20,14) \sqcup (4,7) \sqcup (11,13) \sqcup (5,6) \sqcup (1,3)$ $(9,12,10,11) \sqcup (6,7) \sqcup (0,2) \sqcup (3,4) \sqcup (8,5)$ $(10,1,11,5) \sqcup (9,15) \sqcup (4,12) \sqcup (0,7) \sqcup (8,3)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(11,9,6) \sqcup (0,1,2) \sqcup (18,15) \sqcup (16,19) \sqcup (17,20)$ $(9,7,10) \sqcup (1,3,2) \sqcup (17,18) \sqcup (11,14) \sqcup (8,5)$ $(11,12,14) \sqcup (4,6,7) \sqcup (19,20) \sqcup (13,15) \sqcup (3,5)$ $(11,1,6) \sqcup (16,8,14) \sqcup (0,9) \sqcup (10,3) \sqcup (17,13)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(0,1,2) \sqcup (18,15) \sqcup (9,11) \sqcup (16,19) \sqcup (5,6) \sqcup (10,7)$ $(1,3,2) \sqcup (17,18) \sqcup (9,7) \sqcup (11,14) \sqcup (8,5) \sqcup (16,13)$ $(4,6,7) \sqcup (12,14) \sqcup (3,5) \sqcup (13,15) \sqcup (17,20) \sqcup (18,19)$ $(16,8,14) \sqcup (1,11) \sqcup (0,9) \sqcup (10,3) \sqcup (17,13) \sqcup (2,7)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,1,\infty,2,4,5,3) \sqcup (12,15)$ $(0,2,5,\infty,6,4,1) \sqcup (10,11)$ $(5,7,\infty,3,6,9,10) \sqcup (13,14)$ $(\infty,4,7,10,8,6,5) \sqcup (16,15)$ $(0,4,9,15,8,16,7) \sqcup (1,11)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(3,5,4,2,\infty,8,1) \sqcup (12,15)$ $(4,6,\infty,5,2,0,18) \sqcup (10,11)$ $(10,9,6,3,\infty,0,7) \sqcup (12,14)$ $(5,6,8,10,7,4,9) \sqcup (0,1)$ $(16,8,15,9,4,0,6) \sqcup (1,11)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(3,5,4,2,\infty,1,6) \sqcup (9,10)$ $(0,2,5,\infty,6,4,1) \sqcup (10,11)$ $(5,7,\infty,3,6,9,8) \sqcup (13,14)$ $(\infty,4,7,10,8,6,1) \sqcup (12,15)$ $(7,16,8,15,9,4,12) \sqcup (1,11)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(1,2,4,5,8,0,\infty) \sqcup (11,13)$ $(4,\infty,5,2,3,8,6) \sqcup (16,13)$ $(6,7,\infty,10,13,8,5) \sqcup (19,20)$ $(11,10,7,4,1,8,12) \sqcup (13,15)$ $(7,15,9,4,0,8,6) \sqcup (1,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(5,4,2,3,6,0,1) \sqcup (9,\infty)$ $(2,5,\infty,6,4,8,11) \sqcup (16,13)$ $(10,\infty,7,8,11,5,6) \sqcup (12,13)$ $(4,7,10,8,5,11,12) \sqcup (13,15)$ $(4,9,15,8,12,6,7) \sqcup (1,11)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(8,5,4,2,0,6,\infty) \sqcup (11,13)$ $(3,2,5,\infty,8,1,6) \sqcup (16,13)$ $(5,7,\infty,3,4,8,6) \sqcup (13,14)$ $(\infty,4,7,10,8,1,12) \sqcup (13,15)$ $(0,4,9,15,8,12,6) \sqcup (1,11)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(1,2,4,5,7,0,3) \sqcup (8,11)$ $(11,\infty,6,4,5,8,12) \sqcup (10,13)$ $(6,7,\infty,10,2,8,5) \sqcup (9,12)$ $(11,10,8,5,6,12,7) \sqcup (16,13)$ $(7,15,9,4,13,8,6) \sqcup (1,11)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(1,2,4,6,0,3,5) \sqcup (8,11)$ $(11,\infty,6,5,8,2,12) \sqcup (13,15)$ $(6,7,\infty,10,8,4,5) \sqcup (11,12)$ $(11,10,8,5,12,13,7) \sqcup (9,6)$ $(6,15,9,4,8,11,7) \sqcup (2,12)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(5,4,2,0,1,3,6) \sqcup (9,\infty)$ $(4,6,\infty,1,2,12,13) \sqcup (8,11)$ $(10,\infty,7,5,3,6,9) \sqcup (13,15)$ $(5,8,10,11,\infty,7,4) \sqcup (9,12)$ $(4,9,15,8,12,7,16) \sqcup (1,11)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(5,4,2,3,6,\infty,0) \sqcup (8,7)$ $(13,12,\infty,6,4,10,1) \sqcup (8,11)$ $(10,\infty,7,6,9,2,5) \sqcup (13,15)$ $(5,8,10,7,4,9,11) \sqcup (16,19)$ $(4,9,15,8,12,18,7) \sqcup (1,11)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(3,5,4,2,\infty,1) \sqcup (13,12,15)$ $(0,2,5,\infty,6,4) \sqcup (8,11,10)$ $(5,7,\infty,3,6,9) \sqcup (13,14,15)$ $(\infty,4,7,10,8,6) \sqcup (17,16,15)$ $(0,4,9,15,8,16) \sqcup (1,11,2)$
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(\infty,2,4,5,8,0) \sqcup (11,13,12)$ $(6,\infty,5,2,3,8) \sqcup (13,16,15)$ $(6,3,\infty,7,5,4) \sqcup (13,14,15)$ $(8,10,7,4,\infty,12) \sqcup (18,15,13)$ $(0,4,9,15,8,12) \sqcup (1,11,2)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(5,4,2,3,6,0) \sqcup (9,\infty,11)$ $(4,6,\infty,12,13,1) \sqcup (11,8,7)$ $(10,\infty,7,6,9,5) \sqcup (16,15,13)$ $(5,8,10,7,4,11) \sqcup (16,19,17)$ $(7,0,4,9,15,12) \sqcup (1,11,2)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(5,4,7,2,1,3) \sqcup (8,11,\infty)$ $(12,\infty,8,6,4,5) \sqcup (13,10,7)$ $(10,\infty,2,7,8,5) \sqcup (19,16,14)$ $(11,10,12,8,5,6) \sqcup (16,13,14)$ $(7,0,6,4,9,12) \sqcup (1,11,2)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(1,2,4,5,0,3) \sqcup (8,11,14)$ $(11,\infty,6,4,8,5) \sqcup (10,13,12)$ $(6,7,\infty,3,8,5) \sqcup (9,12,15)$ $(11,10,8,6,12,7) \sqcup (13,16,\infty)$ $(8,0,4,9,6,7) \sqcup (1,11,2)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(1,2,0,3,4,5) \sqcup (11,8,\infty)$ $(2,\infty,3,4,5,6) \sqcup (12,13,15)$ $(6,7,8,4,5,\infty) \sqcup (11,12,15)$ $(11,10,8,12,13,7) \sqcup (9,6,4)$ $(4,0,8,5,6,7) \sqcup (1,11,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(5,4,2,\infty,1) \sqcup (11,13,12,15)$ $(0,2,5,\infty,6) \sqcup (8,11,10,12)$ $(5,7,\infty,3,6) \sqcup (16,13,14,15)$ $(\infty,4,7,10,8) \sqcup (17,16,15,13)$ $(4,9,15,8,16) \sqcup (2,11,1,5)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(\infty,2,4,5,0) \sqcup (11,13,12,15)$ $(6,\infty,5,2,1) \sqcup (8,11,10,12)$ $(6,3,\infty,7,1) \sqcup (16,13,14,15)$ $(10,7,4,\infty,5) \sqcup (17,16,15,13)$ $(16,8,15,9,12) \sqcup (2,11,1,6)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(\infty, 2, 4, 3, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (10, 12, 11, 15)$ $(6, 3, \infty, 7, 1) \sqcup (12, 14, 13, 15)$ $(\infty, 4, 7, 10, 1) \sqcup (17, 16, 13, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(0, 2, 1, 3, 4) \sqcup (11, 8, \infty, 6)$ $(2, \infty, 3, 4, 5) \sqcup (9, 12, 13, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15, 14)$ $(0, 3, 1, 5, 6) \sqcup (16, 13, 11, 