Study of Relaxed Recentered Log-Barrier function based Nonlinear Model Predictive Control
(RRLB NMPC)

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Problem formulation - Regular NMPC

$$
\begin{aligned}
V_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad & \sum_{k=0}^{N-1} l(x_k, u_k) \; + \; F(x_N) \\
\text{s.t.} \quad & x_0 = x \text{ and } x_{k+1} = f(x_k, u_k) \\
& x_k \in \mathcal{X} \\
& u_k \in \mathcal{U}
\end{aligned}
$$

with $l(x, u) = x^T Q x + u^T R u$ and $F(x) = x^T P x$ (P determined later).
$\mathcal{X} = \{x \mid C_x x \leq d_x\}$ and $\mathcal{U} = \{u \mid C_u u \leq d_u\}$.

Problem formulation - RRLB NMPC

$$\tilde{V}_N(x) = \min_{\mathbf{x},\mathbf{u}} \quad \sum_{k=0}^{N-1} \tilde{l}(x_k, u_k) \; + \; \tilde{F}(x_N)$$

$$\text{s.t.} \quad x_0 = x \text{ and } x_{k+1} = f(x_k, u_k)$$

with $\tilde{l}(x, u) = l(x, u) + \epsilon B_x(x) + \epsilon B_u(u)$ (see next slide for RRLBs) and $\tilde{F}(x) = x^T P x$ (P determined later)

Our goal

Stabilize the system at $x^* = 0$ and $u^* = 0$.

Problem formulation - RRLB functions

For $\mathcal{X} = \{x \mid C_x x \leq d_x\}$ we define

$$B_x(x) = \sum_{i=1}^{q_x} (1 + w_{x,i}) B_{x,i}(x)$$

$$\text{with } B_{x,i}(x) = \begin{cases} \log(d_{x,i}) - \log(d_{x,i} - \mathrm{row}_i(C_x)x) \\ \quad \text{if } d_{x,i} - \mathrm{row}_i(C_x)x > \delta \\ \beta(d_{x,i} - \mathrm{row}_i(C_x)x; \delta) \\ \quad \text{otherwise} \end{cases}$$

$$\beta(z;\delta) = \frac{1}{2}\left[\left(\frac{z - 2\delta}{\delta}\right)^2 - 1\right] - \log(\delta)$$

Theoretical results - Nominal stability

Theorem (Nominal stability)

If we suppose:

- *If we denote the objective function as* $J(x, \mathbf{u})$*, we have* $D_{\mathbf{u}} J(0, \mathbf{u}(0)) = D_{\mathbf{u}} J(0, 0) = 0$ *and* $\nabla^2_{\mathbf{uu}} J(0, \mathbf{u}(0)) = \nabla^2_{\mathbf{uu}} J(0, 0) \succ 0$.
- *When* $A = D_x f(0, 0)$, $B = D_u f(0, 0)$*, we suppose that* (A, B) *is stabilizable* ($\implies \exists K$ *such that* $A_K := A + BK$ *Hurwitz*)
- P *solution to* $P = A_K^T P A_K + \mu Q_K$ *where* $\mu > 1$ *and* $Q_K = Q + \epsilon M_x + K^T (R + \epsilon M_u) K$.

then 0 is asymptotically stable for all initial state in a nbh of 0.

Theoretical results - Nominal stability

Sketch of proof.

By our definitions, we have $\tilde{l}(x, Kx) = x^T Q_K x + O(\|x\|^3)$ and $\tilde{F}(A_K x) + \mu x^T Q_K x - \tilde{F}(x) = 0,\ \forall x \in \mathbb{R}^{n_x}$. Now by the proof in the book, we have locally

$$\tilde{F}(f(x, Kx)) + x^T Q_K x - \tilde{F}(x) \leq 0 \iff \tilde{F}(f(x, Kx)) - \tilde{F}(x) + \tilde{l}(x, Kx) = O(\|x\|^3)$$

and now from optimal solutions $\tilde{\mathbf{x}} = \{\tilde{x}_0, \dots, \tilde{x}_N\}$ and $\tilde{\mathbf{u}} = \{\tilde{u}_0, \dots, \tilde{u}_{N-1}\}$ we construct feasible solutions $\mathbf{x}' = (\tilde{x}_1, \dots, \tilde{x}_N, f(\tilde{x}_N, K\tilde{x}_N))$ and $\mathbf{u}' = (\tilde{u}_1, \dots, \tilde{u}_{N-1}, K\tilde{x}_N)$ and we have

$$\tilde{V}_N(\tilde{x}_1) \leq \tilde{J}_N(\mathbf{x}', \mathbf{u}') = \underbrace{\tilde{J}_N(\tilde{\mathbf{x}}, \tilde{\mathbf{u}})}_{=\tilde{V}_N(x)} \underbrace{\underbrace{-\tilde{l}(x, \tilde{u}_0)}_{=O(\|x\|^2)} + \underbrace{\tilde{F}(f(\tilde{x}_N, K\tilde{x}_N)) - \tilde{F}(\tilde{x}_N) + \tilde{l}(\tilde{x}_N, K\tilde{x}_N)}_{=O(\|\tilde{x}_N\|^3)=O(\|x\|^3)}}_{\leq -c\|x\|^2 \text{ in a smaller nbh}}$$

