Def. Fie X o v.a. discretă. Se numește media lui X, valoarea

$$\mathbb{E}[X] = \sum_x x f(x) = \sum_x x \mathbb{P}(X=x)$$

ori de câte ori $\sum_x |x| \cdot f(x) < \infty$

Dacă $\sum |x| \cdot f(x) = \infty$ atunci spunem că X nu are medie

V. Saint Peter.

Curs 8 : 21 noiembrie

Media și momentele de ordin superior

Def. Fie $(\Omega, \mathcal{F}, \mathbb{P})$ c.p. și $X : \Omega \to \mathbb{R}$, v.a. discretă definim media a X

$$E[X] = \sum_x x \mathbb{P}(X=x) = \sum_x x f(x)$$

ori de câte ori $\sum |x| f(x) < \infty$. În cazul în care seria este ∞ atunci spunem că v.a X nu are medie.

Exp. Aruncăm 1 zar

$X \in \{1, 2, 3, 4, 5, 6\}$

$\mathbb{P}(X=x) = \frac{1}{6}$

$\frac{1}{3}$ $\frac{1}{2}$ $\frac{2}{3}$

$2+3+4+5+6$

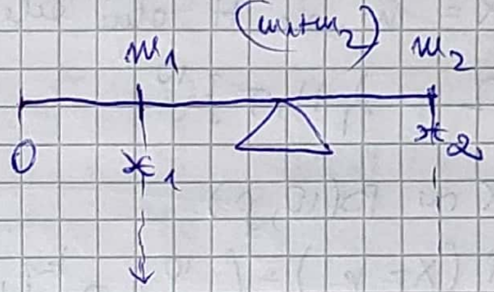
$E[X] = \sum x \mathbb{P}(X=x) = 1 \cdot \frac{1}{6} + 2 \cdot \frac{1}{6} + \dots + 6 \cdot \frac{1}{6}$

$= 3,5$

Exp: $X \sim \begin{pmatrix} x_1 & x_2 & \dots & x_n \\ p_1 & p_2 & \dots & p_n \end{pmatrix}$ $E[X] = x_1 p_1 + x_2 p_2 + \dots + x_n p_n$

$X \sim \begin{pmatrix} -1 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & \frac{1}{4} \end{pmatrix} \Rightarrow \mathbb{E}[X] = -\frac{1}{2} + \frac{1}{4}$

Interpretare fizică:



$xM = x_1 m_1 + x_2 m_2$

$\bar{x} = x_1 \frac{m_1}{M} + x_2 \frac{m_2}{M}$

Proprietăți ale mediei

1) Dacă X este constantă, i.e. $X = c$, atunci $E[X] = c$.

$\boxed{\text{pozitivitate}}$ 2) Dacă $X \geq 0$, atunci $\mathbb{E}[X] \geq 0$.

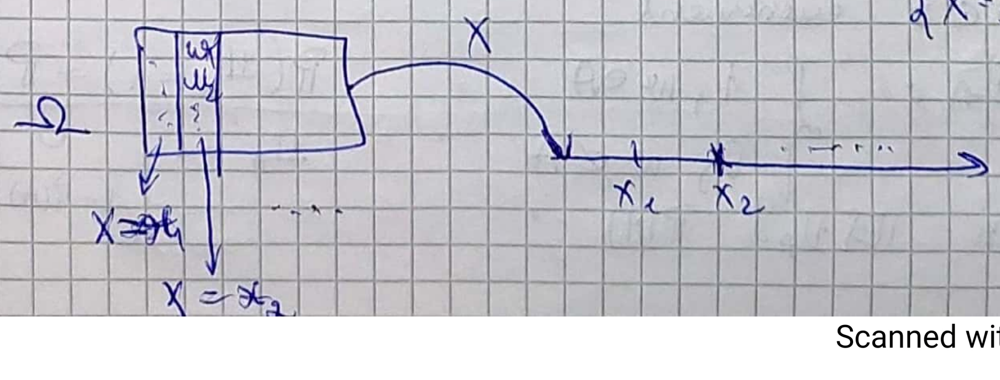
3) Dacă $X \geq Y$, atunci $\mathbb{E}[X] \geq \mathbb{E}[Y] \rightarrow$ prop. de $\boxed{\text{monotonie}}$

$(X(\omega) \geq Y(\omega), \forall\, \omega \in \Omega)$

4) $\boxed{\text{Liniaritate}}$ Dacă X și Y 2 var. aleatoare discrete și $a, b \in \mathbb{R}$ at. $\boxed{\mathbb{E}[aX + bY] = a\mathbb{E}[X] + b\,\mathbb{E}[Y]}$

