Image Restoration

14.05.2014

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visual inference

Image Restoration

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- Now we will shift gears and look at another problem domain: **Image restoration**
  - We will see that we can use a number of techniques that are very similar to what we have used in stereo.
- We will look at 3 applications:
  - Image denoising.
  - Image inpainting.
  - Super-resolution.

Image Denoising

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- Essentially all digital images exhibit **image noise** to some degree.
  - Even very high quality images contain some noise.
  - Really noisy images may look like they are dominated by noise.
- Image noise is both:
  - Visually disturbing for the human viewer.
  - Distracting for vision algorithms.
- Image denoising aims at removing or reducing the amount of noise in the image.

Image Noise

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- Image noise is a very common phenomenon that can come from a variety of sources:
  - Shot noise or photon noise due to the stochastic nature of the photons arriving at the sensor.
    - Cameras actually count photons!
  - Thermal noise (“fake” photon detections).
  - Processing noise within CMOS or CCD, or in camera electronics.
  - If we use “analog” film, there are physical particles of a finite size that cause noise in a digital scan.

Shot Noise

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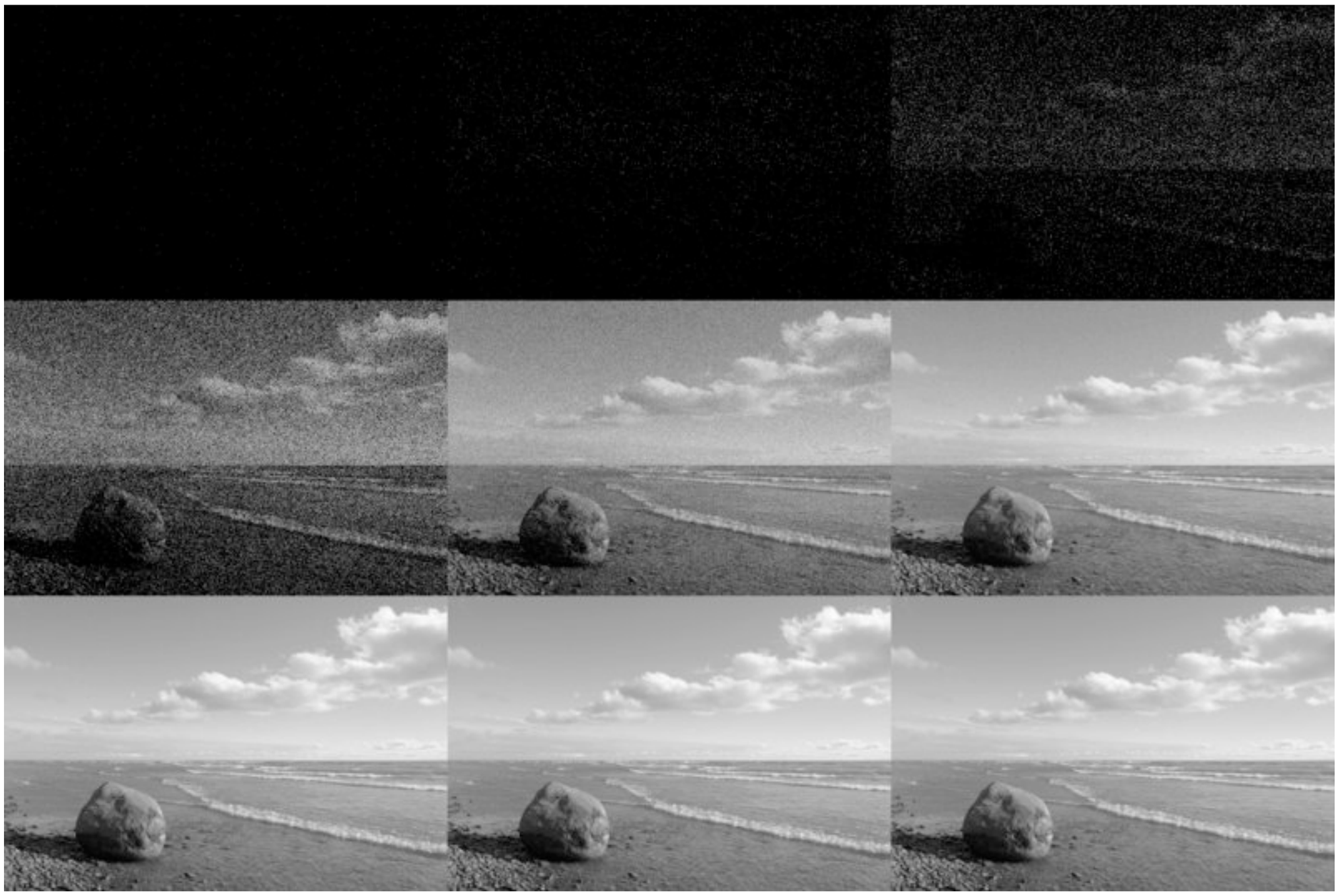

Simulated shot noise (Poisson process)

From Wikipedia

Thermal Noise or “Dark Noise”

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62 minute exposure with no incident light

From Jeff Medkeff (photo.net)

Bias Noise from Amplifiers

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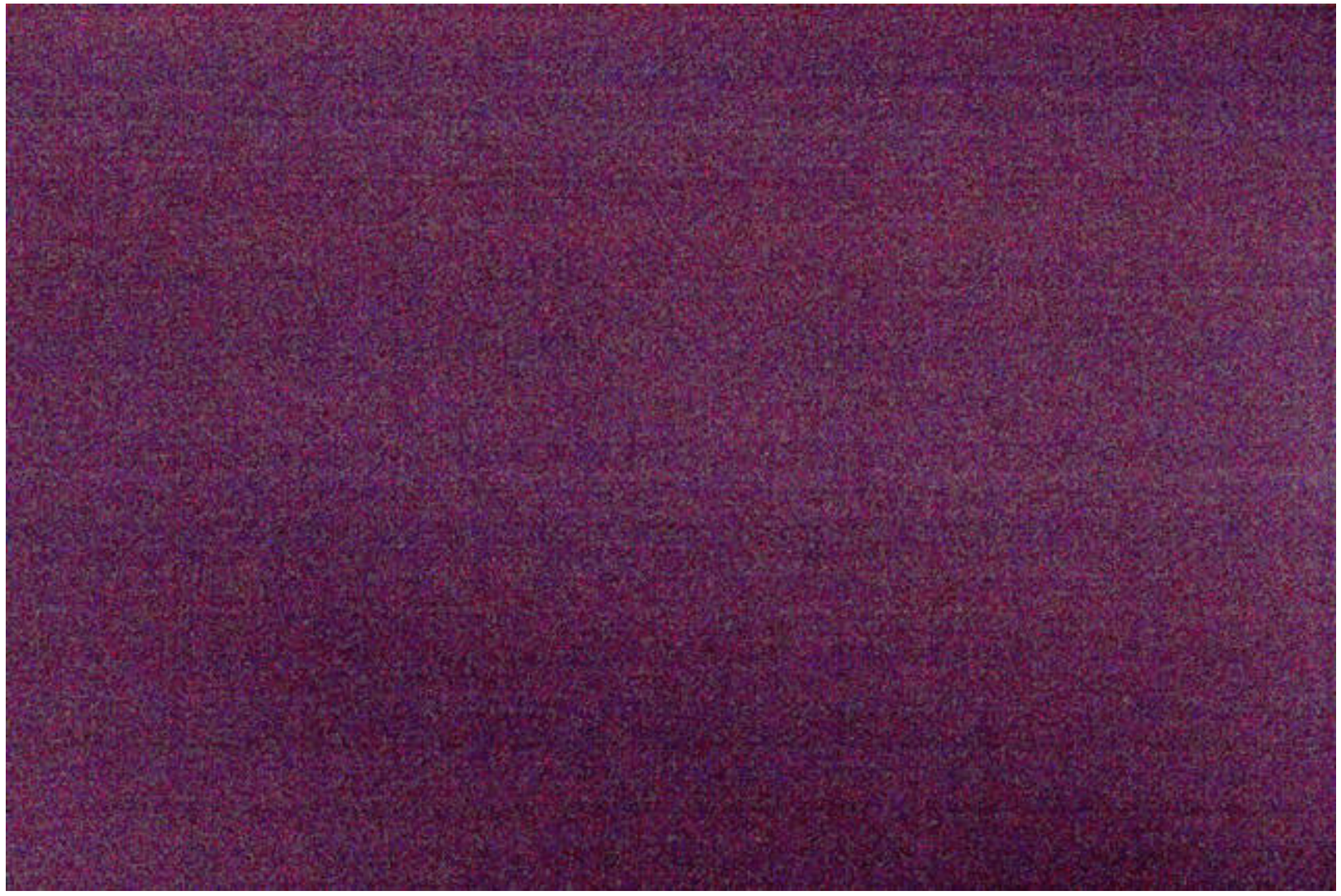

High ISO exposure (3200)

From Jeff Medkeff (photo.net)

Noise in Real Digital Photographs

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- Combination of many different noise sources:

From dcresource.com

Film Grain

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- If we make a digital scan of a film, we get noise from the small silver particles on the film strip:

Das Testament des Dr. Mabuse, 1933

How do we remove image noise?

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- Classical techniques:
  - Linear filtering, e.g. Gauss filtering.
  - Median filtering
  - Wiener filtering
  - Etc.
- Modern techniques:
  - PDE-based techniques
  - Wavelet techniques
  - **MRF-based techniques**
  - Etc.

Linear Filtering

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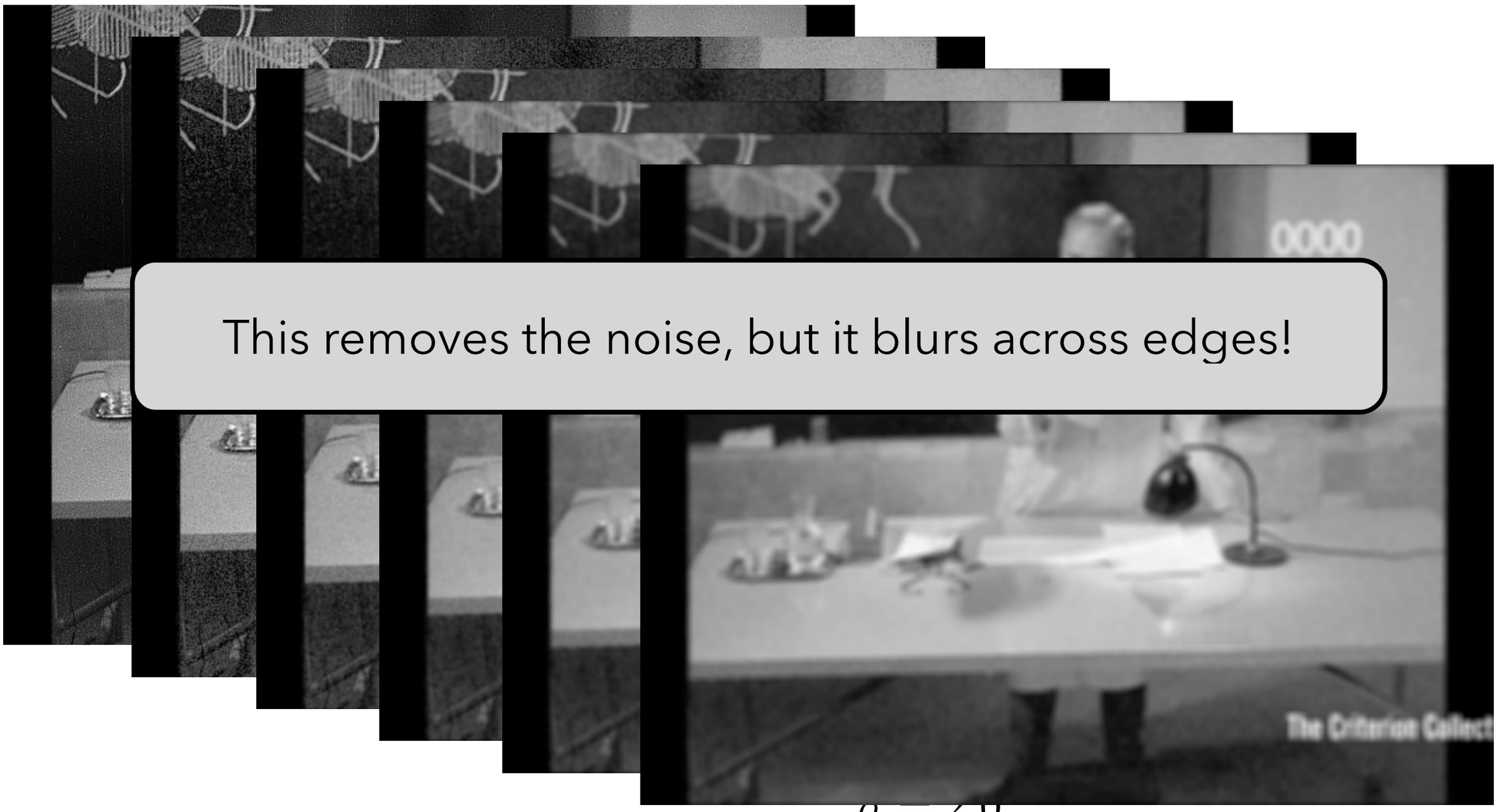
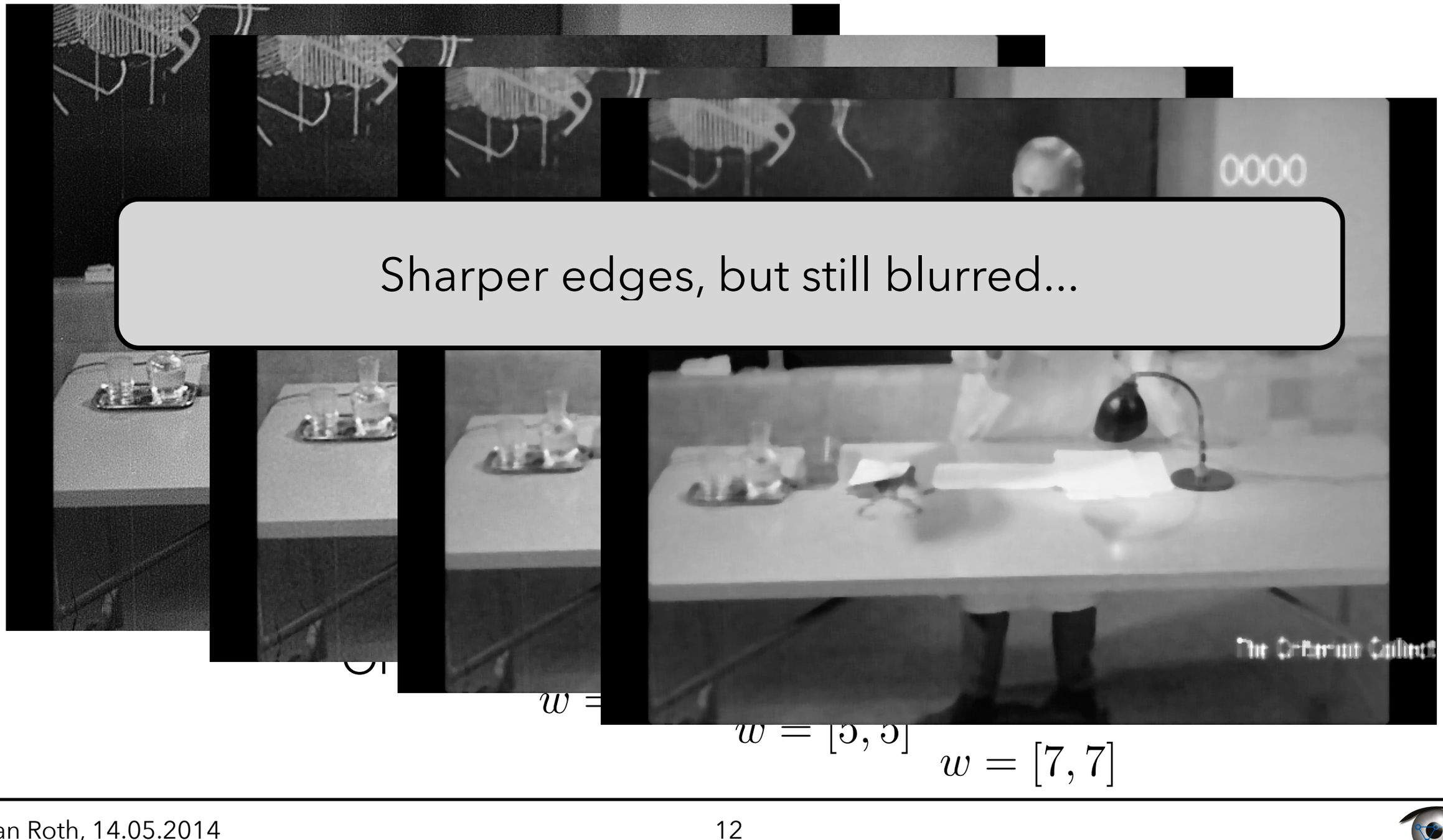
- The simplest idea is to use a Gaussian filter:



This removes the noise, but it blurs across edges!

