$$X|Y=0 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{3/18}{7/18} & 0 & \frac{4/18}{7/18} \end{pmatrix}$$

$$Y|X=1 \sim \begin{pmatrix} -1 & 0 & 2 \\ \frac{\frac{1}{18}}{6/18} & \frac{3/18}{6/18} & \frac{2/18}{6/18} \end{pmatrix}$$

prima linie

X Y nu sunt indep $\Rightarrow$ fără 0 ... (**)

$\Rightarrow \mathbb{E}[XY] = \frac{1}{18}(-1) + 2 \cdot \frac{2}{18} + (-2) \cdot \frac{2}{18} + 4 \cdot \frac{3}{18} + 6 \cdot \frac{3}{18} = \ldots$

$\mathbb{E}[2X+3Y] = 2\mathbb{E}[X] + 3\mathbb{E}[Y]$

$\overset{sau}{=} \sum (2x+3y)\, \mathbb{P}(X=x, Y=y)$

$$\boxed{\mathbb{E}[X|Y=y] = \sum_x x\, \mathbb{P}(X=x|Y=y)} \Leftrightarrow$$ media condiționată a lui $Y=y$

* $\mathbb{E}[X|Y] = g(Y)$

* X e cond la Y ia val. $g(y_i)$ din Y

MAIL LAB

9 feb ora 9

Cursul 13 : 09.01

$f_{X,Y}(x,y)$ densitatea comună

$f_X(x) = \int f_{X,Y}(x,y)\, dy$

$f_Y(y) = \int f_{X,Y}(x,y)\, dx$

$$f_{X|A}(x) = \frac{f_X(x)}{\mathbb{P}(X\in A)}$$

$$\mathbb{P}(X\in B|A) = \int_B f_{X|A}(x)\,dx$$

$$f_{X|Y}(x|y) = \frac{f_{X,Y}(x|y)}{f_Y(y)}$$

Formula probab. totale !

$$f_X(x) = \sum_{i=1}^{n} f_{X|A_i}\,\mathbb{P}(A_i)$$

---

a) Y este o variabilă a. discretă $\{y_1, \dots, y_n\}$
X este o v. a. cont. f_X

$$f_X(x) = \sum_{i=1}^{n} f_{X|Y}(x|y_i)\,\mathbb{P}(Y=y_i)$$

b) Y v. a. cont. f_Y
X v. a. cont. cu dens. f_X

$$f_X(x) = \int f_{X|Y}(x|y)\, f_Y(y)\,dy$$

Independența v. a.

$X \perp Y \iff \mathbb{P}(X\in A, Y\in B) = \mathbb{P}(X\in A)\,\mathbb{P}(Y\in B),\ \forall A, B \subseteq \mathbb{R}$

Pt. $A = (-\infty, x]$; $B = (-\infty, y] \Rightarrow \mathbb{P}(X\le x, Y\le y) = \mathbb{P}(X\le x)\cdot\mathbb{P}(Y\le y)$
$\forall x, y$

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$$\int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du = \int_{-\infty}^{x} f_X(u)\,du \int_{-\infty}^{y} f_Y(v)\,dv$$

derivăm după x și y

$$\frac{\partial^2}{\partial x\,\partial y}\int_{-\infty}^{x}\int_{-\infty}^{y} f_{X,Y}(u,v)\,dv\,du =$$

$$= \frac{d}{dx}\int_{-\infty}^{x} f_X(u)\,du \; \frac{d}{dy}\int_{-\infty}^{y} f_Y(v)\,dv$$

$$f_{X,Y}(x,y) = f_X(x)\, f_Y(y)$$

! (P) Fie X, Y v.a. cu densitățile f_X, respectiv f_Y. Atunci $X \perp\!\!\!\perp Y \Leftrightarrow f_{X,Y}(x,y) = f_X(x)\cdot f_Y(y)$.

