Stability properties of Relaxed Recentered log-barrier function based Nonlinear MPC

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1 Introduction

Given a controlled nonlinear dynamical system of the form $x^+ = f(x,u)$ with state and control constraints $x \in \mathcal{X} \subseteq \mathbb{R}^{n_x},\ u \in \mathcal{U} \subseteq \mathbb{R}^{n_u}$ and a fixed point x^* for these dynamics, i.e. a point such that $f(x^*,0) = x^*$, the problem of *stabilization* is to find a a feedback control law $\mu : \mathcal{X} \to \mathcal{U}$ such that x^* is asymptoticall stable for the dynamical system $x^+ = f(x,\mu(x))$. Such a problem is usually tackled using *optimal control*, and precisely *model predictive control* (MPC), an algorithm that defines this feedback control as the solution to an optimization problem of the following form :

$$
\begin{aligned}
V_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & J_N(\mathbf{x},\mathbf{u}) \\
\text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k,u_k),\ k = 0,\dots,N-1 \\
& x_k \in \mathcal{X},\ k = 0,\dots,N \\
& u_k \in \mathcal{U},\ k = 0,\dots,N-1
\end{aligned} \tag{1}
$$

Here N is called the horizon size, J_N is the cost function that is usually described with *stage costs* $l : \mathcal{X} \times \mathcal{U} \to \mathbb{R}$ and *final cost* $F : \mathcal{X} \to \mathbb{R}$ by

$$
J_N(\mathbf{x},\mathbf{u}) = \sum_{k=0}^{N-1} l(x_k,u_k) \ + \ F(x_N)
$$

The system is then controlled by applying the first value of the optimal control sequence to the system.
In our work the follwoing assumptions will be made :

Assumption 1.1.

- The function f describing the dynamics is a general C^2 function, not necessarily linear.
- Without loss of generality, $x^* = 0$. We can always come back to this case by defining new translated states $\tilde{x} := x - x^*$ and new dynamics and new constraints accordingly by translation.
- $x^* = 0 \in \operatorname{int} \mathcal{X}$ and $0 \in \operatorname{int} \mathcal{U}$
- The stage costs and the final costs are quadratic : $l(x,u) = x^T Q x + u^T Q u,\ F(x) = x^T P x$ with Q, R and P positive definite matrices. This is very usual in MPC.

- The state and constraints sets are polytopic :

$$X = \{x \in \mathbb{R}^{n_x} \mid C_x x \leq d_x \text{ with } C_x \in \mathbb{R}^{q_x \times n_x} \text{ and } d_x \in \mathbb{R}^{q_x}\}$$
$$U = \{u \in \mathbb{R}^{n_u} \mid C_u u \leq d_u \text{ with } C_u \in \mathbb{R}^{q_u \times n_u} \text{ and } d_u \in \mathbb{R}^{q_u}\}$$

  Up to defining additional states and/or controls and modifying the dynamics accordingly, this can always be achieved.

The goal is usually to define the terminal cost $F(x)$ in such a way that the optimal value function V_N is a *Lyapunov function*, which would prove that $x^* = 0$ is asymptotically stable for the system controlled by the MPC. In some cases the authors also include *terminal constraints* on the last state x_N to ensure this stability, but here we are solely focusing on MPC with terminal costs and without terminal constraints.
Our work presents a new formulation that is based on this classical MPC framework and replaces the inequality constraints in the optimization problem (given by the state and control constraints) by some modified log-barrier functions added to the objective function. To properly introduce this new formulation let's introduce the central notion of *relaxed recentered log-barrier function.*

2 Statement of the new formulation

Definition 2.1. Given a constraint of the form $c^T x \leq d$, the associated *log-barrier function* is defined as $-\log(d - c^T x)$. Such a function is defined on the interior of feasible set of the constraint and becomes infinity near its boundary. For a set of polytopic constraints similar to the ones describes above, we can define the log-barrier for the state constraints as the sum of the log-barriers for each constraint :

$$B_x(x) = \sum_{i=1}^{q_x} -\log(d_{x,i} - \text{row}_i(C_x)x)$$

Definition 2.2. A *weight recentered log-barrier function* for a set of polytopic constraints similar to the ones described above is of the form :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) \left[\log(d_{x,i}) - \log(d_{x,i} - \text{row}_i(C_x)x)\right]$$

where the weights $w_{x,i}$ are defined as chosen such that $B_x(0) = 0$ and $\nabla B_x(0) = 0$.

