Stability properties of Relaxed Recentered log-barrier function based Nonlinear MPC

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1 Introduction

Given a controlled nonlinear dynamical system of the form $x^+ = f(x,u)$ with state and control constraints $x \in \mathcal{X} \subseteq \mathbb{R}^{n_x},\ u \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ and a fixed point x^* for these dynamics, i.e. a point such that $f(x^*,0) = x^*$, the problem of *stabilization* is to find a a feedback control law $\mu : \mathcal{X} \to \mathcal{U}$ such that x^* is asymptoticall stable for the dynamical system $x^+ = f(x, \mu(x))$. Such a problem is usually tackled using *optimal control*, and precisely *model predictive control* (MPC), an algorithm that defines this feedback control as the solution to an optimization problem of the following form :

$$
\begin{aligned}
V_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & J_N(\mathbf{x},\mathbf{u}) \\
\text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k,u_k),\ k = 0,\dots,N-1 \\
& x_k \in \mathcal{X},\ k = 0,\dots,N \\
& u_k \in \mathcal{U},\ k = 0,\dots,N-1
\end{aligned} \tag{1}
$$

Here N is called the horizon size, J_N is the cost function that is usually described with *stage costs* $l : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and *final cost* $F : \mathcal{X} \to \mathbb{R}$ by

$$
J_N(\mathbf{x},\mathbf{u}) = \sum_{k=0}^{N-1} l(x_k,u_k) \ + \ F(x_N)
$$

The system is then controlled by applying the first value of the optimal control sequence to the system.
In our work the follwoing assumptions will be made :

Assumption 1.1.

- The function f describing the dynamics is a general C^2 function, not necessarily linear.
- Without loss of generality, $x^* = 0$. We can always come back to this case by defining new translated states $\tilde{x} := x - x^*$ and new dynamics and new constraints accordingly by translation.
- $x^* = 0 \in \operatorname{int} \mathcal{X}$ and $0 \in \operatorname{int} \mathcal{U}$
- The stage costs and the final costs are quadratic : $l(x,u) = x^T Q x + u^T Q u,\ F(x) = x^T P x$ with Q, R and P positive definite matrices. This is very usual in MPC.

- The state and constraints sets are polytopic :

$$X = \{x \in \mathbb{R}^{n_x} \mid C_x x \leq d_x \text{ with } C_x \in \mathbb{R}^{q_x \times n_x} \text{ and } d_x \in \mathbb{R}^{q_x}\}$$
$$U = \{u \in \mathbb{R}^{n_u} \mid C_u u \leq d_u \text{ with } C_u \in \mathbb{R}^{q_u \times n_u} \text{ and } d_u \in \mathbb{R}^{q_u}\}$$

  Up to defining additional states and/or controls and modifying the dynamics accordingly, this can always be achieved.

The goal is usually to define the terminal cost $F(x)$ in such a way that the optimal value function V_N is a *Lyapunov function*, which would prove that $x^* = 0$ is asymptotically stable for the system controlled by the MPC. In some cases the authors also include *terminal constraints* on the last state x_N to ensure this stability, but here we are solely focusing on MPC with terminal costs and without terminal constraints.
Our work presents a new formulation that is based on this classical MPC framework and replaces the inequality constraints in the optimization problem (given by the state and control constraints) by some modified log-barrier functions added to the objective function. To properly introduce this new formulation let's introduce the central notion of *relaxed recentered log-barrier function.*

2 Statement of the new formulation

Definition 2.1. Given a constraint of the form $c^T x \leq d$, the associated *log-barrier function* is defined as $-\log(d - c^T x)$. Such a function is defined on the interior of feasible set of the constraint and becomes infinity near its boundary. For a set of polytopic constraints similar to the ones describes above, we can define the log-barrier for the state constraints as the sum of the log-barriers for each constraint :

$$B_x(x) = \sum_{i=1}^{q_x} -\log(d_{x,i} - \text{row}_i(C_x)x)$$

Definition 2.2. A *weight recentered log-barrier function* for a set of polytopic constraints similar to the ones described above is of the form :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) \left[\log(d_{x,i}) - \log(d_{x,i} - \text{row}_i(C_x)x)\right]$$

where the weights $w_{x,i}$ are defined as chosen such that $B_x(0) = 0$ and $\nabla B_x(0) = 0$.

