Seven Edge Forest Designs

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There are many people that have earned my gratitude for their contribution to my time in graduate school. However, without Professor Bryan Freyberg I would not have had the opportunity to become a graduate student, and Professor Joseph Gallian's courses and guidance has been very influential to the way I organize myself mathematically today.

Dedication

I dedicate this Thesis to my advisor Professor Bryan Freyberg, to my family who has supported me throughout this process, and to Jordi, Ian, TK, and Torta from Tuscarora Ave.

Abstract

Let G be a subgraph of K_n where $n \in \mathbb{N}$. A G-decomposition of K_n, or G-design of order n, is a finite collection $\mathcal{G} = \{G_1, \ldots, G_k\}$ of pairwise edge-disjoint subgraphs of K_n that are all isomorphic to some graph G. We prove that an F-decomposition of K_n exists for every seven-edge forest F if and only if $n \equiv 0, 1, 7, \text{ or } 8 \pmod{14}$.

Along the way, we introduce new methods, constraint programming algorithms in Python, and some bonus results for Galaxy graph decompositions of complete bipartite, and eventually multipartite graphs.

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Chapter 1

Introduction

A *G-decomposition* of a graph K is a set of mutually edge-disjoint subgraphs of K which are isomorphic to a given graph G. If such a set exists we say that K *allows* a G-decomposition, and if $K \cong K_n$ we sometimes call the decomposition a *G-design of order n*.

G-decompositions are a longstanding topic in combinatorics, graph theory, and design theory, with roots tracing back to at least the 19th century. The work of Rosa and Kotzig in the 1960s on what are now known as graph labelings laid the foundation for the modern treatment of such problems. Using adaptations of these labelings alongside techniques from design theory, numerous papers have since been published on G-decompositions. This work is a natural continuation of Freyberg and Peters' recent paper on decomposing complete graphs into forests with six edges [4]. Their paper also includes a summary of G-decompositions for graphs G with less than 7 edges.

Every connected component of a forest with 7 edges is a tree with 6 or less edges. All such trees are cataloged in Figure **??**. We use the naming convention $\mathbf{T^i_j}$ to denote the i^{th} tree with j vertices. For each tree $\mathbf{T^i_j}$, the names of the vertices, v_t for $1 \le t \le j$, will be referred to in the decompositions described in Section **??**.

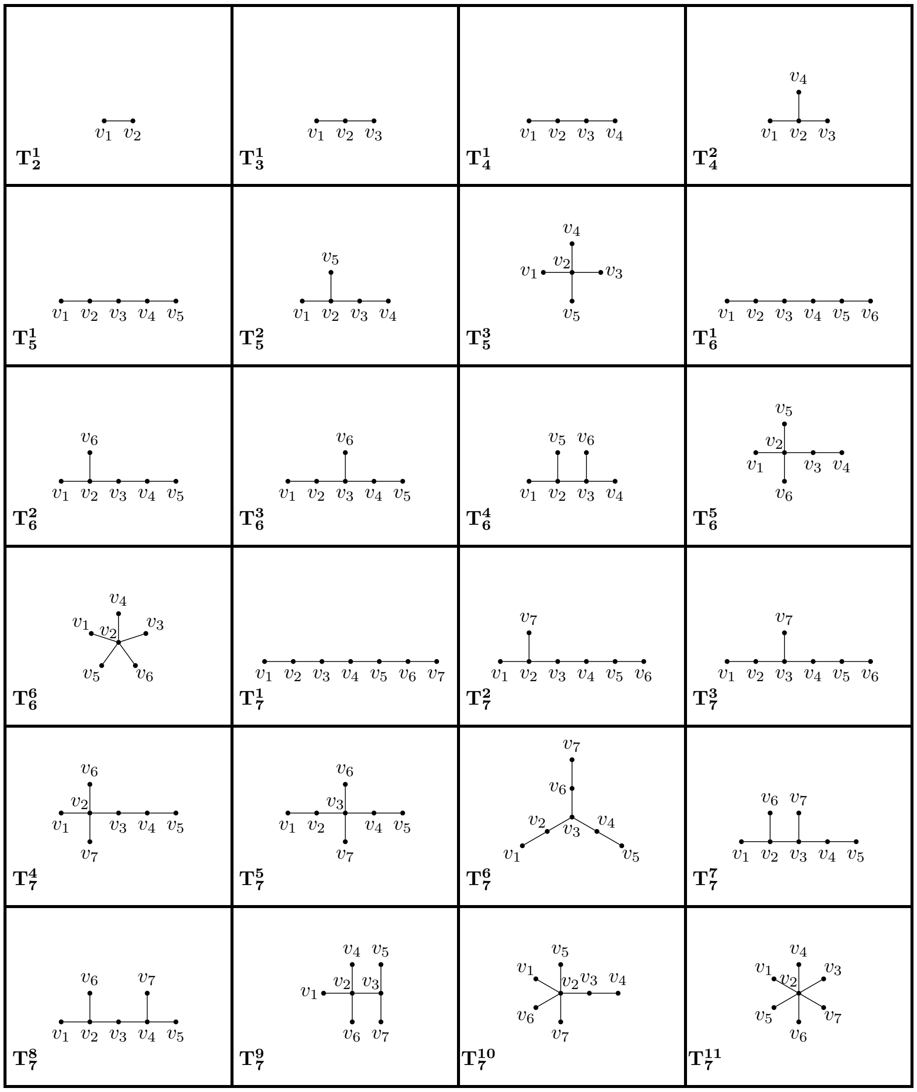

Figure 1.1: trees with less than seven edges

The next theorem gives the necessary conditions for the existence of a G-decomposition of K_n when G is a graph with 7 edges.

Theorem 1.0.1. *If G is a graph with 7 edges and a G-decomposition of K_n exists, then $n \equiv 0, 1, 7,$ or $8 \pmod{14}$.*

Proof. If a G-decomposition exists, then $7 | \binom{n}{2}$ which immediately implies $n \equiv 0, 1, 7,$ or 8 (mod 14).

$\square$

In this article, we only consider simple graphs without isolated vertices. There are 47 non-isomorphic forests with 7 edges. Section **??** treats all 47 forests when $n \equiv 0$ or $1 \pmod{14}$. Section **??** applies to all the forests when $n \equiv 7$ or $8 \pmod{14}$ with the lone exception of $F \cong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$, which is solved for those values of n in Section **??**.

Chapter 2

$\mathbf{n \equiv 0, 1 (mod\ 14)}$

In this section, we use established graph labeling techniques to construct the G-decompositions of K_n when $n \equiv 0$ or $1 \pmod{14}$.