Dem. $\mathbb{E}[X+Y] = \mathbb{E}[X] + \mathbb{E}[Y]$

$\{X = x\} = X^{-1}(x)$



$$\mathbb{E}[X] = \sum_{x} x\,\mathbb{P}(X=x) \qquad \{X=x\} = \{\omega \in \Omega \mid X(\omega)=x\}$$

$$= \sum_{\omega} X(\omega)\cdot\mathbb{P}(\{\omega\}) \qquad \mathbb{P}(X=x) = \sum_{\omega} \mathbb{P}(\{\omega\})$$

Ex. Aruncăm o monedă de 10 ori:

X = nr. de H din cele 10 aruncări

$\Omega = \{H, T\}^{10}$

$X \sim B(10, p)$

$\mathbb{P}(X=k) = \binom{10}{k} p^k (1-p)^{10-k}$

$\{X=k\} = \{(\omega_1, \omega_2, \dots \omega_{10}) \mid \omega_i \in \{H,T\}$ și exact k sunt H

$$\underbrace{HH\dots H}_{k}\underbrace{TT\dots T}_{10-k}$$

$$E[X+Y] = \sum_{\omega} (X+Y)(\omega)\,\mathbb{P}(\{\omega\}) =$$

$$= \sum_{\omega} (X(\omega)+Y(\omega))\,\mathbb{P}(\{\omega\}) =$$

$$= \underbrace{\sum_{\omega} X(\omega)\cdot\mathbb{P}(\{\omega\})}_{\text{media lui } X} + \underbrace{\sum_{\omega} Y(\omega)\cdot\mathbb{P}(\{\omega\})}_{\text{media lui } Y}$$

---

5) <u>Legătura dintre medie și probabilitate</u>

Fie $A \in \mathcal{F}$ eveniment

$$\mathbb{1}_A = \begin{cases} 1, & \omega \in A \\ 0, & \omega \notin A \end{cases} \qquad \mathbb{P}(\mathbb{1}_A = 1) = \mathbb{P}(A),$$

$$\mathbb{1}_A \sim \begin{pmatrix} 0 & 1 \\ 1-\mathbb{P}(A) & \mathbb{P}(A) \end{pmatrix}$$

~~[illegible]~~ $\mathbb{E}[\mathbb{1}_A] = \mathbb{P}(A)$

(6) Fie X o v.a (discretă) și $g: \mathbb{R} \to \mathbb{R}$

$Y = g(X)$. Atunci:

$$\boxed{\mathbb{E}[g(X)] = \sum_x g(x) \cdot \mathbb{P}(X = x)}$$

$$\mathbb{E}[Y] = \sum_y y \mathbb{P}(Y = y)$$

$$\begin{tikzcd} \Omega \arrow[r, "X"] \arrow[rr, bend right, "g \circ X"'] & \mathbb{R} \arrow[r, "g"] & \mathbb{R} \end{tikzcd}$$

$$\mathbb{P}(Y = y) = \mathbb{P}(g(X) = y) = \mathbb{P}(X \in g^{-1}(y)) =$$

$$\boxed{g^{-1}(y) = \{x \mid g(x) = y\}} = \sum_{x \in g^{-1}(y)} \mathbb{P}(X = x)$$

$$\mathbb{E}[g(X)] = \mathbb{E}[Y] = \sum_y y \sum_{x \in g^{-1}(y)} \mathbb{P}(X = x)$$

$$= \sum_x g(x) \mathbb{P}(X = x)$$

Exp: $X \sim \begin{pmatrix} -2 & -1 & 1 & 3 \\ \frac{1}{4} & \frac{1}{8} & \frac{1}{2} & \frac{1}{8} \end{pmatrix}$; $Y = X^2$

Metoda 1: $Y \in \{1, 4, 9\}$

$$\mathbb{P}(Y = 1) = \mathbb{P}(X = -1) + \mathbb{P}(X = 1)$$

$$= \frac{1}{8} + \frac{1}{2} = \frac{5}{8}$$

$$Y \sim \begin{pmatrix} 1 & 4 & 9 \\ \frac{5}{8} & \frac{1}{4} = \frac{2}{8} & \frac{1}{8} \end{pmatrix}$$

$$\mathbb{E}[X^2] = \frac{5}{8} + 4 \cdot \frac{2}{8} + \frac{9}{8} = \frac{22}{8}$$

