# Vehicle models for autonomous driving

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## 1. Intro

This paper serves as a summary of the knowledge I have gained in the last few years working on car models for autonomous driving. I do my best to present a complete and self-sufficient description of these models, their derivation, strengths and weaknesses, and whenever possible the physical intuition behind them. I hope that this paper will be useful to anyone interested in the subject, and I would be happy to receive any feedback or suggestions.

kinematic bicyle model, the different car models used in the `ihm2` project, be it in the simulator or the controllers.

model that is used in the `hess_sim` simulator. We dedicate one section to each In this document, we present the hybrid (kinematic bicyle)-(dynamic four wheel) of these to models, in which we present the equations that define the dynamics of the car and try to provide some insight on the physical intuition behind them. A third section sums up the strengths and weaknesses of each model and the way they are both currently used in the simulator.

As usual in the litterature, we simplify the model by (almost always) ommitting the vertical motions of the car, as well as the pitch and roll dynamics.

We will denote by $X, Y$ the position of the car in an absolute and fixed cartesian reference frame (that we will call the *world* or *inertial* frame), by $\varphi$ the yaw angle of the car, by $v$ the absolute velocity of the car that has components $v_x, v_y$ in the mobile reference frame attached to the car (that we will call the *car* or *body* frame), and finally by $r$ the yaw rate of the car. The car is always actuated by a global torque $T$ (which is then divided into wheel torques by different methods evocated in section …) and a steering angle $\delta$.

All the following models share the following fundamental assumptions:
1. the car drives on a perfect plane, and we can therefore ignore the vertical, pitch and roll motions.
2. the car has a four wheel drive traction system and we can independently model and control each wheel.

