cc2d_elasBPM: cc2d ELAStic Biphasic Porous Media

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This is a short report which shortly presents our problem and its implementation on FEATFLOW2.

1 Equations to solve

First, we respectively write down the momentum balance of solid skeleton, the momentum balance of pore fluid and the mixture volume balance equations taken from Markert and Ehlers [2]:

$$(\mathbf{u}_S)'_S = \mathbf{v}_S \tag{1}$$

$$\rho^S (\mathbf{v}_S)'_S = \mathbf{div}\,(\sigma^S_E - n^S p\mathbf{I}) + \rho^S \mathbf{b} + \frac{(n^F)^2 \gamma^{FR}}{k^F} \mathbf{w}_{FR} - p\,\mathbf{grad}\, n^F \tag{2}$$

$$\rho^F (\mathbf{v}_F)'_S + \rho^F (\mathbf{grad}\,\mathbf{v}_F)\mathbf{w}_{FR} = \mathbf{div}\,(\sigma^F_E - n^F p\mathbf{I}) + \rho^F \mathbf{b} - \frac{(n^F)^2 \gamma^{FR}}{k^F} \mathbf{w}_{FR} \tag{3}$$

$$\mathbf{div}\,(\mathbf{v}_S + n^F \mathbf{w}_{FR}) = 0 \tag{4}$$

we shall drop out the red-colored convective term and include σ^F_E. We assume small deformation in the solid skeleton. We also assume the pore fluid is incompressibile and Newtonian. Accordingly,

$$\mathbf{div}\,\sigma^F_E = \mu_F \Delta \mathbf{v}_F \tag{5}$$

For beautiful derivation of (5), see page 7 of GOLUB [1]

now we respectively multiply the above equations with the displacement test function η , the velocity test function φ and the pressure test function ψ and integrate over the whole domain to get the following weak forms:

$$\underbrace{\int \nabla\eta : \sigma_E^s \, \mathrm{d}\Omega}_{=\mathbf{u}_S^T \mathbf{K}_{u_S,u_S} \mathbf{u}_S} - \underbrace{\int \frac{\left(n_o^F\right)^2 \gamma^{FR}}{k_o^F} \eta \cdot \mathbf{v}_F \, \mathrm{d}\Omega}_{=\mathbf{u}_S^T \mathbf{K}_{u_S,v_F} \mathbf{v}_F} - \underbrace{\int n_o^s \operatorname{div}(\eta) p \, \mathrm{d}\Omega}_{=\mathbf{u}_S^T \mathbf{K}_{u_S,p} P} \tag{6}$$

$$+ \underbrace{\int \frac{\left(n_o^F\right)^2 \gamma^{FR}}{k_o^F} \eta \cdot \mathbf{v_s} \, \mathrm{d}\Omega}_{=\mathbf{u}_S^T \mathbf{k}_{u_S,v_S} \mathbf{v}_S} + \underbrace{\int n^s \rho^{SR} \eta \cdot (\mathbf{v}_S)'_S \, \mathrm{d}\Omega}_{=\mathbf{u}_S^T \mathbf{M}_{u_S,v_S} \mathbf{v}'_S} = \underbrace{\int \eta \cdot \mathbf{t}^s \, \mathrm{d}\Gamma}_{=\mathbf{u}_S^T \mathbf{f}_{u_S}} + \underbrace{\int n_o^s \rho^{sR} \, \eta \cdot \mathbf{b} \, \mathrm{d}\Omega}_{=\mathbf{u}_S^T \mathbf{b}_{u_S}}$$