Metoda 2:

$g(x) = x^2$

$$\mathbb{E}[X^2] = (-2)^2 \cdot \frac{1}{4} + (-1)^2 \cdot \frac{1}{8} + 1^2 \cdot \frac{1}{2} + 3^2 \cdot \frac{1}{8} =$$

$$= \frac{22}{8}$$

Scanned with CamScanner

⑦ Fie X și Y: 2 v.a. independente

$$\boxed{\mathbb{E}[X\cdot Y] = \mathbb{E}[X]\cdot\mathbb{E}[Y]}$$

Dacă g și h sunt 2 funcții atunci $g(X)$ și $h(Y)$ sunt indep. : $\mathbb{E}[g(X)\cdot h(Y)] = \mathbb{E}[g(X)]\cdot\mathbb{E}[h(Y)]$

$$Y \sim \begin{pmatrix} 0 & 2 \\ \frac{1}{2} & \frac{1}{2} \end{pmatrix} \qquad X \perp Y$$

$$\mathbb{E}[X^7\cdot Y^9] = \mathbb{E}[X^7]\cdot\mathbb{E}[Y^9]$$

$\boxed{\text{Obs.}}$ În general, $\mathbb{E}[XY] \neq \mathbb{E}[X]\cdot\mathbb{E}[Y]$

Def. Fie X o v.a (discretă). ~~Num~~ Numim moment de ordin k $(k \geq 1)$ $\mathbb{E}[X^k]$

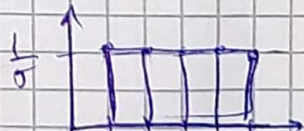
Se numește moment de ordin k centrat în $a \in \mathbb{R}$, $\mathbb{E}[(X-a)^k]$ și moment centrat de ordin k, $\mathbb{E}[(X-\mathbb{E}[X])^k]$

Def: Varianța sau dispersia v.a. X este momentul centrat de ordin 2 și se notează cu

$$\boxed{Var(X) = \mathbb{E}[(X-\mathbb{E}[X])^2]}$$

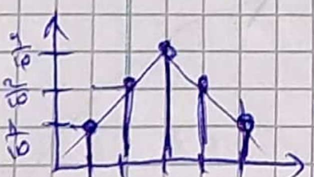
Obs. ~~Arată~~ gradul de împrăștiere ~~a~~ valorilor față de medie.

Exp: $X_1 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} & \frac{1}{5} \end{pmatrix}$



$$V_1 = \mathbb{E}[(X_1-3)^2] = \frac{(1-3)^2}{5} + \frac{(2-3)^2}{5} + \cancel{\frac{(3-3)^2}{5}} + \frac{(4-3)^2}{5} + \frac{(5-3)^2}{5} = 2$$

$X_2 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{10} & \frac{2}{10} & \frac{4}{10} & \frac{2}{10} & \frac{1}{10} \end{pmatrix}$



$$V_2 = \mathbb{E}[(X_2-3)^2] = \frac{(1-3)^2}{10} + \frac{2\cdot(2-3)^2}{10} + \cancel{\frac{4(3-3)^2}{10}} + \frac{2\cdot(4-3)^2}{10} + \frac{(5-3)^2}{10} = \frac{12}{10}$$

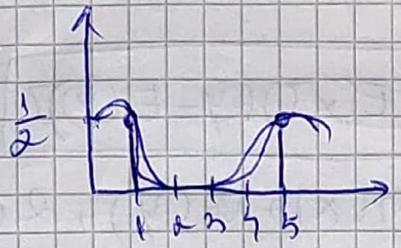
4+2+2+4

$X_3 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ \frac{1}{2} & 0 & 0 & 0 & \frac{1}{2} \end{pmatrix}$

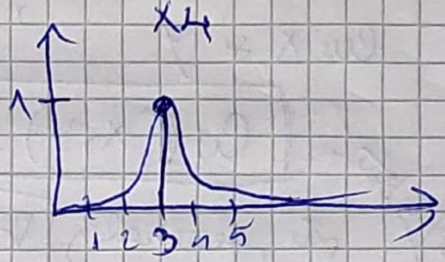
$$X_4 \sim \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix}$$

$$E[X_1] = E[X_2] = E[X_4] = E[X_4] = 3$$

X_3

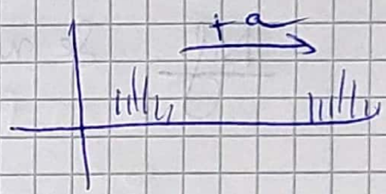


X_4



$$Vr(X_3) = \frac{(1-3)^2}{2} + \frac{(5-3)^2}{2} = 4 \qquad Vr(X_4) = \frac{(3-3)^2}{2} = 0$$

Proprietăți ale variației



1) Dacă X este constantă $\Rightarrow Vr(X) = 0$

2) $Vr(X) \geq 0$!