Definition 2.0.1 ((Rosa [7])). Let G be a graph with m edges. A *ρ-labeling* of G is an injection $f : V(G) \to \{0, 1, 2, \ldots, 2m\}$ that induces a bijective *length function* $\ell : E(G) \to \{1, 2, \ldots, m\}$ where

$$\ell(uv) = \min\{|f(u) - f(v)|, 2m + 1 - |f(u) - f(v)|\},$$

for all $uv \in E(G)$.

Rosa showed that a ρ-labeling of a graph G with m edges and a cyclic G-decomposition of K_{2m+1} are equivalent, which the next thm shows. Later, Rosa, his students, and colleagues began considering more restrictive types of ρ-labeling to address decomposing complete graphs of more orders. Definitions of these labelings and related results follow.

Theorem 2.0.2 ((Rosa [7])). *Let G be a graph with m edges. There exists a cyclic G-decomposition of K_{2m+1} if and only if G admits a ρ-labeling.*

Definition 2.0.3 ((Rosa [7])). A σ-labeling of a graph G is a ρ-labeling such that $\ell(uv) = |f(u) - f(v)|$ for all $uv \in E(G)$.

Definition 2.0.4 ((El-Zanati, Vanden Eynden [2])). A ρ- or σ-labeling of a bipartite graph G with bipartition (A, B) is called an *ordered* ρ- or σ-labeling and denoted ρ^+, σ^+, respectively, if $f(a) < f(b)$ for each edge ab with $a \in A$ and $b \in B$.

Theorem 2.0.5 ((El-Zanati, Vanden Eynden [2])). *Let G be a graph with m edges which has a ρ^+-labeling. Then G decomposes K_{2mk+1} for all positive integers k.*

Definition 2.0.6 ((Freyberg, Tran [5])). A σ^{+-}-*labeling* of a bipartite graph G with m edges and bipartition (A, B) is a σ^+-labeling with the property that $f(a) - f(b) \neq m$ for all $a \in A$ and $b \in B$, and $f(v) \notin \{2m, 2m-1\}$ for any $v \in V(G)$.

Theorem 2.0.7 ((Freyberg, Tran [5])). *Let G be a graph with m edges and a σ^{+-}-labeling such that the edge of length m is a pendant. Then there exists a G-decomposition of both K_{2mk} and K_{2mk+1} for every positive integer k.*

Figure 2 gives a σ^{+-}-labeling of every forest with 7 edges. The vertex labels of each connected component with k vertices are given as a k-tuple, $(v_1, \ldots, v_k)$ corresponding to the vertices $v_1, \ldots, v_k$ given in Figure **??**. We leave it to the reader to infer the bipartition (A, B).

Example 2.0.8. *A σ^{+-}-labeling of $\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$ is shown in Figure 2.1. The vertices labeled* $1, 2$ *and* 9 *belong to A, and the others belong to B. The lengths of each edge are indicated on the edge.*

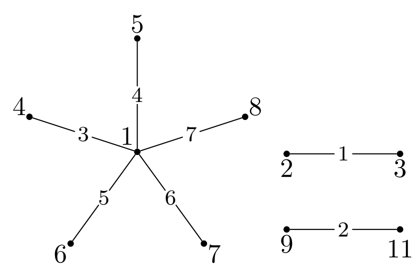

Figure 2.1: σ^{+-}-labeling of $\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$

The labelings given in Figure 2 along with thm 2.0.7 are enough to prove the following thm.

Forest	Vertex Labels
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2,9,7) \sqcup (3,4)$
$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,3) \sqcup (8,7)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(9,2,5,1,6,0,4) \sqcup (8,7)$
$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(5,1,4,2,9,6,7) \sqcup (10,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(3,8,1,4,2,5,7) \sqcup (9,10)$
$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(7,8,1,6,0,4,3) \sqcup (9,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,1,6,3,4,5,7) \sqcup (9,10)$
$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(6,1,5,3,8,4,7) \sqcup (9,10)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(5,11,9,10,6,12,7) \sqcup (8,1)$
$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(4,8,1,6,0,5,3) \sqcup (9,10)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(0,6,1,5,2,9) \sqcup (11,10,12)$
$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(3,6,1,8,4,0) \sqcup (10,9,11)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(5,11,9,12,7,10) \sqcup (1,8,4)$
$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(3,8,4,1,6,7) \sqcup (10,9,11)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(5,1,8,3,4,7) \sqcup (10,9,11)$
$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(4,1,8,5,6,7) \sqcup (10,9,11)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(0,6,1,5,2) \sqcup (9,8,10,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(7,1,8,5,6) \sqcup (0,4,2,3)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(7,1,8,4,6) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(6,0,3,4,5) \sqcup (8,7,9,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^2}$	$(4,8,1,7,2) \sqcup (10,9,11,12)$
$\mathbf{T_5^3} \sqcup \mathbf{T_4^2}$	$(6,0,3,4,5) \sqcup (8,9,2,7)$
$\mathbf{T_6^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5,2,9) \sqcup (8,10) \sqcup (3,4)$
$\mathbf{T_6^2} \sqcup 2\mathbf{T_2^1}$	$(3,6,1,8,4,0) \sqcup (5,7) \sqcup (9,10)$
$\mathbf{T_6^5} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,3,5,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^4} \sqcup 2\mathbf{T_2^1}$	$(5,8,4,1,6,7) \sqcup (0,2) \sqcup (9,10)$
$\mathbf{T_6^3} \sqcup 2\mathbf{T_2^1}$	$(5,11,9,12,7,10) \sqcup (8,1) \sqcup (0,4)$
$\mathbf{T_6^6} \sqcup 2\mathbf{T_2^1}$	$(4,1,8,5,6,7) \sqcup (2,3) \sqcup (9,11)$