## 2. Kinematic models

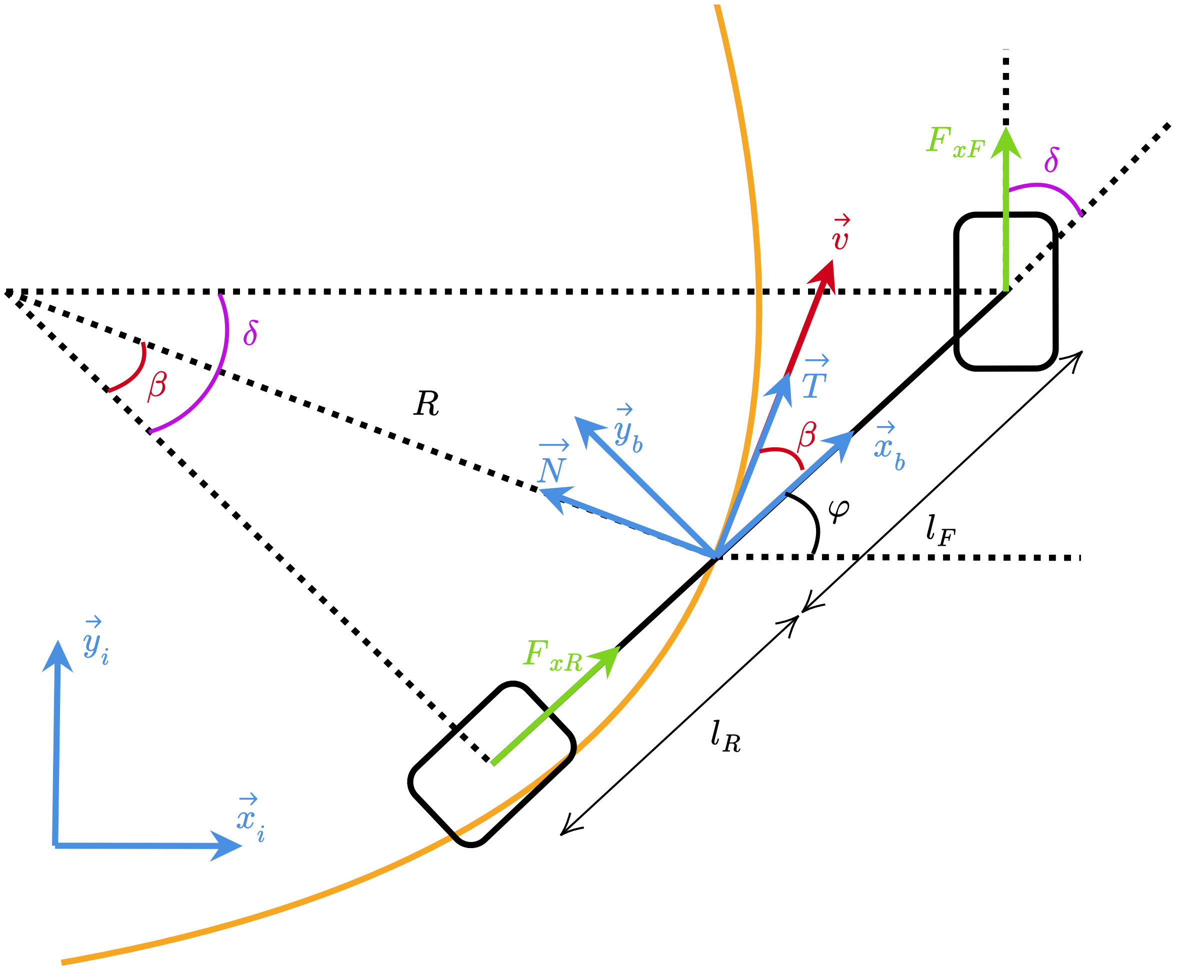


Figure 1: Summary of the kinematic bicycle model.
We ommited the $z$ axis for simplicity.

If we only consider the 4DOF state $x = (X, Y, \varphi, v)^T$ with the absolute velocity $v$, then the dynamics are:

$$\begin{aligned}
\dot{X} &= v\cos(\varphi + \beta), \\
\dot{Y} &= v\sin(\varphi + \beta), \\
\dot{\varphi} &= v\frac{\sin(\beta)}{l_R}, \\
\dot{v} &= \frac{1}{m}(F_{xR}\cos(\beta) + F_{xF}\cos(\delta - \beta)).
\end{aligned}$$

with $\beta = \arctan\left(\frac{l_R}{l_R + l_F}\tan(\delta)\right)$ the kinematic slip angle.

If instead we want to consider the 6DOF state $x = \left(X, Y, \varphi, v_x, v_y, r\right)^T$ that treats separately the longitudinal velocity $v_x = v\cos(\beta)$, the lateral velocity $v_y = v\sin(\beta)$, and the yaw rate $r$, then the dynamics are:

$$\begin{aligned}
\dot{X} &= v_x\cos(\varphi) - v_y\sin(\varphi), \\
\dot{Y} &= v_x\sin(\varphi) + v_y\cos(\varphi), \\
\dot{\varphi} &= r, \\
\dot{v}_x &= \dot{v}\cos(\beta) - \dot{\beta}, \\
\dot{v}_y &= v\sin(\varphi + \beta),
\end{aligned}$$

To both models we can also add the actuator dynamics of the torque command $T$ and the steering angle $\delta$. We choose to model them as first order systems, so that their dynamics read:

$$\dot{T} = \frac{1}{t_T}(u_T - T),$$
$$\dot{\delta} = \frac{1}{t_\delta}(u_\delta - \delta),$$

where for $a \in \{T, \delta\}$, $a$ is the actual value delivered by the actuator, $u_a$ is the actuator input, and $t_a$ is the actuator time constant.

## 2.1. Derivation

### 2.1.1. Initial intuition and assumptions

In the case of the kinematic bicycle model, the car has to be visualized as a system of 2 points corresponding to the front and rear axles, rigidly connected by a weightless rod. These points have masses $m\frac{l_F}{l}$ and $m\frac{l_R}{l}$ respectively, such that the total mass and the center of gravity coincides with the ones of the car.

Each one of these points is subject to a single longitudinal force $F_{\text{lon},F}, F_{\text{lon},R}$ coming from the motors.

### 2.1.2. Transport formula

Consider a rotating refrence frame $\{\vec{x}, \vec{y}, \vec{z}\}$ with an angular velocity $\vec{\omega}$ with respect to the inertial frame $\{\vec{x}_i, \vec{y}_i, \vec{z}_i\}$. By Poisson formula, we have $\frac{\partial \vec{x}}{\partial t} = \vec{\omega} \times \vec{x}$.

