$\Rightarrow$ progresie aritmetică $\Rightarrow p_k = p_0 + k \cdot b_1$

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$\Rightarrow p_k = 1 + k \cdot \underline{b_1}$

Dar, în plus, $p_N = 0$

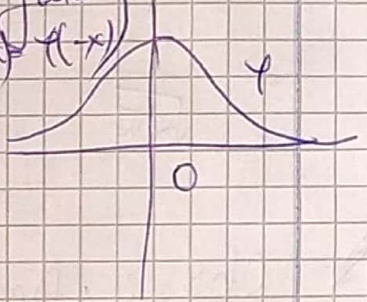
Deci $p_N = 1 + N \cdot b_1 \iff 0 = 1 + N \cdot b_1 \Rightarrow \boxed{b_1 = -\frac{1}{N}}$

$\Rightarrow p_k = 1 + k \cdot \left(-\frac{1}{N}\right)$

Curs 10, 11 (săpt. 11) 12 decembrie

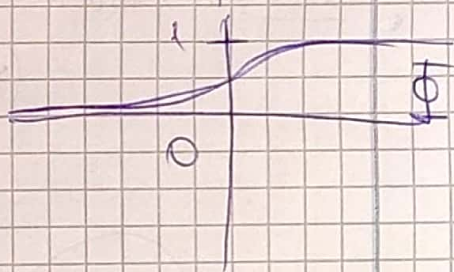
Repartiția normală

(simetrică față de 0: $f(x) = f(-x)$)



1) $f(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}, \; x \in \mathbb{R}$

2) $\Phi(x) = \int_{-\infty}^{x} f(t)\, dt$



f densitate

1) $f(x) \geq 0$

2) $\int_{-\infty}^{+\infty} f(x)\, dx = 1$

$\frac{1}{e^{\frac{x^2}{2}}}$

$X \sim N(0,1)$ normală standard

$\mathbb{E}[X] = \int_{-\infty}^{+\infty} x\, f(x)\, dx = \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}}\, dx$

funcție impară

$= 0$

$$Var(X) = E[X^2] - \underbrace{E[X]^2}_{=0}$$

$$E[X^2] = \int x^2 f(x)\,dx = \int_{-\infty}^{+\infty} \frac{x^2}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} dx =$$

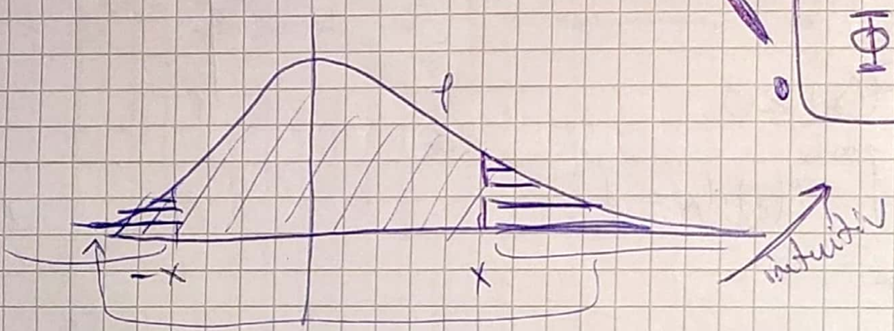
$$= \int_{-\infty}^{+\infty} \frac{x}{\sqrt{2\pi}} \left(-e^{-\frac{x^2}{2}}\right)' dx =$$

$$= \underbrace{-\frac{x \cdot e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} \Bigg|_{-\infty}^{+\infty}}_{0} + \int_{-\infty}^{+\infty} -\frac{e^{-\frac{x^2}{2}}}{\sqrt{2\pi}} dx$$

$$= \frac{1}{\sqrt{2\pi}} \cdot \int_{-\infty}^{+\infty} e^{-\frac{x^2}{2}} dx = 1 \Rightarrow Var(X) = 1$$

! **Obs.**: Dacă $X \sim N(0,1)$ atunci $E[X] = 0$ și $Var(X) = 1$ (X e rep. normal)

! $\boxed{\Phi(x) = 1 - \Phi(-x)}$ (*)



Dem.