$$\underbrace{\int \left\{ \mu_F \nabla\varphi : \nabla\mathbf{v}_F + \frac{\left(n_o^F\right)^2 \gamma^{FR}}{k_o^F} \varphi \cdot \mathbf{v}_F \right\} \mathrm{d}\Omega}_{=\mathbf{v}_F^T \left(\tilde{\mathbf{K}}_{v_F v_F} + \hat{\mathbf{K}}_{v_F v_F}\right) \mathbf{v}_F} + \underbrace{\int -n_o^F \operatorname{div}\varphi \, p \, \mathrm{d}\Omega}_{=\mathbf{v}_F^T \mathbf{K}_{\mathbf{v_F p}} p} + \underbrace{\int -\frac{\left(n_o^F\right)^2 \gamma^{FR}}{k_o^F} \varphi \cdot \mathbf{v}_s \, \mathrm{d}\Omega}_{=\mathbf{v}_F^T \mathbf{k}_{\mathbf{v_F v_S}} \mathbf{v}_S} \tag{7}$$

$$+ \underbrace{\int n_o^F \rho^{FR} \varphi \cdot (v_F)'_S \, \mathrm{d}\Omega}_{\mathbf{v}_F^T \mathbf{M}_{\mathbf{v_F v_F}} \mathbf{v}'_F} = \underbrace{\int \varphi \cdot t_F \, \mathrm{d}\Gamma}_{\mathbf{v}_F^T \mathbf{f_F}} + \underbrace{\int n_o^F \rho^{FR} \varphi \cdot \mathbf{b} \, \mathrm{d}\Omega}_{=\mathbf{v}_F^T \mathbf{b_F}}$$

$$\underbrace{\int n_o^S \psi \, \mathbf{div}\, \mathbf{v}_s \, \mathrm{d}\Omega}_{p \mathbf{K}_{\mathbf{pv_S}} \mathbf{v_S}} + \underbrace{\int n_o^F \psi \, \mathbf{div}\, \mathbf{v}_F \, \mathrm{d}\Omega}_{p \mathbf{K}_{\mathbf{pv_F}} \mathbf{v_F}} = 0 \tag{8}$$

Where

$$\mathbf{t}^S = \left(\sigma_E^s - n^S p \mathbf{I}\right) \mathbf{n}$$

and

$$\mathbf{t}^F = \mu_F \frac{\partial \mathbf{v}_F}{\partial \mathbf{n}} - n^F p \mathbf{n}$$

Note that for small deformation both $\mathbf{grad}\, n^F$ and $\mathbf{grad}\, n^S$ are negligible.

Equations (6), (7) and (8) can be expressed as the following matrix-vector multiplications

$$\underbrace{\begin{pmatrix}\mathbf{u}_S\\ \mathbf{v}_S\\ \mathbf{v}_F\\ p\end{pmatrix}^T}_{\mathbf{y}^T}\underbrace{\begin{pmatrix}\mathbf{0} & \mathbf{M_{u_s v_s}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{M_{v_F v_F}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}}_{\mathbf{M}}\underbrace{\begin{pmatrix}\dot{\mathbf{u}}_S\\ \dot{\mathbf{v}}_S\\ \dot{\mathbf{v}}_F\\ \dot{p}\end{pmatrix}}_{\dot{\mathbf{y}}}+\underbrace{\begin{pmatrix}\mathbf{u}_S\\ \mathbf{v}_S\\ \mathbf{v}_F\\ p\end{pmatrix}^T}_{\mathbf{y}^T}\underbrace{\begin{pmatrix}\mathbf{k}_{u_S u_S} & \mathbf{k}_{u_S v_S} & \mathbf{k}_{u_S v_F} & \mathbf{k}_{u_S p}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{k}_{v_F v_S} & \mathbf{k}_{v_F v_F} & \mathbf{k}_{v_F p}\\ \mathbf{0} & \mathbf{k}_{p v_S} & \mathbf{k}_{p v_F} & \mathbf{0}\end{pmatrix}}_{K}\underbrace{\begin{pmatrix}\mathbf{u}_S\\ \mathbf{v}_S\\ \mathbf{v}_F\\ p\end{pmatrix}}_{\mathbf{y}}$$
$$=\underbrace{\begin{pmatrix}\mathbf{u}_S\\ \mathbf{v}_S\\ \mathbf{v}_F\\ p\end{pmatrix}^T}_{\mathbf{y}^T}\underbrace{\begin{pmatrix}\mathbf{f}_u+\mathbf{b}_u\\ \mathbf{0}\\ \mathbf{f}_F+\mathbf{b}_F\\ \mathbf{0}\end{pmatrix}}_{\mathbf{F}} \tag{9}$$

