Computer Vision I

View-Based Modeling - 15.05.2013

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vi visual inference

Announcements

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- **No class next week!**
  - Project review meeting

- Homework assignment 1:
  - Is out...
  - Signup for the “Testat” current read only -- will go online again soon

- **Exam**
  - 21. August 2013, 10:00 - 12:00

Wrap up - Basics of Digital Image Processing

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- Images
- Filtering
  - Linear filtering
  - Non-linear filtering & morphology
- Multi-scale image representation
  - Gaussian pyramid
  - Laplacian pyramid
- Edge detection
  - ‘Recognition using line drawings’
  - Image derivatives (1st and 2nd order)

Laplacian pyramid

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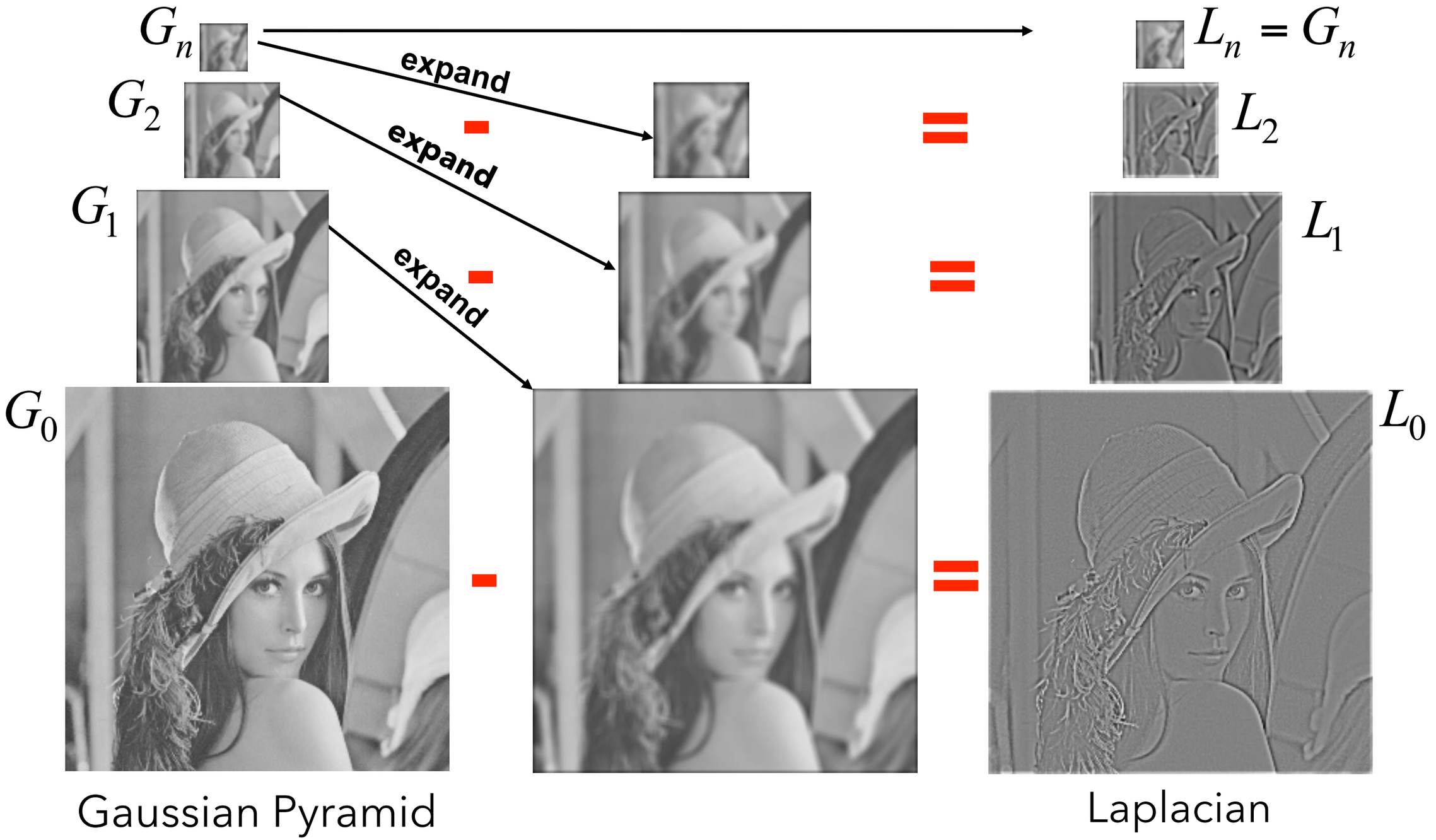
$$L_i = G_i - \text{expand}(G_{i+1}) \qquad G_i = L_i + \text{expand}(G_{i+1})$$



Laplacian pyramid

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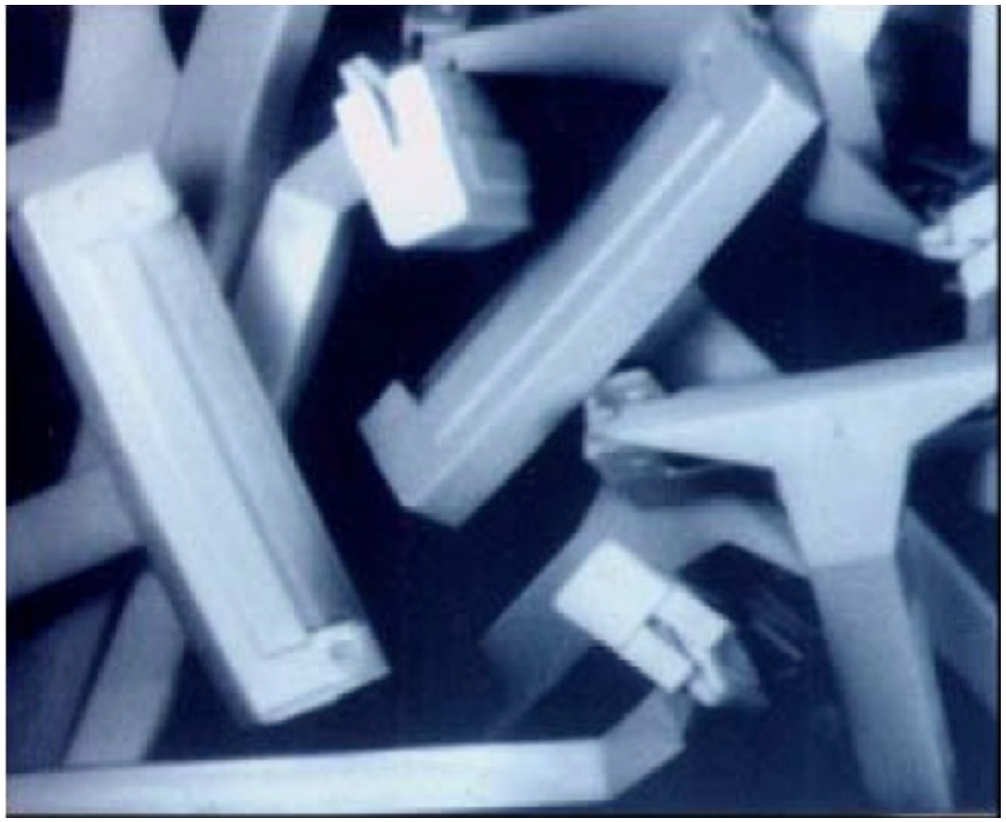
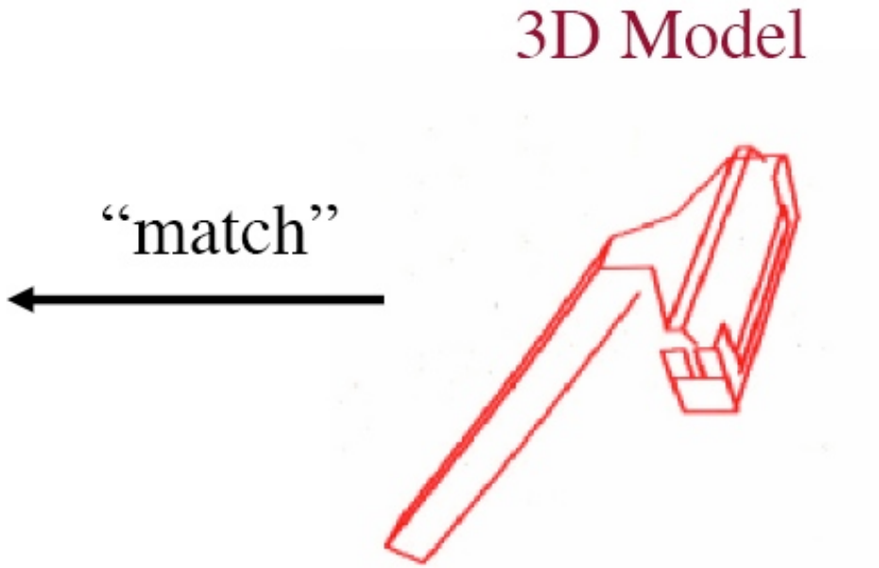
- The Laplacian pyramid provides a simple frequency decomposition into subbands.
- Each level / subband contains only image structure of a particular range of spatial frequencies:
  - E.g., the finest level contains all the high-frequency detail.
- Observation: We can get back the original image by reversing the decomposition.

Example of Recognition & Localization

- Simple approach [David Lowe]





Parameters: 3D position and orientation

Example of Recognition & Localization

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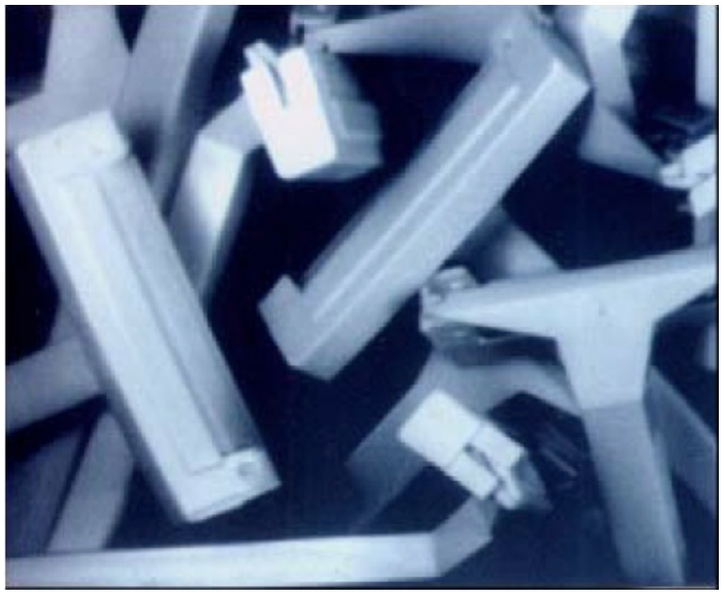
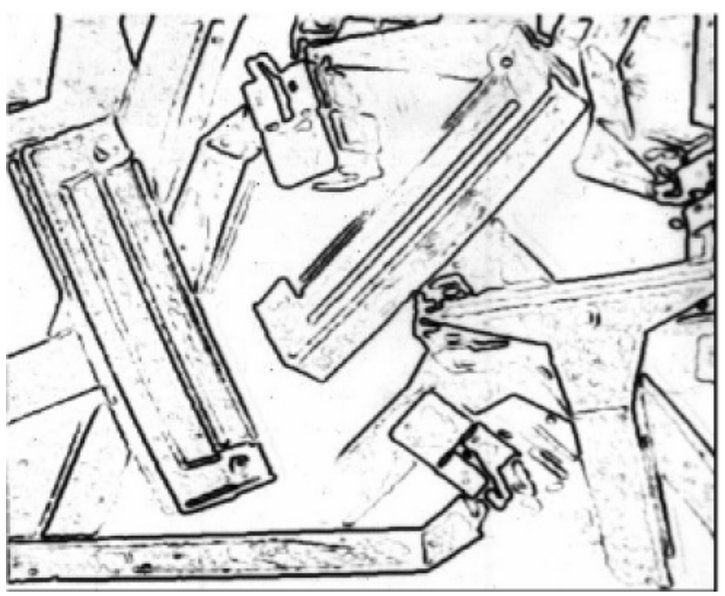
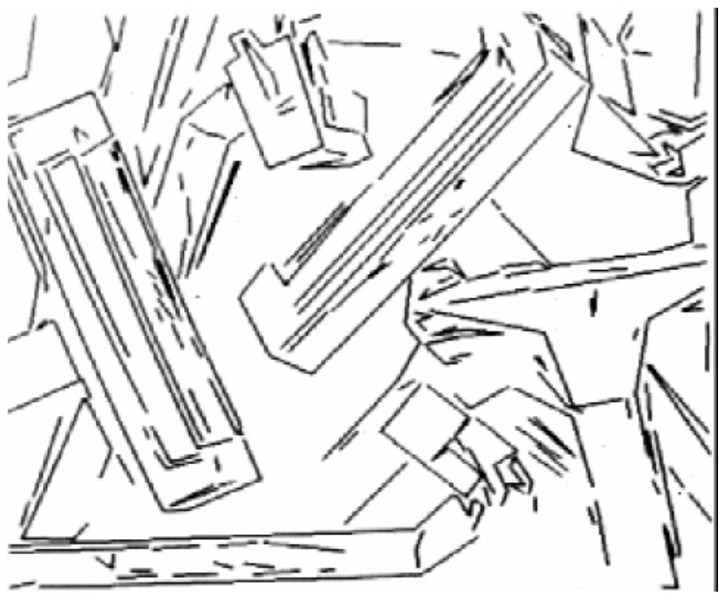
- Simple approach [David Lowe]
  - 1. “Filter” image to **find brightness changes**
  - 2. **“Fit” lines** to the raw measurements







Example of Recognition & Localization

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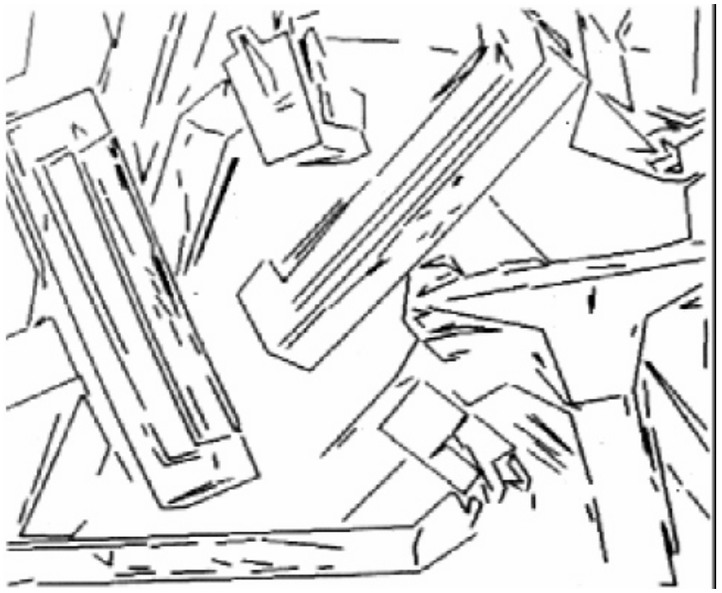
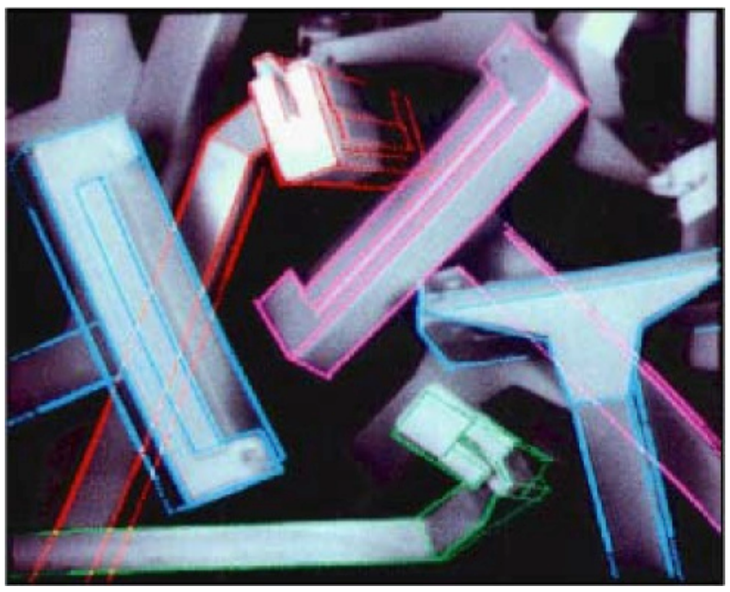
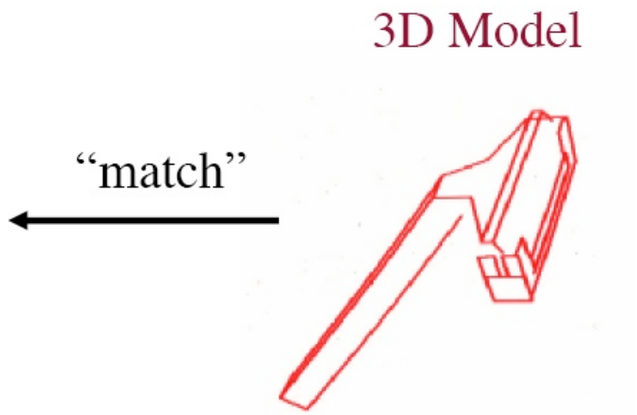
- Simple approach [David Lowe]
  - 3. "Project" model into the image and **"match" to lines** (solving for 3D pose)