$$\Phi(-x) = \int_{-\infty}^{-x} f(t)\,dt \quad \overline{\overline{u = -t}}$$

$$= \int_{+\infty}^{x} f(-u)(-du) = \int_{x}^{+\infty} f(-u)\,du$$

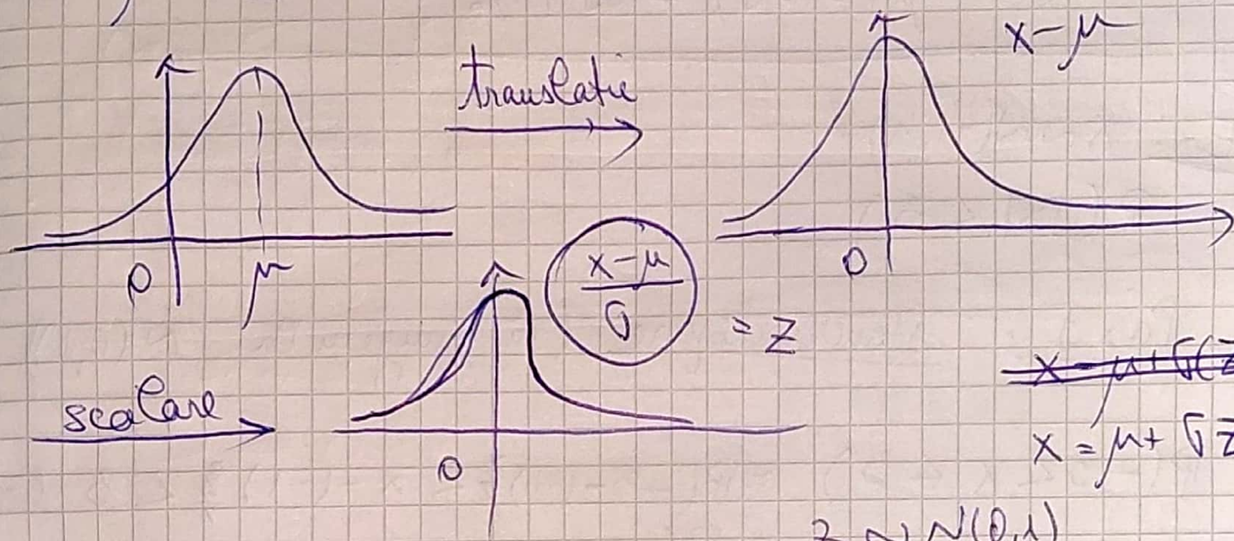
$$= \int_{x}^{\infty} f(u)\,du = 1 - \int_{-\infty}^{x} f(u)\,du$$

$\mathbb{P}(X > x)$ ... "$\mathbb{P}(X \le x)$"

simetrică față de 0

Def.: Spunem că $X \sim N(\mu, \sigma^2)$, dacă admite densitatea de repartiție $f(x) = \frac{1}{\sqrt{2\pi}\cdot\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}}$, $x \in \mathbb{R}$.

(P) Dacă $X \sim N(\mu, \sigma^2)$ atunci $(\exists)\ Z \sim N(0,1)$ a.î. $X = \mu + \sigma Z$



~~$X = \mu + \sigma(Z)$~~

$X = \mu + \sigma Z$

$Z \sim N(0,1)$

$\Rightarrow X \sim N(\underbrace{\mu}_{\text{media}}, \underbrace{\sigma^2}_{\text{varianță}}) \Rightarrow E[X] = E[\mu + \sigma Z] = \mu + \sigma \cdot E[Z] = \mu$

$Var(X) = Var(\mu + \sigma Z) = \sigma^2 Var(Z) = \sigma^2$

$F(x) = P(X \leq x) = P(\mu + \sigma Z \leq x)$
$= P\left(Z \leq \frac{x-\mu}{\sigma}\right) = \Phi\left(\frac{x-\mu}{\sigma}\right)$

$f(x) = \frac{d}{dx} F(x) = \frac{d}{dx} \Phi\left(\frac{x-\mu}{\sigma}\right) = f\left(\frac{x-\mu}{\sigma}\right) \cdot \frac{1}{\sigma}$

$\left(f(x) = \frac{1}{\sqrt{2\pi}} \cdot \frac{1}{\sigma} \cdot e^{-\frac{(x-\mu)^2}{2\sigma^2}} \right)$

