Curs 19.12.2022

a) Cazul discret

X, Y v.a discrete, $X \in \{x_1, \dots x_m\}$
$Y \in \{y_1, \dots y_n\}$

$$f_{X,Y}(x,y) = \mathbb{P}(X=x, Y=y) \leftarrow \text{rep. comună}$$

$$f_X(x) = \mathbb{P}(X=x) = \sum_y f_{X,Y}(x,y)$$

rep. marginală

$$f_Y(y) = \mathbb{P}(Y=y) = \sum_x f_{X,Y}(x,y)$$

rep. condiționată

$$f_{X|Y}(x|y) = \mathbb{P}(X=x | Y=y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$f_{Y|X}(y|x) = \mathbb{P}(Y=y | X=x) = \frac{f_{X,Y}(x,y)}{f_X(x)}$$

Media condiționată

Dacă X v.a discretă și $A \in \mathcal{F}$, $\mathbb{P}(A) > 0$ atunci am văzut că $f_{X|A}(x) = \mathbb{P}(X=x | A) \leftarrow$ este o probabilitate

Media condiționată a lui X la A:

$$\boxed{E[X|A] = \sum_x x\, P(X=x|A)}$$
$$= \sum_x x\, f_{X|A}(x)$$

Dacă g este o funcție atunci $g(X)$ este o v.a. discretă

și $$\boxed{E[g(X)|A] = \sum_x g(x)\, f_{X|A}(x)}$$

Dacă $A=\{Y=y\}$ atunci

$$\boxed{E[X|Y=y] = \sum_x x\, f_{X|Y}(x|y)}$$

media condiționată a lui X la $Y=y$

Exp:

X\Y	-1	0	2	Σ
1	1/18	3/18	2/18	6/18
2	2/18	0	3/18	5/18
3	0	4/18	3/18	7/18
Σ	3/18	7/18	8/18	1

$$E[X] = \frac{6+10+21}{18}$$

$$X \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{6}{18} & \frac{5}{18} & \frac{7}{18} \end{pmatrix}$$

$$Y \sim \begin{pmatrix} -1 & 0 & 2 \\ \frac{3}{18} & \frac{7}{18} & \frac{8}{18} \end{pmatrix}$$

$$X|Y=0 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{3/18}{7/18} & 0 & \frac{4/18}{7/18} \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ \frac{3}{7} & 0 & \frac{4}{7} \end{pmatrix}$$

$$E[X|Y=0] = 1 \times \frac{3}{7} + 2 \times 0 + 3 \times \frac{4}{7} = 15/7$$

$$X \mid Y=1 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{1}{3} & \frac{2}{3} & 0 \end{pmatrix}$$

$$\mathbb{E}[X \mid Y=1] = 1 \times \frac{1}{3} + 2 \times \frac{2}{3} + 3 \times 0 = 5/3$$

$$X \mid Y=2 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{2}{8} & \frac{3}{8} & \frac{3}{8} \end{pmatrix} \Rightarrow \mathbb{E}[X \mid Y=2] = \frac{2}{8} + \frac{6}{8} + \frac{9}{8} = \frac{17}{8}$$

Puteți calcula voi pt $Y \mid X=1 \ldots$

(P) Dacă X și Y sunt v.a discrete atunci

$$\boxed{\mathbb{E}[X] = \sum_y \mathbb{E}[X \mid Y=y]\, \mathbb{P}(Y=y)}$$

$$\mathbb{E}[X \mid Y=y] = \sum_x x f_{X|Y}(x|y), \quad f_{X|Y}(x|y) = \frac{f_{X,Y}(x,y)}{f_Y(y)}$$

$$\sum_y \sum_x x f_{X|Y}(x|y) f_Y(y) = \sum_y \sum_x x f_{X,Y}(x,y)$$

$$= \sum_x x \underbrace{\sum_y f_{X,Y}(x,y)}_{f_X(x)} = \mathbb{E}[X]$$

Def: Fie X și Y două v.a discrete. Se numește media condiționată a lui X la Y, și se notează $\mathbb{E}[X \mid Y]$, v.a de forma $h(Y)$ pt care

$$h(y) = \mathbb{E}[X \mid Y=y], \ \forall y$$

Exp: (cont.)

Am văzut că $\mathbb{E}[X|Y=-1] = 5/3$

$\mathbb{E}[X|Y=0] = 15/7$

$\mathbb{E}[X|Y=2] = 17/8$

$\mathbb{E}[X|Y] = h(Y)$ ce valori ia această v.a.?

