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University of Minnesota Duluth

Driven to Discover

G-decompositions and G-designs for Forests with Seven Edges

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University of Minnesota: Duluth

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Outline

Forest Graphs

Decompositions and Designs

Edge Length and Decompositions

σ^{+-}-labelings

$n \equiv 7$ or 8 (mod 14)

$K_{1,7} \sqcup P_2$

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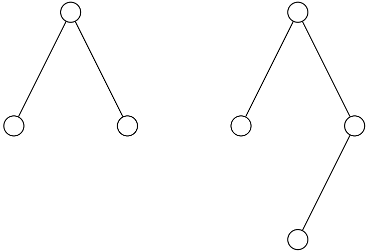

Figure: A Forest graph composed of trees with 3 and 4 vertices, respectively

Decompositions and Designs

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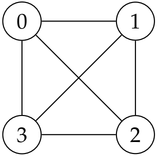

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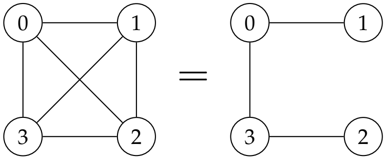

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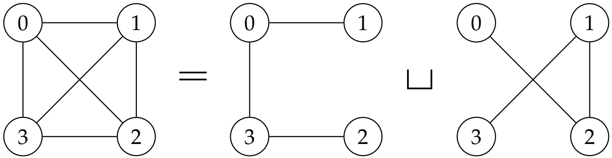

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Proof.

Let $m, n \in \mathbb{N}$. Suppose $m \mid \binom{n}{2}$. Then $\frac{n(n-1)}{2} = mq$ for some $q \in \mathbb{N}$. So then $\frac{n(n-1)}{2m} = q$, and $n(n-1) \equiv 0 \pmod{2m}$. Therefore $n \equiv n^2 \pmod{2m}$, and so n is idempotent modulo $2m$. Suppose n is idempotent modulo $2m$, then $n^2 > n$ and therefore $n^2 - n = n(n-1) = 2mp$ for some $p \in \mathbb{N}$. So then $\frac{n(n-1)}{2} = \binom{n}{2} = mp$ and m divides $\binom{n}{2}$. $\square$

$\square$

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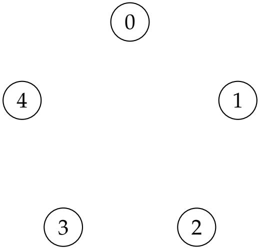
- A G-design is *cyclic* if the permutation $v \mapsto v + 1$ on $V(K_n)$ is an automorphism of the design.
- We call this act of applying permutations to a labeling *clicking*.

An example of a cyclic design

Cyclic P_3-design of order 5



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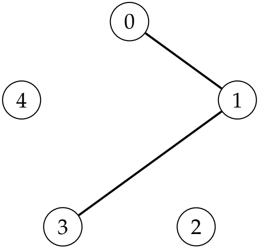

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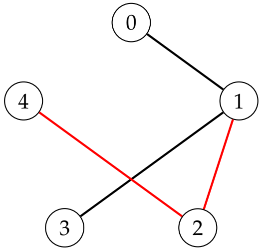

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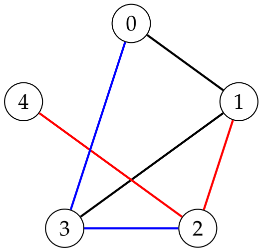

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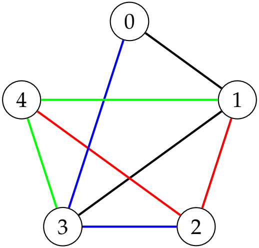

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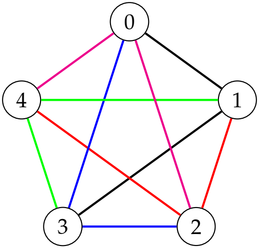

Figure: Cyclic P_3-design of order 5

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- The *length* of edge $uv \in E(K_n)$ is $\min(|u-v|, n-|u-v|)$ and will later be denoted $\ell(uv)$.
- If the length of uv is $n-|u-v|$ or equivalently, if $|u-v| > \lfloor \frac{n}{2} \rfloor$, then we call xy a *wrap-around* edge

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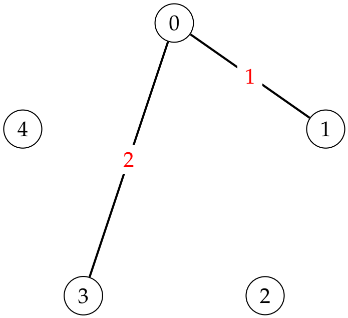