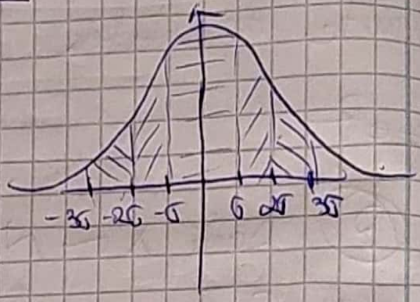
(P) Prop. 68 - 95 - 99.7 %

Dacă $X \sim N(\mu, \sigma^2)$ atunci

$P(|X-\mu| \le \sigma) \simeq 68\%$

$P(|X-\mu| \le 2\sigma) \simeq 95\%$

$P(|X-\mu| \le 3\sigma) \simeq 99.7\%$



Ex. $X \sim N(-1, 4)$

$P(|X| < 3)$

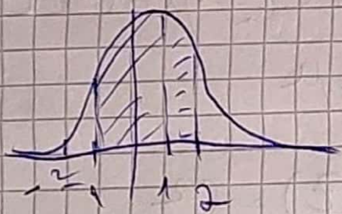
Pas 1: Standardizare (să ajungem la $N(0,1)$)

$P(-3 < X < 3) = P(-3-(-1) < X-(-1) < 3-(-1))$ $\boxed{\begin{array}{l}(-\mu)\\(/\sigma)\end{array}}$

$= P(-2 < X+1 < 4)$

$= P\left(-\frac{2}{2} < \frac{X+1}{2} < \frac{4}{2}\right)$

$= P\left(-1 < \underbrace{\frac{X+1}{2}}_{\sim N(0,1)} < 2\right)$



$P(-1 \le Z \le 1) \simeq 0,68$

$P(-2 \le Z \le 2) \simeq 0,95$

$\Rightarrow P(-1 \le Z \le 1) + P(1 \le Z \le 2)$

$\simeq 0,68 + \frac{0,95 - 0,68}{2}$

Ex. 2: $Y \sim N(0,1)$, $X = |Y|$

$E[X]$, $Var(X)$, $f(x)$

$$E[|Y|] = \int_{-\infty}^{+\infty} f(x)\cdot|x|\,dx = 2\int_0^{\infty} \frac{x}{\sqrt{2\pi}}\, e^{-\frac{x^2}{2}}\,dx$$

$$= \sqrt{\frac{2}{\pi}}\left(-e^{-\frac{x^2}{2}}\right)\Big|_0^{\infty} = \sqrt{\frac{2}{\pi}}$$

$$Var(|Y|) = E[|Y|^2] - E[|Y|]^2$$

$$= E[Y^2] - \frac{2}{\pi} = 1 - \frac{2}{\pi}$$

$$F_X(x) = P(X \le x) = P(|Y| \le x)$$

-1)) $\boxed{(-\mu) \\ (X/\sigma)}$

$$\Rightarrow \begin{cases} F_X(x) = 0, & x < 0 \\ F_X(x) = P(-x \le Y \le x) = \Phi(x) - \Phi(-x) \\ = 2\Phi(x) - 1, & x > 0 \end{cases}$$

(am introdus cu (x) pg. 2)

(!) $\Rightarrow F_X(x) = \begin{cases} 0, & x < 0 \\ 2\Phi(x) - 1, & x > 0 \end{cases}$

$$\Rightarrow f_X(x) = \begin{cases} 0, & x < 0 \\ 2 f(x), & x > 0 \end{cases} \rightarrow$$

Repartiții comune, marginale și condiționate

X, Y două v.a. $(\Omega, \mathcal{F}, \mathbb{P})$ c.p.

$\longrightarrow (X, Y)$ $\Rightarrow \mathbb{P}((X,Y) \in A \times B)$ comună

$\mathbb{P}(X \in A)$ sau $\mathbb{P}(Y \in B)$ marginală

$\mathbb{P}(X \in A \mid Y \in B)$ condiționată

(I) Cazul discret

Fie $(\Omega, \mathcal{F}, \mathbb{P})$ un c.p. și $X: \Omega \to \mathbb{R}$

$Y: \Omega \to \mathbb{R}$

$X(\Omega) = \{x_1, x_2, \dots x_m\}$

$Y(\Omega) = \{y_1, y_2, \dots y_n\}$

$\Rightarrow$ Comună

Perechea $(X, Y): \Omega \longrightarrow \mathbb{R}^2$

$\omega \quad (X(\omega), Y(\omega))$

$(X,Y)(\Omega) = \{(x_i, y_j) \mid i = \overline{1,m}, j = \overline{1,n}\} \Rightarrow$

