Seven Edge Forest Designs

A THESIS
SUBMITTED TO THE FACULTY OF THE GRADUATE SCHOOL
OF THE UNIVERSITY OF MINNESOTA
BY

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IN PARTIAL FULFILLMENT OF THE REQUIREMENTS
FOR THE DEGREE OF
MASTER OF SCIENCE

Professor Bryan Freyberg

May, 2025

Acknowledgements

There are many people that have earned my gratitude for their contribution to my time in graduate school. However, without Professor Bryan Freyberg I would not have had the opportunity to become a graduate student, and Professor Joseph Gallian's courses and guidance has been very influential to the way I organize myself mathematically today.

Dedication

I dedicate this Thesis to my advisor Professor Bryan Freyberg, to my family who has supported me throughout this process, and to Jordi, Ian, TK, and Torta from Tuscarora Ave.

Abstract

Let G be a subgraph of K_n where $n \in \mathbb{N}$. A G-decomposition of K_n, or G-design of order n, is a finite collection $\mathcal{G} = \{G_1, \ldots, G_k\}$ of pairwise edge-disjoint subgraphs of K_n that are all isomorphic to some graph G. We prove that an F-decomposition of K_n exists for every seven-edge forest F if and only if $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$.

Along the way, we introduce new methods, constraint programming algorithms in Python, and some bonus results for Galaxy graph decompositions of complete bipartite, and eventually multipartite graphs.

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Chapter 1

Background

1.1 Fundamentals of Graph Theory

Graph Theory is the study of objects called *vertices* or *nodes* and their relationships which we call *edges*. An edge between vertices u and v is typically denoted uv or (u, v). A graph G is formally defined as an ordered pair $G = (V, E)$ where V is the set of all vertices in G and E is the set of all edges between vertices in G. These sets are sometimes refered to as $V(G)$ and $E(G)$, respectively.

G is called a *simple graph* if: (1) there is at most 1 edge between any two vertices, (2) there are no edges from a vertex to itself and (3) all edges have no directionality to them, meaning $uv = vu$ for any edge $uv \in E(G)$. For the rest of this paper all graphs are finite simple graphs, but note that unions and subgraphs are defined the same way for directed graphs and infinite graphs.

Graphs are more intuitive to work with through their visual representations instead of their formal definitions. Let the simple graph G where $V(G) = \{A, B, C, D, E, a, b, c, d, e\}$ and $E(G) = \{Aa, Bb, Cc, Dd, Ee, AB, BC, CD, DE, EA, ac, ce, eb, bd, da\}$. G is often called the *Petersen* graph. It's unwieldy when described formally, yet its visual representation is very easy to understand.

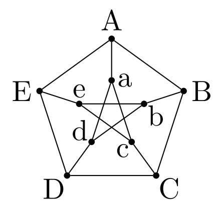

Figure 1.1: The Petersen graph

We say two graphs G and H are *isomorphic* if there exists a bijection from $V(G)$ to $V(H)$ that induces a bijection from $E(G)$ to $E(H)$ and we denote this relationship via $G \equiv H$. In other words, we consider two graphs G, H to be the 'same' if we can relabel and/or move vertices in some fashion (without adding/removing vertices edges) in a visual representations of G and H to go back and forth between the two.

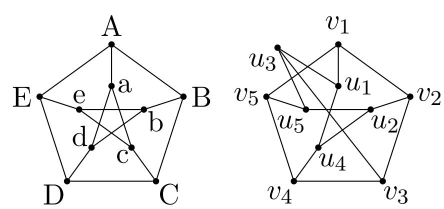

Figure 1.2: $G \cong H$

Graph theorists casually refer to two graphs as the 'same' graph if they are in the same isomorphism class. We will wrap up the fundamentals with a few mdefinitions and some algebraic tools.

Definition 1.1.1 (Subgraph)**.** A subgraph $G \subseteq K$ is a graph whose vertices and edges are subsets of the vertices and edges of K; $G \subseteq K$ if $V(G) \subseteq V(K)$ and $E(G) \subseteq E(K)$.

Definition 1.1.2 (Vertex-induced Subgraph)**.** A *vertex-induced* subgraph $G \subseteq K$ is one whose vertices are some subset of $V(K)$ and whose edges are all edges between those vertices in K; $V(G) \subseteq V(K)$ and $E(G) = \{uv \in E(K) \mid u, v \in E(G)\}$. If G is such a subgraph we say that G is induced by $S = V(G) \subseteq V(K)$.

Definition 1.1.3 (Edge-induced Subgraph)**.** An *edge-induced* subgraph $G \subseteq K$ is one whose edges are some subset of $E(K)$ and whose vertices are all those who appear as an endpoint in that subset of edges; $E(G) \subseteq E(K)$ and $V(G) = \{v \in V(K) \mid vu \in E(G) \text{or } uv \in E(G) \text{ for some } u \in V(K)\}$. If G is such a subgraph we say that G is induced by $S = E(G) \subset E(K)$

Here is a visual example of these types of graphs: Let K be the Petersen graph from Figure 1.1. Now, let

Subgraph: $G \subseteq K$ where $V(G) = \{E, e, b\}$, $E(G) = \{Ee\}$.

Vertex-induced Subgraph: $H \subseteq K$ is induced by $\{a, A, B\} \subseteq V(K)$

Edge-induced Subgraph: $M \subseteq K$ is induced by $\{Dd, DC, Cc\} \subseteq E(K)$

The figure below shows K and it's color-coded subgraphs.

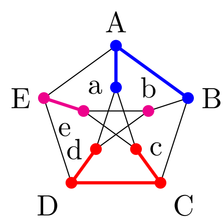

Figure 1.3: K and subgraphs $G, H, M \subseteq K$

Next, we will talk about two important operations done on graphs.

Definition 1.1.4 (Graph Union)**.** The union of two graphs G and H is simply the graph resulting from the union of their vertices and the union of their edges and is denoted $G \cup H$; $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$. If G and H are edge-disjoint, we denote their union via $G \sqcup H$ and call it a *disjoint union* of G and H.

Here is an example of a union and a disjoint union of graphs. Let $G = (\{a, b, c, d\}, \{ab, bc, cd, da\}), H = (\{a, b, c\}, \{ab, bc, ca\}),$ and $K = (\{A, B, C\}, \{AB, BC, CA\})$ Then:

$$G \cup H = (\{a, b, c, d\}, \{ab, bc, cd, da, ca\})$$

$$G \sqcup K = V(G \sqcup K) = (\{a, b, c, d, A, B, C\}, \{ab, bc, cd, da, AB, BC, CA\})$$

These unions are depicted in the following figure.

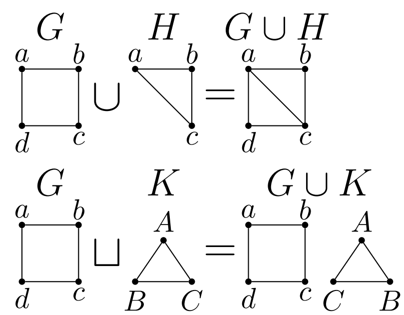

Figure 1.4: (above) $G \cup H$ and (below) $G \sqcup K$

Next, we define another very important operation that combines two graphs in a different manner.

Definition 1.1.5 (Join)**.** Let G and H be vertex disjoint graphs. Their *join*, denoted via $G \vee H$, is the graph obtained by taking the disjoint union of G and H and adding all possible edges between every vertex in G and every vertex in H. Formally:

$$G \vee H \;=\; (V(G) \cup V(H), E(G) \;\cup\; E(H) \;\cup\; \{\, xy \mid x \in V(G),\, y \in V(H)\}).$$

Here is an example. Let $G = (\{a, b, c\}, \{ab, bc, ca\})$ and $H = (\{A, B, C\}, \{AB, BC, CA\})$, then $G \vee H = (\{a, b, c, A, B, C\}, E(G) \sqcup E(H) \sqcup \{aA, aB, aC, bA, bB, bC, cA, cB, cC\})$. This join is depicted in the figure below.

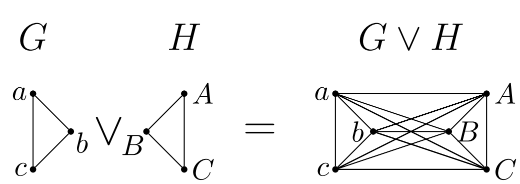

Figure 1.5: (above) $G \cup H$ and (below) $G \sqcup K$

Lastly, we define a few characteristics of graphs and their components. These may or may not be used frequently in this paper, but are important concepts to know in order to be able to talk about graphs comfortably.

Let G be a simple graph. We say two vertices $u, v \in V(G)$ are *adjacent* or *neighbors* if they share an edge $uv \in E(G)$. Similarly, we say a vertex is *incident* with an edge if it is one of it's endpoints; $u \in V(G)$ is incident with $e \in E(G)$ if $e = uv$ for some $v \in V(G)$. The set of all vertices adjacent to u in G is called the *neighborhood* of v denoted $N_G(v)$ or simply $N(v)$. Sometimes this is refered to as the open neighborhood of v in G and then the closed neighborhood is defined via $N_G[v] = N_G(v) \cup \{v\}$. The *degree* of a vertex $v \in V(G)$ is the number of vertices adjacent to it and is denoted via $deg_G(v) = |N_G(v)|$ or simply $deg(v)$. Equivalently, the degree is the number or edges incident to it or the number of neighbors that G has.

The following are three similar types of objects we can form from graphs.

Definition 1.1.6 (Walk)**.** Let G be a graph on n vertices. A *walk* in G is a sequence $(w_0, w_1, \ldots, w_k)$ of vertices in G whose adjacent elements must be adjacent in G. Adjacent elements in a walk much be distinct vertices but a vertex may be repeated multiple times.

Definition 1.1.7 (Path)**.** Let G be a graph on n vertices. A *path* in G is a sequence $(v_0, v_1, \ldots, v_k)$ of distinct vertices in G whose adjacent elements must be adjacent in G, and where no vertex is repeated. This sequence gives the subgraph of G induced by $\{v_0v_1, v_1v_2, \ldots, v_{k-1}v_k\}$.

Definition 1.1.8 (Cycle)**.** Let G be a graph on n vertices. A *cycle* in G is a sequence $(v_0, v_1, \ldots, v_k, v_0)$ of internally distinct vertices (distinct except on the endpoints) that begins and terminates at the same vertex v_0. Often such a cycle is denoted via $(v_0v_1 \cdots v_k)$ and it is understood that the sequence wraps back around to v_0 after v_k. Additionally, the cycle $(v_0v_1 \cdots v_k)$ is equivalent to $(v_1 \cdots v_kv_0)$, $(v_2 \cdots v_kv_0v_1), \ldots$ and so on.

let G be a simple graph. We call G *acyclic* if it contains no cycles. If there exists a path from any vertex to every other vertex in G, then we call G *connected.* If not, we call G *disconnected.* We call the set of connected subgraphs of G whose disjoint union equals G the *connected components* of G.

This concludes the fundamental concepts needed to understand this project. The next and final section of this chapter will introduce all the fundamental families of graphs we refer to in the proceeding chapters.

1.2 Fundamental Families of Graphs

In this section introduce some fundamental families of graphs which we refer to throughout this paper. Often instead of fully defining the graphs being worked with, we simply refer to it as a member of a larger family of graphs. These families are not completely distinct, but sometimes it is helpful to view graphs as a member of one family or another depending on the context.

Recall that a graph is acyclic if it contains no cycles. Similarly, we call a graph *k-cyclic* if it contains exactly k distinct cycles. If $k = 2$ or 3 we call it *bicyclic* or *tricyclic*, respectively. In a similar vein, we call a graph *k-partite* if we can partition it's vertices into k disjoint sets.If $k = 2$ or 3, we call it *bipartite* or *tripartite*, respectively. These are all very broad families of graphs often used to characterize graphs within another family. The following are more nuanced, and more popular families of graphs to work with.