$h(y) = \mathbb{E}[X|Y=y]$

$$\underline{\mathbb{E}[X|Y]} \sim \begin{pmatrix} 5/3 & \frac{15}{7} & \frac{17}{8} \\ \mathbb{P}(Y=-1) & \mathbb{P}(Y=0) & \mathbb{P}(Y=2) \end{pmatrix}$$

$$\sim \begin{pmatrix} 5/3 & 15/7 & 17/8 \\ 3/18 & 7/18 & 8/18 \end{pmatrix}$$

$$\mathbb{E}[\mathbb{E}[X|Y]] = \frac{5}{\cancel{3}} \times \cancel{3}/18 + \frac{15}{\cancel{7}} \times \frac{\cancel{7}}{18} + \frac{17}{\cancel{8}} \times \frac{\cancel{8}}{18}$$

$$= \frac{5+15+17}{18} = \frac{37}{18} = \mathbb{E}[X]$$

(P) Media mediei condiționate este

$$\boxed{\mathbb{E}[\mathbb{E}[X|Y]] = \mathbb{E}[X]}$$

$$\mathbb{E}[X|Y] \sim \begin{pmatrix} \mathbb{E}[X|Y=y_1] & \dots & \mathbb{E}[X|Y=y_n] \\ \mathbb{P}(Y=y_1) & \dots & \mathbb{P}(Y=y_n) \end{pmatrix}$$

<u>Ex</u>: Calculați $\mathbb{E}[Y|X]$

<u>Obs</u>: $$\mathrm{Var}(X|A) = \mathbb{E}\left[\left(X-\mathbb{E}[X|A]\right)^2 \middle| A\right]$$
$$= \mathbb{E}[X^2|A] - \mathbb{E}[X|A]^2$$

$A=\{Y=y\}$

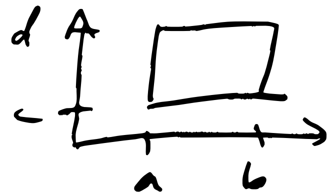
$$\mathrm{Var}(X|Y=y) = \mathbb{E}[X^2|Y=y] - \mathbb{E}[X|Y=y]^2$$

b) <u>Cazul v.a continue</u> : rep. comună, rep. marginală, rep. condiționată

<u>Def</u>: Fie $(\Omega, \mathcal{F}, \mathbb{P})$ c.p și X, Y două v.a cont. Spunem că vectorul (X,Y) formează o pereche de v.a cont. dacă există $f_{(X,Y)}(x,y) \geq 0$ cu prop:

$$\mathbb{P}((X,Y)\in A) = \iint_A f_{(X,Y)}(x,y)\,dx\,dy, \ \forall A \subseteq \mathbb{R}^2$$

A dreptunghi



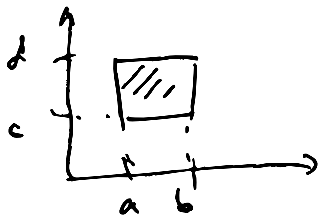
$(a,b)\times(c,d)$

Funcția $f_{(X,Y)}(x,y) : \mathbb{R}^2 \to \mathbb{R}$ se numește densitatea comună a (X,Y)

Dacă $A = [a,b] \times [c,d]$

$$P\big((X,Y) \in [a,b] \times [c,d]\big) = P(a \le X \le b,\ c \le Y \le d)$$

$$= \iint_A f_{(X,Y)}(x,y)\,dx\,dy$$



$$= \int_a^b \int_c^d f_{(X,Y)}(x,y)\,dy\,dx$$

Dacă $A = \mathbb{R}^2$ atunci

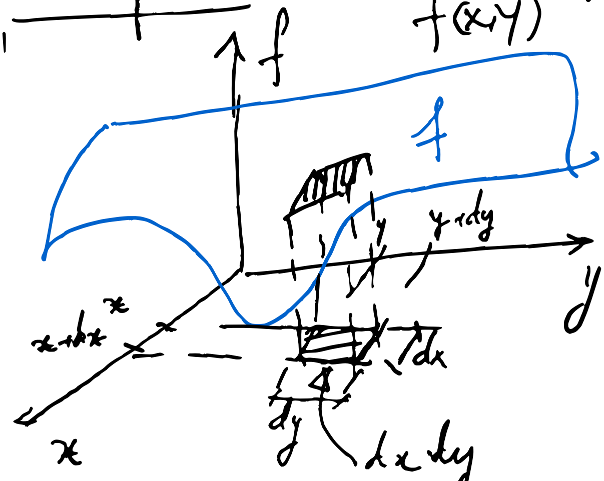
$$\underbrace{P\big((X,Y) \in \mathbb{R}^2\big)}_{=1} = \iint_{\mathbb{R}^2} f_{(X,Y)}(x,y)\,dx\,dy$$

$f_{(X,Y)}$ este densitate $\iff$ a) $f_{(X,Y)} \ge 0$

b) $\displaystyle\iint_{\mathbb{R}^2} f_{(X,Y)}(x,y)\,dx\,dy = 1$

<u>Obs</u>: $\displaystyle\iint_{\mathbb{R}^2} f\,dx\,dy = \int_{-\infty}^{+\infty}\int_{-\infty}^{+\infty} f(x,y)\,dx\,dy$

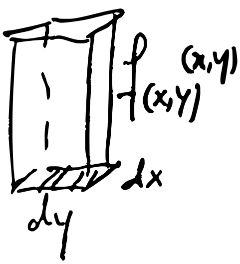
Interpretare: $f_{(X,Y)}(x,y)$



$$\mathbb{P}(X\in(x,x+dx),\ Y\in(y,y+dy)) = \int_x^{x+dx}\int_y^{y+dy} f_{(X,Y)}(u,v)\,dv\,du$$