Median Filter

- Replace each pixel by the median of the pixel values in a window around it:



How do we improve on this?

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- We need denoising techniques that **better preserve boundaries** in images and do not blur across them.
- There is a whole host of modern denoising techniques:
  - We would need a whole semester to go through the important ones in detail.
  - So we will do the area of denoising some major injustice and restrict ourselves to what we can easily understand with what we have learned so far.

Denoising as Probabilistic Inference

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- We formulate the problem of image denoising in a Bayesian fashion as a problem of probabilistic inference.
- For that we model denoising using a suitable posterior distribution:

$$p(\text{true image}|\text{noisy image}) = p(\mathbf{T}|\mathbf{N})$$

- If we model this posterior appropriately, we can use the inference techniques from last class to estimate the true image that we want to recover.

Modeling the Posterior

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- We could model the posterior directly:
  - Again, this is a little more complicated.
  - Secondly, we would need to change the whole model if we want to make it work for a different type or even strength of noise.
- So instead, we apply Bayes' rule again and obtain:

likelihood of noisy given true image (observation model)

image prior for all true images

$$p(\mathbf{T}|\mathbf{N}) = \frac{p(\mathbf{N}|\mathbf{T}) \cdot p(\mathbf{T})}{p(\mathbf{N})}$$

posterior

normalization term (constant)

Modeling the Likelihood

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- Like in stereo, the likelihood expresses a model of the observation:
  - Given the true, noise free image, we assess how likely it is to observe a particular noisy image.
  - If we wanted to model particular real noise phenomena, we could model the likelihood based on real physical properties of the world.
  - Here, we will simplify things and only use a simple, generic noise model.
  - Nevertheless, our formulation allows us to easily adapt the noise model without having to change everything...

Modeling the Likelihood

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- Simplification: We assume that the noise at one pixel is **independent** of the others.

$$p(\mathbf{N}|\mathbf{T}) = \prod_{i,j} p(N_{ij}|T_{ij})$$

  - This is a pretty reasonable assumption, for example since sites of a CCD sensor are relatively independent.

- Then we will assume that the noise at each pixel is **additive and Gaussian distributed**:

$$p(N_{ij}|T_{ij}) = \mathcal{N}(N_{ij} - T_{ij}|0, \sigma^2)$$

  - The variance σ^2 controls the amount of noise.

Gaussian Image Likelihood

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- We can thus write the Gaussian image likelihood as:

$$p(\mathbf{N}|\mathbf{T}) = \prod_{i,j} \mathcal{N}(N_{ij} - T_{ij}|0, \sigma^2)$$

- While this may seem a bit hacky, it works well in many applications.
  - It is suboptimal when the noise is not really independent, such as in some high definition (HD) images.
  - It also is suboptimal when the noise is non-additive, or not really Gaussian, for example as with film grain noise.

Modeling the Prior

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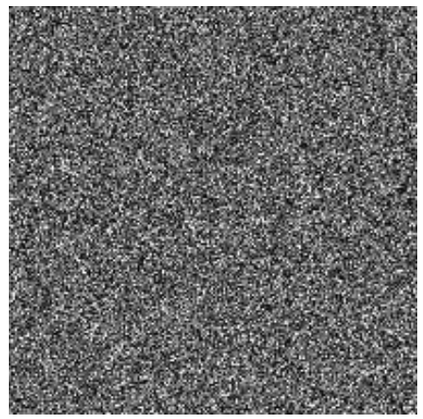
- How do we model the prior distribution of true images?
- What does that even mean?
  - We want the prior to describe how probable it is (a-priori) to have a particular true image among the set of all possible images.

probable



improbable

Natural Images

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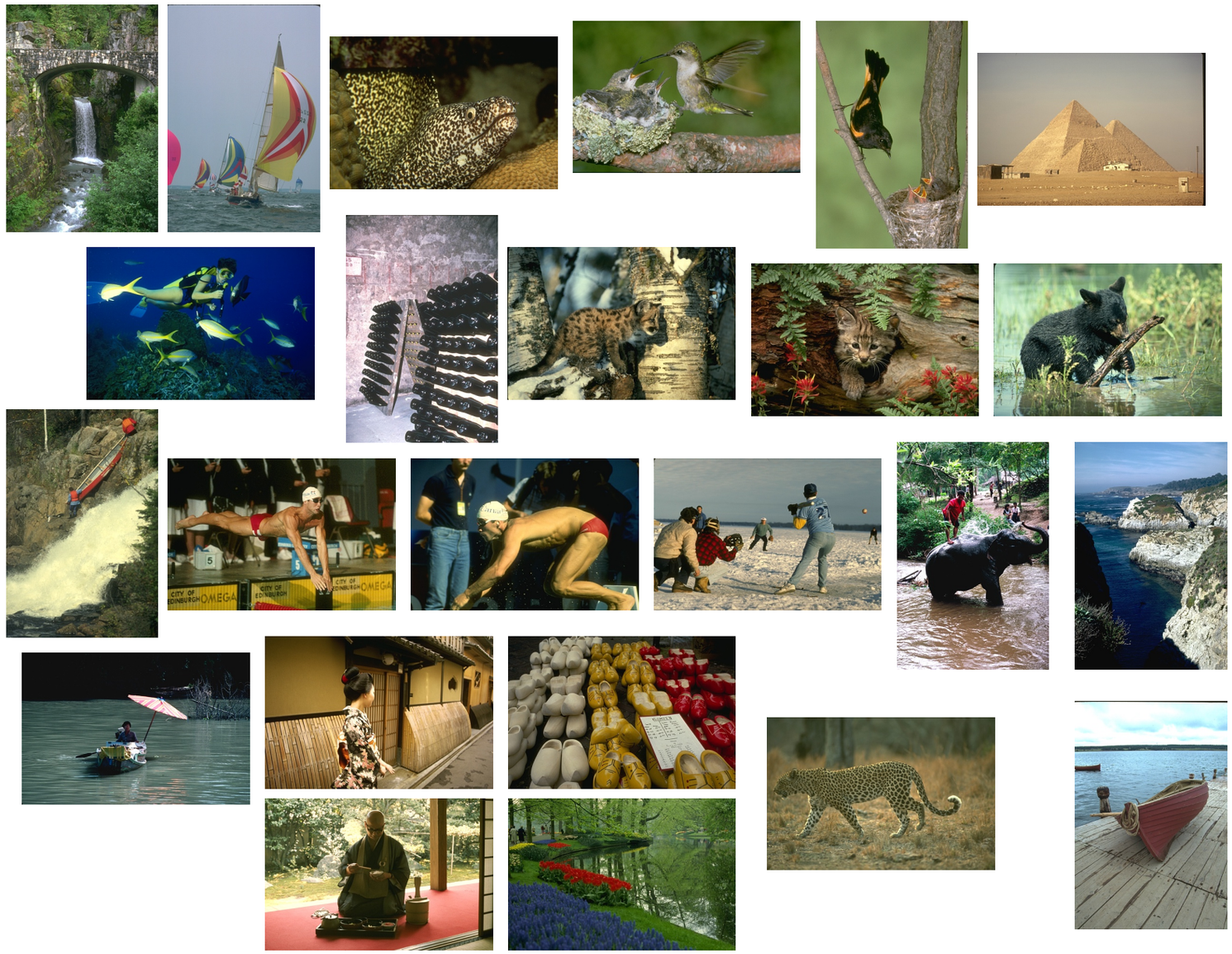
- What distinguishes “natural” images from “fake” ones?
  - We can take a large database of natural images and study them.



Simple Observation

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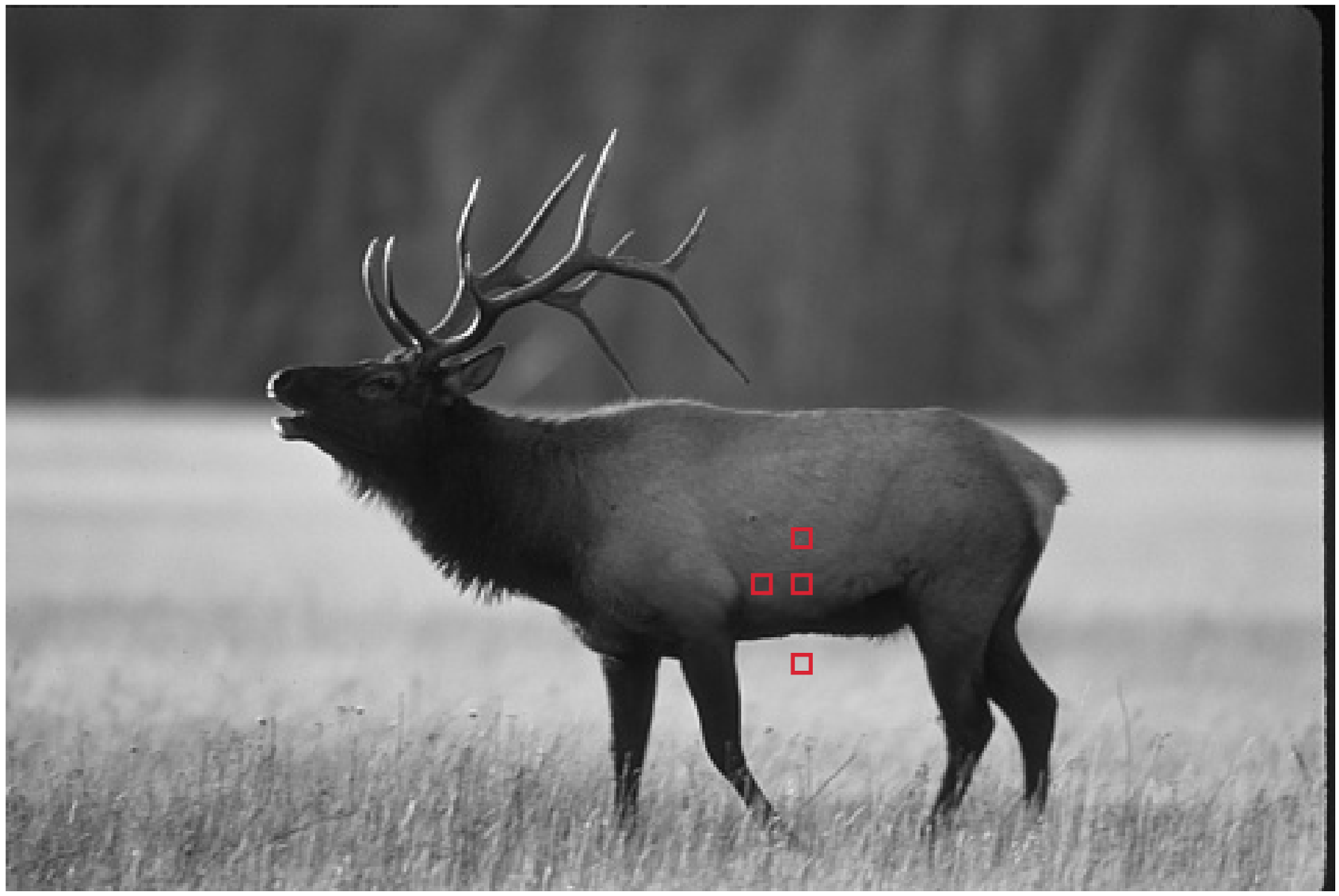
- Nearby pixels often have a similar intensity:



- But sometimes there are large intensity changes.

Statistics of Natural Images

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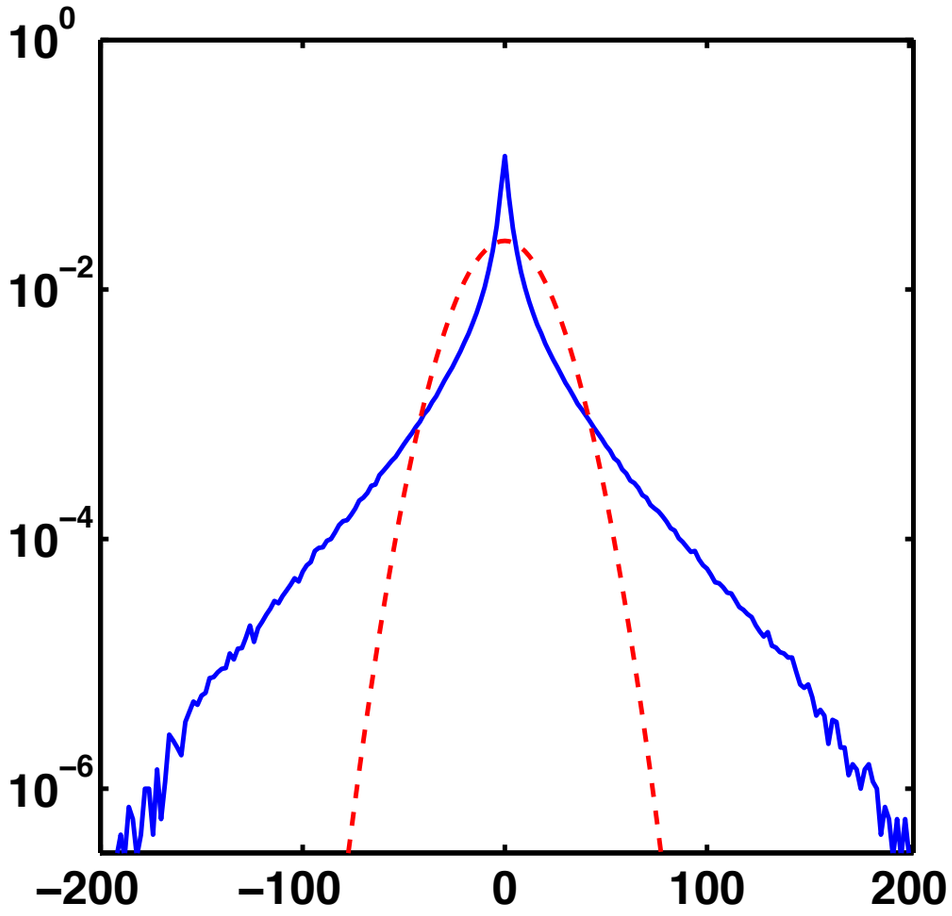
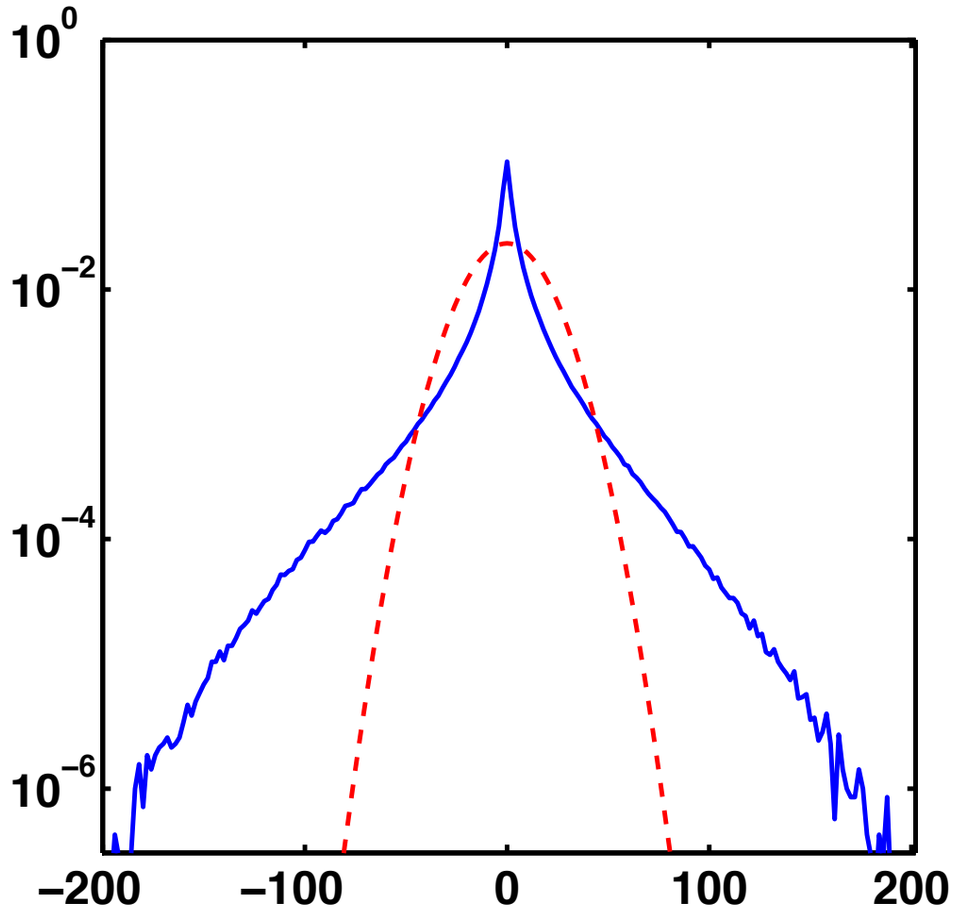
- Compute the image derivative of all images in an image database and plot a histogram:



x-derivative



y-derivative

empirical histogram

fit with a Gaussian

Statistics of Natural Images

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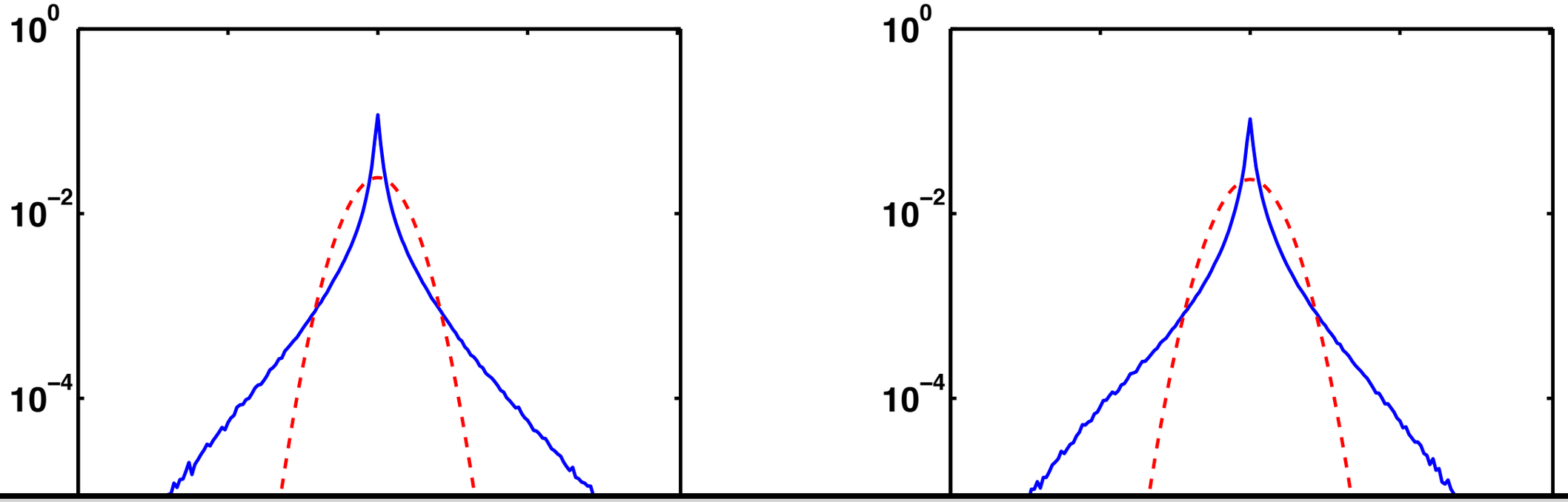
- Compute the image derivative of all images in an image database and plot a histogram:



- Sharp peak at zero: Neighboring pixels most often have identical intensities.
- Heavy tails: Sometimes, there are strong intensity differences due to discontinuities in the image.

MRF Model of Image Prior

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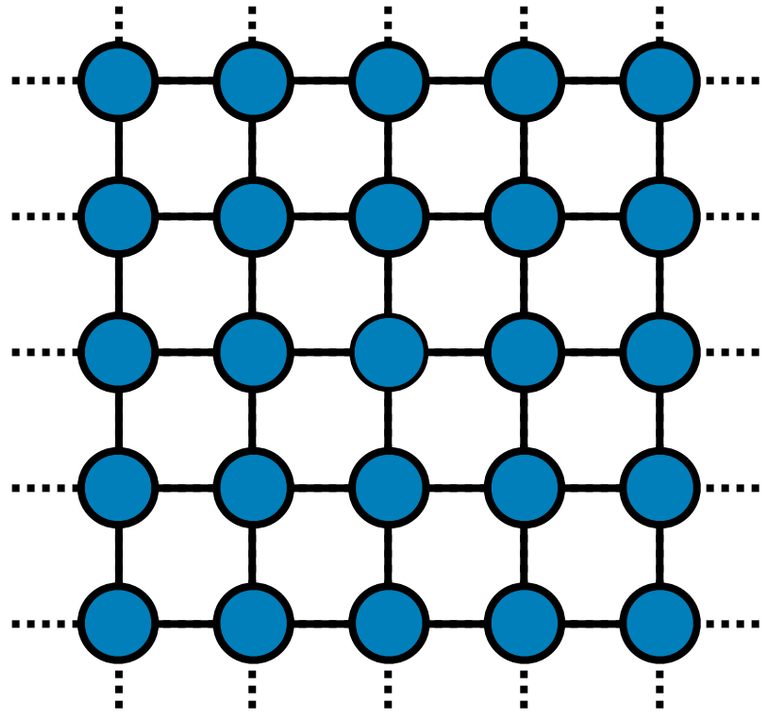
- We have seen something like this before!
- In stereo, we had almost the same challenge in modeling disparity.
  - So why don't we use the same formalism?
- We model the image prior as a Markov random field:

$$p(\mathbf{T}) = \prod_{i,j} f_H(T_{i,j}, T_{i+1,j}) \cdot f_V(T_{i,j}, T_{i,j+1})$$



Modeling the Potentials

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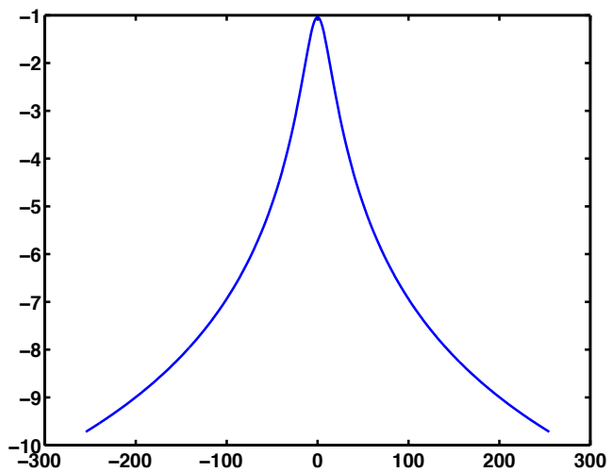
- We face a similar challenge in modeling the potentials (or compatibility functions), e.g.: $f_H(T_{i,j}, T_{i+1,j})$
- Gaussian distributions are inappropriate:
  - They do not match the statistics of natural images well.
  - They would lead to blurred discontinuities.
- We need **discontinuity-preserving potentials**:
  - One possibility: Student-t distribution.

$$f_H(T_{i,j}, T_{i+1,j}) = \left(1 + \frac{1}{2\sigma^2}(T_{i,j} - T_{i+1,j})^2\right)^{-\alpha}$$



log-density

MRF Model of the Posterior

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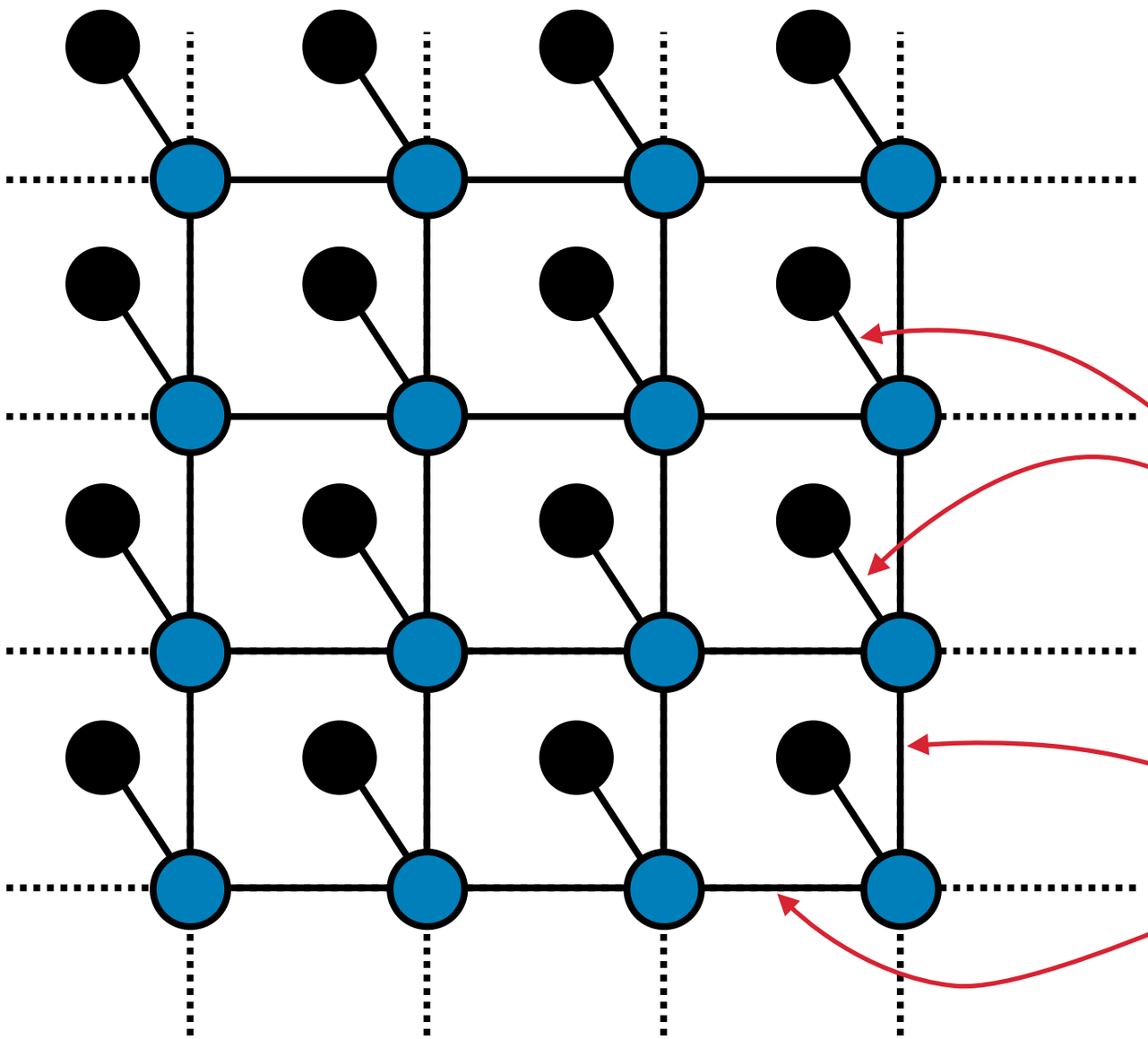
- We can now put the likelihood and the prior together in a single MRF model:



$T_{i,j}$ pixels of the true image (hidden)

$N_{i,j}$ pixels of the noisy image (observed)

Edges representing the likelihood

Edges representing the prior

Denoising as Inference

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- Because the likelihood is simpler here than in the stereo case, gradient techniques work quite well for inference.
  - Very simple to implement.
- Of course, we can also use graph cuts or belief propagation.
  - Can lead to slightly better results (in case of a pairwise prior).

Denoising Results

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original image

noisy image,
σ=20

PSNR 22.49dB
SSIM 0.528

denoised using
gradient ascent

PSNR 27.60dB
SSIM 0.810

Denoising Results

- Very sharp discontinuities. No blurring across boundaries.
- Noise is removed quite well nonetheless.

Denoising Results

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- Because the noisy image is based on synthetic noise, we can measure the performance:
  - PSNR: Peak signal-to-noise ratio
  - SSIM [Wang et al., 04]: Perceptual similarity
    - Gives an estimate of how humans would assess the quality of the image.

original image

noisy image, σ=20

PSNR 22.49dB
SSIM 0.528

denoised using gradient ascent

PSNR 27.60dB
SSIM 0.810

Is this the end of the story?

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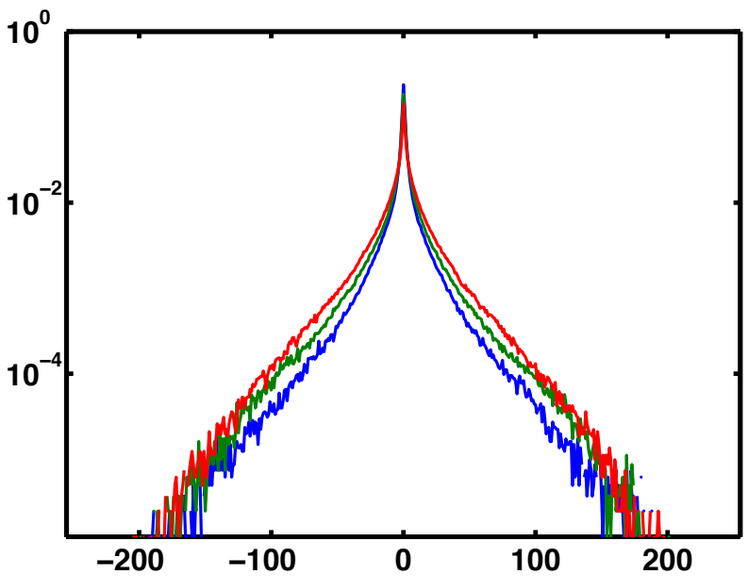
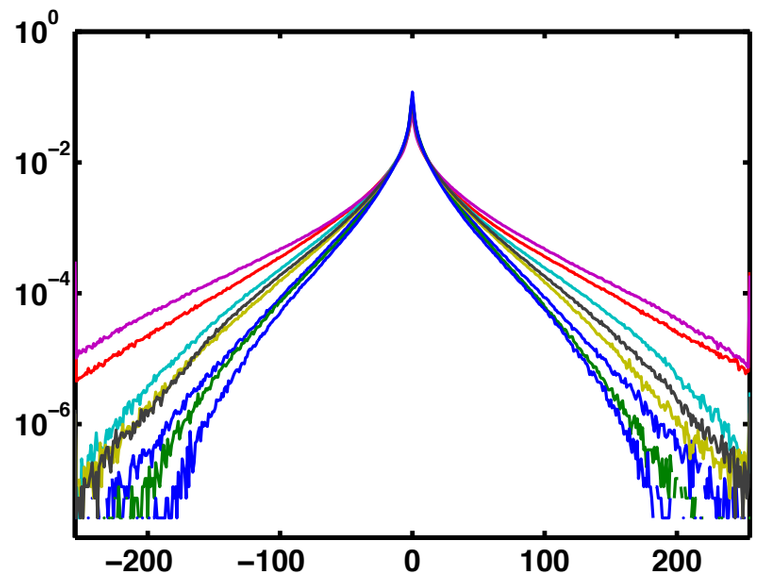
- No, natural images have many complex properties that we have not modeled.
  - For example, they have complex structural properties that are not modeled with a simple MRF based on a 4-connected grid.
  - Natural images have scale-invariant statistics, our model does not.
  - Responses to random linear filters are heavy-tailed.
  - Etc.



