

# **Seven Edge Forest Designs**

**A THESIS**

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# Dedication

I dedicate this Thesis to my advisor Professor Bryan Freyberg, to my family who has supported me throughout this process, and to Jeremy, Aaron, Mike B, Joe, Jordi, Ian, TK, and Torta from Tuscarora and West 7th. Thank you for believing in me when I couldn't and helping me realize what is possible when I apply myself. Especially to the Tusca boyz, we all made it somehow. Who would have thought.

## Abstract

Let  $G$  be a subgraph of  $K_n$  where  $n \in \mathbb{N}$ . A  $G$ -decomposition of  $K_n$ , or  $G$ -design of order  $n$ , is a finite collection  $\mathcal{G} = \{G_1, \dots, G_k\}$  of pairwise edge-disjoint subgraphs of  $K_n$  that are all isomorphic to some graph  $G$ . We prove that an  $F$ -decomposition of  $K_n$  exists for every seven-edge forest  $F$  if and only if  $n \equiv 0, 1, 7$ , or  $8 \pmod{14}$ .

Along the way, we introduce new methods, constraint programming algorithms in Python, and some bonus results for Galaxy graph decompositions of complete bipartite, and eventually multipartite graphs.

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# Chapter 1

## Background

### 1.1 Fundamentals of Graph Theory

Graph Theory is the study of objects called *vertices* or *nodes* and their relationships which we call *edges*. An edge between vertices  $u$  and  $v$  is typically denoted  $uv$  or  $(u, v)$ . A graph  $G$  is formally defined as an ordered pair  $G = (V, E)$  where  $V$  is the set of all vertices in  $G$  and  $E$  is the set of all edges between vertices in  $G$ . These sets are sometimes referred to as  $V(G)$  and  $E(G)$ , respectively.

$G$  is called a *simple graph* if: (1) there is at most 1 edge between any two vertices, (2) there are no edges from a vertex to itself and (3) all edges have no directionality to them, meaning  $uv = vu$  for any edge  $uv \in E(G)$ . For the rest of this paper all graphs are finite simple graphs, but note that unions and subgraphs are defined the same way for directed graphs and infinite graphs.

Graphs are more intuitive to work with through their visual representations instead of their formal definitions. Let the simple graph  $G$  where  $V(G) = \{A, B, C, D, E, a, b, c, d, e\}$  and  $E(G) = \{Aa, Bb, Cc, Dd, Ee, AB, BC, CD, DE, EA, ac, ce, eb, bd, da\}$ .  $G$  is often called the *Petersen* graph. It's unwieldy when described formally, yet its visual representation is very easy to understand.

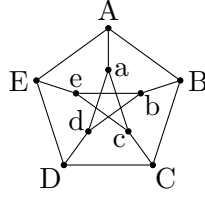
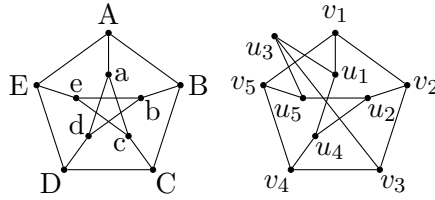


Figure 1.1: The Petersen graph

We say two graphs  $G$  and  $H$  are *isomorphic* if there exists a bijection from  $V(G)$  to  $V(H)$  that induces a bijection from  $E(G)$  to  $E(H)$  and we denote this relationship via  $G \cong H$ . In other words, we consider two graphs  $G, H$  to be the 'same' if we can relabel and/or move vertices in some fashion (without adding/removing vertices edges) in a visual representations of  $G$  and  $H$  to go back and forth between the two.

Figure 1.2:  $G \cong H$ 

Graph theorists casually refer to two graphs as the 'same' graph if they are in the same isomorphism class. We will wrap up the fundamentals with a few mdefinitions and some algebraic tools.

**Definition 1.1.1** (Subgraph). A subgraph  $G \subseteq K$  is a graph whose vertices and edges are subsets of the vertices and edges of  $K$ ;  $G \subseteq K$  if  $V(G) \subseteq V(K)$  and  $E(G) \subseteq E(K)$ .

**Definition 1.1.2** (Vertex-induced Subgraph). A *vertex-induced* subgraph  $G \subseteq K$  is one whose vertices are some subset of  $V(K)$  and whose edges are all edges between those vertices in  $K$ ;  $V(G) \subseteq V(K)$  and  $E(G) = \{uv \in E(K) \mid u, v \in E(G)\}$ . If  $G$  is such a subgraph we say that  $G$  is induced by  $S = V(G) \subseteq V(K)$ .

**Definition 1.1.3** (Edge-induced Subgraph). A *edge-induced* subgraph  $G \subseteq K$  is one whose edges are some subset of  $E(K)$  and whose vertices are all those who appear as

an endpoint in that subset of edges;  $E(G) \subseteq E(K)$  and  $V(G) = \{v \in V(K) \mid vu \in E(G) \text{ or } uv \in E(G) \text{ for some } u \in V(K)\}$ . If  $G$  is such a subgraph we say that  $G$  is induced by  $S = E(G) \subset E(K)$

Here is a visual example of these types of graphs: Let  $K$  be the Petersen graph from Figure 1.1. Now, let

**Subgraph:**  $G \subseteq K$  where  $V(G) = \{E, e, b\}$ ,  $E(G) = \{Ee\}$ .

**Vertex-induced Subgraph:**  $H \subseteq K$  is induced by  $\{a, A, B\} \subseteq V(K)$

**Edge-induced Subgraph:**  $M \subseteq K$  is induced by  $\{Dd, DC, Cc\} \subseteq E(K)$

The figure below shows  $K$  and its color-coded subgraphs.

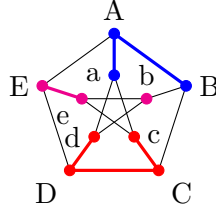


Figure 1.3:  $K$  and subgraphs  $G, H, M \subseteq K$

Next, we will talk about two important operations done on graphs.

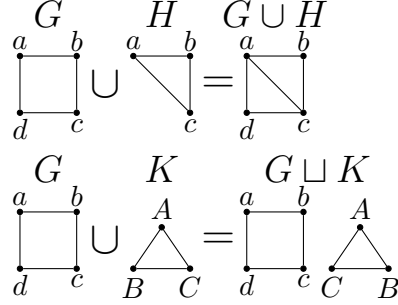
**Definition 1.1.4** (Graph Union). The union of two graphs  $G$  and  $H$  is simply the graph resulting from the union of their vertices and the union of their edges and is denoted  $G \cup H$ ;  $G \cup H = (V(G) \cup V(H), E(G) \cup E(H))$ . If  $G$  and  $H$  are edge-disjoint, we denote their union via  $G \sqcup H$  and call it a *disjoint union* of  $G$  and  $H$ .

Here is an example of a union and a disjoint union of graphs. Let  $G = (\{a, b, c, d\}, \{ab, bc, cd, da\})$ ,  $H = (\{a, b, c\}, \{ab, bc, ca\})$ , and  $K = (\{A, B, C\}, \{AB, BC, CA\})$  Then:

$$G \cup H = (\{a, b, c, d\}, \{ab, bc, cd, da, ca\})$$

$$G \sqcup K = V(G \sqcup K) = (\{a, b, c, d, A, B, C\}, \{ab, bc, cd, da, AB, BC, CA\})$$

These unions are depicted in the following figure.

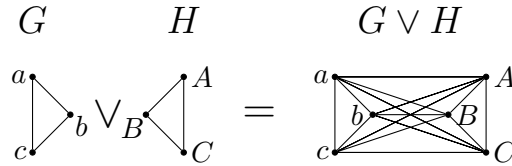
Figure 1.4: (above)  $G \cup H$  and (below)  $G \sqcup K$ 

Next, we define another very important operation that combines two graphs in a different manner.

**Definition 1.1.5** (Join). Let  $G$  and  $H$  be vertex disjoint graphs. Their *join*, denoted via  $G \vee H$ , is the graph obtained by taking the disjoint union of  $G$  and  $H$  and adding all possible edges between every vertex in  $G$  and every vertex in  $H$ . Formally:

$$G \vee H = (V(G) \cup V(H), E(G) \cup E(H) \cup \{xy \mid x \in V(G), y \in V(H)\}).$$

Here is an example. Let  $G = (\{a, b, c\}, \{ab, bc, ca\})$  and  $H = (\{A, B, C\}, \{AB, BC, CA\})$ , then  $G \vee H = (\{a, b, c, A, B, C\}, E(G) \sqcup E(H) \sqcup \{aA, aB, aC, bA, bB, bC, cA, cB, cC\})$ . This join is depicted in the figure below.

Figure 1.5: (above)  $G \cup H$  and (below)  $G \sqcup K$ 

Lastly, we define a few characteristics of graphs and their components. These may or may not be used frequently in this paper, but are important concepts to know in order to be able to talk about graphs comfortably.

Let  $G$  be a simple graph. We say two vertices  $u, v \in V(G)$  are *adjacent* or *neighbors* if they share an edge  $uv \in E(G)$ . Similarly, we say a vertex is *incident* with an edge if it is one of its endpoints;  $u \in V(G)$  is incident with  $e \in E(G)$  if  $e = uv$  for some  $v \in V(G)$ . The set of all vertices adjacent to  $u$  in  $G$  is called the *neighborhood* of  $u$  denoted  $N_G(u)$  or simply  $N(u)$ . Sometimes this is referred to as the open neighborhood of  $u$  in  $G$  and then the closed neighborhood is defined via  $N_G[u] = N_G(u) \cup \{u\}$ . The *degree* of a vertex  $u \in V(G)$  is the number of vertices adjacent to it and is denoted via  $\deg_G(u) = |N_G(u)|$  or simply  $\deg(u)$ . Equivalently, the degree is the number of edges incident to it or the number of neighbors that  $G$  has.

The following are three similar types of objects we can form from graphs.

**Definition 1.1.6** (Walk). Let  $G$  be a graph on  $n$  vertices. A *walk* in  $G$  is a sequence  $(w_0, w_1, \dots, w_k)$  of vertices in  $G$  whose adjacent elements must be adjacent in  $G$ . Adjacent elements in a walk must be distinct vertices but a vertex may be repeated multiple times.

**Definition 1.1.7** (Path). Let  $G$  be a graph on  $n$  vertices. A *path* in  $G$  is a sequence  $(v_0, v_1, \dots, v_k)$  of distinct vertices in  $G$  whose adjacent elements must be adjacent in  $G$ , and where no vertex is repeated. This sequence gives the subgraph of  $G$  induced by  $\{v_0v_1, v_1v_2, \dots, v_{k-1}v_k\}$ .

**Definition 1.1.8** (Cycle). Let  $G$  be a graph on  $n$  vertices. A *cycle* in  $G$  is a sequence  $(v_0, v_1, \dots, v_k, v_0)$  of internally distinct vertices (distinct except on the endpoints) that begins and terminates at the same vertex  $v_0$ . Often such a cycle is denoted via  $(v_0v_1 \cdots v_k)$  and it is understood that the sequence wraps back around to  $v_0$  after  $v_k$ . Additionally, the cycle  $(v_0v_1 \cdots v_k)$  is equivalent to  $(v_1 \cdots v_kv_0)$ ,  $(v_2 \cdots v_kv_0v_1)$ ,  $\dots$  and so on.

Let  $G$  be a simple graph. We call  $G$  *acyclic* if it contains no cycles. If there exists a path from any vertex to every other vertex in  $G$ , then we call  $G$  *connected*. If not, we call  $G$  *disconnected*. We call the set of connected subgraphs of  $G$  whose disjoint union equals  $G$  the *connected components* of  $G$ .

This concludes the fundamental concepts needed to understand this project. The next and final section of this chapter will introduce all the fundamental families of graphs we refer to in the proceeding chapters.

## 1.2 Fundamental Families of Graphs

In this section introduce some fundamental families of graphs which we refer to throughout this paper. Often instead of fully defining the graphs being worked with, we simply refer to it as a member of a larger family of graphs. These families are not completely distinct, but sometimes it is helpful to view graphs as a member of one family or another depending on the context.

Recall that a graph is acyclic if it contains no cycles. Similarly, we call a graph *k-cyclic* if it contains exactly  $k$  distinct cycles. If  $k = 2$  or  $3$  we call it *bicyclic* or *tricyclic*, respectively. In a similar vein, we call a graph *k-partite* if we can partition its vertices into  $k$  disjoint sets. If  $k = 2$  or  $3$ , we call it *bipartite* or *tripartite*, respectively. These are all very broad families of graphs often used to characterize graphs within another family. The following are more nuanced, and more popular families of graphs to work with.

**Definition 1.2.1** (Complete Graph). The *complete graph* on  $n$  vertices, denoted  $K_n$ , is the graph on  $n$  vertices such that every pair of distinct vertices shares an edge.

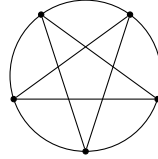


Figure 1.6: The Complete Graph  $K_5$

**Definition 1.2.2** (Complete Bipartite Graph). Let  $m, n \in \mathbb{N}$ . The *complete bipartite graph*  $K_{m,n}$  is the bipartite graph whose vertices can be partitioned into two disjoint sets of sizes  $m$  and  $n$ , respectively, such that every vertex in the one partite set is adjacent to every vertex in the other partite set and there are no edges between vertices in the same partite set.



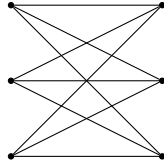


Figure 1.7: The Complete Bipartite Graph  $K_{3,3}$

**Definition 1.2.3** (Complete Multipartite Graph). The *complete  $k$ -partite graph* or *complete multipartite graph*  $K_{n_1, \dots, n_k}$  is the graph whose vertices can be partitioned into  $k$  disjoint sets of sizes  $n_1, n_2, \dots, n_k$ , respectively such that every vertex in the one partite set is adjacent to every vertex in the other  $k - 1$  partite sets and there are no edges between vertices in the same partite set.

If all partite sets are the same size  $n$  we call this graph the *complete equipartite graph*  $K_{n:k}$  or  $K_{n \times m}$ .

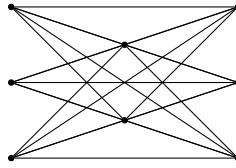


Figure 1.8: The Complete Multipartite Graph  $K_{3,2,3}$

**Definition 1.2.4** (Cycle Graph). The *cycle graph* on  $n$  vertices denoted  $C_n$  is a graph with exactly one cycle containing all of its edges.

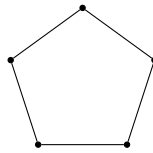


Figure 1.9: The Cycle Graph  $C_5$

**Definition 1.2.5** (Tree). A *tree* is any connected acyclic graph. Trees on  $n$  vertices have  $n - 1$  edges. Equivalently, these graphs are any connected bipartite graphs.

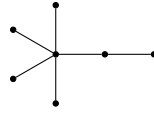
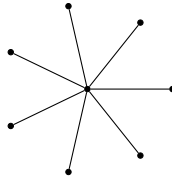


Figure 1.10: A Tree Graph on 6 vertices

**Definition 1.2.6** (Path Graph). The *path* graph on  $n$  vertices, denoted  $P_n$ , is an acyclic graph with exactly one path containing all of its edges. All paths are trees.

Figure 1.11: The Path Graph  $P_4$ 

**Definition 1.2.7** (Star Graph). The *star graph* on  $n + 1$  vertices, denoted  $K_{1,n}$  (or  $S_{n+1}$  which we never use in this paper) consisting of one central *hub* vertex adjacent to  $n$  *outer* vertices, with no other edges. All stars are trees. Sometimes this graph is referred to as an *n-star*.

Figure 1.12: The 7-star ( $K_{1,7}$ )

**Definition 1.2.8** (Forest Graph). Any disjoint union of tree graphs is called a *forest* graph. These graphs are all bipartite and can be equivalently defined as disconnected bipartite graphs.

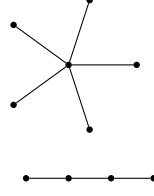
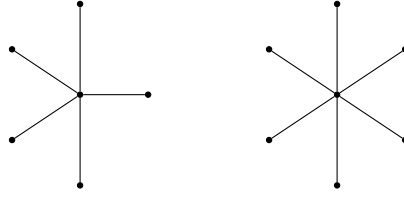


Figure 1.13: A forest on 9 vertices

**Definition 1.2.9** (Galaxy Graph). Any disjoint union of star graphs is called a *galaxy* graph. We refer to  $G = G_1 \sqcup \dots \sqcup G_k$  as a  $(G_1, \dots, G_k)$ -*galaxy* graph if  $G_1, \dots, G_k$  are all stars. This family is a subset of the forest family.