Line drawing vs. edge detection

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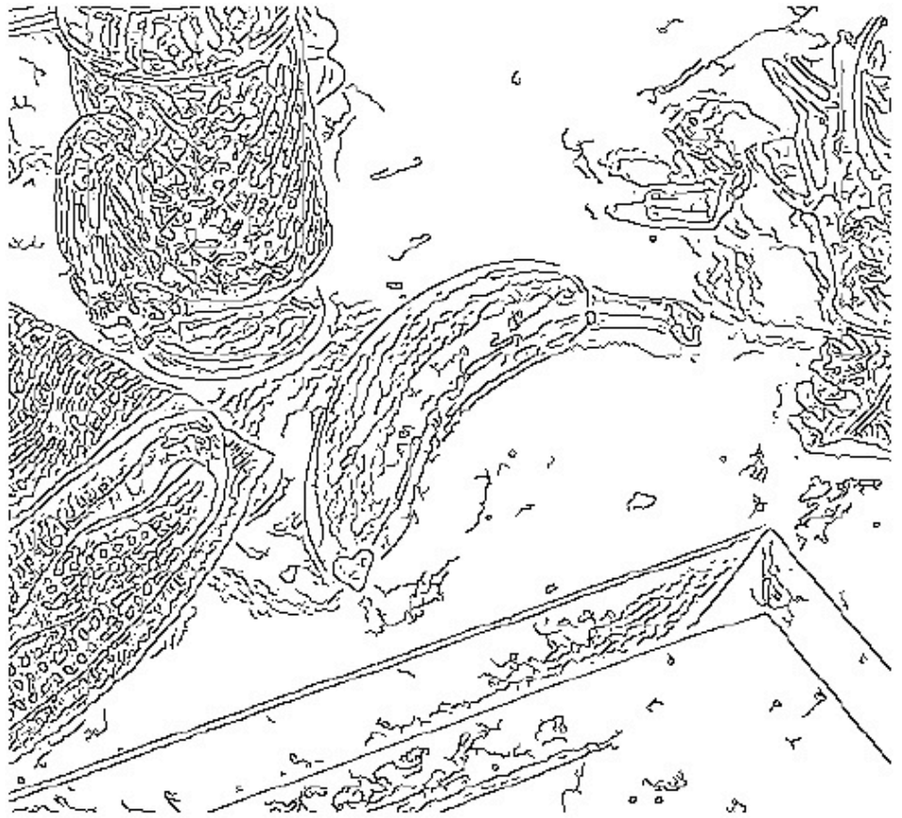

Models vs. edge detection

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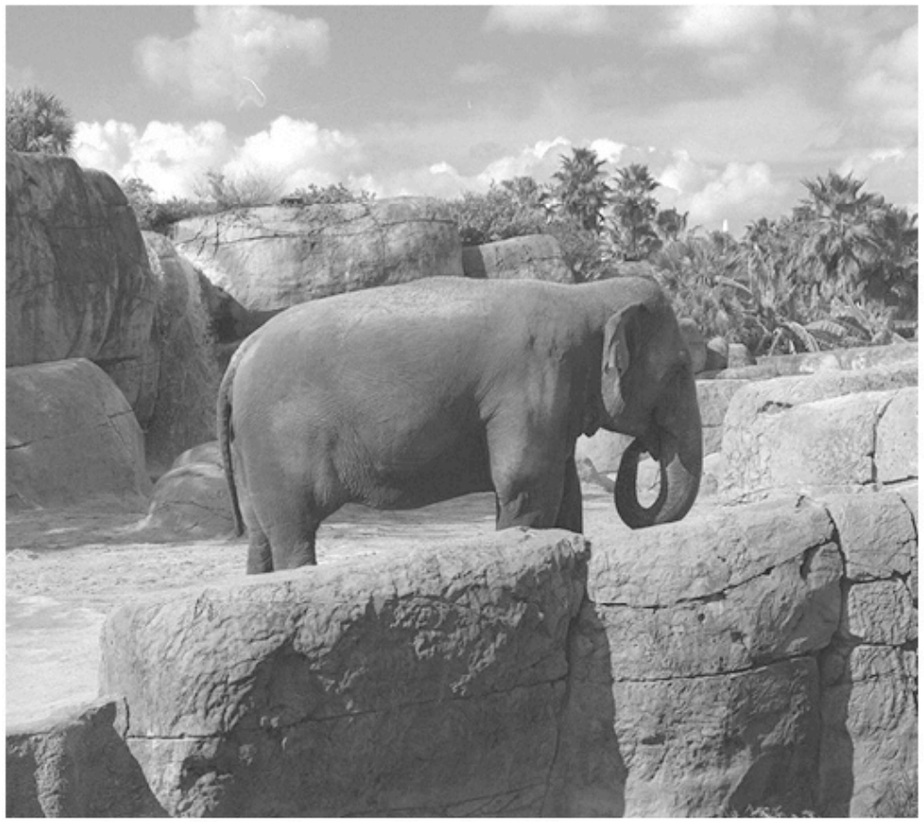

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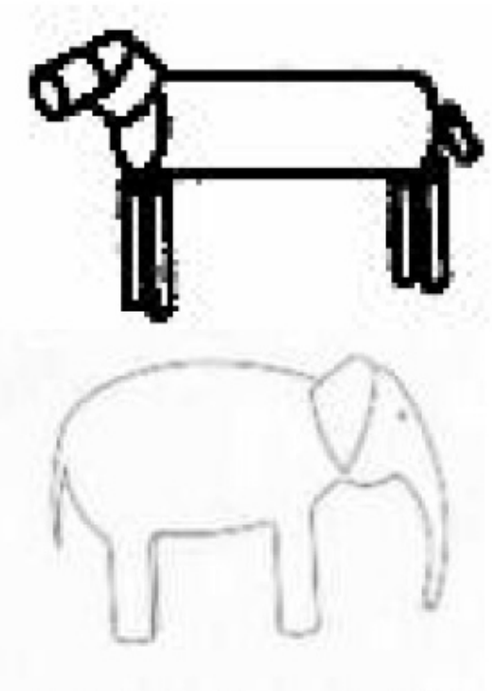

Match “model” to measurements?

Search and Recognition

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- What if we do not have a line-based object model?
  - Edges may not help us.

- How can we find the mouth?
- How can we recognize the “expression”?

Naïve View-Based Approach

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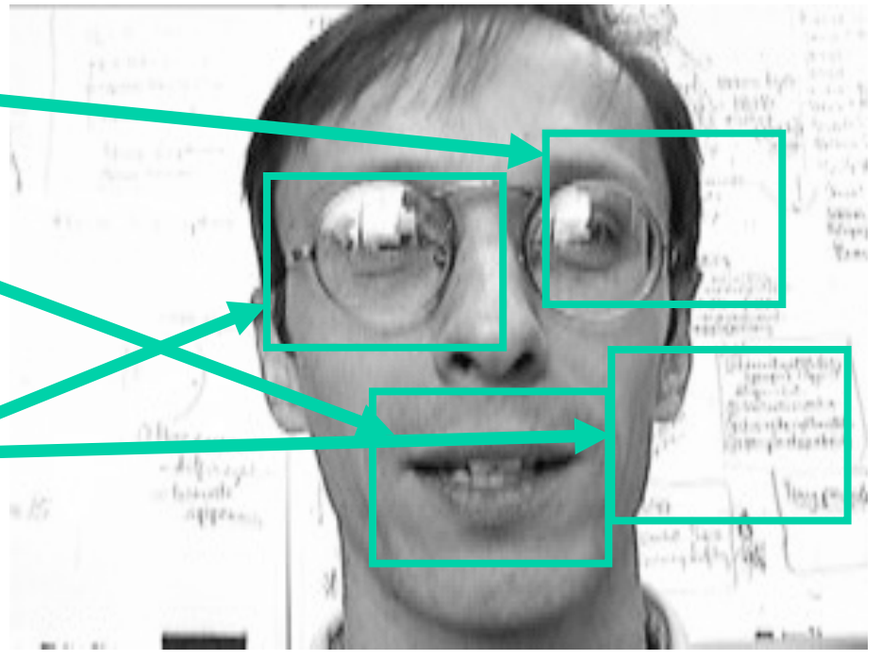
Database of mouth “templates”



- Search every image region (at every scale).
- Compare each template; chose the best match.

View-Based Methods

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- Idea: Represent objects by their appearance in an ensemble of images, including different poses, illuminations, configurations of shape, …
- Approach covered here:
  - Subspace methods (also called “Eigen methods”)

Images as Vectors

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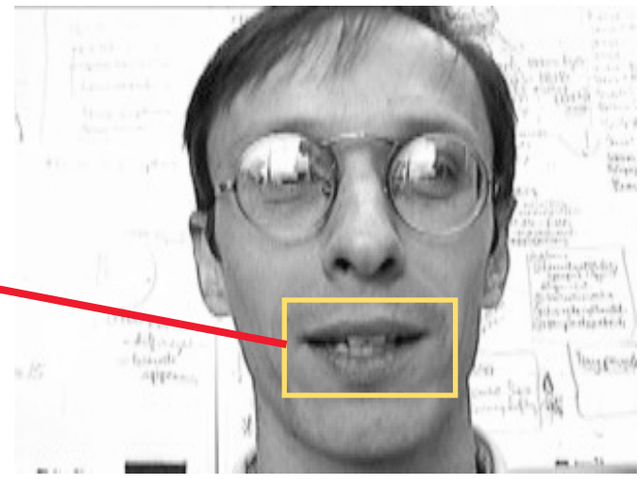
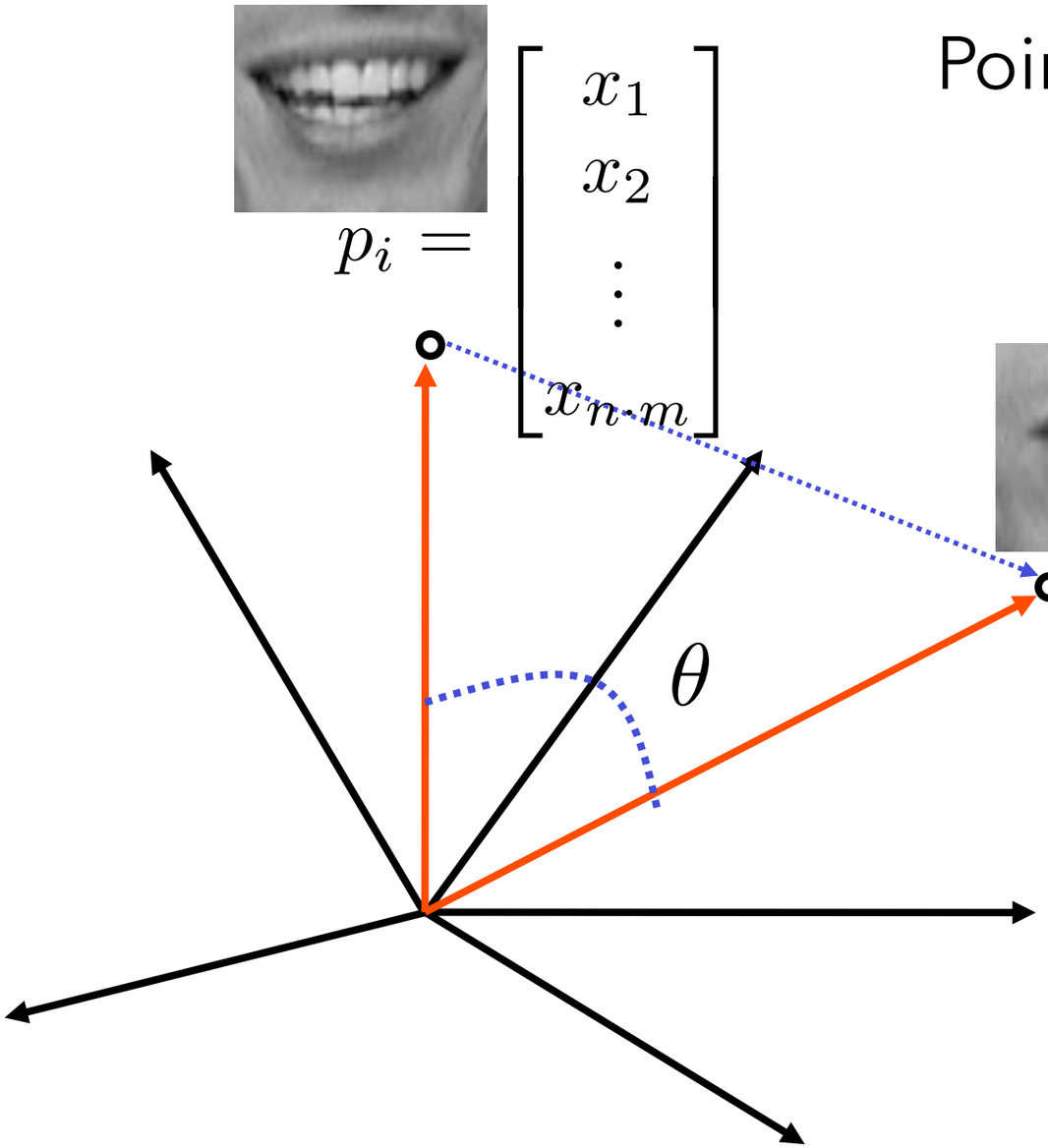
$$= \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n \cdot m} \end{bmatrix}$$

$$= \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n \cdot m} \end{bmatrix}$$



e.g. standard lexicographic ordering

Images as Points



Points in n x m dimensional space

Matching involves deciding how far apart they are in this space.