(P) Fie X, Y două v.a. și g și h două fct. Dacă $f_{X,Y}(x,y) = g(x)\,h(y)$, $\forall x,y$ atunci $X \perp\!\!\!\perp Y$

(P) Dacă X și Y sunt 2 variabile aleatoare independente atunci

! $\mathbb{E}[g(X)\,h(Y)] = \mathbb{E}[g(X)]\,\mathbb{E}[h(Y)]$

Dacă $g(X) = X$ și $h(Y) = Y \Rightarrow$ fct. identitate

atunci $\mathbb{E}[X, Y] = \mathbb{E}[X]\cdot\mathbb{E}[Y]$

$\boxed{!\ Var(X+Y) = Var(X) + Var(Y)}$ nu e reciprocă

Formula lui Bayes

$$f_{X,Y}(x,y) = f_{X|Y}(x|y) \cdot f_Y(y)$$
$$= f_{Y|X}(y|x)\, f_X(x)$$

$$\Rightarrow f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)} = \frac{f_{Y|X} \cdot f_X(x)}{f_Y(y)} =$$

$$= \frac{f_{Y|X}(y|x)\, f_X(x)}{\int f_{Y|X}(y|x')\cdot f_X(x')\, dx'}$$ Form. lui Bayes

<u>Caz hibrid</u>

$$\underset{B}{\{0,1\}} \longrightarrow \boxed{\text{zgomot}}_{+N(0,1)} \xrightarrow{Y} \boxed{\text{inf}} \longrightarrow P(A|Y=y) \quad \underset{0}{\overset{P(Y=y)}{\shortparallel}}$$

$$P(A|Y=y) = \lim_{dy\to 0} P(A|Y\in(y, y+dy))$$
$$\left(y-\frac{dy}{2}, y+\frac{dy}{2}\right)$$

$$= \lim_{dy\to 0} \frac{P(A \cap Y\in(y,y+dy))}{P(Y\in(y,y+dy))} =$$

$$= \lim_{dy\to 0} \frac{P(A)\cdot P(Y\in(y,y+dy)\,|\,A)}{P(Y\in(y,y+dy))}$$

$$= \lim_{dy\to 0} \frac{P(A)\cdot \int_y^{y+dy} f_{Y|A}(u)\,du}{\int_y^{y+dy} f_Y(u)\,du}$$

$$= \lim_{dy\to 0} \frac{P(A)\cdot f_{Y|A}(y)\,\cancel{dy}}{f_Y(y)\,\cancel{dy}}$$

$$P(A|Y=y) = \frac{P(A)\cdot f_{Y|A}(y)}{f_Y(y)}$$

dens. marginală a lui Y

f. lui Bayes

$$= f_{Y|A}(y)\, P(A) + f_{Y|A^c}(y)\cdot P(A^c)$$

Formula probab. totale

X\Y	discret	cont
discret	$\mathbb{P}(X=x)=\sum_y \mathbb{P}(X=x \mid Y=y)\cdot \mathbb{P}(Y=y)$	$\mathbb{P}(X=x)=\int \mathbb{P}(X=x \mid Y=y)\cdot f_Y(y)\,dy$
cont.	$f_X(x)=\sum_y f_{X\mid Y}(x\mid y)\,\mathbb{P}(Y=y)$	$f_X(x)=\int f_{X\mid Y}(x\mid y)\cdot f_Y(y)\,dy$

Formula lui Bayes

X\Y	discret	cont
discret	$\mathbb{P}(Y=y \mid X=x)=\mathbb{P}(X=x \mid Y=y)\cdot \mathbb{P}(Y=y)\cdot \frac{1}{\mathbb{P}(X=x)}$	$f_{Y\mid X}(y\mid x)=\frac{\mathbb{P}(X=x \mid Y=y)\, f_Y(y)}{\mathbb{P}(X=x)}$
cont	$\mathbb{P}(Y=y \mid X=x)=\frac{f_{X\mid Y}(x\mid y)\cdot \mathbb{P}(Y=y)}{f_X(x)}$	$f_{Y\mid X}(y\mid x)=\frac{f_{X\mid Y}(x\mid y)\, f_Y(y)}{f_X(x)}$

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Exp: A și B : durata de viață A $Exp(\lambda_0)$

——— " ——— B $Exp(\lambda_1)$

$\boxed{\lambda_0 < \lambda_1}$

Pp. că primim un telefon de la A cu prob. p_0 și de la B cu prob. $p_1 = 1 - p_0$

Fie T durata de viață a telefonului primit.

a) Fct. de rep. și densitatea lui T

b) Vrem să găsim prob. ca telefonul să fie provenit de la B știind că $T = t$.

