# Study of Relaxed Recentered Log-Barrier function based Nonlinear Model Predictive Control

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**Abstract**

In this project, we investigate the use of relaxed logarithmic barrier functions in the context of nonlinear model predictive control. We base our work on the one of C. Feller and C. Ebenbauer in [1] and use one of their globally stabilizing schemes with terminal costs and without terminal sets. We partially extend the results on the nominal asymptotic stability of the corresponding closed-loop system and the constraint satisfaction guarantees to the case of nonlinear dynamics and show that they are satisfied only in a neighborhood of the target state. The theoretical results are complemented by numerical illustrations based on RTI and SQP algorithms.

## 1 Introduction

*Control theory* can simply be defined as the study of a generally dynamical system whose state evolution that we can influence by the means of external parameters called *controls*, or *inputs*. This theory is a very important one in numerous fields of engineering such as robotics, mechanical or chemical engineering where problems such as stabilizing a chemical reaction at a certain temperature or making an autonomous car follow the road are common.

An important subfield of control theory is *optimal control* (OC), which aims at finding a *control law*, i.e. a way to control a system in order to attain a certain goal, by formulating and solving an optimization problem. For example, if one wants to stabilize the state around a certain target, or reference point that is constant in time, one will want to choose a control law that will minimize the distance between the future state (that depends on the current state and the chosen control) and the target state. This problem of *stabilization* will be the one we will consider in this paper, but it is interesting to know there are other problems for which OC is applicable, such as path tracking or optimal trajectory generation.

One of the most important (family of) algorithm(s) of OC is *Model Predictive Control*, or MPC. This procedure uses the knowledge of the system *dynamic model* (the equation governing its evolution) to compute an control law that also takes the future into account. More precisely, the control law is computed for a certain interval in time called the *horizon*, and the cost along this whole horizon is minized. In some sense, the controller won't take decisions it will regret afterwards. MPC has many advantages over other classical control strategies, such as the ability to handle several control parameters and to take into consideration constraints on the states and the controls (by solving a *constrained* optimization problem). These are very important in practice because any real-life system has physical limitations (e.g. the maximum speed of a car, the maximum torque of a motor, etc.). We will properly introduce the mathematical formulation of MPC and the theoretical tools used to study it in section 2.

In section 3 we will introduce a different formulation in which (most of) the state and control constraints are replaced with a penalty added to the cost function. Such penalties are common in Interior Point methods and are usually log-barrier functions. These functions have several drawbacks, such as the fact that they are only defined in the interior of the feasible set. In this paper we will introduce a relaxed version of these functions called *Relaxed Recentered Log-Barrier functions*, or *RRLB functions*. They extend regular log-barrier functions to the whole space and yield a convex optimization problem that is in general much easier to solve than the original one. More details on the upsides of RRLB functions can be found in the Introduction of [1].

In the theoretical study of this new formulation that we will call *RRLB MPC*, we will prove two major results :

1. When using the control law computed by the RRLB MPC, the corresponding closed-loop system asymptotically stabilizes at the target state. (see subsection 4.1)
2. In this formulation, the optimal solution of the optimization problem might yield states or controls that are not feasible in the original problem. However, if we start sufficiently close to the target state, this will never happen. (see subsection 4.2)

Finally, in section 5 we examine numercial aspects of the RRLB MPC. In practice, the optimization problem is not solved exactly, and we can't necessarily guarantee the satisfaction of the previous results in practice. However, we will show that we can when using *Real Time Iteration* methods (RTI, that correspond to using Sequential Quadratic Programming with only one iteration), which is quite common in practical implementations when the problem has to be solved online and with limited computation power. All our results will in the end be illustrated with a classical benchmark problem for MPC schemes : the Continuously Stirred Tank Reactor (CSTR) system.

# 2 Background material

## 2.1 What is MPC ?

We are given a discrete-time controlled nonlinear dynamical system of the form $x(k+1) = f(x(k), u(k))$ (which we will usually denote $x^+ = f(x, u)$) with state and control constraints $x(k) \in \mathcal{X} \subseteq \mathbb{R}^{n_x},\ u(k) \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$. Our goal is to stabilize the system around a target state $x^*$ and a target control $u^*$, which have to be a fixed point of the system : $x^* = f(x^*, u^*)$. To do that we will try to construct a *state-feedback control law*, i.e. a function $\mu : \mathcal{X} \to \mathcal{U}$ such that $x^*$ is *asymptotically stable* for the dynamical system $x^+ = f(x, \mu(x))$ (which only depends on the state). The exact definition of asymptotica stability and how to prove it is discussed in subsection 2.2.