Filters are Templates

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- Remember image filtering from last lecture.
- Applying a filter at some pixel can be seen as taking a dot-product between the image and some vector.
- Filtering the image is a set of dot products:

$$f[m,n] = (I * g)[m,n] = \sum_{k,l} I[m-k, n-l] g[k,l]$$

Filters as Templates

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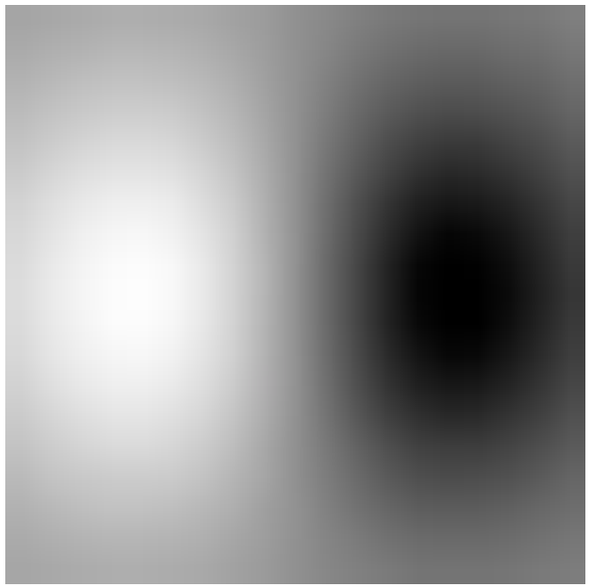
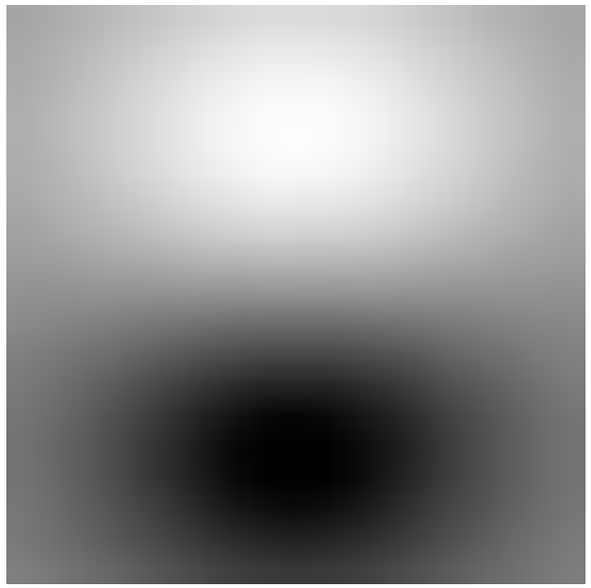
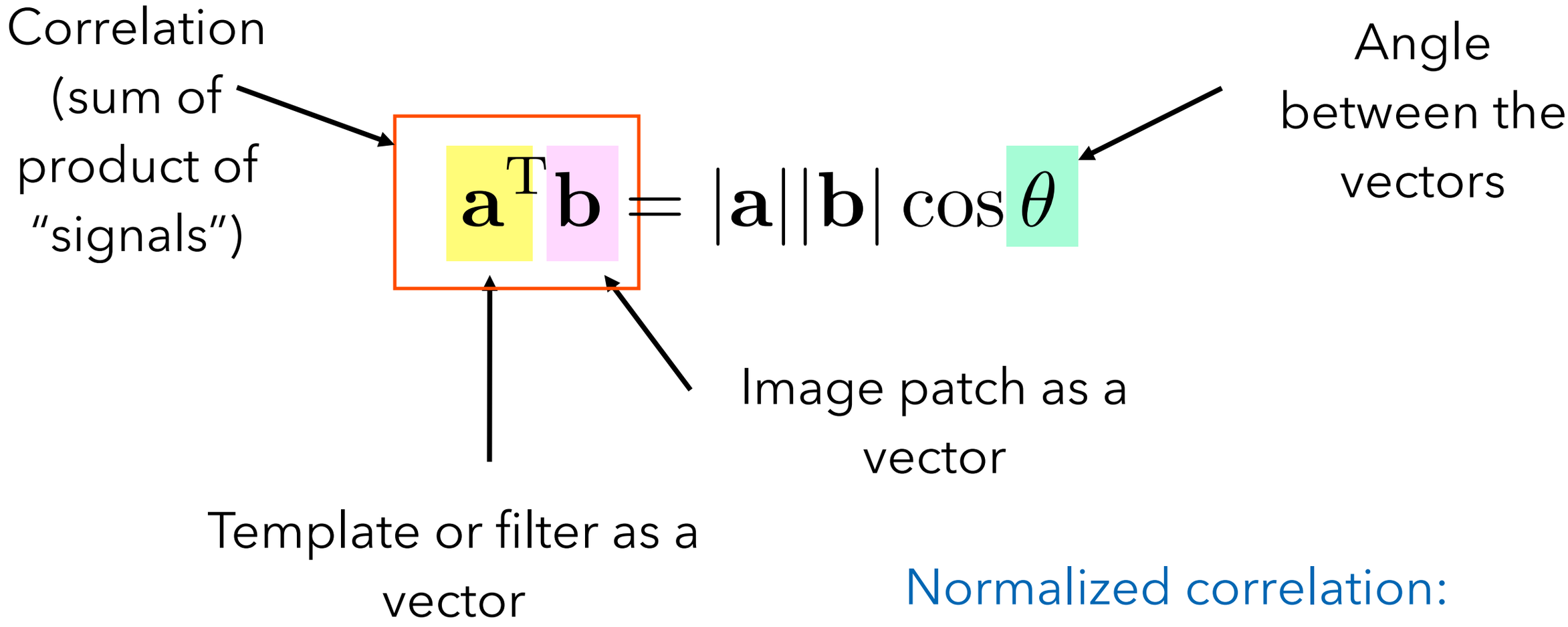
- Insight:
  - Filters look like the effects they are intended to find.
  - Filters find effects they look like.
  - We can use filters to match a template against an image.





Template Matching



Normalized correlation:

$$\frac{\mathbf{a}^T\mathbf{b}}{|\mathbf{a}||\mathbf{b}|} = \cos\theta$$

Dot Product

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- When we filter, we measure the angle (cosine of it, really) between the filter template and the image patch.

$$p_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n \cdot m} \end{bmatrix}$$

$$p_j = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n \cdot m} \end{bmatrix}$$

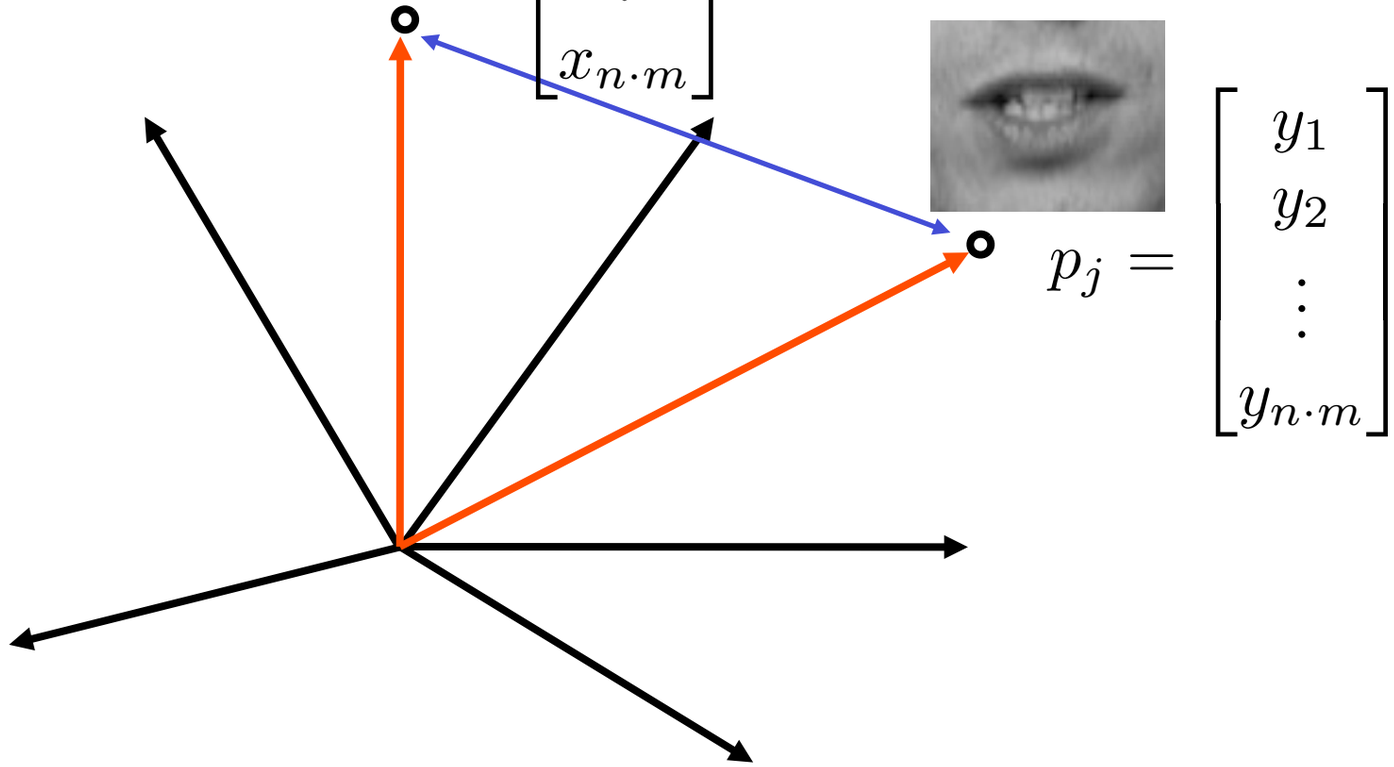
SSD Matching

- An alternative is to minimize the Sum of Squared Differences (SSD):

$$E(I,T) = \sum_{i,j} \left(I(i,j) - T(i,j)\right)^2$$

- Distance metric

$$p_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n \cdot m} \end{bmatrix}$$



$$p_j = \begin{bmatrix} y_1 \\ y_2 \\ \vdots \\ y_{n \cdot m} \end{bmatrix}$$

Template Methods

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- Image templates (simplest view-based method – straw man):
  - Keep an image of every object from different viewing directions, lighting conditions, etc.
  - Nearest neighbor cross-correlation matching with images in model database (or robust matching for clutter & occlusion).
- Obvious problems:
  - Storage and computation costs become unreasonable as the number of objects increases.
  - May require very large ensemble of 'training' images.

Fleet & Szeliski

Subspace Methods

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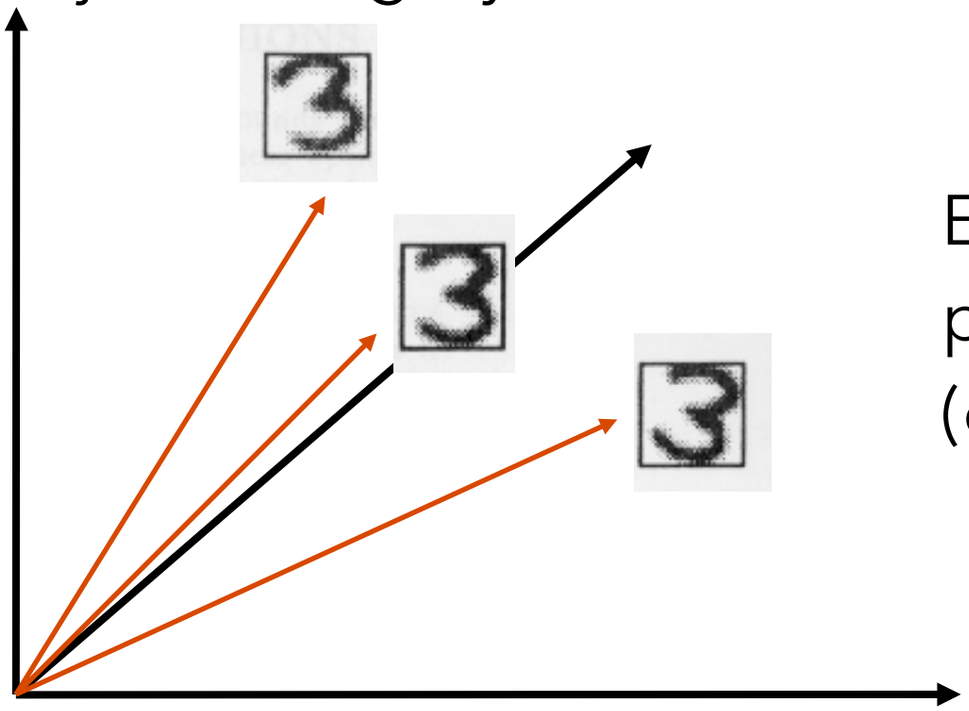
- How can we find more efficient representations for the ensemble of views, and more efficient methods for matching?
  - Idea: images are not random… especially images of the same object (category) have similar appearance



E.g., let images be represented as points in a high-dimensional space (e.g., one dimension per pixel)

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Linear Dimensionality Reduction

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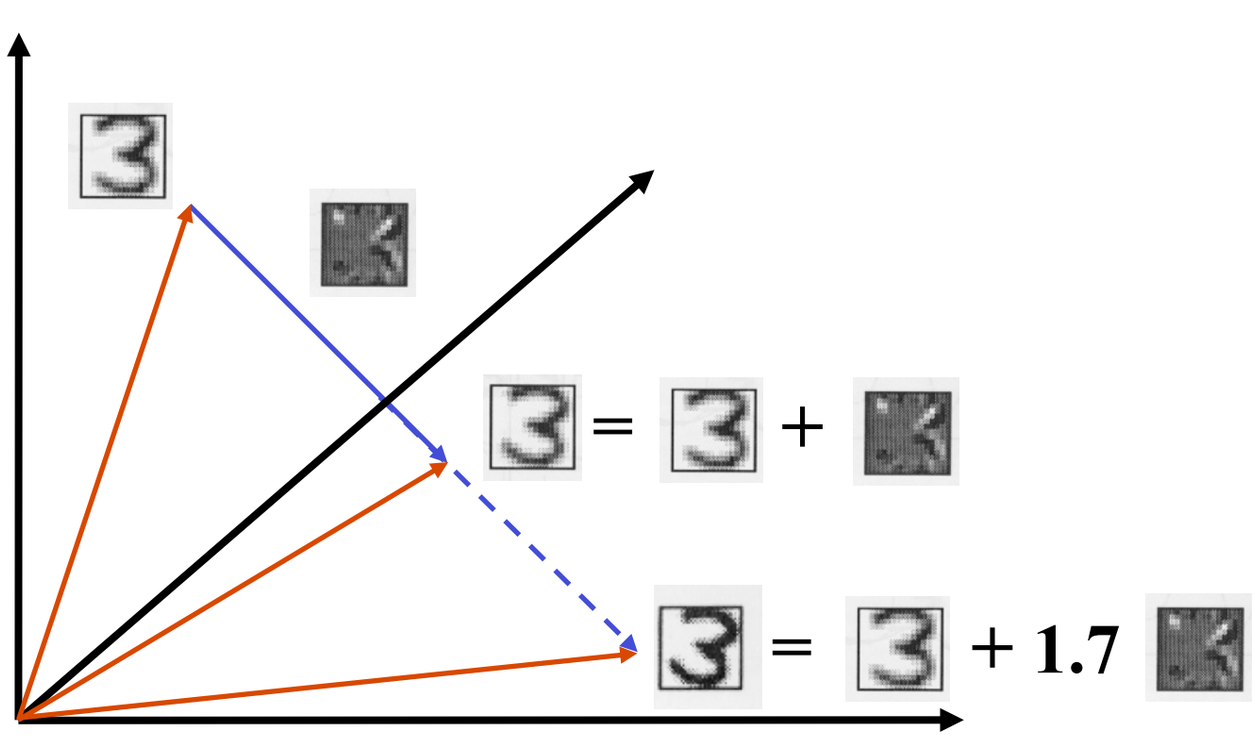
- Given that differences are structured, we can use 'basis images' to transform images into other images in the same space.



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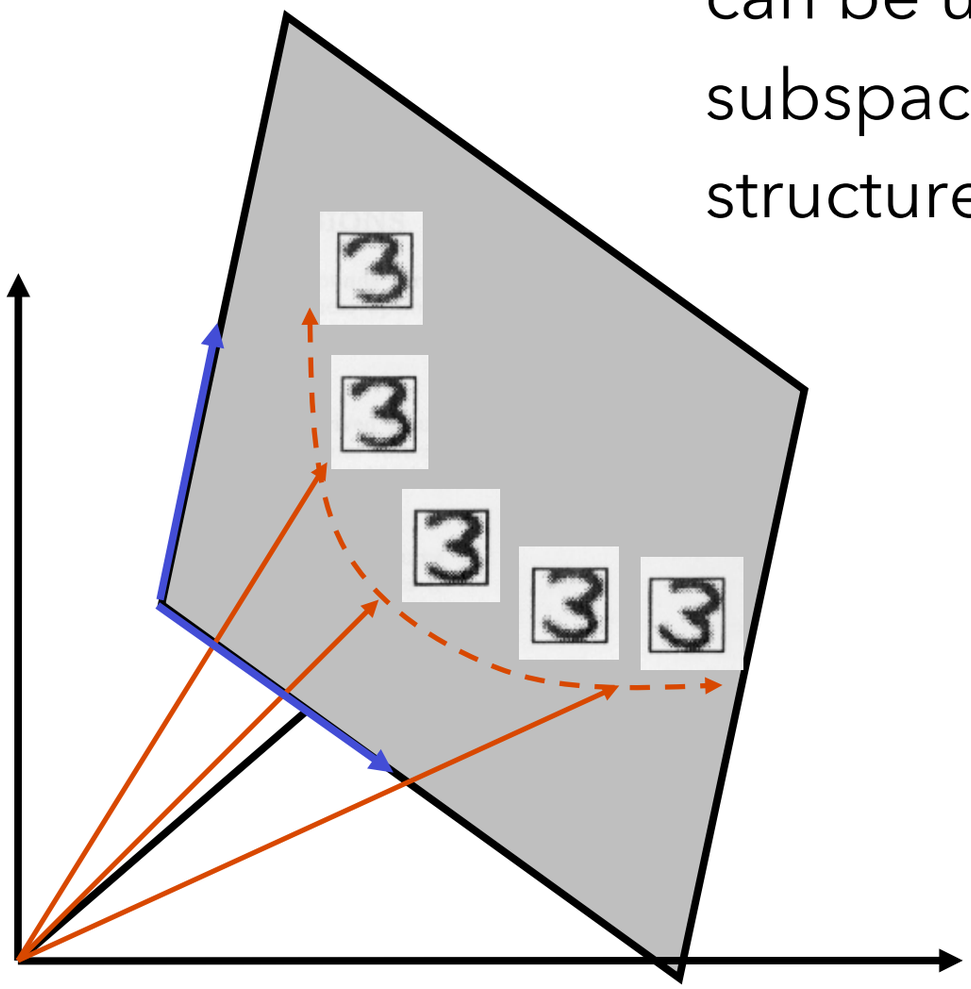
Linear Dimensionality Reduction



What linear transformations of the images can be used to define a lower-dimensional subspace that captures most of the structure in the image ensemble?