T v.a. cont

Fie I v.a. $\begin{cases} 0, \text{ dacă tel. A} \\ 1, \text{ dacă tel. B} \end{cases}$

$$\boxed{\begin{aligned} Exp(\lambda).\ & f(x) = \lambda e^{-\lambda x} \\ & F(x) = 1 - e^{-\lambda x} \end{aligned}}$$

$\mathbb{P}(I = 0) = p_0$

$\mathbb{P}(I = 1) = p_1 = 1 - p_0$

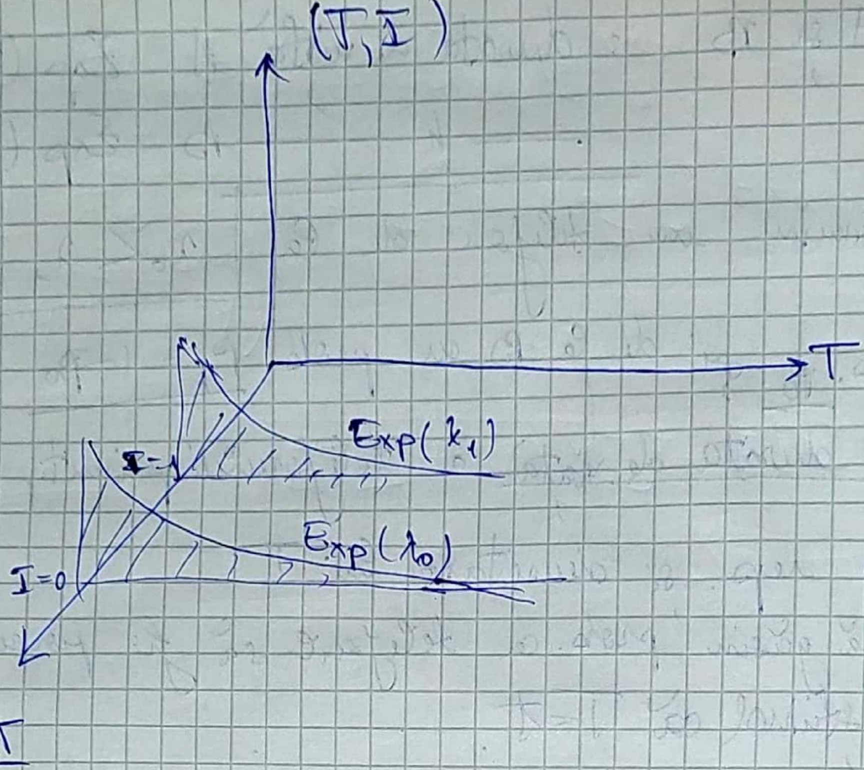
$T \mid I = 0 \sim Exp(\lambda_0)$

$T \mid I = 1 \sim Exp(\lambda_1)$

$$\mathbb{P}(T \le t) = \overbrace{\mathbb{P}(T \le t \mid I = 0)}^{1 - e^{-\lambda_0 t}} \cdot \overbrace{\mathbb{P}(I = 0)}^{p_0} + \underbrace{\mathbb{P}(T \le t \mid I = 1)}_{1 - e^{-\lambda_1 t}} \cdot \underbrace{\mathbb{P}(I = 1)}_{1 - p_0}$$

$$= (1 - e^{-\lambda_0 t}) p_0 + (1 - e^{-\lambda_1 t})(1 - p_0) \qquad t > 0$$

$$f_T(t) = \frac{d}{dt} F_T(t) = \lambda_0 e^{-\lambda_0 t} p_0 + \lambda_1 e^{-\lambda_1 t}(1 - p_0)$$



b) $P(I=1 \mid T=t) = \dfrac{f_{T|I}(t|1) \cdot P(I=1)}{f_T(t)}$

$$= \frac{\lambda_1 \cdot e^{-\lambda_1 t}(1-p_0)}{\lambda_0 \cdot e^{-\lambda_0 t} \cdot p_0 + \lambda_1 \cdot e^{-\lambda_1 t}(1-p_0)}$$

Media unei fct. de v. a

X, Y două v. a $f_{X,Y}(x,y)$ și $g: \mathbb{R}^2 \to \mathbb{R}$

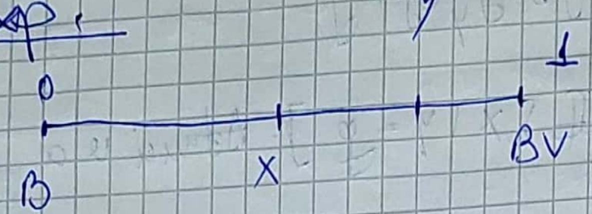
$$\boxed{E[g(X,Y)] = \iint g(x,y) \cdot f_{X,Y}(x,y)\, dx\, dy} \; !$$

În particular

~~E[X]~~

$$E[XY] = \iint xy\, f_{X,Y}(x,y)\, dx\, dy$$

Exp:



$X, Y \sim \mathcal{U}[0,1)$ indep.