Such a problem is usually tackled using *optimal control*, and precisely *model predictive control* (MPC), an algorithm that defines this feedback control as the solution to an optimization problem of the following form :

$$\begin{aligned} V_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & J_N(\mathbf{x}, \mathbf{u}) \\ \text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k, u_k),\ k = 0, \dots, N-1 \\ & x_k \in \mathcal{X},\ k = 0, \dots, N \\ & u_k \in \mathcal{U},\ k = 0, \dots, N-1 \end{aligned} \tag{1}$$

Here $N$ is called the horizon size, $J_N$ is the cost function that is usually described with *stage costs* $l : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and *final cost* $F : \mathcal{X} \to \mathbb{R}$ by

$$J_N(\mathbf{x}, \mathbf{u}) = \sum_{k=0}^{N-1} l(x_k, u_k) \ + \ F(x_N)$$

The system is then controlled by applying the first value of the optimal control sequence to the system.
In our work the follwoing assumptions will be made :

**Assumption 2.1.**

- The function $f$ describing the dynamics is a general $C^2$ function, not necessarily linear.
- Without loss of generality, $x^* = 0$. We can always come back to this case by defining new translated states $\tilde{x} := x - x^*$ and new dynamics and new constraints accordingly by translation.
- $x^* = 0 \in \operatorname{int} \mathcal{X}$ and $0 \in \operatorname{int} \mathcal{U}$
- The stage costs and the final costs are quadratic : $l(x, u) = x^T Q x + u^T Q u,\ F(x) = x^T P x$ with $Q, R$ and $P$ positive definite matrices. This is very usual in MPC.
- The state and constraints sets are polytopic :

$$X = \left\{ x \in \mathbb{R}^{n_x} \mid C_x x \leq d_x \text{ with } C_x \in \mathbb{R}^{q_x \times n_x} \text{ and } d_x \in \mathbb{R}^{q_x} \right\}$$
$$U = \left\{ u \in \mathbb{R}^{n_u} \mid C_u u \leq d_u \text{ with } C_u \in \mathbb{R}^{q_u \times n_u} \text{ and } d_u \in \mathbb{R}^{q_u} \right\}$$

  Up to defining additional states and/or controls and modifying the dynamics accordingly, this can always be achieved.

The goal is usually to define the terminal cost $F(x)$ in such a way that the optimal value function $V_N$ is a *Lyapunov function*, which would prove that $x^* = 0$ is asymptotically stable for the system controlled by the MPC. In some cases the authors also include *terminal constraints* on the last state $x_N$ to ensure this stability, but here we are solely focusing on MPC with terminal costs and without terminal constraints.
Our work presents a new formulation that is based on this classical MPC framework and replaces the inequality constraints in the optimization problem (given by the state and control constraints) by some modified log-barrier functions added to the objective function. To properly introduce this new formulation let's introduce the central notion of *relaxed recentered log-barrier function.*

## 2.2 Lyapunov stability theory

# 3 The RRLB MPC

**Definition 3.1.** Given a constraint of the form $c^T x \leq d$, the associated *log-barrier function* is defined as $-\log(d - c^T x)$. Such a function is defined on the interior of feasible set of the constraint and becomes infinity near its boundary. For a set of polytopic constraints similar to the ones describes above, we can define the log-barrier for the state constraints as the sum of the log-barriers for each constraint :

$$B_x(x) = \sum_{i=1}^{q_x} - \log(d_{x,i} - \operatorname{row}_i(C_x) x)$$

**Definition 3.2.** A *weight recentered log-barrier function* for a set of polytopic constraints similar to the ones described above is of the form :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) \left[ \log(d_{x,i}) - \log(d_{x,i} - \operatorname{row}_i(C_x) x) \right]$$

where the weights $w_{x,i}$ are defined as chosen such that $B_x(0) = 0$ and $\nabla B_x(0) = 0$.