Derivative histogram on 4 spatial scales

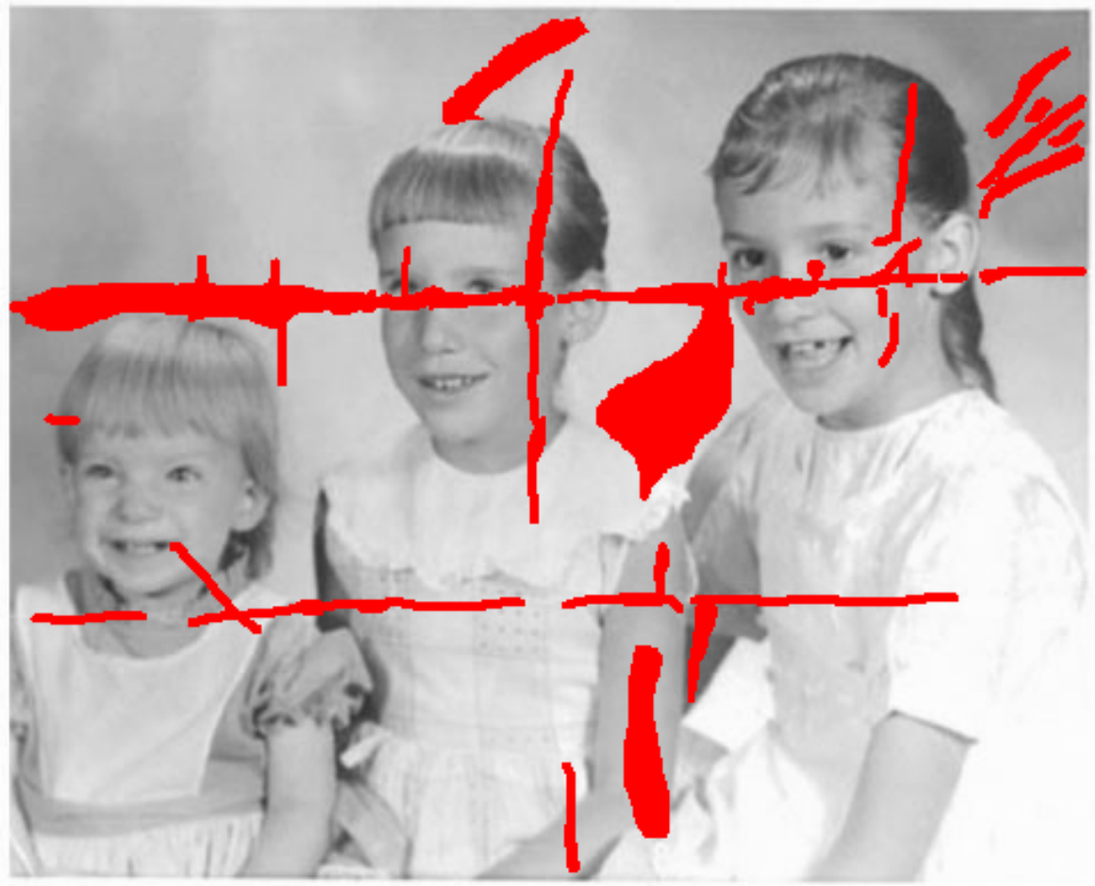


Histograms of random (zero-mean) linear filters

Image Inpainting

- In image inpainting the goal in to fill in a "missing" part of an image:
  - Restoration of old photographs, e.g. scratch removal...

old photograph



user-supplied mask

[Bertalmio et al., 2000]

Image Inpainting

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- There are many different ways to do inpainting:
  - PDEs: [Bertalmio et al, 2000], ...
  - Exemplar-based: [Criminisi et al., 2003], ...
  - And many more.
- But, we can also apply what we already know:
  - We model the problem in a Bayesian fashion, where we regard the inpainted image as the true image. We are given the corrupted image with missing pixels.

$$p(\text{true image}|\text{corrupted image}) = p(\mathbf{T}|\mathbf{C})$$

  - Then we apply probabilistic inference...

Image Inpainting

- We apply Bayes' rule:

$$p(\mathbf{T}|\mathbf{C}) = \frac{p(\mathbf{C}|\mathbf{T}) \cdot p(\mathbf{T})}{p(\mathbf{C})}$$

  - I know this may be boring, but this general approach really is this versatile...
- Modeling the prior:
  - **Important observation:** This is the very same prior that we needed in order to do denoising!
  - We can re-use the prior model from denoising here.
  - Once we have a good probabilistic model of images, we can use it in many applications!

Inpainting Likelihood

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- Again, we assume independence of the pixels:

$$p(\mathbf{C}|\mathbf{T}) = \prod_{i,j} p(C_{ij}|T_{ij})$$

- Desiderata:
  - We want to keep all known pixels fixed.
  - For all unknown pixels all intensities should be equally likely.
- Simple likelihood model:

$$p(C_{ij}|T_{ij}) = \begin{cases} \text{const}, & C_{ij} \text{ is corrupted} \\ \delta(T_{ij} - C_{ij}), & C_{ij} \text{ is not corrupted} \end{cases}$$

MRF Model of the Posterior

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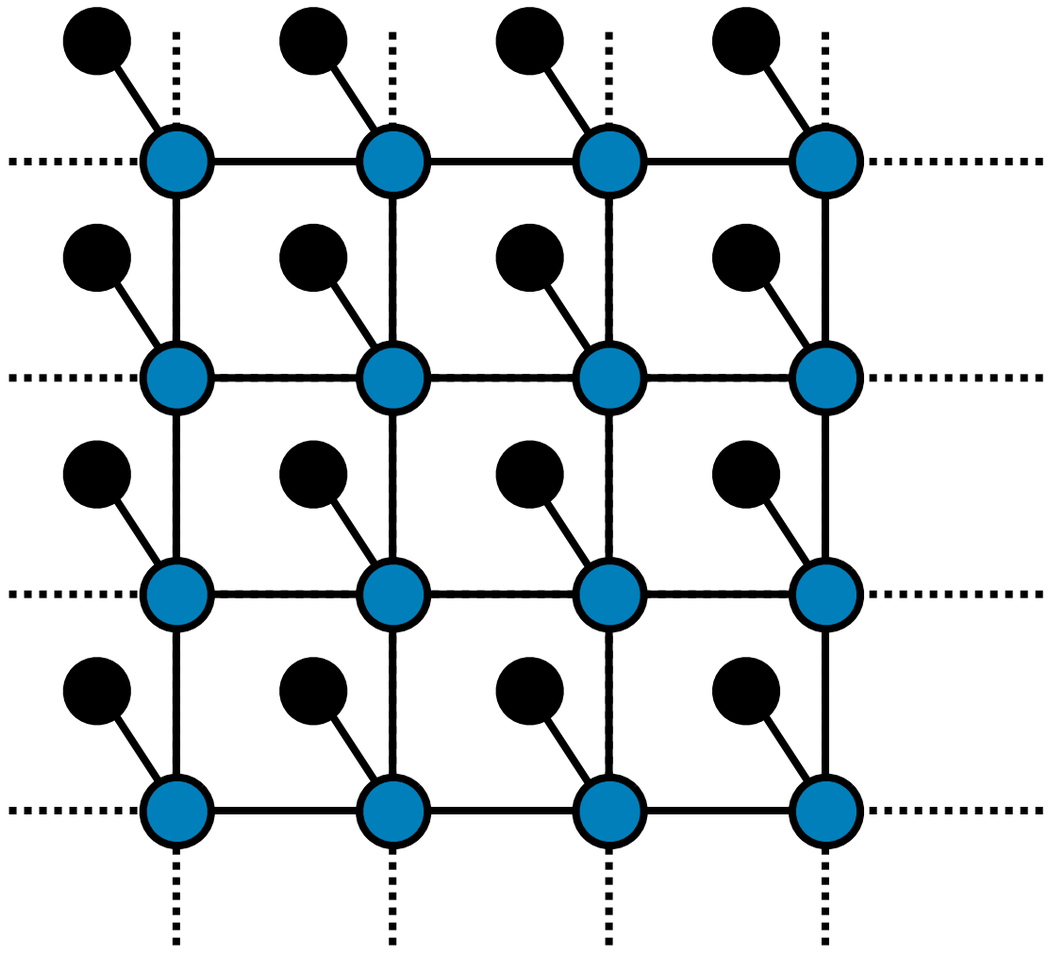
- The posterior for inpainting has the same graphical model structure as the one for denoising.
- Nonetheless, the potentials representing the likelihood are different.



Inpainting Results

"Corrupted" image (artificially corrupted for benchmarking)

Inpainted image obtained using gradient ascent

Other Inpainting Results

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From [Bertalmio et al., 2000]

Image Super-Resolution

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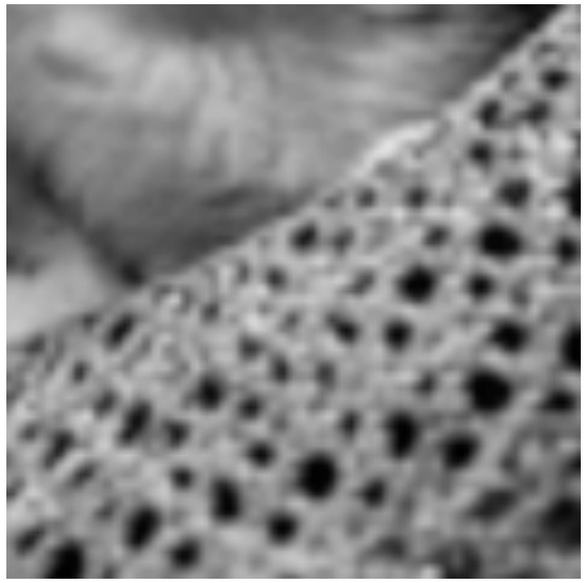
- In super-resolution, the goal is to increase the resolution of an image, while making the high-resolution image seem sharp and natural.

64x64 original image



128x128 super-resolved image (bicubic interpolation)

From [Tappen et al., 2003]

Image Super-Resolution

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- In super-resolution, the goal is to increase the resolution of an image, while making the high-resolution image seem sharp and natural.

- Many applications:
  - Resizing of old, low-res digital imagery.
  - Improving resolution of images from cheap cameras.
  - Up-conversion of standard definition TV and movie footage to high-definition (720p or 1080p).

Image Super-Resolution

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- The story parallels what we have seen before.
- There are many special purpose super-resolution techniques:
  - The simplest ones are bilinear and bicubic interpolation.
  - Example-based methods.
  - Etc.
- But once again, we can formulate the problem in a Bayesian way and super-resolve images using probabilistic inference:

$$p(\text{high res. image}|\text{low res. image}) = p(\mathbf{H}|\mathbf{L})$$

Image Super-Resolution

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- Yet again apply Bayes rule:

$$p(\mathbf{H}|\mathbf{L}) = \frac{p(\mathbf{L}|\mathbf{H}) \cdot p(\mathbf{H})}{p(\mathbf{L})}$$

  - I know, this may really get boring by now...

- Modeling the prior of high-resolution images:
  - We can again use the prior model that we already have!
- Modeling the likelihood:
  - The likelihood has to model how the low-resolution pixels correspond to the high-resolution pixels.
  - For simplicity, assume a fixed zooming factor of 2x.

Super-Resolution Likelihood

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- If we have a zoom factor of 2x, there is a 2x2 patch of high-resolution pixels that corresponds to a single low-resolution pixel.
- Again assume conditional independence of the low-resolution pixels given the high-resolution image.
- We can thus write the likelihood as:

$$p(\mathbf{L}|\mathbf{H}) = \prod_{i,j} p(L_{i,j}|H_{2i,2j}, H_{2i+1,2j}, H_{2i,2j+1}, H_{2i+1,2j+1})$$

- We may for example assume that a low-resolution pixel is the average of the 4 high-resolution pixels plus a small amount of Gaussian noise:

$$p(L_{i,j}|H_{2i,2j}, H_{2i+1,2j}, H_{2i,2j+1}, H_{2i+1,2j+1}) = \mathcal{N}\left(L_{i,j} - \frac{1}{4}(H_{2i,2j} + H_{2i+1,2j} + H_{2i,2j+1} + H_{2i+1,2j+1}); 0, \sigma^2\right)$$

Super-resolution Results

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- Gradient ascent does not work very well here, because the likelihood is more complex than in denoising or inpainting.
- But belief propagation can be made to work (with some tricks...):

original

MRF + BP

bicubic interpolation

From [Tappen et al., 2003]

Super-resolution Results

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original

MRF + BP

bicubic interpolation

From [Tappen et al., 2003]

Summary

- Most relevant image processing problems can be formulated as problems of probabilistic inference.
  - This is only one of many ways of approaching these problems!
- Advantages:
  - Unified approach to many different problems, in which important components (prior) may be re-used.
  - Is is relatively easy to understand what the various parts do.
  - Good application performance, despite generality.
- Disadvantages:
  - Computationally often expensive.
  - Special purpose techniques often have somewhat better application performance.

High-order MRFs & Learning

14.05.2014

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visual inference

"Pairwise" MRF Image Prior

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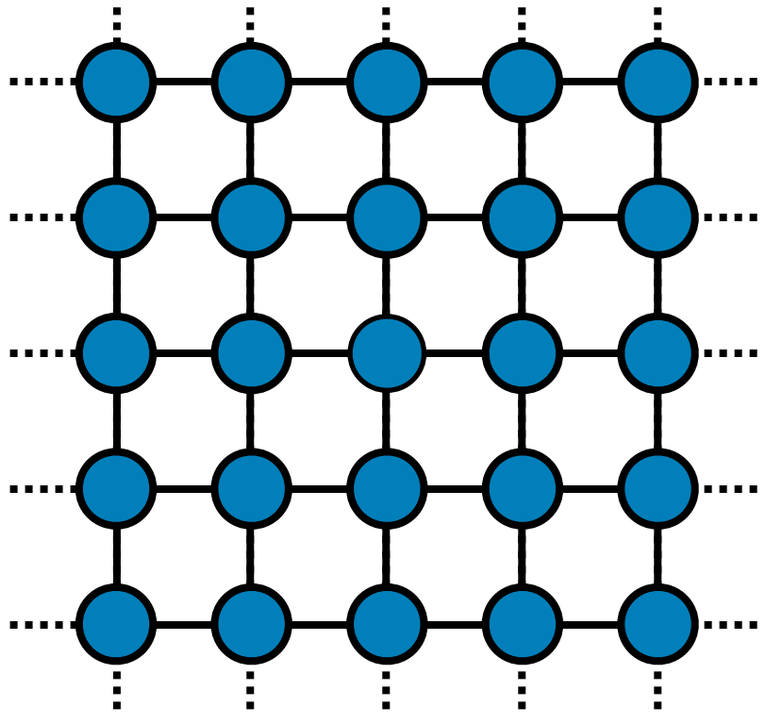
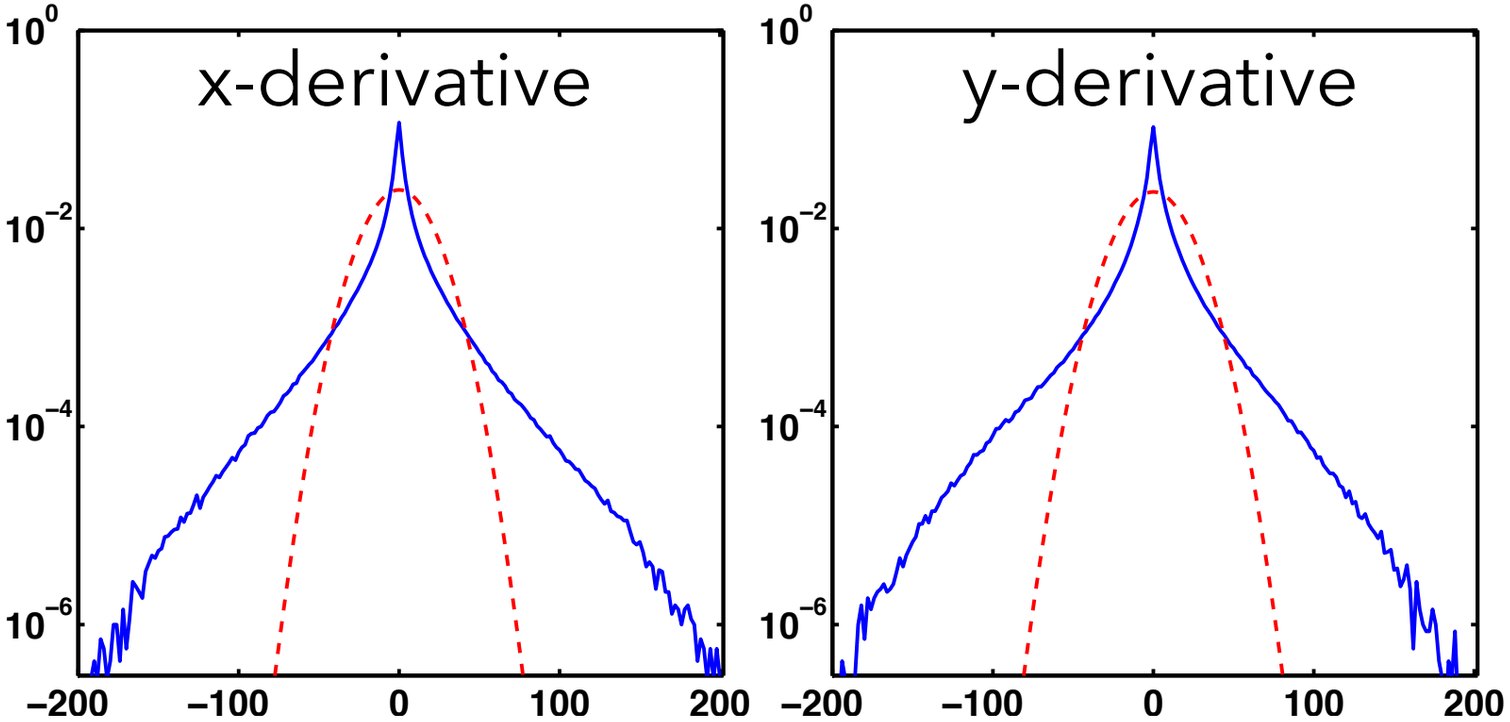
- 4-connected Markov random field