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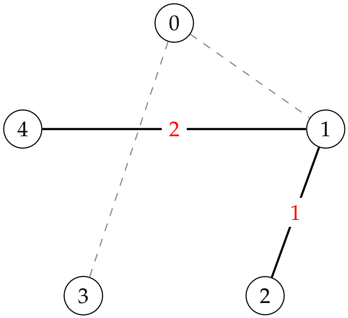

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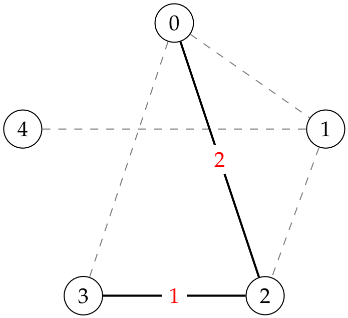

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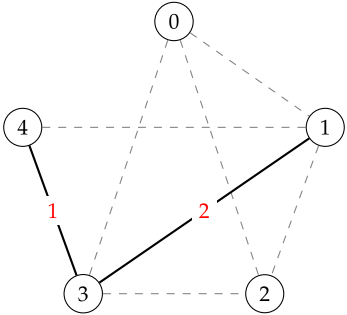

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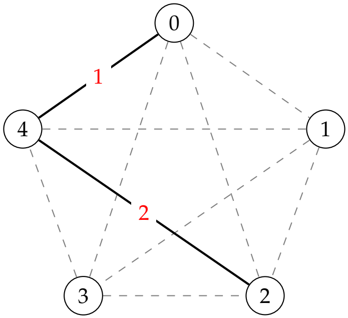

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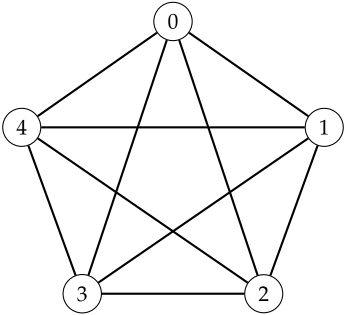

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- A σ-labeling is a labeling such that the length of every edge $uv \in E(K_n)$ is $|u-v|$.
- Freyberg and Tran introduced the following restricted σ-labeling in 2020.

Definition

Let G be a bipartite graph with m edges and bipartition $V(G) = A \cup B$. A σ^{+-}-*labeling* of G is a σ-labeling with:

1. $f(a) < f(b)$ for every edge $ab \in E(G)$ with $a \in A$ and $b \in B$
2. $f(a) - f(b) \neq m$ for all $a, b \in V(G)$
3. $f(v) \notin \{2m-1, 2m\}$ for all $v \in V(G)$

σ^{+-}-labelings

Theorem (Freyberg, Tran, 2020)

Let G be a graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant edge. Then there exists cyclic G-decompositions of K_{2mt} and K_{2mt+1} for every positive integer t.

7 edge forest G-decompositions and G-designs

- Recall that if G has m edges, then there exists a G-design of order n only if n is idempotent modulo $2m$.
- So F is a forest on 7 edges, there exists an F-design of order n only if $n \equiv 0, 1, 7,$ or $8 \pmod{14}$, since those are all the idempotents in $\mathbb{Z}_{14}$.
- So by Freyberg and Tran, if there exists a σ^{+-}-labeling of all forests F on 7 edges, then there exists F-G-decompositions and G-designs of order $14t$ and $14t + 1$ for all $t > 0$ and all forests F on 7 edges.

7 edge forest G-decompositions and G-designs

- The matching on 7-edges: $\bigsqcup_{i=1}^{7} P_2$ was solved by De Werra in 1970.
- So last summer I found a σ^{+-}-labeling of all forests on seven edges, up to isomorphism except $\bigsqcup_{i=1}^{7} P_2$. There are 46 total excluding the matching.

7 edge forest G-decompositions and G-designs

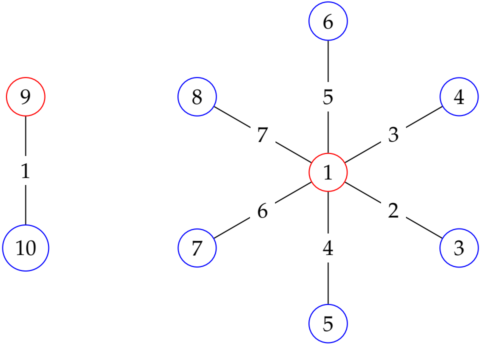

Figure: A σ^{+-}-labeling of $K_{1,7} \sqcup P_2$

σ^{+-}-labelings of seven edge forests

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- This idea also gives G-designs of order $14t + 1$ where the new node is labeled ∞, and the lengths are still $\{0, \ldots, 7\}$ since $\lfloor \frac{15}{2} \rfloor = 7$ plus a new length ∞ for each edge incident to the node ∞.