Fleet & Szeliski

Approach

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- Find a lower dimensional representation that captures the variability in the data.
- Search using this low dimensional model.

Goal

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Data point n:

$$\mathbf{x}^n \in \mathbb{R}^M$$

Low-dimensional representation:

$$\mathbf{a}^n \in \mathbb{R}^D \qquad D << M$$

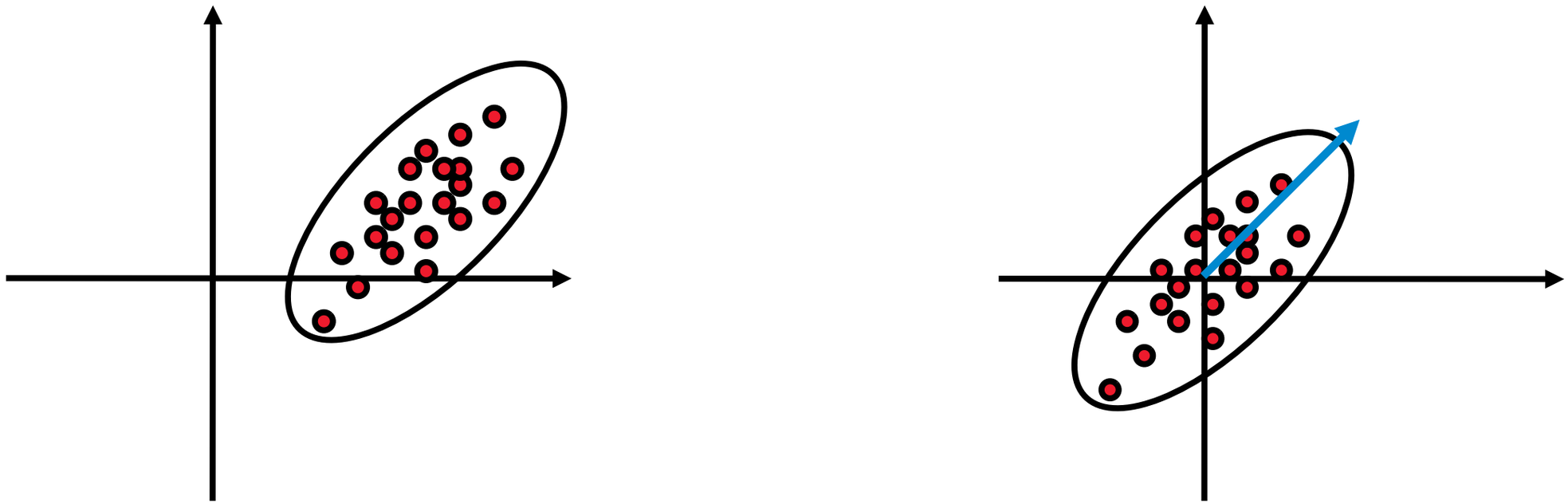
Mapping: $\mathbf{x}^n \rightarrow \mathbf{a}^n$

- Restrict this mapping to be a linear function:

$$\boldsymbol{a}^n = \mathbf{B}\mathbf{x}^n, \qquad \mathbf{B} \in \mathbb{R}^{D \times M}$$

- **Note:**
  - From now on, I will use boldface symbols to denote vectors, e.g.: $\mathbf{x}, \mathbf{y}, \ldots$

Intuition



If I give you the mean and one vector to represent the data, what vector would you choose?

Why?

Recall: Basis Representations

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We can always write a vector as:

$$\mathbf{x} = \sum_{i=1}^{M} a_i \mathbf{u}_i \qquad \text{where} \qquad \mathbf{u}_i^{\mathrm{T}} \mathbf{u}_j = \delta_{ij}$$

Kronecker delta = 1 if $i = j$, 0 otherwise.

orthonormal

Example:

$$\begin{bmatrix} 3 \\ 7 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 0 \end{bmatrix} + 7 \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

Recall: Basis Representations

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$$
\begin{aligned}
\mathbf{x} &= a_1\mathbf{u_1} + a_2\mathbf{u_2} \\
a_1 &= \mathbf{u}_1^{\mathrm{T}}(\mathbf{x} - a_2\mathbf{u}_2) \\
&= \mathbf{u}_1^{\mathrm{T}}\mathbf{x} - a_2\mathbf{u}_1^{\mathrm{T}}\mathbf{u}_2 \\
&= \mathbf{u}_1^{\mathrm{T}}\mathbf{x}
\end{aligned}
$$

More generally:

$$a_i = \underbrace{\mathbf{u}_i^{\mathrm{T}}\mathbf{x}}$$

Scalar coefficient

projection

Decomposition

$$\mathbf{x}^n = \sum_{i=1}^{D} a_i \mathbf{u}_i + \sum_{j=D+1}^{M} b_j \mathbf{u}_j$$

Approximation $\tilde{\mathbf{x}}^n$

Error

Want the D bases that minimize the mean squared error over the training data

$$\arg\min E(\mathbf{u}_1, \ldots, \mathbf{u}_D) = \sum_{n=1}^{N} ||\boldsymbol{x}^n - \tilde{\boldsymbol{x}}^n||^2$$

Minimizing the Error

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- Rewrite the error (assuming a single basis vector):

$$E(\mathbf{u}) = \sum_{n=1}^{N} ||\mathbf{x}^n - \tilde{\mathbf{x}}^n||^2$$

$$= \sum_{n=1}^{N} ||\mathbf{x}^n - (\mathbf{u}^{\mathrm{T}}\mathbf{x}^n)\mathbf{u}||^2$$

$$= \sum_{n=1}^{N} ||\mathbf{x}^n||^2 - 2(\mathbf{u}^{\mathrm{T}}\mathbf{x}^n)^2 + (\mathbf{u}^{\mathrm{T}}\mathbf{x}^n)^2 \cdot \mathbf{u}^{\mathrm{T}}\mathbf{u}$$

$$= \sum_{n=1}^{N} ||\mathbf{x}^n||^2 - (\mathbf{u}^{\mathrm{T}}\mathbf{x}^n)^2 = \sum_{n=1}^{N} ||\mathbf{x}^n||^2 - (a_n)^2$$

Minimizing the Error

- Rewrite the error (assuming a single basis vector):

$$E(\mathbf{u}) = \sum_{n=1}^{N} ||\mathbf{x}^n||^2 - (\mathbf{u}^{\mathrm{T}}\mathbf{x}^n)^2 = \sum_{n=1}^{N} ||\mathbf{x}^n||^2 - (a_n)^2$$

  - Minimizing the error is equivalent to maximizing the variance of the projection:

$$\frac{1}{N}\sum_{n=1}^{N} a_n^2 \qquad \text{(assuming mean 0)}$$

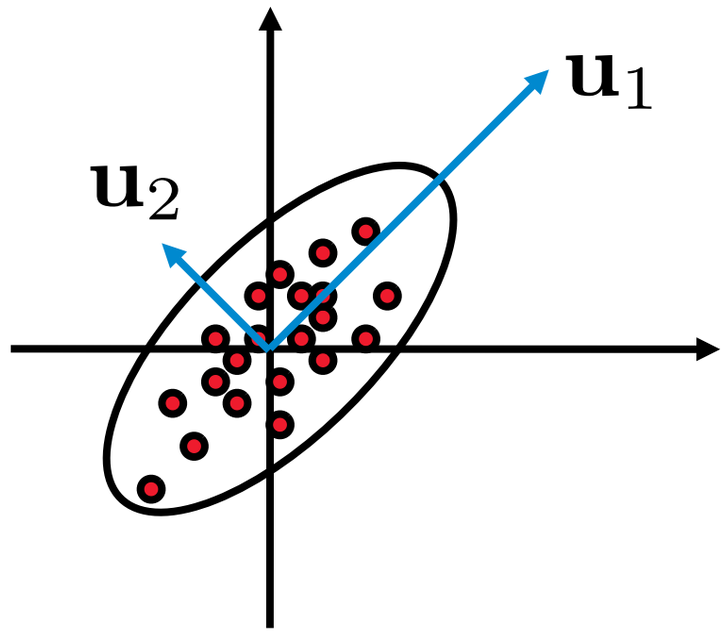
- We can ensure a zero mean projection by subtracting the mean from every data point:

$$\mathbf{x}^n - \bar{\mathbf{x}}$$

Intuition

$$\mathbf{x}^n - \bar{\mathbf{x}} = \sum_{i=1}^{D} a_i \mathbf{u}_i + \sum_{j=D+1}^{M} b_j \mathbf{u}_j$$

$$\tilde{\mathbf{x}}^n = \sum_{i=1}^{D} a_i \mathbf{u}_i + \bar{\mathbf{x}}$$



Projecting onto $\mathbf{u}_1$ captures the majority of the variance and hence projecting onto it minimizes the error

How do we find the axis of largest variance?

Statistics Review: Univariate

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Sample mean:

(univariate case)

$$\bar{x} = \langle x \rangle = E[x] = \frac{1}{N} \sum_{n=1}^{N} x^n$$

Sample variance:

$$\begin{aligned} \hat{\sigma}^2 &= \left\langle (x - \bar{x})^2 \right\rangle = \mathrm{Var}[x] = \frac{1}{N} \sum_{n=1}^{N} (x^n - \bar{x})^2 \\ &= \left\langle x^2 \right\rangle - \langle 2\bar{x}x \rangle + \left\langle \bar{x}^2 \right\rangle = \left\langle x^2 \right\rangle - \bar{x}^2 \end{aligned}$$

Statistics Review: Multivariate

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Sample mean:

(multivariate case)

$$\bar{\mathbf{x}} = \langle \mathbf{x} \rangle = E[\mathbf{x}] = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}^n$$

Sample covariance:

$$\begin{aligned} \mathrm{cov}(x_i, x_j) = \sigma_{ij} &= \langle (x_i - \bar{x}_i)(x_j - \bar{x}_j) \rangle \\ &= \langle x_i x_j \rangle - \langle x_i \bar{x}_j \rangle - \langle \bar{x}_i x_j \rangle + \langle \bar{x}_i \bar{x}_j \rangle \\ &= \langle x_i x_j \rangle - \bar{x}_i \bar{x}_j \\ \mathrm{cov}(x_i, x_i) = \sigma_i^2 &= \langle x_i^2 \rangle - \bar{x}_i^2 \end{aligned}$$

Statistical Correlation

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- The covariance of two variates X_i and X_j of a random vector $\mathbf{X}$ provides a measure of how strongly correlated these variables are, and the derived quantity is:

$$\mathrm{cor}(x_i, x_j) = \frac{\mathrm{cov}(x_i, x_j)}{\sigma_i \sigma_j}$$

Covariance Matrix

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- The covariance matrix describes the covariance between all components of a random vector, e.g.:

$$\mathbf{C} = \begin{bmatrix} \sigma_{11} & \sigma_{12} \\ \sigma_{21} & \sigma_{22} \end{bmatrix}$$

$$\hat{\mathbf{C}} = \frac{1}{N} \sum_{n=1}^{N} (\mathbf{x}^n - \bar{\mathbf{x}})(\mathbf{x}^n - \bar{\mathbf{x}})^{\mathrm{T}}$$

Recall: Outer Product

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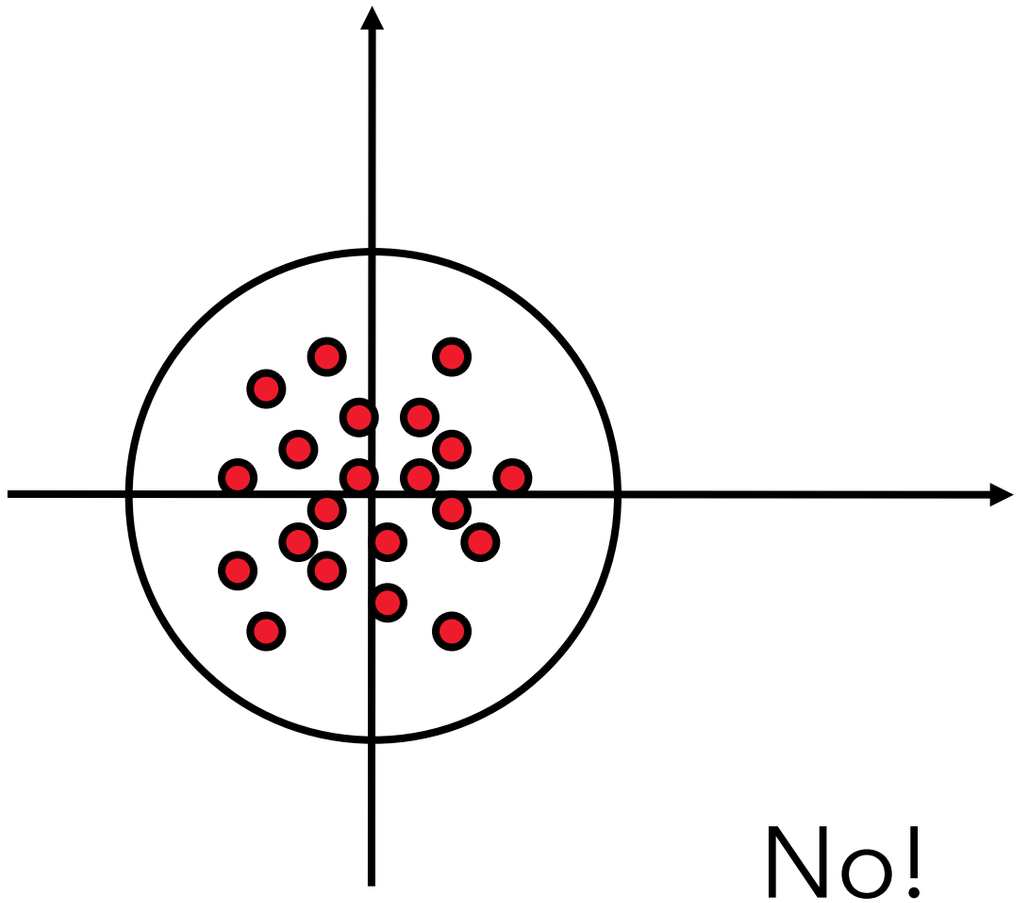
$$\mathbf{x}\mathbf{x}^{\mathrm{T}} = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \begin{bmatrix} x_1 & x_2 \end{bmatrix} = \begin{bmatrix} x_1 x_1 & x_1 x_2 \\ x_2 x_1 & x_2 x_2 \end{bmatrix}$$

Correlated?



No! $\mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & c \end{bmatrix}$

Correlated?

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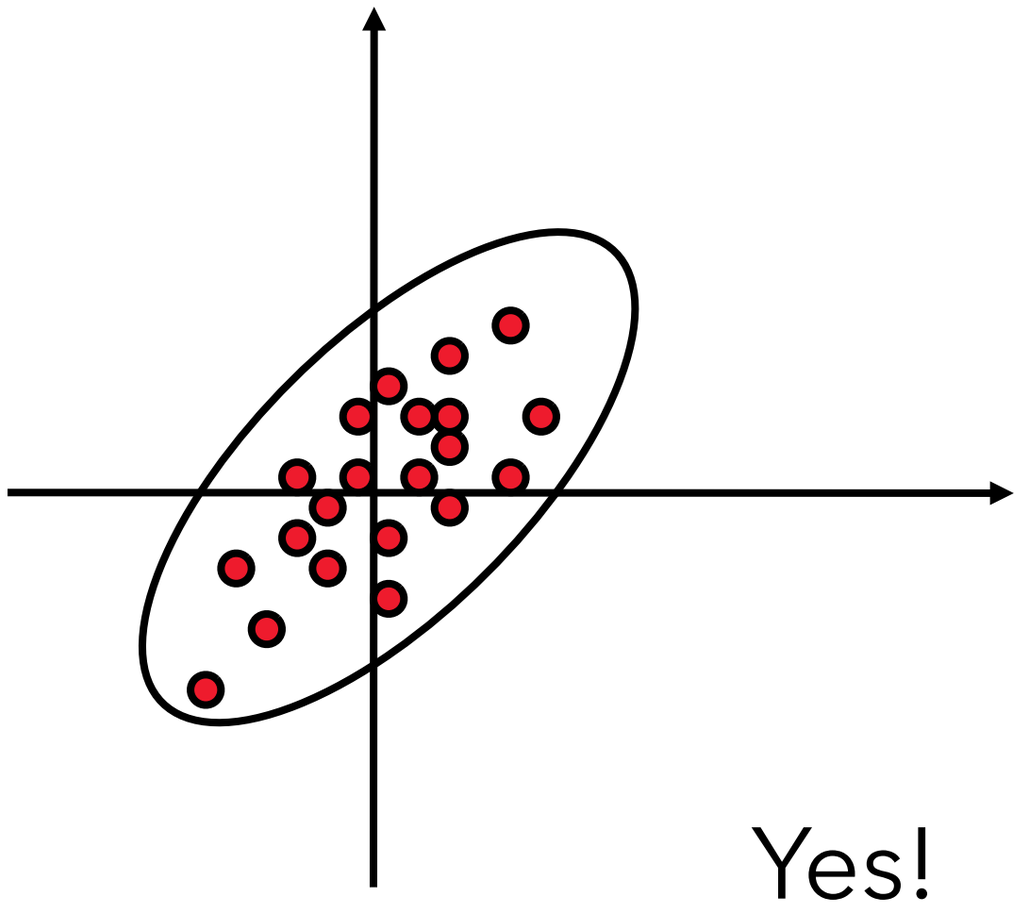

Yes! $\mathbf{C} = \begin{bmatrix} a & b \\ b & c \end{bmatrix}$

Correlated?