**Definition 3.3.** A *relaxed recentered log-barrier function* (RRLB function) is defined by :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) B_{x,i}(x)$$

$$\text{with } B_{x,i}(x) = \begin{cases} \log(d_{x,1}) - \log(d_{x,1} - \mathrm{row}_i(C_x)x) & \text{if } d_{x,i} - \mathrm{row}_i(C_x)x > \delta \\ \beta(d_{x,1} - \mathrm{row}_i(C_x)x; \delta) & \text{otherwise} \end{cases}$$

where $0 < \delta$ is a relaxation parameter and $\beta$ is a function that twice continuously extends the log-barrier function on $(-\infty, \delta]$. The simplest example of such a function is

$$\beta(z; \delta) = \frac{1}{2}\left[\left(\frac{z - 2\delta}{\delta}\right)^2 - 1\right] - \log(\delta)$$

**Lemma 3.4.** *The RRLB functions are upper bounded by quadratic functions.*

*Proof.* The proof is similar to the one of the Lemma 3 of □

Now we can finally define our new MPC formulation as follows :

$$\begin{aligned} \tilde{V}_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & \tilde{J}_N(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k, u_k),\ k = 0, \ldots, N-1 \end{aligned} \tag{2}$$

where the new objective function $\tilde{J}_N(\mathbf{x},\mathbf{u}) = \sum_{k=0}^{N-1} \tilde{l}(x_k, u_k) + \tilde{F}(x_N)$ is defined using the new stage costs $\tilde{l}(x,u) = l(x,u) + \epsilon B_x(x) + \epsilon B_u(u)$ and the new terminal cost $\tilde{F}(x) = x^T P x$ for a certain matrix $P$ that will be determined later. The barrier parameter $\epsilon$ has in theory the following interpretation : when it goes to zero, the solution of problem 2 converges to the one of 1.

# 4 Theoretical properties of RRLB Nonlinear MPC

## 4.1 Nominal asymptotic stability

**Lemma 4.1.** *Consider the problem 2 and re-write it in a simpler way as*

$$\tilde{V}_N(x) = \min_{\mathbf{u}} \quad J(x,\mathbf{u})$$

*where* $J(x,\mathbf{u}) = \tilde{l}(x_0,u_0) + \tilde{l}(f(x_0,u_0),u_1) + \cdots + \tilde{F}(f(f(...),u_{N-1}))$ *. If for a certain value for the initial state* $x$ *we denote by* $\tilde{\mathbf{u}}(x) = (\tilde{u}_0(x), \ldots, \tilde{u}_{N-1}(x))$ *the optimal sequence of controls and we suppose that* $D_{\mathbf{u}} J(x,\tilde{\mathbf{u}}) = 0$ *and* $\nabla^2_{\mathbf{uu}} J(x,\tilde{\mathbf{u}}) \succ 0$ *(the matrix is positive definite) then :*

- $\forall k = 0, \ldots, N-1, \quad \|\tilde{u}_k(x)\| = O(\|x\|)$
- $\forall k = 1, \ldots, N, \quad \|\tilde{x}_k(x)\| = f(f(\ldots, u_{k-2}), u_{k-1}) = O(\|x\|)$

*Proof.* See appendix □

Now the main piece :

**Theorem 4.2.** *Let's consider the problem 2 and assume the following :*

1. *the assumptions of previous lemma*
2. *When linearizing the system dynamics around the equilibrium and letting* $A = D_x f(0,0),\ B = D_u f(0,0)$*, we suppose that the pair* $(A,B)$ *is stabilizable. This implies in particular that there exists a stabilizing cost* $K$*, i.e. a matrix such that* $A_K := A + BK$ *only has eigenvalues in the unit disk.*
3. *The matrix* $P$ *defining the terminal costs is the unique positive definite solution to the following Lyapunov equation :*

$$P = A_K^T P A_K + \mu Q_K$$

*where* $\mu > 1$ *and* $Q_K = Q + \epsilon M_x + K^T(R + \epsilon M_u)K$ *.*

*Then if we use the same notations for the optimal controls and states as in the previous lemma, the origin is asymptotically stable for the dynamical system* $x^+ = f(x, \tilde{u}_0(x))$ *for all initial state in a certain neighborhood of the origin.*

*Remark.* 1. The matrix $K$ can be constructed using the Discrete Algebraic Riccati Equation for the infinite horizon LQR problem :

$$\begin{aligned} \min_{\mathbf{x},\mathbf{u}} \quad & \sum_{k=0}^{\infty} x_k^T(Q+\epsilon M_x)x_k + u_k^T(R+\epsilon M_u)u_k \\ \text{s.t.} \quad & x_{k+1} = Ax_k + Bu_k,\ k=0,\dots,N-1 \end{aligned} \tag{3}$$