$\mathbf{T_5^1} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (8,10,9) \sqcup (11,4)$
$\mathbf{T_5^2} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(7,1,8,5,6) \sqcup (10,9,11) \sqcup (0,4)$
$\mathbf{T_5^3} \sqcup \mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(6,0,3,4,5) \sqcup (1,8,7) \sqcup (9,11)$
$2\mathbf{T_4^1} \sqcup \mathbf{T_2^1}$	$(0,6,1,5) \sqcup (2,9,7,10) \sqcup (3,4)$
$\mathbf{T_4^1} \sqcup \mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(11,9,10,7) \sqcup (4,0,5,6) \sqcup (8,1)$
$2\mathbf{T_4^2} \sqcup \mathbf{T_2^1}$	$(4,0,5,6) \sqcup (10,9,11,12) \sqcup (8,1)$
$\mathbf{T_4^1} \sqcup 2\mathbf{T_3^1}$	$(0,6,1,5) \sqcup (8,10,9) \sqcup (11,4,7)$
$\mathbf{T_4^2} \sqcup 2\mathbf{T_3^1}$	$(4,0,5,6) \sqcup (1,8,7) \sqcup (11,9,12)$
$\mathbf{T_4^1} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (8,10,7) \sqcup (11,4) \sqcup (2,3)$
$\mathbf{T_4^2} \sqcup \mathbf{T_3^1} \sqcup 2\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (11,9,12) \sqcup (2,3) \sqcup (8,1)$
$\mathbf{T_5^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1,5,2) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_5^2} \sqcup 3\mathbf{T_2^1}$	$(6,1,8,4,7) \sqcup (3,5) \sqcup (9,12) \sqcup (10,11)$
$\mathbf{T_5^3} \sqcup 3\mathbf{T_2^1}$	$(3,0,4,5,6) \sqcup (8,1) \sqcup (10,11) \sqcup (9,7)$
$3\mathbf{T_3^1} \sqcup \mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (2,9,7) \sqcup (10,11)$
$\mathbf{T_4^1} \sqcup 4\mathbf{T_2^1}$	$(0,6,1,5) \sqcup (9,2) \sqcup (8,10) \sqcup (4,7) \sqcup (11,12)$
$\mathbf{T_4^2} \sqcup 4\mathbf{T_2^1}$	$(4,0,5,6) \sqcup (2,3) \sqcup (9,11) \sqcup (8,1) \sqcup (10,7)$
$2\mathbf{T_3^1} \sqcup 3\mathbf{T_2^1}$	$(0,6,1) \sqcup (4,8,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$
$\mathbf{T_3^1} \sqcup 5\mathbf{T_2^1}$	$(0,6,1) \sqcup (8,4) \sqcup (2,5) \sqcup (10,3) \sqcup (9,7) \sqcup (11,12)$

Figure 2.2: σ^{+-}-labelings for forests with 7 edges

Theorem 2.0.9. *Let F be a forest with 7 edges. There exists an F-decomposition of K_n whenever $n \equiv 0$ or $1 \pmod{14}$.*

Proof. The proof follows from thm 2.0.7 and the labelings given in Figure 2. $\square$

In this section, we use our own constructions based on the same edge length definition as in the previous section. The first one addresses the $n \equiv 7 \pmod{14}$ case.

Definition 2.0.10. Let G be a graph with 7 edges. A (1-2-3)-*labeling* of $3G$ is an assignment f of the integers $\{0, \dots, 20\}$ to the vertices of $3G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

   and

2. $$\bigcup_{uv \in E(3G)} \{(f(u) \textbf{ mod } 7, f(v) \textbf{ mod } 7)\} = \bigcup_{i=0}^{6} \bigcup_{j=1}^{3} \{(i, i+j \textbf{ mod } 7)\}.$$

Notice that the second condition of a (1-2-3)-labeling says that $3G$ contains exactly 7 edges of each of the lengths $1, 2$, and 3. Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A (1-2-3) labeling of every forest with 7 edges with the exception of $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ is given in Figure **??**. This exceptional forest does not admit such a labeling and is dealt with in Section **??**.

Theorem 2.0.11. *Let G be a bipartite graph with 7 edges. If $3G$ admits a (1-2-3)-labeling and G admits a ρ^+-labeling, then G decomposes K_{14k+7} for every $k \geq 1$.*

Proof. Let $n = 14k + 7$ and notice that K_n has $|E(K_n)| = (7k + 3)(14k + 7)$ edges, which can be partitioned into $14k + 7$ edges of each of the lengths in $\{1, 2, \dots, 7k + 3\}$. We will construct the G-decomposition in two steps. First, we use the 1-2-3-labeling to identify all the edges of lengths $1, 2$, and 3 accounting for $3(2k + 1)$ copies of G. Then, we use the ρ^+-labeling to identify edges of the remaining lengths in $7k(2k + 1)$ copies of G. In total, the decomposition consists of $|E(K_n)|/7 = (7k + 3)(2k + 1)$ copies of G.

Let f_1 be a (1-2-3)-labeling of $3G$ and identify this graph as a block B_0. Then develop B_0 by 7 modulo n. Since the order of the development is $\frac{n}{7} = 2k + 1$ and there are 7 edges of each of the lengths $1, 2$, and 3 in B_0, we have identified $3(2k + 1)$ copies of G containing all $14k + 7 = n$ edges of each length $1, 2$, and 3. Notice (2) of Definition 2.0.10 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \to \{0, \ldots, 14\}$ be a ρ^+-labeling of G with associated vertex partition (A, B). For $i = 1, 2, \ldots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i-1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i-3, 7i-2, \ldots,$ and $7i+3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i-1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \ldots, 7\}$. Developing each block B_i by 1 yields $14k+7$ copies of G per block and accounts for $14k+7$ edges of each of the lengths $4, 5, \ldots,$ and $7k+3$.

Since we have identified

$$3(2k+1) + k(14k+7) = (7k+3)(2k+1)$$

edge-disjoint copies of G, the proof is complete. $\square$

To address the $n \equiv 8 \pmod{14}$ case, we define the following labeling.

Definition 2.0.12. Let G be a graph with 7 edges. A *1-rotational* (1-2-3)*-labeling* of $4G$ is an assignment f of $\{0, \ldots, 20\} \cup \infty$ to the vertices of $4G$ such that

1. $f(u) \neq f(v)$ whenever u and v belong to the same connected component,

   and

2. $$\bigcup_{uv \in E(4G)} \{(f(u) \textbf{ mod } 7, f(v) \textbf{ mod } 7)\} = \bigcup_{i=0}^{6} \bigcup_{j=1}^{3} \{(i, i+j \textbf{ mod } 7), (i, \infty)\}.$$

Notice that the second condition of a 1-rotational (1-2-3)-labeling says that $4G$ contains exactly 7 edges of each of the lengths $1, 2, 3,$ and ∞. Furthermore, no two edges of the same length have the same end labels when reduced modulo 7. A 1-rotational (1-2-3)-labeling of every forest with 7 edges with the exception of $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ is given in Figure **??**.

Theorem 2.0.13. *Let G be a bipartite graph with 7 edges. If $4G$ admits a 1-rotational (1-2-3)-labeling and G admits a ρ^+-labeling, then G decomposes K_{14k+8} for every $k \geq 1$.*

Proof. Let $n = 14k+8$ and notice that K_n has $|E(K_n)| = (7k+4)(14k+7)$ edges, which can be partitioned into $14k+7$ edges of each of the lengths in $\{1, 2, \dots, 7k+3, \infty\}$. We will construct the G-decomposition in two steps. First, we use the 1-rotational (1-2-3)-labeling to identify all the edges of lengths $1, 2, 3$, and ∞ accounting for $4(2k+1)$ copies of G. Then, we use the ρ^+-labeling to identify edges of the remaining lengths in $7k(2k+1)$ copies of G. In total, the decomposition consists of $|E(K_n)|/7 = (7k+4)(2k+1)$ copies of G. Let f_1 be a 1-rotational (1-2-3)-labeling of $4G$ and identify this graph as a block B_0. Then develop B_0 by 7 modulo $n-1$. Since the order of the development is $\frac{n-1}{7} = 2k+1$ and there are 7 edges of each of the lengths $1, 2, 3$ and ∞ in B_0, we have identified $4(2k+1)$ copies of G containing all $14k+7 = n-1$ edges of each length $1, 2, 3$ and ∞. Notice (2) of Definition 2.0.12 ensures no edge has been counted more than once in the development.