10)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(10, 13, \infty, 8, 11) \sqcup (1, 2, 3, 4)$ $(15, 13, 12, 9, 7) \sqcup (3, \infty, 4, 5)$ $(11, 12, 15, 14, 13) \sqcup (4, 7, 5, \infty)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12, 7)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(0, 2, 3, 4, 5) \sqcup (9, 8, 11, \infty)$ $(2, \infty, 3, 4, 5) \sqcup (12, 13, 14, 15)$ $(4, 7, 8, 5, \infty) \sqcup (10, 12, 11, 15)$ $(0, 3, 1, 4, 6) \sqcup (16, 13, 11, \infty)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(3, 5, 4, 2, \infty, 1) \sqcup (19, 20) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (17, 18) \sqcup (8, 11)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14) \sqcup (0, 1)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (16, 15) \sqcup (2, 3)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(\infty, 2, 4, 5, 8, 0) \sqcup (18, 20) \sqcup (12, 13)$ $(13, \infty, 5, 2, 3, 8) \sqcup (9, 6) \sqcup (16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14) \sqcup (0, 1)$ $(15, 17, 14, 11, \infty, 19) \sqcup (8, 6) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(3,2,4,5,0,1) \sqcup (18,15) \sqcup (11,14)$ $(5,\infty,6,4,8,11) \sqcup (10,13) \sqcup (19,20)$ $(8,7,\infty,3,5,6) \sqcup (16,19) \sqcup (12,15)$ $(7,10,8,6,11,12) \sqcup (16,13) \sqcup (9,\infty)$ $(6,0,8,4,5,7) \sqcup (1,11) \sqcup (3,12)$
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(5,4,7,2,1,3) \sqcup (8,11) \sqcup (18,\infty)$ $(12,\infty,8,6,4,5) \sqcup (0,3) \sqcup (10,13)$ $(10,\infty,2,7,8,5) \sqcup (9,6) \sqcup (16,19)$ $(11,10,12,8,5,6) \sqcup (13,14) \sqcup (0,2)$ $(7,0,6,4,9,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(5,4,2,3,6,0) \sqcup (9,12) \sqcup (11,\infty)$ $(4,6,\infty,12,13,15) \sqcup (0,1) \sqcup (8,11)$ $(10,\infty,7,6,9,5) \sqcup (13,15) \sqcup (1,2)$ $(5,8,10,7,4,11) \sqcup (17,19) \sqcup (9,\infty)$ $(7,0,4,9,15,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(1,2,0,3,4,5) \sqcup (\infty,15) \sqcup (8,11)$ $(11,\infty,2,3,5,6) \sqcup (13,15) \sqcup (19,20)$ $(6,7,8,4,5,\infty) \sqcup (18,19) \sqcup (12,15)$ $(11,10,8,12,13,7) \sqcup (18,20) \sqcup (9,6)$ $(11,1,8,9,10,7) \sqcup (0,5) \sqcup (2,6)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(10,13,\infty,8,11) \sqcup (3,2,4) \sqcup (16,15)$ $(15,13,12,9,7) \sqcup (10,\infty,5) \sqcup (11,14)$ $(11,12,15,14,13) \sqcup (4,\infty,7) \sqcup (0,3)$ $(3,4,6,9,\infty) \sqcup (8,10,12) \sqcup (5,7)$ $(0,9,1,8,2) \sqcup (5,10,6) \sqcup (3,13)$
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(8,\infty,13,10,9) \sqcup (3,2,4) \sqcup (14,15)$ $(7,9,12,13,8) \sqcup (10,\infty,5) \sqcup (11,14)$ $(11,12,15,18,14) \sqcup (4,\infty,7) \sqcup (0,3)$ $(9,6,4,3,8) \sqcup (19,17,15) \sqcup (13,14)$ $(1,8,0,9,2) \sqcup (5,10,6) \sqcup (3,13)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(2, \infty, 3, 4, 5) \sqcup (12, 13, 15) \sqcup (16, 19)$