Further consider an arbitrary time-dependent vector $\vec{u}(t)$ that we can expressed as $\vec{u} = u_x\vec{x} + u_y\vec{y} + u_z\vec{z}$. If we apply the Poisson formula to $\vec{u}$ we get:

$$\begin{aligned}
\frac{\partial \vec{u}}{\partial t} &= \frac{\partial(u_x\vec{x})}{\partial t} + \frac{\partial\left(u_y\vec{y}\right)}{\partial t} + \frac{\partial(u_z\vec{z})}{\partial t} \\
&= \frac{\partial u_x}{\partial t}\vec{x} + \frac{\partial u_y}{\partial t}\vec{y} + \frac{\partial u_y}{\partial t}\vec{z} + u_z\frac{\partial \vec{z}}{\partial t} + u_y\frac{\partial \vec{y}}{\partial t} + u_z\frac{\partial \vec{z}}{\partial t} \\
&= \frac{\partial u_x}{\partial t}\vec{x} + \frac{\partial u_y}{\partial t}\vec{y} + \frac{\partial u_y}{\partial t}\vec{z} + \vec{\omega} \times \vec{u}
\end{aligned}$$

Note that in the previous formulae, all the vectors are to be understood as vectors in an abstract mathematical sense (i.e. member of a 3D euclidean space). They are not (yet)expressed in any particular reference frame.

### 2.1.3. Derivation of the yaw rate of $\vec{T}$ solely based on differential geometry

The path taken by the kinematic bicycle is given by $(X(t), Y(t))$.

$$\beta = \arctan\left(\frac{l_R}{l_R + l_F}\tan(\delta)\right) \approx C\delta,\ \ R = \frac{l_R}{\sin(\beta)}$$

$$\dot{\beta} = \frac{C(1 + \tan^2(\delta))\dot{\delta}}{1 + C^2\tan^2(\delta)} \approx C\dot{\delta}((1 + \tan^2(\delta))(1 - C^2\tan^2(\delta))) = C\dot{\delta}(1 + (1 - C^2)\tan^2(\delta) - C^2\tan^4(\delta))$$

### 2.1.4. derive curvature of path taken by a kinematic bicycle

### 2.1.5. derive $\dot{v}_x, \dot{v}_y$ using transport formula

### 2.1.6. on body frame derive $\dot{v}$ from that

### 2.1.7. second derivation of $\dot{v}$ directly by applying transport formula to T,N, which also gives us the lateral acceleration

By the transport formula we get:

$$\dot{v} = \dot{v}_x\vec{x} + \dot{v}_y\vec{y} + \dot{v}_z\vec{z} = v\cos(\beta)\cos(\varphi) - v\sin(\beta)\sin(\varphi)\vec{x} + v\cos(\beta)\sin(\varphi) + v\sin(\beta)\cos(\varphi)\vec{y}$$

### 2.1.8. Car yaw rate

$\varphi + \beta = \text{arctan2}\left(\dot{Y}, \dot{X}\right) \Rightarrow r = \dot{\varphi} = v\kappa - \dot{\beta}$

### 2.1.9. Position dynamics

$$\dot{X} = v_x \cos(\varphi) - v_y \sin(\varphi) = v \cos(\beta) \cos(\varphi) - v \sin(\beta) \sin(\varphi) = v \cos(\varphi + \beta)$$
$$\dot{Y} = v_x \sin(\varphi) + v_y \cos(\varphi) = v \cos(\beta) \sin(\varphi) + v \sin(\beta) \cos(\varphi) = v \sin(\varphi + \beta)$$

# 3. Dynamic models

The distinction between dynamic and *kinematic* and *dynamic* in this paper is not the usual one made in physics, where *kinematics* solely refer to the despcription of the motion of a system, and *dynamics* to the description of the forces and moments that explain this motion. In autonomous car models, the terms *kinematic* and *dynamic* are usually used to describe the degree of consideration of the various forces that act on the car. In the case of the kinematic models described in section Section 2, even if longitudinal forces appear, most of the equations are derived from geometric considerations. In the following subsections, all the equations will be entirely derived from Newton's laws of motion and the empirical modelization of the forces acting on the car.

Technically, the pure kinematic considerations (in the physical sense) would be the description of the

## 3.1. Dynamic bicycle model with 4 wheels (DYN6)

## 3.2. Dynamic model with 4 wheels (DYN6+)

## 3.3. Dynamic model with 4 wheels and their speeds (DYN10)