Let $f_2 : V(G) \to \{0, \dots, 14\}$ be a ρ^+-labeling of G with associated vertex partition (A, B). For $i = 1, 2, \dots, k$, identify blocks $B_i \cong G$ with vertex labels ℓ such that

$$\ell(v) = \begin{cases} f_2(v), & \text{if } v \in A \\ f_2(v) + 3 + 7(i-1), & \text{if } v \in B \end{cases}$$

Notice that the i^{th} block contains exactly one edge of each length $7i-3, 7i-2, \dots,$ and $7i+3$. This is because every edge ab has length

$$\ell(b) - \ell(a) = f_2(b) - f_2(a) + 3 + 7(i-1)$$

and $f_2(b) - f_2(a)$ is a length in $\{1, \dots, 7\}$. Developing each block B_i by 1 yields $14k+7$ copies of G per block and accounts for $14k+7$ edges of each of the lengths $4, 5, \dots,$ and $7k+3$.

Since we have identified

$$4(2k+1) + k(14k+7) = (7k+4)(2k+1)$$

edge-disjoint copies of G, the proof is complete. □

We are now able to state the main thm of this section.

Theorem 2.0.14. *Let F be a forest with 7 edges and $F \ncong \mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$. There exists an F-decomposition of K_n whenever $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$.*

Proof. If $n \equiv 7 \pmod{14}$, a (1-2-3)-labeling of $3F$ can be found in Figure **??**. On the other hand, if $n \equiv 8 \pmod{14}$, then a 1-rotational (1-2-3)-labeling of $4F$ can be found in Figure **??**. In either case, a ρ^+-labeling of F can be found in Figure 2 (recall that a σ^{+-}-labeling is a ρ^+-labeling). The result now follows from thms 2.0.11 and 2.0.13. $\square$

Example 2.0.15. *We illustrate the constructions in the previous two thms by finding an F-decomposition of K_{35} and K_{36} for the forest graph $F \cong \mathbf{T_6^6} \sqcup \mathbf{T_3^1}$.*

Here are excerpts from the preceding tables for $\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$

Labeling Type	Labelings
σ^{+-}	$(4,1,8,5,6,7) \sqcup (10,9,11)$
$7 \pmod{14}$	$(0,2,1,3,4,5) \sqcup (12,11,14)$
	$(4,6,8,9,5,7) \sqcup (14,12,15)$
	$(0,3,1,4,5,6) \sqcup (11,8,7)$
$8 \pmod{14}$	$(1,2,0,3,4,5) \sqcup (11,8,\infty)$
	$(2,\infty,3,4,5,6) \sqcup (12,13,15)$
	$(6,7,8,4,5,\infty) \sqcup (11,12,15)$
	$(11,10,8,12,13,7) \sqcup (9,6,4)$

Figure 2.3: Labelings for $\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$

The ρ^+ *labelings obtained by stretching the* σ^{+-} *labeling are bottommost labelings in the following generating presentations and are developed by* 1*.*

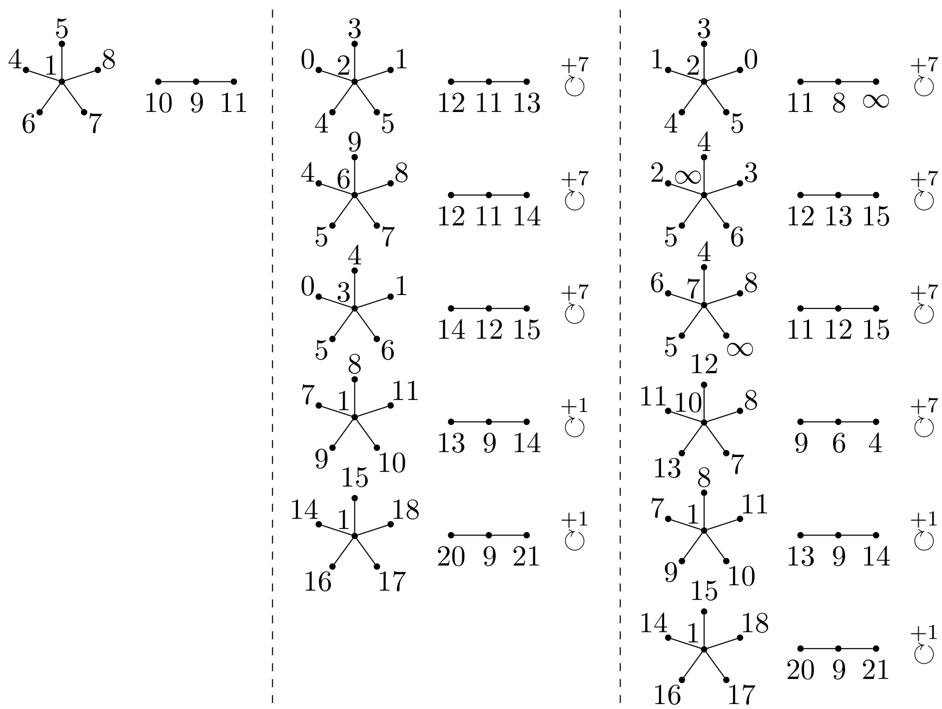

Figure 2.4: A σ^{+-}-labeling of $F \cong \mathbf{T_6^6} \sqcup \mathbf{T_3^1}$ (left) and generating presentations for the F-decomposition of K_n where $n = 35$ (middle) and $n = 36$ (right)

Chapter 3

$\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$

We begin this case by constructing K_n for $n \equiv 7$ or $8 \pmod{14}$ and $n \geq 21$ using *joined* copies of K_{22}, K_{21}, and K_{14}. Recall, the *join* of two graphs G_1 and G_2 is the graph obtained by adding an edge $\{g_1, g_2\}$ for every vertex $g_1 \in V(G_1)$ and every vertex of $g_2 \in V(G_2)$.

Let t be a positive integer and join $t-1$ copies of K_{14} with each other and a lone copy of K_{21}. The resulting graph is $K_{14(t-1)+21} \cong K_{14t+7}$. So we can think of K_{14t+7} as K_t whose t "vertices" consist of $t-1$ copies of K_{14} and 1 copy of K_{21} and whose edges are the join between them. From now on, we will refer to these "vertices" as nodes. Similarly, K_{14t+8} can be constructed as K_t whose nodes are $t-1$ copies of K_{14} and 1 copy of K_{22} and whose edges are the join between them.