$\Rightarrow m \cdot n$ valori

1) Funcția de masă a (X, Y)

$f_{X,Y}(x,y) = \mathbb{P}(X = x, Y = y)$; $\forall x \in \{x_1, \dots x_m\}$

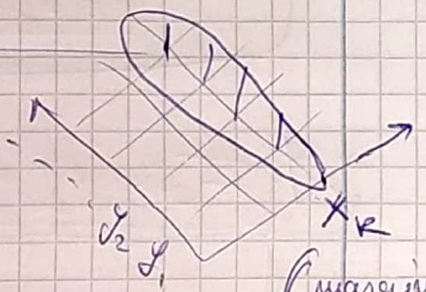
notată și $(p_{X,Y}(x,y))$ $\qquad y \in \{y_1, \dots y_n\}$

Proprietăți:

i) $f_{X,Y}(x,y) \geq 0$; $\forall x, y$

ii) $\sum_{x\in X(\Omega)} \sum_{y\in Y(\Omega)} f_{X,Y}(x,y) = 1$

* Prob. să ∈ unei regiuni ~~într-o regiune~~ * — $P((X,Y)\in C) = \sum_{\substack{(x,y)\in X(\Omega)\times Y(\Omega)\\ (x,y)\in C}} f_{XY}(x,y)$



(marginala de X)

R: Marginală

$$P(X\in A) = P(\{X\in A\}\cap\Omega)$$
$$= P(X\in A,\ Y\in \mathbb{R})$$
$$= P\left(X\in A,\ \bigcup_y \{Y=y\}\right)$$
$$= P\left(\bigcup_y \{X\in A,\ Y=y\}\right) = \sum_y P(X\in A,\ Y=y)$$
$$\Rightarrow \left[P(X=x)=\sum_y P(X=x,\ Y=y)\right]$$

$$\begin{cases} f_X(x) = \sum_y f_{X,Y}(x,y) \\ f_Y(y) = \sum_x f_{X,Y}(x,y) \end{cases}$$