Which is equivalent to the following compact energy form

$$\mathbf{y}^T\mathbf{M}\dot{\mathbf{y}}+\mathbf{y}^T\mathbf{k}\mathbf{y}=\mathbf{y}^T\mathbf{F} \tag{10}$$

We factor out $\mathbf{y}^T$ to get the force balance forms

$$\mathbf{M}\dot{\mathbf{y}}+\mathbf{k}\mathbf{y}=\mathbf{F} \tag{11}$$

Which we write in more detail:

$$\underbrace{\begin{pmatrix}\mathbf{0} & \mathbf{M_{u_s v_s}} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{M_{v_F v_F}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}}_{\mathbf{M}}\underbrace{\begin{pmatrix}\dot{\mathbf{u}}_S\\ \dot{\mathbf{v}}_S\\ \dot{\mathbf{v}}_F\\ \dot{p}\end{pmatrix}}_{\dot{\mathbf{y}}}+\underbrace{\begin{pmatrix}\mathbf{k}_{u_S u_S} & \mathbf{k}_{u_S v_S} & \mathbf{k}_{u_S v_F} & \mathbf{k}_{u_S p}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{k}_{v_F v_S} & \mathbf{k}_{v_F v_F} & \mathbf{k}_{v_F p}\\ \mathbf{0} & \mathbf{k}_{p v_S} & \mathbf{k}_{p v_F} & \mathbf{0}\end{pmatrix}}_{K}\underbrace{\begin{pmatrix}\mathbf{u}_S\\ \mathbf{v}_S\\ \mathbf{v}_F\\ p\end{pmatrix}}_{\mathbf{y}}=\underbrace{\begin{pmatrix}\mathbf{f}_u+\mathbf{b}_u\\ \mathbf{0}\\ \mathbf{f}_F+\mathbf{b}_F\\ \mathbf{0}\end{pmatrix}}_{\mathbf{F}} \tag{12}$$

Now it remains to include equation (1) in (12) to have the following final form which looks like better for Gauss elimaination because the elements of the first column of K are all zeors except the diagonal block whereas the second diagonal block is the identity matrix

$$\underbrace{\begin{pmatrix}\mathbf{0} & \mathbf{M_{u_s v_s}} & \mathbf{0} & \mathbf{0}\\ \mathbf{I} & \mathbf{0} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{M_{v_F v_F}} & \mathbf{0}\\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0}\end{pmatrix}}_{\mathbf{M}}\underbrace{\begin{pmatrix}\dot{\mathbf{u}}_S\\ \dot{\mathbf{v}}_S\\ \dot{\mathbf{v}}_F\\ \dot{p}\end{pmatrix}}_{\dot{\mathbf{y}}}+\underbrace{\begin{pmatrix}\mathbf{k}_{u_S u_S} & \mathbf{k}_{u_S v_S} & \mathbf{k}_{u_S v_F} & \mathbf{k}_{u_S p}\\ \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0}\\ \mathbf{0} & \mathbf{k}_{v_F v_S} & \mathbf{k}_{v_F v_F} & \mathbf{k}_{v_F p}\\ \mathbf{0} & \mathbf{k}_{p v_S} & \mathbf{k}_{p v_F} & \mathbf{0}\end{pmatrix}}_{K}\underbrace{\begin{pmatrix}\mathbf{u}_S\\ \mathbf{v}_S\\ \mathbf{v}_F\\ p\end{pmatrix}}_{\mathbf{y}}=\underbrace{\begin{pmatrix}\mathbf{f}_u+\mathbf{b}_u\\ \mathbf{0}\\ \mathbf{f}_F+\mathbf{b}_F\\ \mathbf{0}\end{pmatrix}}_{\mathbf{F}} \tag{13}$$