We show that $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes K_n for $n \equiv 7$ or $8 \pmod{14}$ by proving that K_{22}, K_{21}, K_{14}, $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ are each $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposable. Notice that these 6 graphs make up the nodes and edges of the K_t representations of K_{14t+7} and K_{14t+8} stated in the constructions above.
The proof of the next theorem was obtained by manipulating a $K_{1,7}$-decomposition of K_{22} by Cain in [1].

Theorem 3.0.1. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* K_{21} *and* K_{22}.

Proof. Figures 8 and 9 show $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decompositions of K_{21} and K_{22}, respectively. $\square$

Theorem 3.0.2. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* $K_{n,7}$ *for all* $n \geq 2$.

Proof. Take the partite set of n nodes to be $\mathbb{Z}_n$ and color them white. Then, take the other partite set of 7 nodes to be $\mathbb{Z}_7$ and color them black. Notice that $|E(K_{n,7})| = |\mathbb{Z}_n \oplus \mathbb{Z}_7| = 7n$. So let us refer to edges of $K_{n,7}$ as elements of $\mathbb{Z}_n \oplus \mathbb{Z}_7$ and vice versa. Note that since $n \geq 2, (1,0) \neq (0,0)$.

Now, let $E_i = (i,0) + \{(0,0),(1,1),(1,2),\ldots,(1,6)\}$ for each $i \in \mathbb{Z}_n$ and F_i be the subgraph induced by E_i. Since each F_i contains a path $(i,0)$ which is vertex disjoint from the star centered at the white $i+1$, it must be isomorphic to $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$.

Suppose that there exist distinct $i, j \in \mathbb{Z}_n$ such that $E_i \cap E_j \neq \emptyset$. But then we have that $(i,0) = (j,0)$ or $(i+1,a) = (j+1,b)$ for some $a, b \in \mathbb{Z}_7$, which is impossible. So all distinct E_i's are pairwise disjoint, and therefore all distinct F_i's are pairwise edge-disjoint. Lastly, $\bigcup_{i \in \mathbb{Z}_n} E_i = \langle(1,0)\rangle + [\{(0,0)\} \cup [(1,0) + \langle(0,1)\rangle] \setminus \{(1,0)\}] = \langle(1,0)\rangle + \langle(0,1)\rangle = \langle(1,0),(0,1)\rangle = \mathbb{Z}_n \oplus \mathbb{Z}_7$. Therefore, $\bigcup_{i \in \mathbb{Z}_n} F_i = K_{n,7}$.

Thus, $\{F_i \mid i \in \mathbb{Z}_n\}$ is a $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}-$decomposition of $K_{n,7}$. Furthermore, This decomposition is generated by developing the white nodes of F_0 by 1. $\square$

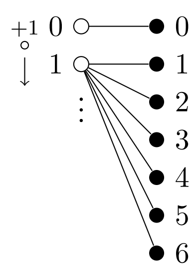

Figure 3.1: A generating presentation of the $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposition of $K_{n,7}$

Corollary 3.0.3. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* $K_{22,14}$, $K_{21,14}$, *and* $K_{14,14}$.

Proof. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes $K_{7,7}$ and $K_{8,7}$ by Theorem 3.0.2. $K_{14,14}$ can be expressed as the edge-disjoint union of four copies of $K_{7,7}$, $K_{21,14}$ can be expressed as the edge-disjoint union of six copies of $K_{7,7}$, and $K_{22,14}$ can be expressed as the edge-disjoint union of two copies of $K_{8,7}$ and four copies of $K_{7,7}$. Therefore, $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes them all. $\square$

Theorem 3.0.4. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ *decomposes* K_{14t+7} *and* K_{14t+8} *where* t *is a positive integer.*

Proof. $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes K_{14} by Theorem 2.0.7, $K_{22,14}$, $K_{21,14}$, and $K_{14,14}$ by Corollary 3.0.3, and lastly K_{22}, K_{21} by Theorem 3.0.1.

Therefore, $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes the join of $(t-1)$ copies of K_{14} with each other and 1 copy of K_{21}, which is isomorphic to K_{14t+7}. Similarly $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$ decomposes the join of $(t-1)$ copies of K_{14} with each other and 1 copy of K_{22} which is isomorphic to K_{14t+8}. $\square$