Definition 2.3. A *relaxed recentered log-barrier function* (RRLB function) is defined by :

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) B_{x,i}(x)$$
$$\text{with } B_{x,i}(x) = \begin{cases} \log(d_{x,1}) - \log(d_{x,1} - \text{row}_i(C_x)x) & \text{if } d_{x,i} - \text{row}_i(C_x)x > \delta \\ \beta(d_{x,1} - \text{row}_i(C_x)x; \delta) & \text{otherwise} \end{cases}$$

where $0 < \delta$ is a relaxation parameter and β is a function that twice continuously extends the log-barrier function on $(-\infty, \delta]$. The simplest example of such a function is

$$\beta(z; \delta) = \frac{1}{2}\left[\left(\frac{z - 2\delta}{\delta}\right)^2 - 1\right] - \log(\delta)$$

Lemma 2.4. *The RRLB functions are upper bounded by quadratic functions.*

Proof. The proof is similar to the one of the Lemma 3 of □

Now we can finally define our new MPC formulation as follows :

$$\begin{aligned} \tilde{V}_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & \tilde{J}_N(\mathbf{x},\mathbf{u}) \\ \text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k, u_k),\ k = 0, \dots, N-1 \end{aligned} \tag{2}$$

where the new objective function $\tilde{J}_N(\mathbf{x},\mathbf{u}) = \sum_{k=0}^{N-1} \tilde{l}(x_k, u_k) + \tilde{F}(x_N)$ is defined using the new stage costs $\tilde{l}(x,u) = l(x,u) + \epsilon B_x(x) + \epsilon B_u(u)$ and the new terminal cost $\tilde{F}(x) = x^T P x$ for a certain matrix P that will be determined later. The barrier parameter ϵ has in theory the following interpretation : when it goes to zero, the solution of problem 2 converges to the one of 1.

3 Theoretical properties of RRLB Nonlinear MPC

3.1 Nominal asymptotic stability

Lemma 3.1. *Consider the problem 2 and re-write it in a simpler way as*

$$\begin{aligned} \tilde{V}_N(x) = \min_{\mathbf{u}} \quad & J(x,\mathbf{u}) \\ s.t. \quad & \end{aligned}$$

where $J(x,\mathbf{u}) = \tilde{l}(x_0,u_0) + \tilde{l}(f(x_0,u_0),u_1) + \dots + \tilde{F}(f(f(...),u_{N-1}))$ *. If for a certain value for the initial state* x *we denote by* $\tilde{\mathbf{u}}(x) = (\tilde{u}_0(x), \dots, \tilde{u}_{N-1}(x))$ *the optimal sequence of controls and we suppose that* $D_{\mathbf{u}}J(x,\tilde{\mathbf{u}}) = 0$ *and* $\nabla^2_{\mathbf{uu}}J(x,\tilde{\mathbf{u}}) \succ 0$ *(the matrix is positive definite) then :*

- $\forall k = 0, \dots, N-1, \quad \|\tilde{u}_k(x)\| = O(\|x\|)$
- $\forall k = 1, \dots, N, \quad \|\tilde{x}_k(x)\| = f(f(\dots, u_{k-2}), u_{k-1}) = O(\|x\|)$

Proof. See appendix □

Now the main piece :

Theorem 3.2. *Let's consider the problem 2 and assume the following :*

1. *When linearizing the system dynamics around the equilibrium and letting* $A = D_x f(0,0),\ B = D_u f(0,0)$*, we suppose that the pair* (A,B) *is stabilizable. This implies in particular that there exists a stabilizing cost* K*, i.e. a matrix such that* $A_K := A + BK$ *only has eigenvalues in the unit disk.*