We shall use the θ-scheme to solve (13), in which

$$\mathbf{M}\frac{\mathbf{y}_{n+1}-\mathbf{y}_n}{\Delta t}+\theta\mathbf{K}\mathbf{y}_{n+1}=-(1-\theta)\mathbf{K}\mathbf{y}_n+\theta\mathbf{f}_{n+1}+(1-\theta)\mathbf{f}_n \tag{14}$$

Recall that θ is either 0, 1 or 0.5.

2 Matrices of CC2d_elastBPM

We shall expand the source codes of cc2d to solve our system. Two important remarks must be considered when designing our code:

$\mathbf{K}_{\mathbf{u}_s\mathbf{u}_s}$ is symmetric and because all the spaces (except the pressure space) share the same discretisation structure we conclude that $\mathbf{K}_{\mathbf{u}_s\mathbf{v}_s} = -\mathbf{K}_{\mathbf{u}_s\mathbf{v}_F} = -\mathbf{K}_{\mathbf{v}_F\mathbf{v}_s} = \hat{\mathbf{K}}_{\mathbf{v}_F\mathbf{v}_F}$. It is worth to know that $\tilde{\mathbf{A}}_{55}$ and $\tilde{\mathbf{A}}_{66}$ are already computed in cc2d and defined as stokes matrix.

The possibility to drop the fluid shear stress must be available. If the shear stress to be neglegted just turn off $\tilde{\mathbf{K}}_{v_F v_F}$.

Our stiffness matrix looks likes:

$$K = \begin{pmatrix} \mathbf{K}_{11} & \mathbf{K}_{12} & \mathbf{K}_{13} & \mathbf{0} & \mathbf{K}_{15} & \mathbf{0} & \mathbf{B}_{S1} \\ \mathbf{K}_{21} & \mathbf{K}_{22} & \mathbf{0} & \mathbf{K}_{24} & \mathbf{0} & \mathbf{K}_{26} & \mathbf{B}_{S2} \\ \mathbf{0} & \mathbf{0} & -\mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{K}_{53} & \mathbf{0} & \mathbf{K}_{55} & \mathbf{0} & \mathbf{B}_{F1} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{K}_{64} & \mathbf{0} & \mathbf{K}_{66} & \mathbf{B}_{F2} \\ \mathbf{0} & \mathbf{0} & -\mathbf{B}_{S1}^T & -\mathbf{B}_{S2}^T & -\mathbf{B}_{F1}^T & -\mathbf{B}_{F2}^T & \mathbf{0} \end{pmatrix} \tag{15}$$

The blocks $\mathbf{K}_{ij}$ are defined below:

$$
\begin{aligned}
\mathbf{K}_{11} &= \{2\mu+\lambda\}\frac{\partial \eta_1}{\partial x}\frac{\partial u_1}{\partial x} + \{\mu\}\frac{\partial \eta_1}{\partial y}\frac{\partial u_1}{\partial y} \\
\mathbf{K}_{12} &= \{\lambda\}\frac{\partial \eta_1}{\partial x}\frac{\partial u_2}{\partial y} + \{\mu\}\frac{\partial \eta_1}{\partial y}\frac{\partial u_2}{\partial x} \\
\mathbf{K}_{21} &= \{\lambda\}\frac{\partial \eta_2}{\partial y}\frac{\partial u_1}{\partial x} + \{\mu\}\frac{\partial \eta_2}{\partial x}\frac{\partial u_1}{\partial y} \\
\mathbf{K}_{22} &= \{\mu\}\frac{\partial \eta_2}{\partial x}\frac{\partial u_2}{\partial x} + \{2\mu+\lambda\}\frac{\partial \eta_2}{\partial y}\frac{\partial u_2}{\partial y} \\
\mathbf{K}_{13} &= \left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\eta_1 v_{s1} \\
\mathbf{K}_{24} &= \left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\eta_2 v_{s2} \\
\mathbf{K}_{15} &= -\left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\eta_1 v_{s1} \\
\mathbf{K}_{26} &= -\left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\eta_2 v_{s2} \\
\mathbf{K}_{33} &= -\eta_1 u_{S1} \\
\mathbf{K}_{44} &= -\eta_2 u_{S2} \\
\hat{\mathbf{K}}_{55} &= \left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\eta_1 v_{s1} \\
\hat{\mathbf{K}}_{66} &= \left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\eta_2 v_{s2} \\
\tilde{\mathbf{K}}_{55} &= \mu_F\frac{\partial \varphi_1}{\partial x}\frac{\partial v_{F1}}{\partial x} + \mu_F\frac{\partial \varphi_1}{\partial y}\frac{\partial v_{F1}}{\partial y} \\
\tilde{\mathbf{K}}_{66} &= \mu_F\frac{\partial \varphi_2}{\partial x}\frac{\partial v_{F2}}{\partial x} + \mu_F\frac{\partial \varphi_2}{\partial y}\frac{\partial v_{F2}}{\partial y} \\
\mathbf{K}_{55} &= \tilde{\mathbf{K}}_{55} + \hat{\mathbf{K}}_{55} \\
\mathbf{K}_{66} &= \tilde{\mathbf{K}}_{66} + \hat{\mathbf{K}}_{66} \\
\mathbf{B}_{S1} &= -n^S\frac{\partial \eta_1}{\partial x}p \\
\mathbf{B}_{S2} &= -n^S\frac{\partial \eta_2}{\partial y}p \\
\mathbf{B}_{F1} &= -n^F\frac{\partial \varphi_1}{\partial x}p \\
\mathbf{B}_{F2} &= -n^F\frac{\partial \varphi_2}{\partial y}p \\
\mathbf{K}_{53} &= -\left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\varphi_1 v_{s1} \\
\mathbf{K}_{64} &= -\left\{\frac{\left(n^F\right)^2\gamma^{FR}}{k^F}\right\}\varphi_2 v_{s2}
\end{aligned}
$$

It remains to show how to form the mass matrix:

$$M = \begin{pmatrix} \mathbf{0} & \mathbf{0} & \mathbf{M}_{13} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{24} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{M} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{55} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{M}_{66} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{pmatrix} \tag{16}$$

and the computation of non-zero blocks are shown below

$$\begin{aligned} \mathbf{M} &= \eta_1 v_{S1} \\ \mathbf{M}_{13} &= n^S \rho^{SR} \eta_1 v_{S1} \\ \mathbf{M}_{24} &= n^S \rho^{SR} \eta_2 v_{S2} \\ \mathbf{M}_{31} &= \eta_1 v_{S1} \\ \mathbf{M}_{42} &= \eta_2 v_{S2} \\ \mathbf{M}_{55} &= n^F \rho^{FR} \varphi_1 v_{F1} \\ \mathbf{M}_{66} &= n^F \rho^{FR} \varphi_2 v_{F2} \end{aligned}$$

3 Precalculated Matrices for ccbasics.f90

Only 10 matrices need to be computed: K_{11}, K_{12}, K_{21}, K_{22}, M, L, B_{S1} , B_{S2}, B_{F1} and B_{F2}.

Where L is the Stoke's matrix and represented by $\tilde{k}_{55}$

However benifiting from $K_{21} = k_{12}^T$ to save memeory will be considered latter.