Chapter 4

Additional charts

Design Name	Graph Decomposition	Design Name	Graph Decomposition
$\mathbf{T_7^1} \sqcup \mathbf{T_2^1}$	$(0,1,2,4,6,9,12) \sqcup (13,14)$ $(3,4,7,9,10,13,15) \sqcup (8,5)$ $(8,11,12,10,7,5,6) \sqcup (1,3)$ $(0,4,9,15,8,16,7) \sqcup (1,11)$	$\mathbf{T_7^3} \sqcup \mathbf{T_2^1}$	$(12,9,6,4,2,1,7) \sqcup (14,15)$ $(15,13,10,9,7,4,11) \sqcup (8,5)$ $(8,11,12,10,7,5,13) \sqcup (1,3)$ $(16,8,15,9,4,0,6) \sqcup (1,11)$
$\mathbf{T_7^2} \sqcup \mathbf{T_2^1}$	$(0,1,2,4,6,9,3) \sqcup (16,19)$ $(15,13,10,9,7,4,14) \sqcup (17,18)$ $(6,5,7,10,12,11,8) \sqcup (18,15)$ $(7,16,8,15,9,4,12) \sqcup (1,11)$	$\mathbf{T_7^4} \sqcup \mathbf{T_2^1}$	$(8,6,4,2,1,9,7) \sqcup (14,15)$ $(8,10,9,7,4,11,13) \sqcup (12,15)$ $(9,12,10,7,5,11,13) \sqcup (1,4)$ $(7,15,9,4,0,8,6) \sqcup (1,11)$
$\mathbf{T_7^5} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12,8,7) \sqcup (11,14)$ $(0,2,3,6,5,1,4) \sqcup (8,7)$ $(0,3,5,4,1,8,7) \sqcup (16,15)$ $(4,9,15,8,12,6,7) \sqcup (1,11)$	$\mathbf{T_7^8} \sqcup \mathbf{T_2^1}$	$(1,2,4,6,8,5,9) \sqcup (12,15)$ $(4,7,9,10,11,8,13) \sqcup (1,3)$ $(5,7,10,12,11,6,13) \sqcup (1,4)$ $(0,4,9,15,8,12,6) \sqcup (1,11)$
$\mathbf{T_7^9} \sqcup \mathbf{T_2^1}$	$(8,6,4,2,5,9,7) \sqcup (12,14)$ $(1,3,2,0,5,4,6) \sqcup (10,12)$ $(9,8,7,10,4,11,5) \sqcup (12,13)$ $(7,15,9,4,13,8,6) \sqcup (1,11)$	$\mathbf{T_7^{10}} \sqcup \mathbf{T_2^1}$	$(7,6,4,2,8,9,5) \sqcup (12,14)$ $(2,3,4,7,0,5,6) \sqcup (9,12)$ $(7,8,5,4,9,10,11) \sqcup (0,2)$ $(6,15,9,4,8,11,7) \sqcup (2,12)$
$\mathbf{T_7^6} \sqcup \mathbf{T_2^1}$	$(2,4,6,8,7,9,12) \sqcup (13,14)$ $(0,2,3,4,7,6,5) \sqcup (8,10)$ $(0,3,5,8,9,4,1) \sqcup (12,14)$ $(4,9,15,8,12,7,16) \sqcup (1,11)$	$\mathbf{T_7^7} \sqcup \mathbf{T_2^1}$	$(2,4,6,9,12,1,8) \sqcup (14,15)$ $(5,6,3,2,0,7,4) \sqcup (8,9)$ $(0,3,5,4,7,1,8) \sqcup (12,14)$ $(4,9,15,8,12,18,7) \sqcup (1,11)$
$\mathbf{T_6^1} \sqcup \mathbf{T_3^1}$	$(1,2,4,6,9,12) \sqcup (13,14,15)$ $(3,4,7,9,10,13) \sqcup (5,8,6)$ $(11,12,10,7,5,6) \sqcup (3,1,4)$ $(0,4,9,15,8,16) \sqcup (1,11,2)$	$\mathbf{T_6^2} \sqcup \mathbf{T_3^1}$	$(1,2,4,6,9,5) \sqcup (13,14,15)$ $(13,10,9,7,4,11) \sqcup (5,8,6)$ $(11,12,10,7,5,13) \sqcup (3,1,4)$ $(0,4,9,15,8,12) \sqcup (1,11,2)$
$\mathbf{T_6^3} \sqcup \mathbf{T_3^1}$	$(0,1,2,4,6,5) \sqcup (16,13,14)$ $(8,6,3,2,0,4) \sqcup (14,12,15)$ $(7,4,5,3,0,6) \sqcup (10,8,11)$ $(7,0,4,9,15,12) \sqcup (1,11,2)$	$\mathbf{T_6^4} \sqcup \mathbf{T_3^1}$	$(1,2,5,4,6,7) \sqcup (16,14,13)$ $(8,6,9,3,2,4) \sqcup (14,12,15)$ $(4,5,6,3,0,1) \sqcup (11,8,7)$ $(7,0,6,4,9,12) \sqcup (1,11,2)$
$\mathbf{T_6^5} \sqcup \mathbf{T_3^1}$	$(0,2,4,7,1,5) \sqcup (12,11,13)$ $(7,6,3,2,8,9) \sqcup (14,12,15)$ $(4,3,5,6,0,1) \sqcup (11,8,7)$ $(8,0,4,9,6,7) \sqcup (1,11,2)$	$\mathbf{T_6^6} \sqcup \mathbf{T_3^1}$	$(0,2,1,3,4,5) \sqcup (12,11,14)$ $(4,6,8,9,5,7) \sqcup (14,12,15)$ $(0,3,1,4,5,6) \sqcup (11,8,7)$ $(4,0,8,5,6,7) \sqcup (1,11,2)$
$\mathbf{T_5^1} \sqcup \mathbf{T_4^1}$	$(2,4,6,9,12) \sqcup (16,15,14,13)$ $(3,4,7,9,10) \sqcup (11,12,15,13)$ $(12,10,7,5,6) \sqcup (18,15,17,20)$ $(4,9,15,8,16) \sqcup (2,11,1,5)$	$\mathbf{T_5^2} \sqcup \mathbf{T_4^1}$	$(12,9,6,4,11) \sqcup (17,16,15,14)$ $(9,7,4,3,6) \sqcup (11,12,15,13)$ $(6,5,7,10,3) \sqcup (18,15,17,20)$ $(16,8,15,9,12) \sqcup (2,11,1,6)$
$\mathbf{T_5^2} \sqcup \mathbf{T_4^2}$	$(4,6,9,11,8) \sqcup (16,15,18,14)$ $(9,7,4,3,6) \sqcup (16,17,20,15)$ $(6,5,7,10,3) \sqcup (9,12,11,15)$ $(16,8,15,9,12) \sqcup (10,1,11,6)$	$\mathbf{T_5^3} \sqcup \mathbf{T_4^1}$	$(13,15,16,18,14) \sqcup (11,9,6,7)$ $(14,17,16,20,15) \sqcup (9,7,4,3)$ $(9,12,10,11,15) \sqcup (4,6,5,7)$ $(5,1,10,11,6) \sqcup (16,8,15,9)$