Figure 1.14: The  $(K_{1,6}, K_{1,7})$ -Galaxy

We have now defined a few important families of graphs which we will refer to throughout the rest of this paper. We generally don't explicitly define every graph by its vertices and edges and simply refer to it as some member of one family or say this it is isomorphic to one. This is much more efficient and concise than listing out all vertices and edges as we did in the beginning of this chapter.

We are now ready to move on and introduce graph decompositions, the objects which are the subject of this project.

# Chapter 2

## Introduction

### 2.1 Decompositions

Suppose you have  $n$  translucent sheets of tracing paper with some points drawn on all  $n$  sheets of paper in the same set arrangement. Now, draw lines connecting points on each sheet of paper, so that no line appears on two distinct sheets of paper.

A graph is depicted when all  $n$  sheets of tracing paper are aligned and stacked on top of each other with some light source present, call this graph  $K$ . Call the graph depicted on the  $i$ th sheet of paper  $G_i$  for  $i = 1, \dots, n$ . The stacking of these sheets of paper depicts  $G_1 \sqcup \dots \sqcup G_n = K$ , and this collection of papers depicts the set  $\{G_1, \dots, G_n\}$  which we call a *graph decomposition* of  $K$ . This is defined formally below.

**Definition 2.1.1** (Graph Decomposition). Let  $K$  be a simple graph. We call a collection  $\mathcal{G} = \{G_1, \dots, G_n\}$  of pairwise edge-disjoint subgraphs  $G_1, \dots, G_n \subseteq K$  of  $K$  a *decomposition* of  $K$  if their disjoint union equals  $K$ ;  $G_1 \sqcup \dots \sqcup G_n = K$  and  $\{G_1, \dots, G_t\}$ .

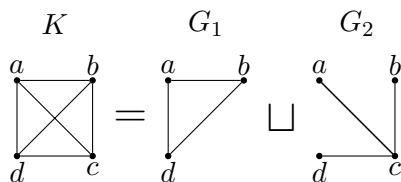


Figure 2.1:  $\{G_1, G_2\}$  is a decomposition of  $K_4$

Graph decompositions are an important topic combinatorics, graph theory, and design theory, with origins dating back to the 1800s. Notably, in 1850 Reverend Thomas Kirkman, a full-time clergyman and legendary mathematician, posed an important problem in *The Lady's and Gentleman's Diary* [?] now known as *the school girl problem*. It goes

*Fifteen young ladies in a school walk out three abreast for  
seven days in succession: it is required to arrange them daily  
so that no two shall walk twice abreast.*

The problem asks if we can form five distinct rows of three school girls on each day of the week so that no two school girls walk in the same row more than once in a week. This is equivalent to finding a decomposition of  $K_{15}$  whose members are all triangles; whose members are isomorphic to  $C_3$  or  $K_3$ . Both Kirkman and Arthur Cayley independently solved the the schoolgirl problem and published their solutions in the 1851 edition of *The Lady's and Gentleman's Diary* [?]. Kirkman's solution is provided below.

*Denoting the ladies by  $a_1, a_2, a_3; b_1, b_2, b_3; c_1, c_2, c_3; d_1, d_2, d_3; e_1, e_2, e_3$ , the following arrangement will be found to answer the question:*

$a_1 a_2 a_3$	$a_1 b_1 c_1$	$a_1 d_1 e_1$	$a_1 b_2 d_2$	$a_1 c_2 e_2$	$a_1 b_3 e_3$	$a_1 c_3 d_3$
$b_1 b_2 b_3$	$a_2 b_2 c_2$	$a_2 d_2 e_2$	$a_2 b_3 d_3$	$a_2 c_3 e_3$	$a_2 b_1 e_1$	$a_2 c_1 d_1$
$c_1 c_2 c_3$	$a_3 d_3 e_3$	$a_3 b_3 c_3$	$a_3 c_1 e_1$	$a_3 b_1 d_1$	$a_3 c_2 d_2$	$a_3 b_2 e_2$
$d_1 d_2 d_3$	$b_3 d_1 e_2$	$b_1 c_1 e_3$	$b_1 c_3 e_1$	$b_2 c_3 d_1$	$c_2 b_3 e_1$	$c_2 b_3 e_1$
$e_1 e_2 e_3$	$c_3 d_2 e_1$	$e_3 b_2 c_1$	$d_1 c_2 e_3$	$c_1 d_2 b_3$	$d_2 b_1 c_2$	$c_1 d_3 b_2$

*This is the symmetrical and only possible solution. All others differ from this only in disturbing the alphabetical order, or that of the three subindices in certain triplets of the first column, or in both these together.*

Each triple in this array above gives a edge-distinct triangle subgraph of  $K_{15}$  whose vertex set we take to be  $\{a_1, a_2, \dots, e_4, e_5\}$ . The set of all these subgraphs is a decomposition of  $K_{15}$ . Since all of these subgraphs are isomorphic to  $C_3$ , we call it a  $C_3$ -decomposition. This is a special type of decomposition which is defined formally on the following page.

**Definition 2.1.2** (*G*-decomposition). A *G*-decomposition of a graph  $K$  is a decomposition  $\mathcal{G} = \{G_1, \dots, G_t\}$  whose members are all isomorphic to some graph  $G$ ;  $\mathcal{G} = \{G_1, \dots, G_t\}$  such that  $K = G_1 \sqcup \dots \sqcup G_t$  and  $G_i \cong G$  for  $i = 1, \dots, t$ . If such a set exists we say that  $K$  allows a *G*-decomposition or equivalently, that  $G$  decomposes  $K$ . If  $K \cong K_n$  we sometimes call the decomposition a *G*-design of order  $n$ .

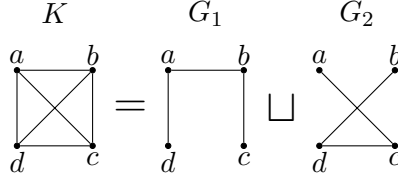


Figure 2.2:  $\{G_1, G_2\}$  is a  $P_3$ -decomposition of  $K_4$  or a  $P_3$ -design of order 4

We know that if a *G*-decomposition of some graph  $K$  exists, that all of its members have the same number of edges and vertices. This allows us to find constraints on the types of bigger graphs  $K$  that can be decomposed by some subgraph  $G \subseteq K$  on  $m$  edges solely based on divisibility properties.

**Lemma 2.1.3** (Necessary Condition (general)). *Let  $G$  be a simple graph on  $m$  edges. There exists a  $G$ -decomposition of a graph  $K$  only if  $|E(G)| = m$  divides  $|E(K)|$ .*

*Proof.* Suppose there exists a  $G$ -decomposition  $\mathcal{G} = \{G_1, \dots, G_n\}$  of  $K$ . Then  $E(G_1) \sqcup \dots \sqcup E(G_t) = E(K)$  and so  $|E(K)| = |E(G_1) \sqcup \dots \sqcup E(G_t)| = |E(G_1)| + \dots + |E(G_t)| = tm$ . So  $|E(G)| = m$  divides  $|E(K)|$ .  $\square$

**Theorem 2.1.4** (Necessary Condition ( $K_n$ )). *Let  $G$  be a simple graph on  $m$  edges. There exists a  $G$ -decomposition of  $K_n$  only if  $n$  is idempotent modulo  $2m$ ; only if  $n^2 \equiv n \pmod{2m}$ .*

*Proof.* Suppose there exists a  $G$ -decomposition of  $K_n$ . Then  $|E(G)| = m$  divides  $|E(K_n)| = \binom{n}{2} = \frac{n(n-1)}{2}$  by Lemma 2.1.3. Therefore,  $\frac{n^2-n}{2} = mt$  for some  $t \in \mathbb{N}$ . Observe.

$$n^2 - n = 2mt \implies n^2 - n \equiv 0 \pmod{2m} \implies n^2 \equiv n \pmod{2m}.$$

$\square$

By the previous theorem, any graph on  $m$  edges decomposes  $K_n$  only if  $n$  is idempotent modulo  $2m$ . Note that the converse isn't necessarily true. However, for a graph  $G$  on  $m$  edges, this finite set of constraints allows us to ask:

For what  $n$  is  $K_n$   $G$ -decomposable?

This question is known as the *spectrum problem* for decompositions. Pioneering work by Rosa and Kotzig in the 1960s—especially in the development of graph labeling—helped shape the modern treatment of  $G$ -decomposition problems. Since then, labeling-based techniques and tools from design theory have driven significant progress. In particular, graph labeling methods have played a central role in addressing the spectrum problem for small graphs. This directly continues contributions by Freyberg and Peters, who recently solved the spectrum problem for forests with six edges [?]. Their paper provides a comprehensive summary of known decompositions for graphs  $G$  with fewer than seven edges.

Using graph labelings to solve  $G$ -decomposition problems is basically about doing algebra on graphs in order to generate edge-disjoint subgraphs. If we take the vertices of a graph  $K$  to be elements of a group, we can use the structure of the group to our advantage. Specifically, when  $K \cong K_n$ , and we take its vertices to be  $\mathbb{Z}_n$ , and then we label the vertices of  $G$  with some subset of  $\mathbb{Z}_n$ . There are various labeling techniques of this kind stemming from Rosa's work in the 1960s that allow us to permute or act on the labels of the vertices of  $G$  with subgroups of  $\mathbb{Z}_n$  to generate new isomorphic copies of  $G$  which are pairwise edge-disjoint. In the next section, we provide an example which outlines in some detail how this machinery works for  $G$ -decompositions of Complete Graphs.

## 2.2 Graph labeling

Take the vertices of  $K_5$  to be  $\mathbb{Z}_5$  and arrange it in the same manner as in 1.6. Notice that every vertex shares an edge with two vertices directly adjacent to it and two vertices that are 'two adjacencies away' on the outer cycle (01234). We call this idea *length* denoted  $\ell$  where edges joining two vertices  $u, v$  have length  $\ell(uv) = l$  if they are ' $l$  adjacencies' away from each other on the outer cycle.

Formally, for  $K_n$  we define the edge length function  $\ell$  as follows:

$$\ell(uv) = \min\{|u - v|, n - |u - v|\}$$

Notice that for  $K_5$ , we only have lengths 1 or 2 as previously observed. Color the length 1 edges **blue**, and the length 2 edges **red**. This is depicted in the figure below.

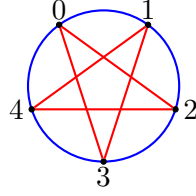


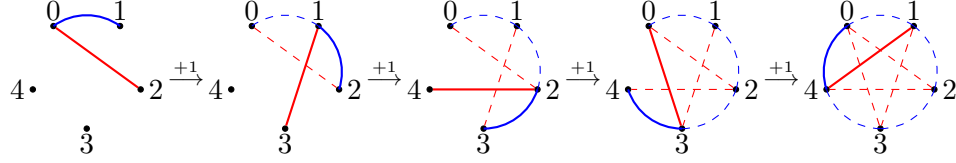
Figure 2.3:  $K_5$  with lengths colored

Now, consider  $P_3$ . It has 2 edges, and  $K_5$  has  $\binom{5}{2} = 10$  edges. Since  $2|10$ , by Lemma 2.1.3 it's *possible* that a  $P_3$ -decomposition of  $K_5$  exists. Now since  $P_3$  has 2 edges and there are 2 lengths in  $K_5$ , what if we can just make sure each copy of  $P_3$  has both a **blue** edge and a **red** edge? How can we do that while ensuring that no edge is repeated?

It turns out that if we take the vertices of  $K_n$  to be  $\mathbb{Z}_n$ , adding 1 (and therefore anything) modulo  $n$  to the endpoints of an edge preserves its length. We call the act of permuting vertices in this manner *clicking* or *developing*.

In the context of our problem with  $P_3$  and  $K_5$ , this means that if we label  $P_3$  with elements of  $\mathbb{Z}_5$  such that just have one **blue** edge of length 1, and one **red** edge of length 2, that we can simply generate all edges of length 1 and 2 in  $K_n$  (which is just all edges of  $K_n$ ) while preserving the structure of the graph by developing all vertices of our labeling by 1 at the same time repeatedly. Label the defining path of  $P_3$  via  $(2,0,1)$ . Developing the vertices 1 modulo 5 will give all members of a  $P_3$ -decomposition of  $K_5$ . A decomposition that can be generated by permuting all vertices of one labeling repeatedly in this fashion is called a *cyclic decomposition*. This is depicted in the following figure.



Figure 2.4: A cyclic  $P_3$ -decomposition of  $K_5$ 

Nice and easy right? But that's just one complete graph that  $P_3$  can decompose. Remember, that it is *possible* that  $P_3$  can decompose any  $K_n$  where  $n \equiv n^2 \pmod{4}$  by Theorem 2.1.4. This equivalent to saying  $n = 4t + r$  where  $r$  is an idempotent in the ring  $\mathbb{Z}_4$  and  $t \geq 1$ . The idempotents in  $\mathbb{Z}_4$  are 0, 1. So this means  $\mathbb{Z}_5$  is just a special case of  $n$  where  $n = 4t + 1$  where  $t = 1$ . Luckily, even though these are infinite families, it is known that for each step  $t \mapsto t + 1$ , new lengths come 2 at a time. This means if we can somehow transform our labeling at each step to include the new lengths, we can maybe take care of the entire family  $K_{4t+1}$ . We want to fine tune our labeling to be able to weather this process. This is what graph labeling is all about. Note that if  $r$  was not 0 or 1, we would need multiple labelings to take care of the whole family. This is explained later in this paper.

Lastly, some basic observations about a general  $G$  with  $m$  edges and  $K_n$ . The maximal length in  $K_n$  is  $\lfloor \frac{n}{2} \rfloor$ . This is intuitive, since when you travel halfway across the outer cycle from some vertex, the lengths start going back down again and you are nearing that vertex. Now,  $n$  must be of the form  $2mt + r$  where  $t \geq 1$  and  $r$  is an idempotent in the ring  $\mathbb{Z}_{2m}$ . This means that in  $K_{2m+r}$  if  $\ell(uv) = |u - v| < \lfloor \frac{2m+r}{2} \rfloor < \lfloor \frac{2mt+r}{2} \rfloor$  for  $t > 1$ , then  $\ell(uv) = |u - v|$  in all  $K_{2mt+r}$  for  $t \geq 1$ . this is important, because at each step  $t \mapsto t + 1$ , new lengths come  $m$  at a time. This means that some wraparound edges  $xy$  in  $K_{2m+r}$  are short edges of length  $|x - y|$  in  $K_{2mt+r}$  for  $t > 1$ .

Now, for  $r = 0$  or  $1$ , if a certain labeling of a graph  $G$  on  $m$  edges exists, there exists a  $G$ -decomposition of  $K_{2mt+r}$  for  $t \geq 1$ . However, if  $r \neq 0, 1$ , one labeling will not suffice and other techniques are needed to prove that  $G$  decomposes  $K_{2mt+r}$  for  $t \geq 1$ . These labelings and techniques are defined as they are needed in the proceeding chapters. Finally, we are ready to introduce the focus of this project.

## 2.3 Seven edge forests

This project continues on Freyberg and Peters' work on six edge forests by asking the same question about seven edge forests:

Let  $F$  be a forest on seven edges. For which  $n$  does  $F$  decompose  $K_n$ ?

The spectrum problem for the matching  $7\mathbf{T}_2^{11}$  was solved by de Werra in 1970. Every component of a forest on seven edges is a tree on six or less edges which are cataloged in Figure 2.5. We use the naming convention  $\mathbf{T}_j^i$  to denote the  $i^{\text{th}}$  tree with  $j$  vertices and we index the vertices  $v_1$  through  $v_j$  for each tree as specified below.

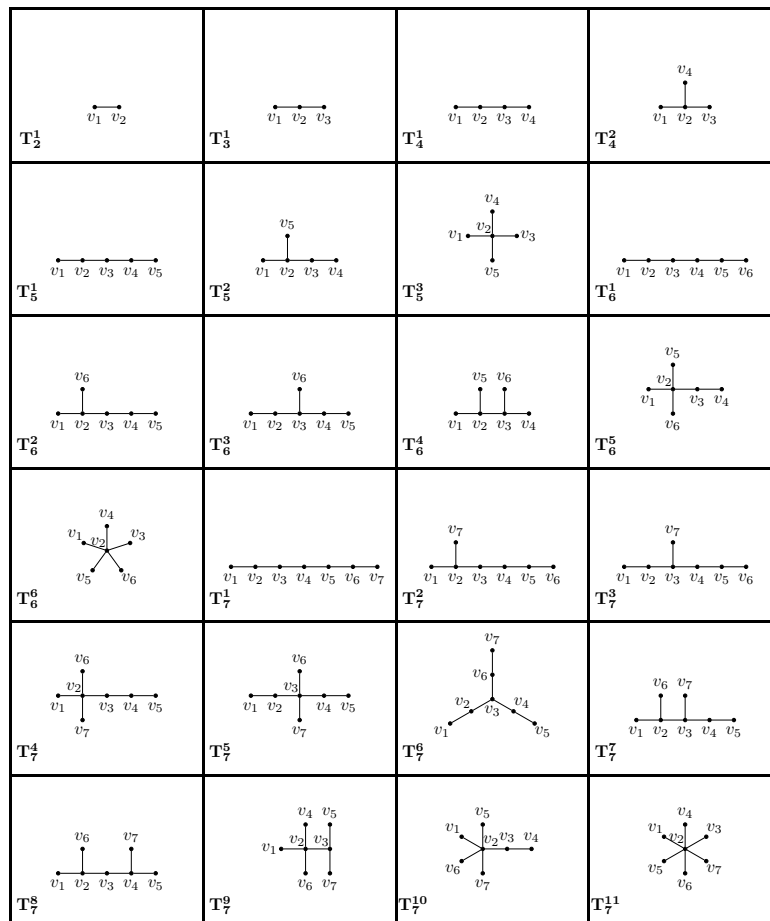


Figure 2.5: Trees with less than seven edges

The next theorem gives the necessary conditions for the existence of a  $G$ -decomposition of  $K_n$  when  $G$  is a graph with 7 edges.