3) Dacă X v.a și $a \in \mathbb{R}$ atunci $\boxed{Vr(a+X) = Vr(X)}$
la translație

4) Dacă X v.a și $b \in \mathbb{R}^*$ atunci $\boxed{Vr(bX) = b^2 Vr(X)}$
o scalăm

$$E[(bx - E(bx))^2]$$
$$E[b^2(x - E(x))^2]$$
$$b^2 E[(x - E(x))^2]$$

$$\boxed{Vr(a+bx) = b^2 Vr(X)} \quad \forall a, b \in \mathbb{R}$$

---

5) $\boxed{Vr(X) = E[x^2] - E[x]^2}$

$$E[(x - E[x])^2] = E[x^2 - 2x E[x] + E[x]^2]$$
$$= E[x^2] - 2E[x E[x]] + E[x]^2$$
$$= E[x^2] - 2E[x]^2 + E[x]^2$$
$$= E[x^2] - E[x]^2$$

6) X și Y indep.

$$\boxed{Vr(X+Y) = Vr(X) + Vr(Y)}$$

7) Fie X și Y 2 v.a. se numește **covarianța** lui X și Y

$$\boxed{Cov(X,Y) = \mathbb{E}[(X-E[X])(Y-E(Y))]}$$

utilitate

În general, $\boxed{Var(X+Y) = Vr(X) + Vr(Y) + 2\,Cov(X,Y)}$

→ creștere sau descreștere simultană, dar doar din persp. liniară

<u>Def</u>. Se numește abatere standard

$$\boxed{SD(X) = \sqrt{Vr(X)}}$$

~~X este~~ <u>Notații</u>: $\sigma^2 \rightarrow$ varianța

$\sigma \rightarrow$ abatere.

<u>Exemple de calcul al mediei și varianței</u>
<u>pt. v.a. discrete</u>

1) $X \sim B(p)$; $X \sim \begin{pmatrix} 0 & 1 \\ 1-p & p \end{pmatrix}$

$\mathbb{E}[X] = p$

$Vr(X) = \mathbb{E}[X^2] - E[X]^2$

$= p - p^2 = p(1-p)$

2) $X \sim B(n,p)$; $\mathbb{P}(X=k) = \binom{n}{k} p^k (1-p)^{n-k}$

$\mathbb{E}[X] = \sum_{k=0}^{n} k\,\mathbb{P}(X=k)$

$$= \sum_{k=0}^{n} k \cdot \binom{n}{k} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \cancel{k} \cdot \frac{n!}{(n-k)! \cdot \cancel{k!}_{(k-1)!}} p^k (1-p)^{n-k}$$

$$= \sum_{k=1}^{n} \frac{n!}{(k-1)!\,(n-k)!} p^k (1-p)^{n-k}$$

$$\cancel{= \sum_{k=1}^{n} \frac{n!}{(n-k-1)(k-1)!} \frac{1}{(n-k)} p^k (1-p)^{n-k}}$$

$$= n \sum_{k=1}^{n} \frac{(n-1)!}{(n-k-1+1)!\,(k-1)} p^k (1-p)^{n-k}$$

$$= np \underbrace{\sum_{k=1}^{n} \binom{n-1}{k-1} p^{k-1} (1-p)^{n-k}}_{=1} = \boxed{np}$$

$$\sum_{k=1}^{n} \binom{n-1}{\underbrace{k-1}_{\ell}} p^{k-1} (1-p)^{n-k}$$

$$\Rightarrow \sum_{\ell=0}^{n-1} \binom{n-1}{\ell} p^\ell (1-p)^{n-1-\ell}$$

$$\Rightarrow \sum_{\ell=0}^{m} \binom{m}{\ell} p^\ell (1-p)^{m-\ell} = (p+1-p)^m = 1$$

$X = X_1 + X_2 + \dots + X_n$ indep.