*Proof.* By writing the Taylor expansion of the stage costs $\tilde{l}$ we can see that

$$\begin{aligned} \tilde{l}(x,u) &= x^T[\nabla^2_{xx}\tilde{l}(0,0)]x + u^T[\nabla^2_{uu}\tilde{l}(0,0)]u + O(\|x\|^3+\|u\|^3) \\ &= x^T[Q+\epsilon M_x]x + u^T[R+\epsilon M_u]u + O(\|x\|^3+\|u\|^3) \end{aligned}$$

so $\tilde{l}(x,Kx) = x^TQ_Kx + O(\|x\|^3)$ . By the second assumption, we also have $\forall x \in \mathbb{R}^{n_x}$:

$$\tilde{F}(A_Kx) + \mu x^TQ_Kx - \tilde{F}(x) = 0$$

We can use the same reasoning as in the paragraph 2.5.5 of [2, MPC Theory, Computation and Design] to show that $\tilde{F}$ is a local Lyapunov funnction, i.e. there exists a neighborhood of the origin such that $\forall x$ in it :

$$\begin{aligned} &\tilde{F}(f(x,Kx)) + x^TQ_Kx - \tilde{F}(x) \leq 0 \\ \iff &\tilde{F}(f(x,Kx)) - \tilde{F}(x) + \tilde{l}(x,Kx) = O(\|x\|^3) \end{aligned}$$

What's more, by previous lemma, the predicted states are all Lipschitz with respect to the initial state, so we can choose an even smaller neighborhood such that $\tilde{x}_N(x)$ is also in it.

Now using the optimal sequence of states and controls of the problem 2 starting at $x$, we can construct new feasible sequences for the problem starting at $\tilde{x}_1(x)$ as $\mathbf{x}' = (\tilde{x}_1,\dots,\tilde{x}_N, f(\tilde{x}_N, K\tilde{x}_N))$ and $\mathbf{u}' = (\tilde{u}_1,\dots,\tilde{u}_{N-1}, K\tilde{x}_N)$ . Then we have :

$$\tilde{V}_N(\tilde{x}_1) \leq \tilde{J}_N(\mathbf{x}',\mathbf{u}') = \underbrace{\tilde{J}_N(\tilde{\mathbf{x}},\tilde{\mathbf{u}})}_{=\tilde{V}_N(x)} \underbrace{\underbrace{-\tilde{l}(x,\tilde{u}_0)}_{=O(\|x\|^2)} + \underbrace{\tilde{F}(f(\tilde{x}_N,K\tilde{x}_N)) - \tilde{F}(\tilde{x}_N) + \tilde{l}(\tilde{x}_N,K\tilde{x}_N)}_{=O(\|\tilde{x}_N\|^3)=O(\|x\|^3)}}_{\leq -c\|x\|^2 \text{ in a smaller nbh}} \checkmark$$

It is easy to show that $\tilde{V}_N$ is lower and upper bounded by coercive quadratic functions (see appendix), so that $\tilde{V}_N$ is a Lyapunov function in a small neighborhood around the origin. ✓ □

## 4.2 Constraints satisfaction guarantees

This section follows closely the section IV.D of [1] but gives more loose results that cannot take us as far. In particular, will show that we can ensure the existence of a neighborhood around the origin such that if we start in it, the states and controls along the closed-loop simulation will never violate the constraints. However, this neighborhood cannot be easily computed because of the generality of the considered dynamics.

**Lemma 4.3.** *Consider problem 2 and an initial state $x(0)$ in the neighborhood given by 4.2. Let's denote by $\{x(0), x(1), \dots\}$ and $\{u(0), u(1), \dots\}$ the closed-loop state and control trajectories (given by $u(k) = \tilde{u}_0(x(k))$, $x(k+1) = f(x(k), u(k))$). Then $\forall k \geq 0$:*

$$\begin{aligned} B_x(x(k)) &\leq \frac{1}{\epsilon}\left(\tilde{V}_N(x(0)) - x(0)^TP_{LQR}x(0) - \sum_{k=0}^{\infty}\eta(x(k))\right) \\ B_u(u(k)) &\leq \frac{1}{\epsilon}\left(\tilde{V}_N(x(0)) - x(0)^TP_{LQR}x(0) - \sum_{k=0}^{\infty}\eta(x(k))\right) \end{aligned}$$

*where $P_{LQR}$ is the solution to the DARE associated 3 and $\eta(x) = \tilde{l}(\tilde{x}_N(x), K\tilde{x}_N(x)) + \tilde{F}(\tilde{x}_N(x)) - \tilde{F}(f(\tilde{x}_N(x), K\tilde{x}_N(x)))$.*