Definition 2.3. A *relaxed recentered log-barrier function* (RRLB function) is defined by :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) B_{x,i}(x)$$
$$\text{with } B_{x,i}(x) = \begin{cases} \log(d_{x,1}) - \log(d_{x,1} - \text{row}_i(C_x)x) & \text{if } d_{x,i} - \text{row}_i(C_x)x > \delta \\ \beta(d_{x,1} - \text{row}_i(C_x)x; \delta) & \text{otherwise} \end{cases}$$

where $0 < \delta$ is a relaxation parameter and β is a function that twice continuously extends the log-barrier function on $(-\infty, \delta]$. The simplest example of such a function is

$$\beta(z; \delta) = \frac{1}{2}\left[\left(\frac{z - 2\delta}{\delta}\right)^2 - 1\right] - \log(\delta)$$

Lemma 2.4. *The RRLB functions are upper bounded by quadratic functions.*

Proof. The proof is similar to the one of the Lemma 3 of $\square$

Now we can finally define our new MPC formulation as follows :

$$
\begin{aligned}
\tilde{V}_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & \tilde{J}_N(\mathbf{x}, \mathbf{u}) \\
\text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k, u_k),\ k = 0, \dots, N-1
\end{aligned}
\tag{2}
$$

where the new objective function $\tilde{J}_N(\mathbf{x}, \mathbf{u}) = \sum_{k=0}^{N-1} \tilde{l}(x_k, u_k) + \tilde{F}(x_N)$ is defined using the new stage costs $\tilde{l}(x, u) = l(x, u) + \epsilon B_x(x) + \epsilon B_u(u)$ and the new terminal cost $\tilde{F}(x) = x^T P x$ for a certain matrix P that will be determined later. The barrier parameter ϵ has in theory the following interpretation : when it goes to zero, the solution of problem 2 converges to the one of 1.

3 Theoretical properties of RRLB Nonlinear MPC

3.1 Nominal asymptotic stability

Lemma 3.1. *Consider the problem 2 and re-write it in a simpler way as*

$$
\tilde{V}_N(x) = \min_{\mathbf{u}} \quad J(x, \mathbf{u})
$$

where $J(x, \mathbf{u}) = \tilde{l}(x_0, u_0) + \tilde{l}(f(x_0, u_0), u_1) + \dots + \tilde{F}(f(f(\dots), u_{N-1}))$ *. If for a certain value for the initial state x we denote by $\tilde{\mathbf{u}}(x) = (\tilde{u}_0(x), \dots, \tilde{u}_{N-1}(x))$ the optimal sequence of controls and we suppose that $D_{\mathbf{u}} J(x, \tilde{\mathbf{u}}) = 0$ and $\nabla^2_{\mathbf{uu}} J(x, \tilde{\mathbf{u}}) \succ 0$ (the matrix is positive definite) then :*

- $\forall k = 0, \dots, N-1, \quad \|\tilde{u}_k(x)\| = O(\|x\|)$
- $\forall k = 1, \dots, N, \quad \|\tilde{x}_k(x)\| = f(f(\dots, u_{k-2}), u_{k-1}) = O(\|x\|)$

Proof. See appendix □

Now the main piece :

Theorem 3.2. *Let's consider the problem 2 and assume the following :*

1. *the assumptions of previous lemma*
2. *When linearizing the system dynamics around the equilibrium and letting $A = D_x f(0, 0),\ B = D_u f(0, 0)$, we suppose that the pair (A, B) is stabilizable. This implies in particular that there exists a stabilizing cost K, i.e. a matrix such that $A_K := A + BK$ only has eigenvalues in the unit disk.*
3. *The matrix P defining the terminal costs is the unique positive definite solution to the following Lyapunov equation :*

$$
P = A_K^T P A_K + \mu Q_K
$$

where $\mu > 1$ and $Q_K = Q + \epsilon M_x + K^T (R + \epsilon M_u) K$.

Then if we use the same notations for the optimal controls and states as in the previous lemma, the origin is asymptotically stable for the dynamical system $x^+ = f(x, \tilde{u}_0(x))$ for all initial state in a certain neighborhood of the origin.