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- Well, what if we can collect edges of some lengths $\{a, b, c\}$ from $\{1, \ldots, 10\}$ in some other way for each? Then we can simply take some variation of our σ^{+-}-labelings accounting for lengths $\{1, \ldots, 10\} \setminus \{a, b, c\}$ and develop them by 1 to get the remaining lengths.

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- In the case of $n \equiv 8 \pmod{14}$, we look at $\{a, b, c, \infty\}$. If we can successfully do this, we can once again just 'stretch' the B partite set of some variation of our σ^{+-}-labelings to scale up (since new lengths still come 7 at a time).

(1-2-3)-labelings and 1-rotational (1-2-3)-labelings

We present the following labeling, which allows us to develop 3 labelings of each forest G by 7 to collect all edges of lengths in $\{1, 2, 3\}$ in copies of G:

Definition

Let G be a graph with 7 edges. A (1-2-3)-*labeling* of $3G$ is an assignment f of the integers $\{0, \ldots, 20\}$ to the vertices of $3G$ such that:

1. $f(u) \neq f(v)$ whenever u and v belong to the same component
2. $$\bigcup_{uv \in E(3G)} \{(f(u) \bmod 7, f(v) \bmod 7)\} = \bigcup_{i=0}^{6} \bigcup_{j=1}^{3} \{(i, i+j \bmod 7)\}$$

(1-2-3)-labelings and 1-rotational (1-2-3)-labelings

Similarly, we present the 1-rotational version of this labeling which allows us to develop 4 labelings of each forest G by 7 to collect all edges of lengths in $\{1, 2, 3, \infty\}$ in copies of G

Definition

Let G be a graph with 7 edges. A *1-rotational* (1-2-3)-*labeling* of $4G$ is an assignment f of $\{0, \ldots, 20\} \cup \infty$ to the vertices of $4G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component
2. $$\bigcup_{uv \in E(4G)} \{(f(u) \textbf{ mod } 7, f(v) \textbf{ mod } 7)\} = \bigcup_{i=0}^{6} \bigcup_{j=1}^{3} \{(i, i+j \textbf{ mod } 7), (i, \infty)\}$$

(1-2-3)-labelings and 1-rotational (1-2-3)-labelings

How do these work?

- We once again first focus on the $n \equiv 7 \pmod{14}$ case. Essentially, we introduce a new length function on top of the previously defined length function ℓ ; $\ell_7^+ := uv \mapsto (u+v) \textbf{ mod } 7$. We have labeled all vertices in $3G$ via $\mathbb{Z}_{21}$ such that only non-wraparound lengths $\{1, 2, 3\}$ appear, but also we have another crucial constraint.

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- When the vertices are reduced modulo 7, each of the 7 edges total across the labeling originally each length $i \in 1, 2, 3$ has a distinct length via ℓ_7^+ in $\{0, 1, \ldots, 6\}$. This allows us to develop the labelings by 7 to generate smaller partite sets (equivalence classes) of edges with lengths $(i, j) \in \{1, 2, 3\} \times \{0, 1, \ldots, 6\}$.

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- When the vertices are reduced modulo 7, each of the 7 edges total across the labeling originally each length $i \in 1, 2, 3$ has a distinct length via ℓ_7^+ in $\{0, 1, \ldots, 6\}$. This allows us to develop the labelings by 7 to generate smaller partite sets (equivalence classes) of edges with lengths $(i, j) \in \{1, 2, 3\} \times \{0, 1, \ldots, 6\}$.

An example:
Consider the non-wraparound edge $\{1, 2\}$ in K_{21}. $\ell(\{1, 2\}) = 1$ and $\ell_7^+(\{1, 2\}) = 3$. Developing $\{1, 2\}$ by 7 modulo 21 repeatedly, we get the whole partite set $\{\{1, 2\}, \{8, 9\}, \{15, 16\}\}$ of edges with standard length ℓ of 1 and new ℓ_7^+ length of 3.

7-edge forest designs of order $14t + 7$ and $14t + 8$

- We found (1-2-3)-labelings of $3G$ and 1-rotational (1-2-3)-labelings of $4G$ for each seven edge forest G, except $K_{1,6} \sqcup P_2$.