$\mathbb{E}[X] = \mathbb{E}[\;\;] = \mathbb{E}[X_1] + \mathbb{E}[X_2] + \dots + \mathbb{E}[X_n] = np$

$$\mathrm{Vr}(X) = \mathrm{Vr}(X_1 + \dots + X_n)$$

$$= \sum_{i=1}^{n} \mathrm{Vr}(X_i) = \boxed{np(1-p)}$$

3) Hipergeometrica : fără întoarcere

$HG(n, N, M)$

$$P(X=k) = \frac{\binom{M}{k}\binom{N-M}{n-k}}{\binom{N}{n}}$$

$x_j = \{0, 1\}$

x_j la extragerea j avem bilă neagră $x_j = 1$ $P(x_j = 1) = \frac{M}{N}$

albă $x_j = 0$

$X = X_1 + X_2 + \dots + X_n$

$$E[X] = E[X_1] + E[X_2] + \dots + E[X_n] = n\frac{M}{N}$$

4) $X \sim Poiss(\lambda)$

$$P(X=k) = e^{-\lambda}\frac{\lambda^k}{k!}$$

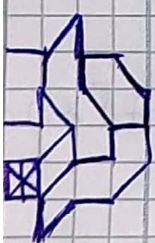
$$E[X] = \sum_{k=0}^{\infty} k \cdot P(X=k) = \sum_{k=0}^{\infty} k \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!}$$

$$= \sum_{k=1}^{\infty} \cancel{k} \cdot e^{-\lambda} \frac{\lambda^k}{\cancel{k!}\,(k-1)!} = \sum_{k=1}^{\infty} e^{-\lambda} \cdot \frac{\lambda^k}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=1}^{\infty} \frac{\lambda^{k-1}}{(k-1)!} = \lambda e^{-\lambda} \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}$$

$\lambda \cdot e^{-\lambda} \cdot e^{\lambda}$

$$= \boxed{\lambda}$$



$$Var(X) = E[X^2] - E[X]^2$$

$$E[X^2] = \sum_{k=0}^{\infty} k^2 P(X=k)$$

$$= \sum_{k=0}^{\infty} k^2 \cdot e^{-\lambda} \cdot \frac{\lambda^k}{k!} = \sum_{k\geq 1}^{\infty} k^{\cancel{2}} \cdot e^{-\lambda} \frac{\lambda^k}{\cancel{k!}\,(k-1)!}$$

$$= \sum_{k=1}^{\infty} k \cdot e^{-\lambda} \frac{\lambda^k}{(k-1)!} = \lambda e^{-\lambda} \sum_{k=1}^{\infty} k \frac{\lambda^{k-1}}{(k-1)!}$$

$$= \lambda e^{-\lambda} \sum_{k=0}^{\infty} (k+1) \cdot \frac{\lambda^k}{k!}$$

$$= \lambda e^{-\lambda} \left\{ \underbrace{\sum_{k=0}^{\infty} k \frac{\lambda^k}{k!}}_{\lambda e^{\lambda}} + \underbrace{\sum_{k=0}^{\infty} \frac{\lambda^k}{k!}}_{e^{\lambda}} \right\} = \lambda^2 + \boxed{\lambda}$$

$$\underline{Vr(X) = \lambda}$$

Obs.: $\mathbb{E}[X] = Vr(X)$

5) $X \sim Geom(p)$

$X \in \{1, 2, \dots\}$

$\mathbb{P}(X = k) = (1-p)^{k-1} p$

$$\mathbb{E}[X] = \sum_{k=1}^{\infty} k(1-p)^{k-1} \cdot p = \sum_{k=1}^{\infty} k q^{k-1} p = p \sum_{k=1}^{\infty} \underbrace{k q^{k-1}}_{(q^k)'}$$

$q = 1-p$

$$= p \sum_{k=1}^{\infty} (q^k)' = p \left(\sum_{k=1}^{\infty} q^k \right)'$$

$$= p \left(\sum_{k=0}^{\infty} q^k - 1 \right)' = p \left(\frac{1}{1-q} - 1 \right)'$$

$$= \frac{p}{(\underbrace{1-q}_{1-p})^2} = \frac{1}{p}$$

Tema

$Vr(X) = ?$ $\frac{1-p}{p^2}$ $\mathbb{E}[X(X-1)]$

Variabile continue

Def.: Fie $(\Omega, \mathcal{F}, \mathbb{P})$ un cp și $X: \Omega \to \mathbb{R}$ o.v.a.
v.a. X este continuă (absolut continuă) dacă

$(\exists)$ o funcție $f: \mathbb{R} \to \mathbb{R}_+$ cu prop.