**Theorem 2.3.1.** *If  $G$  is a graph with 7 edges and a  $G$ -decomposition of  $K_n$  exists, then  $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$ .*

*Proof.* If a  $G$ -decomposition of  $K_n$  exists, then  $n$  is idempotent modulo  $2(7) = 14$  by Theorem 2.1.4 which immediately implies that  $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$  since those are all the idempotents in  $\mathbb{Z}_{14}$ .  $\square$

For this project, we do not define the graph on one vertex to be a tree. This means that any connected component in a forest has at least one edge and we also require there to be at least two connected components. There are 47 such forests with 7 edges up to isomorphism. As stated previously, the matching on seven edges is solved, so only the remaining 46 trees need be considered in the subsequent chapters. Chapter 3 handles all decomposing  $K_n$  into all 47 forests when  $n \equiv 0 \text{ or } 1 \pmod{14}$ . Chapter 4 applies to all the forests when  $n \equiv 7 \text{ or } 8 \pmod{14}$  with the lone exception of  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ , which is solved for those values of  $n$  in Chapter 5.

After proving the main result of this project, we provide a couple of additional results in Chapter ??: (1) An edge mapping depending on  $t$  that preserves length for wraparound edges in  $2m+r$  to  $2mt+r$  at each step  $t \mapsto t+1$ . (2) Galaxy decompositions of complete multipartite graphs. These two results conclude this thesis project, and produce some open questions.

Lastly, we present some *Python* programs that were created as a result of this thesis project. One is *tikzgrapher*: my graph visualization software, which was built from scratch only using *Pygame* and *NetworkX*. It allows one to visualize simple NetworkX graphs in an interactive Pygame window that allows for colorings and custom labelings along with dragging and moving components of the graph. The main feature however, is that the user can save the graph layout depicted in the Pygame window as a tikZ graph in a standalone  $\text{\LaTeX}$ file. *tikzgrapher* is paired with a graph labeling solver: a constraint programming project that can find several labelings on graphs if they exist. Chapter ?? concludes the results of this project.

## Chapter 3

### $n \equiv 0, 1 \pmod{14}$

To begin this chapter, we extend intuition developed in the Introduction to present some machinery specific to  $K_n$  where  $n \equiv 0, 1 \pmod{14}$ . This informs the formal definitions and theorems we use for this case.

#### 3.1 Construction for $n \equiv 0, 1 \pmod{14}$

$K_1$  and  $K_2$  don't have enough vertices to contain a forest on seven edges. So  $K_{14}$  and  $K_{15}$  are the base graphs for  $K_n$  where  $n \equiv 0$  or  $1 \pmod{14}$ , respectively. We first show how to decompose  $K_{14}$  and  $K_{15}$  in subsection 3.1.1 and then show how to generalize this to their entire families in subsection 3.1.2

To be absolutely clear, **these are not original ideas as a result of this project**. Rosa and his colleagues developed this decades ago. This is just an explanation from our perspective, of the amazing work of Rosa and his successors in creating graph labelings techniques to solve decomposition problems.

### 3.1.1 $K_{14}$ and $K_{15}$

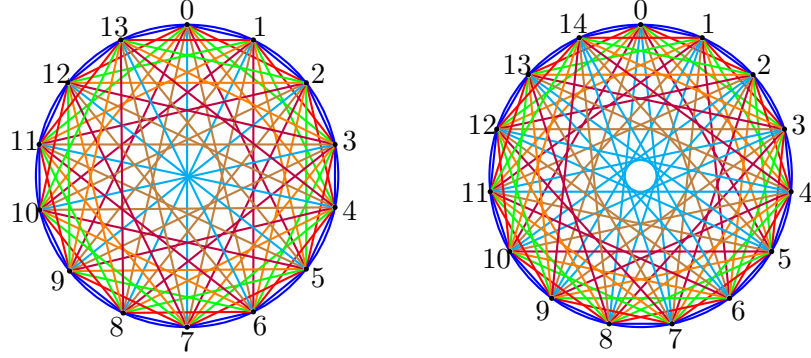


Figure 3.1:  $K_{14}$  (left) and  $K_{15}$  (right) with edges colored by length

Both  $K_{14}$  and  $K_{15}$  both only have edges with lengths 1 through 7, however, there is important distinction in counting edges of one length  $K_{14}$  versus counting them in  $K_{15}$ . The cyan edges have length 7, and conveniently because of the way these graphs are drawn in the figure above, we can easily count them as they form the innermost spoke of the graphs. Because these graphs are so similar we can use one labeling to deal with both at the same time! However, we must be very careful.

In  $K_{14}$  there are only 7 edges of length 7 and 14 edges of all other lengths, yet in  $K_{15}$  there are 15 edges of length 7 along with 15 edges of all other lengths. it turns out this pattern generalizes. For  $K_n$  where

**n is odd:** There are  $n$  edges of lengths 1 through  $\frac{n-1}{2}$

**n is even:** There are  $n$  edges of lengths 1 up to  $\frac{n}{2}$ , and then  $\frac{n}{2}$  edges of length  $\frac{n}{2}$

This is why the labeling of  $P_3$  in Figure 2.4 used to decompose  $K_5$  in the Introduction worked so easily.  $K_5$  has odd order, so there are 5 edges of each length in  $\{1, 2\}$ . The same applies here for  $K_{15}$ , we just label each 7-edge forest  $F$  so that it contains all seven lengths in  $K_{15}$ , and then developing the vertices by 1 repeatedly generates the entire  $F$ -decomposition of  $K_{15}$ .

But notice that if we repeatedly develop nodes of the same labeling of  $F$  in  $K_{14}$  (assuming we don't use the vertex 14) we would *overcount* length 7 edges. There is a

simple remedy for this, but it requires a shift in perspective.

Take  $V(K_{14})$  to be  $\mathbb{Z}_{13} \cup \{\infty\}$ , and label all edges via the length function modulo 13 except for edges incident to  $\infty$  which will refer to as length  $\infty$ . Now redefine developing nodes by 1 only to effectively fix the  $\infty$  vertex so that  $\infty \mapsto \infty + 1 = \infty$ . Formally, let:

$$\ell(uv) = \begin{cases} \min\{|u - v|, 13 - |u - v|\}, & u, v \neq \infty, \\ \infty, & u \text{ or } v = \infty \end{cases} \quad \text{and } v \mapsto \begin{cases} v + 1, & v \in \mathbb{Z}_{13}, \\ \infty, & v = \infty. \end{cases}$$

The reason for doing this is that we will now have 13 edges of lengths 1 through 6 as well as  $\infty$ , since the infinity node will have all 13 edges of length  $\infty$  to nodes in  $\mathbb{Z}_{13}$  incident to it. Now if we develop the endpoints of an edge with length  $i$  by 1 repeatedly, we will get all 13 of a distinct edges of length  $i$ .

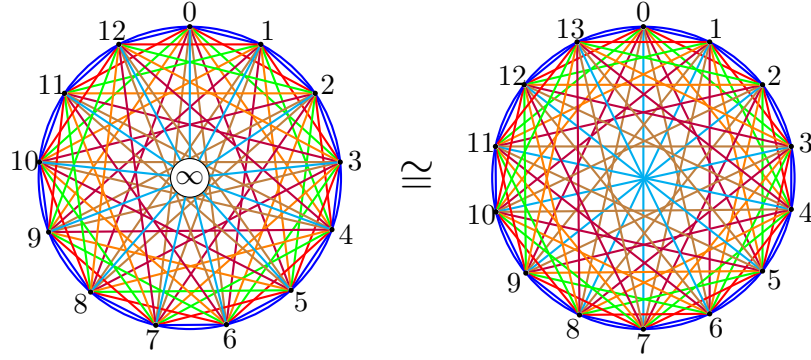


Figure 3.2:  $K_{13} \vee K_1$  is isomorphic to  $K_{14}$

Now, notice that if we use this new construction for  $K_{14}$ , we must ensure that we only use nodes in  $\mathbb{Z}_{13}$ . Here is an example.

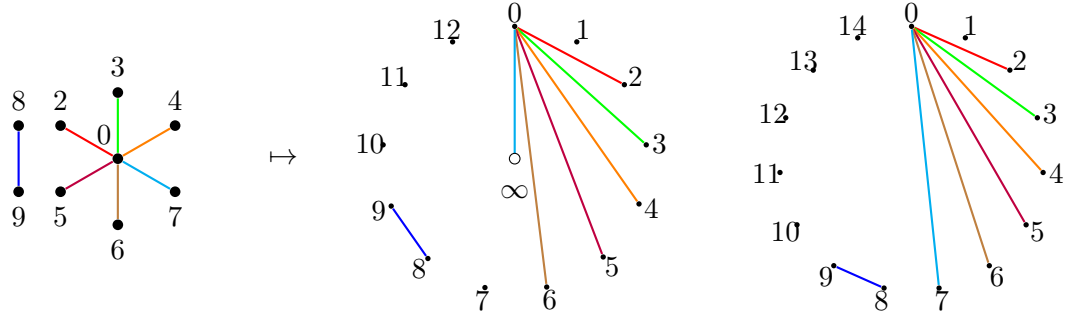


Figure 3.3: A labeling (left) that gives the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of  $K_{14}$  (middle) and  $K_{15}$  (right), respectively, when developed by 1. The leaf 7 in the pendant edge  $(0, 7)$  (of length 7 modulo 14 and 15) is relabeled as  $\infty$  for  $K_{14}$

### 3.1.2 Stretching a labeling

Now, we have learned how to decompose  $K_{14}$  and  $K_{15}$ , but what about the rest of the family? Well,  $K_{14t}$  and  $K_{14t+1}$  actually contain edges of lengths in

$$\bigcup_{0 \leq i < t} \{1 + 7i, \dots, 7 + 7i\} \text{ for } t \geq 1$$

This is because the maximal length in  $K_{14t}$  and  $K_{14t+1}$  is  $\lfloor \frac{14t+1}{2} \rfloor = 7t$ . Luckily, it turns out that we can actually just replicate this process we did for lengths 1 through 7 for each step  $t \mapsto t+1$  for  $K_{14t}$  and  $K_{14t+1}$  for the new lengths that come 7 at a time if we add some restrictions.

There are two types of edges in complete graphs. We call edges  $uv$  of length  $n - |u - v|$  *wraparound edges*, and edges  $ab$  of length  $|a - b|$  *short edges*. If we want to build a labeling that can generalize to an entire family of complete graphs, we need to understand how they would generalize as the order of the complete graph increases. Observe.

$$\text{In } K_{14(1)}: \ell((0, 8)) = \min\{|0 - 8|, 14 - |0 - 8|\} = \min\{8, 6\} = 6.$$

$$\text{In } K_{14(2)}: \ell((0, 8)) = \min\{|0 - 8|, 28 - |0 - 8|\} = \min\{8, 20\} = 8.$$

There is a way to map labelings at each step to preserve length which is described in Chapter ??, but it can get ugly. Luckily, there is a way to avoid this problem entirely. If we only use *short edges*  $uv$  which have length  $|u - v|$  in  $K_{13}$  for our labelings, then

that edge will have length  $|u - v|$  in every complete graph  $K_n$  where  $n > 13$ . Let  $k > 2$ . We show another example below.

$$\text{In } K_{14(1)}: \ell((0, 6)) = \min\{|0 - 6|, 14 - |0 - 6|\} = \min\{6, 8\} = 6.$$

$$\text{In } K_{14(2)}: \ell((0, 6)) = \min\{|0 - 6|, 28 - |0 - 6|\} = \min\{6, 22\} = 6.$$

$\vdots$

$$\text{In } K_{14(k)}: \ell((0, 6)) = \min\{|0 - 6|, 14k - |0 - 6|\} = \min\{6, 22 + 14(k - 2)\} = 6.$$

So we see that if we only use short edges, the length of edges in our labeling will be preserved as we scale up. So for that reason, we simply *need* to use them. However, another important feature is that if we only use short edges  $uv$ , we know WLOG that  $v > u$  and so  $|u - v| = v - u$ . This introduces another extremely important mechanism at play in the labelings we define in the proceeding section of this chapter.

If we bundle the short edge requirement and the requirement that the maximal edge is a pendant edge with a new ordered bipartition  $V(F) = A \sqcup B$  requirement on the vertices of a labeling  $F$  such that all vertices  $a \in A$  only have neighbors in  $B$  which are larger than  $a$ , then we get a really nice property. All edges are now of the form  $ab$  where  $a \in A$  and  $b \in B$  so that  $a < b$  and so all such edges  $ab$  have length  $b - a$  modulo 13. This means that if  $\ell(ab) = l$  in  $\mathbb{Z}_{13}$ , then  $b - a = l$  and so  $(b + c) - a = l + c$ . More importantly, if we then add any  $c \in \mathbb{Z}_{14t}$  to all vertices in the larger partite set  $B$  of our labeling, we will simply increase the lengths of all edges in our labeling by  $c$  in  $\mathbb{Z}_{14t}$  and  $\mathbb{Z}_{14t+1}$  as a result of bundling these restrictions together!! We call the act of adding some constant to all vertices of the larger partite set  $B$  of a labeling of this type *stretching*.

Let  $t > 1$ . Then for a labeling  $F$  for  $K_{14t+1}$  we simply develop our original labeling by 1 to generate lengths in  $\{1, \dots, 7\}$ , then stretch that labeling by  $7i$  and develop it by 1 for each  $0 < i < t$ , to generate all edges of  $K_{14t+1}$ .

Now, in the same labeling  $F$  for  $K_{14t}$ , recall that there are only  $7t$  edges of the maximal length  $7t$ . This means that we want to take it's vertices to be  $\mathbb{Z}_{14t-1} \cup \{\infty\}$  and so now there are now  $14t - 1$  edges of length 7 along with all lengths less than  $7t$ . So then we simply develop our original labeling by 1 to generate lengths in  $\{1, \dots, 7\}$ , then stretch that labeling by  $7i$  and develop it by 1 for each  $0 < i < t$  **except** the



last labeling which was stretched by  $7(t-1)$  with lengths in  $\{7t-6, \dots, 7t\}$ . Since the pendant edge of length 7 was stretched to be one of maximal length  $7t$ , it is still a pendant edge. So now we relabel the leaf as  $\infty$  and so its length becomes  $\infty$ . We then develop it by 1 and have collected all edges in  $K_{14t}$ .

We show how the labeling from Figure 3.3 gives the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of  $K_{14(2)} = K_{28}$  and  $K_{14(2)+1} = K_{29}$ .