$$p(\mathbf{T}) = \prod_{i,j} f_H(T_{i,j}, T_{i+1,j}) \cdot f_V(T_{i,j}, T_{i,j+1})$$



empirical histogram — fit with a Gaussian

MRF Model of the Posterior

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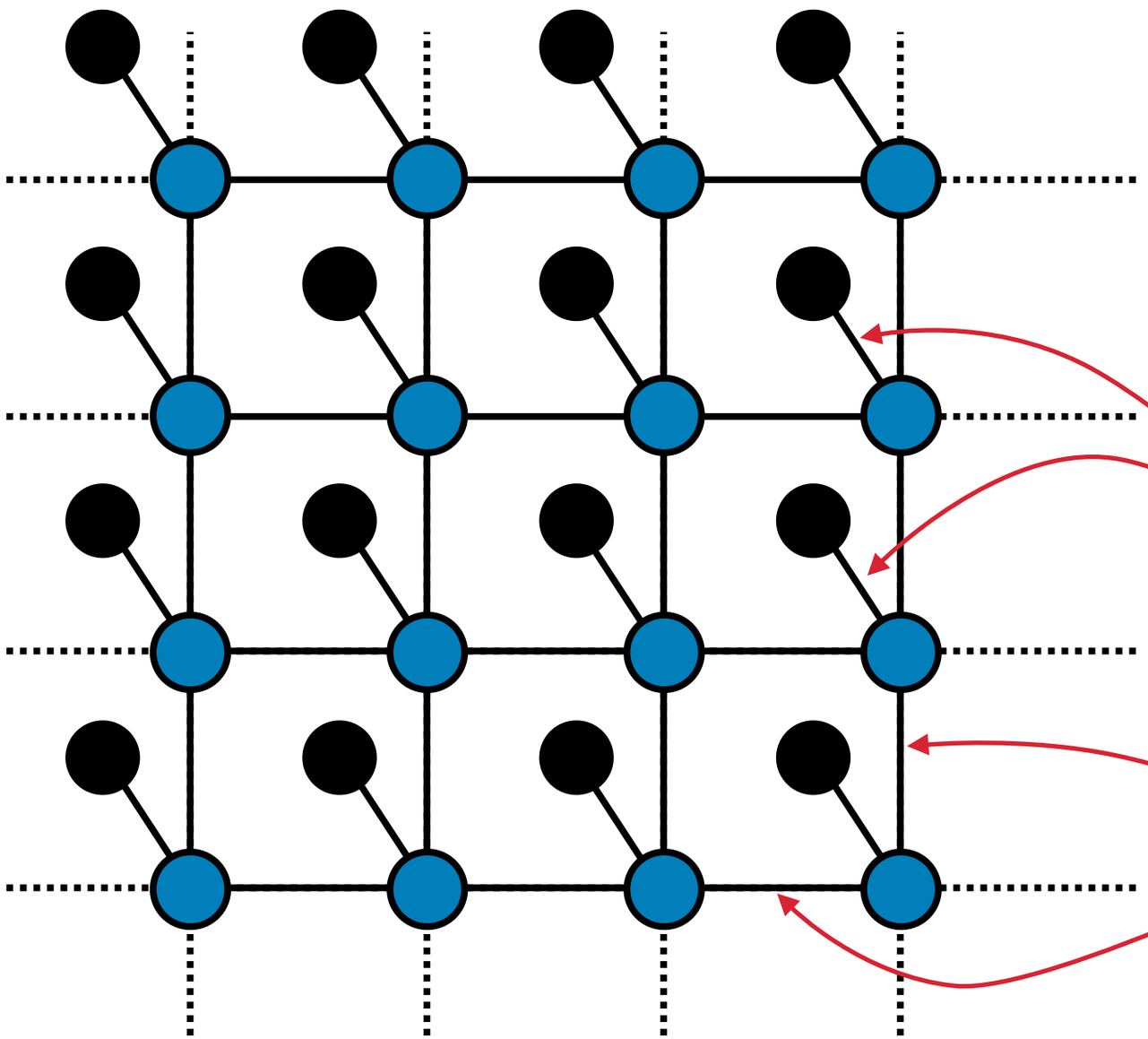
- Likelihood and the prior together in a single MRF model:



$T_{i,j}$ pixels of the true image (hidden)

$N_{i,j}$ pixels of the noisy image (observed)

Edges representing the likelihood

Edges representing the prior

Denoising Results

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original image

noisy image,
σ=20

PSNR 22.49dB
SSIM 0.528

denoised using
gradient ascent

PSNR 27.60dB
SSIM 0.810

Can we do better?

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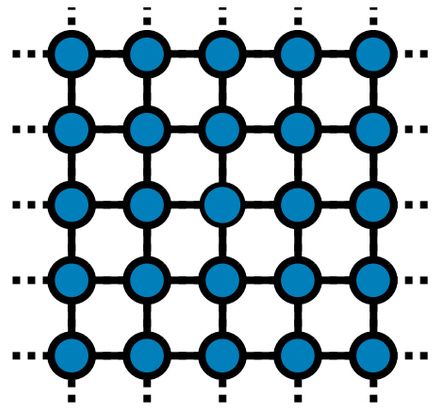
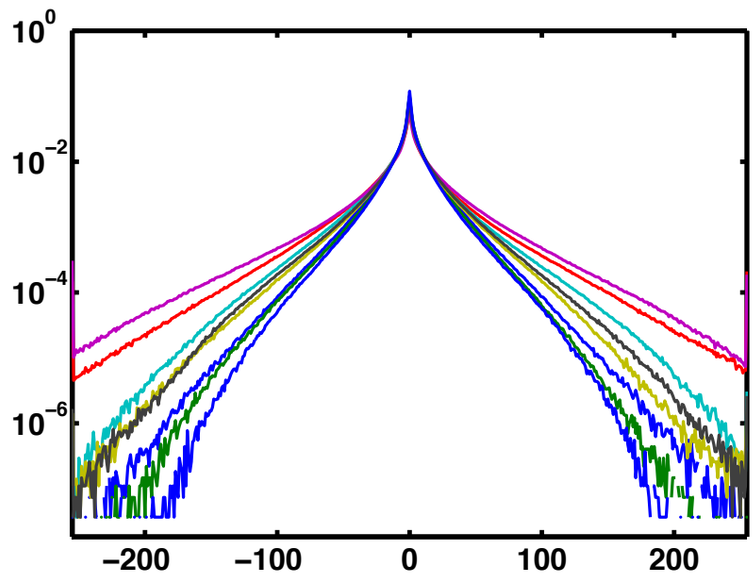
- Relax strong conditional independence (Markov) assumption
  - MRF based on a 4-connected grid restrictive (more complex properties cannot be modeled)

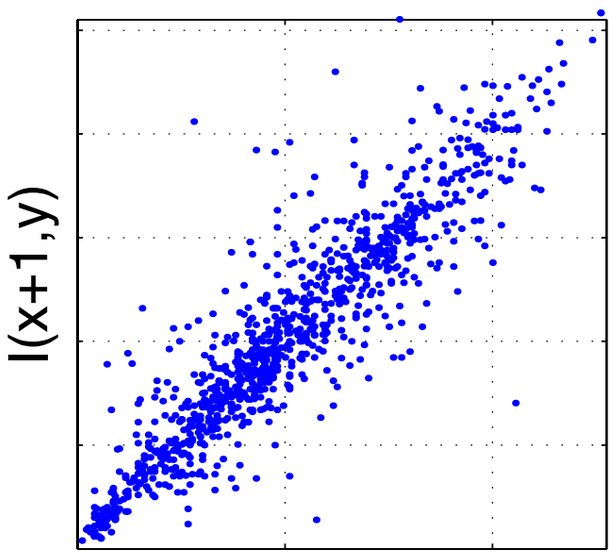


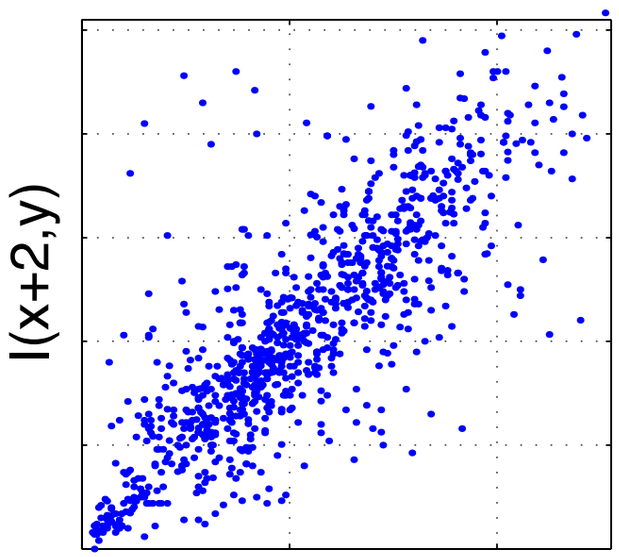
- Motivated by image statistics
  - Responses to larger (random) linear filters are also heavy-tailed

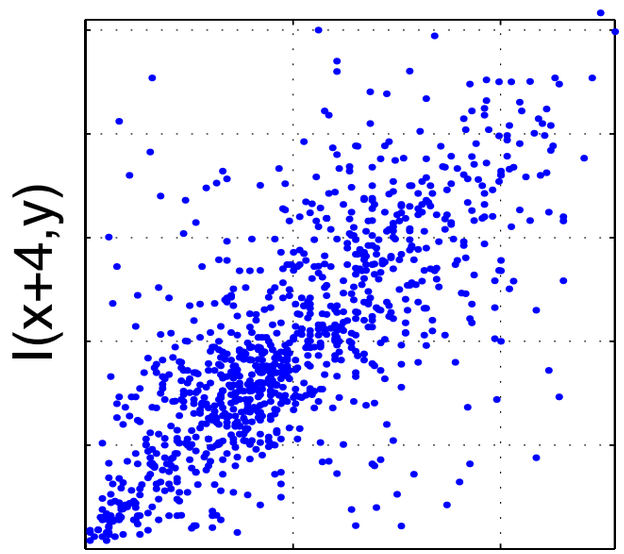


Histograms of random (zero-mean) linear filters

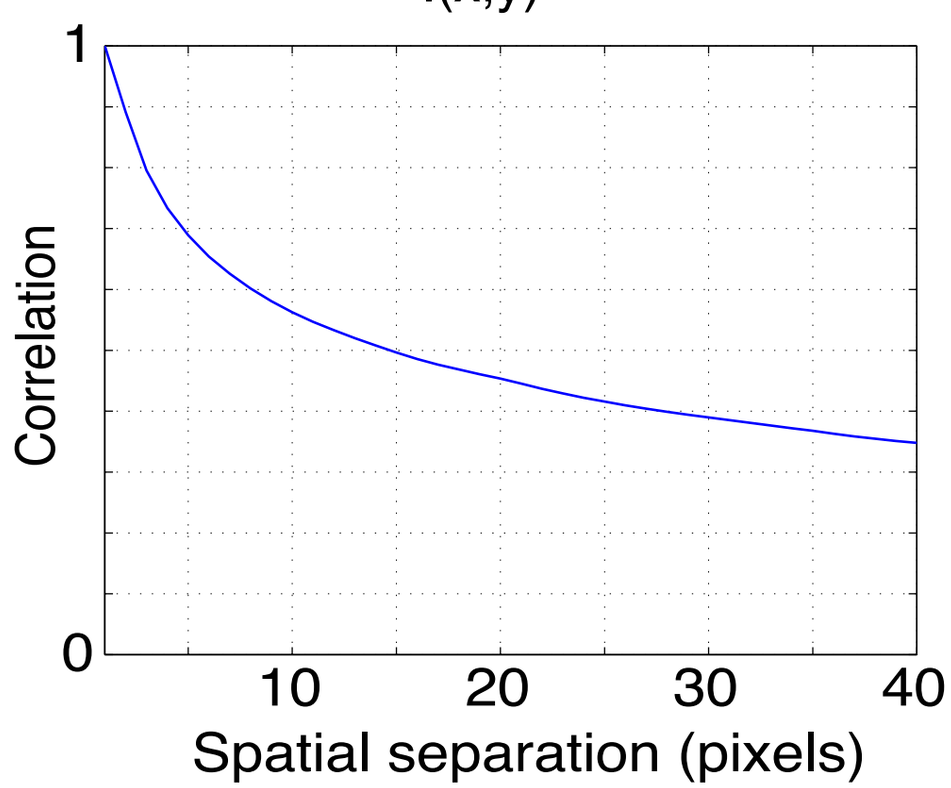
Pixel - joint







second-order correlation



High-order MRF Prior

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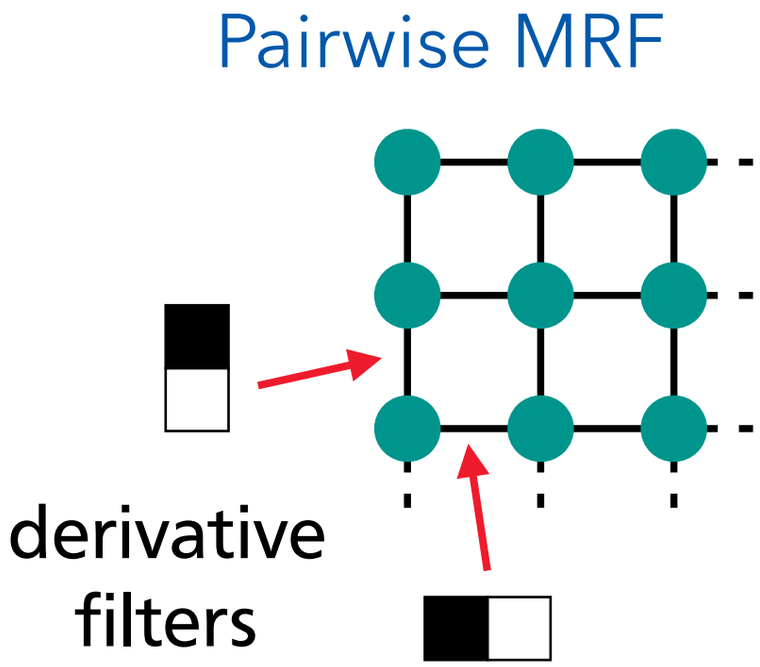
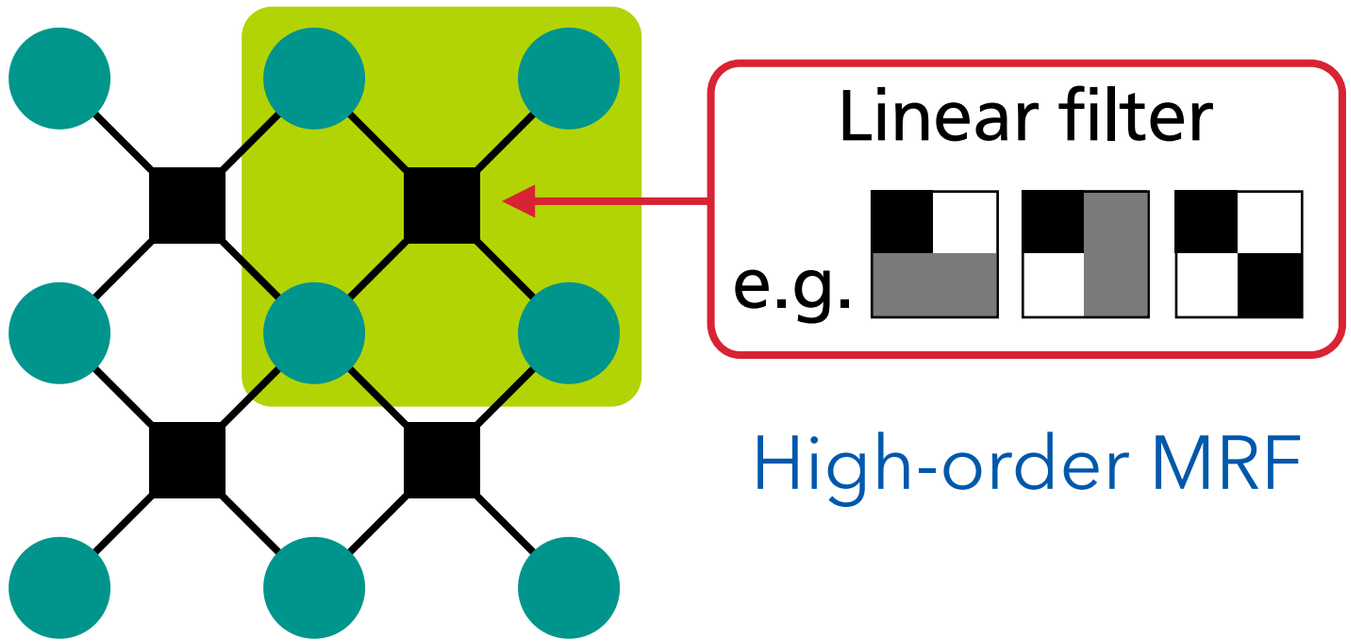
- Directly model extended pixel neighborhoods
  - Weaker conditional independence assumption (→ Markov blanket)
- High-dimensional potential functions
  - Model product of linear filter responses