4 Assembly procedure for ccmatvecassembly.f90

The assembly procedure in ccmatvecassembly.f90 of our non-stationary problem will be shown. The red-colored parts are comming from the mass matrix and the blue-colored symbol are those coefficients used by cc2d to be multiplied by A_{11}, ... B_2. and their values depends on the type of the time integration scheme, time step and whether the RHS or LHS to be assembeled.

$$\mathbf{A}_{11} = \theta\mathbf{K}_{11}$$
$$\mathbf{A}_{12} = \theta\mathbf{K}_{12}$$
$$\mathbf{A}_{21} = \theta\mathbf{K}_{21}$$
$$\mathbf{A}_{22} = \theta\mathbf{K}_{22}$$

$$\mathbf{A}_{13} = \theta\left\{\frac{(n^F)^2\gamma^{FR}}{k^F}\right\}M + \alpha\, n^S\rho^{SR}\mathbf{M}$$
$$\mathbf{A}_{24} = \theta\left\{\frac{(n^F)^2\gamma^{FR}}{k^F}\right\}M + \alpha\, n^S\rho^{SR}\mathbf{M}$$

Note that both A_{13} and A_{24} are identical and hence they share the contents and the structure.

$$\mathbf{A}_{15} = -\theta\left\{\frac{(n^F)^2\gamma^{FR}}{k^F}\right\}\mathbf{M}$$
$$\mathbf{A}_{26} = -\theta\left\{\frac{(n^F)^2\gamma^{FR}}{k^F}\right\}\mathbf{M}$$

A_{15} and A_{26} share the contents and the structure.

$$\mathbf{A}_{33} = -\theta\mathbf{M}$$
$$\mathbf{A}_{44} = -\theta\mathbf{M}$$

$$\mathbf{K}_{55} = \theta\left\{\frac{(n^F)^2\gamma^{FR}}{k^F}\right\}\mathbf{M} + \theta L + \alpha\, n^F\rho^{FR}\mathbf{M}$$
$$\mathbf{K}_{66} = \theta\left\{\frac{(n^F)^2\gamma^{FR}}{k^F}\right\}\mathbf{M} + \theta L + \alpha\, n^F\rho^{FR}\mathbf{M}$$

where L is the Stoke's matrix. $\mathbf{K}_{55}$ and $\mathbf{K}_{66}$ share also the same contents and structure

$$\mathbf{B}_{S1} = -\eta n^S \frac{\partial\eta_1}{\partial x} p$$
$$\mathbf{B}_{S2} = -\eta n^S \frac{\partial\eta_2}{\partial y} p$$
$$\mathbf{B}_{F1} = -\eta n^F \frac{\partial\varphi_1}{\partial x} p$$
$$\mathbf{B}_{F2} = -\eta n^F \frac{\partial\varphi_2}{\partial y} p$$

$\mathbf{A}_{53}$ and $\mathbf{A}_{53}$ share also the contents and the structure and computed as below

$$\mathbf{A}_{53} = -\theta\Big\{\frac{(n^F)^2\gamma^{FR}}{k^F}\Big\}\mathbf{M}$$
$$\mathbf{A}_{64} = -\theta\Big\{\frac{(n^F)^2\gamma^{FR}}{k^F}\Big\}\mathbf{M}$$

Finally

$$\mathbf{A}_{31} = \alpha\mathbf{M}$$
$$\mathbf{A}_{42} = \alpha\mathbf{M}$$

the divergent matrices D_{S1}, D_{S2}, D_{F1} and D_{F2} are computed by transposing (and multiplying by τ) B_{S1}, B_{S2}, B_{F1} and B_{F2} respectively.

D_{S1}, D_{S2}, D_{F1} and D_{F2} are stored in A_{73}, A_{74}, A_{75} and A_{76} respectively.

