EXAMEN CALCUL DIFERENTIAL SI INTEGRAL
SERIA 13

OFICIU: **1 punct**

SUBIECTUL 1. *(2 puncte)*

Sa se studieze natura seriei $\sum\limits_{n=0}^{+\infty} \frac{a^n (n!)^2}{(2n)!}$, unde $a > 0$.

SUBIECTUL 2. *(2 puncte)*

Sa se determine punctele de extrem local ale functiei $f : \mathbb{R}^2 \to \mathbb{R}, f(x,y) = x^3 + y^3 + 21xy + 36x + 36y \ \forall (x,y) \in \mathbb{R}^2$.

SUBIECTUL 3. *(2 puncte)*

Sa se demonstreze inegalitatea $\ln(x-1) < x - 2 + \frac{(x-2)^2}{2} + \frac{(x-2)^3}{3} \ \forall x \in (2, +\infty)$.

SUBIECTUL 4. *(3 puncte)*

a) Sa se calculeze $\iint\limits_D \sqrt{x^2+y^2}dxdy$, unde $D = \left\{(x,y) \in \mathbb{R}^2 \mid 2x \le x^2 + y^2 \le 4x, y \ge 0\right\}$.

b) Fie $\sum\limits_{n=0}^{+\infty} x_n$ o serie convergenta de numere reale pozitive. Sa se arate ca $\liminf nx_n = 0$.

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Examen

Subiectul I

$$\sum_{n=0}^{\infty} \frac{a^n (n!)^2}{(2n)!}, a>0$$

$$x_n = \frac{a^n (n!)^2}{(2n)!}$$

$$\frac{x_{n+1}}{x_n} = \frac{a^{n+1}[(n+1)!]^2}{(2n+2)!} \cdot \frac{2n!}{a^n (n!)^2} = \frac{a(n+1)^2}{(2n+1)(2n+2)} = \frac{a(n^2+2n+1)}{4n^2+6n+2}$$

$$\lim_{n\to\infty} \frac{x_{n+1}}{x_n} = \frac{a}{4} = L$$

I $a<4 \Rightarrow L<1 \Rightarrow$ seria convergentă

II $a>4 \Rightarrow L>1 \Rightarrow$ seria divergentă

III $a=4 \Rightarrow L=1 \Rightarrow$ aplicăm Raabe-Duhamel

$$\lim_{n\to\infty} n\left(\frac{x_n}{x_{n+1}} - 1\right) = \lim_{n\to\infty} n\left(\frac{4n^2+6n+2}{4n^2+8n+4} - 1\right) = \lim_{n\to\infty} n \cdot \frac{-2n-2}{4n^2+8n+4} =$$

$$= \lim_{n\to\infty} \frac{-2n^2-2n}{4n^2+8n+4} = -\frac{2}{4} = -\frac{1}{2} < 1 \xRightarrow{C.R.D.} \text{serie divergentă}$$

Subiectul II

$f:\mathbb{R}^2 \to \mathbb{R}, f(x,y) = x^3+y^3+21xy+36x+36y, \forall (x,y)\in\mathbb{R}^2$

f continuă pe $\mathbb{R}^2$

$$\frac{\partial f}{\partial x}(x,y) = (x^3+y^3+21xy+36x+36y)'_x = 3x^2+21y+36, \forall (x,y)\in\mathbb{R}^2$$

$$\frac{\partial f}{\partial y}(x,y) = (x^3+y^3+21xy+36x+36y) = 3y^2+21x+36, \forall (x,y)\in\mathbb{R}^2$$

$\cancel{\frac{\partial f}{\partial x}}$

$\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}$ funcții continue pe $\mathbb{R}$

$\mathbb{R}^2$ mulțime deschisă

f diferențiabilă pe $\mathbb{R}^2$

$\cancel{\Delta = f \times 6}$

$$(\mathbb{R}^2) \begin{cases} \frac{\partial f}{\partial x}(x,y)=0 \\ \frac{\partial f}{\partial y}(x,y)=0 \end{cases} \Rightarrow \begin{cases} 3x^2+21y+36=0 \\ 3y^2+21x+36=0 \end{cases}$$

