Stability properties of Relaxed Recentered log-barrier function based Nonlinear MPC

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1 Introduction

Given a controlled nonlinear dynamical system of the form $x^+ = f(x, u)$ with state and control constraints $x \in \mathcal{X} \subseteq \mathbb{R}^{n_x},\ u \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ and a fixed point x^* for these dynamics, i.e. a point such that $f(x^*, 0) = x^*$, the problem of *stabilization* is to find a a feedback control law $\mu : \mathcal{X} \to \mathcal{U}$ such that x^* is asymptoticall stable for the dynamical system $x^+ = f(x, \mu(x))$. Such a problem is usually tackled using *optimal control*, and precisely *model predictive control* (MPC), an algorithm that defines this feedback control as the solution to an optimization problem of the following form :

$$
\begin{aligned}
V_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & J_N(\mathbf{x}, \mathbf{u}) \\
\text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k, u_k),\ k = 0, \dots, N-1 \\
& x_k \in \mathcal{X},\ k = 0, \dots, N \\
& u_k \in \mathcal{U},\ k = 0, \dots, N-1
\end{aligned}
\tag{1}
$$

Here N is called the horizon size, J_N is the cost function that is usually described with *stage costs* $l : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and *final cost* $F : \mathcal{X} \to \mathbb{R}$ by

$$
J_N(\mathbf{x}, \mathbf{u}) = \sum_{k=0}^{N-1} l(x_k, u_k) \ + \ F(x_N)
$$

The system is then controlled by applying the first value of the optimal control sequence to the system.
In our work the follwoing assumptions will be made :

Assumption 1.1.

- The function f describing the dynamics is a general C^2 function, not necessarily linear.
- Without loss of generality, $x^* = 0$. We can always come back to this case by defining new translated states $\tilde{x} := x - x^*$ and new dynamics and new constraints accordingly by translation.
- $x^* = 0 \in \operatorname{int} \mathcal{X}$ and $0 \in \operatorname{int} \mathcal{U}$
- The stage costs and the final costs are quadratic : $l(x, u) = x^T Q x + u^T Q u,\ F(x) = x^T P x$ with Q, R and P positive definite matrices. This is very usual in MPC.

- The state and constraints sets are polytopic :

$$X = \{x \in \mathbb{R}^{n_x} \mid C_x x \le d_x \text{ with } C_x \in \mathbb{R}^{q_x \times n_x} \text{ and } d_x \in \mathbb{R}^{q_x}\}$$
$$U = \{u \in \mathbb{R}^{n_u} \mid C_u u \le d_u \text{ with } C_u \in \mathbb{R}^{q_u \times n_u} \text{ and } d_u \in \mathbb{R}^{q_u}\}$$

  Up to defining additional states and/or controls and modifying the dynamics accordingly, this can always be achieved.

The goal is usually to define the terminal cost $F(x)$ in such a way that the optimal value function V_N is a *Lyapunov function*, which would prove that $x^* = 0$ is asymptotically stable for the system controlled by the MPC. In some cases the authors also include *terminal constraints* on the last state x_N to ensure this stability, but here we are solely focusing on MPC with terminal costs and without terminal constraints.
Our work presents a new formulation that is based on this classical MPC framework and replaces the inequality constraints in the optimization problem (given by the state and control constraints) by some modified log-barrier functions added to the objective function. To properly introduce this new formulation let's introduce the central notion of *relaxed recentered log-barrier function.*

2 Statement of the new formulation

Definition 2.1. Given a constraint of the form $c^T x \le d$, the associated *log-barrier function* is defined as $-\log(d - c^T x)$. Such a function is defined on the interior of feasible set of the constraint and becomes infinity near its boundary. For a set of polytopic constraints similar to the ones describes above, we can define the log-barrier for the state constraints as the sum of the log-barriers for each constraint :

$$B_x(x) = \sum_{i=1}^{q_x} -\log(d_{x,i} - \mathrm{row}_i(C_x)x)$$

Definition 2.2. A *weight recentered log-barrier function* for a set of polytopic constraints similar to the ones described above is of the form :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) \left[\log(d_{x,i}) - \log(d_{x,i} - \mathrm{row}_i(C_x)x)\right]$$

where the weights $w_{x,i}$ are defined as chosen such that $B_x(0) = 0$ and $\nabla B_x(0) = 0$.