$\square$

Theoretical results - Constraint satisfaction guarantees

Theorem (Constraint satisfaction guarantees)

Let $\{x(k)\}_{k\geq 0}$ *and* $\{u(k)\}_{k\geq 0}$ *be the closed-loop trajectories of the system controlled by the RRLB MPC law. Under the assumptions of last theorem, there exists a nbh of 0 such that if* $x(0)$ *is in it, there is no constraint violation along the closed-loop trajectory.*

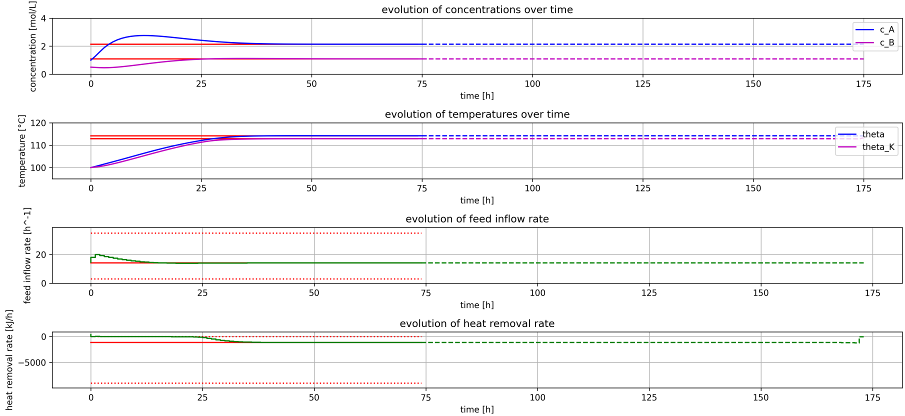
Numerical experiments - CSTR

Setting:

- Discretization via RK4 and multi-shooting
- NLP solved with IPOPT through CasADi
- Horizon size 100
- max 1 iteration for RRLB MPC (equivalent to RTI) and 10 for regular MPC
- ✓all the assumptions of the nominal stability theorem are satisfied

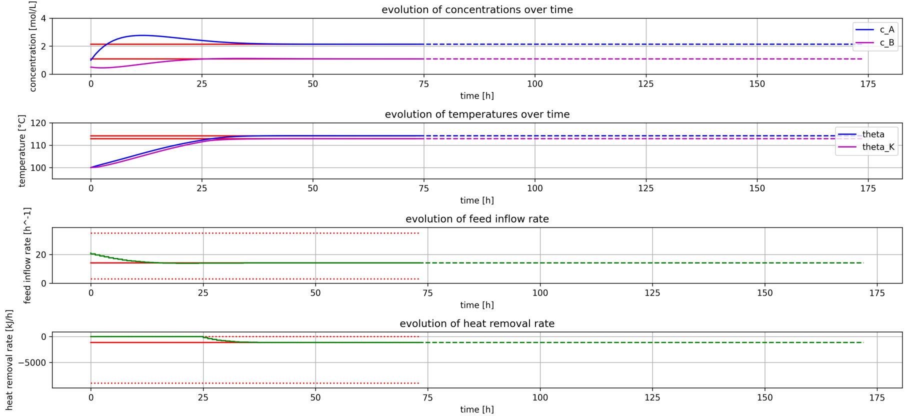
Results: Both converged after 74 time steps with similar total costs along the closed-loop trajectory.

Numerical experiments - CSTR



Closed-loop trajectories of RRLB MPC

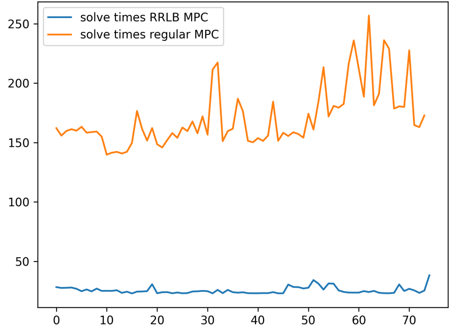
Numerical experiments - CSTR



Closed-loop trajectories of MPC

Numerical experiments - CSTR

Here are the solve times in ms :



Solve times for both schemes

	mean	stdev
RRLB MPC	25.538	2.863
MPC	170.896	25.234