$$\mathbb{E}[\,|X-Y|\,] = \iint |x-y|\, f_{X,Y}(x,y)\, dx\, dy$$

$$\mathbb{E}[\,|X-Y|\,] = \iint |x-y| \cdot \mathbb{1}_{[0,1]}(x) \cdot \mathbb{1}_{[0,1)}(y)\, dx\, dy$$

$$= \int_0^1 \int_y^1 (x-y)\, dx\, dy + \int_0^1 \int_0^y (y-x)\, dx\, dy$$

$$= \int_0^1 \left. \frac{x^2}{2} - yx \right|_y^1 dy + \int_0^1 \left. yx - \frac{x^2}{2} \right|_0^y dy = \frac{1}{3}$$

Media condiționată

X v.a. cont și A un ev. $\mathbb{P}(A) > 0$

$$\mathbb{E}[X \mid A] = \int x\, f_{X|A}(x)\, dx$$

Dacă $A = \{Y = y\}$

$$\mathbb{E}[X \mid Y = y] = \int x\, f_{X|Y}(x|y)\, dx$$

Formula probab. totale

$$f_X(x) = \sum_{i=1}^{n} f_{X|A_i} \cdot \mathbb{P}(A_i)$$

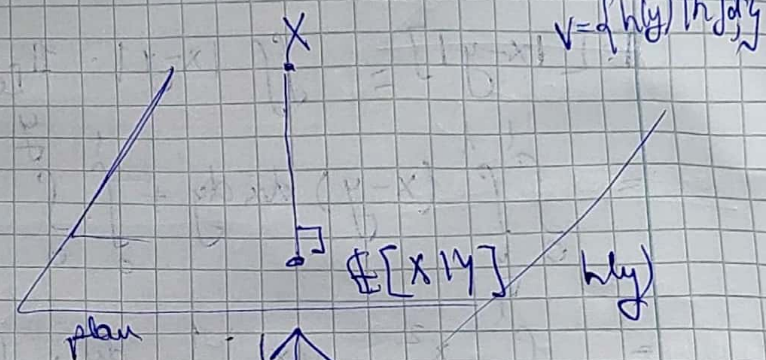
$\cdot\, x$ apoi integrez

$$\mathbb{E}[X] = \sum_{i=1}^{n} \mathbb{E}[X \mid A_i] \cdot \mathbb{P}(A_i)$$

$$\mathbb{E}[X] = \int \mathbb{E}[X \mid Y = y]\, f_Y(y)\, dy$$

<u>Def.</u> Fie $g(y) = \mathbb{E}[X \mid Y = y]$. Atunci v.a.
$\mathbb{E}[X \mid Y] = g(Y)$

$\boxed{\text{Prop.}}$ a) $\mathbb{E}\big[\mathbb{E}[X \mid Y]\big] = \mathbb{E}[X]$



cea mai bună predicție

$\mathbb{E}[X \mid Y] = \arg\min_{g} \mathbb{E}[(X - g(Y))^2]$

$\mathbb{E}[(X-a)^2]$

ct. care minimizează $(X-a)^2$ este media

$$\mathrm{Var}(X) = \mathrm{Var}(\mathbb{E}[X \mid Y]) + \mathbb{E}[\mathrm{Var}(X \mid Y)]$$

N

$X_1, X_2 \ldots$

$T = X_1 + X_2 + \ldots + X_N$

$\mathbb{E}[T]$

$\omega_1 \quad N(\omega_1) = 10$

$T(\omega_1) = X_1^{(\omega_1)} + \ldots + X_{10}^{(\omega_1)}$

Covarianță și corelație

Def. Fie X și Y două v.a. S.n. covarianța dintre X și Y.

$$Cov(X,Y) = E[(X - E[X])(Y - E[Y])]$$

În particular, $X = Y \Rightarrow Cov(X,X) = Var(X)$

(P) $\boxed{Cov(X,Y) = E[XY] - E[X]E[Y]}$!!

Def. Spunem că v.a. X și Y sunt necorelate dacă $Cov(X,Y) = 0$.

Altfel spus, $E[XY] = E[X]E[Y]$

Obs $\boxed{\text{Dacă } X \perp Y \Rightarrow X \text{ și } Y \text{ sunt necorelate}}$ Reciproca e falsă

Exp: $\left.\begin{array}{l} X \sim N(0,1) \\ Y = X^2 \end{array}\right\} \Rightarrow \left.\begin{array}{l} E[X]E[Y] = 0 \\ E[XY] = E[X^3] = 0 \end{array}\right\} \Rightarrow$

$\Rightarrow$ X și Y sunt necorelate

X și Y nu sunt indep!