Remark. 1. The matrix K can be constructed using the Discrete Algebraic Riccati Equation for the infinite horizon LQR problem :

$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \quad & \sum_{k=0}^{\infty} x_k^T(Q+\epsilon M_x)x_k + u_k^T(R+\epsilon M_u)u_k \\ \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k,\ k=0,\dots,N-1 \end{aligned} \tag{3}$$

Proof. By writing the Taylor expansion of the stage costs $\tilde{l}$ we can see that

$$\begin{aligned} \tilde{l}(x,u) &= x^T[\nabla^2_{xx}\tilde{l}(0,0)]x + u^T[\nabla^2_{uu}\tilde{l}(0,0)]u + O(\|x\|^3+\|u\|^3) \\ &= x^T[Q+\epsilon M_x]x + u^T[R+\epsilon M_u]u + O(\|x\|^3+\|u\|^3) \end{aligned}$$

so $\tilde{l}(x,Kx) = x^TQ_Kx + O(\|x\|^3)$. By the second assumption, we also have $\forall x \in \mathbb{R}^{n_x}$:

$$\tilde{F}(A_Kx) + \mu x^TQ_Kx - \tilde{F}(x) = 0$$

We can use the same reasoning as in the paragraph 2.5.5 of [2, MPC Theory, Computation and Design] to show that $\tilde{F}$ is a local Lyapunov funnction, i.e. there exists a neighborhood of the origin such that $\forall x$ in it :

$$\begin{aligned} & \tilde{F}(f(x,Kx)) + x^TQ_Kx - \tilde{F}(x) \le 0 \\ \iff & \tilde{F}(f(x,Kx)) - \tilde{F}(x) + \tilde{l}(x,Kx) = O(\|x\|^3) \end{aligned}$$

What's more, by previous lemma, the predicted states are all Lipschitz with respect to the initial state, so we can choose an even smaller neighborhood such that $\tilde{x}_N(x)$ is also in it.

Now using the optimal sequence of states and controls of the problem 2 starting at x, we can construct new feasible sequences for the problem starting at $\tilde{x}_1(x)$ as $\mathbf{x}' = (\tilde{x}_1,\dots,\tilde{x}_N, f(\tilde{x}_N,K\tilde{x}_N))$ and $\mathbf{u}' = (\tilde{u}_1,\dots,\tilde{u}_{N-1},K\tilde{x}_N)$. Then we have :

$$\tilde{V}_N(\tilde{x}_1) \le \tilde{J}_N(\mathbf{x}',\mathbf{u}') = \underbrace{\tilde{J}_N(\tilde{\mathbf{x}},\tilde{\mathbf{u}})}_{=\tilde{V}_N(x)} \underbrace{\underbrace{-\tilde{l}(x,\tilde{u}_0)}_{=O(\|x\|^2)} + \underbrace{\tilde{F}(f(\tilde{x}_N,K\tilde{x}_N)) - \tilde{F}(\tilde{x}_N) + \tilde{l}(\tilde{x}_N,K\tilde{x}_N)}_{=O(\|\tilde{x}_N\|^3)=O(\|x\|^3)}}_{\le -c\|x\|^2 \text{ in a smaller nbh}} \checkmark$$

It is easy to show that $\tilde{V}_N$ is lower and upper bounded by coercive quadratic functions (see appendix), so that $\tilde{V}_N$ is a Lyapunov function in a small neighborhood around the origin. ✓ □

3.2 Constraints violation guarantees

This section follows closely the section IV.D of [1] but gives more loose results that cannot take us as far. In particular, will show that we can ensure the existence of a neighborhood around the origin such that if we start in it, the states and controls along the closed-loop simulation will never violate the constraints. However, this neighborhood cannot be easily computed because of the generality of the considered dynamics.

Lemma 3.3. *Consider problem 2 and an initial state $x(0)$ in the neighborhood given by 3.2. Let's denote by $\{x(0),x(1),\dots\}$ and $\{u(0),u(1),\dots\}$ the closed-loop state and control trajectories (given by $u(k)=\tilde{u}_0(x(k))$, $x(k+1)=f(x(k),u(k))$). Then $\forall k \ge 0$:*

$$\begin{aligned} B_x(x(k)) &\le \frac{1}{\epsilon}\left(\tilde{V}_N(x(0)) - x(0)^TP_{LQR}x(0) - \sum_{k=0}^{\infty}\eta(x(k))\right) \\ B_u(u(k)) &\le \frac{1}{\epsilon}\left(\tilde{V}_N(x(0)) - x(0)^TP_{LQR}x(0) - \sum_{k=0}^{\infty}\eta(x(k))\right) \end{aligned}$$

where P_{LQR} is the solution to the DARE associated 3 and $\eta(x) = \tilde{l}(\tilde{x}_N(x), K\tilde{x}_N(x)) + \tilde{F}(\tilde{x}_N(x)) - \tilde{F}(f(\tilde{x}_N(x), K\tilde{x}_N(x)))$.