R: Condiționată

• Fie X o v.a. discretă și $A\in\mathcal{F}$, $P(A)>0$

$$P(X=x\mid A) = \frac{P(\{X=x\}\cap A)}{P(A)}$$

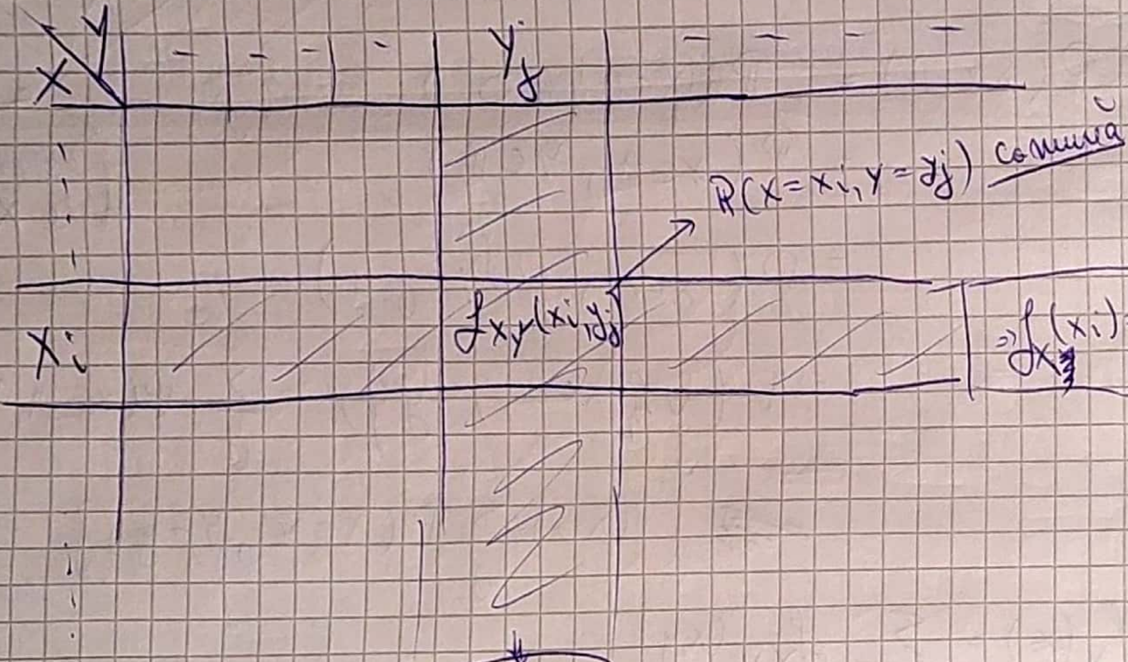
$\overset{not.}{=} f_{X\mid A}(x)$

Dacă $A=\{Y=y\}$ atunci $P(X=x\mid Y=y) = f_{X,Y}(x,y)$

$$= \frac{P(X=x,\ Y=y)}{P(Y=y)} = \frac{f_{XY}(x,y)}{f_Y(y)}$$

$$\Rightarrow \boxed{f_{X|Y}(x|\overset{\text{fixat}}{y}) = \frac{f_{X,Y}(x,y)}{f_Y(y)}}$$ X la Y

$$\boxed{f_{Y|X}(y|\overset{\text{fixat}}{x}) = \frac{f_{X,Y}(x,y)}{f_X(x)}}$$ Y la X



$$\Rightarrow \boxed{f_X(x_i)} = \sum_y f_{X,Y}(x_i,y)$$

$$\boxed{f_Y(y_j)} = \sum_x f_{X,Y}(x,y_j)$$ marginala lui Y

$$X \sim \begin{pmatrix} x_1 \; \dots \; x_i \; \dots \; x_m \\ f_X(x_i) \end{pmatrix}$$

$$X|Y \overset{\text{not.}}{=} y_j \sim \begin{pmatrix} x_1 \; \dots \; x_i \; \dots \; x_m \\ f_{X|Y}(x_i|y_j) \end{pmatrix}$$

$$\boxed{= \frac{f_{X,Y}(x_i,y_j)}{f_Y(y_j)}}$$

condiționată

împărțim la prob. marginalei (suma lor)

ca adunate să dea 1

Ex. 1: Prof. răspunde greșit 1/4 din cazuri independent de întrebări.

Avem (0, 1 sau 2) întrebări cu prob. 1/3

$\Leftrightarrow$ Prob. să am 0 întreb. e 1/3 ; 1 întreb. e 1/3 ; 2 întreb e 1/3

X – nr. întrebări $\in \{0,1,2\}$

Y – nr. răsp. greșite $\in \{0,1,2\}$

$(X,Y) : \Omega \to \mathbb{R}^2$

$(X,Y) \in \{0,1,2\}^2$

niciо întreb. $\Rightarrow$ niciun răsp. greșit

$X \backslash Y$	0	1	2	
0	$\frac{1}{3}$	0	0	1/3
1	$\frac{1}{4}$	$\frac{1}{12}$	0	1/3
2	3/16	2/16	$\frac{1}{3}\left(\frac{3}{4}\right)^2$	1/3

$P($ 2 întreb. 0 răsp. greșit

$\left(1-\frac{1}{4}\right)\left(1-\frac{1}{4}\right)$

$= \left(\frac{3}{4}\right)^2$

răsp. greșit la 1 din 2

$\frac{1}{3} \cdot \frac{3}{4} \cdot 1$

2 înt + 2 răsp. greșite

$= \frac{1}{3}\left(\frac{3}{4}\right)^2$

$\Rightarrow P(X=0, Y=0) = P(X=0) \cdot P(Y=0 \mid X=0)$

$= \frac{1}{3} \cdot 1 = \frac{1}{3}$

$\Rightarrow P(X=1, Y=1) = P(X=1) \cdot P(Y=1 \mid X=1)$

$= \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{12}$

o întreb și un răsp. greșit

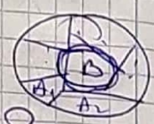
$P(X=1, Y=0) = \frac{1}{3} \cdot \frac{3}{4} = \frac{1}{4}$ (o întrebare și 0 răsp. greșite)

