Probabilități și statistică
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Exercițiul 1

a) $\mathbb{P}(X=-1, Y=7) = 0.045$

$\Rightarrow \mathbb{P}(X=-1)\,\mathbb{P}(Y=7 \mid X=-1) = 0.045$

$0.18 \cdot \mathbb{P}(Y=7 \mid X=-1) = 0.045$

$0.18 \cdot p_2 = 0.045 \Leftrightarrow p_2 = 0.25$

$p_1 = 1 - p_2 = 0.75$

$$Y \sim \begin{pmatrix} -5 & 7 \\ 0.75 & 0.25 \end{pmatrix}$$

- $X+Y \sim \begin{pmatrix} -6 & 6 & 2 & 14 \\ 0.135 & 0.045 & 0.615 & 0.205 \end{pmatrix}$

$\mathbb{P}(X=x, Y=y) = p_x \cdot p_y$

- $X-Y \sim \begin{pmatrix} 4 & -8 & 12 & 0 \\ 0.135 & 0.045 & 0.615 & 0.205 \end{pmatrix}$

- $X^2 \sim \begin{pmatrix} 1 & 49 \\ 0.18 & 0.82 \end{pmatrix}$ $\quad Y^2 \sim \begin{pmatrix} 25 & 49 \\ 0.75 & 0.25 \end{pmatrix}$

$4X^2+7Y^2 \sim \begin{pmatrix} 179 & 347 & 371 & 539 \\ 0.135 & 0.045 & 0.615 & 0.205 \end{pmatrix}$

- $\mathbb{E}[X] = (-1) \cdot 0.18 + 7 \cdot 0.82 = 5.56$
- $\mathbb{E}[Y] = (-5) \cdot 0.75 + 7 \cdot 0.25 = -2$

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- $Var(X) = E[X^2] - E[X]^2 = 9.4464$
- $Var(Y) = 27$
- $Var(7X - 2Y + 8) = Var(7X) + Var(-2Y) + Var(8)$

$= 49 Var(X) + 4 Var(Y)$

$= 570.8736$

- $\rho = \dfrac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$

$Cov(X,Y) = 0 \Rightarrow \rho(X,Y) = 0.$

Exercitiul 2

1. $E[\log(X)] \leq \log(E[X])$

~~log este definit (X pozitiv)~~

~~$X > 0 \Rightarrow E[X] > 0$~~

~~Notăm $X = E[X] \Rightarrow X > \log X$~~

$\log(X)$ - concavă

din ineg lui Jensen $\Rightarrow E[\log(X)] \leq \log E[X]$

2. $E[X] \geq \sqrt{E[X]}$

~~$X > 0 \Rightarrow E[X] > 0 \Rightarrow \sqrt{\ }$ e definit~~

~~Notăm $X = E[X] \Rightarrow X \geq \sqrt{X}, \forall X > 0$~~

$\sqrt{\ }$ - convex pt $E[X] \geq 1$

din ineg lui Jensen $\Rightarrow E[X] \geq \sqrt{E[X]}$

3. $\mathbb{E}[\sin^2(X)] + \mathbb{E}[\cos^2(X)] = 1$

$\mathbb{E}[\sin^2(X)] + \mathbb{E}[\cos^2(X)] = \mathbb{E}[\sin^2(X) + \cos^2(X)] = \mathbb{E}[1] = 1$

4. $\mathbb{P}(X > c) \leq \frac{\mathbb{E}[X^3]}{c^3}$

din inegalitatea lui Markov

5. $\mathbb{P}(X \leq Y) = \mathbb{P}(X \geq Y)$

$X \perp\!\!\!\perp Y$, și nu avem altă inf despre ele, deci probabilitatea e aceeași