	$(0, 2, 1, 3, 4) \sqcup (8, \infty, 6) \sqcup (18, 15)$
	$(4, 7, 5, 6, \infty) \sqcup (11, 12, 15) \sqcup (0, 1)$
	$(8, 10, 12, 13, 7) \sqcup (9, 6, 4) \sqcup (17, 18)$
	$(9, 0, 8, 6, 7) \sqcup (11, 1, 5) \sqcup (10, 15)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(1, \infty, 16, 18) \sqcup (11, 13, 12, 15) \sqcup (4, 5)$
	$(2, 5, \infty, 6) \sqcup (8, 11, 10, 12) \sqcup (9, 7)$
	$(0, \infty, 3, 6) \sqcup (16, 13, 14, 15) \sqcup (5, 7)$
	$(10, 7, 4, \infty) \sqcup (17, 16, 15, 13) \sqcup (1, 3)$
	$(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11, 9, \infty, 1) \sqcup (10, 12, 13, 15) \sqcup (4, 5)$
	$(2, 5, \infty, 6) \sqcup (8, 11, 10, 13) \sqcup (9, 7)$
	$(0, \infty, 17, 20) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$
	$(10, 7, 4, \infty) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$
	$(2, 12, 6, 15) \sqcup (8, 0, 5, 7) \sqcup (9, 13)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(18, 16, 19, \infty) \sqcup (10, 12, 13, 15) \sqcup (3, 6)$
	$(1, \infty, 12, 6) \sqcup (8, 11, 10, 13) \sqcup (4, 5)$
	$(0, \infty, 3, 4) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$
	$(9, 7, 10, 4) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$
	$(9, 0, 8, 7) \sqcup (11, 1, 5, 6) \sqcup (10, 4)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$
	$(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$
	$(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$
	$(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$
	$(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (5, 3, 6)$
	$(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (3, 4, 5)$
	$(0, \infty, 3, 4) \sqcup (12, 14, 13) \sqcup (6, 8, 7)$
	$(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (2, 1, 3)$
	$(9, 0, 8, 7) \sqcup (5, 1, 6) \sqcup (10, 4, 14)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(11,13,12,15) \sqcup (9,\infty,1) \sqcup (4,5) \sqcup (16,18)$ $(8,11,10,12) \sqcup (19,\infty,6) \sqcup (2,5) \sqcup (16,14)$ $(8,10,7,4) \sqcup (0,\infty,11) \sqcup (16,17) \sqcup (9,6)$ $(5,7,8,6) \sqcup (20,17,\infty) \sqcup (13,14) \sqcup (1,2)$ $(3,10,5,11) \sqcup (0,9,1) \sqcup (2,12) \sqcup (17,13)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(18,16,19,\infty) \sqcup (13,12,15) \sqcup (3,5) \sqcup (17,20)$ $(1,\infty,12,6) \sqcup (8,11,13) \sqcup (4,5) \sqcup (17,18)$ $(3,\infty,4,7) \sqcup (12,14,13) \sqcup (8,6) \sqcup (1,2)$ $(9,7,10,4) \sqcup (17,16,13) \sqcup (1,3) \sqcup (14,15)$ $(9,0,8,7) \sqcup (11,1,6) \sqcup (18,12) \sqcup (10,14)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(4,1,\infty,13,10) \sqcup (2,3) \sqcup (16,15) \sqcup (9,11)$ $(5,\infty,10,11,13) \sqcup (4,7) \sqcup (0,2) \sqcup (9,12)$ $(7,\infty,4,5,8) \sqcup (17,19) \sqcup (0,3) \sqcup (12,14)$ $(7,8,6,9,\infty) \sqcup (13,14) \sqcup (1,3) \sqcup (19,20)$ $(1,11,2,10,3) \sqcup (0,6) \sqcup (9,4) \sqcup (8,12)$