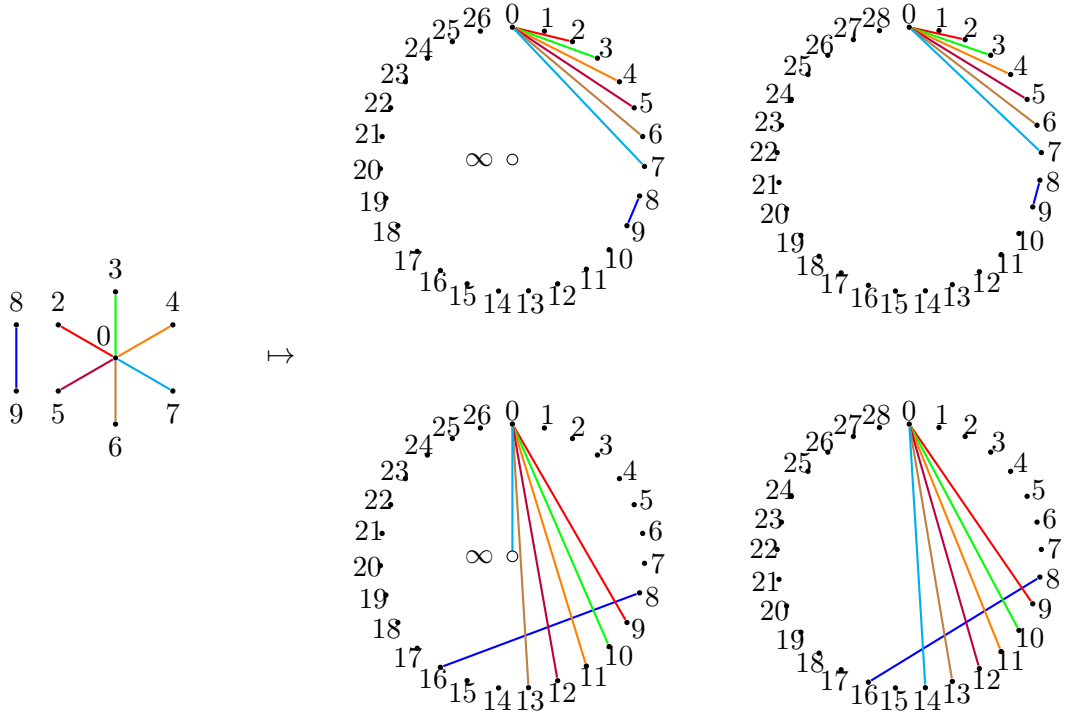


Figure 3.4: A single labeling (left) gives four labelings which when developed by 1 give the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^{11}$ -decomposition of (middle)  $K_{28}$  and (right)  $K_{29}$ .

We will refer to this act of stretching a labeling in later sections. The ideas in this section are now put together formally in the next section.

### 3.2 Results for $n \equiv 0, 1 \pmod{14}$

We begin with the original  $\rho$ -labeling of a bipartite graph  $G$  on  $m$  edges which uses lengths modulo  $2m$  (in  $K_{2m}$ ).

**Definition 3.2.1** ((Rosa [?])). Let  $G$  be a graph with  $m$  edges. A  $\rho$ -labeling of  $G$  is an injection  $f : V(G) \rightarrow \{0, 1, 2, \dots, 2m\}$  that induces a bijective *length function*  $\ell : E(G) \rightarrow \{1, 2, \dots, m\}$  where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all  $uv \in E(G)$ .

Rosa showed that if a  $\rho$ -labeling of a graph  $G$  with  $m$  edges exists, then a cyclic  $G$ -decomposition of  $K_{2m+1}$  exists, which is presented formally later. Later, Rosa and his peers began studying more restrictive types of  $\rho$ -labelings to decompose more complete graphs. Next, we define some of these labelings and theorems associated with them.

**Theorem 3.2.2** ((Rosa [?])). *Let  $G$  be a graph with  $m$  edges. There exists a cyclic  $G$ -decomposition of  $K_{2m+1}$  if and only if  $G$  admits a  $\rho$ -labeling.*

**Definition 3.2.3** ((Rosa [?])). A  $\sigma$ -labeling of a graph  $G$  is a  $\rho$ -labeling such that  $\ell(uv) = |f(u) - f(v)|$  for all  $uv \in E(G)$ .

**Definition 3.2.4** ((El-Zanati, Vanden Eynden [?])). A  $\rho$ - or  $\sigma$ -labeling of a bipartite graph  $G$  with bipartition  $(A, B)$  is called an *ordered*  $\rho$ - or  $\sigma$ -labeling and denoted  $\rho^+, \sigma^+$ , respectively, if  $f(a) < f(b)$  for each edge  $ab$  with  $a \in A$  and  $b \in B$ .

**Theorem 3.2.5** ((El-Zanati, Vanden Eynden [?])). *Let  $G$  be a graph with  $m$  edges which has a  $\rho^+$ -labeling. Then  $G$  decomposes  $K_{2mk+1}$  for all positive integers  $k$ .*

**Definition 3.2.6** ((Freyberg, Tran [?])). A  $\sigma^{+-}$ -labeling of a bipartite graph  $G$  with  $m$  edges and bipartition  $(A, B)$  is a  $\sigma^+$ -labeling with the property that  $f(a) - f(b) \neq m$  for all  $a \in A$  and  $b \in B$ , and  $f(v) \notin \{2m, 2m - 1\}$  for any  $v \in V(G)$ .

**Theorem 3.2.7** ((Freyberg, Tran [?])). *Let  $G$  be a graph with  $m$  edges and a  $\sigma^{+-}$ -labeling such that the edge of length  $m$  is a pendant. Then there exists a  $G$ -decomposition of both  $K_{2mk}$  and  $K_{2mk+1}$  for every positive integer  $k$ .*

Table 3.1 gives  $\sigma^{+-}$ -labelings of all forests on 7 edges except the matching. The vertex labels of each connected component with  $k$  vertices are given as a  $k$ -tuple,  $(v_1, \dots, v_k)$  corresponding to the vertices  $v_1, \dots, v_k$  positioned as shown in Figure 2.5. We leave it to the reader to infer the bipartition  $(A, B)$ .

**Example 3.2.8.** A  $\sigma^{+-}$ -labeling of  $\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$  is shown in Figure 3.5. The vertices labeled 1, 2 and 9 belong to  $A$ , and the others belong to  $B$ . The lengths of each edge are indicated on the edge.

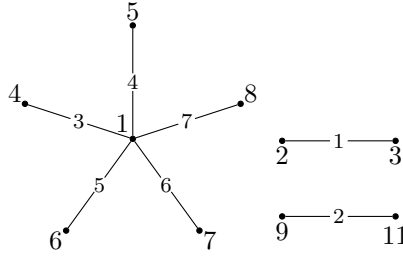


Figure 3.5:  $\sigma^{+-}$ -labeling of  $\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$

The labelings given in Figure 3.1 along with thm 3.2.7 are enough to conclude this case.

Forest	Vertex Labels
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5, 2, 9, 7) \sqcup (3, 4)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(9, 2, 5, 1, 6, 0, 3) \sqcup (8, 7)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(9, 2, 5, 1, 6, 0, 4) \sqcup (8, 7)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(5, 1, 4, 2, 9, 6, 7) \sqcup (10, 11)$
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(3, 8, 1, 4, 2, 5, 7) \sqcup (9, 10)$
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(7, 8, 1, 6, 0, 4, 3) \sqcup (9, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(8, 1, 6, 3, 4, 5, 7) \sqcup (9, 10)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(6, 1, 5, 3, 8, 4, 7) \sqcup (9, 10)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(5, 11, 9, 10, 6, 12, 7) \sqcup (8, 1)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(4, 8, 1, 6, 0, 5, 3) \sqcup (9, 10)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(0, 6, 1, 5, 2, 9) \sqcup (11, 10, 12)$
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(3, 6, 1, 8, 4, 0) \sqcup (10, 9, 11)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(5, 11, 9, 12, 7, 10) \sqcup (1, 8, 4)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(3, 8, 4, 1, 6, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(5, 1, 8, 3, 4, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(4, 1, 8, 5, 6, 7) \sqcup (10, 9, 11)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(0, 6, 1, 5, 2) \sqcup (9, 8, 10, 3)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(7, 1, 8, 5, 6) \sqcup (0, 4, 2, 3)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(7, 1, 8, 4, 6) \sqcup (10, 9, 11, 12)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(6, 0, 3, 4, 5) \sqcup (8, 7, 9, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(4, 8, 1, 7, 2) \sqcup (10, 9, 11, 12)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(6, 0, 3, 4, 5) \sqcup (8, 9, 2, 7)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(0, 6, 1, 5, 2, 9) \sqcup (8, 10) \sqcup (3, 4)$
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(3, 6, 1, 8, 4, 0) \sqcup (5, 7) \sqcup (9, 10)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(4, 1, 8, 3, 5, 7) \sqcup (0, 2) \sqcup (9, 10)$
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(5, 8, 4, 1, 6, 7) \sqcup (0, 2) \sqcup (9, 10)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(5, 11, 9, 12, 7, 10) \sqcup (8, 1) \sqcup (0, 4)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(4, 1, 8, 5, 6, 7) \sqcup (2, 3) \sqcup (9, 11)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5, 2) \sqcup (8, 10, 9) \sqcup (11, 4)$

Table 3.1:  $\sigma^{+-}$ -labelings for forests with seven edges

Forest	Vertex Labels
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(7, 1, 8, 5, 6) \sqcup (10, 9, 11) \sqcup (0, 4)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(6, 0, 3, 4, 5) \sqcup (1, 8, 7) \sqcup (9, 11)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (2, 9, 7, 10) \sqcup (3, 4)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, 10, 7) \sqcup (4, 0, 5, 6) \sqcup (8, 1)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (10, 9, 11, 12) \sqcup (8, 1)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(0, 6, 1, 5) \sqcup (8, 10, 9) \sqcup (11, 4, 7)$
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(4, 0, 5, 6) \sqcup (1, 8, 7) \sqcup (11, 9, 12)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (8, 10, 7) \sqcup (11, 4) \sqcup (2, 3)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (11, 9, 12) \sqcup (2, 3) \sqcup (8, 1)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(0, 6, 1, 5, 2) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(6, 1, 8, 4, 7) \sqcup (3, 5) \sqcup (9, 12) \sqcup (10, 11)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(3, 0, 4, 5, 6) \sqcup (8, 1) \sqcup (10, 11) \sqcup (9, 7)$
$3\mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(0, 6, 1) \sqcup (4, 8, 5) \sqcup (2, 9, 7) \sqcup (10, 11)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(0, 6, 1, 5) \sqcup (9, 2) \sqcup (8, 10) \sqcup (4, 7) \sqcup (11, 12)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(4, 0, 5, 6) \sqcup (2, 3) \sqcup (9, 11) \sqcup (8, 1) \sqcup (10, 7)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(0, 6, 1) \sqcup (4, 8, 5) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(0, 6, 1) \sqcup (8, 4) \sqcup (2, 5) \sqcup (10, 3) \sqcup (9, 7) \sqcup (11, 12)$

Table 3.1:  $\sigma^{+-}$ -labelings for forests with seven edges

**Theorem 3.2.9.** *Let  $F$  be a forest with 7 edges. There exists an  $F$ -decomposition of  $K_n$  whenever  $n \equiv 0$  or  $1 \pmod{14}$ .*

*Proof.* The proof follows from thm 3.2.7 and the labelings given in Figure 3.1.  $\square$

# Chapter 4

## $n \equiv 7, 8 \pmod{14}$

In this chapter, we will use our own constructions based on edge lengths in  $K_n$  where  $n \equiv 7$  or  $8 \pmod{14}$ . We begin by describing our construction and then later formalize these ideas in the results section.

### 4.1 Construction

The number of vertices in  $K_7$  and  $K_8$  is less than 9, the minimum number of vertices of a seven edge forest. So neither are decomposable by seven edge forests and our base graphs are  $K_{21}$  and  $K_{22}$  for  $n \equiv 7$  and  $8 \pmod{14}$ , respectively. We show these base graphs in the figure below.

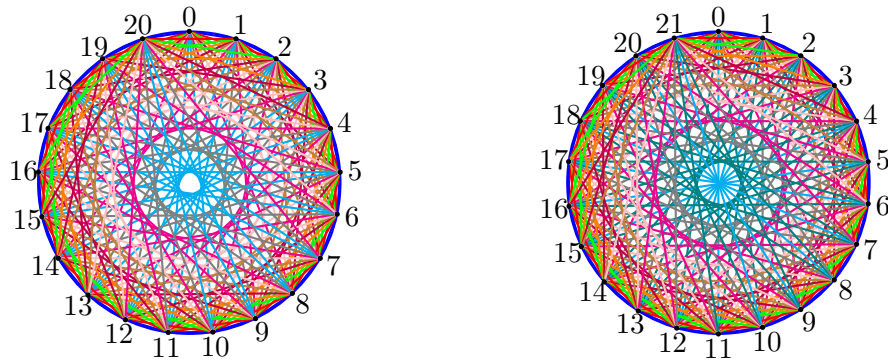


Figure 4.1:  $K_{21}$  (left) and  $K_{22}$  (right) with edges colored by length.

#### 4.1.1 $K_{21}$ and $K_{22}$

We have lost a lot of luxuries present in the  $n \equiv 0, 1 \pmod{14}$  case. There are really two main problems in this case:

- (1) In  $K_{22}$  there are 11 total maximal length 11 edges but 22 of every other length.
- (2)  $K_{21}$  and  $K_{22}$  have lengths  $\{1, \dots, 10\}$  and  $\{1, \dots, 11\}$ , respectively.

(1) is a problem because we want to be able to develop the vertices of our labelings to collect all edges of  $K_{22}$  in the same number of steps. (2) is a problem because this means we cannot just fit 7 distinct lengths on a single labeling and collect all edges when we develop the vertices by 1. Furthermore, since  $K_{22}$  has one more length than  $K_{21}$ , we don't have a single labeling strategy in this project that takes care of both cases where  $n \equiv 7$  and  $8 \pmod{14}$  at once like  $\sigma^{+-}$  did for  $n \equiv 0$  and  $1 \pmod{14}$ . We address these problems in order.

We have a remedy for length 11 edges in  $K_{22}$  which is similar to what we did for  $K_{14}$ . We take the vertices of  $K_{22}$  to be  $\mathbb{Z}_{21} \cup \{\infty\}$  and redefine length of edges and development for vertices in  $K_{22}$ :

$$\ell(uv) = \begin{cases} \min\{|u - v|, 21 - |u - v|\}, & u, v \neq \infty, \\ \infty, & u \text{ or } v = \infty \end{cases} \quad \text{and } v \mapsto \begin{cases} v + 1, & v \in \mathbb{Z}_{n-1}, \\ \infty, & v = \infty. \end{cases}$$

Now, we have 21 edges of lengths 1 through 6 as well as  $\infty$ , since the infinity node will have all 21 edges of length  $\infty$  to nodes in  $\mathbb{Z}_{21}$  incident to it. Now we can theoretically cyclically generate all edges of  $K_{21}$  and  $K_{22}$  in the same number of steps with labelings.

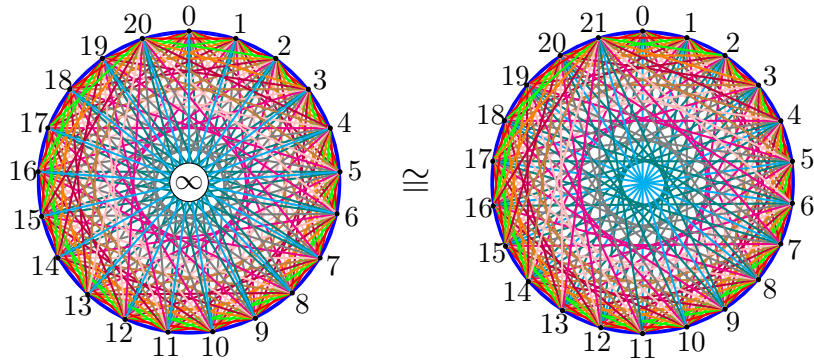


Figure 4.2:  $K_{21} \vee K_1$  is isomorphic to  $K_{22}$  (right)

Let's begin solving (2) by just looking at  $K_{21}$ . If we can generate all 21 distinct edges of lengths 1, 2 and 3 separately, then we can generate the remaining 7 lengths with one labeling. But how can we collect 21 edges of length 1, 2, and 3 in isomorphic copies? By imposing a new edge length on top of the standard length  $\ell$ , we can achieve this. We define  $\ell_7^+$  from  $\mathbb{Z}_{21} \cup \{\infty\}$  to  $\mathbb{Z}_7$  as follows

$$\ell_7^+(uv) = \begin{cases} u + v \bmod 14, & u, v \neq \infty \\ v, & u = \infty \end{cases}$$

Now, every edge has a standard length  $\ell$  and an additive length  $\ell_7^+$  modulo 7. Previously, we partitioned the edges into sets  $E_i$  of edges of length  $i$  for each length  $i \in \{1, \dots, 10\}$  via the standard length function  $\ell$ . Now, within each partite set  $E_i$ , we have further partitioned the edges into sets  $E_{i,j}$  of standard length  $i$  and additive length  $j$  modulo 7. For example: the edge  $(1, 8)$  has length  $\ell((1, 8)) = 7$  and  $\ell_7^+((1, 8)) = 8 + 1 \bmod 7 = 2$ , so  $(1, 8) \in E_{7,2}$ . Here is a bundle of labelings of  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$  that utilizes this new partition.