Definition 2.3. A *relaxed recentered log-barrier function* (RRLB function) is defined by :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) B_{x,i}(x)$$
$$\text{with } B_{x,i}(x) = \begin{cases} \log(d_{x,1}) - \log(d_{x,1} - \mathrm{row}_i(C_x)x) & \text{if } d_{x,i} - \mathrm{row}_i(C_x)x > \delta \\ \beta(d_{x,1} - \mathrm{row}_i(C_x)x; \delta) & \text{otherwise} \end{cases}$$

where $0 < \delta$ is a relaxation parameter and β is a function that twice continuously extends the log-barrier function on $(-\infty, \delta]$. The simplest example of such a function is

$$\beta(z; \delta) = \frac{1}{2}\left[\left(\frac{z - 2\delta}{\delta}\right)^2 - 1\right] - \log(\delta)$$

Lemma 2.4. *The RRLB functions are upper bounded by quadratic functions.*

Proof. The proof is similar to the one of the Lemma 3 of [**RRLB-linear-MPC**]. □

Now we can finally define our new MPC formulation as follows :

$$\begin{aligned} \tilde{V}_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & \tilde{J}_N(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k,u_k),\ k = 0,\dots,N-1 \end{aligned} \tag{2}$$

where the new objective function $\tilde{J}_N(\mathbf{x},\mathbf{u}) = \sum_{k=0}^{N-1} \tilde{l}(x_k,u_k) + \tilde{F}(x_N)$ is defined using the new stage costs $\tilde{l}(x,u) = l(x,u) + \epsilon B_x(x) + \epsilon B_u(u)$ and the new terminal cost $\tilde{F}(x) = x^T P x$. The barrier parameter ϵ has in theory the following interpretation : when it goes to zero, the solution of problem 2 converges to the one of 1. The matrix P is chosen as the unique solution to the modified Riccati equation

$$P = A^T PA - A^T PB(R + \epsilon M_u + B^T PB)^{-1} B^T PA + Q + \epsilon M_x \tag{3}$$

3 Stability property of RRLB Nonlinear MPC

We are going to show the following stability result :

Theorem 3.1. *Let's denote by $\tilde{\mathbf{u}}(x) = (\tilde{u}_0(x),\dots,\tilde{u}_{N-1}(x))$ and $\tilde{\mathbf{x}}(x) = (\tilde{x}_0(x) = x, \tilde{x}_1(x),\dots,\tilde{x}_N(x))$ the sequences of optimal controls and states given by 2. Then the origin is asymptotically stable for the dynamical system $x^+ = f(x,\tilde{u}_0(x))$ for all initial state in a neighborhood of the origin.*

Proof.

- Step 1 : show that $\tilde{V}_\infty(x) = F(x) + O(\|x\|^3)$
  where $\tilde{V}_\infty$ is the infinite horizon version of 2 defined by :

$$\begin{aligned} \tilde{V}_\infty(x) = \min_{\mathbf{x},\mathbf{u}} \quad & \sum_{k=0}^{\infty} \tilde{l}(x_k,u_k) \\ \text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k,u_k),\ k \geq 0 \end{aligned}$$

  By **???** we know that $\tilde{V}_\infty \in \mathcal{C}^2$ and we can write its Taylor expansion :

$$\tilde{V}_\infty(x) = \tilde{V}_\infty(0) + \nabla \tilde{V}_\infty(0)^T x + x^T \left(\underbrace{\frac{1}{2}\nabla^2 \tilde{V}_\infty(0)}_{=:\tilde{P}} \right) x + O(\|x\|^3)$$

  It is easy to see that $\tilde{V}_\infty(0) = 0$ and $\nabla \tilde{V}_\infty(0)$, so if show that $\tilde{P} = P$ then we will have shown that $\tilde{V}_\infty(x) = F(x) + O(\|x\|^3)$.