5 Example

First we drop the body force $(\mathbf{b} \approx \mathbf{0})$ and perform analytic simulation for purely Dirichlet problem. Our system of equations are written below:

$$\rho^S(\mathbf{v}_S)'_S - \mathbf{div}\,(\sigma^S_E) - \frac{(n^F)^2\gamma^{FR}}{k^F}(\mathbf{v}_F - \mathbf{v}_S) + n^S\nabla p = \mathbf{f}_u \tag{17}$$

$$(\mathbf{u}_S)'_S - \mathbf{v}_S = \mathbf{0} \tag{18}$$

$$\rho^F(\mathbf{v}_F)'_S - \nu\Delta\mathbf{v}_F + \frac{(n^F)^2\gamma^{FR}}{k^F}(\mathbf{v}_F - \mathbf{v}_S) + n^F\nabla p = \mathbf{f}_v \tag{19}$$

$$n^S\nabla\cdot\mathbf{v}_S + n^F\nabla\cdot\mathbf{v}_F = 0 \tag{20}$$

Which we write in the following scalar form

$$\rho^S\frac{\partial v_{S1}}{\partial t} - (2\mu+\lambda)u_{1xx} - \mu\,u_{1yy} - (\lambda+\mu)u_{2xy} - \frac{(n^F)^2\gamma^{FR}}{k^F}(v_{F1} - v_{S1}) + n^S p_x = f_{u1} \tag{21}$$

$$\rho^S\frac{\partial v_{S2}}{\partial t} - \mu\,u_{2xx} - (2\mu+\lambda)\,u_{2yy} - (\lambda+\mu)u_{1xy} - \frac{(n^F)^2\gamma^{FR}}{k^F}(v_{F2} - v_{S2}) + n^S p_y = f_{u2} \tag{22}$$

$$\frac{\partial u_{S1}}{\partial t} - v_{S1} = 0 \tag{23}$$

$$\frac{\partial u_{S2}}{\partial t} - v_{S2} = 0 \tag{24}$$

$$\frac{\partial v_{F1}}{\partial t} - \nu(v_{F1xx} + v_{F1yy}) + \frac{\left(n^F\right)^2 \gamma^{FR}}{k^F}(v_{F1} - v_{S1}) + n^F p_x = f_{vF1} \tag{25}$$

$$\frac{\partial v_{F2}}{\partial t} - \nu(v_{F2xx} + v_{F2yy}) + \frac{\left(n^F\right)^2 \gamma^{FR}}{k^F}(v_{F2} - v_{S2}) + n^F p_y = f_{vF2} \tag{26}$$

$$n^S(v_{S1x} + v_{S2y}) + n^F(v_{F1x} + v_{F2y}) = 0 \tag{27}$$

For the debugging purpose, the following analtical functions were used

$$\begin{aligned}
u_{S1} &= yt \\
u_{S2} &= xt \\
v_{S1} &= y \\
v_{S2} &= x \\
v_{F1} &= y \\
v_{F2} &= x \\
p &= (1 - x - y)t
\end{aligned}$$

these functions are found in cccallback.f90 in the subroutines ffunction_TargetuSx, ffunction_TargetuSy, ffunction_TargetvSx, ffunction_TargetvSy and ffunction_Targetp respectively. The RHS source functions resulting from substituting the above 7 functions in (21), (22), (23) , (24), (25), (26) and (27) to get the following

$$\begin{aligned}
f_{uS1} &= -\frac{3}{4}t \\
f_{uS2} &= -\frac{3}{4}t \\
f_{vS1} &= 0 \\
f_{vS2} &= 0 \\
f_{vF1} &= -\frac{1}{4}t \\
f_{vF2} &= -\frac{1}{4}t \\
f_p &= 0
\end{aligned}$$

the above functions are also in cccallback.f90 in the subroutines coeff_RHS_uSx, coeff_RHS_uSy, coeff_RHS_vSx, coeff_RHS_vSy, coeff_RHS_vFx, coeff_RHS_vFy, coeff_RHS_p. and finally the boundary values are purely direchlet and defined in cccallback.f90 in the subroutine getBoundaryValues where component 1 ... component 6 are $u_{S1} \ldots u_{F2}$.

References

[1] G. H. GOLUB, editor. *Numerical Mathematics and Scientific Computation*. Oxford University Press Inc., New York, New York, 2006.

[2] Heider Markert and Ehlers. Comparison of monolithic and splitting solution schemes for dynamic porous media. *International Journal for Numerical Methods in Engineering*, 82, 2009.