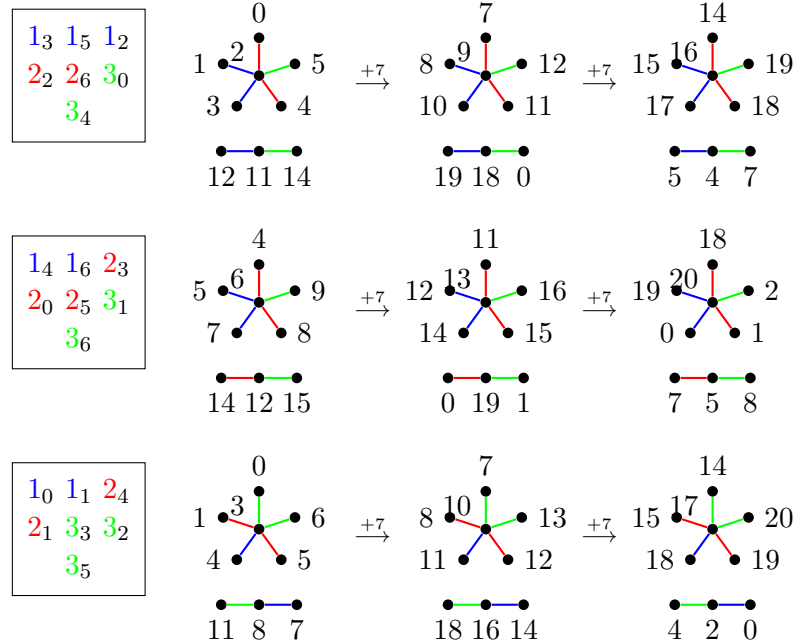


Figure 4.3: Three labelings of  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$  (left column) that generate all edges of lengths 1, 2, and 3 in  $K_{21}$  when developed by 7



Each square in the figure above contains the edge length pairs of the edges in the labeling to its right. Standard lengths  $\ell$  are colored in the same manner as in Figure 4.3 and the new length  $\ell_7^+$  appears in black as the subscript for each of these lengths. Across all three labelings, there is exactly one representative for all 7 distinct equivalence classes modulo  $\ell_7^+$  for each length in  $\{1, 2, 3\}$ . Now, we already know developing the nodes by 7 preserves  $\ell$ , but it also preserves  $\ell_7^+$  since  $u + 7 + v + 7 \bmod 7 = u + v \bmod 7$ . So then developing all labelings by 7 will give us all edges of lengths in  $\{1, 2, 3\}$  across 9 isomorphic copies of  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ .

An equivalent way of understanding development is through *group actions*. Let  $S$  be the set of all labelings in Figure 4.3. The cyclic subgroup  $\langle 7 \rangle = \{0, 7, 14\} \subseteq \mathbb{Z}_{21}$  acts on  $S$ ;  $\langle 7 \rangle \curvearrowright S$  via developing the vertices by 7. Then if we call the labelings in column 1:  $F_1, F_2, F_3$  from top to bottom, the set of labelings in Row 1 is simply  $\text{Orb}_{\langle 7 \rangle}(F_1)$ , Row 2 is  $\text{Orb}_{\langle 7 \rangle}(F_2)$  and Row 3 is  $\text{Orb}_{\langle 7 \rangle}(F_3)$ . We employ the same technique for  $K_{22}$ , we just need four labelings  $F_1, F_2, F_3, F_4$  with lengths  $1, 2, 3, \infty$ .

Now have a technique to collect all edges of lengths  $\{1, 2, 3\}$  and  $\{1, 2, 3, \infty\}$  in  $K_{21}$  and  $K_{22}$ , respectively, in isomorphic copies of graphs. So we just need to collect edges of lengths in  $\{3, \dots, 10\}$ . Well, if we just stretch our  $\sigma^{+-}$ -labelings by 3, we will now have lengths  $\{4, \dots, 10\}$  instead of  $\{1, \dots, 7\}$ . So developing those stretched labelings by 1 will give use the remaining edges of  $K_{21}$  and  $K_{22}$  in isomorphic copies of our forests.

#### 4.1.2 Generalizing to $K_{14t+7}$ and $K_{14t+8}$ where $t > 1$

Recall our labeling constructions for collecting edges of lengths  $\{1, 2, 3\}$  and  $\{1, 2, 3, \infty\}$  in  $K_{21}$  and  $K_{22}$ . As long as we only use short edges, we can actually just generate all edges of these lengths for the entire families with the same set of labelings we used for  $K_{14t+7}$  and  $K_{14t+8}$  where  $t > 1$ . The only thing that changes at each step  $t \mapsto t + 1$  is that more steps of development is needed to collect all the edges, and therefore the number of isomorphic copies.

Equivalently, the orbits of the labelings will simply grow at each step  $t \mapsto t + 1$ . Let  $\langle 7 \rangle_n$  denote the subgroup  $\langle 7 \rangle \subset K_n$ . Then for each forest and its set of labelings  $F_1, F_2, F_3$  used for  $K_{21}$ ,  $\text{Orb}_{\langle 7 \rangle_{21}}(F_i) \subseteq \text{Orb}_{14t+7}(F_i)$  for each  $i = 1, 2, 3$  and  $t > 1$ .

We formalize these constructions in the next section.

## 4.2 Results

We begin with a formal definition for the set of labelings used to generate edges in  $K_{14t+7}$  of lengths in  $\{1, 2, 3\}$  in isomorphic copies of a graph  $G$  on seven edges for  $t \geq 1$ . Then we prove that if a graph  $G$  admits such a labeling and a  $\rho^+$ -labeling, then it decomposes  $K_{14t+7}$  for  $t \geq 1$ .

**Definition 4.2.1.** Let  $G$  be a graph with 7 edges. A (1-2-3)-labeling of  $3G$  is an assignment  $f$  of the integers  $\{0, \dots, 20\}$  to the vertices of  $3G$  such that

- (1)  $f(u) \neq f(v)$  whenever  $u$  and  $v$  belong to the same connected component,

and

- (2)

$$\bigcup_{uv \in E(3G)} \{(f(u) \bmod 7, f(v) \bmod 7)\} = \bigcup_{i=0}^6 \bigcup_{j=1}^3 \{(i, i+j \bmod 7)\}.$$

Notice that the second condition of a (1-2-3)-labeling demands that  $3G$  contains exactly 7 edges of each length in  $\{1, 2, 3\}$ . Additionally, the second condition requires that no two edges of the same length have the same end labels modulo 7. A (1-2-3)-labeling of every forest with 7 edges except  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  is given in Figure A.1. This exceptional graph does not admit such a labeling, and we deal with it in Chapter 5.

**Theorem 4.2.2.** *Let  $G$  be a bipartite graph with 7 edges. If  $3G$  admits a (1-2-3)-labeling and  $G$  admits a  $\rho^+$ -labeling, then  $G$  decomposes  $K_{14k+7}$  for every  $k \geq 1$ .*

*Proof.* Let  $n = 14k + 7$  and notice that  $K_n$  has  $|E(K_n)| = (7k + 3)(14k + 7)$  edges, which can be partitioned into  $14k + 7$  edges of each of the lengths in  $\{1, 2, \dots, 7k + 3\}$ . We will construct the  $G$ -decomposition in two steps. First, we use the 1-2-3-labeling to identify all the edges of lengths 1, 2, and 3 accounting for  $3(2k + 1)$  copies of  $G$ . Then, we use the  $\rho^+$ -labeling to identify edges of the remaining lengths in  $7k(2k + 1)$  copies of  $G$ . In total, the decomposition consists of  $|E(K_n)|/7 = (7k + 3)(2k + 1)$  copies of  $G$ .

Let  $f_1$  be a (1-2-3)-labeling of  $3G$  and identify this graph as a block  $B_0$ . Then develop  $B_0$  by 7 modulo  $n$ . Since the order of the development is  $\frac{n}{7} = 2k + 1$  and there are 7 edges of each of the lengths 1, 2, and 3 in  $B_0$ , we have identified  $3(2k + 1)$  copies

of  $G$  containing all  $14k + 7 = n$  edges of each length 1, 2, and 3. Notice (2) of Definition 4.2.1 ensures no edge has been counted more than once in the development.

Let  $f_2 : V(G) \rightarrow \{0, \dots, 14\}$  be a  $\rho^+$ -labeling of  $G$  with associated vertex partition  $(A, B)$ . For  $i = 1, 2, \dots, k$ , identify blocks  $B_i \cong G$  with vertex labels  $\ell$  such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i - 1), & \text{if } v \in B \end{cases}$$

Notice that the  $i^{\text{th}}$  block contains exactly one edge of each length  $7i - 3, 7i - 2, \dots$ , and  $7i + 3$ . This is because every edge  $ab$  has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i - 1)$$

and  $f_2(b) - f_2(a)$  is a length in  $\{1, \dots, 7\}$ . Developing each block  $B_i$  by 1 yields  $14k + 7$  copies of  $G$  per block and accounts for  $14k + 7$  edges of each of the lengths 4, 5,  $\dots$ , and  $7k + 3$ .

Since we have identified

$$3(2k + 1) + k(14k + 7) = (7k + 3)(2k + 1)$$

edge-disjoint copies of  $G$ , the proof is complete. □

Next we formalize the set of labelings used to generate edges in  $K_{14t+8}$  of lengths in  $\{1, 2, 3, \infty\}$  in isomorphic copies of a graph  $G$  on seven edges for  $t \geq 1$ . Then we prove that if a graph  $G$  admits such a labeling and a  $\rho^+$ -labeling, then it decomposes  $K_{14t+8}$  for  $t \geq 1$ .

**Definition 4.2.3.** Let  $G$  be a graph with 7 edges. A *1-rotational (1-2-3)-labeling* of  $4G$  is an assignment  $f$  of  $\{0, \dots, 20\} \cup \infty$  to the vertices of  $4G$  such that

- (1)  $f(u) \neq f(v)$  whenever  $u$  and  $v$  belong to the same connected component,

and

- (2)

$$\bigcup_{uv \in E(4G)} \{(f(u) \bmod 7, f(v) \bmod 7)\} = \bigcup_{i=0}^6 \bigcup_{j=1}^3 \{(i, i + j \bmod 7), (i, \infty)\}.$$

Notice that the second condition of a 1-rotational (1-2-3)-labeling demands that  $4G$  contains exactly 7 edges of each length in  $\{1, 2, 3, \infty\}$ . Additionally, the second condition requires that no two edges of the same length have the same end labels modulo 7. A 1-rotational (1-2-3)-labeling of every forest with 7 edges except  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  is given in Figure A.2. This exceptional graph does not admit this labeling either, and we deal with it for this case as well in Chapter 5.

**Theorem 4.2.4.** *Let  $G$  be a bipartite graph with 7 edges. If  $4G$  admits a 1-rotational (1-2-3)-labeling and  $G$  admits a  $\rho^+$ -labeling, then  $G$  decomposes  $K_{14k+8}$  for every  $k \geq 1$ .*

*Proof.* Let  $n = 14k + 8$  and notice that  $K_n$  has  $|E(K_n)| = (7k + 4)(14k + 7)$  edges, which can be partitioned into  $14k + 7$  edges of each of the lengths in  $\{1, 2, \dots, 7k + 3, \infty\}$ . We will construct the  $G$ -decomposition in two steps. First, we use the 1-rotational (1-2-3)-labeling to identify all the edges of lengths 1, 2, 3, and  $\infty$  accounting for  $4(2k + 1)$  copies of  $G$ . Then, we use the  $\rho^+$ -labeling to identify edges of the remaining lengths in  $7k(2k + 1)$  copies of  $G$ . In total, the decomposition consists of  $|E(K_n)|/7 = (7k + 4)(2k + 1)$  copies of  $G$ . Let  $f_1$  be a 1-rotational (1-2-3)-labeling of  $4G$  and identify this graph as a block  $B_0$ . Then develop  $B_0$  by 7 modulo  $n - 1$ . Since the order of the development is  $\frac{n-1}{7} = 2k + 1$  and there are 7 edges of each of the lengths 1, 2, 3 and  $\infty$  in  $B_0$ , we have identified  $4(2k + 1)$  copies of  $G$  containing all  $14k + 7 = n - 1$  edges of each length 1, 2, 3 and  $\infty$ . Notice (2) of Definition 4.2.3 ensures no edge has been counted more than once in the development.

Let  $f_2 : V(G) \rightarrow \{0, \dots, 14\}$  be a  $\rho^+$ -labeling of  $G$  with associated vertex partition  $(A, B)$ . For  $i = 1, 2, \dots, k$ , identify blocks  $B_i \cong G$  with vertex labels  $\ell$  such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i - 1), & \text{if } v \in B \end{cases}$$

Notice that the  $i^{\text{th}}$  block contains exactly one edge of each length  $7i - 3, 7i - 2, \dots$ , and  $7i + 3$ . This is because every edge  $ab$  has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i - 1)$$

and  $f_2(b) - f_2(a)$  is a length in  $\{1, \dots, 7\}$ . Developing each block  $B_i$  by 1 yields  $14k + 7$  copies of  $G$  per block and accounts for  $14k + 7$  edges of each of the lengths 4, 5,  $\dots$ , and  $7k + 3$ .

Since we have identified

$$4(2k + 1) + k(14k + 7) = (7k + 4)(2k + 1)$$

edge-disjoint copies of  $G$ , the proof is complete.  $\square$

We are now able to state the main theorem of this chapter.

**Theorem 4.2.5.** *Let  $F$  be a forest with 7 edges and  $F \not\cong \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ . There exists an  $F$ -decomposition of  $K_n$  whenever  $n \equiv 7$  or  $8 \pmod{14}$  and  $n \geq 21$ .*

*Proof.* If  $n \equiv 7 \pmod{14}$ , a (1-2-3)-labeling of  $3F$  can be found in Figure A.1. On the other hand, if  $n \equiv 8 \pmod{14}$ , then a 1-rotational (1-2-3)-labeling of  $4F$  can be found in Figure A.2. In either case, a  $\rho^+$ -labeling of  $F$  can be found in Figure 3.1 (recall that a  $\sigma^{+-}$ -labeling is a  $\rho^+$ -labeling). The result now follows from Theorems 4.2.2 and 4.2.4.  $\square$

We illustrate how to interpret the tables of labelings and realize the constructions from the last two chapters by building an  $F$ -decomposition of  $K_{35}$  and  $K_{36}$  for the forest graph  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$ .

**Example 4.2.6.** *Here are excerpts from tables 3.1, A.1, and A.2 for  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$*

Labeling Type	Labelings
$\sigma^{+-}$	$(4, 1, 8, 5, 6, 7) \sqcup (10, 9, 11)$
(1-2-3)	$(0, 2, 1, 3, 4, 5) \sqcup (12, 11, 14)$ $(4, 6, 8, 9, 5, 7) \sqcup (14, 12, 15)$ $(0, 3, 1, 4, 5, 6) \sqcup (11, 8, 7)$
1-rotational (1-2-3)	$(1, 2, 0, 3, 4, 5) \sqcup (11, 8, \infty)$ $(2, \infty, 3, 4, 5, 6) \sqcup (12, 13, 15)$ $(6, 7, 8, 4, 5, \infty) \sqcup (11, 12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (9, 6, 4)$

Figure 4.4: Labelings for  $\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$

The  $\rho^+$  labelings obtained by stretching the  $\sigma^{+-}$  labeling are bottommost labelings in the following generating presentations and are developed by 1.

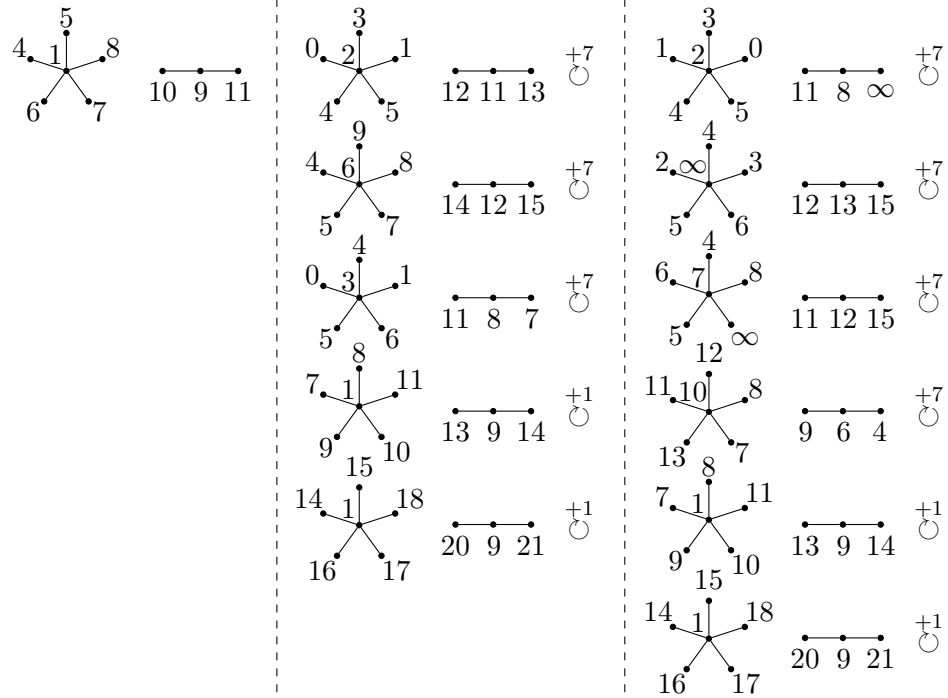


Figure 4.5: A  $\sigma^{+-}$ -labeling of  $F \cong \mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$  (left) and generating presentations for the  $F$ -decomposition of  $K_n$  where  $n = 35$  (middle) and  $n = 36$  (right)

We have now proven that every seven edge forest except  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_n$  if and only if  $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$ . As stated earlier, we deal with the this exceptional forest  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  in the next chapter.