Proof. In the proof of theorem 3.2 we showed that $\forall k \geq 0$:

$$\begin{aligned}\tilde{V}_N(x(k+1)) - \tilde{V}_N(x(k)) \leq -\tilde{l}(x(k), u(k)) + \tilde{l}(\tilde{x}_N(x(k)), K\tilde{x}_N(x(k))) \\ - \tilde{F}(\tilde{x}_N(x(k))) + \tilde{F}(f(\tilde{x}_N(x(k)), \tilde{x}_N(x(k))))\end{aligned}$$

so by summing everything we get a telescopic sum that we can compute using the fact that the system is asymptotically stable so $\lim_{k\to\infty} \tilde{V}_N(x(k)) = 0$, we get :

$$\begin{aligned}\tilde{V}_N(x(0)) &\geq \sum_{k=0}^{\infty} \tilde{l}(x(k), u(k)) - \underbrace{\tilde{l}(\tilde{x}_N(x(k)), K\tilde{x}_N(x(k))) + \tilde{F}(\tilde{x}_N(x(k))) - \tilde{F}(f(\tilde{x}_N(x(k)), \tilde{x}_N(x(k))))}_{\eta(x(k)) = O(\|x(0)\|^3)} \\ &= \sum_{k=0}^{\infty} l(x(k), u(k)) + \epsilon B_x(x(k)) + \epsilon B_u(u(k)) + \eta(x(k)) \\ &\geq x(0)^T P_{LQR} x(0) + \sum_{k=0}^{\infty} \eta(x(k)) + \epsilon \sum_{k=0}^{\infty} B_x(x(k)) + \epsilon \sum_{k=0}^{\infty} B_u(u(k))\end{aligned}$$

Note that even if we don't know a closed form for $\eta(x(k))$, we know that $\sum_{k=0}^{\infty} \eta(x(k))$ must be finite because it is bounded by $\tilde{V}_N(x(0))$. Now since the RRLB functions are all positive definite, we can easily conclude. □

We define for ease of notation the following quantities

$$\alpha(x(0)) = \tilde{V}_N(x(0)) - x(0)^T P_{LQR} x(0) - \sum_{k=0}^{\infty} \eta(x(k)) \tag{4}$$

$$\beta_x = \min_{i,x} \{B_x(x) \mid \mathrm{row}_i(C_x)x = d_{x,i}\} \tag{5}$$

$$\beta_u = \min_{i,u} \{B_u(u) \mid \mathrm{row}_i(C_u)u = d_{u,i}\} \tag{6}$$

Note that the bounds β_x, β_u depend on the relaxation parameter δ .

Lemma 3.4. *For all relaxation parameter δ:*

$$\{x \in \mathbb{R}^{n_x} \mid B_x(x) \leq \beta_x\} \subseteq \mathcal{X} \; \{u \in \mathbb{R}^{n_u} \mid B_u(u) \leq \beta_u\} \subseteq \mathcal{U}$$

Proof. See Lemma 1 in [1]. □

Theorem 3.5. *In the same setting as lemma 3.3, for any initial state $x(0)$ in the set*

$$\mathcal{X}_N(\delta) := \{x \in \mathbb{R}^{n_x} \mid \alpha(x(0)) \leq \epsilon \min\{\beta_x, \beta_u\}\}$$

there is no state or control constraint violation along the closed loop trajectories.

Proof. For any $x(0) \in \mathcal{X}_N(\delta)$ it holds due to 3.3 that for all $k \geq 0$, $\epsilon B_x(x(k)) \leq \alpha(x(0)) \leq \epsilon \beta_x$ so $B_x(x(k)) \leq \beta_x$ and $x(k) \in \mathcal{X}$. The same reasoning applies on the controls. □

4 Numerical extensions

4.1 Real Time Iteration methods

4.1.1 Asymptotic stability of RTI scheme

4.1.2 Numerical experiments

4.2 Parallelization

4.2.1 Numerical experiments

5 Appendix

References

[1] Christian Feller and Christian Ebenbauer. "Relaxed Logarithmic Barrier Function Based Model Predictive Control of Linear Systems". In: *IEEE Transactions on Automatic Control* 62.3 (2017), pp. 1223–1238. DOI: 10.1109/TAC.2016.2582040.

[2] J.B. Rawlings, D.Q. Mayne, and M. Diehl. *Model Predictive Control: Theory, Computation, and Design.* Nob Hill Publishing, 2017. ISBN: 9780975937730. URL: https://books.google.ch/books?id=MrJctAEACAAJ.

Proof of lemma 3.1. [1] □

Lemma 5.1 (Existence an Unicity of solution to the Lyapunov equation)**.**

Proof. □

Lemma 5.2 (Coercive quadratic lower and upper bounds for objective value function)**.**