$\cancel{(+)}$ $(-)$

$\cancel{3x^2+3y^2+21y+21x+72=0}$

$\cancel{3(x^2+y^2)+21(y+x}$

$3x^2-3y^2+21y-21x+36-36=0$

$3(x^2-y^2)+21(y-x)=0$

$3(x-y)(x+y)+21(y-x)=0$

$-3(y-x)(x+y)+21(y-x)=0$

$(y-x)[-3(y-x)+21]=0$

I $y-x=0 \Rightarrow y=x$

$3x^2+21x+36=0$

$\Delta = 21\cdot 21 - 4\cdot 3\cdot 36 = 441-432=9$

$x_{1,2}=\frac{-21\pm 3}{6}$ — $x_1=-\frac{24}{6}=-4 \Rightarrow y=-4 \Rightarrow (-4;-4)$

— $x_2=\frac{-18}{6}=-3 \Rightarrow y=-3 \Rightarrow (-3;-3)$

II $-3x+3y+21=0$

$-x+y+7=0 \Rightarrow y=x-7$

$3x^2+21(x-7)+36=0$

$3x^2+21x-147+36=0$

$3x^2+21x-111=0$

$\Delta<0 \Rightarrow$ nu are soluții reale

$C=\{(-4,-4);(-3,-3)\}$

(1) $\frac{\partial^2 f}{\partial x^2}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial x}\right)(x,y) = (3x^2+21y+36)'_x = 6x, \forall (x,y)\in \mathbb{R}^2$

(2) $\frac{\partial^2 f}{\partial x \partial y}(x,y) = \frac{\partial}{\partial x}\left(\frac{\partial f}{\partial y}\right)(x,y) = (3y^2+21x+36)'_x = 21, \forall (x,y)\in \mathbb{R}^2$

(3) $\frac{\partial^2 f}{\partial y \partial x}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial x}\right)(x,y) = (3x^2+21y+36)'_y = 21, \forall (x,y)\in \mathbb{R}^2$

(4) $\frac{\partial^2 f}{\partial y^2}(x,y) = \frac{\partial}{\partial y}\left(\frac{\partial f}{\partial y}\right)(x,y) = (3y^2+21x+36)'_y = 6y, \forall (x,y)\in \mathbb{R}^2$

$\left.\begin{array}{l} \frac{\partial^2 f}{\partial x^2}, \frac{\partial^2 f}{\partial x \partial y}, \frac{\partial^2 f}{\partial y \partial x}, \frac{\partial^2 f}{\partial y^2} \text{ funcții continue pe } \mathbb{R}^2 \\ \mathbb{R}^2 \text{ mulțime deschisă } \cancel{\text{și (1),(2),(3) și (4) } f \text{ cont}} \end{array}\right\} \Rightarrow$

$\Rightarrow f$ este diferențiabilă de 2 ori pe $\mathbb{R}^2$

$$Hf(-4,-4) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(-4,-4) & \frac{\partial^2 f}{\partial x \partial y}(-4,-4) \\ \frac{\partial^2 f}{\partial y \partial x}(-4,-4) & \frac{\partial^2 f}{\partial y^2}(-4,-4) \end{pmatrix} = \begin{pmatrix} -24 & 21 \\ 21 & -24 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

$\Delta_1 = a_{11} = -24$

$\Delta_2 = \begin{vmatrix} -24 & 21 \\ 21 & -24 \end{vmatrix} = 576 - 441 = 135$

$\left.\begin{array}{l} \Delta_1 < 0 \\ \Delta_2 > 0 \end{array}\right\} \Rightarrow (-4,-4)$ pct de maxim local al funcției f

$$Hf(-3,-3) = \begin{pmatrix} \frac{\partial^2 f}{\partial x^2}(-3,-3) & \frac{\partial^2 f}{\partial x \partial y}(-3,-3) \\ \frac{\partial^2 f}{\partial y \partial x}(-3,-3) & \frac{\partial^2 f}{\partial y^2}(-3,-3) \end{pmatrix} = \begin{pmatrix} -18 & 21 \\ 21 & -18 \end{pmatrix} \in \mathcal{M}_2(\mathbb{R})$$