High-order MRF Prior

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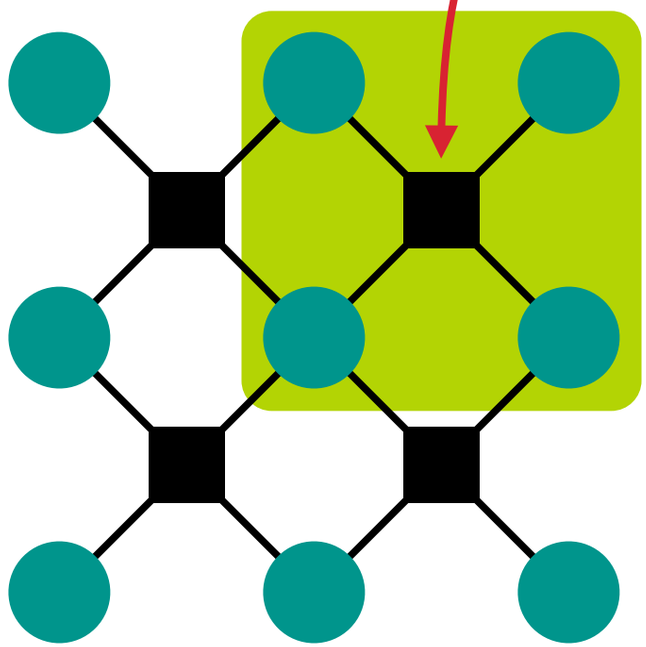
- Fields of Experts (FoE) [Roth & Black '05, '09]
  - Includes various pairwise & high-order MRFs

$$p(\mathbf{x};\boldsymbol{\Theta}) = \frac{1}{Z(\boldsymbol{\Theta})} e^{-\epsilon\|\mathbf{x}\|^2/2} \prod_{c\in\mathcal{C}} \prod_{i=1}^{N} \phi\left(\mathbf{J}_i^{\mathrm{T}} \mathbf{x}_{(c)}; \boldsymbol{\alpha}_i\right)$$

Image

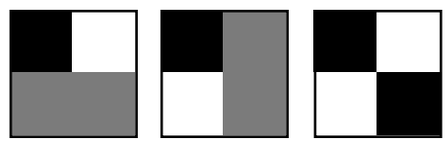
Parameters

$$\boldsymbol{\Theta} = \{\mathbf{J}_i, \boldsymbol{\alpha}_i\}_{i=1,\ldots,N}$$



Expert function

Linear filter

e.g. 

Vector of nodes in clique c

High-order MRF Prior

TECHNISCHE UNIVERSITÄT DARMSTADT

- Fields of Experts (FoE) [Roth & Black '05, '09]
  - Includes various pairwise & high-order MRFs

$$p(\mathbf{x};\boldsymbol{\Theta}) = \frac{1}{Z(\boldsymbol{\Theta})} e^{-\epsilon\|\mathbf{x}\|^2/2} \prod_{c\in\mathcal{C}} \prod_{i=1}^{N} \phi\left(\mathbf{J}_i^{\mathrm{T}}\mathbf{x}_{(c)};\boldsymbol{\alpha}_i\right)$$

Intractable partition function

Guarantees model to be normalizable [Weiss & Freeman '07]

High-order MRF Prior

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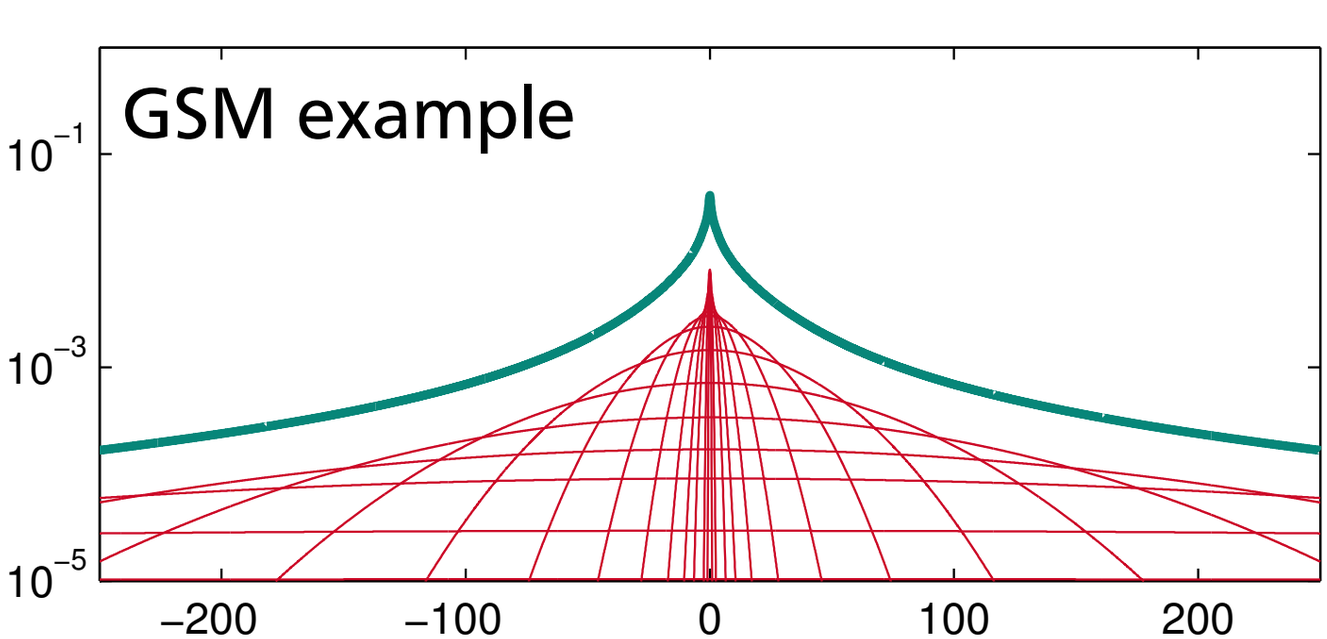
- Fields of Experts (FoE) [Roth & Black '05, '09]
  - Includes various pairwise & high-order MRFs

$$p(\mathbf{x};\boldsymbol{\Theta}) = \frac{1}{Z(\boldsymbol{\Theta})} e^{-\epsilon\|\mathbf{x}\|^2/2} \prod_{c\in\mathcal{C}} \prod_{i=1}^{N} \phi\left(\mathbf{J}_i^{\mathrm{T}} \mathbf{x}_{(c)}; \boldsymbol{\alpha}_i\right)$$

Mixture Weights



Gaussian Scale Mixture (GSM) Distribution

[Wainwright & Simoncelli '99, Weiss & Freeman '07]

$$\phi\left(\mathbf{J}_i^{\mathrm{T}} \mathbf{x}_{(c)}; \boldsymbol{\alpha}_i\right) = \sum_{j=1}^{J} \alpha_{ij} \cdot \mathcal{N}\left(\mathbf{J}_i^{\mathrm{T}} \mathbf{x}_{(c)}; 0, \sigma_i^2/s_j\right)$$

High-order MRF Prior

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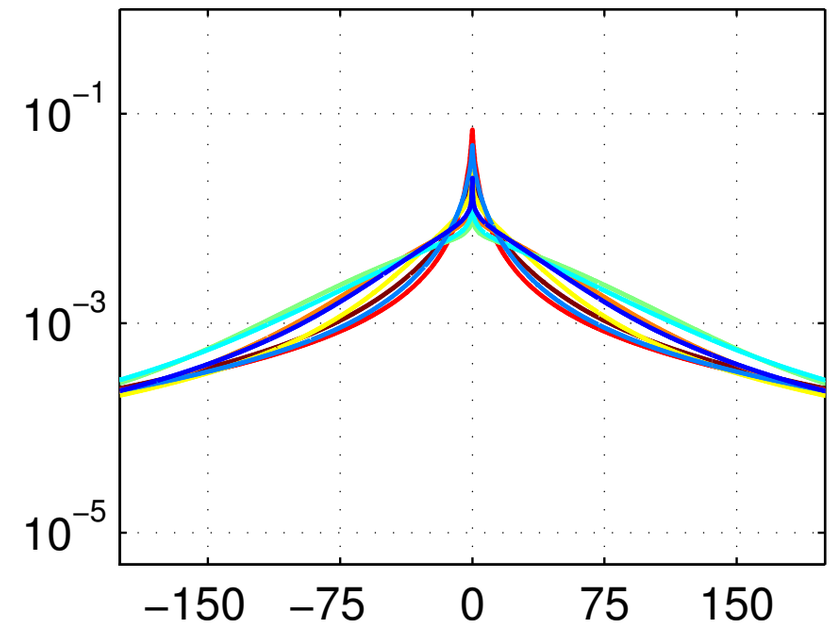
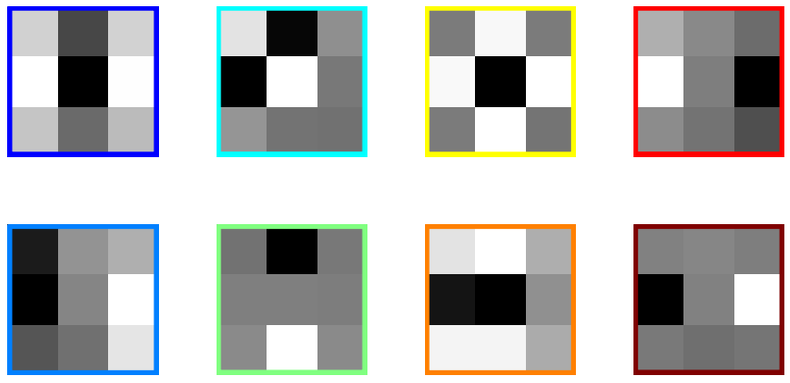
- More flexibility → more parameters
  - difficult to choose manually
- Parameter learning necessary to obtain good results



Potential (expert) functions and associated 3x3 filters of a learned fields of experts prior



Maximum Likelihood

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- Training data $\mathcal{D} = \{x_1, \ldots, x_n\}$
- Goal: estimate $p(x|\theta)$

- Aim to estimate one single θ
- Likelihood of θ

$$\mathcal{L}(\theta) = p(\mathcal{D}|\theta)$$

- Assume that the data is independent and identically distributed (iid)

$$\mathcal{L}(\theta) = p(\mathcal{D}|\theta) = \prod_{i=1}^{n} p(x_i|\theta)$$

- Now choose θ such that it maximizes this likelihood

$$\theta_{ML} = \underset{\theta}{\operatorname{argmax}}\, \mathcal{L}(\theta; \mathcal{D})$$

Maximum Likelihood

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- Maximum Likelihood

$$\theta_{ML} = \underset{\theta}{\operatorname{argmax}}\, \mathcal{L}(\theta; \mathcal{D})$$

  is equivalent to

- minimizing the negative log-Likelihood

$$\begin{aligned} \theta_{ML} &= \underset{\theta}{\operatorname{argmax}} \ln \mathcal{L}(\theta) \\ &= \underset{\theta}{\operatorname{argmin}} - \ln \mathcal{L}(\theta) \\ &= \underset{\theta}{\operatorname{argmin}} - \sum_{i=1}^{n} \ln p(x_i|\theta) \end{aligned}$$

- Numerically more stable

MRF Learning – Likelihood

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- Markov Network defined on (not necessarily maximal) cliques

$$p(x \mid \theta) = \frac{1}{Z(\theta)} \prod_c \Phi_c(x_{(c)} \mid \theta_c)$$

- Clique variables $x_{(c)}$
- Partition function

$$Z(\theta) = \sum_x \prod_c \Phi_c(x_{(c)} \mid \theta_c)$$

- Training data $\mathcal{D} = \{x^1, \ldots, x^N\}$
- Log-Likelihood

$$\mathcal{L}(\theta; \mathcal{D}) = \sum_n \sum_c \log \Phi_c(x^n_{(c)} \mid \theta_c) - N \log Z(\theta)$$

MRF Learning – Likelihood

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- The Likelihood (repeated from before)

$$L(\theta;\mathcal{D}) = \sum_n \sum_c \log \Phi_c(x^n_{(c)} \mid \theta_c) - N \log Z(\theta)$$

$$= N \left\langle \sum_c \log \Phi_c(x_{(c)} \mid \theta_c) \right\rangle_{q(x;\mathcal{D})} - N \log Z(\theta)$$

- with the empirical distribution

$$q(x;\mathcal{D}) = \frac{1}{N} \sum_n \delta(x - x^n)$$

Likelihood Gradient

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- First step:

$$\begin{aligned}
\frac{\partial}{\partial \theta_d} \mathcal{L}(\theta; \mathcal{D}) &= N \frac{\partial}{\partial \theta_d} \left\langle \sum_c \log \Phi_c(x_{(c)} \mid \theta_c) \right\rangle_{q(x;\mathcal{D})} - N \frac{\partial}{\partial \theta_d} \log Z(\theta) \\
&= N \left\langle \sum_c \frac{\partial}{\partial \theta_d} \log \Phi_c(x_{(c)} \mid \theta_c) \right\rangle_{q(x;\mathcal{D})} - N \frac{\partial}{\partial \theta_d} \log Z(\theta) \\
&= N \left\langle \frac{\partial}{\partial \theta_d} \log \Phi_d(x_{(d)} \mid \theta_d) \right\rangle_{q(x;\mathcal{D})} - N \frac{\partial}{\partial \theta_d} \log Z(\theta)
\end{aligned}$$

- Second step:

$$\frac{\partial}{\partial \theta_d} \log Z(\theta) = \frac{\partial}{\partial \theta_d} \log \left(\int \prod_c \Phi_c(x_{(c)} \mid \theta_c) \, dx \right)$$

Likelihood Gradient

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$$\frac{\partial}{\partial x}\log f(x) = \frac{1}{f(x)}\cdot\frac{\partial}{\partial x}f(x)$$

$$\begin{aligned}
\frac{\partial}{\partial\theta_d}\log Z(\theta) &= \frac{\partial}{\partial\theta_d}\log\left(\int\prod_c \Phi_c(x_{(c)}\mid\theta_c)\;dx\right)\\
&= \frac{\frac{\partial}{\partial\theta_d}\int\prod_c \Phi_c(x_{(c)}\mid\theta_c)\;dx}{Z(\theta)}\\
&= \frac{1}{Z(\theta)}\int\left[\frac{\partial}{\partial\theta_d}\Phi_d(x_{(d)}\mid\theta_d)\right]\prod_{c\neq d}\Phi_c(x_{(c)}\mid\theta_c)\;dx\\
&= \frac{1}{Z(\theta)}\int\left[\frac{\partial}{\partial\theta_d}\log\Phi_d(x_{(d)}\mid\theta_d)\right]\prod_{c}\Phi_c(x_{(c)}\mid\theta_c)\;dx\\
&= \int\left[\frac{\partial}{\partial\theta_d}\log\Phi_d(x_{(d)}\mid\theta_d)\right]p(x\mid\theta)dx
\end{aligned}$$

Likelihood Gradient

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$$
\begin{aligned}
\frac{\partial}{\partial \theta_d} \log Z(\theta) &= \int \left[\frac{\partial}{\partial \theta_d} \log \Phi_d(x_{(d)} \mid \theta_d) \right] p(x \mid \theta) dx \\
&= \int \left[\frac{\partial}{\partial \theta_d} \log \Phi_d(x_{(d)} \mid \theta_d) \right] p(x_{(d)} \mid \theta) dx_{(d)} \\
&= \left\langle \frac{\partial}{\partial \theta_d} \log \Phi_d(x_{(d)} \mid \theta_d) \right\rangle_{p(x_{(d)} \mid \theta)}
\end{aligned}
$$

Likelihood Gradient

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- Finally – Likelihood gradient:

$$\frac{\partial}{\partial \theta_d}\mathcal{L}(\theta;\mathcal{D}) = N\left\langle \frac{\partial}{\partial \theta_d}\log \Phi_d(x_{(d)} \mid \theta_d)\right\rangle_{q(x;\mathcal{D})} - N\frac{\partial}{\partial \theta_d}\log Z(\theta)$$

$$= N\left\langle \frac{\partial}{\partial \theta_d}\log \Phi_d(x_{(d)} \mid \theta_d)\right\rangle_{q(x;\mathcal{D})} - N\left\langle \frac{\partial}{\partial \theta_d}\log \Phi_d(x_{(d)} \mid \theta_d)\right\rangle_{p(x_{(d)}|\theta)}$$

- Problem: last term depends on $p(x_{(d)} \mid \theta)$
- Need to take an average over the model distribution $p(x \mid \theta)$
- Learning requires inference!