Prop. a) $Cov(X,X) = Var(X)$

b) $Cov(X,a) = 0$; a const.

c) $Cov(a+bX, Y) = b\,Cov(X,Y)$

d) $Cov(X,Y) = Cov(Y,X)$

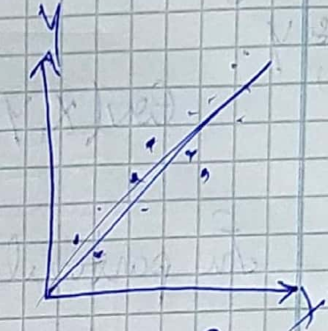
e) $Var(X+Y) = Var(X) + Var(Y) + 2\,Cov(X,Y)$

$$Var(X_1 + X_2 + \dots + X_n) = \sum_{i=1}^{n} Var(X_i) + 2\sum_{i<j} Cov(X_i, X_j)$$

f) $Cov(X+Y, Z) = Cov(X, Z) + Cov(Y, Z)$

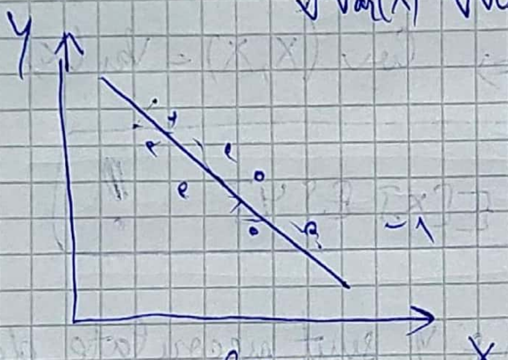
Def. Fie X și Y două v.a. și definim coeficientul de corelație dintre X și Y.

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)} \cdot \sqrt{Var(Y)}}$$

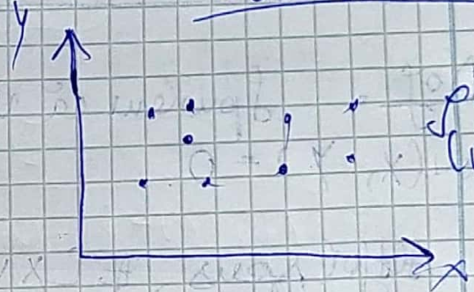


$\rho > 0$

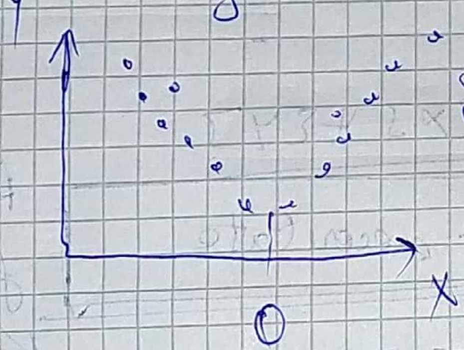
cresc simultan



$\rho < 0$



$\rho = 0$ (indep)



$\rho = 0$ necorelate

Prop. $\rho \in [-1, 1]$. Dacă $\rho = 1$ (sau -1) atunci $X = a + bY$ ($Y = a + bX$) a.s. (aproape sigur)

$P(X = a + bY) = 1$

Dem. X, Y: $E[X] = \mu_X$, $Var(X) = \sigma_X^2$,

$E[Y] = \mu_Y$, $Var(Y) = \sigma_Y^2$

Pp. că $\mu_X = \mu_Y = 0$ și $\sigma_X^2 = \sigma_Y^2 = 1$ $\rightarrow \rho(X,Y) = E[X, Y]$

$$\rho(X,Y) = \frac{Cov(X,Y)}{\sqrt{Var(X)\,Var(Y)}} = \frac{E[(X-\mu_X)(Y-\mu_Y)]}{\sigma_X \sigma_Y}$$

$$= \mathbb{E}\Big[\underbrace{\Big(\frac{X-\mu_X}{\sigma_X}\Big)\Big(\frac{Y-\mu_Y}{\sigma_Y}\Big)}_{}\Big]$$

v.a. normalizate
obținute prin translație
și scalare

Știm că $\mathbb{E}[(X+\lambda Y)^2] \geq 0, \forall \lambda \in \mathbb{R}$

$\Leftrightarrow$

$$\lambda^2 \mathbb{E}[Y^2] + 2\lambda \mathbb{E}[XY] + \mathbb{E}[X^2] \geq 0, \forall \lambda$$

$$\Delta = 4\mathbb{E}[XY]^2 - 4\mathbb{E}[X^2]\mathbb{E}[Y^2] \leq 0$$

$$\mathbb{E}[XY]^2 \leq \underbrace{\mathbb{E}[X^2]}_{=1}\underbrace{\mathbb{E}[Y^2]}_{=1} \quad !$$

$$\Rightarrow \mathbb{E}[XY]^2 \leq 1 \Rightarrow \rho(X,Y)^2 \leq 1 \Rightarrow |\rho(X,Y)| \leq 1$$