Definition 1.2.1 (Complete Graph)**.** The *complete graph* on n vertices, denoted K_n, is the graph on n vertices such that every pair of distinct vertices shares an edge.

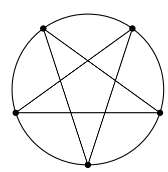

Figure 1.6: The Complete Graph K_5

Definition 1.2.2 (Complete Bipartite Graph)**.** Let $m, n \in \mathbb{N}$. The *complete bipartite graph $K_{m,n}$* is the bipartite graph whose vertices can be partitioned into two disjoint sets of sizes m and n, respectively, such that every vertex in the one partite set is adjacent to every vertex in the other partite set and there no edges between vertices in the same partite set.

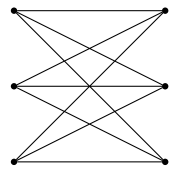

Figure 1.7: The Complete Bipartite Graph $K_{3,3}$

Definition 1.2.3 (Complete Multipartite Graph)**.** The *complete k-partite graph* or *complete multipartite graph* $K_{n_1,\dots,n_k}$ is the graph whose vertices can be partitioned into k disjoint sets of sizes n_1, n_2, $\dots$, n_k, respectively such that such that every vertex in the one partite set is adjacent to every vertex in the other $k-1$ partite sets and there no edges between vertices in the same partite set.

If all partite sets are the same size n we call this graph the *complete equipartite graph* $K_{n:k}$ or $K_{n\times m}$.

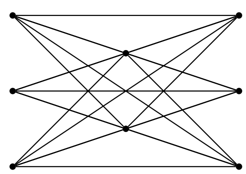

Figure 1.8: The Complete Multipartite Graph $K_{3,2,3}$

Definition 1.2.4 (Cycle Graph)**.** The *cycle graph* on n vertices denoted C_n is a graph with exactly one cycle containing all of it's edges.

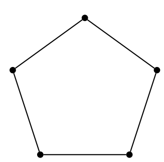

Figure 1.9: The Cycle Graph C_5

Definition 1.2.5 (Tree)**.** A *tree* is any connected acyclic graph. Trees on n vertices have $n-1$ edges. Equivalently, these graphs are any connected bipartite graphs.

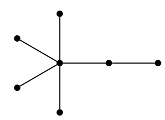

Figure 1.10: A Tree Graph on 6 vertices

Definition 1.2.6 (Path Graph)**.** The *path* graph on n vertices, denoted P_n, is an acyclic graph with exactly one path containing all of it's edges. All paths are trees.

Figure 1.11: The Path Graph P_4

Definition 1.2.7 (Star Graph)**.** The *star graph* on $n + 1$ vertices, denoted $K_{1,n}$ (or S_{n+1} which we never use in this paper) consisting of one central *hub* vertex adjacent to n *outer* vertices, with no other edges. All stars are trees. Sometimes this graph is refered to as an *n-star*.

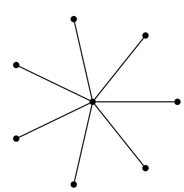

Figure 1.12: The 7-star ($K_{1,7}$)

Definition 1.2.8 (Forest Graph)**.** Any disjoint union of tree graphs is called a *forest* graph. These graphs are all bipartite and can be equivalently defined as disconnected bipartite graphs.

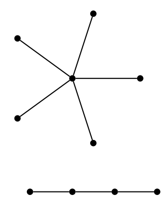

Figure 1.13: A forest on 9 vertices

Definition 1.2.9 (Galaxy Graph)**.** Any disjoint union of star graphs is called a *galaxy* graph. We refer to $G = G_1 \sqcup \cdot \sqcup G_k$ as a $(G_1, \ldots, G_k)$-*galaxy* graph if $G_1, \ldots, G_k$ are all stars. This family is a subset of the forest family.

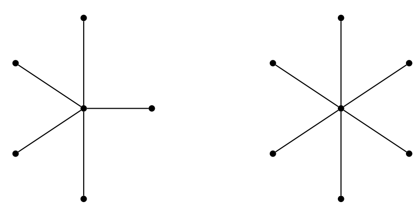

Figure 1.14: The $(K_{1,6}, K_{1,7})$-Galaxy

We have now defined a few important families of graphs which we will refer to throughout the rest of this paper. We generally don't explicitly define every graph by its vertices and edges and simply refer to it as some member of one family or say this it is isomorphic to one. This is much more efficient and concise than listing out all vertices and edges as we did in the beginning of this chapter.

We are now ready to move on and introduce graph decompositions, the objects which are the subject of this project.

Chapter 2

Introduction

2.1 Decompositions

Suppose you have n translucent sheets of tracing paper with some points drawn on all n sheets of paper in the same set arrangement. Now, draw lines connecting points on each sheet of paper, so that no line appears on two distinct sheets of paper.

A graph is depicted when all n sheets of tracing paper are aligned and stacked on top of each other with some light source present, call this graph K. Call the graph depicted on the ith sheet of paper G_i for $i = 1, \ldots, n$. The stacking of these sheets of paper depicts $G_1 \sqcup \cdots \sqcup G_n = K$, and this collection of papers depicts the set $\{G_1, \ldots, G_n\}$ which we call a *graph decomposition* of K. This is defined formally below.

Definition 2.1.1 (Graph Decomposition)**.** Let K be a simple graph. We call a collection $\mathcal{G} = \{G_1, \ldots, G_n\}$ of pairwise edge-disjoint subgraphs $G_1, \ldots, G_n \subseteq K$ of K a *decomposition* of K if their disjoint union equals K; $G_1 \sqcup \cdots \sqcup G_n = K$ and $\{G_1, \ldots, G_t\}$.

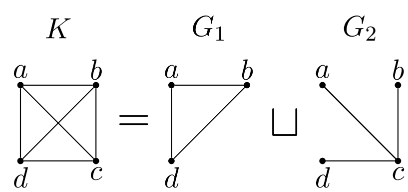

Figure 2.1: $\{G_1, G_2\}$ is a decomposition of K_4

Graph decompositions are an important topic combinatorics, graph theory, and design theory, with origins dating back to the 1800s. Notably, in 1850 Reverend Thomas Kirkman, a full-time clergyman and legendary mathematician, posed an important problem in *The Lady's and Gentleman's Diary* [8] now known as *the school girl problem.* It goes

Fifteen young ladies in a school walk out three abreast for seven days in succession: it is required to arrange them daily so that no two shall walk twice abreast.

The problem asks if we can form five distinct rows of three school girls on each day of the week so that no two school girls walk in the same row more than once in a week. This is equivalent to finding a decomposition of K_{15} whose members are all triangles; whose members are isomorphic to C_3 or K_3. Both Kirkman and Arthur Cayley independently solved the the schoolgirl problem and published their solutions in the 1851 edition of *The Lady's and Gentleman's Diary* [9]. Kirkman's solution is provided below.

Denoting the ladies by a_1, a_2, a_3*;* b_1, b_2, b_3*;* c_1, c_2, c_3*;* d_1, d_2, d_3*;* e_1, e_2, e_3*, the following arrangement will be found to answer the question:*

$a_1\, a_2\, a_3$	$a_1\, b_1\, c_1$	$a_1\, d_1\, e_1$	$a_1\, b_2\, d_2$	$a_1\, c_2\, e_2$	$a_1\, b_3\, e_3$	$a_1\, c_3\, d_3$
$b_1\, b_2\, b_3$	$a_2\, b_2\, c_2$	$a_2\, d_2\, e_2$	$a_2\, b_3\, d_3$	$a_2\, c_3\, e_3$	$a_2\, b_1\, e_1$	$a_2\, c_1\, d_1$
$c_1\, c_2\, c_3$	$a_3\, d_3\, e_3$	$a_3\, b_3\, c_3$	$a_3\, c_1\, e_1$	$a_3\, b_1\, d_1$	$a_3\, c_2\, d_2$	$a_3\, b_2\, e_2$
$d_1\, d_2\, d_3$	$b_3\, d_1\, e_2$	$b_1\, c_1\, e_3$	$b_1\, c_3\, e_1$	$b_2\, c_3\, d_1$	$c_2\, b_3\, e_1$	$c_2\, b_3\, e_1$
$e_1\, e_2\, e_3$	$c_3\, d_2\, e_1$	$e_3\, b_2\, c_1$	$d_1\, c_2\, e_3$	$c_1\, d_2\, b_3$	$d_2\, b_1\, c_2$	$c_1\, d_3\, b_2$

This is the symmetrical and only possible solution. All others differ from this only in disturbing the alphabetical order, or that of the three subindices in certain triplets of the first column, or in both these together.

Each triple in this array above gives a edge-distinct triangle subgraph of K_{15} whose vertex set we take to be $\{a_1, a_2, \ldots e_4, e_5\}$. The set of all these subgraphs is a decomposition of K_{15}. Since all of these subgraphs are isomorphic to C_3, we call it a C_3-*decomposition.* This is a special type of decomposition which is defined formally on the following page.

Definition 2.1.2 (G-decomposition)**.** A *G-decomposition* of a graph K is a decomposition $\mathcal{G} = \{G_1, \ldots, G_t\}$ whose members are all isomorphic to some graph G; $\mathcal{G} = \{G_1, \ldots, G_t\}$ such that $K = G_1 \sqcup \cdots \sqcup G_t$ and $G_i \cong G$ for $i = 1, \ldots, t$. If such a set exists we say that K *allows* a G-decomposition or equivalently, that G *decomposes* K. If $K \cong K_n$ we sometimes call the decomposition a *G-design of order n*.

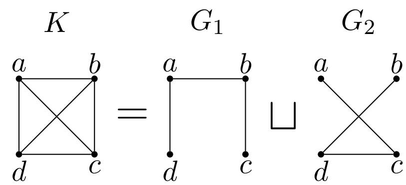

Figure 2.2: $\{G_1, G_2\}$ is a P_3-decomposition of K_4 or a P_3-design of order 4

We know that if a G-decomposition of some graph K exists, that all of its members have the same number of edges and vertices. This allows us to find constraints on the types of bigger graphs K that can be decomposed by some subgraph $G \subseteq K$ on m edges solely based on divisiblity properties.

Lemma 2.1.3 (Necessary Condition (general))**.** *Let G be a simple graph on m edges. There exists a G-decomposition of a graph K only if $|E(G)| = m$ divides $|E(K)|$.*

Proof. Suppose there exists a G-decomposition $\mathcal{G} = \{G_1, \ldots, G_n\}$ of K. Then $E(G_1) \sqcup \cdots \sqcup E(G_t) = E(K)$ and so $|E(K)| = |E(G_1) \sqcup \cdots \sqcup E(G_t)| = |E(G_1)| + \cdots + |E(G_t)| = tm$. So $|E(G)| = m$ divides $|E(K)|$. $\square$

Theorem 2.1.4 (Necessary Condition (K_n))**.** *Let G be a simple graph on m edges. There exists a G-decomposition of K_n only if n is idempotent modulo $2m$; only if $n^2 \equiv n \pmod{2m}$.*

Proof. Suppose there exists a G-decomposition of K_n. Then $|E(G)| = m$ divides $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$ by Lemma 2.1.3. Therefore, $\frac{n^2-n}{2} = mt$ for some $t \in \mathbb{N}$. Observe.

$$n^2 - n = 2mt \implies n^2 - n \equiv 0 \pmod{2m} \implies n^2 \equiv n \pmod{2m}.$$

$\square$

By the previous theorem, any graph on m edges decomposes K_n only if n is idempotent modulo $2m$. Note that the converse isn't necessarily true. However, for a graph G on m edges, this finite set of constraints allows us to ask:

For what n is K_n G-decomposable?

