11.01.2023

Probabilități și Statistică
– Săpt 13 – lab –

dbinom $(7, \_\_) \to \mathbb{P}(BIN(\_\_)=7)$
funcția de masă

dexp $(15, \_\_) \to f\,exp(\_)$ 15 val. densității

funcț. de repartiție X v.a.
$F_X : \mathbb{R} \to [0,1]$
$F_X(y) = \mathbb{P}(X \leq y)$, $\forall y \in \mathbb{R}$
$\rightsquigarrow$ pexp
pnorm

Th. de universalitate (p. 106)

$U \sim Unif(\{x_1, x_2 \ldots x_n\})$
$\mathbb{P}(U = x_i) = \frac{1}{n} = \frac{1}{\#\{x_1 \ldots x_n\}}$

$U \sim Unif([a,b])$
$f_U(x) = \begin{cases} \frac{1}{b-a}, & \text{pt. } x \in [a,b] \\ 0, & \text{dacă } x \notin [a,b] \end{cases}$

X va. continuă

$E(X) = \int x f_X(x)\, dx$
$Var(X) = E[(X - E(X))^2] =$
sau
$Var(X) = \underbrace{E(X^2)}_{\int x^2 \cdot f_X(x)\, dx} - E(X^2)$

COVARIANȚA

$Cov(X,Y) = E[(X - E(X)) \cdot (Y - E(Y))]$ $\boxed{Cov(X,X) = Var(X)}$
$\stackrel{sau}{=} E(X \cdot Y) - E(X) \cdot E(Y)$

–1–

Exercitiu: $X \sim Unif([2,7])$

$\mathbb{P}(X=6) = 0$ → prob. să ia o val. exactă e zero.

$\mathbb{P}(X \in \{6,7\}) = ?$

$\mathbb{P}(\{X=6\} \cup \{X=7\}) = 0$] ac. lucru

- $\mathbb{P}(X \in \{1,2,3,4,5,6,7\}) = 0$

  ↳ mulțime finită

Y v.a cont. cu densit. f_Y

$$\mathbb{P}(Y=a) = \lim_{\varepsilon \to 0} \mathbb{P}(Y \in (a-\varepsilon, a+\varepsilon)) =$$

$$= \lim_{\varepsilon \to 0} \int_{a-\varepsilon}^{a+\varepsilon} f_Y(y)\,dy =$$

↳ $\int_a^a$

$$= 0$$

$\mathbb{P}(Y \le a)$

$U \sim Unif((2,7))$

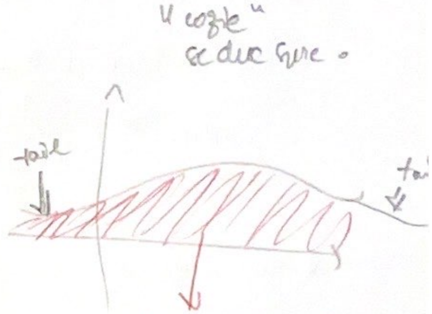
$$f_U(x) = \begin{cases} \frac{1}{5}, & dc\ x \in [2,7] \\ 0, & dc\ x \notin [2,7] \end{cases}$$

$$\mathbb{P}(U \le 5) = \int_{-\infty}^{5} f(x)\,dx = \int_2^5 f_U(x)\,dx$$

$\overset{\shortparallel}{\mathbb{P}}(U < 5)$

$U \le 5 = U < 5 \cup U = 5$

$\mathbb{P}(U \le 5) = \mathbb{P}(U<5) + \cancel{\mathbb{P}(U=5)}$



pt a fi densit, aria subgr. trebuie să fie 1

! $X \sim \mathcal{N}(\mu, \sigma^2)$

$E(X) = \mu$; $Var(X) = \sigma^2$

(media) (varianta)

⇒ definim $Y = \frac{X-\mu}{\sigma}$. Obs. că

$$E\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma} \cdot E(X-\mu) = \frac{1}{\sigma}(E(X)-\mu) = 0$$

$$Var(Y) = Var\left(\frac{X-\mu}{\sigma}\right) = \frac{1}{\sigma^2} Var(X) = \frac{1}{\sigma^2} \cdot \sigma^2 = 1$$

Recapitulare.

$X : \Omega \to \mathbb{R}$ v.a.

- media : $E(X) = \begin{cases} \sum x_i \, P(X = x_i), & \text{pt. } X \text{ discretă cu val. } \{x_1, \ldots x_n \ldots\} \\ \int_{\mathbb{R}} x \, f_X(x) \, dx, & \text{dacă } X \text{ e cont. și are densit. } f_X \end{cases}$

În general, formula de calcul pentru medie e un caz particular de apl. a unei reguli:

$$\boxed{E(h(X)) = \int_{\mathbb{R}} h(x) \cdot f_X(x) \, dx}$$ formula de transport

- varianța : $Var(X) = E((X - E(X))^2) =$
$= E(X^2) - E(X)^2$

- dispersia $= \sqrt{\text{varianța}}$: $\sigma = \sqrt{Var(X)}$

- covarianța : $Cov(X, Y) = E(X, Y) - E(X) \cdot E(Y)$

Obs: $X \perp\!\!\!\perp Y \Rightarrow E(X \cdot Y) = E(X) \cdot E(Y)$
$\Rightarrow Cov(X, Y) = 0$

$Cov(X, Y) \not\Rightarrow X \perp\!\!\!\perp Y$

distrib. comună a (X,Y)

$X \backslash Y$	-1	0	1	
-1	0	$1/4$	0	$\to 1/4$
0	$1/4$	0	$1/4$	$\to 1/2$
1	0	$1/4$	0	$\to 1/4$

$P(X=-1, Y=0) = 1/4$

Vreau doar distrib. lui X din cuplul (X,Y)

Folosesc marginalizarea:

$$P(X=x) = \sum_{y} P(X=x, Y=y)$$

$$P(X=-1) = P(X=-1, Y=-1) + P(X=-1, Y=0) + P(X=-1, Y=0) = \frac{1}{4}$$

1) Sunt X, Y indep?