$dx, dy \to 0$

$$\simeq f_{(X,Y)}(x,y)\,dx\,dy$$



$$f_{(X,Y)}(x,y) \triangleq \frac{\mathbb{P}(X\in(x,x+dx),\ Y\in(y,y+dy))}{dx\,dy}$$

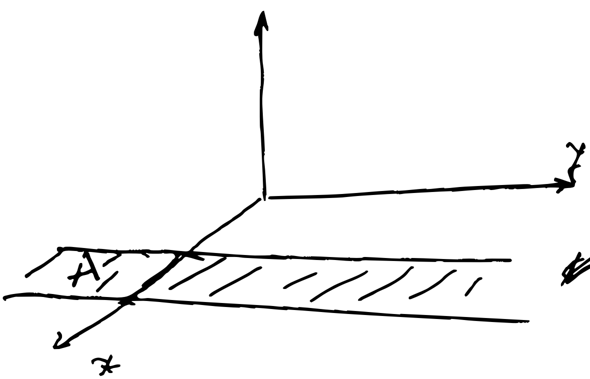
$$\simeq \frac{\text{probabilitate}}{\text{unitatea de arie}}$$

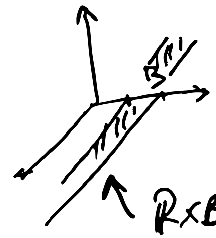
Obs: Dacă știm $f_{(X,Y)}(x,y)$ atunci putem calcula orice probabilitate de tipul $\mathbb{P}(X\in A, Y\in B)$

$f_{(X,Y)}(x,y)$ – conține toată informația despre X și Y

Vrem să calculăm

$$\mathbb{P}(X\in A) = \mathbb{P}(X\in A, Y\in \mathbb{R}) = \int_A\int_{\mathbb{R}} f_{(X,Y)}(x,y)\,dy\,dx$$





$A\times\mathbb{R}$

$\mathbb{R}\times B$

$$\mathbb{P}(Y\in B) = \mathbb{P}(X\in\mathbb{R}, Y\in B) = \int_{\mathbb{R}}\int_B f_{(X,Y)}(x,y)\,dy\,dx$$

Pp că X și Y v.a continue cu densitățile f_X și respectiv f_Y

$$\mathbb{P}(X\in A) = \int_A f_X(x)\,dx$$

Avem

$$\int_A \underbrace{f_X}(x)\,dx = \int_A \overbrace{\int_{\mathbb{R}} f_{(X,Y)}(x,y)\,dy}\,dx$$

$$\Rightarrow \boxed{f_X(x) = \int_{\mathbb{R}} f_{(X,Y)}(x,y)\,dy}$$

densitatea marginală a lui X

Similar,

$$\boxed{f_Y(y) = \int_{\mathbb{R}} f_{(X,Y)}(x,y)\,dx}$$ - densitatea marginală a lui Y

Cazul discret	Cazul cont.
$f_{(X,Y)}(x,y) = \mathbb{P}(X=x, Y=y)$	$f_{(X,Y)}(x,y)$
$f_X(x) = \mathbb{P}(X=x) = \sum_y f_{(X,Y)}(x,y)$	$f_X(x) = \int_{\mathbb{R}} f_{(X,Y)}(x,y)\,dy$
$f_Y(y) = \mathbb{P}(Y=y) = \sum_x f_{(X,Y)}(x,y)$	$f_Y(y) = \int_{\mathbb{R}} f_{(X,Y)}(x,y)\,dx$

<u>Repartiția uniformă pe $S \subseteq \mathbb{R}^2$</u>

Fie $S \subseteq \mathbb{R}^2$ mărginită (triunghi, dreptunghi...)

$(X,Y) \sim \mathcal{U}(S)$ dacă $\exists f_{(X,Y)}(x,y) \geq 0$ a.î.

$$f_{(X,Y)}(x,y) = \begin{cases} c, & (x,y) \in S \\ 0, & \text{altfel} \end{cases}$$

<u>Cât este c?</u>

Cum $f(x,y)$ este densitate $\Rightarrow c \geq 0$ și

$$\iint_{\mathbb{R}^2} f(x,y)\,dx\,dy = 1 \Rightarrow \iint_{\mathbb{R}^2} c \cdot \mathbb{1}_S(x,y)\,dx\,dy = 1$$