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(1,\infty,13,10,7) \sqcup (2,3) \sqcup (16,15) \sqcup (9,11)$ $(5,\infty,10,11,16) \sqcup (4,7) \sqcup (0,2) \sqcup (9,12)$ $(6,4,5,8,\infty) \sqcup (17,19) \sqcup (0,3) \sqcup (12,14)$ $(7,8,6,9,11) \sqcup (13,14) \sqcup (1,3) \sqcup (19,20)$ $(3,10,2,11,5) \sqcup (0,6) \sqcup (4,8) \sqcup (17,7)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(1,\infty,13,5,7) \sqcup (2,3) \sqcup (16,15) \sqcup (9,11)$ $(0,3,1,4,\infty) \sqcup (2,5) \sqcup (9,7) \sqcup (10,13)$ $(12,11,13,14,\infty) \sqcup (17,19) \sqcup (5,7) \sqcup (9,6)$ $(5,8,11,6,7) \sqcup (13,14) \sqcup (2,\infty) \sqcup (19,20)$ $(6,0,8,9,7) \sqcup (1,11) \sqcup (10,5) \sqcup (16,12)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(11,13,12,15) \sqcup (9,\infty,1) \sqcup (2,4,5)$ $(8,11,10,12) \sqcup (19,\infty,6) \sqcup (0,2,5)$ $(0,\infty,3,6) \sqcup (16,13,14) \sqcup (8,7,5)$ $(17,16,15,13) \sqcup (\infty,4,7) \sqcup (0,3,1)$ $(9,15,8,16) \sqcup (11,1,5) \sqcup (3,12,7)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(9,\infty,8,6) \sqcup (12,15) \sqcup (16,17) \sqcup (1,2) \sqcup (19,20)$ $(5,\infty,13,14) \sqcup (9,6) \sqcup (0,2) \sqcup (1,4) \sqcup (17,19)$ $(0,\infty,4,3) \sqcup (10,7) \sqcup (16,18) \sqcup (2,5) \sqcup (11,14)$ $(18,20,17,\infty) \sqcup (4,5) \sqcup (12,14) \sqcup (8,10) \sqcup (0,1)$ $(0,9,1,11) \sqcup (10,3) \sqcup (12,6) \sqcup (19,14) \sqcup (17,13)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(8,\infty,9,5) \sqcup (12,15) \sqcup (16,17) \sqcup (1,2) \sqcup (3,4)$ $(15,13,14,\infty) \sqcup (9,6) \sqcup (0,2) \sqcup (1,4) \sqcup (17,19)$ $(0,\infty,3,4) \sqcup (10,7) \sqcup (16,18) \sqcup (2,5) \sqcup (11,14)$ $(17,20,18,19) \sqcup (4,5) \sqcup (12,14) \sqcup (8,10) \sqcup (0,1)$ $(9,0,8,7) \sqcup (1,11) \sqcup (12,6) \sqcup (10,5) \sqcup (16,20)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(8,\infty,9) \sqcup (13,12,15) \sqcup (4,5) \sqcup (16,18) \sqcup (1,2)$ $(19,\infty,6) \sqcup (11,10,12) \sqcup (2,5) \sqcup (18,20) \sqcup (1,4)$ $(11,\infty,14) \sqcup (10,7,4) \sqcup (16,17) \sqcup (0,2) \sqcup (1,3)$ $(20,17,\infty) \sqcup (14,13,15) \sqcup (5,7) \sqcup (9,6) \sqcup (0,1)$ $(0,9,4) \sqcup (2,10,3) \sqcup (12,6) \sqcup (17,7) \sqcup (1,5)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(8,\infty,9) \sqcup (12,15) \sqcup (4,5) \sqcup (16,18) \sqcup (1,2) \sqcup (19,20)$ $(5,\infty,13) \sqcup (9,6) \sqcup (0,2) \sqcup (18,20) \sqcup (1,4) \sqcup (17,19)$ $(11,\infty,14) \sqcup (4,7) \sqcup (16,17) \sqcup (2,5) \sqcup (8,10) \sqcup (0,3)$ $(20,17,\infty) \sqcup (13,14) \sqcup (5,7) \sqcup (10,11) \sqcup (0,1) \sqcup (8,6)$ $(0,9,4) \sqcup (2,10,3) \sqcup (12,6) \sqcup (17,7) \sqcup (1,5)$