Ineg. Cauchy Schwartz

$$\Big(\sum_{i=1}^{n} a_i b_i\Big)^2 \leq \Big(\sum a_i^2\Big)\Big(\sum b_i^2\Big)$$

$$|\langle x, y\rangle| \leq \|x\| \cdot \|y\|$$

Ineg. și teoreme limită

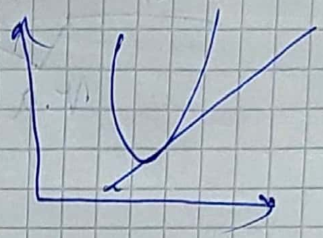
1. Ineg. Cauchy - Schwartz

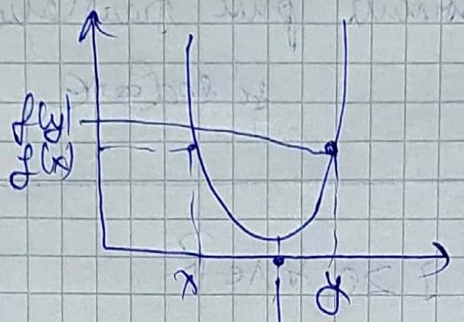
Fie X și Y v.a. cu $Var(X) < \infty$, $Var(Y) < \infty$. Atunci

$$|\mathbb{E}[XY]| \leq \sqrt{\mathbb{E}[X^2]\,\mathbb{E}[Y^2]}$$

② Inegalitatea Jensen

a) Fct. convexă





$tx+(1-t)y \quad t\in[0,1]$

$$\boxed{f(tx+(1-t)y) \le tf(x)+(1-t)f(y)}$$

Fct. concavă: $\forall x,y$; $\forall t\in[0,1]$

$$\boxed{f(tx+(1-t)y)\ge t\cdot f(x)+(1-t)f(y)}$$

(T) (Ineg. Jensen)

Fie X v.a. și g o fct. convexă. Atunci

$$E[g(X)] \ge g(E[X])$$

Dacă g este concavă: $E[g(X)] \le g(E[X])$

! Obs. $Var(X)\ge 0$

$$E[X^2]\ge E[X]^2$$

---

③ Ineg. Markov

Fie X v.a. pozitivă. Atunci

$$P(X\ge a)\le \frac{E[X]}{a}$$

Dem. $Y = \begin{cases} 0, & X < a \\ a, & X \geq a \end{cases}$

$\mathbb{E}[Y] = a\,\mathbb{P}(X \geq a)$

$Y \leq X \Rightarrow \mathbb{E}[Y] \leq \mathbb{E}[X]$

Exp. $X \sim \mathcal{U}([0,1])$

$\mathbb{P}(X > \alpha) \leq \frac{1}{4}$

$\mathbb{P}(X \geq 1) \leq \frac{1}{\alpha}$

$\mathbb{P}(X \geq \frac{1}{\alpha}) \leq 1$

---

④ (Ineg. ~~de~~ Chebyshev)

Fie X v.a. $\mathbb{E}[X] = \mu < \infty$; $Var(X) = \sigma^2 < \infty$

$\mathbb{P}(|X-\mu| \geq a) = \frac{Var(X)}{a^2}$; $\forall a > 0$

$Y = (X-\mu)^2$

$\mathbb{P}(Y \geq a^2) \leq \frac{\mathbb{E}[Y]}{a^2} = \frac{Var(X)}{a^2}$

Obs. $a = k\sigma$

$\mathbb{P}(|X-\mu| \geq k\sigma) \leq \frac{1}{k^2}$

---

⑤ (Ineg. Chernoff) ! f.p. (foarte puternic)

Fie X v.a. $a > 0$, $t > 0$

$\mathbb{P}(X \geq a) \leq \frac{\mathbb{E}[e^{tx}]}{e^{ta}}$ $\forall a, t$

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