This question is known as the *spectrum problem* for decompositions. Pioneering work by Rosa and Kotzig in the 1960s—especially in the development of graph labeling—helped shape the modern treatment of G-decomposition problems. Since then, labeling-based techniques and tools from design theory have driven significant progress. In particular, graph labeling methods have played a central role in addressing the spectrum problem for small graphs. This directly continues contributions by Freyberg and Peters, who recently solved the spectrum problem for forests with six edges [4]. Their paper provides a comprehensive summary of known decompositions for graphs G with fewer than seven edges.

Using graph labelings to solve G-decomposition problems is basically about doing algebra on graphs in order to generate edge-disjoint subgraphs. If we take the vertices of a graph K to be elements of a group, we can use the structure of the group to our advantage. Specifically, when $K \cong K_n$, and we take it's vertices to be $\mathbb{Z}_n$, and then we label the vertices of G with some subset of $\mathbb{Z}_n$. There are various labeling techniques of this kind stemming from Rosa's work in the 1960s that that allow us permute or act on the labels of the vertices of G with subgroups of $\mathbb{Z}_n$ to generate new isomorphic copies of G which are pairwise edge-disjoint. In the next section, we provide a example which outlines in some detail how this machinery works for G-decompositions of Complete Graphs.

2.2 Graph labeling

Take the vertices of K_5 to be $\mathbb{Z}_5$ and arrange it in the same manner as in 1.6. Notice that every vertex shares an edge with two vertices directly adgacent to it and two vertices that are 'two adjacencies away' on the outer cycle (01234). We call this idea *length* denoted ℓ where edges joining two vertices u, v have length $\ell(uv) = l$ if they are 'l adjacencies' away from each other on the outer cycle.

Formally, for K_n we define the edge length function ℓ as follows:

$$\ell(uv) = \min\{|u - v|, n - |u - v|\}$$

Notice that for K_5, we only have lengths 1 or 2 as previously observed. Color the length 1 edges blue, and the length 2 edges red. This is depicted in the figure below.

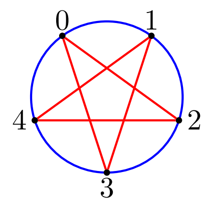

Figure 2.3: K_5 with lengths colored

Now, consider P_3. It has 2 edges, and K_5 has $\binom{5}{2} = 10$ edges. Since $2|10$, by Lemma 2.1.3 its *possible* that a P_3-decomposition of K_5 exists. Now since P_3 it has 2 edges and there are 2 lengths in K_5, what if we can just make sure each copy of P_3 has both a blue edge and a red edge? How can we do that while ensuring that no edge is repeated?

It turns out that if we take the vertices of K_n to be $\mathbb{Z}_n$, adding 1 (and therefore anything) modulo n to the endpoints of an edge preserves it's length. We call the act of permuting vertices in this manner *clicking* or *developing*.

In the context of our problem with P_3 and K_5, this means that if we label P_3 with elements of $\mathbb{Z}_5$ such that just have one blue edge of length 1, and one red edge of length 2, that we can simply generate all edges of length 1 and 2 in K_n (which is just all edges of K_n) while preserving the structure of the graph by developing all vertices of our labeling by 1 at the same time repeatedly. Label the defining path of P_3 via (2,0,1). Developing the vertices 1 modulo 5 will give all members of a P_3-decomposition of K_5. A decomposition that can generated by permuting all vertices of one labeling repeatedly in this fashion is called a *cyclic decomposition*. This is depicted in the following figure.

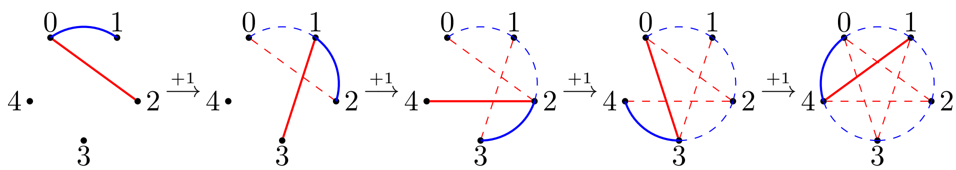

Figure 2.4: A cyclic P_3-decomposition of K_5

Nice and easy right? But that's just one complete graph that P_3 can decompose. Remember, that it is *possible* that P_3 can decompose any K_n where $n \equiv n^2 \pmod 4$ by Theorem 2.1.4. This equivalent to saying $n = 4t + r$ where r is an idempotent in the ring $\mathbb{Z}_4$ and $t \geq 1$. The idempotents in $\mathbb{Z}_4$ are $0, 1$. So this means $\mathbb{Z}_5$ is just a special case of n where $n = 4t + 1$ where $t = 1$. Luckily, even though these are infinite families, it is known that for each step $t \mapsto t + 1$, new lengths come 2 at a time. This means if we can somehow transform our labeling at each step to include the new lengths, we can maybe take care of the entire family K_{4t+1}. We want to fine tune our labeling to be able to weather this process. This is what graph labeling is all about. Note that if r was not 0 or 1, we would need multiple labelings to take care of the whole family. This is explained later in this paper.

Lastly, some basic observations about a general G with m edges and K_n. The maximal length in K_n is $\lfloor \frac{n}{2} \rfloor$. This is intuitive, since when you travel halfway across the outer cycle from some vertex, the lengths start going back down again and you are nearing that vertex. Now, n must be of the form $2mt + r$ where $t \geq 1$ and r is an idempotent in the ring $\mathbb{Z}_{2m}$. This means that in K_{2m+r} if $\ell(uv) = |u - v| < \lfloor \frac{2m+r}{2} \rfloor < \lfloor \frac{2mt+r}{2} \rfloor$ for $t > 1$, then $\ell(uv) = |u - v|$ in all K_{2mt+r} for $t \geq 1$. this is important, because at each step $t \mapsto t + 1$, new lengths come m at a time. This means that some wraparound edges xy in K_{2m+r} are short edges of length $|x - y|$ in K_{2mt+r} for $t > 1$.

Now, for $r = 0$ or 1, if a certain labeling of a graph G on m edges exists, there exists a G-decomposition of K_{2mt+r} for $t \geq 1$. However, if $r \neq 0, 1$, one labeling will not suffice and other techniques are needed to prove that G decomposes K_{2mt+r} for $t \geq 1$. These labelings and techniques are defined as they are needed in the proceeding chapters. Finally, we are ready to introduce the focus of this project.

2.3 Seven edge forests

As stated earlier, this project continues on Freyberg and Peters' work on six edge forests by asking the same question about seven edge forests:

> Let F be a forest on seven edges. For which n does F decompose K_n?

Every component of a forest on seven edges is a tree on six or less edges. These trees are cataloged in Figure 2.5. We use the naming convention $\mathbf{T_j^i}$ to denote the i^{th} tree with j vertices and we index the vertices v_1 through v_j for each tree as specified below.

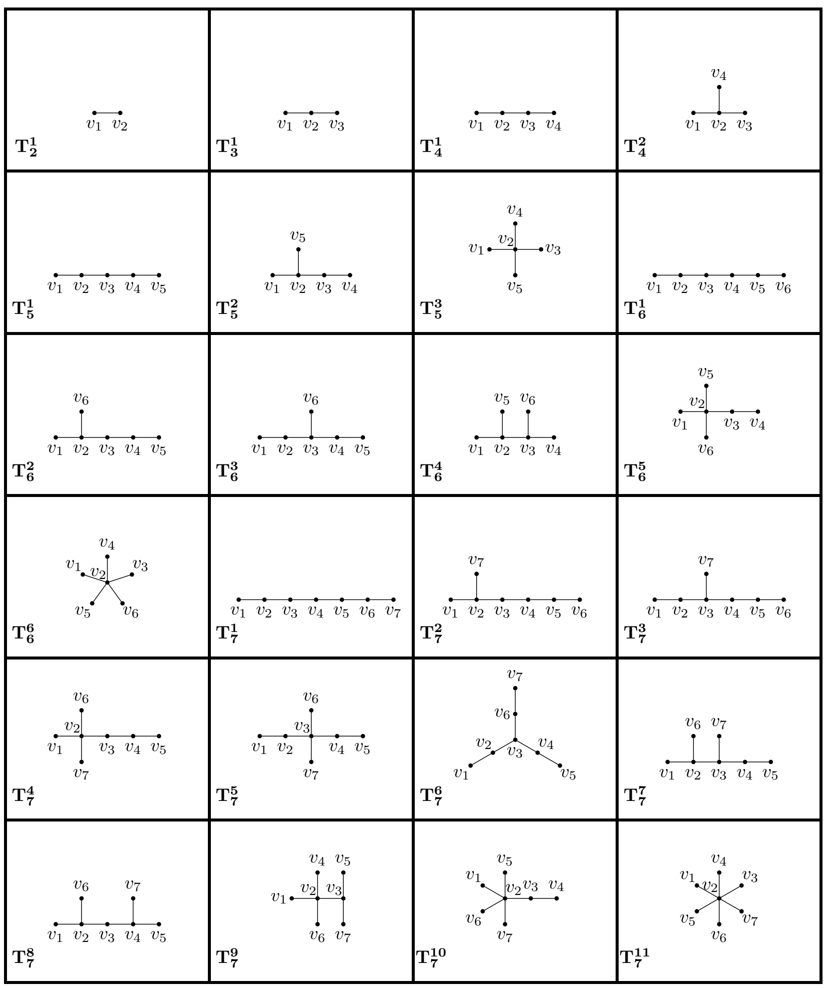

Figure 2.5: Trees with less than seven edges

The next theorem gives the necessary conditions for the existence of a G-decomposition of K_n when G is a graph with 7 edges.

Theorem 2.3.1. *If G is a graph with 7 edges and a G-decomposition of K_n exists, then $n \equiv 0, 1, 7,$ or $8 \pmod{14}$.*

Proof. If a G-decomposition of K_n exists, then n is idempotent modulo $2(7) = 14$ by Theorem 2.1.4 which immediately implies that $n \equiv 0, 1, 7, \text{or } 8 \pmod{14}$ since those are all the idempotents in $\mathbb{Z}_{14}$. □

For this project, we do not define the graph on one vertex to be a tree. This means that any connected component in a forest has at least one edge and we also require there to be at least two connected components. There are 47 such forests with 7 edges up to isomorphism. The the spectrum problem for the matching $\mathbf{7T_2^{11}}$ on 7 edges was solved by de Werra in 1970. Therefore, we need only consider the remaining 46 trees in the subsequent chapters.

Chapter 3 handles all decomposing K_n into all 47 forests when $n \equiv 0$ or $1 \pmod{14}$. Chapter 4 applies to all the forests when $n \equiv 7$ or $8 \pmod{14}$ with the lone exception of $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^{1}}$, which is solved for those values of n in Chapter 5.

After proving the main result of this project, we provide a couple of additional results in Chapter 6: (1) An edge mapping depending on t that preserves length for wraparound edges in $2m+r$ to $2mt+r$ at each step $t \mapsto t+1$. (2) Galaxy decompositions of complete multipartite graphs. These two results conclude this thesis project, and produce some open questions.

Lastly, we present some *Python* programs that were created as a result of this thesis project. One is *tikzgrapher*: my graph visualization software, which was built from scratch only using *Pygame* and *NetworkX*. It allows one to visualize simple NetworkX graphs in an interactive Pygame window that allows for colorings and custom labelings along with dragging and moving components of the graph. The main feature however, is that the user can save the graph layout depicted in the Pygame window as a tikZ graph in a standalone LaTeXfile. tikzgrapher is paired with a graph labeling solver: a constraint programming project that can find several labelings on graphs if they exist. Chapter 6 concludes the results of this project.

Chapter 3

$\mathbf{n \equiv 0, 1 (mod\ 14)}$

To begin this section, we define some established graph labelings and theorems that go along with them in order to construct the seven edge forest decompositions of K_n where $n \equiv 0$ or $1 \pmod{14}$.