## Chapter 5

### $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$

To begin we construct  $K_{14t+7}$  and  $K_{14t+8}$  using joined copies of  $K_{22}$ ,  $K_{21}$ , and  $K_{14}$ . Let  $t$  be a positive integer. Now join  $t - 1$  copies of  $K_{14}$  with each other and a lone copy of  $K_{21}$ . The resulting graph is  $K_{14(t-1)+21} \cong K_{14t+7}$ . Similarly,  $K_{14t+8}$  can be constructed by joining  $t - 1$  copies of  $K_{14}$  with each other and 1 copy of  $K_{22}$ .

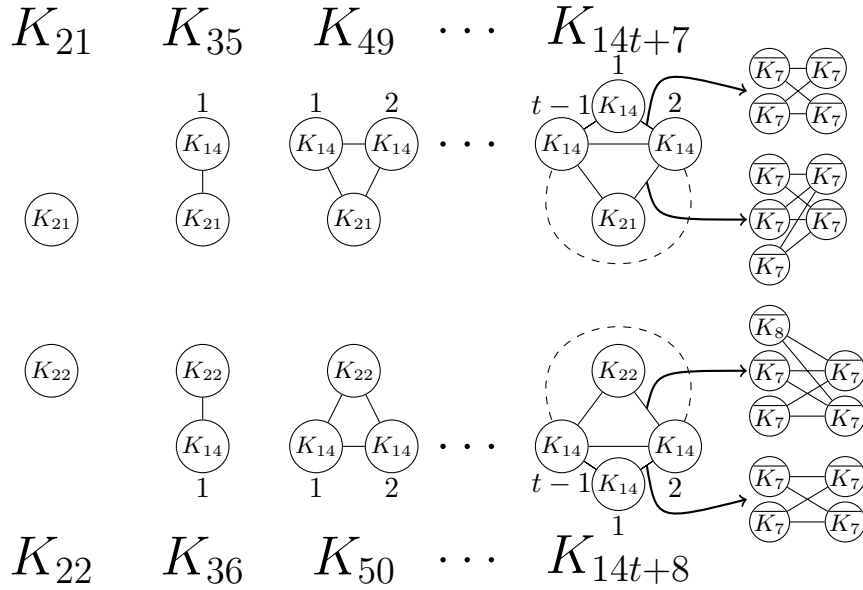


Figure 5.1: A new construction for  $K_{14t+7}$  and  $K_{14t+8}$

Equivalently, We can view  $K_{14t+7}$  as  $K_t$  whose 'vertices' are  $K_{14}$  except one which is  $K_{21}$ , and whose edges are the join between them. We will refer to these 'vertices' as nodes. Similarly, we can view  $K_{14t+8}$  as  $K_t$  whose nodes are  $K_{14}$  except one which is  $K_{22}$ , and whose edges are the join between them. Notice in Figure 5.1 that all edges in the  $K_t$  constructions of these families are then the edges of  $K_{14,14}$ ,  $K_{21,14}$ , and  $K_{22,14}$ .

We show that  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_n$  for  $n \equiv 7$  or  $8 \pmod{14}$  where  $n \geq 21$  by proving that  $K_{22}$ ,  $K_{21}$ ,  $K_{14}$ ,  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$  can each be decomposed by  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  since these six graphs make up the nodes and edges of the  $K_t$  representations of  $K_{14t+7}$  and  $K_{14t+8}$  where  $t \geq 1$ .

We begin with  $K_{21}$  and  $K_{22}$ . The proof of the next theorem was obtained by manipulating a  $K_{1,7}$ -decomposition of  $K_{21}$  by Cain in [?]. We 'plucked' edges off of every 7-edge star in the decomposition, put them to the side, and then sent them to 6-edge stars that they were vertex disjoint from. This gave us a  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{21}$  which can be found in Figure A.3. Then, we just added three 6-edge stars centered at  $\infty$ , and three distinct paths from  $\infty$  to the remaining three neighbors of  $\infty$  in  $K_{22}$ . We once again put all the lone paths aside, and sent them to 6-edge stars that they were vertex disjoint from them. This gave us a  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{22}$  which can be found in Figure A.4.

**Theorem 5.0.1.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{21}$  and  $K_{22}$ .

*Proof.* Figures 8 and 9 give  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decompositions of  $K_{21}$  and  $K_{22}$ , respectively.  $\square$

We didn't need to develop anything formally to do this as we only performed this process once for the given 7-edge star decomposition of  $K_{21}$ . Initially, we were interested in proving a more stronger statement resulting from Cain's work, but found it difficult. Investigating implications of Cain's work [?] from 1974 is something we are interested in for future work.

Next, we address  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$ .



**Theorem 5.0.2.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{n,7}$  for all  $n \geq 2$ .

*Proof.* Consider  $K_{n,7}$  where  $n \geq 2$ . Take the partite set of  $n$  vertices to be  $\mathbb{Z}_n$  and color them white. Similarly, take the partite set of 7 vertices to be  $K_7$  and color them black. Naturally we refer to *white-black* vertices  $uv$  in  $K_{n,7}$  via  $(u, v) \in \mathbb{Z}_n \times \mathbb{Z}_7$  and vice versa. Finally, let  $E_i = \{(i, 0)\} \sqcup (\{i+1\} \times \{1, \dots, 6\})$  and  $G_i \subset K_{n,7}$  be the subgraph induced by  $E_i$  for each  $i \in \mathbb{Z}_n$ . Note that  $G_i \cong \mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  for all  $i \in \mathbb{Z}_n$ .

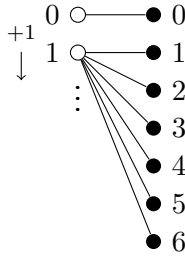


Figure 5.2:  $G_0$  in a generating presentation of the  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{n,7}$ .

Notice that  $E_i \cap E_j = \emptyset$  if  $i \neq j$ , so by definition all distinct  $G_i$ 's are pairwise edge disjoint. Lastly,

$$\bigcup_{i \in \mathbb{Z}_n} E_i = [\bigcup_{i \in \mathbb{Z}_n} \{(i, 0)\}] \sqcup [\bigcup_{i \in \mathbb{Z}_n} (\{i+1\} \times \{1, \dots, 6\})] = [\mathbb{Z}_n \times \{0\}] \sqcup [\mathbb{Z}_n \times \{1, \dots, 6\}] = \mathbb{Z}_n \times \mathbb{Z}_7$$

So  $G_0 \sqcup \dots \sqcup G_{n-1} = K_{n,7}$  and  $\{G_i \mid i \in \mathbb{Z}_n\}$  is a  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{n,7}$ . Furthermore, it is generated by developing the white nodes of  $G_0$  by 1.  $\square$

**Corollary 5.0.3.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$ .

*Proof.*  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{7,7}$  and  $K_{8,7}$  by Theorem 5.0.2.  $K_{14,14}$  can be expressed as the edge-disjoint union of four copies of  $K_{7,7}$ ,  $K_{21,14}$  can be expressed as the edge-disjoint union of six copies of  $K_{7,7}$ , and  $K_{22,14}$  can be expressed as the edge-disjoint union of two copies of  $K_{8,7}$  and four copies of  $K_{7,7}$ . Therefore,  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes them all.  $\square$

Recall that we proved  $K_{14}$  and  $K_{15}$  are  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposable in Chapter 3. So we are now ready to put everything together to state the main theorem of this Chapter, completing the main result of this thesis.

**Theorem 5.0.4.**  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{14t+7}$  and  $K_{14t+8}$  where  $t$  is a positive integer.

*Proof.*  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes  $K_{14}$  by Theorem 3.2.7,  $K_{22,14}$ ,  $K_{21,14}$ , and  $K_{14,14}$  by Corollary 5.0.3, and lastly  $K_{22}, K_{21}$  by Theorem 5.0.1.

Therefore,  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes the join of  $(t - 1)$  copies of  $K_{14}$  with each other and 1 copy of  $K_{21}$ , which is isomorphic to  $K_{14t+7}$ . Similarly  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$  decomposes the join of  $(t - 1)$  copies of  $K_{14}$  with each other and 1 copy of  $K_{22}$  which is isomorphic to  $K_{14t+8}$ .  $\square$

We now present some additional results.

## Chapter 6

# Programming

By the end of my research I found around 600 labelings of various forests, essentially by hand. I realized 200 labelings in that just due to the sheer number of labelings I was doing I was bound to (1) have typos (2) compute incorrectly (3) violate some constraint of the labeling I was doing just like probabilistically.

So I started all this programming because I just wanted some sort of local program that could display my labelings, so I could check with some level of certainty that they were isomorphic to the forest I was working with. However, everything I found displayed graphs but didn't allow for dragging nodes and interacting with graphs. So I decided to make my own program that did exactly what I wanted.

In total there are 7 programs I will add links for in this Chapter. However, some of these projects have

In order, these are the features I added to *graph\_visualization.py*:

- (1) Outputs a list of NetworkX graphs together on one page, starting from top to bottom.
- (2) Reduces vertices modulo  $n$  and computes the standard edge length for each edge modulo  $n$ , and has the subscript as the additive edge length  $\ell 7^+$ .
- (3) Uses longest path search algorithm and by default displays the longest path of graph in the center row of a grid of coordinates, then displays nodes coming off of that row.

- (4) Has a tab on the left that displays all standard edge lengths  $\ell$  and a chart for the subscript labels of the labelings in order. The window with the tab open looks similar to Figure 4.3 except it doesn't have colored edges.
- (5) Allows user to save displayed graphs as a tikz graph in a standalone L<sup>A</sup>T<sub>E</sub>Xfile to a specified path

## Chapter 7

# Conclusion and Discussion

# References

- [1] P. Cain. Decomposition of complete graphs into stars. *Bull. Austral. Math. Soc.*, 10:23–30, 1974.
- [2] S. I. El-Zanati and C. Vanden Eynden. On the cyclic decomposition of complete graphs into bipartite graphs. *Australas. J. Combin.*, 24:209–219, 2001.
- [3] S. I. El-Zanati and C. Vanden Eynden. On rosa-type labelings and cyclic graph decompositions. *Math. Slovaca*, 59:1–18, 2009.
- [4] B. Freyberg and R. Peters. Decomposition of complete graphs into forests with six edges. *Discuss. Math. Graph Theory*, 34, 2024. in press.
- [5] B. Freyberg and N. Tran. Decomposition of complete graphs into bipartite unicyclic graphs with eight edges. *J. Combin. Math. Combin. Comput.*, 114:133–142, 2020.
- [6] D. Froncek and M. Kubesa. Decomposition of complete graphs into connected unicyclic bipartite graphs with seven edges. *Bull. Inst. Combin. Appl.*, 93:52–80, 2021.
- [7] A. Rosa. On certain valuations of the vertices of a graph. In *Theory of Graphs (Intl. Symp. Rome 1966)*, pages 349–355. Gordon and Breach, Dunod, Paris, 1967.
- [8] T. P. Kirkman. On a problem in combinations. *The Lady’s and Gentleman’s Diary*, pages 48–50, 1850.
- [9] T. P. Kirkman. Answer to query vi. *The Lady’s and Gentleman’s Diary*, page 48, 1851.

# Appendix A

## Labelings

### A.1 (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 1, 2, 4, 6, 9, 12) \sqcup (13, 14)$ $(3, 4, 7, 9, 10, 13, 15) \sqcup (8, 5)$ $(8, 11, 12, 10, 7, 5, 6) \sqcup (1, 3)$ $(0, 4, 9, 15, 8, 16, 7) \sqcup (1, 11)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(12, 9, 6, 4, 2, 1, 7) \sqcup (14, 15)$ $(15, 13, 10, 9, 7, 4, 11) \sqcup (8, 5)$ $(8, 11, 12, 10, 7, 5, 13) \sqcup (1, 3)$ $(16, 8, 15, 9, 4, 0, 6) \sqcup (1, 11)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(0, 1, 2, 4, 6, 9, 3) \sqcup (16, 19)$ $(15, 13, 10, 9, 7, 4, 14) \sqcup (17, 18)$ $(6, 5, 7, 10, 12, 11, 8) \sqcup (18, 15)$ $(7, 16, 8, 15, 9, 4, 12) \sqcup (1, 11)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(8, 6, 4, 2, 1, 9, 7) \sqcup (14, 15)$ $(8, 10, 9, 7, 4, 11, 13) \sqcup (12, 15)$ $(9, 12, 10, 7, 5, 11, 13) \sqcup (1, 4)$ $(7, 15, 9, 4, 0, 8, 6) \sqcup (1, 11)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12, 8, 7) \sqcup (11, 14)$ $(0, 2, 3, 6, 5, 1, 4) \sqcup (8, 7)$ $(0, 3, 5, 4, 1, 8, 7) \sqcup (16, 15)$ $(4, 9, 15, 8, 12, 6, 7) \sqcup (1, 11)$
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 6, 8, 5, 9) \sqcup (12, 15)$ $(4, 7, 9, 10, 11, 8, 13) \sqcup (1, 3)$ $(5, 7, 10, 12, 11, 6, 13) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12, 6) \sqcup (1, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(8, 6, 4, 2, 5, 9, 7) \sqcup (12, 14)$ $(1, 3, 2, 0, 5, 4, 6) \sqcup (10, 12)$ $(9, 8, 7, 10, 4, 11, 5) \sqcup (12, 13)$ $(7, 15, 9, 4, 13, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(7, 6, 4, 2, 8, 9, 5) \sqcup (12, 14)$ $(2, 3, 4, 7, 0, 5, 6) \sqcup (9, 12)$ $(7, 8, 5, 4, 9, 10, 11) \sqcup (0, 2)$ $(6, 15, 9, 4, 8, 11, 7) \sqcup (2, 12)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 8, 7, 9, 12) \sqcup (13, 14)$ $(0, 2, 3, 4, 7, 6, 5) \sqcup (8, 10)$ $(0, 3, 5, 8, 9, 4, 1) \sqcup (12, 14)$ $(4, 9, 15, 8, 12, 7, 16) \sqcup (1, 11)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12, 1, 8) \sqcup (14, 15)$ $(5, 6, 3, 2, 0, 7, 4) \sqcup (8, 9)$ $(0, 3, 5, 4, 7, 1, 8) \sqcup (12, 14)$ $(4, 9, 15, 8, 12, 18, 7) \sqcup (1, 11)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 6, 9, 12) \sqcup (13, 14, 15)$ $(3, 4, 7, 9, 10, 13) \sqcup (5, 8, 6)$ $(11, 12, 10, 7, 5, 6) \sqcup (3, 1, 4)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11, 2)$