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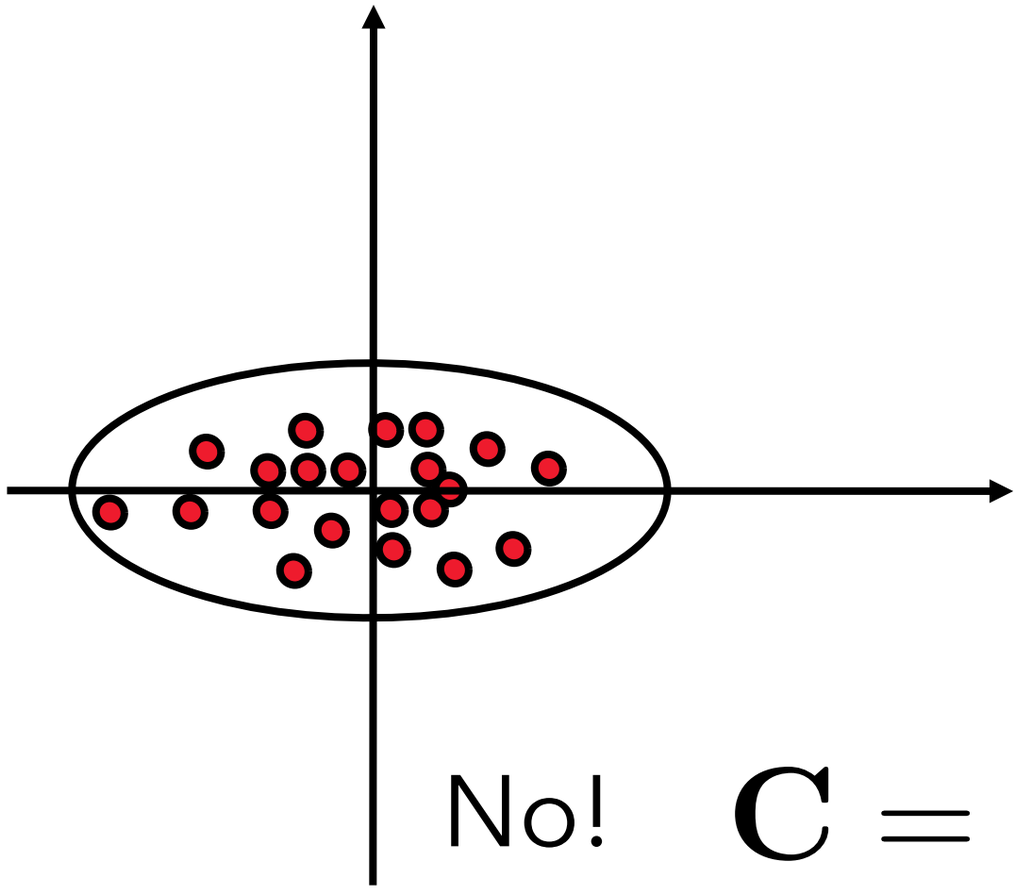
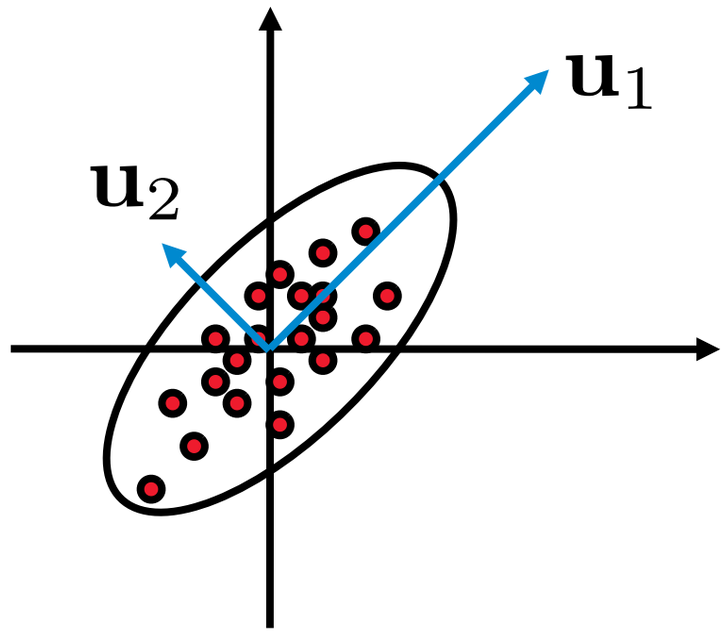


No! $\mathbf{C} = \begin{bmatrix} c & 0 \\ 0 & d \end{bmatrix}, \quad c > d$

Intuition

$$\mathbf{x}^n - \bar{\mathbf{x}} = \sum_{i=1}^{D} a_i \mathbf{u}_i + \sum_{j=D+1}^{M} b_j \mathbf{u}_j$$

$$\tilde{\mathbf{x}}^n = \sum_{i=1}^{D} a_i \mathbf{u}_i + \bar{\mathbf{x}}$$



Data is correlated in its original coordinate system!

Note that axes of the projection are orthogonal and **decorrelate** the data; i.e. in the coordinate frame of these axes, the data is uncorrelated.

(side note: this only works for Gaussians).

Formulation of problem

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- Let $\mathbf{X} = [\mathbf{x}^1, \ldots, \mathbf{x}^N] \in \mathbb{R}^{M \times N}$ be a matrix of N vectors in M-dimensional input space.
- Let $\mathbf{u} \in \mathbb{R}^M$ be a direction (a vector of length 1) in the input space.
- The projection of the n-th vector $\mathbf{x}^n$ onto the vector $\mathbf{u}$ can be calculated as:

$$a_n = \mathbf{u}^{\mathrm{T}} \mathbf{x}^n = \sum_{n=1}^{M} x_i^n u_i$$

- The goal is to find a direction $\mathbf{u}$ that maximizes the variance of the projections of all input vectors $\mathbf{x}^n, \quad n = 1 \ldots N$

Mean & Variance

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- The mean of the projection

$$\bar{a} = \frac{1}{N}\sum_{n=1}^{N} a_n = \frac{1}{N}\sum_{n=1}^{N} \mathbf{u}^{\mathrm{T}}\mathbf{x}^n = \frac{1}{N}\mathbf{u}^{\mathrm{T}}\left[\sum_{n=1}^{N} \mathbf{x}^n\right] = \mathbf{u}^{\mathrm{T}}\bar{\mathbf{x}}$$

- ... is the projection of the mean.

Mean & Variance

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- Variance of the projection:

$$\sigma^2 = \frac{1}{N}\sum_{n=1}^{N}(a_n - \bar{a})^2 = \frac{1}{N}\sum_{n=1}^{N}(\mathbf{u}^{\mathrm{T}}\mathbf{x}^n - \mathbf{u}^{\mathrm{T}}\bar{\mathbf{x}})^2$$

$$= \frac{1}{N}\sum_{n=1}^{N}\left(\mathbf{u}^{\mathrm{T}}(\mathbf{x}^n - \bar{\mathbf{x}})\right)^2 = \frac{1}{N}\sum_{n=1}^{N}\mathbf{u}^{\mathrm{T}}(\mathbf{x}^n - \bar{\mathbf{x}})(\mathbf{x}^n - \bar{\mathbf{x}})^{\mathrm{T}}\mathbf{u}$$

$$= \mathbf{u}^{\mathrm{T}}\left[\frac{1}{N}\sum_{n=1}^{N}(\mathbf{x}^n - \bar{\mathbf{x}})(\mathbf{x}^n - \bar{\mathbf{x}})^{\mathrm{T}}\right]\mathbf{u} = \mathbf{u}^{\mathrm{T}}\mathbf{C}\mathbf{u}$$

covariance matrix

Maximizing variance

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- Maximize σ^2 under the constraint that $||\mathbf{u}|| = 1$.
  - Maximize the following function (Langrangian):

$$F(u;\lambda) = \mathbf{u}^{\mathrm{T}}\mathbf{C}\mathbf{u} - \lambda(||\mathbf{u}||^2 - 1) = \mathbf{u}^{\mathrm{T}}\mathbf{C}\mathbf{u} - \lambda(\mathbf{u}^{\mathrm{T}}\mathbf{u} - 1)$$

$$\frac{\partial F(u;\lambda)}{\partial \mathbf{u}} = 2\mathbf{C}\mathbf{u} - 2\lambda\mathbf{u} = 0$$

$$\Leftrightarrow \quad \mathbf{C}\mathbf{u} = \lambda\mathbf{u}$$

with $||\mathbf{u}|| = 1$

- Compute eigenvectors and eigenvalues of $\mathbf{C}$:
- The largest eigenvalue = maximal variance.
- Corresponding eigenvector = direction with the max. variance

Eigendecomposition

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- Let the eigenvectors and eigenvalues of $\mathbf{C}$ be $\mathbf{u}_k$ and λ_k for $k \leq M$, i.e., $\mathbf{C}\mathbf{u}_k = \lambda_k \mathbf{u}_k$ with $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_M$

- In matrix form: $\mathbf{C}\mathbf{U} = \mathbf{U}\Lambda$, where $\Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_M)$ and $\mathbf{U} = [\mathbf{u}_1, \ldots, \mathbf{u}_M]$.

- Because $\mathbf{U}$ is orthonormal (we usually assume that the eigenvectors have unit norm) we know that $\mathbf{U}\mathbf{U}^{\mathrm{T}} = \mathbf{I}$.

- This means that we can decompose $\mathbf{C}$ as $\boxed{\mathbf{C} = \mathbf{U}\Lambda\mathbf{U}^{\mathrm{T}}}$

Principal Component Analysis

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- $\mathbf{C}$ is real, symmetric, and positive-definite. Thus we can decompose it as:

$$\mathbf{C} = \mathbf{U}\Lambda\mathbf{U}^{\mathrm{T}} = \begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_M \end{bmatrix} \begin{bmatrix} \lambda_1 & & & \\ & \lambda_2 & & \\ & & \ddots & \\ & & & \lambda_M \end{bmatrix} \begin{bmatrix} \mathbf{u}_1^{\mathrm{T}} \\ \mathbf{u}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{u}_M^{\mathrm{T}} \end{bmatrix}$$

Orthonormal columns

eigenvectors

eigenvalues

- This is the eigendecomposition of $\mathbf{C}$

Principal Component Analysis

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- Observation: If $\lambda_k \approx 0$ for $k > D$ for some $D << M$ then we can use the subset of the first D eigenvectors to define a basis for approximating the data vectors.

$$\mathbf{x}^n - \bar{\mathbf{x}} = \sum_{i=1}^{D} a_i \mathbf{u}_i + \sum_{j=D+1}^{M} b_j \mathbf{u}_j$$

$$\mathbf{x}^n \approx \tilde{\mathbf{x}}^n = \bar{\mathbf{x}} + \sum_{i=1}^{D} a_i \mathbf{u}_i \quad \text{where} \quad a_i = \mathbf{u}_i^{\mathrm{T}}(\mathbf{x}^n - \bar{\mathbf{x}})$$

- This representation has the minimal mean squared error (MSE) of all linear representations of dimension D :

$$\min E(\mathbf{u}_1, \ldots, \mathbf{u}_D) = \sum_{n=1}^{N} ||\mathbf{x}^n - \tilde{\mathbf{x}}^n||^2$$

Principal Component Analysis

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- Now we know how we can represent our data in a lower dimensional space in a principled fashion.
  - We compute the mean of the data, and subtract it.
  - We compute the covariance matrix, decompose it, and choose a the first D eigenvalues.
  - This gives us an (eigen)basis for representing the data:

$$\mathbf{a}^n = \mathbf{B}^{\mathrm{T}}(\mathbf{x}^n - \bar{\mathbf{x}}) \qquad \text{where} \qquad \mathbf{B} = [\mathbf{u}_1, \ldots, \mathbf{u}_D]$$
$$\tilde{\mathbf{x}}^n = \bar{\mathbf{x}} + \mathbf{B}\mathbf{a}^n$$

  - It is common to also normalize the variance of each dimension.
- Question:
  - How do we choose D?

Choosing D

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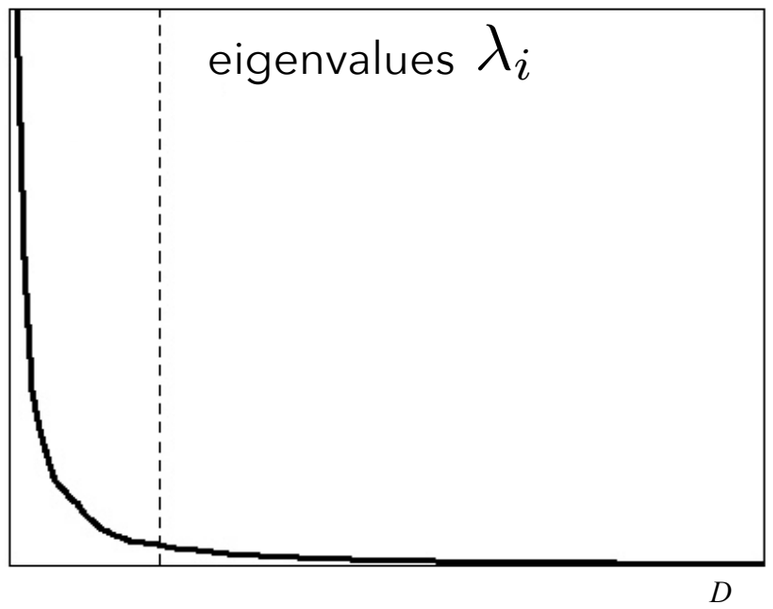
- Larger D leads to better approximation.
- There are at least 2 good possibilities for choosing D:
  - First, choose D based on application performance, i.e. choose the smallest D that makes the application work well enough.
  - Second, choose D so that the eigenbasis captures some fraction of the variance, say $\eta = 0.9$
    - Remember: Eigenvalue λ_i describes the marginal variance captured by $\mathbf{u}_i$
    - Choose D using

$$\sum_{i=1}^{D} \lambda_i \geq \eta \sum_{i=1}^{M} \lambda_i$$



PCA in Practice

$$p_i = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_{n \times m} \end{bmatrix}$$

- Suppose we want to find a low-dimensional representation for mouth images.
- Let's assume the images are of moderate size: 128×128
- How "big" is the covariance matrix?
  - The data vector has $2^7 \cdot 2^7 = 2^{14}$ dimensions.
  - So the covariance matrix has 2^{28} entries.
  - Takes up 2^{31} Bytes, i.e. 2GB.
  - Too large to handle efficiently.
- What can we do to reduce this?

More Efficiency

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- Observation 1:
  - While we may work in spaces that have 100000s of dimensions, we often only have 100s or 1000s of examples.
- Observation 2:
  - We do not need to compute all eigenvectors and eigenvalues. Computing a few dozen of the largest ones usually suffices.
- But how do we exploit this?