$$c = \frac{1}{\iint_S 1\,dx\,dy}$$

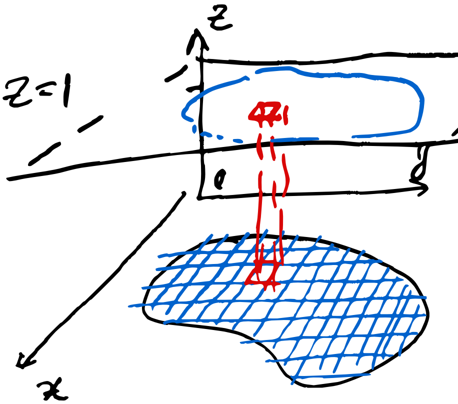
$S = [a,b] \times [c,d]$

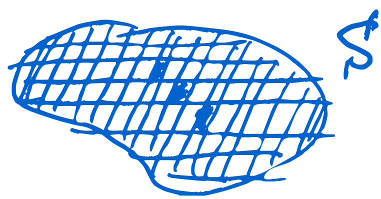
$(b-a)(d-c)$

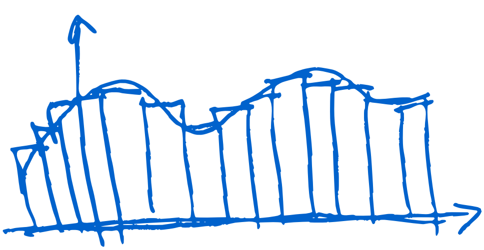
$$\iint_S 1\,dx\,dy = \sum 1 \times (\ )$$

cu

aria drept mic





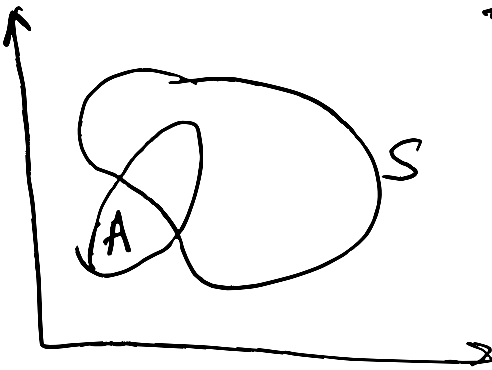


$$\iint_S 1\,dx\,dy = \text{Aria}(S)$$

$$c = \frac{1}{\text{Aria}(S)}$$

$$f(x,y) = \begin{cases} \frac{1}{\text{Aria}(S)}, & (x,y) \in S \\ 0, & \text{altfel} \end{cases}$$

$$\mathbb{P}((X,Y)\in A) = \iint_A f_{(X,Y)}(x,y)\,dx\,dy$$



$$= \iint_A \frac{1}{\text{Aria}(S)} \mathbb{1}_S(x,y)\,dx\,dy$$

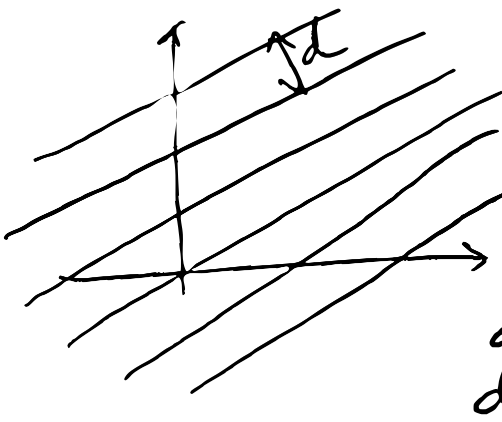
$$= \frac{\iint_A \mathbb{1}_S(x,y)\,dx\,dy}{\text{Aria}(S)}$$

$$= \frac{\iint_{A\cap S} 1\,dx\,dy}{\text{Aria}(S)}$$

$$= \frac{\text{Aria}(A\cap S)}{\text{Aria}(S)}$$

Exp: (Problema acului lui Buffon)

O suprafață marcată cu linii paralele aflate la distanța d una față de cealaltă.

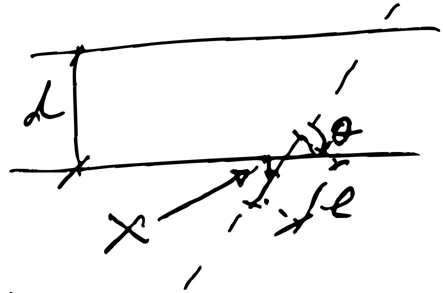


Presupunem că aruncăm un ac de lungime $l < d$

Care este prob. ca acul să intersecteze una dintre linii?

Poziția acului:

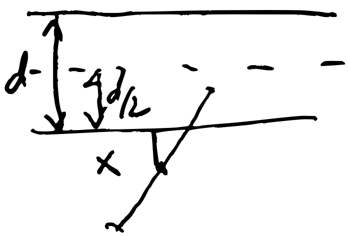
θ - unghiul ascuțit format de axa acului cu dr. ||



X - distanța de la mij. acului la cea mai apropiată dreaptă

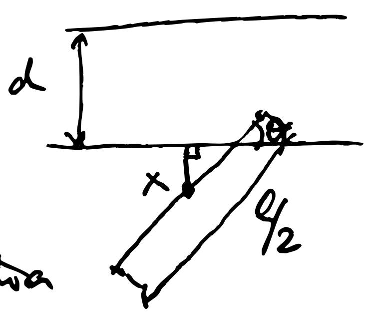
$(X,\theta) \sim \mathcal{U}(S)$

$$S = \left\{ (x,\theta) \;\middle|\; 0 \le \theta \le \pi/2,\ 0 \le x \le d/2 \right\}$$