Table A.1: (1-2-3)-labelings



Forest	Labeling
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14, 15)$ $(13, 10, 9, 7, 4, 11) \sqcup (5, 8, 6)$ $(11, 12, 10, 7, 5, 13) \sqcup (3, 1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(0, 1, 2, 4, 6, 5) \sqcup (16, 13, 14)$ $(8, 6, 3, 2, 0, 4) \sqcup (14, 12, 15)$ $(7, 4, 5, 3, 0, 6) \sqcup (10, 8, 11)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(1, 2, 5, 4, 6, 7) \sqcup (16, 14, 13)$ $(8, 6, 9, 3, 2, 4) \sqcup (14, 12, 15)$ $(4, 5, 6, 3, 0, 1) \sqcup (11, 8, 7)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(0, 2, 4, 7, 1, 5) \sqcup (12, 11, 13)$ $(7, 6, 3, 2, 8, 9) \sqcup (14, 12, 15)$ $(4, 3, 5, 6, 0, 1) \sqcup (11, 8, 7)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(0, 2, 1, 3, 4, 5) \sqcup (12, 11, 14)$ $(4, 6, 8, 9, 5, 7) \sqcup (14, 12, 15)$ $(0, 3, 1, 4, 5, 6) \sqcup (11, 8, 7)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(2, 4, 6, 9, 12) \sqcup (16, 15, 14, 13)$ $(3, 4, 7, 9, 10) \sqcup (11, 12, 15, 13)$ $(12, 10, 7, 5, 6) \sqcup (18, 15, 17, 20)$ $(4, 9, 15, 8, 16) \sqcup (2, 11, 1, 5)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(12, 9, 6, 4, 11) \sqcup (17, 16, 15, 14)$ $(9, 7, 4, 3, 6) \sqcup (11, 12, 15, 13)$ $(6, 5, 7, 10, 3) \sqcup (18, 15, 17, 20)$ $(16, 8, 15, 9, 12) \sqcup (2, 11, 1, 6)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(4, 6, 9, 11, 8) \sqcup (16, 15, 18, 14)$ $(9, 7, 4, 3, 6) \sqcup (16, 17, 20, 15)$ $(6, 5, 7, 10, 3) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 6, 7)$ $(14, 17, 16, 20, 15) \sqcup (9, 7, 4, 3)$ $(9, 12, 10, 11, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(7, 6, 9, 11, 8) \sqcup (16, 15, 13, 14)$ $(9, 7, 4, 3, 5) \sqcup (16, 17, 20, 15)$ $(4, 6, 5, 7, 10) \sqcup (9, 12, 11, 15)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(13, 15, 16, 18, 14) \sqcup (11, 9, 12, 6)$ $(18, 17, 16, 20, 15) \sqcup (9, 7, 10, 4)$ $(10, 12, 11, 14, 15) \sqcup (4, 6, 5, 7)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 4, 6, 9, 12) \sqcup (13, 14) \sqcup (8, 7)$ $(3, 4, 7, 9, 10, 13) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 6) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 4, 6, 9, 5) \sqcup (13, 14) \sqcup (8, 7)$ $(13, 10, 9, 7, 4, 11) \sqcup (8, 6) \sqcup (12, 15)$ $(11, 12, 10, 7, 5, 13) \sqcup (1, 4) \sqcup (17, 15)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(0, 1, 2, 4, 7, 5) \sqcup (9, 6) \sqcup (8, 10)$ $(8, 6, 3, 2, 0, 4) \sqcup (5, 7) \sqcup (12, 13)$ $(6, 4, 5, 3, 0, 8) \sqcup (13, 14) \sqcup (18, 15)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11) \sqcup (5, 14)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 5, 4, 6, 7) \sqcup (13, 14) \sqcup (12, 15)$ $(8, 6, 9, 3, 2, 4) \sqcup (12, 14) \sqcup (18, 15)$ $(4, 5, 6, 3, 0, 1) \sqcup (8, 7) \sqcup (16, 14)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(0, 2, 4, 7, 1, 5) \sqcup (11, 13) \sqcup (12, 15)$ $(7, 6, 3, 2, 8, 9) \sqcup (11, 12) \sqcup (1, 4)$ $(4, 3, 5, 6, 0, 1) \sqcup (8, 7) \sqcup (12, 14)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(0, 2, 1, 3, 4, 5) \sqcup (12, 14) \sqcup (18, 19)$ $(4, 6, 8, 9, 5, 7) \sqcup (12, 15) \sqcup (11, 14)$ $(0, 3, 1, 4, 5, 6) \sqcup (8, 11) \sqcup (14, 15)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(2, 4, 6, 9, 12) \sqcup (13, 14, 15) \sqcup (18, 19)$ $(3, 4, 7, 9, 10) \sqcup (12, 15, 13) \sqcup (1, 2)$ $(12, 10, 7, 5, 6) \sqcup (20, 17, 15) \sqcup (1, 4)$ $(4, 9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(12, 9, 6, 4, 11) \sqcup (17, 16, 15) \sqcup (0, 1)$ $(9, 7, 4, 3, 6) \sqcup (12, 15, 13) \sqcup (18, 19)$ $(6, 5, 7, 10, 3) \sqcup (20, 17, 15) \sqcup (1, 4)$ $(16, 8, 15, 9, 12) \sqcup (1, 11, 2) \sqcup (0, 5)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(13, 15, 16, 18, 14) \sqcup (9, 6, 7) \sqcup (2, 4)$ $(14, 17, 16, 20, 15) \sqcup (3, 4, 7) \sqcup (11, 13)$ $(9, 12, 10, 11, 15) \sqcup (6, 5, 7) \sqcup (0, 2)$ $(5, 1, 10, 11, 6) \sqcup (8, 15, 9) \sqcup (4, 12)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(4, 6, 9, 12) \sqcup (16, 15, 14, 13) \sqcup (19, 20)$ $(9, 7, 4, 3) \sqcup (11, 12, 15, 13) \sqcup (16, 17)$ $(12, 10, 7, 5) \sqcup (18, 15, 17, 20) \sqcup (9, 11)$ $(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, 6, 7) \sqcup (16, 15, 13, 14) \sqcup (1, 4)$ $(5, 3, 4, 7) \sqcup (16, 17, 20, 15) \sqcup (0, 2)$ $(4, 6, 5, 7) \sqcup (9, 12, 11, 15) \sqcup (0, 3)$ $(16, 8, 15, 9) \sqcup (10, 1, 11, 6) \sqcup (0, 4)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(18, 15, 13, 14) \sqcup (11, 9, 12, 6) \sqcup (1, 2)$ $(18, 17, 20, 15) \sqcup (9, 7, 10, 4) \sqcup (2, 3)$ $(11, 12, 14, 15) \sqcup (4, 6, 5, 7) \sqcup (17, 19)$ $(11, 1, 5, 6) \sqcup (16, 8, 14, 15) \sqcup (0, 9)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(16, 15, 14, 13) \sqcup (0, 3, 5) \sqcup (12, 9, 6)$ $(11, 12, 15, 13) \sqcup (10, 9, 7) \sqcup (16, 18, 20)$ $(18, 15, 17, 20) \sqcup (10, 11, 14) \sqcup (6, 5, 7)$ $(2, 12, 3, 11) \sqcup (8, 1, 7) \sqcup (4, 0, 5)$
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(11, 9, 12, 6) \sqcup (18, 15, 13) \sqcup (0, 1, 2)$ $(9, 7, 10, 4) \sqcup (18, 17, 20) \sqcup (1, 3, 2)$ $(11, 12, 14, 15) \sqcup (4, 6, 7) \sqcup (17, 19, 20)$ $(16, 8, 14, 15) \sqcup (11, 1, 6) \sqcup (9, 0, 4)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(8, 6, 9, 11) \sqcup (0, 1, 2) \sqcup (16, 19) \sqcup (18, 15)$ $(8, 10, 7, 9) \sqcup (18, 17, 20) \sqcup (11, 14) \sqcup (2, 3)$ $(13, 11, 12, 14) \sqcup (17, 19, 20) \sqcup (6, 7) \sqcup (8, 5)$ $(0, 5, 1, 7) \sqcup (3, 10, 2) \sqcup (4, 13) \sqcup (16, 6)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(11, 9, 12, 6) \sqcup (0, 1, 2) \sqcup (18, 15) \sqcup (13, 14)$ $(9, 7, 10, 4) \sqcup (18, 17, 20) \sqcup (11, 13) \sqcup (2, 3)$ $(11, 12, 14, 15) \sqcup (17, 19, 20) \sqcup (8, 6) \sqcup (1, 3)$ $(4, 0, 5, 6) \sqcup (8, 1, 9) \sqcup (3, 12) \sqcup (17, 7)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(2, 4, 6, 9, 12) \sqcup (13, 14) \sqcup (18, 19) \sqcup (0, 1)$ $(3, 4, 7, 9, 10) \sqcup (13, 15) \sqcup (1, 2) \sqcup (8, 5)$ $(6, 5, 7, 10, 12) \sqcup (17, 20) \sqcup (8, 11) \sqcup (1, 3)$ $(4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12) \sqcup (2, 6)$

Table A.1: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(11, 9, 6, 4, 12) \sqcup (16, 15) \sqcup (8, 10) \sqcup (2, 3)$ $(6, 7, 4, 3, 9) \sqcup (13, 15) \sqcup (18, 19) \sqcup (8, 5)$ $(3, 5, 7, 10, 6) \sqcup (17, 20) \sqcup (8, 11) \sqcup (0, 1)$ $(12, 8, 15, 9, 16) \sqcup (2, 11) \sqcup (0, 5) \sqcup (3, 13)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(13, 15, 16, 18, 14) \sqcup (9, 6) \sqcup (2, 4) \sqcup (5, 7)$ $(14, 17, 16, 20, 15) \sqcup (4, 7) \sqcup (11, 13) \sqcup (5, 6)$ $(9, 12, 10, 11, 15) \sqcup (6, 7) \sqcup (0, 2) \sqcup (3, 4)$ $(5, 1, 10, 11, 6) \sqcup (9, 15) \sqcup (4, 12) \sqcup (0, 7)$
$3\mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(18, 15, 13) \sqcup (11, 9, 6) \sqcup (0, 1, 2) \sqcup (16, 19)$ $(18, 17, 20) \sqcup (9, 7, 10) \sqcup (1, 3, 2) \sqcup (11, 14)$ $(11, 12, 14) \sqcup (4, 6, 7) \sqcup (17, 19, 20) \sqcup (8, 5)$ $(11, 1, 6) \sqcup (16, 8, 14) \sqcup (9, 0, 4) \sqcup (10, 3)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(9, 6, 4, 2) \sqcup (13, 14) \sqcup (18, 19) \sqcup (0, 1) \sqcup (10, 12)$ $(9, 7, 4, 3) \sqcup (13, 15) \sqcup (1, 2) \sqcup (8, 5) \sqcup (16, 17)$ $(10, 7, 5, 6) \sqcup (17, 20) \sqcup (8, 11) \sqcup (1, 3) \sqcup (9, 12)$ $(9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12) \sqcup (2, 6) \sqcup (0, 5)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(16, 15, 18, 13) \sqcup (9, 6) \sqcup (2, 4) \sqcup (5, 7) \sqcup (0, 1)$ $(16, 17, 20, 14) \sqcup (4, 7) \sqcup (11, 13) \sqcup (5, 6) \sqcup (1, 3)$ $(9, 12, 10, 11) \sqcup (6, 7) \sqcup (0, 2) \sqcup (3, 4) \sqcup (8, 5)$ $(10, 1, 11, 5) \sqcup (9, 15) \sqcup (4, 12) \sqcup (0, 7) \sqcup (8, 3)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(11, 9, 6) \sqcup (0, 1, 2) \sqcup (18, 15) \sqcup (16, 19) \sqcup (17, 20)$ $(9, 7, 10) \sqcup (1, 3, 2) \sqcup (17, 18) \sqcup (11, 14) \sqcup (8, 5)$ $(11, 12, 14) \sqcup (4, 6, 7) \sqcup (19, 20) \sqcup (13, 15) \sqcup (3, 5)$ $(11, 1, 6) \sqcup (16, 8, 14) \sqcup (0, 9) \sqcup (10, 3) \sqcup (17, 13)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(0, 1, 2) \sqcup (18, 15) \sqcup (9, 11) \sqcup (16, 19) \sqcup (5, 6) \sqcup (10, 7)$ $(1, 3, 2) \sqcup (17, 18) \sqcup (9, 7) \sqcup (11, 14) \sqcup (8, 5) \sqcup (16, 13)$ $(4, 6, 7) \sqcup (12, 14) \sqcup (3, 5) \sqcup (13, 15) \sqcup (17, 20) \sqcup (18, 19)$ $(16, 8, 14) \sqcup (1, 11) \sqcup (0, 9) \sqcup (10, 3) \sqcup (17, 13) \sqcup (2, 7)$

Table A.1: (1-2-3)-labelings

## A.2 1-rotational (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^1 \sqcup \mathbf{T}_2^1$	$(0, 1, \infty, 2, 4, 5, 3) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4, 1) \sqcup (10, 11)$ $(5, 7, \infty, 3, 6, 9, 10) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 6, 5) \sqcup (16, 15)$ $(0, 4, 9, 15, 8, 16, 7) \sqcup (1, 11)$
$\mathbf{T}_7^3 \sqcup \mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 8, 1) \sqcup (12, 15)$ $(4, 6, \infty, 5, 2, 0, 18) \sqcup (10, 11)$ $(10, 9, 6, 3, \infty, 0, 7) \sqcup (12, 14)$ $(5, 6, 8, 10, 7, 4, 9) \sqcup (0, 1)$ $(16, 8, 15, 9, 4, 0, 6) \sqcup (1, 11)$
$\mathbf{T}_7^2 \sqcup \mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 1, 6) \sqcup (9, 10)$ $(0, 2, 5, \infty, 6, 4, 1) \sqcup (10, 11)$ $(5, 7, \infty, 3, 6, 9, 8) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 6, 1) \sqcup (12, 15)$ $(7, 16, 8, 15, 9, 4, 12) \sqcup (1, 11)$
$\mathbf{T}_7^4 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 5, 8, 0, \infty) \sqcup (11, 13)$ $(4, \infty, 5, 2, 3, 8, 6) \sqcup (16, 13)$ $(6, 7, \infty, 10, 13, 8, 5) \sqcup (19, 20)$ $(11, 10, 7, 4, 1, 8, 12) \sqcup (13, 15)$ $(7, 15, 9, 4, 0, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^5 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 3, 6, 0, 1) \sqcup (9, \infty)$ $(2, 5, \infty, 6, 4, 8, 11) \sqcup (16, 13)$ $(10, \infty, 7, 8, 11, 5, 6) \sqcup (12, 13)$ $(4, 7, 10, 8, 5, 11, 12) \sqcup (13, 15)$ $(4, 9, 15, 8, 12, 6, 7) \sqcup (1, 11)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_7^8 \sqcup \mathbf{T}_2^1$	$(8, 5, 4, 2, 0, 6, \infty) \sqcup (11, 13)$ $(3, 2, 5, \infty, 8, 1, 6) \sqcup (16, 13)$ $(5, 7, \infty, 3, 4, 8, 6) \sqcup (13, 14)$ $(\infty, 4, 7, 10, 8, 1, 12) \sqcup (13, 15)$ $(0, 4, 9, 15, 8, 12, 6) \sqcup (1, 11)$
$\mathbf{T}_7^9 \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 5, 7, 0, 3) \sqcup (8, 11)$ $(11, \infty, 6, 4, 5, 8, 12) \sqcup (10, 13)$ $(6, 7, \infty, 10, 2, 8, 5) \sqcup (9, 12)$ $(11, 10, 8, 5, 6, 12, 7) \sqcup (16, 13)$ $(7, 15, 9, 4, 13, 8, 6) \sqcup (1, 11)$
$\mathbf{T}_7^{10} \sqcup \mathbf{T}_2^1$	$(1, 2, 4, 6, 0, 3, 5) \sqcup (8, 11)$ $(11, \infty, 6, 5, 8, 2, 12) \sqcup (13, 15)$ $(6, 7, \infty, 10, 8, 4, 5) \sqcup (11, 12)$ $(11, 10, 8, 5, 12, 13, 7) \sqcup (9, 6)$ $(6, 15, 9, 4, 8, 11, 7) \sqcup (2, 12)$
$\mathbf{T}_7^6 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 0, 1, 3, 6) \sqcup (9, \infty)$ $(4, 6, \infty, 1, 2, 12, 13) \sqcup (8, 11)$ $(10, \infty, 7, 5, 3, 6, 9) \sqcup (13, 15)$ $(5, 8, 10, 11, \infty, 7, 4) \sqcup (9, 12)$ $(4, 9, 15, 8, 12, 7, 16) \sqcup (1, 11)$
$\mathbf{T}_7^7 \sqcup \mathbf{T}_2^1$	$(5, 4, 2, 3, 6, \infty, 0) \sqcup (8, 7)$ $(13, 12, \infty, 6, 4, 10, 1) \sqcup (8, 11)$ $(10, \infty, 7, 6, 9, 2, 5) \sqcup (13, 15)$ $(5, 8, 10, 7, 4, 9, 11) \sqcup (16, 19)$ $(4, 9, 15, 8, 12, 18, 7) \sqcup (1, 11)$
$\mathbf{T}_6^1 \sqcup \mathbf{T}_3^1$	$(3, 5, 4, 2, \infty, 1) \sqcup (13, 12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (8, 11, 10)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14, 15)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (17, 16, 15)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11, 2)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^2 \sqcup \mathbf{T}_3^1$	$(\infty, 2, 4, 5, 8, 0) \sqcup (11, 13, 12)$ $(6, \infty, 5, 2, 3, 8) \sqcup (13, 16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14, 15)$ $(8, 10, 7, 4, \infty, 12) \sqcup (18, 15, 13)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^3 \sqcup \mathbf{T}_3^1$	$(5, 4, 2, 3, 6, 0) \sqcup (9, \infty, 11)$ $(4, 6, \infty, 12, 13, 1) \sqcup (11, 8, 7)$ $(10, \infty, 7, 6, 9, 5) \sqcup (16, 15, 13)$ $(5, 8, 10, 7, 4, 11) \sqcup (16, 19, 17)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^4 \sqcup \mathbf{T}_3^1$	$(5, 4, 7, 2, 1, 3) \sqcup (8, 11, \infty)$ $(12, \infty, 8, 6, 4, 5) \sqcup (13, 10, 7)$ $(10, \infty, 2, 7, 8, 5) \sqcup (19, 16, 14)$ $(11, 10, 12, 8, 5, 6) \sqcup (16, 13, 14)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11, 2)$
$\mathbf{T}_6^5 \sqcup \mathbf{T}_3^1$	$(1, 2, 4, 5, 0, 3) \sqcup (8, 11, 14)$ $(11, \infty, 6, 4, 8, 5) \sqcup (10, 13, 12)$ $(6, 7, \infty, 3, 8, 5) \sqcup (9, 12, 15)$ $(11, 10, 8, 6, 12, 7) \sqcup (13, 16, \infty)$ $(8, 0, 4, 9, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_6^6 \sqcup \mathbf{T}_3^1$	$(1, 2, 0, 3, 4, 5) \sqcup (11, 8, \infty)$ $(2, \infty, 3, 4, 5, 6) \sqcup (12, 13, 15)$ $(6, 7, 8, 4, 5, \infty) \sqcup (11, 12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (9, 6, 4)$ $(4, 0, 8, 5, 6, 7) \sqcup (1, 11, 2)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^1$	$(5, 4, 2, \infty, 1) \sqcup (11, 13, 12, 15)$ $(0, 2, 5, \infty, 6) \sqcup (8, 11, 10, 12)$ $(5, 7, \infty, 3, 6) \sqcup (16, 13, 14, 15)$ $(\infty, 4, 7, 10, 8) \sqcup (17, 16, 15, 13)$ $(4, 9, 15, 8, 16) \sqcup (2, 11, 1, 5)$