Scanned with CamScanner

$$\mathbb{P}(X=2, Y=1) = \mathbb{P}(X=2)\cdot\mathbb{P}(Y=1 \mid X=2)$$
$$= \frac{1}{3} \times \underbrace{\binom{2}{1}\cdot\frac{1}{4}\cdot\frac{3}{4}}$$

$\Rightarrow$ o întrebare greșită și una corectă
indiferent de ordine *

Formula probabilității totale



B, $A_1, A_2, \dots A_n \in \mathcal{F}$ formează o partiție pe Ω

$$\mathbb{P}(B) = \sum_{i=1}^{n} \mathbb{P}(B \mid A_i)\cdot\mathbb{P}(A_i)$$

Dacă $B = \{X = x\} \Rightarrow \mathbb{P}(X=x) = \sum_{i=1}^{n} \underbrace{\mathbb{P}(X=x \mid A_i)}_{f_{X|A_i}(x)}\cdot\mathbb{P}(A_i)$

$$A = \{Y = y_i\} = \mathbb{P}(X=x) = \sum_{i=1}^{n} \mathbb{P}(X=x \mid Y=y_i)\cdot\mathbb{P}(Y=y_i)$$

$$f_X(x) = \sum_{i=1}^{n} f_{X|Y}(x \mid y_i)\cdot \mathbb{P}_Y(y_i)$$

$$\mathbb{P}(A\cap B) = \mathbb{P}(A)\cdot\mathbb{P}(B \mid A)$$
$$= \mathbb{P}(B)\cdot\mathbb{P}(A \mid B)$$

unde $A = \{X = x\}$, $B = \{Y = y\} \Rightarrow$

$$\Rightarrow \mathbb{P}(X=x, Y=y) = \mathbb{P}(X=x)\cdot\mathbb{P}(Y=y \mid X=x)$$
$$= \mathbb{P}(Y=y)\cdot\mathbb{P}(X=x \mid Y=y)$$

$\Leftrightarrow$

$$\boxed{\Rightarrow f_{XY}(x,y) = f_X(x)\cdot f_{Y|X}(y \mid x) = f_Y(y)\cdot f_{X|Y}(x \mid y)}$$!

Formula lui Bayes:

$$\cancel{\mathbb{P}(X=x \mid Y=y) = \mathbb{P}(X=x)\cdot\mathbb{P}(Y=y \mid X=x)}$$

$$\mathbb{P}(X=x \mid Y=y) = \frac{\mathbb{P}(X=x, Y=y)}{\mathbb{P}(Y=y)}$$

$$= \cancel{\frac{\mathbb{P}(Y=y)\cdot\mathbb{P}(X=x \mid Y=y)}{\sum_{x'} \mathbb{P}(X=x')\cdot\mathbb{P}(Y=y \mid X=x')}}$$

$\Downarrow$

$$\cancel{f_{X|Y}(x \mid y) = \frac{f_Y(y)\cdot f_{X|Y}(x \mid y)}{\sum_{x'} f_X(x')\cdot f_{Y|X}(y \mid x')}}$$

$$\boxed{= \frac{\mathbb{P}(X=x)\cdot\mathbb{P}(Y=y \mid X=x)}{\sum_{x'} \mathbb{P}(X=x')\cdot\mathbb{P}(Y=y \mid X=x')} \quad \Rightarrow f_{X|Y}(x \mid y) = \frac{f_X(x)\cdot f_{Y|X}(y \mid x)}{\sum_{x'} f_X(x')\cdot f_{Y|X}(y \mid x')}}$$

Ex: N ouă ; pp. că fiecare eclozează cu $p \in (0,1)$ indep. de celelalte

X nr. de ouă care au eclozat

Y nr. de ouă care nu au eclozat.

$$X + Y = N$$

Vreau să det. rep. (X, Y) și să verif. $X \perp\!\!\!\perp Y$

$$\mathbb{P}(N=n) = e^{-\lambda} \cdot \frac{\lambda^n}{n!}$$

$$\left.\begin{array}{l} X \mid N=n \sim \mathcal{B}(n, p) \\ Y \mid N=n \sim \mathcal{B}(n, 1-p) \end{array}\right\} (*)$$

$$\mathbb{P}(X=i, Y=j) = \sum_{n=0}^{\infty} \mathbb{P}(X=i, Y=j \mid N=n)\, \mathbb{P}(N=n)$$

unde Dacă $i+j \neq n \Rightarrow \mathbb{P}(X=i, Y=j \mid N=n) = 0$

$$= \cancel{\Sigma}\ \mathbb{P}(X=i, Y=j \mid N=i+j) \cdot \mathbb{P}(N=i+j)$$

unde $\mathbb{P}(X=i, Y=j \mid N=i+j) = \mathbb{P}(X=i \mid N=i+j) \overset{(*)}{=}$ binomiala