6. $\mathbb{P}(X+Y > 10) \cancel{\leq} \mathbb{P}(X > 5 \text{ sau } Y > 5)$

Există cazuri în care $X > 5$ sau $Y = 5$, dar $X + Y$ nu e mai mare decât 10

7. $\mathbb{E}[\min(X, Y)]$? $\min \mathbb{E}[X], \mathbb{E}[Y]$

depinde și de $\mathbb{P}(X), \mathbb{P}(Y)$

8. $\mathbb{E}\left[\frac{X}{Y}\right] \geq \frac{\mathbb{E}[X]}{\mathbb{E}[Y]}$

$$\frac{\mathbb{E}[X]}{\mathbb{E}[Y]} = \frac{X\,\mathbb{P}(X)}{Y\,\mathbb{P}(Y)} = \frac{X}{Y} \cdot \frac{\mathbb{P}(X)}{\mathbb{P}(Y)} \Big| \cdot \mathbb{E} \Rightarrow$$

$$\Rightarrow = \mathbb{E}\left[\frac{X}{Y}\right] \cdot \mathbb{E}\underbrace{\left[\frac{\mathbb{P}(X)}{\mathbb{P}(Y)}\right]}_{= \text{const}} = \mathbb{E}\left[\frac{X}{Y}\right] \cdot \underbrace{\frac{\mathbb{P}(X)}{\mathbb{P}(Y)}}_{< 0} \geq \mathbb{E}\left[\frac{X}{Y}\right]$$

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9. $\mathbb{E}[X^2(X^2+1)]$? $\mathbb{E}[X^2[Y^2+1]]$

$$\mathbb{E}[X^2(X^2+1)] = \cancel{\mathbb{E}[X^2]} \cdot \mathbb{E}[X^2+1]$$
$$\mathbb{E}[X^2(Y^2+1)] = \cancel{\mathbb{E}[X^2]}\,\mathbb{E}[Y^2+1]$$

$\mathbb{E}[X^2]$ se reduce și rămâne de comparat $\mathbb{E}[X^2+1]$ și $\mathbb{E}[Y^2+1]$, dar nu știm relația dintre X și Y deci nu le putem compara.

10. $\mathbb{E}\left[\frac{1}{X}\right]$? $\frac{1}{\mathbb{E}[X]}$

$\mathbb{E}[X^{-1}]$? $\mathbb{E}[X]^{-1}$

"moment ~~central~~ de ordin -1" - nu ~~s~~există

Exercițiul 3

a) $2 \le X+Y \le 6$

H_i - telefonul i este defect

$$\mathbb{P}(X=1, Y=1) = \frac{2}{7} \cdot \frac{1}{6} = \frac{2}{42} = \frac{1}{21}$$
$$\mathbb{P}(X=1, Y=2) = \frac{2}{7} \cdot \frac{\cancel{5}}{6} \cdot \frac{1}{\cancel{5}} = \frac{2}{42} = \frac{1}{21}$$
$$\mathbb{P}(X=1, Y=3) = \frac{2}{7} \cdot \frac{\cancel{5}}{6} \cdot \frac{\cancel{4}}{\cancel{5}} \cdot \frac{1}{\cancel{4}} = \frac{2}{42} = \frac{1}{21}$$
$$\mathbb{P}(X=1, Y=4) = \frac{2}{7} \cdot \frac{\cancel{5}}{\cancel{6}_2} \cdot \frac{\cancel{3}^1}{\cancel{5}} = \frac{1}{7} = \frac{1}{7}$$
$$\mathbb{P}(X=1, Y=5) = \frac{2}{7} \cdot \frac{\cancel{5}}{6} \cdot \frac{4}{\cancel{5}} = \frac{8}{42}.$$

Restul rezultatelor: (rep comună):

X\Y	0	1	2	3	4	5
1	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$
2	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$	0
3	0	$\frac{1}{21}$	$\frac{1}{21}$	$\frac{2}{21}$	0	0
4	0	$\frac{1}{21}$	$\frac{2}{21}$	0	0	0
5	$\frac{1}{21}$	$\frac{2}{21}$	0	0	0	0

Rep marginale:

$$X \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{3}{21} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} 0 & 1 & 2 & 3 & 4 & 5 \\ \frac{1}{21} & \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} \end{pmatrix}$$

b) $\mathbb{E}[X] = \frac{6+10+12+12+15}{21} = \frac{55}{21} = 2{,}61$

$\mathbb{E}[Y] = \frac{6+10+12+12+10}{21} = \frac{50}{21} = 2{,}38$

$$X^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 \\ \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{3}{21} \end{pmatrix}$$

$$Y^2 \sim \begin{pmatrix} 0 & 1 & 4 & 9 & 16 & 25 \\ \frac{1}{21} & \frac{6}{21} & \frac{5}{21} & \frac{4}{21} & \frac{3}{21} & \frac{2}{21} \end{pmatrix}$$