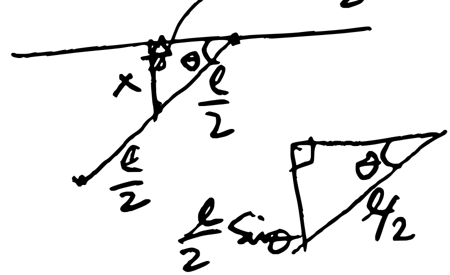


$$f_{(X,\theta)}(x,\theta) = \begin{cases} \dfrac{1}{\frac{d}{2} \times \frac{\pi}{2}}, & (x,\theta) \in S \\ 0, & \text{altfel} \end{cases}$$

Condiția ca acul să intersecteze o linie:



în cazul extrem $X = \frac{l}{2}\sin\theta$



Condiția

$x \le \frac{l}{2}\sin\theta$

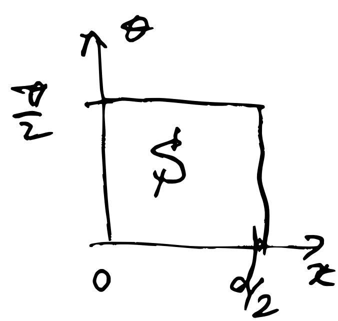
Vrem să calculăm $\mathbb{P}(\underbrace{X \le \frac{\ell}{2}\sin\theta}_{v.a}) = ?$

$$= \mathbb{P}((X,\theta)\in A)$$

unde $A = \{(x,\theta)\in S \mid x \le \frac{\ell}{2}\sin\theta\}$

$$= \iint_A f_{(X,\theta)}(x,\theta)\,dx\,d\theta = \iint_A \frac{1}{\underbrace{\frac{\pi d}{4}}_{\text{Aria}(S)}} \mathbb{1}_S(x,\theta)\,dx\,d\theta$$

$$= \iint_A \frac{4}{\pi d}\,\mathbb{1}_{[0,\frac{d}{2}]\times[0,\frac{\pi}{2}]}(x,\theta)\,dx\,d\theta$$



$$= \frac{4}{\pi d}\int_0^{\pi/2}\int_0^{\frac{\ell}{2}\sin\theta} dx\,d\theta$$

$$= \frac{4}{\pi d}\int_0^{\pi/2}\frac{\ell}{2}\sin\theta\,d\theta = \frac{2\ell}{\pi d}\int_0^{\pi/2}\sin\theta\,d\theta \quad \boxed{\frac{2\ell}{\pi d}}$$

Obs: Fct. de rep. (X,Y)

$$F_{(X,Y)}(x,y) = \mathbb{P}(X\le x, Y\le y) = \int_{-\infty}^{x}\int_{-\infty}^{y} f_{(X,Y)}(u,v)\,dv\,du$$

Densitatea comună se obține:

$$\boxed{f_{(X,Y)}(x,y) = \frac{\partial^2}{\partial x\,\partial y}F_{(X,Y)}(x,y)}$$

Repartiții condiționate

Fie $(\Omega, \mathcal{F}, \mathbb{P})$ un c.p, X o v.a cont și $A \in \mathcal{F}$ cu $\mathbb{P}(A) > 0$.

Definim densitatea condiționată a lui X la A, $f_{X|A}(x)$, funcția $f_{X|A}(x) \geq 0$ care verifică

$$\mathbb{P}(X \in B | A) = \int_B f_{X|A}(x)\, dx, \ \forall B \subseteq \mathbb{R}$$

Lebesgue

Obs: $B = \mathbb{R} \Rightarrow \mathbb{P}(X \in \mathbb{R} | A) = 1$

astfel $\left.\begin{array}{l} \int_{\mathbb{R}} f_{X|A}(x) dx = 1 \\ f_{X|A}(x) \geq 0 \end{array}\right\} \Rightarrow f_{X|A}$ este o densitate de prob.

Obs: În locul lui A considerăm ev. $\{X \in A\}$ cu $\mathbb{P}(X \in A) > 0$

$$\mathbb{P}(X \in B | X \in A) = \frac{\mathbb{P}(X \in B, X \in A)}{\mathbb{P}(X \in A)}$$

$$= \frac{\mathbb{P}(X \in A \cap B)}{\mathbb{P}(X \in A)}$$

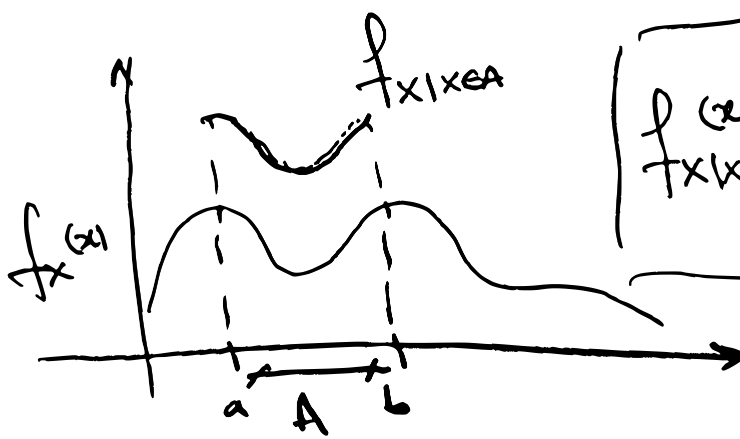
Avem X este v.a. cont. cu densitatea f_X avem