$= \mathbb{P}(Y=j \mid N=i+j)$

$$= \binom{i+j}{i} p^i (1-p)^j$$

$$\mathbb{P}(X=i, Y=j) = \binom{i+j}{i} p^i (1-p)^j \cdot e^{-\lambda} \cdot \frac{\lambda^{n=(i+j)}}{\underset{(i+j)}{n!}}$$

$$= \frac{\cancel{(i+j)!}}{i! \cdot j!} \cdot p^i (1-p)^j \cdot e^{-\lambda(p+1-p)} \cdot \frac{\lambda^{i+j}}{\cancel{(i+j)!}}$$

$e^{-\lambda(p+1-p)} \to e^{-\lambda p} \cdot e^{-\lambda(1-p)}$

$\lambda^{i+j} \to \lambda^i \cdot \lambda^j$

$$= \frac{e^{-\lambda p} \cdot p^i \lambda^i}{i!} \cdot e^{-\lambda(1-p)} \cdot \frac{(1-p)^j \lambda^j}{j!}$$ (?) calcule incomplete

Media unei fct. de v.a

X v.a $: \Omega \to \mathbb{R}$ $\quad g: \mathbb{R} \to \mathbb{R}$

$$\mathbb{E}[g(x)] = \sum_x g(x) \cdot P(X=x)$$

$\Rightarrow$ Pentru

$X, Y: \Omega \to \mathbb{R}$, $g: \mathbb{R}^2 \to \mathbb{R}$

$$\boxed{\mathbb{E}[g(x,y)] = \sum_{x,y} g(x,y) \cdot \mathbb{P}(X=x \mid Y=y)}$$

Obs. $\Rightarrow$ când 2 ev. nu sunt indep. $\Rightarrow$ $\mathbb{E}$ se calc. cu rep. comună

$$\mathbb{E}[X \cdot Y] = \sum_{x,y} xy \, \underline{\mathbb{P}(X=x \mid Y=y)} \quad (**)$$

Ex a) Rep. marginale

$X \backslash Y$	-1	0	2	X
1	1/18	3/18	2/18	6/18
2	2/18	0	3/18	5/18
3	0	4/18	3/18	7/18
Y	3/18	7/18	8/18	

$$\begin{cases} X \sim \begin{pmatrix} 1 & 2 & 3 \\ 6/18 & 5/18 & 7/18 \end{pmatrix} \\ Y \sim \begin{pmatrix} -1 & 0 & 2 \\ 3/18 & 7/18 & 8/18 \end{pmatrix} \end{cases}$$

$X|Y=0 \sim \begin{pmatrix} 1 & 2 & 3 \\ \frac{3/18}{7/18} & 0 & \frac{4/18}{7/18} \end{pmatrix}$

$Y|X=1 \sim \begin{pmatrix} -1 & 0 & 2 \\ \frac{\frac{1}{18}}{6/18} & \frac{3/18}{6/18} & \frac{2/18}{6/18} \end{pmatrix}$

prima linie

X Y nu sunt indep $\Rightarrow$ fără 0 ... (**)

$\Rightarrow \mathbb{E}[XY] = \frac{1}{18}(-1) + 2 \cdot \frac{2}{18} + (-2) \cdot \frac{2}{18} + 4 \cdot \frac{3}{18} + 6 \cdot \frac{3}{18} = \dots$

$\mathbb{E}[2X+3Y] = 2\mathbb{E}[X] + 3\mathbb{E}[Y] \overset{\text{sau}}{=} \sum (2x+3y)\, P(X=x, Y=y)$

$\boxed{\mathbb{E}[X|Y=y] = \sum_x x\, P(X=x|Y=y)}$ $\Leftrightarrow$ media condiționată a lui $Y=y$

* $\mathbb{E}[X|Y] = g(Y)$

X e cond la Y ia val. $g(y_i)$ din Y