Definition 3.0.1 ((Rosa [7]))**.** Let G be a graph with m edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, 2, \ldots, 2m\}$ that induces a bijective *length function* $\ell : E(G) \to \{1, 2, \ldots, m\}$ where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all $uv \in E(G)$.

Rosa showed that if a ρ-labeling of a graph G with m edges exists, then a cyclic G-decomposition of K_{2m+1} exists, which is presented formally later. Later, Rosa and his peers began studying more restrictive types of ρ-labelings to decompose more complete graphs. Next, we define some of these labelings and associated with them.

Theorem 3.0.2 ((Rosa [7]))**.** *Let G be a graph with m edges. There exists a cyclic G-decomposition of K_{2m+1} if and only if G admits a ρ-labeling.*

Definition 3.0.3 ((Rosa [7]))**.** A σ-labeling of a graph G is a ρ-labeling such that $\ell(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$.

Definition 3.0.4 ((El-Zanati, Vanden Eynden [2]))**.** A ρ- or σ-labeling of a bipartite graph G with bipartition (A, B) is called an *ordered* ρ- or σ-labeling and denoted ρ^+, σ^+, respectively, if $f(a) < f(b)$ for each edge ab with $a \in A$ and $b \in B$.

Theorem 3.0.5 ((El-Zanati, Vanden Eynden [2]))**.** *Let G be a graph with m edges which has a ρ^+-labeling. Then G decomposes K_{2mk+1} for all positive integers k.*

Definition 3.0.6 ((Freyberg, Tran [5]))**.** A σ^{+-}-*labeling* of a bipartite graph G with m edges and bipartition (A, B) is a σ^+-labeling with the property that $f(a) - f(b) \neq m$ for all $a \in A$ and $b \in B$, and $f(v) \notin \{2m, 2m-1\}$ for any $v \in V(G)$.

Theorem 3.0.7 ((Freyberg, Tran [5]))**.** *Let G be a graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant. Then there exists a G-decomposition of both K_{2mk} and K_{2mk+1} for every positive integer k.*

Table 3.1 gives σ^{+-}-labelings of all forests on 7 edges except the matching. The vertex labels of each connected component with k vertices are given as a k-tuple, $(v_1, \ldots, v_k)$ corresponding to the vertices $v_1, \ldots, v_k$ given in Figure 2.5. We leave it to the reader to infer the bipartition (A, B).

Example 3.0.8. *A σ^{+-}-labeling of $\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$ is shown in Figure 3.1. The vertices labeled $1, 2$ and 9 belong to A, and the others belong to B. The lengths of each edge are indicated on the edge.*

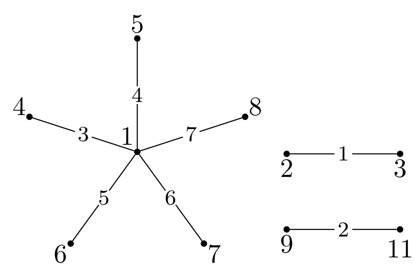

Figure 3.1: σ^{+-}-labeling of $\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$

The labelings given in Figure 3.1 along with thm 3.0.7 are enough to conclude this case.

Forest	Vertex Labels
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2,9,7) \sqcup (3,4)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,3) \sqcup (8,7)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,4) \sqcup (8,7)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(5,1,4,2,9,6,7) \sqcup (10,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(3,8,1,4,2,5,7) \sqcup (9,10)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(7,8,1,6,0,4,3) \sqcup (9,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,1,6,3,4,5,7) \sqcup (9,10)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(6,1,5,3,8,4,7) \sqcup (9,10)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(5,11,9,10,6,12,7) \sqcup (8,1)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(4,8,1,6,0,5,3) \sqcup (9,10)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(0,6,1,5,2,9) \sqcup (11,10,12)$
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(3,6,1,8,4,0) \sqcup (10,9,11)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(5,11,9,12,7,10) \sqcup (1,8,4)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(3,8,4,1,6,7) \sqcup (10,9,11)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(5,1,8,3,4,7) \sqcup (10,9,11)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(4,1,8,5,6,7) \sqcup (10,9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(0,6,1,5,2) \sqcup (9,8,10,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(7,1,8,5,6) \sqcup (0,4,2,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(7,1,8,4,6) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(6,0,3,4,5) \sqcup (8,7,9,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(4,8,1,7,2) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(6,0,3,4,5) \sqcup (8,9,2,7)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5,2,9) \sqcup (8,10) \sqcup (3,4)$
$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(3,6,1,8,4,0) \sqcup (5,7) \sqcup (9,10)$
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,3,5,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(5,8,4,1,6,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(5,11,9,12,7,10) \sqcup (8,1) \sqcup (0,4)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,5,6,7) \sqcup (2,3) \sqcup (9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (8,10,9) \sqcup (11,4)$

Table 3.1: σ^{+-}-labelings for forests with seven edges

Forest	Vertex Labels
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(7,1,8,5,6) \sqcup (10,9,11) \sqcup (0,4)$
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(6,0,3,4,5) \sqcup (1,8,7) \sqcup (9,11)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5) \sqcup (2,9,7,10) \sqcup (3,4)$
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11,9,10,7) \sqcup (4,0,5,6) \sqcup (8,1)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(4,0,5,6) \sqcup (10,9,11,12) \sqcup (8,1)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(0,6,1,5) \sqcup (8,10,9) \sqcup (11,4,7)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(4,0,5,6) \sqcup (1,8,7) \sqcup (11,9,12)$
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (8,10,7) \sqcup (11,4) \sqcup (2,3)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (11,9,12) \sqcup (2,3) \sqcup (8,1)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(6,1,8,4,7) \sqcup (3,5) \sqcup (9,12) \sqcup (10,11)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(3,0,4,5,6) \sqcup (8,1) \sqcup (10,11) \sqcup (9,7)$
$3\mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (2,9,7) \sqcup (10,11)$
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (9,2) \sqcup (8,10) \sqcup (4,7) \sqcup (11,12)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (2,3) \sqcup (9,11) \sqcup (8,1) \sqcup (10,7)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(0,6,1) \sqcup (8,4) \sqcup (2,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$

Table 3.1: σ^{+-}-labelings for forests with seven edges

Theorem 3.0.9. *Let F be a forest with 7 edges. There exists an F-decomposition of K_n whenever $n \equiv 0$ or $1 \pmod{14}$.*

Proof. The proof follows from thm 3.0.7 and the labelings given in Figure 3.1. $\square$

Chapter 4

$\mathbf{n \equiv 7, 8 (mod\ 14)}$

In this section, we use our own constructions based on the same edge length definition as in the previous section. The first one addresses the $n \equiv 7 \pmod{14}$ case.

Definition 4.0.1. Let G be a graph with 7 edges. A (1-2-3)-*labeling* of $3G$ is an assignment f of the integers $\{0, \ldots, 20\}$ to the vertices of $3G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

   and

2. $$\bigcup_{uv \in E(3G)} \{(f(u) \textbf{ mod } 7, f(v) \textbf{ mod } 7)\} = \bigcup_{i=0}^{6} \bigcup_{j=1}^{3} \{(i, i+j \textbf{ mod } 7)\}.$$

Notice that the second condition of a (1-2-3)-labeling says that $3G$ contains exactly 7 edges of each of the lengths $1, 2,$ and 3. Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A (1-2-3) labeling of every forest with 7 edges with the exception of $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ is given in Figure 8.1. This exceptional forest does not admit such a labeling and is dealt with in Section 5.

Theorem 4.0.2. *Let G be a bipartite graph with 7 edges. If $3G$ admits a (1-2-3)-labeling and G admits a ρ^+-labeling, then G decomposes K_{14k+7} for every $k \geq 1$.*

Proof. Let $n = 14k + 7$ and notice that K_n has $|E(K_n)| = (7k + 3)(14k + 7)$ edges, which can be partitioned into $14k + 7$ edges of each of the lengths in $\{1, 2, \ldots, 7k + 3\}$.

We will construct the G-decomposition in two steps. First, we use the 1-2-3-labeling to identify all the edges of lengths $1, 2$, and 3 accounting for $3(2k+1)$ copies of G. Then, we use the ρ^+-labeling to identify edges of the remaining lengths in $7k(2k+1)$ copies of G. In total, the decomposition consists of $|E(K_n)|/7 = (7k+3)(2k+1)$ copies of G.

Let f_1 be a (1-2-3)-labeling of $3G$ and identify this graph as a block B_0. Then develop B_0 by 7 modulo n. Since the order of the development is $\frac{n}{7} = 2k+1$ and there are 7 edges of each of the lengths $1, 2$, and 3 in B_0, we have identified $3(2k+1)$ copies of G containing all $14k+7 = n$ edges of each length $1, 2$, and 3. Notice (2) of Definition 4.0.1 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \to \{0, \ldots, 14\}$ be a ρ^+-labeling of G with associated vertex partition (A, B). For $i = 1, 2, \ldots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i-1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i-3, 7i-2, \ldots,$ and $7i+3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i-1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \ldots, 7\}$. Developing each block B_i by 1 yields $14k+7$ copies of G per block and accounts for $14k+7$ edges of each of the lengths $4, 5, \ldots,$ and $7k+3$.

Since we have identified

$$3(2k+1) + k(14k+7) = (7k+3)(2k+1)$$

edge-disjoint copies of G, the proof is complete. □

To address the $n \equiv 8 \pmod{14}$ case, we define the following labeling.

Definition 4.0.3. Let G be a graph with 7 edges. A *1-rotational* (1-2-3)*-labeling* of $4G$ is an assignment f of $\{0, \ldots, 20\} \cup \infty$ to the vertices of $4G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

   and

2.

$$\bigcup_{uv\in E(4G)} \{(f(u) \textbf{ mod } 7, f(v) \textbf{ mod } 7)\} = \bigcup_{i=0}^{6}\bigcup_{j=1}^{3}\{(i, i+j \textbf{ mod } 7), (i,\infty)\}.$$

Notice that the second condition of a 1-rotational (1-2-3)-labeling says that $4G$ contains exactly 7 edges of each of the lengths $1, 2, 3$, and ∞. Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A 1-rotational (1-2-3)-labeling of every forest with 7 edges with the exception of $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ is given in Figure 8.2.

Theorem 4.0.4. *Let G be a bipartite graph with 7 edges. If $4G$ admits a 1-rotational (1-2-3)-labeling and G admits a ρ^+-labeling, then G decomposes K_{14k+8} for every $k \geq 1$.*

Proof. Let $n = 14k+8$ and notice that K_n has $|E(K_n)| = (7k+4)(14k+7)$ edges, which can be partitioned into $14k+7$ edges of each of the lengths in $\{1, 2, \ldots, 7k+3, \infty\}$. We will construct the G-decomposition in two steps. First, we use the 1-rotational (1-2-3)-labeling to identify all the edges of lengths $1, 2, 3$, and ∞ accounting for $4(2k+1)$ copies of G. Then, we use the ρ^+-labeling to identify edges of the remaining lengths in $7k(2k+1)$ copies of G. In total, the decomposition consists of $|E(K_n)|/7 = (7k+4)(2k+1)$ copies of G. Let f_1 be a 1-rotational (1-2-3)-labeling of $4G$ and identify this graph as a block B_0. Then develop B_0 by 7 modulo $n-1$. Since the order of the development is $\frac{n-1}{7} = 2k+1$ and there are 7 edges of each of the lengths $1, 2, 3$ and ∞ in B_0, we have identified $4(2k+1)$ copies of G containing all $14k+7 = n-1$ edges of each length $1, 2, 3$ and ∞. Notice (2) of Definition 4.0.3 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \to \{0, \ldots, 14\}$ be a ρ^+-labeling of G with associated vertex partition (A, B). For $i = 1, 2, \ldots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i-1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i-3, 7i-2, \ldots,$ and $7i+3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i-1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \ldots, 7\}$. Developing each block B_i by 1 yields $14k+7$ copies of G per block and accounts for $14k+7$ edges of each of the lengths $4, 5, \ldots,$ and $7k+3$.