MRF Sampling – Example

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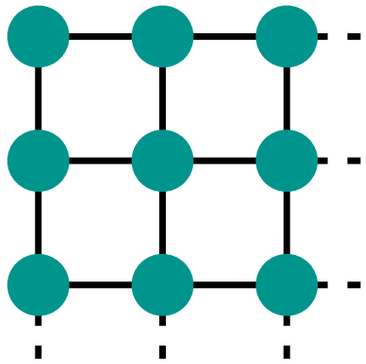
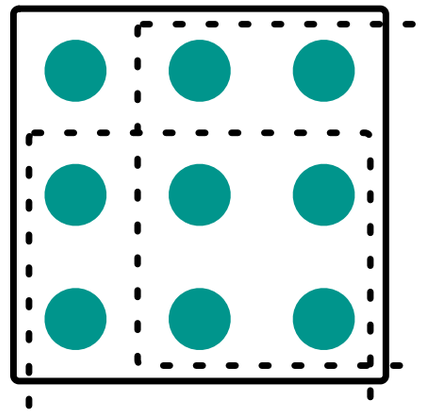
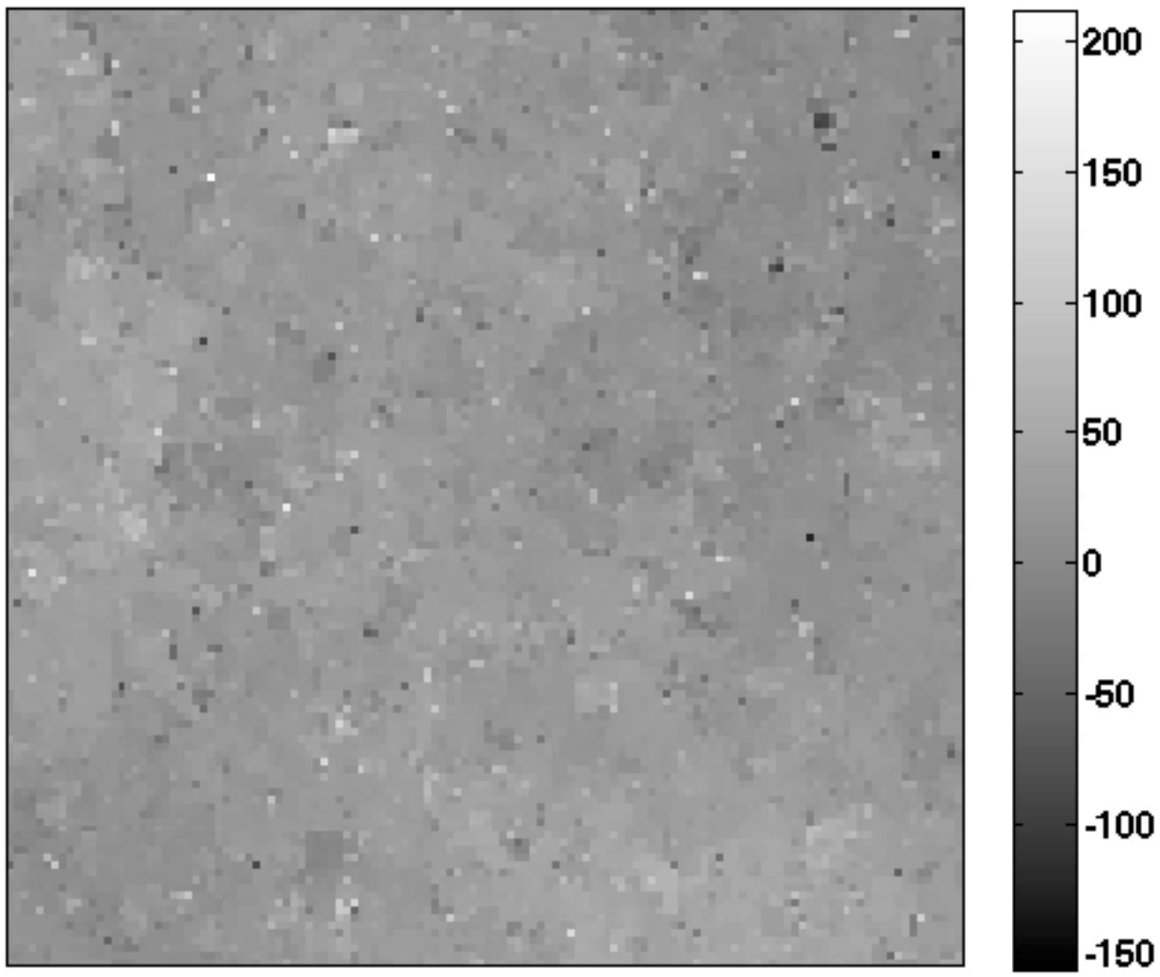
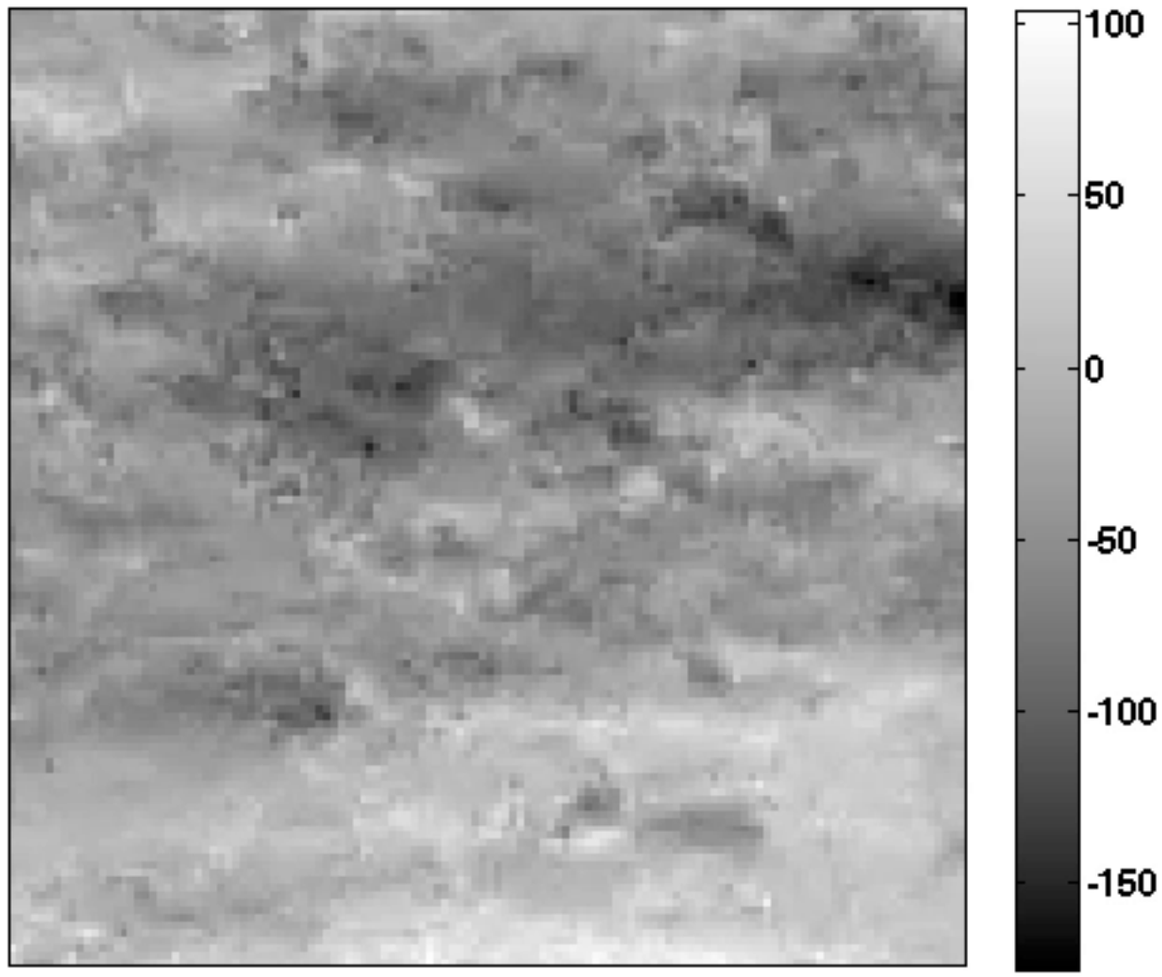


Pairwise MRF

High-order MRF
with 3×3 cliques







Generative Learning

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- We have discussed so far “generative” learning/training
- Image prior independent of application
  - Same prior can be used for denoising, super-resolution, etc.

- Training only requires “good” natural images
- Learning unfortunately difficult due to intractable normalization constant

Probabilistic Inference

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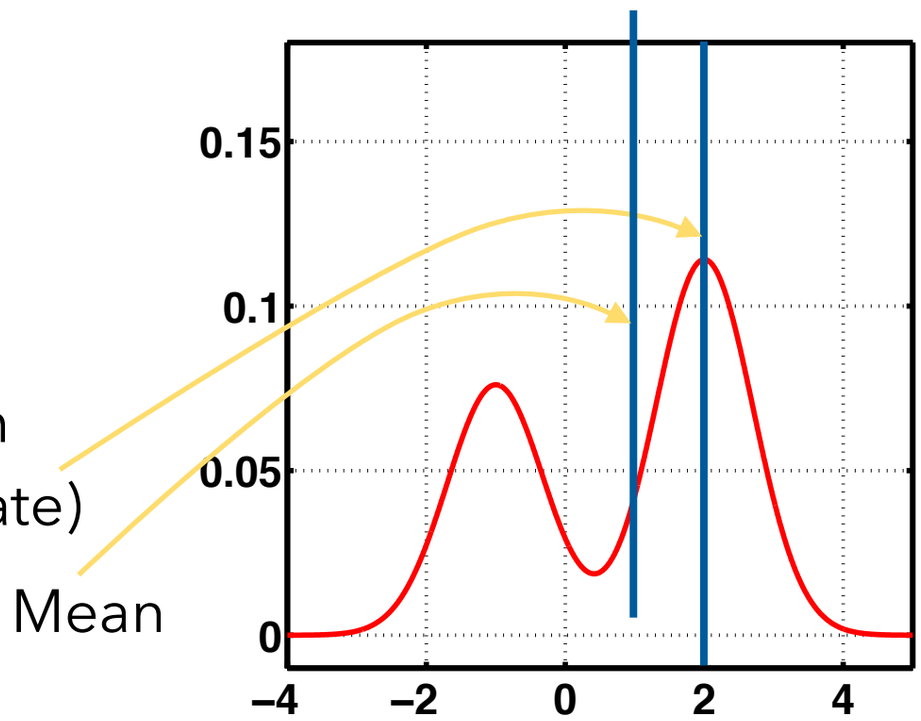
- We learned a prior and combined it with the likelihood
- How do we infer a single solution from the posterior?
- Common choices:
  - Maximum
  - Mean
- How do these differ?
  - Assume we have the following posterior distribution:
  - The posterior may be multi-modal!



Bayesian Decision Theory

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- Your physician tells you there's a 1% chance of you having a disease. Should you get treated now?
  - A drug to treat the disease *now* costs 10 €.
  - *Later* treatment requires an operation that costs 10000 €.

- Decision to get treated *now*. Expected cost/loss:

$$\Delta(\text{now}|\text{healthy}) \cdot p(\text{healthy}) + \Delta(\text{now}|\text{sick}) \cdot p(\text{sick}) = 10 \cdot 0.99 + 10 \cdot 0.01 = 10$$

- Decision to get treated *later*. Expected cost/loss:

$$\Delta(\text{later}|\text{healthy}) \cdot p(\text{healthy}) + \Delta(\text{later}|\text{sick}) \cdot p(\text{sick}) = 0 \cdot 0.99 + 10000 \cdot 0.01 = 100$$

Bayesian Decision Theory

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- Minimize expected loss to choose restored image

$$\hat{\mathbf{x}} = \arg\min_{\mathbf{x}'} \int_{\mathbf{x}} \Delta(\mathbf{x}', \mathbf{x}) \cdot p(\mathbf{x}|\mathbf{y}) \, \mathrm{d}\mathbf{x}$$

- Loss function: $\Delta(\mathbf{x}', \mathbf{x})$
- Posterior distribution: $p(\mathbf{x}|\mathbf{y}) \propto p(\mathbf{y}|\mathbf{x})p(\mathbf{x})$

- Choice of loss function determines decision of restored image (called estimator)
  - in turn, each estimator implies a loss function

Loss Functions & Estimators

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- Maximum a posteriori (MAP) estimation implies 0-1 loss

$$\mathbf{x}_{\mathrm{MAP}} = \underset{\mathbf{x}}{\operatorname{argmax}}\, p(\mathbf{x}|\mathbf{y}) \qquad \Delta(\mathbf{x}', \mathbf{x}) = \begin{cases} 0 & \text{if } \mathbf{x}' = \mathbf{x} \\ 1 & \text{if } \mathbf{x}' \neq \mathbf{x} \end{cases}$$

- Minimum mean squared error (MMSE) estimation

$$\mathbf{x}_{\mathrm{MMSE}} = \underset{\mathbf{x}'}{\operatorname{argmin}} \int_{\mathbf{x}} \underbrace{\|\mathbf{x}' - \mathbf{x}\|^2}_{\Delta(\mathbf{x}', \mathbf{x})} p(\mathbf{x}|\mathbf{y}) d\mathbf{x}$$

$$= \underset{\mathbf{x}'}{\operatorname{argmin}} \left[\|\mathbf{x}'\|^2 - 2 \left(\int_{\mathbf{x}} \mathbf{x} \cdot p(\mathbf{x}|\mathbf{y}) d\mathbf{x} \right)^{\mathsf{T}} \mathbf{x}' \right] = E(\mathbf{x}|\mathbf{y})$$

Can we do better? Yes.