Table A.2: (1-2-3)-labelings



Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^1$	$(\infty, 2, 4, 5, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (8, 11, 10, 12)$ $(6, 3, \infty, 7, 1) \sqcup (16, 13, 14, 15)$ $(10, 7, 4, \infty, 5) \sqcup (17, 16, 15, 13)$ $(16, 8, 15, 9, 12) \sqcup (2, 11, 1, 6)$
$\mathbf{T}_5^2 \sqcup \mathbf{T}_4^2$	$(\infty, 2, 4, 3, 0) \sqcup (11, 13, 12, 15)$ $(6, \infty, 5, 2, 1) \sqcup (10, 12, 11, 15)$ $(6, 3, \infty, 7, 1) \sqcup (12, 14, 13, 15)$ $(\infty, 4, 7, 10, 1) \sqcup (17, 16, 13, 15)$ $(16, 8, 15, 9, 12) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^1$	$(0, 2, 1, 3, 4) \sqcup (11, 8, \infty, 6)$ $(2, \infty, 3, 4, 5) \sqcup (9, 12, 13, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15, 14)$ $(0, 3, 1, 5, 6) \sqcup (16, 13, 11, 10)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 15, 9)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_4^2$	$(10, 13, \infty, 8, 11) \sqcup (1, 2, 3, 4)$ $(15, 13, 12, 9, 7) \sqcup (3, \infty, 4, 5)$ $(11, 12, 15, 14, 13) \sqcup (4, 7, 5, \infty)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12, 7)$ $(16, 8, 15, 9, 5) \sqcup (10, 1, 11, 6)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_4^2$	$(0, 2, 3, 4, 5) \sqcup (9, 8, 11, \infty)$ $(2, \infty, 3, 4, 5) \sqcup (12, 13, 14, 15)$ $(4, 7, 8, 5, \infty) \sqcup (10, 12, 11, 15)$ $(0, 3, 1, 4, 6) \sqcup (16, 13, 11, \infty)$ $(5, 1, 10, 11, 6) \sqcup (16, 8, 14, 15)$
$\mathbf{T}_6^1 \sqcup 2\mathbf{T}_2^1$	$(3, 5, 4, 2, \infty, 1) \sqcup (19, 20) \sqcup (12, 15)$ $(0, 2, 5, \infty, 6, 4) \sqcup (17, 18) \sqcup (8, 11)$ $(5, 7, \infty, 3, 6, 9) \sqcup (13, 14) \sqcup (0, 1)$ $(\infty, 4, 7, 10, 8, 6) \sqcup (16, 15) \sqcup (2, 3)$ $(0, 4, 9, 15, 8, 16) \sqcup (1, 11) \sqcup (3, 12)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_6^2 \sqcup 2\mathbf{T}_2^1$	$(\infty, 2, 4, 5, 8, 0) \sqcup (18, 20) \sqcup (12, 13)$ $(13, \infty, 5, 2, 3, 8) \sqcup (9, 6) \sqcup (16, 15)$ $(6, 3, \infty, 7, 5, 4) \sqcup (13, 14) \sqcup (0, 1)$ $(15, 17, 14, 11, \infty, 19) \sqcup (8, 6) \sqcup (1, 4)$ $(0, 4, 9, 15, 8, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(3, 2, 4, 5, 0, 1) \sqcup (18, 15) \sqcup (11, 14)$ $(5, \infty, 6, 4, 8, 11) \sqcup (10, 13) \sqcup (19, 20)$ $(8, 7, \infty, 3, 5, 6) \sqcup (16, 19) \sqcup (12, 15)$ $(7, 10, 8, 6, 11, 12) \sqcup (16, 13) \sqcup (9, \infty)$ $(6, 0, 8, 4, 5, 7) \sqcup (1, 11) \sqcup (3, 12)$
$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(5, 4, 7, 2, 1, 3) \sqcup (8, 11) \sqcup (18, \infty)$ $(12, \infty, 8, 6, 4, 5) \sqcup (0, 3) \sqcup (10, 13)$ $(10, \infty, 2, 7, 8, 5) \sqcup (9, 6) \sqcup (16, 19)$ $(11, 10, 12, 8, 5, 6) \sqcup (13, 14) \sqcup (0, 2)$ $(7, 0, 6, 4, 9, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(5, 4, 2, 3, 6, 0) \sqcup (9, 12) \sqcup (11, \infty)$ $(4, 6, \infty, 12, 13, 15) \sqcup (0, 1) \sqcup (8, 11)$ $(10, \infty, 7, 6, 9, 5) \sqcup (13, 15) \sqcup (1, 2)$ $(5, 8, 10, 7, 4, 11) \sqcup (17, 19) \sqcup (9, \infty)$ $(7, 0, 4, 9, 15, 12) \sqcup (1, 11) \sqcup (5, 14)$
$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(1, 2, 0, 3, 4, 5) \sqcup (\infty, 15) \sqcup (8, 11)$ $(11, \infty, 2, 3, 5, 6) \sqcup (13, 15) \sqcup (19, 20)$ $(6, 7, 8, 4, 5, \infty) \sqcup (18, 19) \sqcup (12, 15)$ $(11, 10, 8, 12, 13, 7) \sqcup (18, 20) \sqcup (9, 6)$ $(11, 1, 8, 9, 10, 7) \sqcup (0, 5) \sqcup (2, 6)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(10, 13, \infty, 8, 11) \sqcup (3, 2, 4) \sqcup (16, 15)$ $(15, 13, 12, 9, 7) \sqcup (10, \infty, 5) \sqcup (11, 14)$ $(11, 12, 15, 14, 13) \sqcup (4, \infty, 7) \sqcup (0, 3)$ $(3, 4, 6, 9, \infty) \sqcup (8, 10, 12) \sqcup (5, 7)$ $(0, 9, 1, 8, 2) \sqcup (5, 10, 6) \sqcup (3, 13)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(8, \infty, 13, 10, 9) \sqcup (3, 2, 4) \sqcup (14, 15)$ $(7, 9, 12, 13, 8) \sqcup (10, \infty, 5) \sqcup (11, 14)$ $(11, 12, 15, 18, 14) \sqcup (4, \infty, 7) \sqcup (0, 3)$ $(9, 6, 4, 3, 8) \sqcup (19, 17, 15) \sqcup (13, 14)$ $(1, 8, 0, 9, 2) \sqcup (5, 10, 6) \sqcup (3, 13)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(2, \infty, 3, 4, 5) \sqcup (12, 13, 15) \sqcup (16, 19)$ $(0, 2, 1, 3, 4) \sqcup (8, \infty, 6) \sqcup (18, 15)$ $(4, 7, 5, 6, \infty) \sqcup (11, 12, 15) \sqcup (0, 1)$ $(8, 10, 12, 13, 7) \sqcup (9, 6, 4) \sqcup (17, 18)$ $(9, 0, 8, 6, 7) \sqcup (11, 1, 5) \sqcup (10, 15)$
$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(1, \infty, 16, 18) \sqcup (11, 13, 12, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 12) \sqcup (9, 7)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14, 15) \sqcup (5, 7)$ $(10, 7, 4, \infty) \sqcup (17, 16, 15, 13) \sqcup (1, 3)$ $(9, 15, 8, 16) \sqcup (2, 11, 1, 5) \sqcup (12, 7)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11, 9, \infty, 1) \sqcup (10, 12, 13, 15) \sqcup (4, 5)$ $(2, 5, \infty, 6) \sqcup (8, 11, 10, 13) \sqcup (9, 7)$ $(0, \infty, 17, 20) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(10, 7, 4, \infty) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(2, 12, 6, 15) \sqcup (8, 0, 5, 7) \sqcup (9, 13)$
$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(18, 16, 19, \infty) \sqcup (10, 12, 13, 15) \sqcup (3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 10, 13) \sqcup (4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13, 15) \sqcup (8, 6)$ $(9, 7, 10, 4) \sqcup (17, 16, 13, 15) \sqcup (1, 3)$ $(9, 0, 8, 7) \sqcup (11, 1, 5, 6) \sqcup (10, 4)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$ $(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$ $(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (5, 3, 6)$ $(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (3, 4, 5)$ $(0, \infty, 3, 4) \sqcup (12, 14, 13) \sqcup (6, 8, 7)$ $(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (2, 1, 3)$ $(9, 0, 8, 7) \sqcup (5, 1, 6) \sqcup (10, 4, 14)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (4, 5) \sqcup (16, 18)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (2, 5) \sqcup (16, 14)$ $(8, 10, 7, 4) \sqcup (0, \infty, 11) \sqcup (16, 17) \sqcup (9, 6)$ $(5, 7, 8, 6) \sqcup (20, 17, \infty) \sqcup (13, 14) \sqcup (1, 2)$ $(3, 10, 5, 11) \sqcup (0, 9, 1) \sqcup (2, 12) \sqcup (17, 13)$
$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(18, 16, 19, \infty) \sqcup (13, 12, 15) \sqcup (3, 5) \sqcup (17, 20)$ $(1, \infty, 12, 6) \sqcup (8, 11, 13) \sqcup (4, 5) \sqcup (17, 18)$ $(3, \infty, 4, 7) \sqcup (12, 14, 13) \sqcup (8, 6) \sqcup (1, 2)$ $(9, 7, 10, 4) \sqcup (17, 16, 13) \sqcup (1, 3) \sqcup (14, 15)$ $(9, 0, 8, 7) \sqcup (11, 1, 6) \sqcup (18, 12) \sqcup (10, 14)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(4, 1, \infty, 13, 10) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(5, \infty, 10, 11, 13) \sqcup (4, 7) \sqcup (0, 2) \sqcup (9, 12)$ $(7, \infty, 4, 5, 8) \sqcup (17, 19) \sqcup (0, 3) \sqcup (12, 14)$ $(7, 8, 6, 9, \infty) \sqcup (13, 14) \sqcup (1, 3) \sqcup (19, 20)$ $(1, 11, 2, 10, 3) \sqcup (0, 6) \sqcup (9, 4) \sqcup (8, 12)$
$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(1, \infty, 13, 10, 7) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(5, \infty, 10, 11, 16) \sqcup (4, 7) \sqcup (0, 2) \sqcup (9, 12)$ $(6, 4, 5, 8, \infty) \sqcup (17, 19) \sqcup (0, 3) \sqcup (12, 14)$ $(7, 8, 6, 9, 11) \sqcup (13, 14) \sqcup (1, 3) \sqcup (19, 20)$ $(3, 10, 2, 11, 5) \sqcup (0, 6) \sqcup (4, 8) \sqcup (17, 7)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(1, \infty, 13, 5, 7) \sqcup (2, 3) \sqcup (16, 15) \sqcup (9, 11)$ $(0, 3, 1, 4, \infty) \sqcup (2, 5) \sqcup (9, 7) \sqcup (10, 13)$ $(12, 11, 13, 14, \infty) \sqcup (17, 19) \sqcup (5, 7) \sqcup (9, 6)$ $(5, 8, 11, 6, 7) \sqcup (13, 14) \sqcup (2, \infty) \sqcup (19, 20)$ $(6, 0, 8, 9, 7) \sqcup (1, 11) \sqcup (10, 5) \sqcup (16, 12)$

Table A.2: (1-2-3)-labelings

Forest	Labeling
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(11, 13, 12, 15) \sqcup (9, \infty, 1) \sqcup (2, 4, 5)$ $(8, 11, 10, 12) \sqcup (19, \infty, 6) \sqcup (0, 2, 5)$ $(0, \infty, 3, 6) \sqcup (16, 13, 14) \sqcup (8, 7, 5)$ $(17, 16, 15, 13) \sqcup (\infty, 4, 7) \sqcup (0, 3, 1)$ $(9, 15, 8, 16) \sqcup (11, 1, 5) \sqcup (3, 12, 7)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(9, \infty, 8, 6) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13, 14) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 4, 3) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(18, 20, 17, \infty) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(0, 9, 1, 11) \sqcup (10, 3) \sqcup (12, 6) \sqcup (19, 14) \sqcup (17, 13)$
$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(8, \infty, 9, 5) \sqcup (12, 15) \sqcup (16, 17) \sqcup (1, 2) \sqcup (3, 4)$ $(15, 13, 14, \infty) \sqcup (9, 6) \sqcup (0, 2) \sqcup (1, 4) \sqcup (17, 19)$ $(0, \infty, 3, 4) \sqcup (10, 7) \sqcup (16, 18) \sqcup (2, 5) \sqcup (11, 14)$ $(17, 20, 18, 19) \sqcup (4, 5) \sqcup (12, 14) \sqcup (8, 10) \sqcup (0, 1)$ $(9, 0, 8, 7) \sqcup (1, 11) \sqcup (12, 6) \sqcup (10, 5) \sqcup (16, 20)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(8, \infty, 9) \sqcup (13, 12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2)$ $(19, \infty, 6) \sqcup (11, 10, 12) \sqcup (2, 5) \sqcup (18, 20) \sqcup (1, 4)$ $(11, \infty, 14) \sqcup (10, 7, 4) \sqcup (16, 17) \sqcup (0, 2) \sqcup (1, 3)$ $(20, 17, \infty) \sqcup (14, 13, 15) \sqcup (5, 7) \sqcup (9, 6) \sqcup (0, 1)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$
$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(8, \infty, 9) \sqcup (12, 15) \sqcup (4, 5) \sqcup (16, 18) \sqcup (1, 2) \sqcup (19, 20)$ $(5, \infty, 13) \sqcup (9, 6) \sqcup (0, 2) \sqcup (18, 20) \sqcup (1, 4) \sqcup (17, 19)$ $(11, \infty, 14) \sqcup (4, 7) \sqcup (16, 17) \sqcup (2, 5) \sqcup (8, 10) \sqcup (0, 3)$ $(20, 17, \infty) \sqcup (13, 14) \sqcup (5, 7) \sqcup (10, 11) \sqcup (0, 1) \sqcup (8, 6)$ $(0, 9, 4) \sqcup (2, 10, 3) \sqcup (12, 6) \sqcup (17, 7) \sqcup (1, 5)$

Table A.2: (1-2-3)-labelings

### A.3 $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decompositions of $K_{21}$ and $K_{22}$

No.	Block	No.	Block
1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (18, 7)$	8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	10	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
11	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	12	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
13	$(1, 7, 2, 3, 4, 5, 6) \sqcup (0, 14)$	14	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
15	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	16	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$
17	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	18	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
19	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	20	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (1, 3)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$

Table A.3: A  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{21}$

No.	Block	No.	Block
1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (6, \infty)$	8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	10	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
11	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	12	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
13	$(1, 7, 2, 3, 4, 5, 6) \sqcup (13, \infty)$	14	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
15	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	16	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$

Table A.4: A  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{22}$

No.	Block	No.	Block
17	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	18	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
19	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	20	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$
21	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	22	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
23	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	24	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
25	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	26	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
27	$(9, 7, 0, 10, 11, 12, 13) \sqcup (20, \infty)$	28	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
29	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	30	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$
31	$(0, \infty, 1, 2, 3, 4, 5) \sqcup (18, 7)$	32	$(14, \infty, 15, 16, 17, 18, 19) \sqcup (1, 3)$
33	$(7, \infty, 8, 9, 10, 11, 12) \sqcup (0, 14)$		

Table A.4: A  $\mathbf{T}_7^{11} \sqcup \mathbf{T}_2^1$ -decomposition of  $K_{22}$