7-edge forest designs of order $14t + 7$ and $14t + 8$

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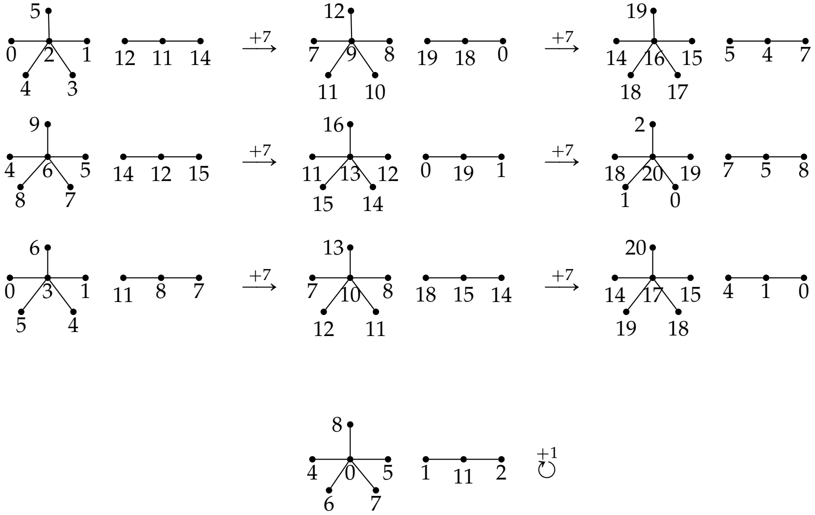
- For the remaining lengths $\{4, 5, 6, 7, 8, 9, 10\}$ we simply 'stretched' σ^{+-}–labelings for each forest except $K_{1,6} \sqcup P_2$ via $b \mapsto b + 3$ so that the edges go from lengths $\{1, 2, \ldots, 7\}$ to lengths $\{1 + 3, 2 + 3, \ldots, 7 + 3\} = \{4, 5, \ldots, 10\}$.
- Combining these two labeling techniques, we proved that there exists a seven edge forest design of orders $14t + 7$ and $14t + 8$ where $t > 0$ for all forests except $K_{1,6} \sqcup P_2$.

An example of such a $S_6 \sqcup P_3$-design of order 21



$K_{1,7} \sqcup P_2$

We were unable to find (1-2-3) and 1-rotational (1-2-3) labelings for exactly one graph (by hand and using a constraint programming algorithm). So we took a completely different approach for this exceptional graph $K_{1,7} \sqcup P_2$.

- In 1974, Pauline Cain proved that there exists an $K_{1,7}-$design of order 21. She provided a construction, which is based on partitioning K_{21} into three joined copies of K_7.

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- By 'peeling' an edge off each 7-edge star in this decomposition, we seperated the decomposition into resulting in 30 6-edge stars and 30 single edge paths. We paired the paths with a vertex disjoint 6-edge star, and these 30 unions resulted in a $K_{1,7} \sqcup P_2$-design of order 21.

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- We obtained the $K_{1,7} \sqcup P_2$-design of order 21 by adding the ∞ node and it's new edges to this $K_{1,7} \sqcup P_2$-design of order 21 and changing some pairings.

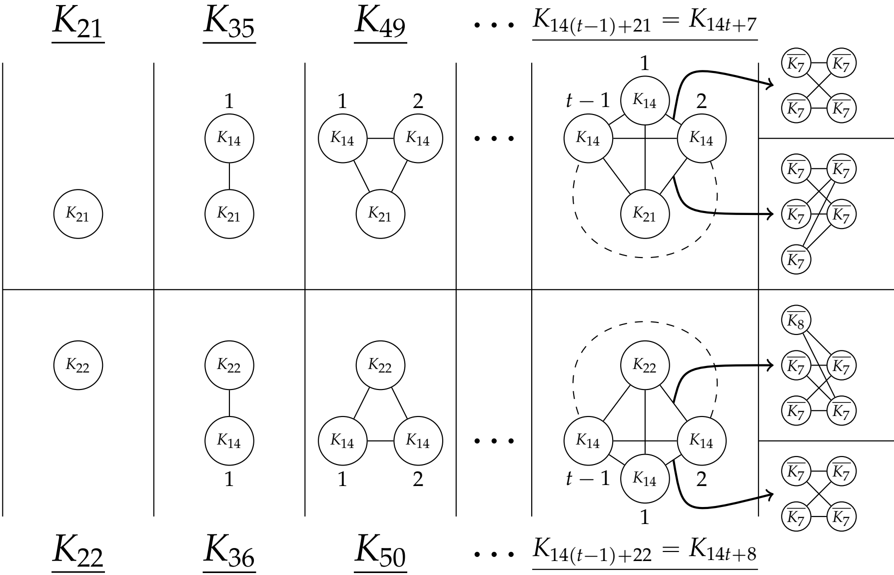
$K_{1,7} \sqcup P_2$-decomposition of $K_{7:7}$ and $K_{8:7}$

Why did we do this?