2) Calculați $Cov(X,Y)$

$$X \begin{pmatrix} -1 & 0 & 1 \\ 1/4 & 1/2 & 1/4 \end{pmatrix}$$ Analog pt Y

Știu că $X \perp\!\!\!\perp Y \Leftrightarrow P(X=x, Y=y) = P(X=x) \cdot P(Y=y)$ $\forall x,y$

$$P(X,Y)=(0,0) = \underbrace{0 \Rightarrow P(X=0)\cdot(P(Y=0)) = \frac{1}{2}\cdot\frac{1}{2} = \frac{1}{4}}_{\neq}$$

deci $X \not\perp\!\!\!\perp Y$

$$Cov(X,Y) = E(X\cdot Y) - E(X)\cdot E(Y)$$

$$\left.\begin{aligned} E(X\cdot Y) &= \sum_{(x,y)} x\cdot y \cdot P(X=x, Y=y) = \boxed{0} \\ E(X) &= 0, \quad E(Y) = 0 \end{aligned}\right\} \Rightarrow Cov(X,Y) = 0.$$

$= 42$

Covarianța:

$$Cor(x,y) = \frac{Cov(x,y)}{\sigma_x \cdot \sigma_y}$$

Notăm $z = x - y$.

$$z : \begin{pmatrix} -- & 0 & -- \end{pmatrix}$$

$$P(x-y=0) = P(z=0) = P(x=y) =$$

$$= P((x,y) \in \{(-1,-1), (0,0), (1,1)\}$$

$$= P(x=-1, y=-1) + P(x=0, y=0) + P(x=1, y=1)$$

$$= 0$$

dacă sunt indep, facem doar prod. probabilit.

$$P(f(x) - g(y) = ceva) = \_\_ P((x,y) \in \{ - \}$$

$f: \mathbb{R} \to \mathbb{R}$ $\quad f(t) = \frac{c}{1+t^2}$

a) $c = ?$ a.î. f să fie densitate

b) $\exists x$, cu x având densit. f

c) $P(x \in [-1, 1])$

<u>Sol.</u>: Impun ca $\int_{\mathbb{R}} f(t)\,dt = 1$, deci $1 = \int_{\mathbb{R}} \frac{c}{1+t^2}\,dt =$

$$= c \cdot \int_{\mathbb{R}} \frac{1}{1+t^2}\,dt = c \cdot \operatorname{arctg} t \Big| = c \cdot \left(\frac{\pi}{2} - \left(-\frac{\pi}{2}\right)\right) = c \cdot \pi$$

de unde $c = \frac{1}{\pi}$. În concluzie $f(t) = \frac{1}{\pi \cdot (1+t^2)}$

Repartiția Cauchy: $f_X(x) = \frac{1}{\pi} \cdot \frac{1}{1+x^2}$, $\forall x \in \mathbb{R}$

b) $X \sim f$

$F_X(x) = P(X \le x) = \int_{a}^{b} f(t)\,dt \rightsquigarrow \int_{-\infty}^{x} \frac{1}{\dots}$

$\underline{\int_{-\infty}^{x} f_X\,dx}$ $\underline{NU}$

! $F_X(x) = \frac{1}{\pi} \cdot \operatorname{arctg}(x) - \frac{1}{2}$

a) $P(X \in [-1,1]) = P(-1 \le X \le 1) = P(X \le 1) - P(X \le -1) =$

$= F_X(1) - F_X(-1) = \left(\frac{1}{\pi} \cdot \operatorname{arctg} 1 - \frac{1}{2}\right) - \left(\frac{1}{\pi} \cdot \operatorname{arctg}(-1) - \frac{1}{2}\right)$

Altfel $P(X \in [-1,1]) = \int_{-1}^{1} f_X(t)\,dt = \int_{-1}^{1} \frac{1}{\pi(1+t^2)}\,dt$

$\underline{\text{Obs}}$ trb. ca $f(t) \geq 0$ pe orice t

$\underline{\text{Exercițiu.}}$

$X = (X_1, X_2)$ v.a 2-dim.

Y_1, X_2 indep dc $E(X \cdot Y) = E(X)E(Y)$

$f_X(x_1, x_2) = \begin{cases} c(x_1^2 + x_2^2), & 0 \le x_1, x_2 \le 1 \\ 0, & \text{altfel} \end{cases}$

a) $c = ?$ a.î. f densit. ‖ b) $X_1 \perp\!\!\!\perp X_2$? $U\,O \in (X_1, X_2)$

$\underline{\text{Sol}}$ a) $1 = \int_{\mathbb{R}} f_X(x_1, x_2)\,dx_1\,dx_2 = \text{— —} = c$ ✓

b) vreau f_{X_1} și f_{X_2} (densit. marginale)

Pentru $x \notin [0,1] \to f_{X_1}(x_1) = 0$

în rap. cu cealaltă var.

$x \in [0,1) \to f_{X_1}(x_1) = \int_{\mathbb{R}} f_X(x_1, x_2)\,\underline{\underline{dx_2}}$