Table 8.2: (1-2-3)-labelings

No.	Block	No.	Block
1	$(15,14,16,17,18,19,20) \sqcup (0,2)$	2	$(13,15,16,17,18,19,20) \sqcup (0,6)$
3	$(8,16,12,17,18,19,20) \sqcup (9,3)$	4	$(8,17,9,11,18,19,20) \sqcup (16,0)$
5	$(8,18,9,11,13,19,20) \sqcup (0,1)$	6	$(8,19,10,11,12,13,20) \sqcup (0,15)$
7	$(8,1,9,10,11,12,13) \sqcup (18,7)$	8	$(1,2,9,10,11,12,13) \sqcup (14,7)$
9	$(0,3,2,6,11,12,13) \sqcup (8,7)$	10	$(0,4,2,3,11,12,13) \sqcup (8,9)$
11	$(0,5,2,3,4,12,13) \sqcup (9,10)$	12	$(1,6,2,4,5,12,13) \sqcup (15,7)$
13	$(1,7,2,3,4,5,6) \sqcup (0,14)$	14	$(3,8,4,5,6,14,20) \sqcup (12,15)$
15	$(4,9,5,6,14,15,20) \sqcup (16,7)$	16	$(15,10,4,5,6,16,20) \sqcup (0,18)$
17	$(15,11,0,5,6,16,20) \sqcup (17,1)$	18	$(14,12,0,11,17,18,20) \sqcup (8,2)$
19	$(16,13,0,11,12,17,20) \sqcup (1,19)$	20	$(1,14,2,3,4,5,6) \sqcup (20,7)$
21	$(1,15,2,3,4,5,6) \sqcup (19,7)$	22	$(1,16,2,3,4,5,6) \sqcup (17,7)$
23	$(0,17,2,3,4,5,6) \sqcup (11,14)$	24	$(1,18,2,3,4,5,6) \sqcup (10,14)$
25	$(0,19,2,3,4,5,6) \sqcup (13,14)$	26	$(0,20,2,3,4,5,6) \sqcup (10,11)$
27	$(9,7,0,10,11,12,13) \sqcup (1,3)$	28	$(10,8,0,11,12,13,15) \sqcup (1,4)$
29	$(11,9,0,12,13,16,19) \sqcup (1,5)$	30	$(12,10,0,3,13,17,18) \sqcup (1,20)$

Table 8.3: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^{1}}$-decomposition of K_{21}

No.	Block	No.	Block
1	$(15,14,16,17,18,19,20) \sqcup (0,2)$	2	$(13,15,16,17,18,19,20) \sqcup (0,6)$
3	$(8,16,12,17,18,19,20) \sqcup (9,3)$	4	$(8,17,9,11,18,19,20) \sqcup (16,0)$
5	$(8,18,9,11,13,19,20) \sqcup (0,1)$	6	$(8,19,10,11,12,13,20) \sqcup (0,15)$
7	$(8,1,9,10,11,12,13) \sqcup (6,\infty)$	8	$(1,2,9,10,11,12,13) \sqcup (14,7)$
9	$(0,3,2,6,11,12,13) \sqcup (8,7)$	10	$(0,4,2,3,11,12,13) \sqcup (8,9)$
11	$(0,5,2,3,4,12,13) \sqcup (9,10)$	12	$(1,6,2,4,5,12,13) \sqcup (15,7)$
13	$(1,7,2,3,4,5,6) \sqcup (13,\infty)$	14	$(3,8,4,5,6,14,20) \sqcup (12,15)$
15	$(4,9,5,6,14,15,20) \sqcup (16,7)$	16	$(15,10,4,5,6,16,20) \sqcup (0,18)$
17	$(15,11,0,5,6,16,20) \sqcup (17,1)$	18	$(14,12,0,11,17,18,20) \sqcup (8,2)$
19	$(16,13,0,11,12,17,20) \sqcup (1,19)$	20	$(1,14,2,3,4,5,6) \sqcup (20,7)$

Table 8.4: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^{1}}$-decomposition of K_{22}

No.	Block	No.	Block
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (20, \infty)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$
31	$(0, \infty, 1, 2, 3, 4, 5) \sqcup (18, 7)$	32	$(14, \infty, 15, 16, 17, 18, 19) \sqcup (1, 3)$
33	$(7, \infty, 8, 9, 10, 11, 12) \sqcup (0, 14)$		

Table 8.4: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^{1}}$-decomposition of K_{22}

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Appendix A

Glossary and Acronyms

Care has been taken in this thesis to minimize the use of jargon and acronyms, but this cannot always be achieved. This appendix defines jargon terms in a glossary, and contains a table of acronyms and their meaning.

A.1 Glossary

- **Cosmic-Ray Muon** (**CR** μ) – A muon coming from the abundant energetic particles originating outside of the Earth's atmosphere.

A.2 Acronyms

Table A.1: Acronyms

Acronym	Meaning
CRμ	Cosmic-Ray Muon