Since we have identified

$$4(2k+1) + k(14k+7) = (7k+4)(2k+1)$$

edge-disjoint copies of G, the proof is complete. □

We are now able to state the main thm of this section.

Theorem 4.0.5. *Let F be a forest with 7 edges and $F \ncong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$. There exists an F-decomposition of K_n whenever $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$.*

Proof. If $n \equiv 7 \pmod{14}$, a (1-2-3)-labeling of $3F$ can be found in Figure 8.1. On the other hand, if $n \equiv 8 \pmod{14}$, then a 1-rotational (1-2-3)-labeling of $4F$ can be found in Figure 8.2. In either case, a ρ^+-labeling of F can be found in Figure 3.1 (recall that a σ^{+-}-labeling is a ρ^+-labeling). The result now follows from Theorems 4.0.2 and 4.0.4. □

Example 4.0.6. *We illustrate the constructions in the previous two thms by finding an F-decomposition of K_{35} and K_{36} for the forest graph $F \cong \mathbf{T_6^6} \sqcup \mathbf{T_3^1}$.*

Here are excerpts from the preceding tables for $\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$

Labeling Type	Labelings
σ^{+-}	$(4,1,8,5,6,7) \sqcup (10,9,11)$
(1-2-3)	$(0,2,1,3,4,5) \sqcup (12,11,14)$
	$(4,6,8,9,5,7) \sqcup (14,12,15)$
	$(0,3,1,4,5,6) \sqcup (11,8,7)$
1-rotational (1-2-3)	$(1,2,0,3,4,5) \sqcup (11,8,\infty)$
	$(2,\infty,3,4,5,6) \sqcup (12,13,15)$
	$(6,7,8,4,5,\infty) \sqcup (11,12,15)$
	$(11,10,8,12,13,7) \sqcup (9,6,4)$

Figure 4.1: Labelings for $\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$

The ρ^+ *labelings obtained by stretching the* σ^{+-} *labeling are bottommost labelings in the following generating presentations and are developed by* 1*.*

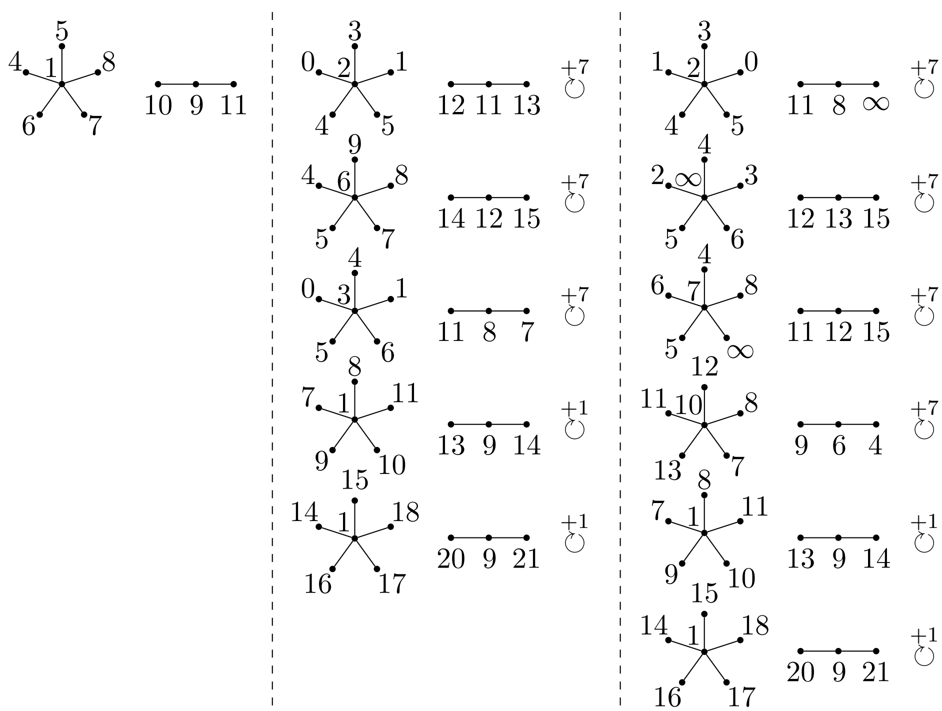

Figure 4.2: A σ^{+-}-labeling of $F \cong \mathbf{T_6^6} \sqcup \mathbf{T_3^1}$ (left) and generating presentations for the F-decomposition of K_n where $n = 35$ (middle) and $n = 36$ (right)

Chapter 5

$\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$

We begin this case by constructing K_n for $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$ using *joined* copies of K_{22}, K_{21}, and K_{14}. Recall, the *join* of two graphs G_1 and G_2 is the graph obtained by adding an edge $\{g_1, g_2\}$ for every vertex $g_1 \in V(G_1)$ and every vertex of $g_2 \in V(G_2)$.

Let t be a positive integer and join $t-1$ copies of K_{14} with each other and a lone copy of K_{21}. The resulting graph is $K_{14(t-1)+21} \cong K_{14t+7}$. So we can think of K_{14t+7} as K_t whose t "vertices" consist of $t-1$ copies of K_{14} and 1 copy of K_{21} and whose edges are the join between them. From now on, we will refer to these "vertices" as nodes. Similarly, K_{14t+8} can be constructed as K_t whose nodes are $t-1$ copies of K_{14} and 1 copy of K_{22} and whose edges are the join between them.

We show that $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes K_n for $n \equiv 7$ or $8 \pmod{14}$ by proving that K_{22}, K_{21}, K_{14}, $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ are each $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposable. Notice that these 6 graphs make up the nodes and edges of the K_t representations of K_{14t+7} and K_{14t+8} stated in the constructions above.
The proof of the next theorem was obtained by manipulating a $K_{1,7}$-decomposition of K_{22} by Cain in [1].

Theorem 5.0.1. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* K_{21} *and* K_{22}.

Proof. Figures 8 and 9 give $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decompositions of K_{21} and K_{22}, respectively. $\square$

Theorem 5.0.2. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* $K_{n,7}$ *for all* $n \geq 2$.

Proof. Consider $K_{n,7}$ where $n \geq 2$. Take the partite set of n vertices to be $\mathbb{Z}_n$ and color them white. Similarly, take the partite set of 7 vertices to be K_7 and color them black. Naturally we refer to *white-black* vertices uv in $K_{n,7}$ via $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_7$ and vice versa. Finally, let $E_i = \{(i, 0)\} \sqcup (\{i+1\} \times \{1, \ldots, 6\})$ and $G_i \subset K_{n,7}$ be the subgraph induced by E_i for each $i \in \mathbb{Z}_n$. Note that $G_i \cong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ for all $i \in \mathbb{Z}_n$.

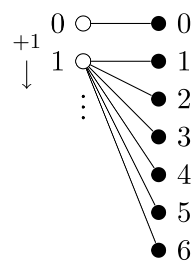

Figure 5.1: G_0 in a generating presentation of the $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$−decomposition of $K_{n,7}$.

Notice that $E_i \cap E_j = \emptyset$ if $i \neq j$, so by definition all distinct G_i's are pairwise edge disjoint. Lastly,

$$\bigcup_{i\in\mathbb{Z}_n} E_i = [\bigcup_{i\in\mathbb{Z}_n} \{(i,0)\}] \sqcup [\bigcup_{i\in\mathbb{Z}_n} (\{i+1\}\times\{1,\ldots,6\})] = [\mathbb{Z}_n\times\{0\}] \sqcup [\mathbb{Z}_n\times\{1,\ldots,6\}] = \mathbb{Z}_n\times\mathbb{Z}_7$$

So $G_0 \sqcup \cdots \sqcup G_{n-1} = K_{n,7}$ and $\{G_i \mid i \in \mathbb{Z}_n\}$ is a $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$−decomposition of $K_{n,7}$. Furthermore, it is generated by developing the white nodes of G_0 by 1. $\square$

Corollary 5.0.3. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* $K_{22,14}$, $K_{21,14}$, *and* $K_{14,14}$.

Proof. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes $K_{7,7}$ and $K_{8,7}$ by Theorem 5.0.2. $K_{14,14}$ can be expressed as the edge-disjoint union of four copies of $K_{7,7}$, $K_{21,14}$ can be expressed as the edge-disjoint union of six copies of $K_{7,7}$, and $K_{22,14}$ can be expressed as the edge-disjoint union of two copies of $K_{8,7}$ and four copies of $K_{7,7}$. Therefore, $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes them all. $\square$

Theorem 5.0.4. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* K_{14t+7} *and* K_{14t+8} *where* t *is a positive integer.*

Proof. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes K_{14} by Theorem 3.0.7, $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ by Corollary 5.0.3, and lastly K_{22}, K_{21} by Theorem 5.0.1.

Therefore, $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes the join of $(t-1)$ copies of K_{14} with each other and 1 copy of K_{21}, which is isomorphic to K_{14t+7}. Similarly $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes the join of $(t-1)$ copies of K_{14} with each other and 1 copy of K_{22} which is isomorphic to K_{14t+8}. $\square$

Chapter 6

Additional Results

We present two main additional results produced by work done on this project. (1) Wraparound edge mappings that preserve length and (2) Galaxy graph decompositions of multipartite graphs.

6.1 Wraparound edge mappings that preserve length

Initially when I found labelings to take care of the $n \equiv 7, 8 \pmod{14}$ cases, we used lengths $8, 9, 10$, and I mistakenly did not have all edges be short edges (non-wraparound edges). This meant they didn't generalize via the permutation $v \mapsto v + 1$ for K_{14t+7} and K_{14t+8} where $t > 1$. In an effort to not have to find 90 completely new labelings (I did later have to anyways), I tried to find a new way to map wraparound edges in my labelings. I was successful but it was very messy, and this caused us to pivot to using lengths $1, 2, 3$ since a larger percent of edges with those lengths are short edges.

Note that this is very preliminary. In order to implement these mappings sucessfully, one has to ensure that new vertices wont collide with currently existing vertices elsewhere in the labelings. Because there was a pivot, this was not investigated and perhaps remains an open problem as a result of this project.