$$\mathbb{E}[X^2] = \frac{6+20+36+48+75}{21} = 8{,}80$$

$$\mathbb{E}[Y^2] = \frac{6+20+36+48+50}{21} = 7{,}61$$

$$Var(X) = \mathbb{E}[X^2] - \mathbb{E}[X]^2 = 1{,}99$$

$$Var(Y) = 1{,}95.$$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}}$$

$$\mathbb{E}[XY] = \sum xy\ \mathbb{P}(X=x,\ Y=y)$$

$$= 1\cdot 0 \cdot 0 + 1\cdot 1\cdot \frac{1}{21} + 1\cdot 2\cdot \frac{1}{21} + 1\cdot 3\cdot \frac{1}{21} + \ \ldots$$

$$= 5$$

$$Cov(X,Y) = 5 - 1{,}99\cdot 1{,}95 = 1{,}12$$

$$\rho(X,Y) = \frac{1{,}12}{\sqrt{1{,}99}\cdot\sqrt{1{,}95}} = 0{,}57$$

c) $\mathbb{E}[X \mid Y=2] =$

$$X \mid Y=2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1/21}{5/21} & \frac{1/21}{5/21} & \frac{1/21}{5/21} & \frac{2/21}{5/21} & \frac{0}{5/21} \end{pmatrix} =$$

$$= \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix}$$

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$$\mathbb{E}[X|Y=2] = \frac{1}{5} + \frac{2}{5} + \frac{3}{5} + \frac{8}{5} + 0 = \frac{14}{5} = 2{,}8$$

$$Var(X|Y=2) = E[(X|Y=2)^2] - \mathbb{E}[X|Y=2]^2$$

$$(X|Y=2)^2 \sim \begin{pmatrix} 1 & 4 & 9 & 16 & 25 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{2}{5} & 0 \end{pmatrix}$$

$$\mathbb{E}[(X|Y=2)^2] = \frac{1+4+9+32}{5} = 9{,}2$$

$$Var(X|Y=2) = 9{,}2 - 7{,}84 = 1{,}36$$

Exercițiul 4

$$f(x) = \frac{x}{64} e^{-\frac{x^2}{128}} 1_{\{x \geq 0\}}$$

$$\left.\begin{array}{l} F(x) = \int_{-\infty}^{x} f(t)\,dt \\ x \geq 0 \end{array}\right\} \Rightarrow F(x) = \int_0^x f(t)\,dt =$$

$$= \int_0^x \frac{t}{64} e^{-\frac{t^2}{128}}\,dt = -e^{-\frac{t^2}{128}}\Big|_0^x$$

$$= -e^{-\frac{x^2}{128}} + 1 = 1 - e^{-\frac{x^2}{128}}$$

$$F^{-1}(x) = y \mid F \Rightarrow x = F(y) = 1 - e^{-\frac{y^2}{128}}$$

$$e^{-\frac{y^2}{128}} = 1 - x$$

$$-\frac{y^2}{128} = \ln(1-x)$$

$$y = \sqrt{128 \ln\left(\frac{1}{1-x}\right)} = 8\sqrt{2 \ln\left(\frac{1}{1-x}\right)} \Rightarrow$$

$$\Rightarrow F^{-1}(x) = 8\sqrt{2\ln\left(\frac{1}{1-x}\right)}$$

$$F^{-1}\left(\frac{3}{4}\right) = 8\sqrt{2\ln 4}, \quad F^{-1}\left(\frac{1}{4}\right) = 8\sqrt{2\ln\frac{4}{3}}$$

$$\mathrm{Var}(X) = E[X^2] - E[X]^2$$

$$E[X] = \int_0^\infty x f(x)\,dx = \int_0^\infty x \frac{x}{64} e^{-\frac{x^2}{128}}\,dx$$

$$= \int_0^\infty (-x)\left(e^{-\frac{x^2}{128}}\right)' dx = \underbrace{-xe^{-\frac{x^2}{128}}\Big|_0^\infty}_{0} + \int_0^\infty x e^{-\frac{x^2}{128}}\,dx$$

$$= \int_0^\infty (-64)\left(e^{-\frac{x^2}{128}}\right)' dx = -64 e^{-\frac{x^2}{128}}\Big|_0^\infty + 64\int_0^\infty e^{-\frac{x^2}{128}}\,dx$$