Design Name	Graph Decomposition	Design Name	Graph Decomposition
$\mathbf{T}_6^3 \sqcup 2\mathbf{T}_2^1$	$(0,1,2,4,7,5) \sqcup (9,6) \sqcup (8,10)$ $(8,6,3,2,0,4) \sqcup (5,7) \sqcup (12,13)$ $(6,4,5,3,0,8) \sqcup (13,14) \sqcup (18,15)$ $(7,0,4,9,15,12) \sqcup (1,11) \sqcup (5,14)$	$\mathbf{T}_6^4 \sqcup 2\mathbf{T}_2^1$	$(1,2,5,4,6,7) \sqcup (13,14) \sqcup (12,15)$ $(8,6,9,3,2,4) \sqcup (12,14) \sqcup (18,15)$ $(4,5,6,3,0,1) \sqcup (8,7) \sqcup (16,14)$ $(7,0,6,4,9,12) \sqcup (1,11) \sqcup (5,14)$
$\mathbf{T}_6^5 \sqcup 2\mathbf{T}_2^1$	$(0,2,4,7,1,5) \sqcup (11,13) \sqcup (12,15)$ $(7,6,3,2,8,9) \sqcup (11,12) \sqcup (1,4)$ $(4,3,5,6,0,1) \sqcup (8,7) \sqcup (12,14)$ $(8,0,4,9,6,7) \sqcup (1,11) \sqcup (5,14)$	$\mathbf{T}_6^6 \sqcup 2\mathbf{T}_2^1$	$(0,2,1,3,4,5) \sqcup (12,14) \sqcup (18,19)$ $(4,6,8,9,5,7) \sqcup (12,15) \sqcup (11,14)$ $(0,3,1,4,5,6) \sqcup (8,11) \sqcup (14,15)$ $(4,0,8,5,6,7) \sqcup (1,11) \sqcup (3,12)$
$\mathbf{T}_5^1 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(2,4,6,9,12) \sqcup (13,14,15) \sqcup (18,19)$ $(3,4,7,9,10) \sqcup (12,15,13) \sqcup (1,2)$ $(12,10,7,5,6) \sqcup (20,17,15) \sqcup (1,4)$ $(4,9,15,8,16) \sqcup (11,1,5) \sqcup (3,12)$	$\mathbf{T}_5^2 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(12,9,6,4,11) \sqcup (17,16,15) \sqcup (0,1)$ $(9,7,4,3,6) \sqcup (12,15,13) \sqcup (18,19)$ $(6,5,7,10,3) \sqcup (20,17,15) \sqcup (1,4)$ $(16,8,15,9,12) \sqcup (1,11,2) \sqcup (0,5)$
$\mathbf{T}_5^3 \sqcup \mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(13,15,16,18,14) \sqcup (9,6,7) \sqcup (2,4)$ $(14,17,16,20,15) \sqcup (3,4,7) \sqcup (11,13)$ $(9,12,10,11,15) \sqcup (6,5,7) \sqcup (0,2)$ $(5,1,10,11,6) \sqcup (8,15,9) \sqcup (4,12)$	$2\mathbf{T}_4^1 \sqcup \mathbf{T}_2^1$	$(4,6,9,12) \sqcup (16,15,14,13) \sqcup (19,20)$ $(9,7,4,3) \sqcup (11,12,15,13) \sqcup (16,17)$ $(12,10,7,5) \sqcup (18,15,17,20) \sqcup (9,11)$ $(9,15,8,16) \sqcup (2,11,1,5) \sqcup (12,7)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(11,9,6,7) \sqcup (16,15,13,14) \sqcup (1,4)$ $(5,3,4,7) \sqcup (16,17,20,15) \sqcup (0,2)$ $(4,6,5,7) \sqcup (9,12,11,15) \sqcup (0,3)$ $(16,8,15,9) \sqcup (10,1,11,6) \sqcup (0,4)$	$2\mathbf{T}_4^2 \sqcup \mathbf{T}_2^1$	$(18,15,13,14) \sqcup (11,9,12,6) \sqcup (1,2)$ $(18,17,20,15) \sqcup (9,7,10,4) \sqcup (2,3)$ $(11,12,14,15) \sqcup (4,6,5,7) \sqcup (17,19)$ $(11,1,5,6) \sqcup (16,8,14,15) \sqcup (0,9)$
$\mathbf{T}_4^1 \sqcup 2\mathbf{T}_3^1$	$(16,15,14,13) \sqcup (0,3,5) \sqcup (12,9,6)$ $(11,12,15,13) \sqcup (10,9,7) \sqcup (16,18,20)$ $(18,15,17,20) \sqcup (10,11,14) \sqcup (6,5,7)$ $(2,12,3,11) \sqcup (8,1,7) \sqcup (4,0,5)$	$\mathbf{T}_4^2 \sqcup 2\mathbf{T}_3^1$	$(11,9,12,6) \sqcup (18,15,13) \sqcup (0,1,2)$ $(9,7,10,4) \sqcup (18,17,20) \sqcup (1,3,2)$ $(11,12,14,15) \sqcup (4,6,7) \sqcup (17,19,20)$ $(16,8,14,15) \sqcup (11,1,6) \sqcup (9,0,4)$
$\mathbf{T}_4^1 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(8,6,9,11) \sqcup (0,1,2) \sqcup (16,19) \sqcup (18,15)$ $(8,10,7,9) \sqcup (18,17,20) \sqcup (11,14) \sqcup (2,3)$ $(13,11,12,14) \sqcup (17,19,20) \sqcup (6,7) \sqcup (8,5)$ $(0,5,1,7) \sqcup (3,10,2) \sqcup (4,13) \sqcup (16,6)$	$\mathbf{T}_4^2 \sqcup \mathbf{T}_3^1 \sqcup 2\mathbf{T}_2^1$	$(11,9,12,6) \sqcup (0,1,2) \sqcup (18,15) \sqcup (13,14)$ $(9,7,10,4) \sqcup (18,17,20) \sqcup (11,13) \sqcup (2,3)$ $(11,12,14,15) \sqcup (17,19,20) \sqcup (8,6) \sqcup (1,3)$ $(4,0,5,6) \sqcup (8,1,9) \sqcup (3,12) \sqcup (17,7)$
$\mathbf{T}_5^1 \sqcup 3\mathbf{T}_2^1$	$(2,4,6,9,12) \sqcup (13,14) \sqcup (18,19) \sqcup (0,1)$ $(3,4,7,9,10) \sqcup (13,15) \sqcup (1,2) \sqcup (8,5)$ $(6,5,7,10,12) \sqcup (17,20) \sqcup (8,11) \sqcup (1,3)$ $(4,9,15,8,16) \sqcup (1,11) \sqcup (3,12) \sqcup (2,6)$	$\mathbf{T}_5^2 \sqcup 3\mathbf{T}_2^1$	$(11,9,6,4,12) \sqcup (16,15) \sqcup (8,10) \sqcup (2,3)$ $(6,7,4,3,9) \sqcup (13,15) \sqcup (18,19) \sqcup (8,5)$ $(3,5,7,10,6) \sqcup (17,20) \sqcup (8,11) \sqcup (0,1)$ $(12,8,15,9,16) \sqcup (2,11) \sqcup (0,5) \sqcup (3,13)$
$\mathbf{T}_5^3 \sqcup 3\mathbf{T}_2^1$	$(13,15,16,18,14) \sqcup (9,6) \sqcup (2,4) \sqcup (5,7)$ $(14,17,16,20,15) \sqcup (4,7) \sqcup (11,13) \sqcup (5,6)$ $(9,12,10,11,15) \sqcup (6,7) \sqcup (0,2) \sqcup (3,4)$ $(5,1,10,11,6) \sqcup (9,15) \sqcup (4,12) \sqcup (0,7)$	$3\mathbf{T}_3^1 \sqcup \mathbf{T}_2^1$	$(18,15,13) \sqcup (11,9,6) \sqcup (0,1,2) \sqcup (16,19)$ $(18,17,20) \sqcup (9,7,10) \sqcup (1,3,2) \sqcup (11,14)$ $(11,12,14) \sqcup (4,6,7) \sqcup (17,19,20) \sqcup (8,5)$ $(11,1,6) \sqcup (16,8,14) \sqcup (9,0,4) \sqcup (10,3)$
$\mathbf{T}_4^1 \sqcup 4\mathbf{T}_2^1$	$(9,6,4,2) \sqcup (13,14) \sqcup (18,19) \sqcup (0,1) \sqcup (10,12)$ $(9,7,4,3) \sqcup (13,15) \sqcup (1,2) \sqcup (8,5) \sqcup (16,17)$ $(10,7,5,6) \sqcup (17,20) \sqcup (8,11) \sqcup (1,3) \sqcup (9,12)$ $(9,15,8,16) \sqcup (1,11) \sqcup (3,12) \sqcup (2,6) \sqcup (0,5)$	$\mathbf{T}_4^2 \sqcup 4\mathbf{T}_2^1$	$(16,15,18,13) \sqcup (9,6) \sqcup (2,4) \sqcup (5,7) \sqcup (0,1)$ $(16,17,20,14) \sqcup (4,7) \sqcup (11,13) \sqcup (5,6) \sqcup (1,3)$ $(9,12,10,11) \sqcup (6,7) \sqcup (0,2) \sqcup (3,4) \sqcup (8,5)$ $(10,1,11,5) \sqcup (9,15) \sqcup (4,12) \sqcup (0,7) \sqcup (8,3)$
$2\mathbf{T}_3^1 \sqcup 3\mathbf{T}_2^1$	$(11,9,6) \sqcup (0,1,2) \sqcup (18,15) \sqcup (16,19) \sqcup (17,20)$ $(9,7,10) \sqcup (1,3,2) \sqcup (17,18) \sqcup (11,14) \sqcup (8,5)$ $(11,12,14) \sqcup (4,6,7) \sqcup (19,20) \sqcup (13,15) \sqcup (3,5)$ $(11,1,6) \sqcup (16,8,14) \sqcup (0,9) \sqcup (10,3) \sqcup (17,13)$	$\mathbf{T}_3^1 \sqcup 5\mathbf{T}_2^1$	$(0,1,2) \sqcup (18,15) \sqcup (9,11) \sqcup (16,19) \sqcup (5,6) \sqcup (10,7)$ $(1,3,2) \sqcup (17,18) \sqcup (9,7) \sqcup (11,14) \sqcup (8,5) \sqcup (16,13)$ $(4,6,7) \sqcup (12,14) \sqcup (3,5) \sqcup (13,15) \sqcup (17,20) \sqcup (18,19)$ $(16,8,14) \sqcup (1,11) \sqcup (0,9) \sqcup (10,3) \sqcup (17,13) \sqcup (2,7)$