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4-connected MRF
MAP estimation

3x3 Fields of Experts
MMSE estimation

Generative vs. Discriminative

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Generative

- model $p(\mathbf{x})$
- train on natural images

✔ application neutral

✘ learning & inference is more difficult

Discriminative

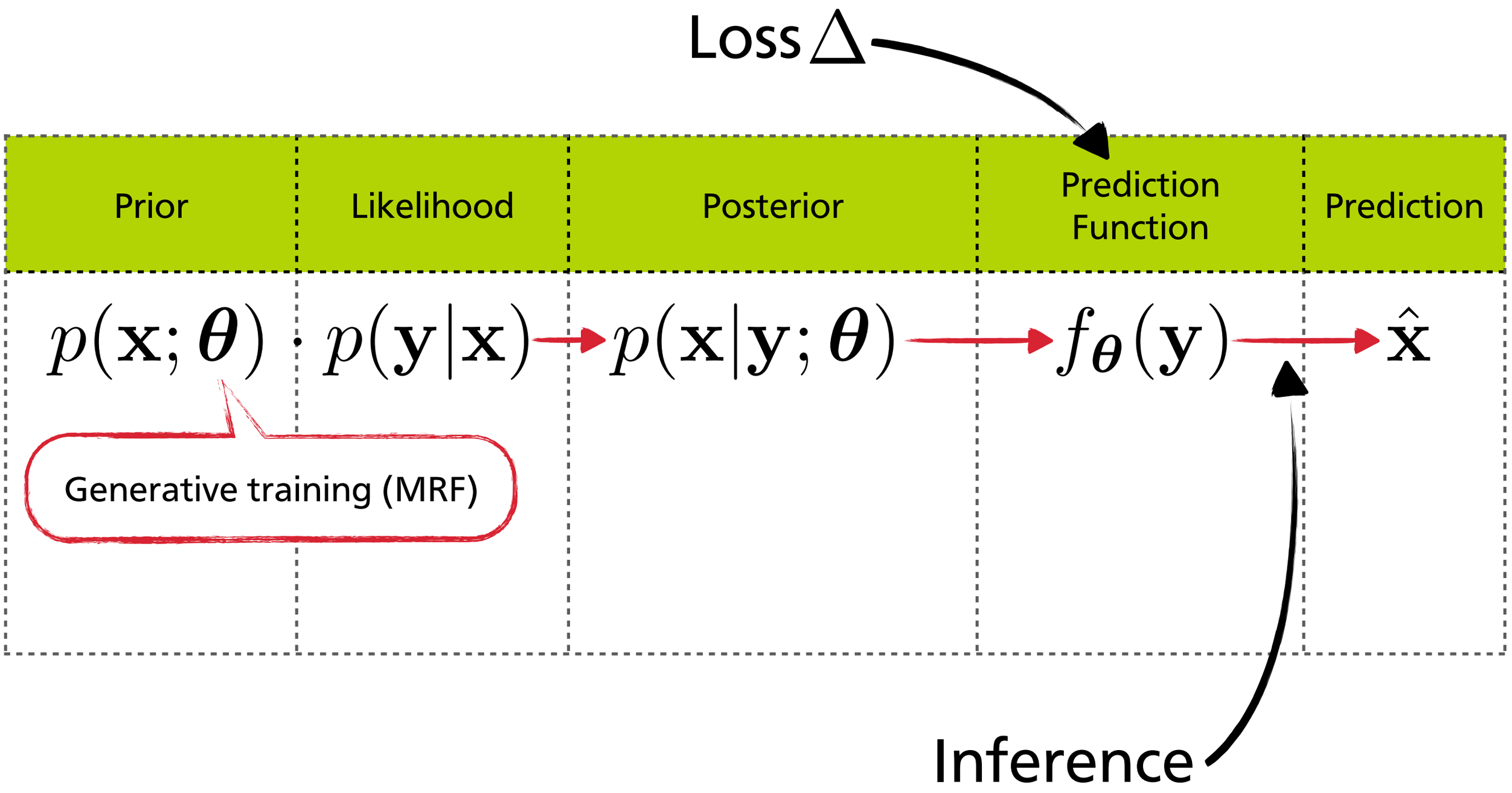
- model $p(\mathbf{x}|\mathbf{y})$
- train on input/output pairs

✘ application specific

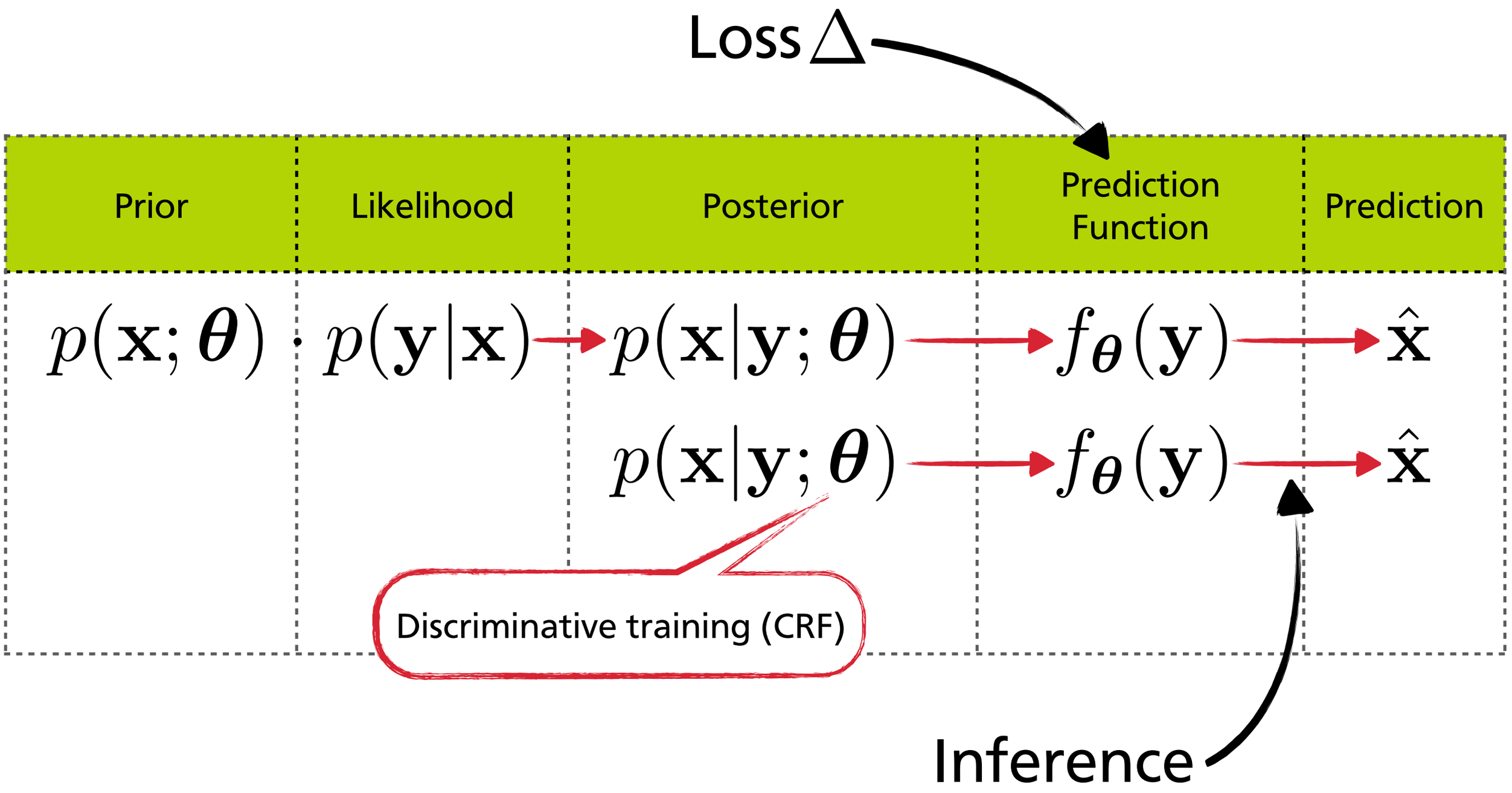
✔ learning & inference are generally easier

✔ can lead to better performance

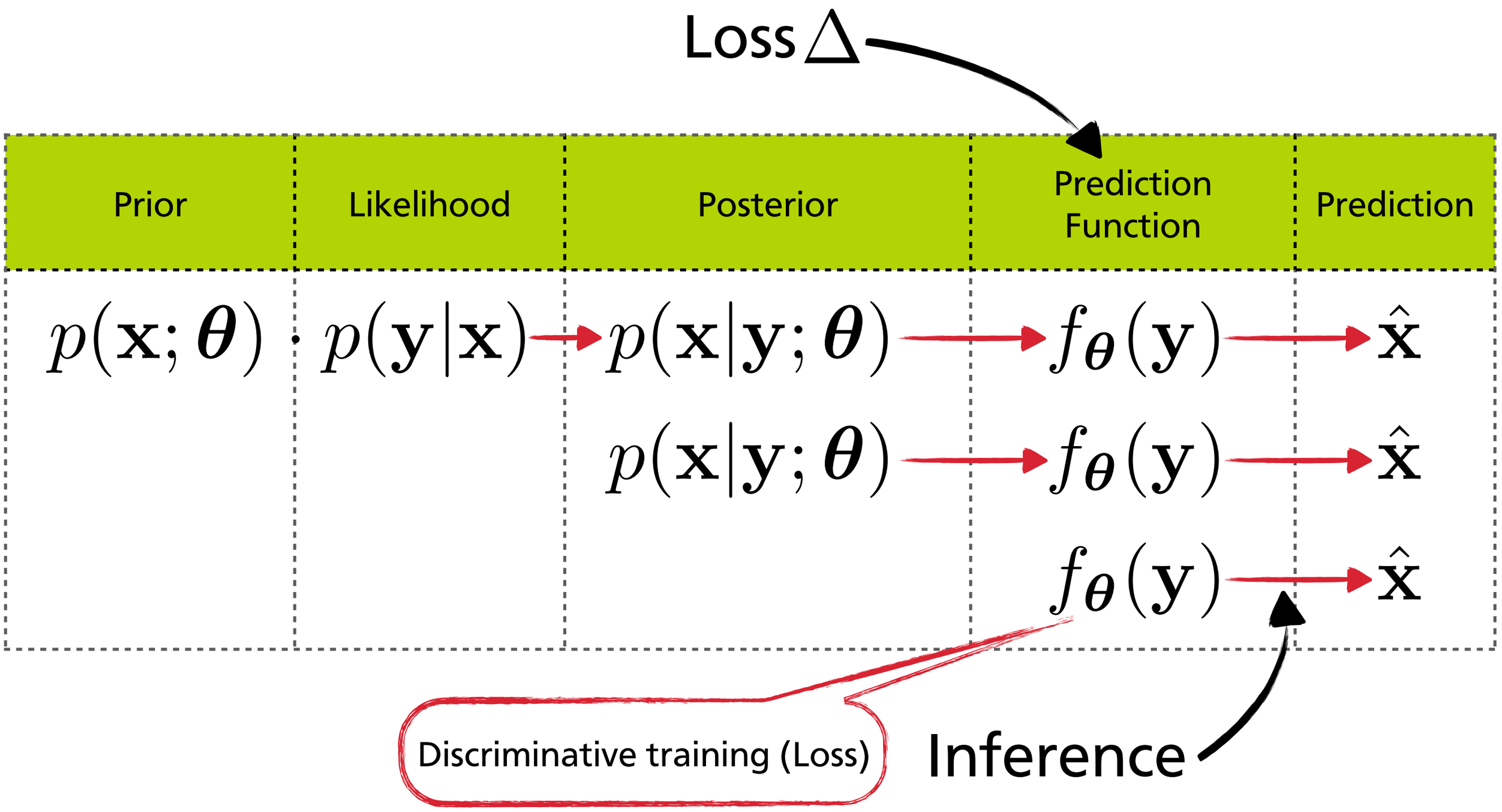
Generative vs. Discriminative Training



Generative vs. Discriminative Training



Generative vs. Discriminative Training



Discriminative Training

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- Two types of discriminative training
  - probabilistic CRF training of conditional likelihood
  - loss-based training (not probabilistic)

- Maximum Conditional likelihood learning also has problem of intractable normalization constant

Loss-based Training

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- Define "posterior" in terms of an energy function $E(\mathbf{x}|\mathbf{y})$

$$p(\mathbf{x}|\mathbf{y};\boldsymbol{\theta}) \propto \exp\left(-E(\mathbf{x}|\mathbf{y};\boldsymbol{\theta})\right)$$

- Prediction via energy minimization:

$$f_{\boldsymbol{\theta}}(\mathbf{y}) = \arg\min_{\mathbf{x}} E(\mathbf{x}|\mathbf{y};\boldsymbol{\theta})$$

- Parameter learning with $\mathcal{D} = \{(\mathbf{x}_1, \mathbf{y}_1), \ldots, (\mathbf{x}_n, \mathbf{y}_n)\}$:

$$J(\boldsymbol{\theta}) = \sum_{i=1}^{n} \Delta(\mathbf{x}_i, f_{\boldsymbol{\theta}}(\mathbf{y}_i))$$

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} J(\boldsymbol{\theta})$$

Loss-based Training

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- Nested optimization problems. How to solve?

$$\hat{\boldsymbol{\theta}} = \arg\min_{\boldsymbol{\theta}} \sum_{i=1}^{n} \Delta(\mathbf{x}_i, f_{\boldsymbol{\theta}}(\mathbf{y}_i))$$

$$f_{\boldsymbol{\theta}}(\mathbf{y}) = \arg\min_{\mathbf{x}} E(\mathbf{x}|\mathbf{y}; \boldsymbol{\theta})$$

- Options:
  - Prediction $f_{\boldsymbol{\theta}}(\mathbf{y})$ and gradient $\frac{\partial f_{\boldsymbol{\theta}}(\mathbf{y})}{\partial \boldsymbol{\theta}}$ computable in closed form
  - Bilevel optimization with implicit differentiation
    - requires Hessian-vector products, often slow
  - "Truncated" optimization

Can we do better? Yes.

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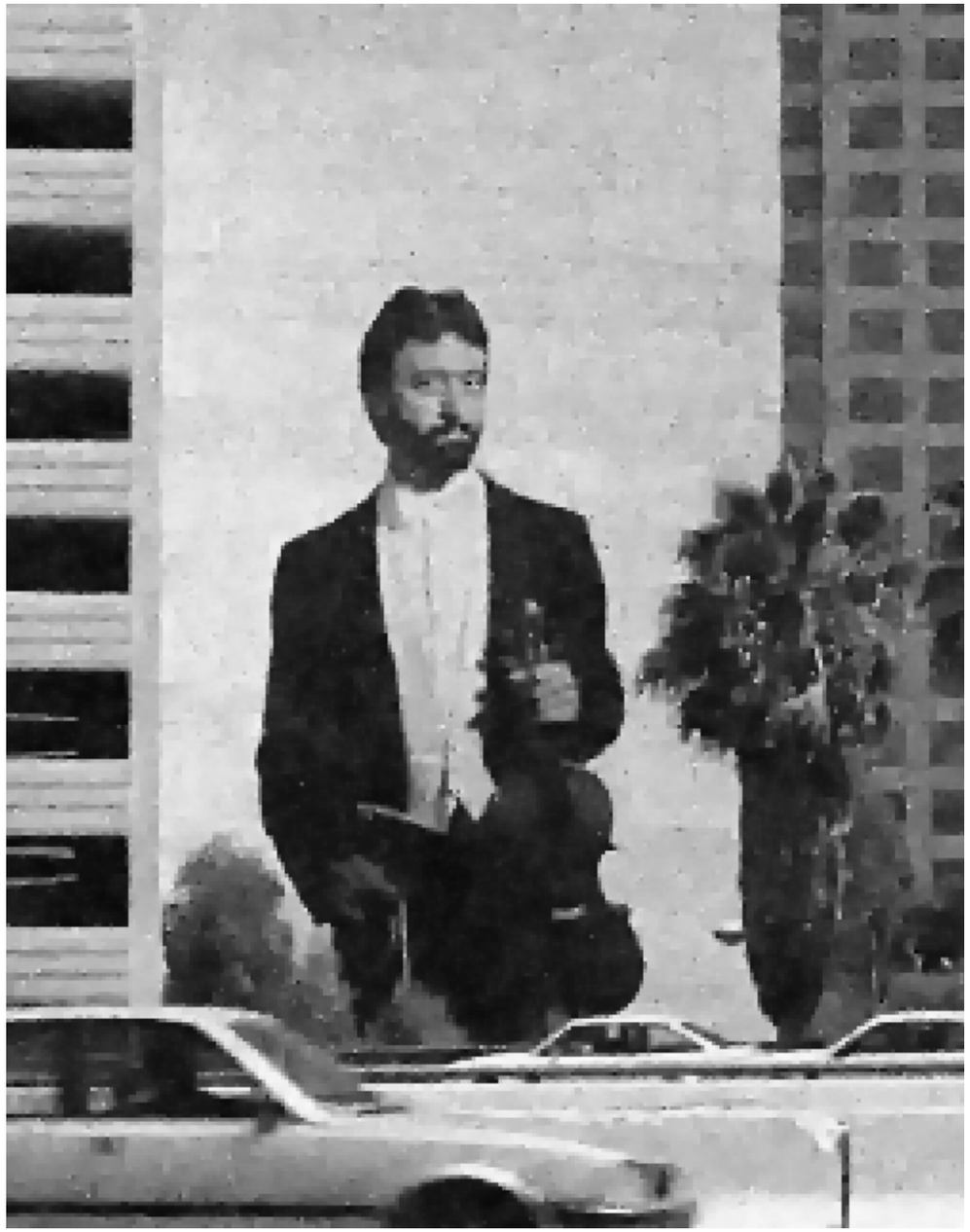
4-connected MRF, MAP
Hand-defined



4-connected MRF, MAP
Loss-based training

Can we do better? Yes.

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3x3 Fields of Experts, MMSE
Generative training

3x3 Fields of Experts, MAP
Loss-based training

Generative vs. Discriminative

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Generative

- model $p(\mathbf{x})$
- train on natural images

✔ application neutral

✘ learning & inference is more difficult

Discriminative

- model $p(\mathbf{x}|\mathbf{y})$
- train on input/output pairs

✘ application specific

✔ learning & inference are generally easier

✔ can lead to better performance

Probabilistic vs. Loss-based

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- **Probabilistic**
  - ✘ model misspecification
  - ✔ handle unobserved variables
  - ✔ not restricted to specific loss
  - ✘ often expensive computation
  - ✔ able to draw samples
- **Loss-based** (discriminative)
  - ✔ model misspecification
  - ✘ handle unobserved variables
  - ✘ re-training for other losses
  - ✔ often faster runtime
  - ✘ labeled data for training