$$y = \frac{x}{8}$$

$$= -64 + 64\cdot 8 \cdot \underbrace{\int_0^\infty e^{-\frac{y^2}{2}}\,dx}_{\frac{\sqrt{2\pi}}{2}}$$

$$= -64 + \frac{512\sqrt{2\pi}}{2}$$

$$E[X^2] = \int_0^\infty x^2 f(x)\,dx = \int_0^\infty (-x^2)\cdot\left(e^{-\frac{x^2}{128}}\right)' dx$$

$$= \underbrace{-x^2 \cdot e^{-\frac{x^2}{128}}\Big|_0^\infty}_{0} + 64^2\int_0^\infty (-1)\left(e^{-\frac{x^2}{128}}\right)' dx$$

$$= 64^2\cdot(-1)\cdot e^{-\frac{x^2}{128}}\Big|_0^\infty + 64^2 \underbrace{\int_0^\infty e^{-\frac{x^2}{128}}\,dx}_{8\cdot\frac{\sqrt{2\pi}}{2}}$$

$$= -64^2 + 64^2\cdot 8 \cdot \frac{\sqrt{2\pi}}{2}$$

$Var(X) = -64^2 + 64^2 \cdot 8 \cdot \frac{\sqrt{2\pi}}{2} + 64^2 - 64 \cdot 8 \frac{\sqrt{2\pi}}{2} +$

$+ \frac{2 \cdot 64^2 \cdot 8 \cdot \sqrt{2\pi}}{2}$

$Var(X) = -2 \cdot 64^2 + \frac{\sqrt{2\pi}}{2}\left(64^2 \cdot 8 - 512 + 2 \cdot 64^2 \cdot 8\right)$

Înlocuim și calculăm raportul din cerință

Exercițiul 5

N - nr alegătorilor, $N \sim Pois(906)$

$906 = \lambda$

X - alegătorii care votează Orban

Y - alegătorii care votează Cițu

p - probabilitatea ca un aleg să voteze Cițu

a) $P(X=i, Y=j) = P(X=i \mid Y=j \mid N=i+j)\, P(N=i+j)$

$P(X=i = Y=j \mid N=i+j) = P(X=i \mid N=i+j) = P(Y=j \mid N=i+j)$

$P(X=i, Y=j) = \binom{j+i}{i} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{(i+j)!} =$

$= \frac{\cancel{(i+j)!}}{i!\, j!} p^i (1-p)^j e^{-\lambda} \frac{\lambda^{i+j}}{\cancel{(i+j)!}} =$

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$$= \frac{e^{-\lambda p}(\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)} \cdot \frac{\lambda(1-p)^j}{j!}$$

$$= \frac{e^{-906\cdot 0.64}(906\cdot 0.64)^i}{i!} \cdot e^{-906\cdot 0.36} \cdot \frac{906\ 0.36^j}{j!}$$

Rep marginale:

$$\mathbb{P}(X=i) = \sum_{j\geq 0} \mathbb{P}(X=i, Y=j)$$

$$= \sum_{j\geq 0} \frac{e^{-\lambda p}(\lambda p)^i}{i!} \cdot e^{-\lambda(1-p)} \cdot \frac{\lambda(1-p)^j}{j!} = e^{-\lambda p}\frac{(\lambda p)^i}{i!}$$

$\Rightarrow X \sim Pois(\lambda p) \Leftrightarrow X \sim Pois(906 \cdot 0.64)$

Analogo $Y \sim Pois(\lambda(1-p)) \Leftrightarrow Y \sim Pois(906\cdot 0.36)$

b) $\mathbb{P}(X=i, Y=j) = \mathbb{P}(X=i)\,\mathbb{P}(Y=j) \Rightarrow X \perp\!\!\!\perp Y$

c) $\mathbb{E}[\cancel{X}V] = \lambda = 906$

$Var(V) = \lambda = 906$

Exercițiul 6

X - nr de succese înainte de al 4-lea eșec

$$P(X=k) = \binom{k-1}{3} \cdot (1-p)^{k-4} \cdot p^4 \quad , k \geq 4$$

$$E[X] = \frac{0.33 \cdot 4}{0.67} = 1,97$$

$$E[15X-9] = 15 \cdot 1,97 - 9 = 20,55$$

$$Var(X) = \frac{0,33 \cdot 4}{(0,67)^2} = 3$$

$$Var(2X+7) = 4\,Var(X) = 4 \cdot 3 = 12$$