Figure 4.1: (1-2-3)-labelings

4.1 $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decompositions of K_{21} and K_{22}

Element	Graph	Element	Graph
G_1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	G_2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
G_3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	G_4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
G_5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	G_6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
G_7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (18, 7)$	G_8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
G_9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	G_{10}	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
G_{11}	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	G_{12}	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
G_{13}	$(1, 7, 2, 3, 4, 5, 6) \sqcup (0, 14)$	G_{14}	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
G_{15}	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	G_{16}	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$
G_{17}	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	G_{18}	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
G_{19}	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	G_{20}	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$
G_{21}	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	G_{22}	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
G_{23}	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	G_{24}	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
G_{25}	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	G_{26}	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
G_{27}	$(9, 7, 0, 10, 11, 12, 13) \sqcup (1, 3)$	G_{28}	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
G_{29}	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	G_{30}	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$

Figure 4.2: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposition of K_{21}

Element	Graph	Element	Graph
G_1	$(15, 14, 16, 17, 18, 19, 20) \sqcup (0, 2)$	G_2	$(13, 15, 16, 17, 18, 19, 20) \sqcup (0, 6)$
G_3	$(8, 16, 12, 17, 18, 19, 20) \sqcup (9, 3)$	G_4	$(8, 17, 9, 11, 18, 19, 20) \sqcup (16, 0)$
G_5	$(8, 18, 9, 11, 13, 19, 20) \sqcup (0, 1)$	G_6	$(8, 19, 10, 11, 12, 13, 20) \sqcup (0, 15)$
G_7	$(8, 1, 9, 10, 11, 12, 13) \sqcup (6, \infty)$	G_8	$(1, 2, 9, 10, 11, 12, 13) \sqcup (14, 7)$
G_9	$(0, 3, 2, 6, 11, 12, 13) \sqcup (8, 7)$	G_{10}	$(0, 4, 2, 3, 11, 12, 13) \sqcup (8, 9)$
G_{11}	$(0, 5, 2, 3, 4, 12, 13) \sqcup (9, 10)$	G_{12}	$(1, 6, 2, 4, 5, 12, 13) \sqcup (15, 7)$
G_{13}	$(1, 7, 2, 3, 4, 5, 6) \sqcup (13, \infty)$	G_{14}	$(3, 8, 4, 5, 6, 14, 20) \sqcup (12, 15)$
G_{15}	$(4, 9, 5, 6, 14, 15, 20) \sqcup (16, 7)$	G_{16}	$(15, 10, 4, 5, 6, 16, 20) \sqcup (0, 18)$
G_{17}	$(15, 11, 0, 5, 6, 16, 20) \sqcup (17, 1)$	G_{18}	$(14, 12, 0, 11, 17, 18, 20) \sqcup (8, 2)$
G_{19}	$(16, 13, 0, 11, 12, 17, 20) \sqcup (1, 19)$	G_{20}	$(1, 14, 2, 3, 4, 5, 6) \sqcup (20, 7)$
G_{21}	$(1, 15, 2, 3, 4, 5, 6) \sqcup (19, 7)$	G_{22}	$(1, 16, 2, 3, 4, 5, 6) \sqcup (17, 7)$
G_{23}	$(0, 17, 2, 3, 4, 5, 6) \sqcup (11, 14)$	G_{24}	$(1, 18, 2, 3, 4, 5, 6) \sqcup (10, 14)$
G_{25}	$(0, 19, 2, 3, 4, 5, 6) \sqcup (13, 14)$	G_{26}	$(0, 20, 2, 3, 4, 5, 6) \sqcup (10, 11)$
G_{27}	$(9, 7, 0, 10, 11, 12, 13) \sqcup (20, \infty)$	G_{28}	$(10, 8, 0, 11, 12, 13, 15) \sqcup (1, 4)$
G_{29}	$(11, 9, 0, 12, 13, 16, 19) \sqcup (1, 5)$	G_{30}	$(12, 10, 0, 3, 13, 17, 18) \sqcup (1, 20)$
G_{31}	$(0, \infty, 1, 2, 3, 4, 5) \sqcup (18, 7)$	G_{32}	$(14, \infty, 15, 16, 17, 18, 19) \sqcup (1, 3)$
G_{33}	$(7, \infty, 8, 9, 10, 11, 12) \sqcup (0, 14)$		

Figure 4.3: A $\mathbf{T_7^{11}} \sqcup \mathbf{T_2^1}$-decomposition of K_{22}

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Chapter 5

Conclusion and Discussion

Appendix A

Glossary and Acronyms

Care has been taken in this thesis to minimize the use of jargon and acronyms, but this cannot always be achieved. This appendix defines jargon terms in a glossary, and contains a table of acronyms and their meaning.

A.1 Glossary

- **Cosmic-Ray Muon** (**CR** μ) – A muon coming from the abundant energetic particles originating outside of the Earth's atmosphere.

A.2 Acronyms

Table A.1: Acronyms

Acronym	Meaning
CRμ	Cosmic-Ray Muon