Chapter 7

Conclusion and Discussion

Chapter 8

Appendix

Forest	Labeling
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,1,2,4,6,9,12) \sqcup (13,14)$ $(3,4,7,9,10,13,15) \sqcup (8,5)$ $(8,11,12,10,7,5,6) \sqcup (1,3)$ $(0,4,9,15,8,16,7) \sqcup (1,11)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(12,9,6,4,2,1,7) \sqcup (14,15)$ $(15,13,10,9,7,4,11) \sqcup (8,5)$ $(8,11,12,10,7,5,13) \sqcup (1,3)$ $(16,8,15,9,4,0,6) \sqcup (1,11)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(0,1,2,4,6,9,3) \sqcup (16,19)$ $(15,13,10,9,7,4,14) \sqcup (17,18)$ $(6,5,7,10,12,11,8) \sqcup (18,15)$ $(7,16,8,15,9,4,12) \sqcup (1,11)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(8,6,4,2,1,9,7) \sqcup (14,15)$ $(8,10,9,7,4,11,13) \sqcup (12,15)$ $(9,12,10,7,5,11,13) \sqcup (1,4)$ $(7,15,9,4,0,8,6) \sqcup (1,11)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12,8,7) \sqcup (11,14)$
	$(0,2,3,6,5,1,4) \sqcup (8,7)$
	$(0,3,5,4,1,8,7) \sqcup (16,15)$
	$(4,9,15,8,12,6,7) \sqcup (1,11)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(1,2,4,6,8,5,9) \sqcup (12,15)$
	$(4,7,9,10,11,8,13) \sqcup (1,3)$
	$(5,7,10,12,11,6,13) \sqcup (1,4)$
	$(0,4,9,15,8,12,6) \sqcup (1,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,6,4,2,5,9,7) \sqcup (12,14)$
	$(1,3,2,0,5,4,6) \sqcup (10,12)$
	$(9,8,7,10,4,11,5) \sqcup (12,13)$
	$(7,15,9,4,13,8,6) \sqcup (1,11)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(7,6,4,2,8,9,5) \sqcup (12,14)$
	$(2,3,4,7,0,5,6) \sqcup (9,12)$
	$(7,8,5,4,9,10,11) \sqcup (0,2)$
	$(6,15,9,4,8,11,7) \sqcup (2,12)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(2,4,6,8,7,9,12) \sqcup (13,14)$
	$(0,2,3,4,7,6,5) \sqcup (8,10)$
	$(0,3,5,8,9,4,1) \sqcup (12,14)$
	$(4,9,15,8,12,7,16) \sqcup (1,11)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12,1,8) \sqcup (14,15)$
	$(5,6,3,2,0,7,4) \sqcup (8,9)$
	$(0,3,5,4,7,1,8) \sqcup (12,14)$
	$(4,9,15,8,12,18,7) \sqcup (1,11)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(1,2,4,6,9,12) \sqcup (13,14,15)$
	$(3,4,7,9,10,13) \sqcup (5,8,6)$
	$(11,12,10,7,5,6) \sqcup (3,1,4)$
	$(0,4,9,15,8,16) \sqcup (1,11,2)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14, 15)$
	$(13, 10, 9, 7, 4, 11) \sqcup (5, 8, 6)$
	$(11, 12, 10, 7, 5, 13) \sqcup (3, 1, 4)$
	$(0, 4, 9, 15, 8, 12) \sqcup (1, 11, 2)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(0, 1, 2, 4, 6, 5) \sqcup (16, 13, 14)$
	$(8, 6, 3, 2, 0, 4) \sqcup (14, 12, 15)$
	$(7, 4, 5, 3, 0, 6) \sqcup (10, 8, 11)$
	$(7, 0, 4, 9, 15, 12) \sqcup (1, 11, 2)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(1, 2, 5, 4, 6, 7) \sqcup (16, 14, 13)$
	$(8, 6, 9, 3, 2, 4) \sqcup (14, 12, 15)$
	$(4, 5, 6, 3, 0, 1) \sqcup (11, 8, 7)$
	$(7, 0, 6, 4, 9, 12) \sqcup (1, 11, 2)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(0, 2, 4, 7, 1, 5) \sqcup (12, 11, 13)$
	$(7, 6, 3, 2, 8, 9) \sqcup (14, 12, 15)$
	$(4, 3, 5, 6, 0, 1) \sqcup (11, 8, 7)$
	$(8, 0, 4, 9, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(0, 2, 1, 3, 4, 5) \sqcup (12, 11, 14)$
	$(4, 6, 8, 9, 5, 7) \sqcup (14, 12, 15)$
	$(0, 3, 1, 4, 5, 6) \sqcup (11, 8, 7)$
	$(4, 0, 8, 5, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(2, 4, 6, 9, 12) \sqcup (16, 15, 14, 13)$
	$(3, 4, 7, 9, 10) \sqcup (11, 12, 15, 13)$
	$(12, 10, 7, 5, 6) \sqcup (18, 15, 17, 20)$
	$(4, 9, 15, 8, 16) \sqcup (2, 11, 1, 5)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(12, 9, 6, 4, 11) \sqcup (17, 16, 15, 14)$
	$(9, 7, 4, 3, 6) \sqcup (11, 12, 15, 13)$
	$(6, 5, 7, 10, 3) \sqcup (18, 15, 17, 20)$
	$(16, 8, 15, 9, 12) \sqcup (2, 11, 1, 6)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(4, 6, 9, 11, 8) \sqcup (16, 15, 18, 14)$ $(9, 7, 4, 3, 6) \sqcup (16, 17, 20, 15)$ $(6, 5, 7, 10, 3) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 6, 7)$ $(14, 17, 16, 20, 15) \sqcup (9, 7, 4, 3)$ $(9, 12, 10, 11, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(7, 6, 9, 11, 8) \sqcup (16, 15, 13, 14)$ $(9, 7, 4, 3, 5) \sqcup (16, 17, 20, 15)$ $(4, 6, 5, 7, 10) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 12, 6)$ $(18, 17, 16, 20, 15) \sqcup (9, 7, 10, 4)$ $(10, 12, 11, 14, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(1, 2, 4, 6, 9, 12) \sqcup (13, 14) \sqcup (8, 7)$ $(3, 4, 7, 9, 10, 13) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 6) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14) \sqcup (8, 7)$ $(13, 10, 9, 7, 4, 11) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 13) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(0, 1, 2, 4, 7, 5) \sqcup (9, 6) \sqcup (8, 10)$ $(8, 6, 3, 2, 0, 4) \sqcup (5, 7) \sqcup (12, 13)$ $(6, 4, 5, 3, 0, 8) \sqcup (13, 14) \sqcup (18, 15)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11) \sqcup (5, 14)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(1,2,5,4,6,7) \sqcup (13,14) \sqcup (12,15)$ $(8,6,9,3,2,4) \sqcup (12,14) \sqcup (18,15)$ $(4,5,6,3,0,1) \sqcup (8,7) \sqcup (16,14)$ $(7,0,6,4,9,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(0,2,4,7,1,5) \sqcup (11,13) \sqcup (12,15)$ $(7,6,3,2,8,9) \sqcup (11,12) \sqcup (1,4)$ $(4,3,5,6,0,1) \sqcup (8,7) \sqcup (12,14)$ $(8,0,4,9,6,7) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(0,2,1,3,4,5) \sqcup (12,14) \sqcup (18,19)$ $(4,6,8,9,5,7) \sqcup (12,15) \sqcup (11,14)$ $(0,3,1,4,5,6) \sqcup (8,11) \sqcup (14,15)$ $(4,0,8,5,6,7) \sqcup (1,11) \sqcup (3,12)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12) \sqcup (13,14,15) \sqcup (18,19)$ $(3,4,7,9,10) \sqcup (12,15,13) \sqcup (1,2)$ $(12,10,7,5,6) \sqcup (20,17,15) \sqcup (1,4)$ $(4,9,15,8,16) \sqcup (11,1,5) \sqcup (3,12)$
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(12,9,6,4,11) \sqcup (17,16,15) \sqcup (0,1)$ $(9,7,4,3,6) \sqcup (12,15,13) \sqcup (18,19)$ $(6,5,7,10,3) \sqcup (20,17,15) \sqcup (1,4)$ $(16,8,15,9,12) \sqcup (1,11,2) \sqcup (0,5)$
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(13,15,16,18,14) \sqcup (9,6,7) \sqcup (2,4)$ $(14,17,16,20,15) \sqcup (3,4,7) \sqcup (11,13)$ $(9,12,10,11,15) \sqcup (6,5,7) \sqcup (0,2)$ $(5,1,10,11,6) \sqcup (8,15,9) \sqcup (4,12)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(4,6,9,12) \sqcup (16,15,14,13) \sqcup (19,20)$ $(9,7,4,3) \sqcup (11,12,15,13) \sqcup (16,17)$ $(12,10,7,5) \sqcup (18,15,17,20) \sqcup (9,11)$ $(9,15,8,16) \sqcup (2,11,1,5) \sqcup (12,7)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11,9,6,7) \sqcup (16,15,13,14) \sqcup (1,4)$ $(5,3,4,7) \sqcup (16,17,20,15) \sqcup (0,2)$ $(4,6,5,7) \sqcup (9,12,11,15) \sqcup (0,3)$ $(16,8,15,9) \sqcup (10,1,11,6) \sqcup (0,4)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(18,15,13,14) \sqcup (11,9,12,6) \sqcup (1,2)$ $(18,17,20,15) \sqcup (9,7,10,4) \sqcup (2,3)$ $(11,12,14,15) \sqcup (4,6,5,7) \sqcup (17,19)$ $(11,1,5,6) \sqcup (16,8,14,15) \sqcup (0,9)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(16,15,14,13) \sqcup (0,3,5) \sqcup (12,9,6)$ $(11,12,15,13) \sqcup (10,9,7) \sqcup (16,18,20)$ $(18,15,17,20) \sqcup (10,11,14) \sqcup (6,5,7)$ $(2,12,3,11) \sqcup (8,1,7) \sqcup (4,0,5)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(11,9,12,6) \sqcup (18,15,13) \sqcup (0,1,2)$ $(9,7,10,4) \sqcup (18,17,20) \sqcup (1,3,2)$ $(11,12,14,15) \sqcup (4,6,7) \sqcup (17,19,20)$ $(16,8,14,15) \sqcup (11,1,6) \sqcup (9,0,4)$
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(8,6,9,11) \sqcup (0,1,2) \sqcup (16,19) \sqcup (18,15)$ $(8,10,7,9) \sqcup (18,17,20) \sqcup (11,14) \sqcup (2,3)$ $(13,11,12,14) \sqcup (17,19,20) \sqcup (6,7) \sqcup (8,5)$ $(0,5,1,7) \sqcup (3,10,2) \sqcup (4,13) \sqcup (16,6)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(11,9,12,6) \sqcup (0,1,2) \sqcup (18,15) \sqcup (13,14)$ $(9,7,10,4) \sqcup (18,17,20) \sqcup (11,13) \sqcup (2,3)$ $(11,12,14,15) \sqcup (17,19,20) \sqcup (8,6) \sqcup (1,3)$ $(4,0,5,6) \sqcup (8,1,9) \sqcup (3,12) \sqcup (17,7)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(2,4,6,9,12) \sqcup (13,14) \sqcup (18,19) \sqcup (0,1)$ $(3,4,7,9,10) \sqcup (13,15) \sqcup (1,2) \sqcup (8,5)$ $(6,5,7,10,12) \sqcup (17,20) \sqcup (8,11) \sqcup (1,3)$ $(4,9,15,8,16) \sqcup (1,11) \sqcup (3,12) \sqcup (2,6)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(11,9,6,4,12) \sqcup (16,15) \sqcup (8,10) \sqcup (2,3)$ $(6,7,4,3,9) \sqcup (13,15) \sqcup (18,19) \sqcup (8,5)$ $(3,5,7,10,6) \sqcup (17,20) \sqcup (8,11) \sqcup (0,1)$ $(12,8,15,9,16) \sqcup (2,11) \sqcup (0,5) \sqcup (3,13)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(13,15,16,18,14) \sqcup (9,6) \sqcup (2,4) \sqcup (5,7)$ $(14,17,16,20,15) \sqcup (4,7) \sqcup (11,13) \sqcup (5,6)$ $(9,12,10,11,15) \sqcup (6,7) \sqcup (0,2) \sqcup (3,4)$ $(5,1,10,11,6) \sqcup (9,15) \sqcup (4,12) \sqcup (0,7)$
$3\mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(18,15,13) \sqcup (11,9,6) \sqcup (0,1,2) \sqcup (16,19)$ $(18,17,20) \sqcup (9,7,10) \sqcup (1,3,2) \sqcup (11,14)$ $(11,12,14) \sqcup (4,6,7) \sqcup (17,19,20) \sqcup (8,5)$ $(11,1,6) \sqcup (16,8,14) \sqcup (9,0,4) \sqcup (10,3)$
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(9,6,4,2) \sqcup (13,14) \sqcup (18,19) \sqcup (0,1) \sqcup (10,12)$ $(9,7,4,3) \sqcup (13,15) \sqcup (1,2) \sqcup (8,5) \sqcup (16,17)$ $(10,7,5,6) \sqcup (17,20) \sqcup (8,11) \sqcup (1,3) \sqcup (9,12)$ $(9,15,8,16) \sqcup (1,11) \sqcup (3,12) \sqcup (2,6) \sqcup (0,5)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(16,15,18,13) \sqcup (9,6) \sqcup (2,4) \sqcup (5,7) \sqcup (0,1)$ $(16,17,20,14) \sqcup (4,7) \sqcup (11,13) \sqcup (5,6) \sqcup (1,3)$ $(9,12,10,11) \sqcup (6,7) \sqcup (0,2) \sqcup (3,4) \sqcup (8,5)$ $(10,1,11,5) \sqcup (9,15) \sqcup (4,12) \sqcup (0,7) \sqcup (8,3)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(11,9,6) \sqcup (0,1,2) \sqcup (18,15) \sqcup (16,19) \sqcup (17,20)$ $(9,7,10) \sqcup (1,3,2) \sqcup (17,18) \sqcup (11,14) \sqcup (8,5)$ $(11,12,14) \sqcup (4,6,7) \sqcup (19,20) \sqcup (13,15) \sqcup (3,5)$ $(11,1,6) \sqcup (16,8,14) \sqcup (0,9) \sqcup (10,3) \sqcup (17,13)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(0,1,2) \sqcup (18,15) \sqcup (9,11) \sqcup (16,19) \sqcup (5,6) \sqcup (10,7)$ $(1,3,2) \sqcup (17,18) \sqcup (9,7) \sqcup (11,14) \sqcup (8,5) \sqcup (16,13)$ $(4,6,7) \sqcup (12,14) \sqcup (3,5) \sqcup (13,15) \sqcup (17,20) \sqcup (18,19)$ $(16,8,14) \sqcup (1,11) \sqcup (0,9) \sqcup (10,3) \sqcup (17,13) \sqcup (2,7)$