Rewrite PCA

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- Data vectors in matrix form: $\mathbf{X} = [\mathbf{x}^1, \ldots, \mathbf{x}^N]$

- First subtract the mean: $\bar{\mathbf{x}} = \frac{1}{N} \sum_{n=1}^{N} \mathbf{x}^n$

$$\hat{\mathbf{X}} = \mathbf{X} - [\bar{\mathbf{x}}, \ldots, \bar{\mathbf{x}}]$$

- Express the covariance matrix differently:

$$\hat{\mathbf{C}} = \frac{1}{N} \sum_{n=1}^{N} (\mathbf{x}^n - \bar{\mathbf{x}})(\mathbf{x}^n - \bar{\mathbf{x}})^{\mathrm{T}} = \frac{1}{N} \hat{\mathbf{X}} \hat{\mathbf{X}}^{\mathrm{T}}$$

Singular Value Decomposition

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- SVD is a generalization of the eigendecomposition to rectangular matrices $\hat{\mathbf{X}} \in \mathbb{R}^{M \times N}$

$$\hat{\mathbf{X}} = \mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}} = \underbrace{\begin{bmatrix} \mathbf{u}_1 & \mathbf{u}_2 & \dots & \mathbf{u}_M \end{bmatrix}}_{M \times M} \underbrace{\begin{bmatrix} s_1 & & & \\ & s_2 & & \\ & & \ddots & \\ & & & \end{bmatrix}}_{M \times N} \underbrace{\begin{bmatrix} \mathbf{v}_1^{\mathrm{T}} \\ \mathbf{v}_2^{\mathrm{T}} \\ \vdots \\ \mathbf{v}_N^{\mathrm{T}} \end{bmatrix}}_{N \times N}$$

Orthonormal columns
left-singular vectors

singular values
(non-negative)

Orthonormal rows
right-singular vectors

How to use SVD for PCA

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- Decompose the data matrix $\hat{\mathbf{X}}$ with SVD:

$$
\begin{aligned}
\frac{1}{N}\hat{\mathbf{X}}\hat{\mathbf{X}}^{\mathrm{T}} &= \frac{1}{N}\mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}\left(\mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}\right)^{\mathrm{T}} \\
&= \frac{1}{N}\mathbf{U}\mathbf{S}\mathbf{V}^{\mathrm{T}}\mathbf{V}\mathbf{S}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}} \\
&= \frac{1}{N}\mathbf{U}\mathbf{S}\mathbf{S}^{\mathrm{T}}\mathbf{U}^{\mathrm{T}} \\
&= \mathbf{U}(\frac{1}{N}\mathbf{S}^2)\mathbf{U}^{\mathrm{T}} \\
&= \mathbf{U}\Lambda\mathbf{U}^{\mathrm{T}}
\end{aligned}
$$

How to use SVD for PCA

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- If we perform SVD on the data matrix (after subtracting the mean) then:
  - The left-singular vectors give us the eigenvectors of the covariance matrix.
  - The singular values let us easily compute the eigenvalues of the covariance matrix:

$$\lambda_i = \frac{1}{N} s_i^2$$

- **Advantage:** We never have to explicitly build and store the covariance matrix!

How does this help us?

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- $\mathbf{S} \in \mathbb{R}^{M \times N}$ is of manageable size now, and so is $\mathbf{V} \in \mathbb{R}^{N \times N}$
- But $\mathbf{U} \in \mathbb{R}^{M \times M}$ is still much too large.
- **Observation 1:** We only need the first N left-singular vectors, because only N singular values can be non-zero.
  - "Economy" decomposition, makes it all manageable memory-wise and computationally.
- Observation 2: We will typically need even fewer than N singular values and left-singular vectors.
  - Special variants that only compute a partial decomposition.

Generic Image Ensembles

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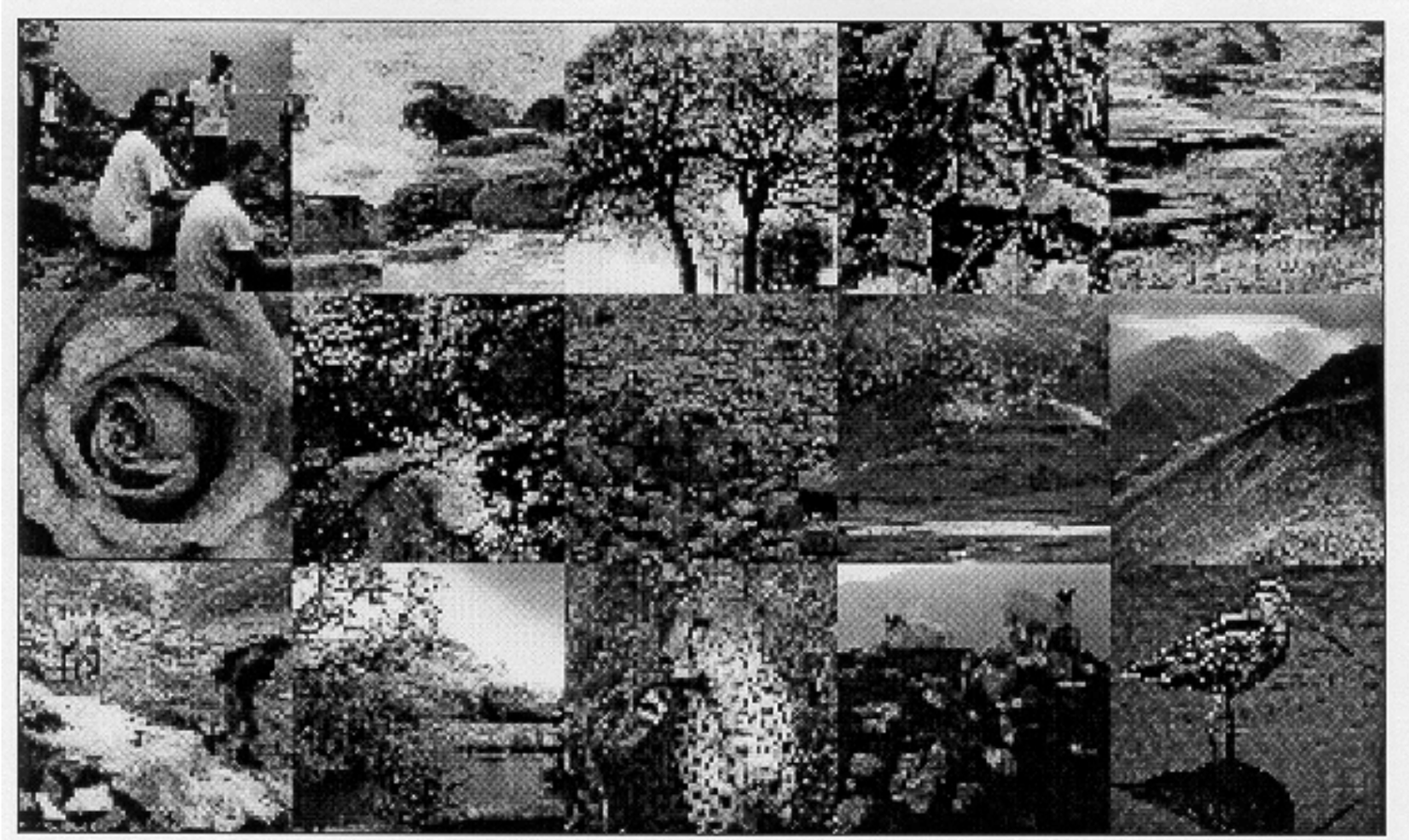

Fleet & Szeliski

Generic Image Ensembles

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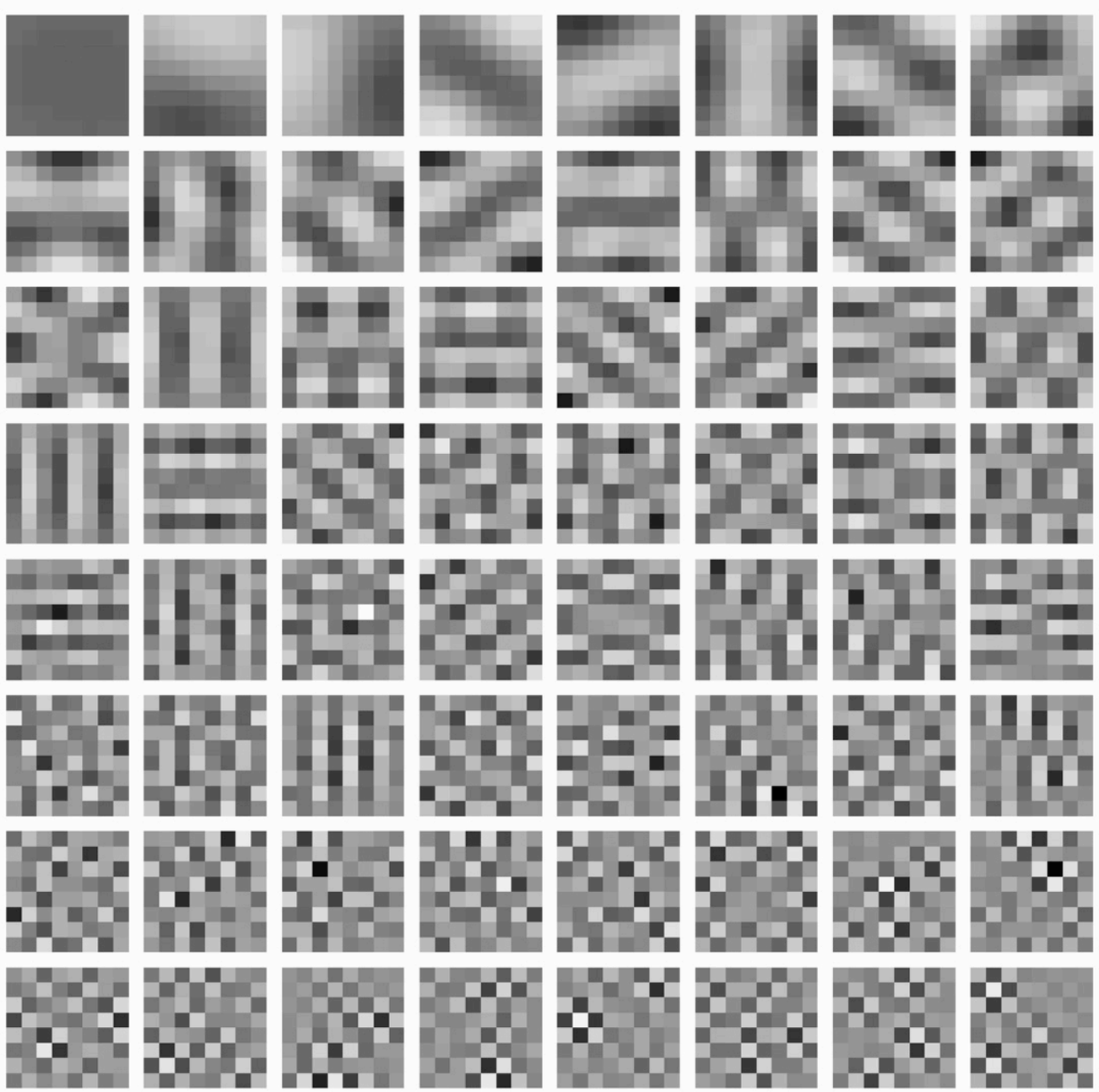
Is there a low dimensional model describing natural images?

Fleet & Szeliski

PCA of Natural Image Patches



What do these bases look like?

8X8 image patches

Fleet & Szeliski

Search and Recognition

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- How can we find the mouth?
- How can we recognize the “expression”?

Naïve View-Based Approach

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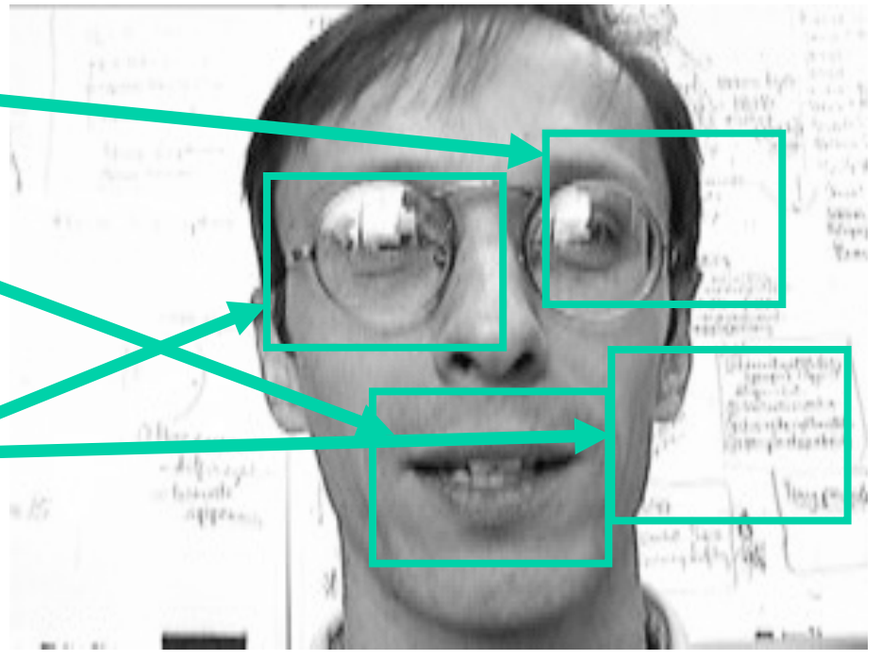
Database of mouth “templates”



- Search every image region (at every scale).
- Compare each template; chose the best match.

View-Based Approach

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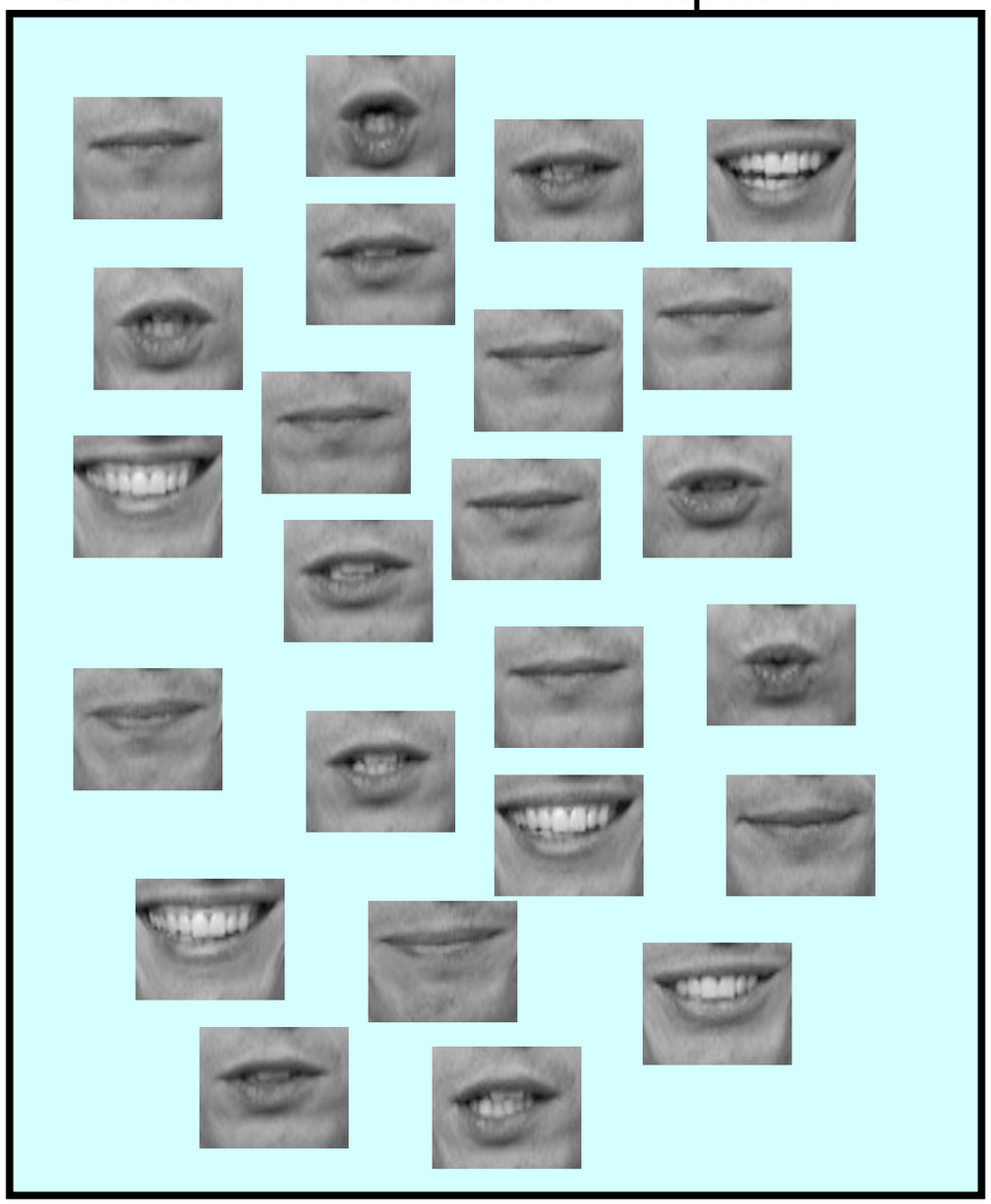
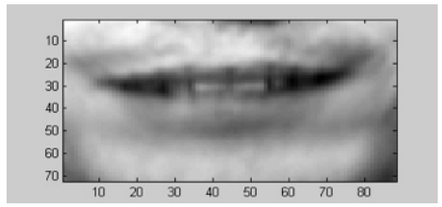
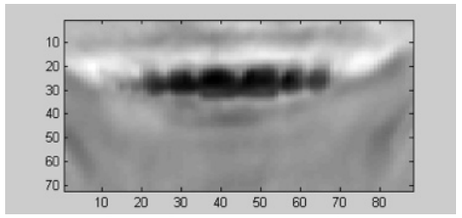
Database of mouth “templates”