$$\left.\begin{aligned} \mathbb{P}(X\in B \mid X\in A) &= \frac{\int_{A\cap B} f_X(x)\,dx}{\mathbb{P}(X\in A)} \\ \mathbb{P}(X\in B \mid X\in A) &= \int_B f_{X\mid\{X\in A\}}(x)\,dx \end{aligned}\right\} \Rightarrow$$

$$\int_B \underbrace{f_{X\mid\{X\in A\}}}(x)\,dx = \int_{A\cap B} \frac{f_X(x)\,dx}{\mathbb{P}(X\in A)} = \int_B \boxed{\frac{f_X(x)\,\mathbb{1}_A(x)}{\mathbb{P}(X\in A)}}\,dx$$

$$\int_{A\cap B} f\,dx = \int f\,\mathbb{1}_{A\cap B}(x)\,dx$$

$$= \int f(x)\,\mathbb{1}_A(x)\,\mathbb{1}_B(x)\,dx = \int_B f(x)\,\mathbb{1}_A(x)\,dx$$



$$\boxed{f_{X\mid X\in A}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(X\in A)}, & x\in A \\ 0, & \text{altfel} \end{cases}}$$

Exp: $X \sim \mathcal{U}([a,b])$, $[c,d] \subseteq [a,b]$

$$f_{X|X\in[c,d]}(x) = ?$$

Avem: $$f_{X|X\in[c,d]}(x) = \begin{cases} \dfrac{f_X(x)}{\mathbb{P}(X\in[c,d])}, & x \in [c,d] \\ 0, & \text{altfel} \end{cases}$$

$$f_X(x) = \frac{1}{b-a} \mathbb{1}_{[a,b]}(x)$$

$$\mathbb{P}(X\in[c,d]) = \int_c^d f_X(x)\,dx = \frac{d-c}{b-a}$$

$$f_{X|X\in[c,d]}(x) = \begin{cases} \dfrac{\cancel{\frac{1}{b-a}} \mathbb{1}_{[a,b]}(x)}{\frac{d-c}{\cancel{b-a}}}, & x \in [c,d] \\ 0, & \text{altfel} \end{cases}$$

$[c,d] \subseteq [a,b]$

$$f_{X|X\in[c,d]}(x) = \frac{1}{d-c} \mathbb{1}_{[c,d]}(x)$$

adică $X \mid X\in[c,d] \sim \mathcal{U}([c,d])$

Formula prob. totale:

Fie $A_1, A_2, \dots A_n \in \mathcal{F}$ care formează o partiție pe Ω

și $B \in \mathcal{F}$ atunci

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B|A_i)\mathbb{P}(A_i)$$

Dacă $B = \{X \leq x\}$ unde X v.a cont f_X

$$\mathbb{P}(X \leq x) = \sum_{i=1}^{n} \mathbb{P}(X \leq x | A_i)\mathbb{P}(A_i)$$

$$\int_{-\infty}^{x} f_X(t)\,dt = \sum_{i=1}^{n} \int_{-\infty}^{x} f_{X|A_i}(t)\,dt\, \mathbb{P}(A_i)$$

$$= \int_{-\infty}^{x} \sum_{i=1}^{n} f_{X|A_i}(t)\,\mathbb{P}(A_i)\,dt$$

derivăm după x

$$\frac{d}{dx}\int_{-\infty}^{x} f_X(t)\,dt = \frac{d}{dx}\int_{-\infty}^{x} \sum_{i=1}^{n} f_{X|A_i}(t)\,\mathbb{P}(A_i)\,dt$$

$$\boxed{f_X(x) = \sum_{i=1}^{n} f_{X|A_i}(x)\,\mathbb{P}(A_i)}$$

Obs: $A \in \mathcal{F}$, $\mathbb{P}(A) > 0$, X v.a. cont. f_X

$$f_X(x) = f_{X|A}(x)\,\mathbb{P}(A) + f_{X|A^c}(x)\,\mathbb{P}(A^c)$$

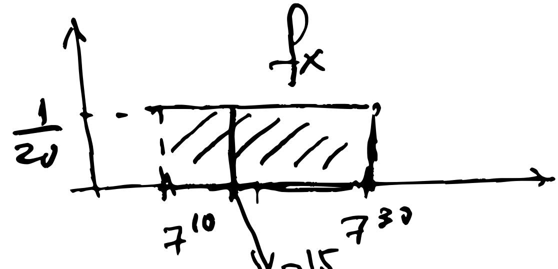
Exp: Pp că metroul circulă la intervale de 15 min începând cu ora 5^{00} a.m. Pp că ajungem în stație în intervalul $7^{10} - 7^{30}$ în mod aleator (uniform pe acest interval)

Ne propunem să determinăm rep. timpului de așteptare până la sosirea primului metrou.