Table 8.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,1,\infty,2,4,5,3) \sqcup (12,15)$ $(0,2,5,\infty,6,4,1) \sqcup (10,11)$ $(5,7,\infty,3,6,9,10) \sqcup (13,14)$ $(\infty,4,7,10,8,6,5) \sqcup (16,15)$ $(0,4,9,15,8,16,7) \sqcup (1,11)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(3,5,4,2,\infty,8,1) \sqcup (12,15)$ $(4,6,\infty,5,2,0,18) \sqcup (10,11)$ $(10,9,6,3,\infty,0,7) \sqcup (12,14)$ $(5,6,8,10,7,4,9) \sqcup (0,1)$ $(16,8,15,9,4,0,6) \sqcup (1,11)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(3,5,4,2,\infty,1,6) \sqcup (9,10)$ $(0,2,5,\infty,6,4,1) \sqcup (10,11)$ $(5,7,\infty,3,6,9,8) \sqcup (13,14)$ $(\infty,4,7,10,8,6,1) \sqcup (12,15)$ $(7,16,8,15,9,4,12) \sqcup (1,11)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(1,2,4,5,8,0,\infty) \sqcup (11,13)$ $(4,\infty,5,2,3,8,6) \sqcup (16,13)$ $(6,7,\infty,10,13,8,5) \sqcup (19,20)$ $(11,10,7,4,1,8,12) \sqcup (13,15)$ $(7,15,9,4,0,8,6) \sqcup (1,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(5,4,2,3,6,0,1) \sqcup (9,\infty)$ $(2,5,\infty,6,4,8,11) \sqcup (16,13)$ $(10,\infty,7,8,11,5,6) \sqcup (12,13)$ $(4,7,10,8,5,11,12) \sqcup (13,15)$ $(4,9,15,8,12,6,7) \sqcup (1,11)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(8,5,4,2,0,6,\infty) \sqcup (11,13)$ $(3,2,5,\infty,8,1,6) \sqcup (16,13)$ $(5,7,\infty,3,4,8,6) \sqcup (13,14)$ $(\infty,4,7,10,8,1,12) \sqcup (13,15)$ $(0,4,9,15,8,12,6) \sqcup (1,11)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(1,2,4,5,7,0,3) \sqcup (8,11)$
	$(11,\infty,6,4,5,8,12) \sqcup (10,13)$
	$(6,7,\infty,10,2,8,5) \sqcup (9,12)$
	$(11,10,8,5,6,12,7) \sqcup (16,13)$
	$(7,15,9,4,13,8,6) \sqcup (1,11)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(1,2,4,6,0,3,5) \sqcup (8,11)$
	$(11,\infty,6,5,8,2,12) \sqcup (13,15)$
	$(6,7,\infty,10,8,4,5) \sqcup (11,12)$
	$(11,10,8,5,12,13,7) \sqcup (9,6)$
	$(6,15,9,4,8,11,7) \sqcup (2,12)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(5,4,2,0,1,3,6) \sqcup (9,\infty)$
	$(4,6,\infty,1,2,12,13) \sqcup (8,11)$
	$(10,\infty,7,5,3,6,9) \sqcup (13,15)$
	$(5,8,10,11,\infty,7,4) \sqcup (9,12)$
	$(4,9,15,8,12,7,16) \sqcup (1,11)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(5,4,2,3,6,\infty,0) \sqcup (8,7)$
	$(13,12,\infty,6,4,10,1) \sqcup (8,11)$
	$(10,\infty,7,6,9,2,5) \sqcup (13,15)$
	$(5,8,10,7,4,9,11) \sqcup (16,19)$
	$(4,9,15,8,12,18,7) \sqcup (1,11)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(3,5,4,2,\infty,1) \sqcup (13,12,15)$
	$(0,2,5,\infty,6,4) \sqcup (8,11,10)$
	$(5,7,\infty,3,6,9) \sqcup (13,14,15)$
	$(\infty,4,7,10,8,6) \sqcup (17,16,15)$
	$(0,4,9,15,8,16) \sqcup (1,11,2)$
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(\infty,2,4,5,8,0) \sqcup (11,13,12)$
	$(6,\infty,5,2,3,8) \sqcup (13,16,15)$
	$(6,3,\infty,7,5,4) \sqcup (13,14,15)$
	$(8,10,7,4,\infty,12) \sqcup (18,15,13)$
	$(0,4,9,15,8,12) \sqcup (1,11,2)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
	$(5,4,2,3,6,0) \sqcup (9,\infty,11)$
	$(4,6,\infty,12,13,1) \sqcup (11,8,7)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(10,\infty,7,6,9,5) \sqcup (16,15,13)$
	$(5,8,10,7,4,11) \sqcup (16,19,17)$
	$(7,0,4,9,15,12) \sqcup (1,11,2)$
	$(5,4,7,2,1,3) \sqcup (8,11,\infty)$
	$(12,\infty,8,6,4,5) \sqcup (13,10,7)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(10,\infty,2,7,8,5) \sqcup (19,16,14)$
	$(11,10,12,8,5,6) \sqcup (16,13,14)$
	$(7,0,6,4,9,12) \sqcup (1,11,2)$
	$(1,2,4,5,0,3) \sqcup (8,11,14)$
	$(11,\infty,6,4,8,5) \sqcup (10,13,12)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(6,7,\infty,3,8,5) \sqcup (9,12,15)$
	$(11,10,8,6,12,7) \sqcup (13,16,\infty)$
	$(8,0,4,9,6,7) \sqcup (1,11,2)$
	$(1,2,0,3,4,5) \sqcup (11,8,\infty)$
	$(2,\infty,3,4,5,6) \sqcup (12,13,15)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(6,7,8,4,5,\infty) \sqcup (11,12,15)$
	$(11,10,8,12,13,7) \sqcup (9,6,4)$
	$(4,0,8,5,6,7) \sqcup (1,11,2)$
	$(5,4,2,\infty,1) \sqcup (11,13,12,15)$
	$(0,2,5,\infty,6) \sqcup (8,11,10,12)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(5,7,\infty,3,6) \sqcup (16,13,14,15)$
	$(\infty,4,7,10,8) \sqcup (17,16,15,13)$
	$(4,9,15,8,16) \sqcup (2,11,1,5)$
	$(\infty,2,4,5,0) \sqcup (11,13,12,15)$
	$(6,\infty,5,2,1) \sqcup (8,11,10,12)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(6,3,\infty,7,1) \sqcup (16,13,14,15)$
	$(10,7,4,\infty,5) \sqcup (17,16,15,13)$
	$(16,8,15,9,12) \sqcup (2,11,1,6)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(\infty, 2, 4, 3, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (10, 12, 11, 15)$ $(6, 3, \infty, 7, 1) \sqcup (12, 14, 13, 15)$ $(\infty, 4, 7, 10, 1) \sqcup (17, 16, 13, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(0, 2, 1, 3, 4) \sqcup (11, 8, \infty, 6)$ $(2, \infty, 3, 4, 5) \sqcup (9, 12, 13, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15, 14)$ $(0, 3, 1, 5, 6) \sqcup (16, 13, 11, 10)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(10, 13, \infty, 8, 11) \sqcup (1, 2, 3, 4)$ $(15, 13, 12, 9, 7) \sqcup (3, \infty, 4, 5)$ $(11, 12, 15, 14, 13) \sqcup (4, 7, 5, \infty)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12, 7)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(0, 2, 3, 4, 5) \sqcup (9, 8, 11, \infty)$ $(2, \infty, 3, 4, 5) \sqcup (12, 13, 14, 15)$ $(4, 7, 8, 5, \infty) \sqcup (10, 12, 11, 15)$ $(0, 3, 1, 4, 6) \sqcup (16, 13, 11, \infty)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(3, 5, 4, 2, \infty, 1) \sqcup (19, 20) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (17, 18) \sqcup (8, 11)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14) \sqcup (0, 1)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (16, 15) \sqcup (2, 3)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(\infty, 2, 4, 5, 8, 0) \sqcup (18, 20) \sqcup (12, 13)$ $(13, \infty, 5, 2, 3, 8) \sqcup (9, 6) \sqcup (16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14) \sqcup (0, 1)$ $(15, 17, 14, 11, \infty, 19) \sqcup (8, 6) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(3,2,4,5,0,1) \sqcup (18,15) \sqcup (11,14)$ $(5,\infty,6,4,8,11) \sqcup (10,13) \sqcup (19,20)$ $(8,7,\infty,3,5,6) \sqcup (16,19) \sqcup (12,15)$ $(7,10,8,6,11,12) \sqcup (16,13) \sqcup (9,\infty)$ $(6,0,8,4,5,7) \sqcup (1,11) \sqcup (3,12)$
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(5,4,7,2,1,3) \sqcup (8,11) \sqcup (18,\infty)$ $(12,\infty,8,6,4,5) \sqcup (0,3) \sqcup (10,13)$ $(10,\infty,2,7,8,5) \sqcup (9,6) \sqcup (16,19)$ $(11,10,12,8,5,6) \sqcup (13,14) \sqcup (0,2)$ $(7,0,6,4,9,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(5,4,2,3,6,0) \sqcup (9,12) \sqcup (11,\infty)$ $(4,6,\infty,12,13,15) \sqcup (0,1) \sqcup (8,11)$ $(10,\infty,7,6,9,5) \sqcup (13,15) \sqcup (1,2)$ $(5,8,10,7,4,11) \sqcup (17,19) \sqcup (9,\infty)$ $(7,0,4,9,15,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(1,2,0,3,4,5) \sqcup (\infty,15) \sqcup (8,11)$ $(11,\infty,2,3,5,6) \sqcup (13,15) \sqcup (19,20)$ $(6,7,8,4,5,\infty) \sqcup (18,19) \sqcup (12,15)$ $(11,10,8,12,13,7) \sqcup (18,20) \sqcup (9,6)$ $(11,1,8,9,10,7) \sqcup (0,5) \sqcup (2,6)$
$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(10,13,\infty,8,11) \sqcup (3,2,4) \sqcup (16,15)$ $(15,13,12,9,7) \sqcup (10,\infty,5) \sqcup (11,14)$ $(11,12,15,14,13) \sqcup (4,\infty,7) \sqcup (0,3)$ $(3,4,6,9,\infty) \sqcup (8,10,12) \sqcup (5,7)$ $(0,9,1,8,2) \sqcup (5,10,6) \sqcup (3,13)$
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(8,\infty,13,10,9) \sqcup (3,2,4) \sqcup (14,15)$ $(7,9,12,13,8) \sqcup (10,\infty,5) \sqcup (11,14)$ $(11,12,15,18,14) \sqcup (4,\infty,7) \sqcup (0,3)$ $(9,6,4,3,8) \sqcup (19,17,15) \sqcup (13,14)$ $(1,8,0,9,2) \sqcup (5,10,6) \sqcup (3,13)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(2, \infty, 3, 4, 5) \sqcup (12, 13, 15) \sqcup (16, 19)$ $(0, 2, 1, 3, 4) \sqcup (8, \infty, 6) \sqcup (18, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15) \sqcup (0, 1)$ $(8, 10, 12, 13, 7) \sqcup (9, 6, 4) \sqcup (17, 18)$ $(9, 0, 8, 6, 7) \sqcup (11, 1, 5) \sqcup (10, 15)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(1, \infty, 16, 18) \sqcup (11, 13, 12, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 12) \sqcup (9, 7)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14, 15) \sqcup (5, 7)$ $(10, 7, 4, \infty) \sqcup (17, 16, 15, 13) \sqcup (1, 3)$ $(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11, 9, \infty, 1) \sqcup (10, 12, 13, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 13) \sqcup (9, 7)$ $(0, \infty, 17, 20) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(10, 7, 4, \infty) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(2, 12, 6, 15) \sqcup (8, 0, 5, 7) \sqcup (9, 13)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(18, 16, 19, \infty) \sqcup (10, 12, 13, 15) \sqcup (3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 10, 13) \sqcup (4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(9, 7, 10, 4) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(9, 0, 8, 7) \sqcup (11, 1, 5, 6) \sqcup (10, 4)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$ $(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$ $(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (5, 3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (3, 4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13) \sqcup (6, 8, 7)$ $(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (2, 1, 3)$ $(9, 0, 8, 7) \sqcup (5, 1, 6) \sqcup (10, 4, 14)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(11,13,12,15) \sqcup (9,\infty,1) \sqcup (4,5) \sqcup (16,18)$ $(8,11,10,12) \sqcup (19,\infty,6) \sqcup (2,5) \sqcup (16,14)$ $(8,10,7,4) \sqcup (0,\infty,11) \sqcup (16,17) \sqcup (9,6)$ $(5,7,8,6) \sqcup (20,17,\infty) \sqcup (13,14) \sqcup (1,2)$ $(3,10,5,11) \sqcup (0,9,1) \sqcup (2,12) \sqcup (17,13)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(18,16,19,\infty) \sqcup (13,12,15) \sqcup (3,5) \sqcup (17,20)$ $(1,\infty,12,6) \sqcup (8,11,13) \sqcup (4,5) \sqcup (17,18)$ $(3,\infty,4,7) \sqcup (12,14,13) \sqcup (8,6) \sqcup (1,2)$ $(9,7,10,4) \sqcup (17,16,13) \sqcup (1,3) \sqcup (14,15)$ $(9,0,8,7) \sqcup (11,1,6) \sqcup (18,12) \sqcup (10,14)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(4,1,\infty,13,10) \sqcup (2,3) \sqcup (16,15) \sqcup (9,11)$ $(5,\infty,10,11,13) \sqcup (4,7) \sqcup (0,2) \sqcup (9,12)$ $(7,\infty,4,5,8) \sqcup (17,19) \sqcup (0,3) \sqcup (12,14)$ $(7,8,6,9,\infty) \sqcup (13,14) \sqcup (1,3) \sqcup (19,20)$ $(1,11,2,10,3) \sqcup (0,6) \sqcup (9,4) \sqcup (8,12)$
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(1,\infty,13,10,7) \sqcup (2,3) \sqcup (16,15) \sqcup (9,11)$ $(5,\infty,10,11,16) \sqcup (4,7) \sqcup (0,2) \sqcup (9,12)$ $(6,4,5,8,\infty) \sqcup (17,19) \sqcup (0,3) \sqcup (12,14)$ $(7,8,6,9,11) \sqcup (13,14) \sqcup (1,3) \sqcup (19,20)$ $(3,10,2,11,5) \sqcup (0,6) \sqcup (4,8) \sqcup (17,7)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(1,\infty,13,5,7) \sqcup (2,3) \sqcup (16,15) \sqcup (9,11)$ $(0,3,1,4,\infty) \sqcup (2,5) \sqcup (9,7) \sqcup (10,13)$ $(12,11,13,14,\infty) \sqcup (17,19) \sqcup (5,7) \sqcup (9,6)$ $(5,8,11,6,7) \sqcup (13,14) \sqcup (2,\infty) \sqcup (19,20)$ $(6,0,8,9,7) \sqcup (1,11) \sqcup (10,5) \sqcup (16,12)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(11,13,12,15) \sqcup (9,\infty,1) \sqcup (2,4,5)$ $(8,11,10,12) \sqcup (19,\infty,6) \sqcup (0,2,5)$ $(0,\infty,3,6) \sqcup (16,13,14) \sqcup (8,7,5)$ $(17,16,15,13) \sqcup (\infty,4,7) \sqcup (0,3,1)$ $(9,15,8,16) \sqcup (11,1,5) \sqcup (3,12,7)$