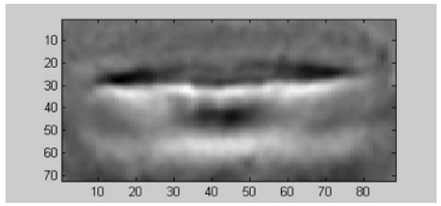


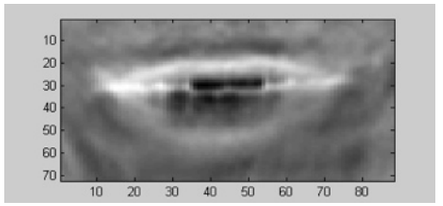


Mean

First three eigenvectors:







View-Based Approach

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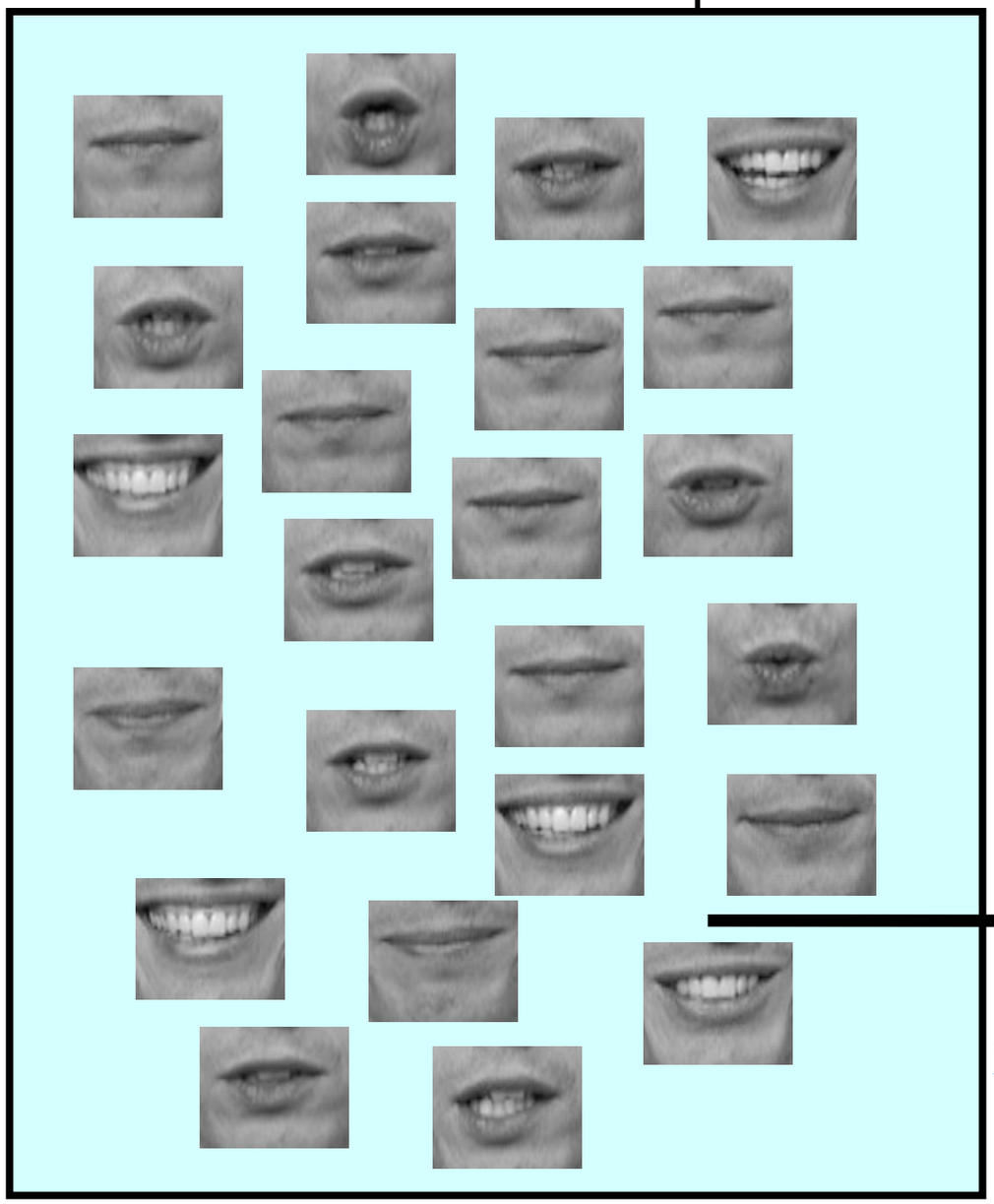
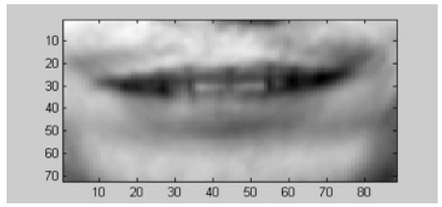
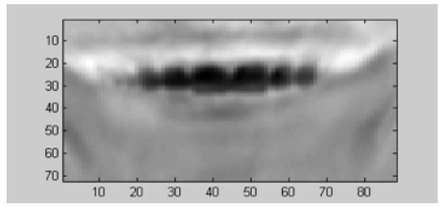
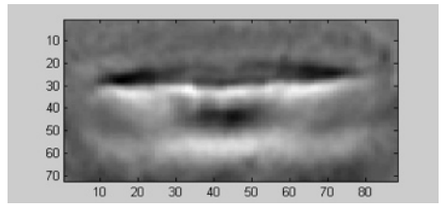
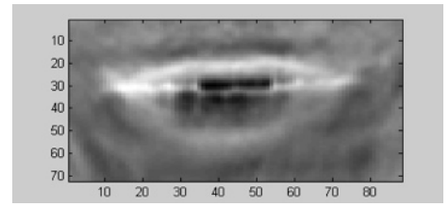
Database of mouth “templates”



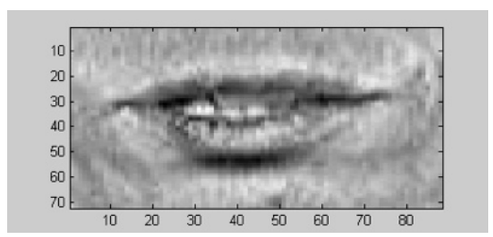








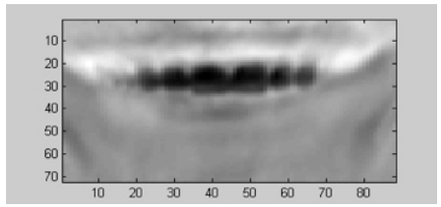
project



Subtract mean

$$\begin{bmatrix} 715.2646 \\ -376.2406 \\ -331.9533 \end{bmatrix}$$

Mouth Space


10
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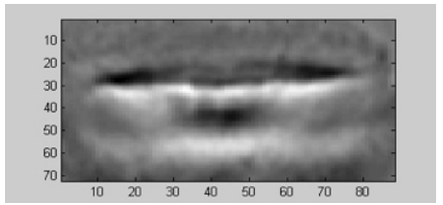
10 20 30 40 50 60 70 80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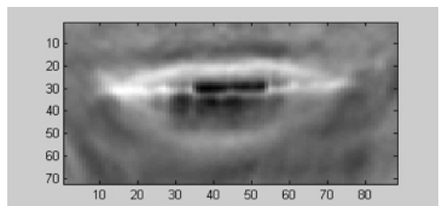
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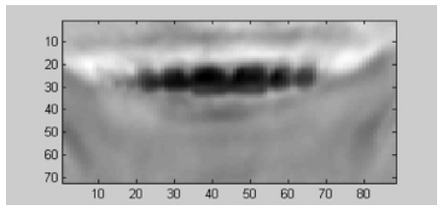
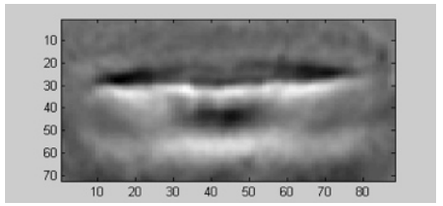
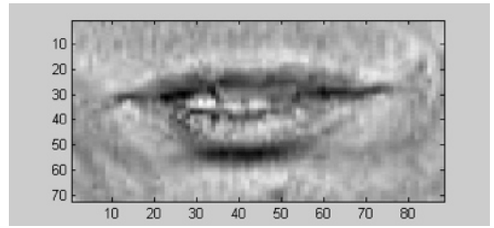
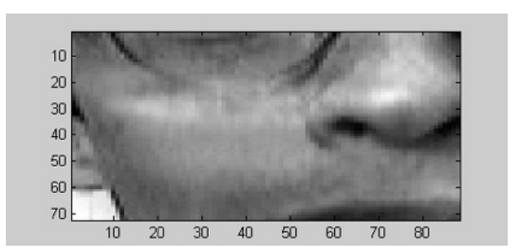
Mouth Space

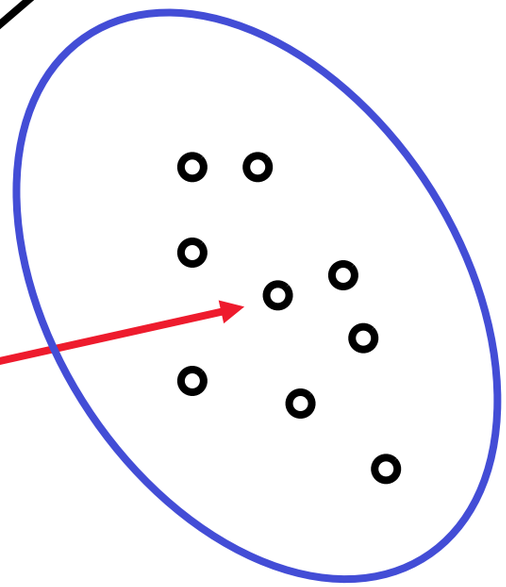




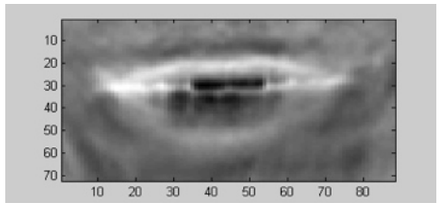




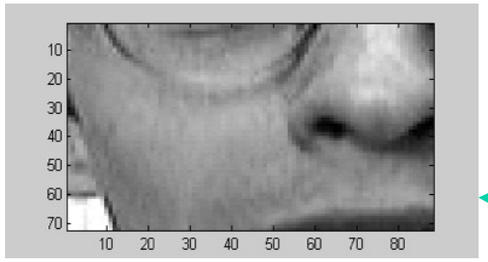
$$\begin{bmatrix} -725.1966 \\ 500.9761 \\ 872.4541 \end{bmatrix}$$

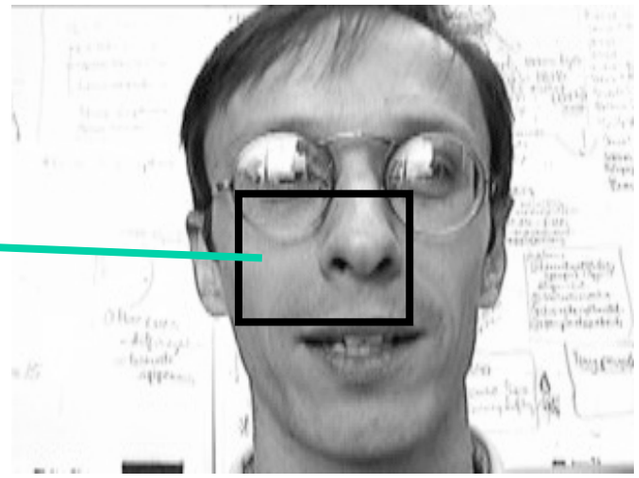


$$\begin{bmatrix} 715.2646 \\ -376.2406 \\ -331.9533 \end{bmatrix}$$



Simple Search Strategy





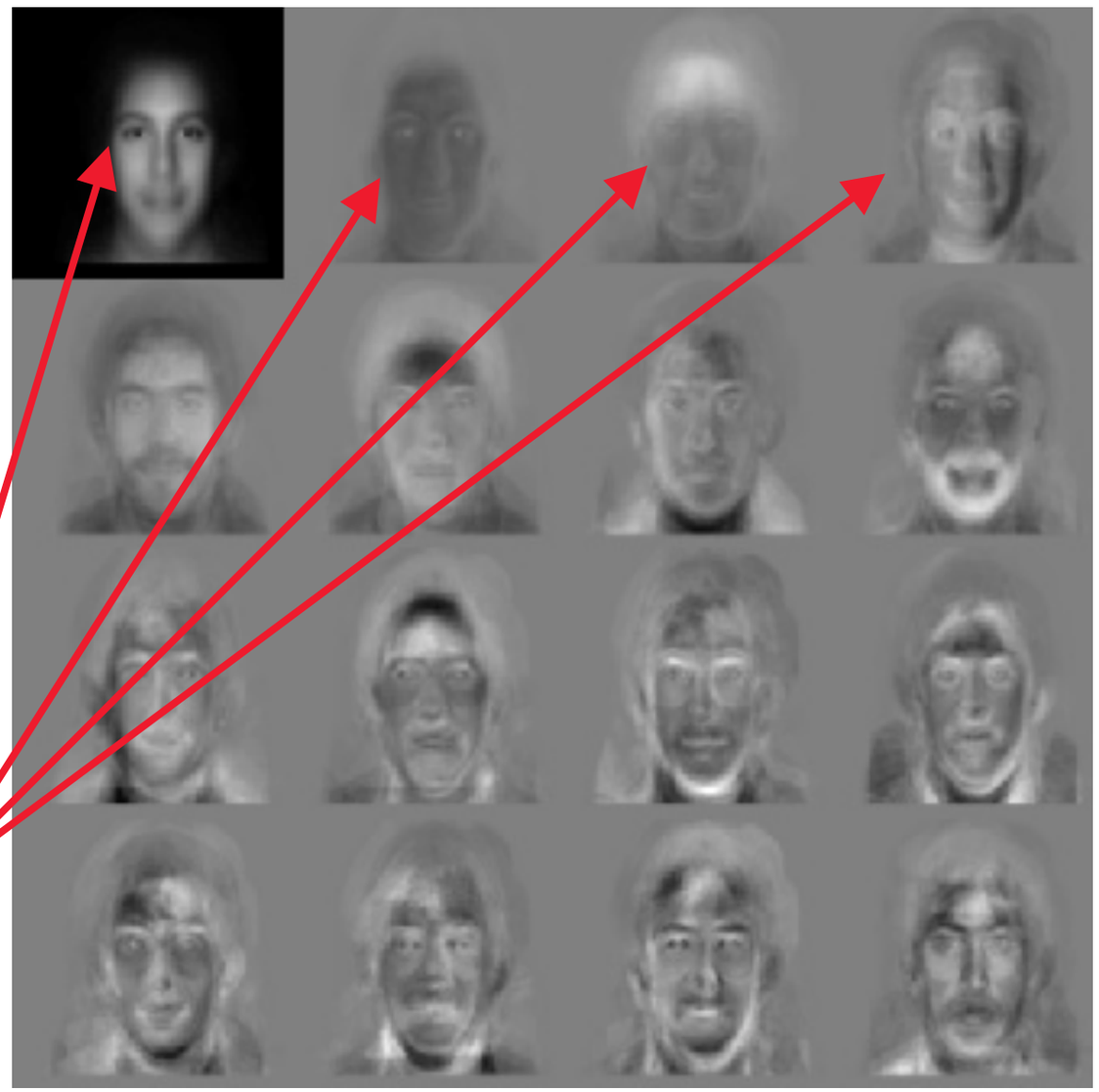
- Project each training image onto the low-dimensional subspace. Store the vectors of coefficients.
- For each image region:
  - Project it onto the low-dimensional subspace.
  - Compare this to each stored coefficient vector (cheap).
  - If the smallest distance is less than some threshold, then it is a mouth.