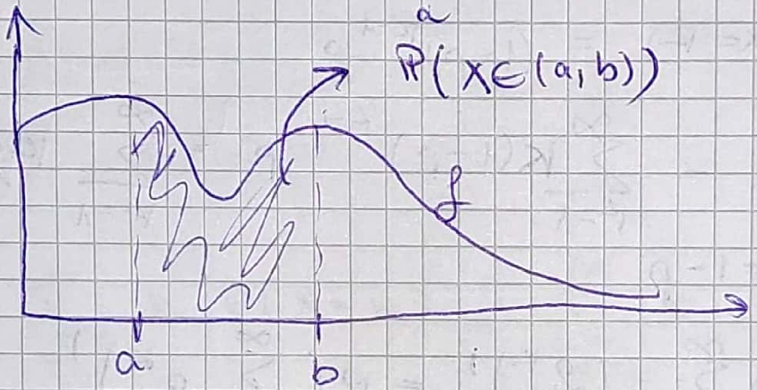
$$\mathbb{P}(X \in A) = \int_A f(x)\,dx, \ (\forall) A \subseteq \mathbb{R}$$

$\hookrightarrow$ interval

Obs.: Dacă $A = (a, b)$

$$\mathbb{P}(a < X < b) = \int_a^b f(x)\,dx$$

$\mathbb{P}(X \in (a,b))$



Obs. În def. mai sus f s.n. densitate de repartiție.

(P) Dacă f este densitate de repartiție atunci

1) $f \geq 0$

2) $\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = 1$

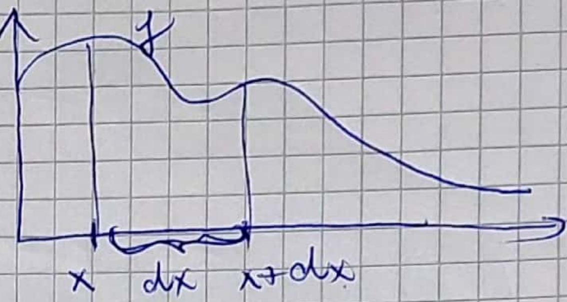
$$\mathbb{P}(X \in \mathbb{R}) = \mathbb{P}(\Omega) = 1$$

$\boxed{\text{Obs}}$: $\mathbb{P}(X = a) = \int_a^a f(x)\,dx = 0$

$A = \{a\}$

prob. ca x să ia o val const.

$\mathbb{P}(a<X<b) = \mathbb{P}(a \le X < b) = \mathbb{P}(a \le X \le b) =$
$= \mathbb{P}(a < X \le b)$



Interpretare:

$\mathbb{P}(X \in \{x, x+dx)\})$

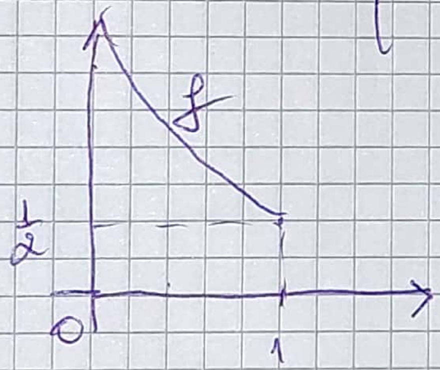
$= \int_x^{x+dx} f(t)\,dt \cong f(x)\,(x+dx-x) = f(x)\,dx$

$f(x) \simeq \dfrac{\mathbb{P}(X \in (x, x+dx))}{dx} = \dfrac{\text{probabilitatea}}{\text{unitate de lungime}}$

$\Downarrow$

densitate

Exp.: $f(x) = \begin{cases} \dfrac{1}{2\sqrt{x}}, & 0 < x \le 1 \\ 0, & \text{altfel} \end{cases}$



$f(x) \ge 0$ ✓

$\int_{\mathbb{R}} f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx =$

$= \int_0^1 \dfrac{1}{2\sqrt{x}}\,dx = 1$

disc. cont.

$\sum \to \int$

$f(x) = \mathbb{P}(X = x) \longrightarrow 0$

~~$\mathbb{P}(X \in (x, x+dx)) = f(x)$~~

$\dfrac{\mathbb{P}(X \in (x, x+dx))}{dx} \longrightarrow f(x)$

$\mathbb{P}(X \in A) = \sum_{x \in A} \mathbb{P}(X = x) \longrightarrow \int_A f(x)\,dx$