2. *The matrix* P *defining the terminal costs is the unique positive definite solution to the following Lyapunov equation :*

$$P = A_K^T P A_K + \mu Q_K \tag{3}$$

where $\mu > 1$ *and* $Q_K = Q + \epsilon M_x + K^T(R + \epsilon M_u)K$ *.*

3. *The matrix* P *defining the terminal costs is the unique positive definite solution to the following modified Discrete Alegbraic Riccati Equation (DARE) :*

$$P = A^T P A - A^T P B(\underbrace{R + \epsilon M_u}_{=:\tilde{R}} + B^T P B)^{-1} B^T P A + \underbrace{Q + \epsilon M_x}_{=:\tilde{Q}} \tag{4}$$

where $M_x := \nabla^2_{xx} B_x(0), M_u := \nabla^2_{uu} B_u(0)$*. This equation can be re-written as :*

$$\begin{cases} P = (\underbrace{A + BK}_{=:A_K})^T P (\underbrace{A + BK}_{=:A_K}) + \underbrace{\tilde{Q} + K^T \tilde{R} K}_{=:\tilde{Q}_K} \\ K = -(\tilde{R} + B^T P B)^{-1} B^T P A \end{cases} \tag{5}$$

then if we use the same notations for the optimal controls and states as in the previous lemma, the origin is asymptotically stable for the dynamical system $x^+ = f(x, \tilde{u}_0(x))$ for all initial state in a certain neighborhood of the origin.

Remark. The proof of the existence and unicity of such a matrix P is based on results on the Continuous Algebraic Riccati equation and the proof can be found in the appendix.

Proof. Let's define the matrices $\hat{Q} = \nabla^2_{xx}\tilde{l}(0,0) = Q + \epsilon M_x, \hat{R} = \nabla^2_{uu}\tilde{l}(0,0) = R + \epsilon M_u$ and $\hat{l}(x,u) = x^T\hat{Q}x + u^T\hat{R}u = \tilde{l}(x,u) + O(\|x\|^3 + \|u\|^3)$. With this definitions we have $\tilde{l}(x, Kx) = x^T Q_K x + O(\|x\|^3)$.

By the second assumption, we have

$$\tilde{F}(A_K x) + \mu x^T Q_K x - \tilde{F}(x) = 0, \ \forall x \in \mathbb{R}^{n_x}$$

We can use the same reasoning as in the book to show that $\tilde{F}$ is a local Lyapunov funnction, i.e. there exists a neighborhood V of the origin such that $\forall x \in V$:

$$\begin{gathered}\tilde{F}(f(x, Kx)) + x^T Q_K x - \tilde{F}(x) \leq 0 \\ \iff \tilde{F}(f(x, Kx)) - \tilde{F}(x) + \tilde{l}(x, Kx) = O(\|x\|^3)\end{gathered}$$

Then if the initial state in problem 2 is in V **(and this set is forward invariant)**, every state $\tilde{x}_k(x)$ will be in V too.

Now using the optimal sequence of states and controls of the problem 2 starting at x, we can construct a new feasible sequence for the problem starting at $\tilde{x}_1(x)$ as $\mathbf{x}' = (\tilde{x}_1, \dots, \tilde{x}_N, f(\tilde{x}_N, K\tilde{x}_N))$ and $\mathbf{u}' = (\tilde{u}_1, \dots, \tilde{u}_{N-1}, K\tilde{x}_N)$. Then we have :

$$\tilde{V}_N(\tilde{x}_1) \leq \tilde{J}_N(\mathbf{x}', \mathbf{u}') = \underbrace{\tilde{J}_N(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})}_{=\tilde{V}_N(x)} \underbrace{\underbrace{-\tilde{l}(x, \tilde{u}_0)}_{=O(\|x\|^2)} + \underbrace{\tilde{F}(f(\tilde{x}_N, K\tilde{x}_N)) - \tilde{F}(\tilde{x}_N) + \tilde{l}(\tilde{x}_N, K\tilde{x}_N)}_{=O(\|\tilde{x}_N\|^3)=O(\|x\|^3)}}_{\leq -c\|x\|^2 \text{ in a nbh } W \subseteq V} \checkmark$$

It is easy to show that $\tilde{V}_N$ is lower and upper bounded by coercive quadratic functions (see appendix), so that $\tilde{V}_N$ is a Lyapunov function in W . ✓

□

3.2 Constraints violation guarantees

4 Numerical extensions

4.1 Real Time Iteration methods

4.1.1 Asymptotic stability of RTI scheme

4.1.2 Numerical experiments

4.2 Parallelization

4.2.1 Numerical experiments

5 Appendix

Proof of lemma 3.1. □

Lemma 5.1 (Existence an Unicity of solution to the Lyapunov equation).

Proof. □

Lemma 5.2 (Coercive quadratic lower and upper bounds for objective value function).