  By the dynamic programming principle we have:

$$\tilde{V}_\infty(x) = \min_{u_0} \quad \tilde{l}(x,u_0) + \tilde{V}_\infty(f(x,u_0)) \tag{4}$$

  Let $u_0^*(x)$ be the minimizer of 4, which we are going to denote as u^* for simplicity. By **???** we know that $u^* = O(\|x\|)$. In our case we can show exactly that the minimizer is a stationnary point so :

$$
\begin{aligned}
0 &= \nabla_u \tilde{l}(x;u^*) + \nabla_u f(x,u^*)\nabla\tilde{V}_\infty(f(x,u^*)) \\
&= R^* + \epsilon\nabla B_u(u^*) + \nabla_u f(x,u^*)\nabla\tilde{V}_\infty(f(x,u^*)) \\
&= Ru^* + \epsilon\left(\underbrace{\nabla B_u(0)}_{=0} + \nabla^2 B_u(0)u^* + O(\|u^*\|^2)\right) \\
&+ (B^T + O(\|u^*\|^2))\left(\tilde{P}(Ax + Bu^* + O(\|x\|^2 + \|u^*\|^2)) + O\left(\left\|Ax + Bu^* + O(\|x\|^2 + \|u^*\|^2)\right\|^2\right)\right) \\
&= R^* + \epsilon M_u u^* + B^T\tilde{P}(Ax + Bu^*) + O(\|x\|^2)
\end{aligned}
$$

where we use that

$$
\begin{aligned}
&f(x,u) + Ax + Bu + O(\|x\|^2 + \|u\|^2) \implies \nabla_u f(x,u) = B^T + O(\|u\|) \\
&\tilde{V}_\infty(x) = x^T\tilde{P}x + O(\|x\|^3) \implies \nabla\tilde{V}_\infty(x) = \tilde{P}x + O(\|x\|^2)
\end{aligned}
$$

This implies that $u^* = \underbrace{-(R + \epsilon M_u + B^T\tilde{P}B)^{-1}}_{=:\tilde{K}} B^T\tilde{P}Ax + O(\|x\|^2)$. By plugging this expression back in the definition of $\tilde{V}_\infty$ given by 4 :

$$
\tilde{V}_\infty(x) = x^T(Q + \epsilon M_x + \tilde{K}^T(R + \epsilon M_u)\tilde{K} + A_{\tilde{K}}^T\tilde{P}A_{\tilde{K}})x + O(\|x\|^3)
$$

and by unicity of the Taylor expansion we conclude that $\tilde{P}$ verifies the equation 3. Since P was the unique solution to this equation, we conclude that $P = \tilde{P}$ as wanted. ✓

- Step 2 : show the descent property for $\tilde{J}_N$

  Using the dynamic programming principle for $\tilde{V}_\infty$, we know there exists a feedback control $\mu : \mathbb{R}^{n_x} \to \mathbb{R}^{n_u}$ (given by the solution to 4) such that :

$$
\tilde{V}_\infty(f(x,\mu(x))) - \tilde{V}_\infty(x) + \tilde{l}(x,\mu(x)) = 0
$$

  hence by the approximation shown at last step :

$$
\tilde{F}(f(x,\mu(x))) - \tilde{F}(x) + \tilde{l}(x,\mu(x)) = O(\|x\|^3)
$$

  Now given $\tilde{\mathbf{x}}(x)$ and $\tilde{\mathbf{u}}(x)$ we can construct some feasible sequences $\mathbf{x}' = (\tilde{x}_1, \ldots, \tilde{x}_N, f(\tilde{x}_N, \mu(\tilde{x}_N)))$ and $\mathbf{u}' = (\tilde{u}_1, \ldots, \tilde{u}_{N-1}, \mu(\tilde{x}_N))$ for the problem 2 with initial condition $\tilde{x}_1$. Then we have :

$$
\tilde{V}_N(\tilde{x}_1) \leq \tilde{J}_N(\mathbf{x}',\mathbf{u}') = \underbrace{\tilde{J}_N(\tilde{\mathbf{x}},\tilde{\mathbf{u}})}_{=\tilde{V}_N(x)} \underbrace{\underbrace{-\tilde{l}(x,\tilde{u}_0)}_{=O(\|x\|^2)} + \underbrace{\tilde{F}(f(\tilde{x}_N,\mu(\tilde{x}_N))) - \tilde{F}(\tilde{x}_N) + \tilde{l}(\tilde{x}_N,\mu(\tilde{x}_N))}_{=O(\|\tilde{x}_N\|^3)=O(\|x\|^3)}}_{\leq -c\|x\|^2 \text{ locally}} \checkmark
$$

- Step 3 : show the quadratic lower and upper bounds of $\tilde{V}_N$

  The upper bound comes from the fact that the costs l and F are quadratic and that the RRLB functions can also be quadraticallt upper bounded as shown in 2.4. Since these functions are also positive, we can lower bound $\tilde{J}_N$ by J_N and the quadratic lower bound is easy to find given that Q, R and P are positive definite matrices.

□

Appendix