Sol: Fie Y - timpul de așteptare până la sosirea primului metrou

Vrem să det. $f_Y = ?$

Fie X - timpul de sosire în stație $\mathcal{U}([7^{10} - 7^{30}])$



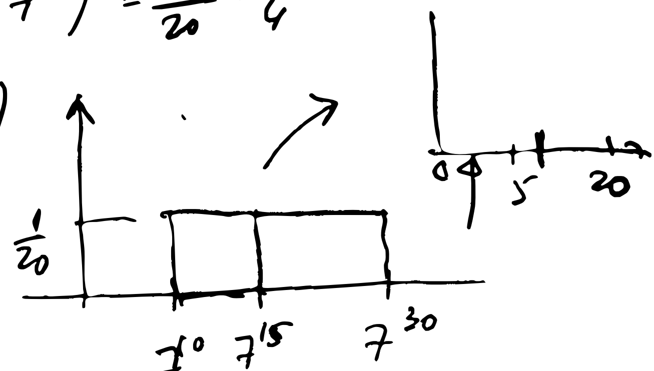
$A = \{7^{10} \leq X \leq 7^{15}\} \rightarrow$ urcăm în metroul de 7^{15}

$B = \{7^{15} < X \leq 7^{30}\} \rightarrow$ —"— 7^{30}

$$f_Y(y) = f_{Y|A}(y)\,\mathbb{P}(A) + f_{Y|B}(y)\,\mathbb{P}(B)$$

$$\mathbb{P}(A) = \mathbb{P}(7^{10} \le X \le 7^{15}) = \frac{5}{20} = \frac{1}{4}$$

$$\mathbb{P}(B) = \mathbb{P}(7^{15} < X \le 7^{30}) = \frac{15}{20} = \frac{3}{4}$$

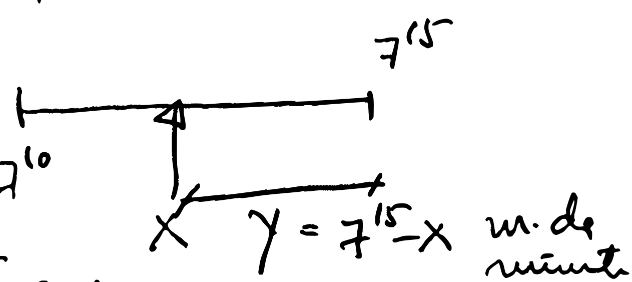


$$f_Y(y) = f_{Y|\underline{A}}(y)\cdot\frac{1}{4} + f_{Y|\underline{B}}(y)\cdot\frac{3}{4} \qquad (*)$$

Amintiri: $U \sim \mathcal{U}([a,b])$, $[c,d] \subseteq [a,b]$

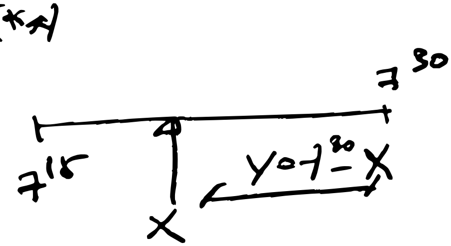
$U \mid U \in [c,d] \sim \mathcal{U}([c,d])$

Dacă A s-a realizat:



$$f_{Y|A}(y) = \frac{1}{5}, \quad 0 \le y \le 5 \qquad (**)$$

Dacă B s-a realizat:



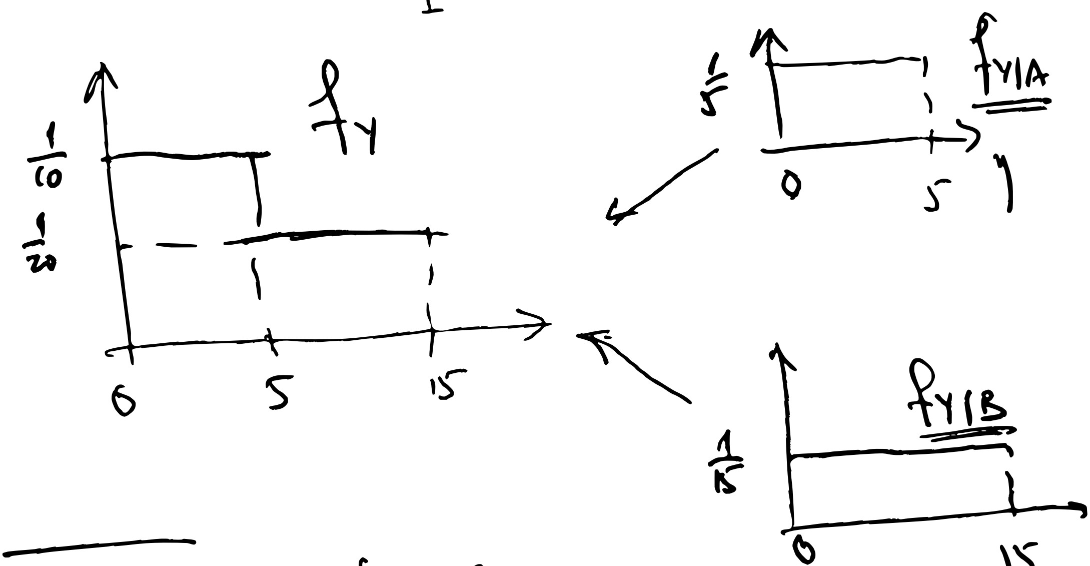
$$f_{Y|B}(y) = \frac{1}{15}, \quad 0 \le y \le 15 \qquad (***)$$