Table 8.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(9, \infty, 8, 6) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13, 14) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 4, 3) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(18, 20, 17, \infty) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(0, 9, 1, 11) \sqcup (10, 3) \sqcup (12, 6) \sqcup (19, 14) \sqcup (17, 13)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(8, \infty, 9, 5) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (3, 4)$ $(15, 13, 14, \infty) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 3, 4) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(17, 20, 18, 19) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(9, 0, 8, 7) \sqcup (1, 11) \sqcup (12, 6) \sqcup (10, 5) \sqcup (16, 20)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(8, \infty, 9) \sqcup (13, 12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2)$ $(19, \infty, 6) \sqcup (11, 10, 12) \sqcup (2, 5) \sqcup (18, 20) \sqcup (1, 4)$ $(11, \infty, 14) \sqcup (10, 7, 4) \sqcup (16, 17) \sqcup (0, 2) \sqcup (1, 3)$ $(20, 17, \infty) \sqcup (14, 13, 15) \sqcup (5, 7) \sqcup (9, 6) \sqcup (0, 1)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(8, \infty, 9) \sqcup (12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13) \sqcup (9, 6) \sqcup (0, 2) \sqcup (18, 20) \sqcup (1, 4) \sqcup (17, 19)$ $(11, \infty, 14) \sqcup (4, 7) \sqcup (16, 17) \sqcup (2, 5) \sqcup (8, 10) \sqcup (0, 3)$ $(20, 17, \infty) \sqcup (13, 14) \sqcup (5, 7) \sqcup (10, 11) \sqcup (0, 1) \sqcup (8, 6)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$

Table 8.2: (1-2-3)-labelings

No.	Block	No.	Block
1	$(15,14,16,17,18,19,20) \sqcup (0,2)$	2	$(13,15,16,17,18,19,20) \sqcup (0,6)$
3	$(8,16,12,17,18,19,20) \sqcup (9,3)$	4	$(8,17,9,11,18,19,20) \sqcup (16,0)$
5	$(8,18,9,11,13,19,20) \sqcup (0,1)$	6	$(8,19,10,11,12,13,20) \sqcup (0,15)$
7	$(8,1,9,10,11,12,13) \sqcup (18,7)$	8	$(1,2,9,10,11,12,13) \sqcup (14,7)$
9	$(0,3,2,6,11,12,13) \sqcup (8,7)$	10	$(0,4,2,3,11,12,13) \sqcup (8,9)$
11	$(0,5,2,3,4,12,13) \sqcup (9,10)$	12	$(1,6,2,4,5,12,13) \sqcup (15,7)$
13	$(1,7,2,3,4,5,6) \sqcup (0,14)$	14	$(3,8,4,5,6,14,20) \sqcup (12,15)$
15	$(4,9,5,6,14,15,20) \sqcup (16,7)$	16	$(15,10,4,5,6,16,20) \sqcup (0,18)$
17	$(15,11,0,5,6,16,20) \sqcup (17,1)$	18	$(14,12,0,11,17,18,20) \sqcup (8,2)$
19	$(16,13,0,11,12,17,20) \sqcup (1,19)$	20	$(1,14,2,3,4,5,6) \sqcup (20,7)$
21	$(1,15,2,3,4,5,6) \sqcup (19,7)$	22	$(1,16,2,3,4,5,6) \sqcup (17,7)$
23	$(0,17,2,3,4,5,6) \sqcup (11,14)$	24	$(1,18,2,3,4,5,6) \sqcup (10,14)$
25	$(0,19,2,3,4,5,6) \sqcup (13,14)$	26	$(0,20,2,3,4,5,6) \sqcup (10,11)$
27	$(9,7,0,10,11,12,13) \sqcup (1,3)$	28	$(10,8,0,11,12,13,15) \sqcup (1,4)$
29	$(11,9,0,12,13,16,19) \sqcup (1,5)$	30	$(12,10,0,3,13,17,18) \sqcup (1,20)$

Table 8.3: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposition of K_{21}

No.	Block	No.	Block
1	$(15,14,16,17,18,19,20) \sqcup (0,2)$	2	$(13,15,16,17,18,19,20) \sqcup (0,6)$
3	$(8,16,12,17,18,19,20) \sqcup (9,3)$	4	$(8,17,9,11,18,19,20) \sqcup (16,0)$
5	$(8,18,9,11,13,19,20) \sqcup (0,1)$	6	$(8,19,10,11,12,13,20) \sqcup (0,15)$
7	$(8,1,9,10,11,12,13) \sqcup (6,\infty)$	8	$(1,2,9,10,11,12,13) \sqcup (14,7)$
9	$(0,3,2,6,11,12,13) \sqcup (8,7)$	10	$(0,4,2,3,11,12,13) \sqcup (8,9)$
11	$(0,5,2,3,4,12,13) \sqcup (9,10)$	12	$(1,6,2,4,5,12,13) \sqcup (15,7)$
13	$(1,7,2,3,4,5,6) \sqcup (13,\infty)$	14	$(3,8,4,5,6,14,20) \sqcup (12,15)$
15	$(4,9,5,6,14,15,20) \sqcup (16,7)$	16	$(15,10,4,5,6,16,20) \sqcup (0,18)$
17	$(15,11,0,5,6,16,20) \sqcup (17,1)$	18	$(14,12,0,11,17,18,20) \sqcup (8,2)$
19	$(16,13,0,11,12,17,20) \sqcup (1,19)$	20	$(1,14,2,3,4,5,6) \sqcup (20,7)$

Table 8.4: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposition of K_{22}

No.	Block	No.	Block
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (20, \infty)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$
31	$(0, \infty, 1, 2, 3, 4, 5) \sqcup (18, 7)$	32	$(14, \infty, 15, 16, 17, 18, 19) \sqcup (1, 3)$
33	$(7, \infty, 8, 9, 10, 11, 12) \sqcup (0, 14)$		

Table 8.4: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^{1}}$-decomposition of K_{22}

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Appendix A

Glossary and Acronyms

Care has been taken in this thesis to minimize the use of jargon and acronyms, but this cannot always be achieved. This appendix defines jargon terms in a glossary, and contains a table of acronyms and their meaning.

A.1 Glossary

- **Cosmic-Ray Muon** (**CR** μ) – A muon coming from the abundant energetic particles originating outside of the Earth's atmosphere.

A.2 Acronyms

Table A.1: Acronyms

Acronym	Meaning
CRμ	Cosmic-Ray Muon