*Proof.* In the proof of theorem 4.2 we showed that $\forall k \geq 0$:

$$\begin{aligned}\tilde{V}_N(x(k+1)) - \tilde{V}_N(x(k)) \leq -\tilde{l}(x(k), u(k)) + \tilde{l}(\tilde{x}_N(x(k)), K\tilde{x}_N(x(k))) \\ - \tilde{F}(\tilde{x}_N(x(k))) + \tilde{F}(f(\tilde{x}_N(x(k)), \tilde{x}_N(x(k))))\end{aligned}$$

so by summing everything we get a telescopic sum that we can compute using the fact that the system is asymptotically stable so $\lim_{k\to\infty} \tilde{V}_N(x(k)) = 0$, we get :

$$\begin{aligned}\tilde{V}_N(x(0)) &\geq \sum_{k=0}^{\infty} \tilde{l}(x(k), u(k)) - \underbrace{\tilde{l}(\tilde{x}_N(x(k)), K\tilde{x}_N(x(k))) + \tilde{F}(\tilde{x}_N(x(k))) - \tilde{F}(f(\tilde{x}_N(x(k)), \tilde{x}_N(x(k))))}_{\eta(x(k))=O(\|x(0)\|^3)} \\ &= \sum_{k=0}^{\infty} l(x(k), u(k)) + \epsilon B_x(x(k)) + \epsilon B_u(u(k)) + \eta(x(k)) \\ &\geq x(0)^T P_{LQR} x(0) + \sum_{k=0}^{\infty} \eta(x(k)) + \epsilon \sum_{k=0}^{\infty} B_x(x(k)) + \epsilon \sum_{k=0}^{\infty} B_u(u(k))\end{aligned}$$

Note that even if we don't know a closed form for $\eta(x(k))$, we know that $\sum_{k=0}^{\infty} \eta(x(k))$ must be finite because it is bounded by $\tilde{V}_N(x(0))$. Now since the RRLB functions are all positive definite, we can easily conclude. □

We define for ease of notation the following quantities

$$\alpha(x(0)) = \tilde{V}_N(x(0)) - x(0)^T P_{LQR} x(0) - \sum_{k=0}^{\infty} \eta(x(k)) \tag{4}$$

$$\beta_x = \min_{i,x} \{B_x(x) \mid \mathrm{row}_i(C_x)x = d_{x,i}\} \tag{5}$$

$$\beta_u = \min_{i,u} \{B_u(u) \mid \mathrm{row}_i(C_u)u = d_{u,i}\} \tag{6}$$

Note that the bounds $\beta_x, \beta_u$ depend on the relaxation parameter $\delta$.

**Lemma 4.4.** *For all relaxation parameter $\delta$:*

$$\{x \in \mathbb{R}^{n_x} \mid B_x(x) \leq \beta_x\} \subseteq \mathcal{X} \; \{u \in \mathbb{R}^{n_u} \mid B_u(u) \leq \beta_u\} \subseteq \mathcal{U}$$

*Proof.* See Lemma 1 in [1]. □

**Theorem 4.5.** *In the same setting as lemma 4.3, for any initial state $x(0)$ in the set*

$$\mathcal{X}_N(\delta) := \{x \in \mathbb{R}^{n_x} \mid \alpha(x(0)) \leq \epsilon \min\{\beta_x, \beta_u\}\}$$

*there is no state or control constraint violation along the closed loop trajectories.*

*Proof.* For any $x(0) \in \mathcal{X}_N(\delta)$ it holds due to 4.3 that for all $k \geq 0$, $\epsilon B_x(x(k)) \leq \alpha(x(0)) \leq \epsilon\beta_x$ so $B_x(x(k)) \leq \beta_x$ and $x(k) \in \mathcal{X}$. The same reasoning applies on the controls. □

# 5 Numerical extensions

## 5.1 Real Time Iteration methods

### 5.1.1 Asymptotic stability of RTI scheme

### 5.1.2 Numerical experiments

## 5.2 Parallelization

### 5.2.1 Numerical experiments

# 6 Appendix

# References

[1] Christian Feller and Christian Ebenbauer. "Relaxed Logarithmic Barrier Function Based Model Predictive Control of Linear Systems". In: *IEEE Transactions on Automatic Control* 62.3 (2017), pp. 1223–1238. DOI: `10.1109/TAC.2016.2582040`.

[2] J.B. Rawlings, D.Q. Mayne, and M. Diehl. *Model Predictive Control: Theory, Computation, and Design*. Nob Hill Publishing, 2017. ISBN: 9780975937730. URL: `https://books.google.ch/books?id=MrJctAEACAAJ`.

*Proof of lemma 4.1.* [1] □

**Lemma 6.1** (Existence an Unicity of solution to the Lyapunov equation).

*Proof.* □

**Lemma 6.2** (Coercive quadratic lower and upper bounds for objective value function).