$$f_Y(y) = f_{Y|A}(y)\,\mathbb{P}(A) + f_{Y|B}(y)\,\mathbb{P}(B)$$

F. prob. totale

$$= \frac{1}{5}\,\mathbb{1}_{[0,5]}(y)\,\frac{1}{4} + \frac{1}{15}\,\mathbb{1}_{[0,15)}(y)\,\frac{3}{4}$$

$$= \begin{cases} \frac{1}{10}, & 0 \le y \le 5 \\ \frac{1}{20}, & 5 < y \le 15 \end{cases}$$



Obs: $A \to \{Y = y\}$

Y v.a. cont $f_Y(y) > 0$

Definim densitatea condiționată a lui X la $Y = y$ prin

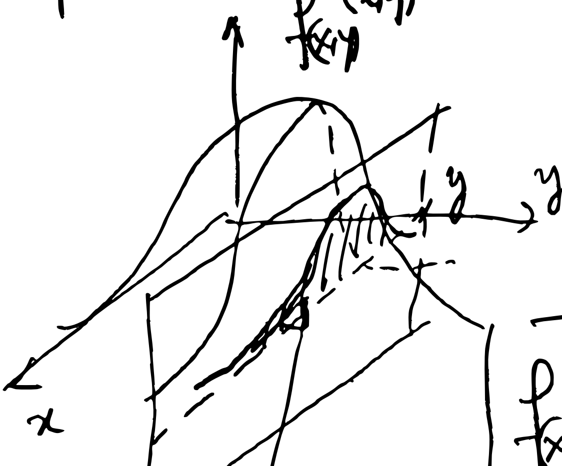
$$\boxed{f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)}}$$

$$f_Y(y) = \int f_{(X,Y)}(x,y)\,dx$$

$$\int_{-\infty}^{+\infty} f_{X|Y}(x|y)\,dx = \int_{-\infty}^{\infty} \frac{f_{(X,Y)}(x,y)}{f_Y(y)}\,dx = \frac{\int f_{(X,Y)}(x,y)\,dx}{f_Y(y)}$$

$$= 1$$

$f_{X|Y}(x|y)$ este o densitate de rep.



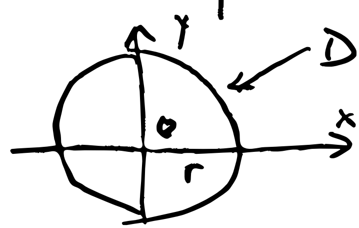
$$\boxed{\begin{aligned} f_{(X,Y)}(x,y) &= f_{X|Y}(x|y)\, f_Y(y) \\ &= f_{Y|X}(y|x)\, f_X(x) \end{aligned}}$$

<u>Exp</u>: $D = \{ (x,y) \in \mathbb{R}^2 \mid x^2+y^2 \le r^2 \}$

$(X,Y) \sim \mathcal{U}(D)$

Vrem să calculăm:

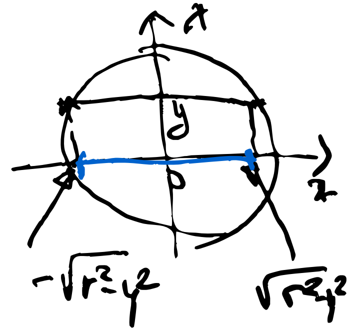
$f_{X|Y}(x|y) = ?$



Stim $f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{Aria(D)}, & (x,y)\in D \\ 0, & altfel \end{cases}$

$$f_{(X,Y)}(x,y) = \begin{cases} \frac{1}{\pi r^2}, & x^2+y^2 \le r^2 \\ 0, & altfel \end{cases}$$

$$f_{X|Y}(x|y) = \frac{f_{(X,Y)}(x,y)}{f_Y(y)}$$



Densitatea marginală a lui Y:

$$f_Y(y) = \int f_{(X,Y)}(x,y)\,dx$$

$$= \int \frac{1}{\pi r^2} \mathbb{1}_D(x,y)\,dx$$

$$= \int_{-\sqrt{r^2-y^2}}^{\sqrt{r^2-y^2}} \frac{1}{\pi r^2}\,dx = \frac{2\sqrt{r^2-y^2}}{\pi r^2},\ y\in[-r,r]$$

$\frac{1}{2a}\mathbb{1}_{[-a,a]}(x)$

$$f_{X|Y}(x|y) = \frac{\frac{1}{\pi r^2}\mathbb{1}_D(x,y)}{\frac{2\sqrt{r^2-y^2}}{\pi r^2}\mathbb{1}_{[-r,r]}(y)} = \frac{1}{2\sqrt{r^2-y^2}}\mathbb{1}_{[-\sqrt{r^2-y^2},\sqrt{r^2-y^2}]}(x)$$