- We begin this case by constructing K_n for $n \equiv 7$ or $8 \pmod{14}$ and $n \geqslant 21$ using *joined* copies of K_{22}, K_{21}, and K_{14}. The *join* of two graphs G_1 and G_2 is the graph obtained by adding an edge $\{g_1, g_2\}$ for every vertex $g_1 \in V(G_1)$ and every vertex of $g_2 \in V(G_2)$.
- Let t be a positive integer and join $t-1$ copies of K_{14} with each other and a lone copy of K_{21}. The resulting graph is $K_{14(t-1)+21} \cong K_{14t+7}$. So we can think of K_{14t+7} as K_t whose t "vertices" consist of $t-1$ copies of K_{14} and 1 copy of K_{21} and whose edges are the join between them. From now on, we will refer to these "vertices" as nodes. Similarly, K_{14t+8} can be constructed as K_t whose nodes are $t-1$ copies of K_{14} and 1 copy of K_{22} and whose edges are the join between them.

We show a figure to illustrate this construction on the next slide.

Special Construction for K_{14t+7} and K_{14t+8}



Building the decompositions piece-by-piece

- We show that $K_{1,7} \sqcup P_2$ decomposes K_n for $n \equiv 7 \text{ or } 8 \pmod{14}$ by proving $K_{22}, K_{21}, K_{14}, K_{22,14}, K_{21,14},$ and $K_{14,14}$ are each $K_{1,7} \sqcup P_2$-decomposable. Notice that these 6 graphs make up the nodes and edges of the K_t representations of K_{14t+7} and K_{14t+8} stated in the constructions above.

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- We have already proven that K_{14}, K_{21}, and K_{22} are $K_{1,7} \sqcup P_2$-decomposable. So all we need to do is prove $K_{22,14}, K_{21,14},$ and $K_{14,14}$ are also $K_{1,7} \sqcup P_2$-decomposable.

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We do so in style. See the figure below.

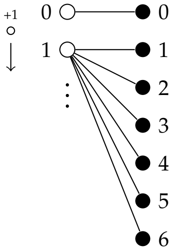

Figure: A generating presentation of the $K_{1,7} \sqcup P_2$-decomposition of $K_{n,7}$ for $n \geqslant 2$

Finally!

We get these theorems as a result of this fun left generated labeling:

Theorem
$K_{1,7} \sqcup P_2$-decomposes $K_{n,7}$ for all $n \geqslant 2$.

Theorem
$K_{1,7} \sqcup P_2$ decomposes $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$.

Proof.
Notice $K_{14,14}$ can be expressed as the edge-disjoint union of four copies of $K_{7,7}$, $K_{21,14}$ can be expressed as the edge-disjoint union of six copies of $K_{7,7}$, and $K_{22,14}$ can be expressed as the edge-disjoint union of two copies of $K_{8,7}$ and four copies of $K_{7,7}$. Therefore, $K_{1,7} \sqcup P_2$ decomposes them all. This is visualized in a previous figure. □

So finally...

Theorem
There exists a seven edge forest design of order n if and only if $n \equiv 0, 1, 7$ or $8 \pmod{14}$.

A bonus result!

From this one-sided cyclic permutation idea, we can generalize this bipartite idea to a special subset of forest graphs called *Galaxy* graphs whose connected components are all star graphs.

Theorem

Let $C = \{G_1, \ldots, G_n\}$ be a set of vertex disjoint star graphs, $x_i = k$ if $G_i \cong K_{1,k}$, and $m = \sum_{i=1}^{n} x_i$. The Galaxy graph $\mathcal{G} = \bigsqcup_{G \in C} G$ decomposes $K_{N,m}$ for all $N > n$.

Proof.

This works the same way the $K_{1,7} \sqcup P_2$-labeling works. Simply color the centers of $G_1, \ldots, G_n$ and label them $0, \ldots, n-1$, respectively. Now simply develop the white vertices by 1 modulo N and you get the decomposition. □

Naturally what follows is that such a Galaxy $\mathcal{G}$ also decomposes $K_{N,M}$ where $N \geqslant n$ and M is a multiple of m, and we can also extend this to $\overline{K_N} \vee (\overline{K_{M_1}} \sqcup \cdots \sqcup \overline{K_{M_c}})$ where all M_i's are mutliples of M... and so on.

Thank you!

Questions? – Thank you all for coming!