EigenFaces

- First popular use of PCA for object recognition was for the detection and recognition of faces

  [Turk and Pentland, 1991]

  - Collect a face ensemble
  - Normalize for contrast, scale, & orientation.
  - Remove backgrounds
  - Apply PCA & choose the first D eigen-images that account for most of the variance of the data.



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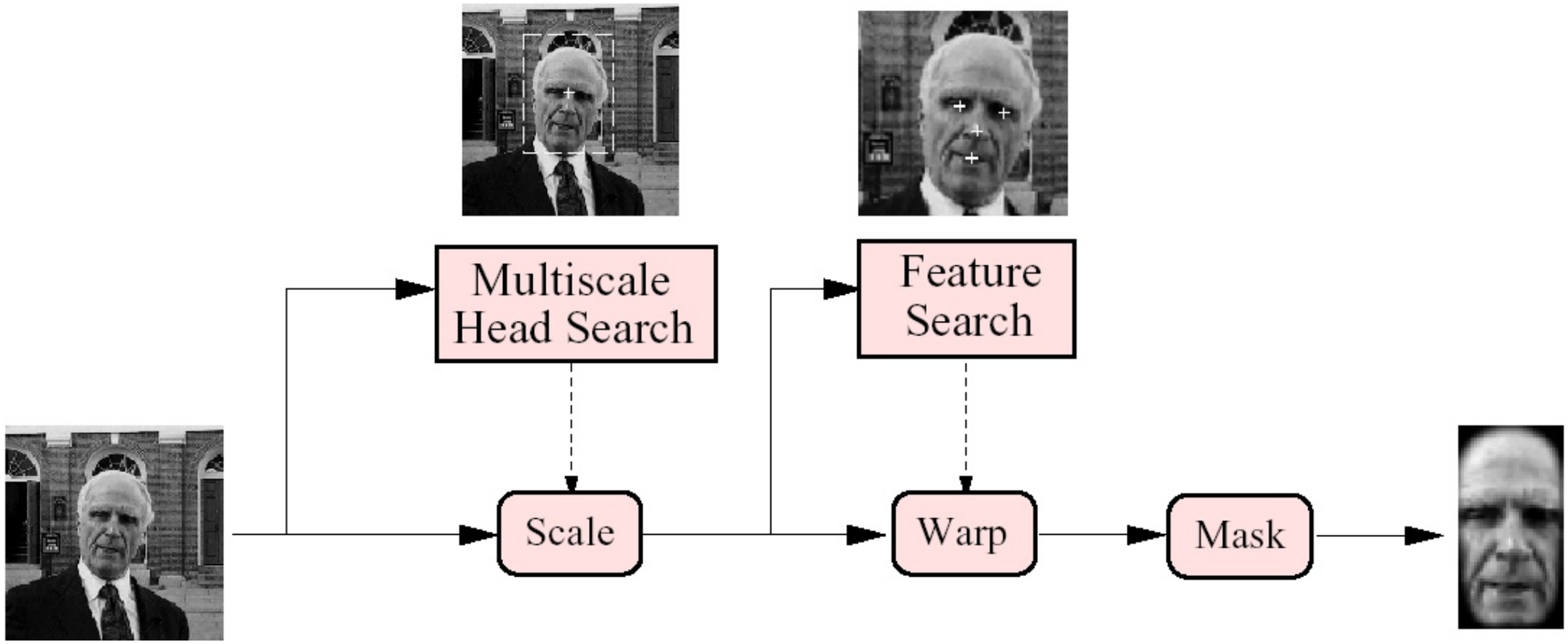
Face Recognition



[Moghaddam, Jebara & Pentland, 2000]

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EigenFeatures

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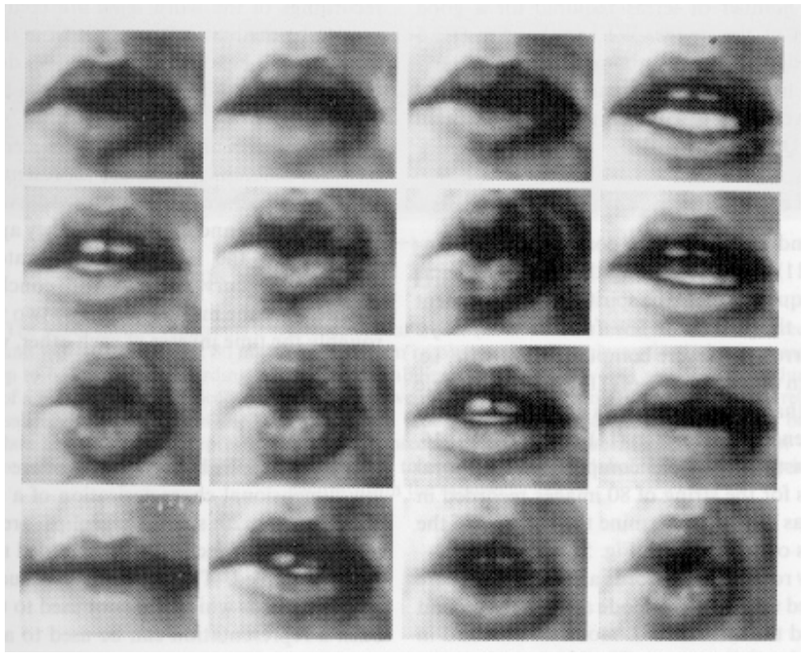
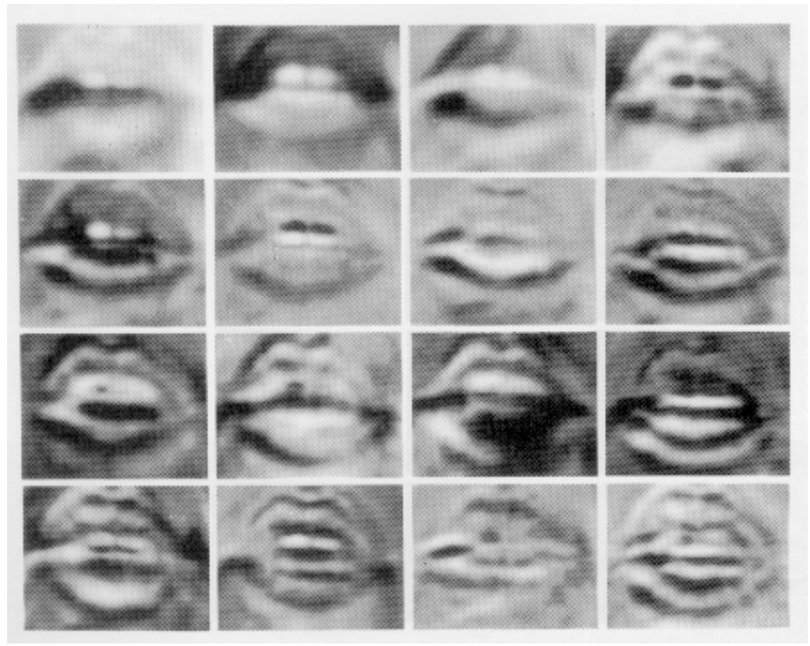
Data:



PCA

New Basis Vectors



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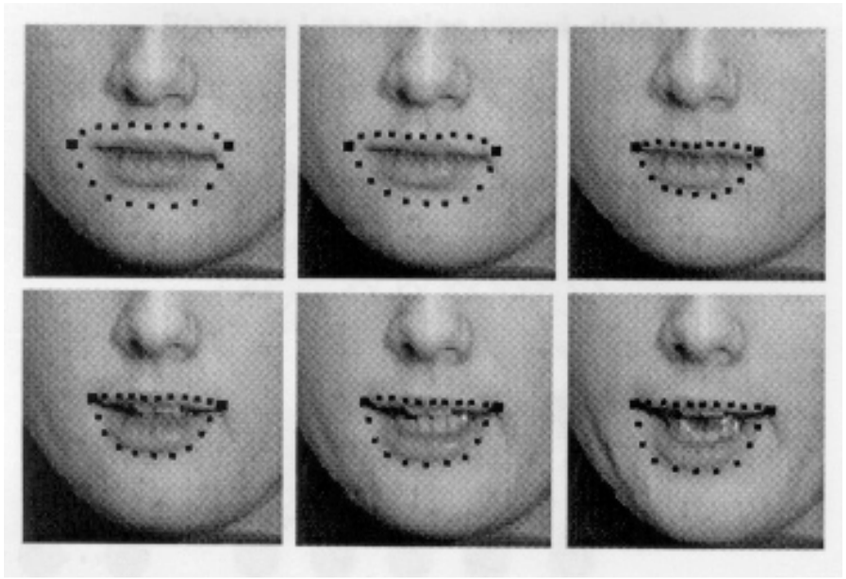
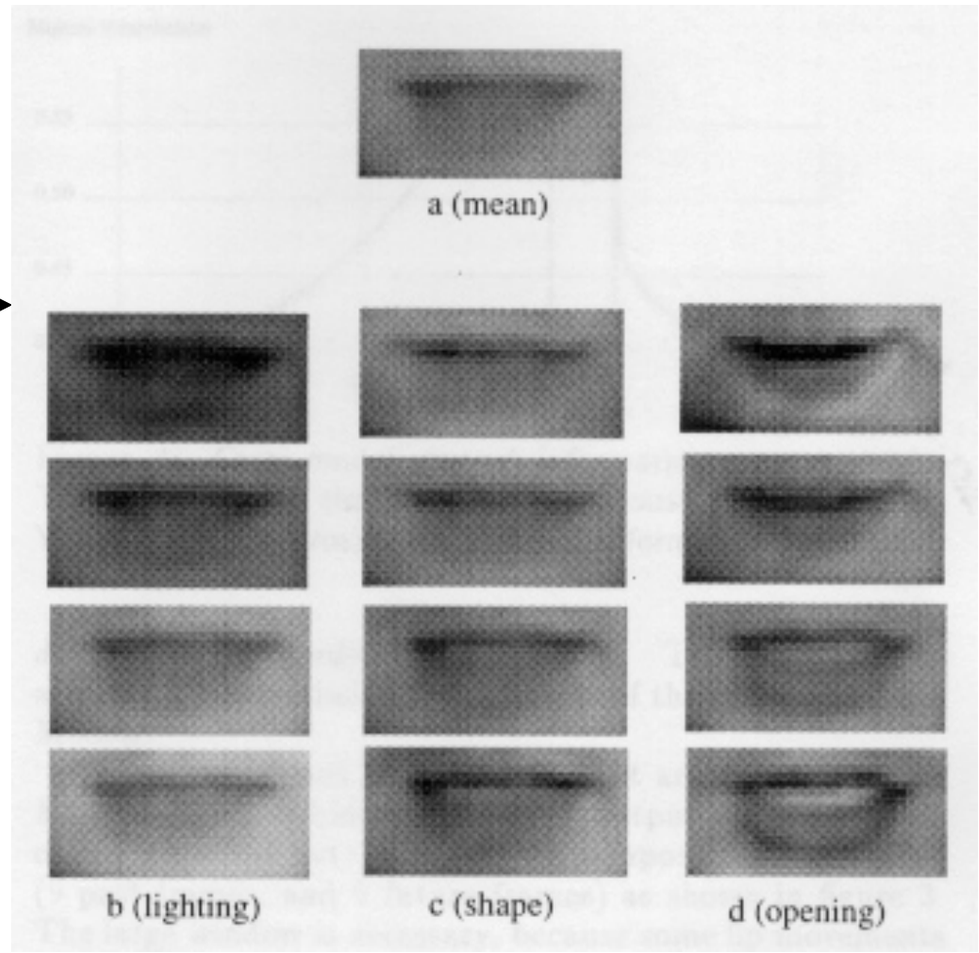
EigenFeatures

Data:



PCA

EigenLips



Fleet & Szeliski

PCA Model of Body Shapes

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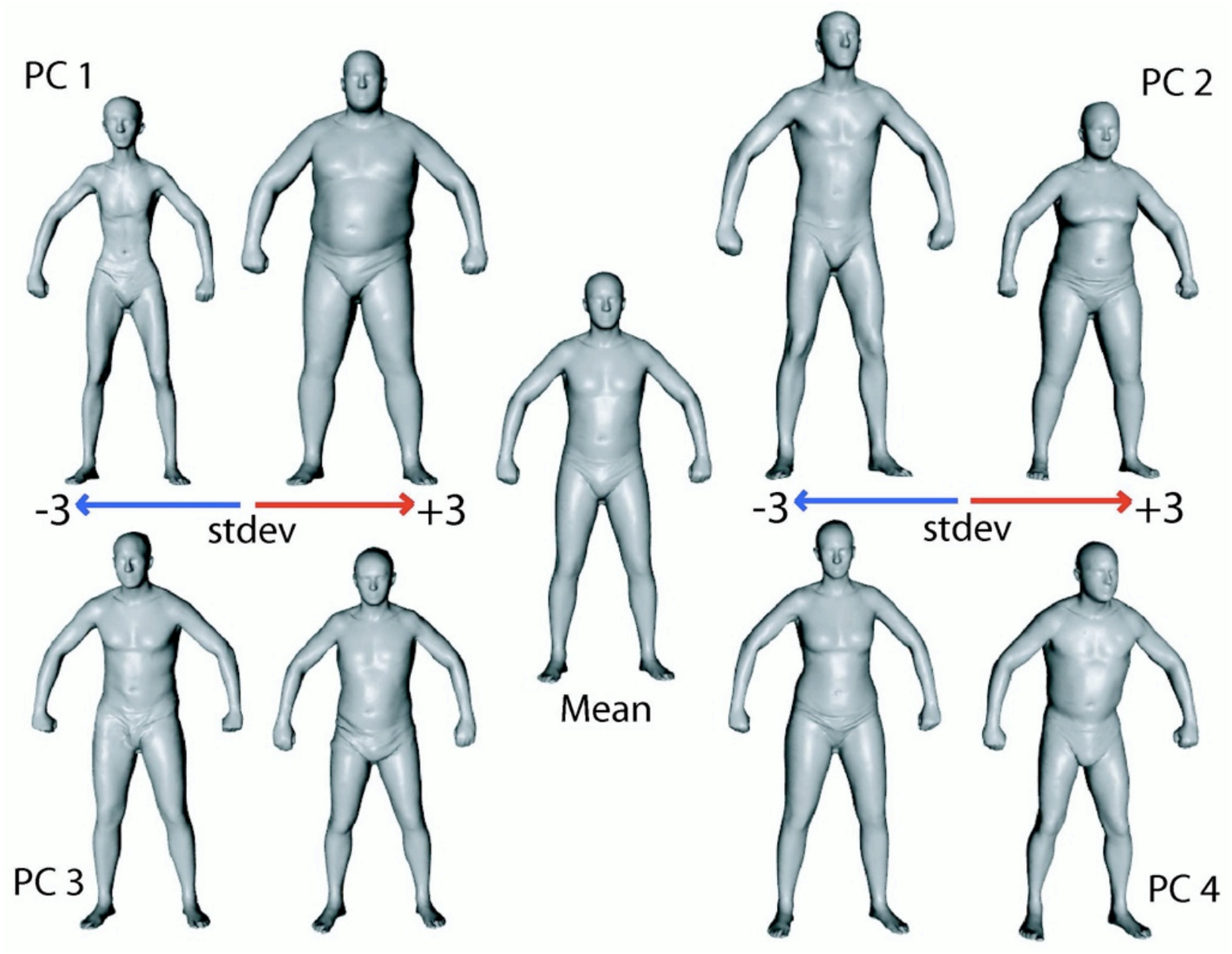
- PCA on a detailed triangle model of human bodies:



[Anguelov et al. 05]

Readings

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- Edge detection, sec. 4.2
- PCA and face recognition, sec. 14.2

- For next time:
  - Feature points, sec. 4.1
  - Bag of word models, sec. 14.4.1

Announcements

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- **No class next week!**
  - Project review meeting

- Homework assignment 1:
  - Is out...
  - Signup for the “Testat” current read only -- will go online again soon

- **Exam**
  - 21. August 2013, 10:00 - 12:00