$\Delta_1 = a_{11} = -18$

$\Delta_2 = \begin{vmatrix} -18 & 21 \\ 21 & -18 \end{vmatrix} = 324 - 441 = \cancel{83} -117$

$\left.\begin{array}{l} \Delta_1 < 0 \\ \Delta_2 < 0 \end{array}\right\}$ $\cancel{(-18}$ $(-3,-3)$ nu este punct de extrem

Subiectul 3

$\ln(x-1) < x-2+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}, \forall x \in (2,\infty)$

fie $x-2=t \Rightarrow x-1=t+1$; $x\in(2,\infty) \Rightarrow t\in(0,\infty)$

$\ln(t+1) < t+\frac{t^2}{2}+\frac{t^3}{3}$

fie $f(t):(0,\infty)\to\mathbb{R}$

$f(t)=t+\frac{t^2}{2}+\frac{t^3}{3}-\ln(t+1)$ ~~$= 1+t+t^2-\frac{1}{t+1} =$~~

$f'(t)=1+t+t^2-\frac{1}{t+1}=\frac{t+\cancel{1}+t^2+t+t^3+t^2-\cancel{1}}{t+1}=$

$\lim\limits_{\substack{t\to 0\\ t>0}} f(t)=0$ $\qquad =\frac{2t+2t^2+t^3}{t+1}=\frac{t(2+2t+t^2)}{t+1}$

deci $f'(t)=\frac{t((t+1)^2+1)}{t+1}>0, (\forall) t\in(0,\infty)$

deci $f(t)$ $\underbrace{\text{crescator}}_{\text{monoton}}$ $\Leftrightarrow \inf\limits_{t>0} f(t)=\lim\limits_{\substack{t\to 0\\ t>0}} f(t)=0$

$\Rightarrow f(t)=$ pozitiv $\forall t>0$ deci

$t+\frac{t^2}{2}+\frac{t^3}{3}>\ln(t+1) \Rightarrow$

$\Rightarrow \ln(x-1)<x-2+\frac{(x-2)^2}{2}+\frac{(x-2)^3}{3}$

Subiectul 4

a) $\iint_D \sqrt{x^2+y^2}\,dx\,dy$, $D=\{(x,y)\in\mathbb{R}^2 \mid 2x\le x^2+y^2\le 4x, y\ge 0\}$

dacă $2x\le x^2+y^2 \Leftrightarrow (x-1)^2-y^2=1 \Rightarrow \mathcal{C}(1,0)$ cu $r=1$

$4x\ge x^2+y^2 \Leftrightarrow (x-2)^2+y^2=4 \Rightarrow \mathcal{C}(2,0)$ cu $r=2$

fie $x=r\cos\theta$
$y=r\sin\theta$

$2r\cos\theta \le r^2 \le 4r\cos\theta$

$\cos\theta \le r \le 2\cos\theta$

$r \in [\cos\theta, 2\cos\theta] \Rightarrow$ $\cos\theta$ = monoton crescător deci $\theta \in [\pi, 2\pi]$

$\varphi(r,\theta) = (\underbrace{r\cos\theta}_{x}, \underbrace{r\sin\theta}_{y})$

Și $r \in [-1, 1]$ dar $r \ge 0$
$\Rightarrow r \in [0, 1]$

cu $\varphi : [0,1] \times [\pi, 2\pi]$

$\iint_\Delta f(\varphi(r,\theta)) \cdot |\underbrace{\det \cdot \varphi'(x)}_{r^2}|$ unde $|\det \varphi'(x)| = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$

$\iint_\Delta \sqrt{r^2}\, dr\, d\theta = \iint_\Delta r\, dr\, d\theta = \iint_{[0,1]\times[\pi,2\pi]} r\, dr\, d\theta$

$\int_0^1 \left(\int_\pi^{2\pi} r\, d\theta \right) dr = \int_0^1 \left(r\theta \Big|_\pi^{2\pi} \right) dr = \int_0^1 r(2\pi - \pi)\, dr = \int_0^1 r\pi\, dr =$

$= \pi \cdot \frac{r^2}{2} \Big|_0^1 = \frac{\pi}{2}$